Project Report

Quantum Gates, Superposition, and Entanglement: A Qiskit Implementation of Bell State

Introduction to Quantum Gates: Building Basic Quantum Circuits
A project report submitted by

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1 Project Overview

Introduction

Quantum computation is based on the quantum bit, or qubit, which generalizes the classical bit. While a classical bit can only take the values 0 or 1, a qubit can exist in a superposition of the computational basis states $|0\rangle$ and $|1\rangle$. Mathematically, the state of a single qubit is written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where α and β are complex amplitudes satisfying $|\alpha|^2 + |\beta|^2 = 1$. [1] This linear combination captures one of the key differences between classical and quantum information.

Quantum gates are physical or logical operations that transform qubit states through unitary evolution. Single-qubit gates, such as the Pauli-X and Hadamard gates, act on individual qubits and can create superposition or flip the logical value of the state. [1] Two-qubit gates, such as the controlled-NOT (CNOT), couple qubits and enable the creation of entangled states, in which the joint state cannot be written as a simple product of single-qubit states.

Quantum circuits provide an abstract model for applying these gates in sequence. A circuit specifies the number of qubits, the gates applied to each qubit, and the final measurements that map quantum states to classical outcomes. By designing and simulating such circuits, it becomes possible to visualize superposition, bit flipping, and entanglement in a controlled setting, which is the central goal of this project.

Objective

This project focuses on designing simple quantum circuits using basic quantum gates (Hadamard, CNOT, and Pauli-X) and observing their behavior, with emphasis on superposition and entanglement. The circuits are implemented and simulated in Python, and the resulting outputs are summarized and analyzed.

Problem Statement

The problem addressed in this project is the difficulty of visualizing and interpreting the behavior of qubits under basic quantum gates. Superposition and entanglement are fundamental yet abstract concepts, and the project aims to clarify them through Python-based circuit simulation.

2 Background and Literature Review

Background

In classical computation, information is encoded in bits that take definite values 0 or 1 at any given time. Quantum computation extends this notion by introducing qubits, which are represented as normalized vectors in a two-dimensional complex Hilbert space. [1] The computational basis states $|0\rangle$ and $|1\rangle$ form an orthonormal basis for this space, and arbitrary qubit states are expressed as superpositions of these basis states. Measurement in the computational basis projects the qubit state onto either $|0\rangle$ or $|1\rangle$ with probabilities given by the squared magnitudes of the corresponding amplitudes.

Quantum gates are modeled as unitary operators acting on one or more qubits. For a single qubit, the Pauli-X gate acts as a quantum analogue of the classical NOT gate, exchanging the amplitudes of $|0\rangle$ and $|1\rangle$. [1] The Hadamard gate is particularly important because it maps a computational basis state into an equal superposition, for example

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

Geometrically, this action can be interpreted on the Bloch sphere as rotating the state from the north pole to a point on the equator, illustrating how gates correspond to rotations and reflections of qubit states.

Multi-qubit systems are described by tensor products of single-qubit Hilbert spaces. For two qubits, the joint state lives in a four-dimensional complex vector space spanned by the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. [1] The controlled-NOT (CNOT) gate is a two-qubit gate that flips the target qubit when the control qubit is in the state $|1\rangle$. When a Hadamard gate is applied to the control qubit before the CNOT, the resulting two-qubit state can become entangled.

Entanglement arises when a composite quantum state cannot be written as a tensor product of single-qubit states. Historically, Einstein, Podolsky, and Rosen used such correlations to argue that the quantum-mechanical description of reality might be incomplete, emphasizing the tension between quantum theory and classical notions of locality and realism. [2] Shortly afterwards, Schrödinger identified entanglement as the characteristic feature of quantum mechanics and discussed how interactions can cause initially independent systems to become entangled in a way that cannot be undone without losing information. [3] Bell states are canonical examples of maximally entangled two-qubit states and play a central role in quantum communication and quantum information theory.

Literature Review

The theoretical foundations of this project draw on both early conceptual work in quantum mechanics and modern treatments of quantum computation. Einstein, Podolsky, and Rosen formulated a famous argument questioning whether quantum mechanics provides a complete description of physical reality. [2] They considered pairs of correlated particles and showed that, under certain assumptions about locality and realism, quantum predictions lead to paradoxical conclusions. This discussion motivated the search for a deeper understanding of non-classical correlations.

Schrödinger responded by analyzing the behavior of two systems that interact and later separate. [3] He introduced the term *entanglement* for the resulting joint states and emphasized that once systems become entangled, their individual descriptions are no longer sufficient. He also pointed out that disentangling such systems generally requires a measurement process that irreversibly reduces information about the total system. This perspective highlights why entangled states, such as the Bell states studied in this project, are fundamentally different from classical correlated states.

The modern circuit model and the formal description of qubits, quantum gates, and quantum algorithms are presented in detail by Nielsen and Chuang. [1] Their treatment provides the mathematical language used in this project, including the representation of single-qubit states, unitary gates, and tensor-product spaces, as well as explicit examples of the Pauli-X, Hadamard, and CNOT gates and the construction of Bell states. The methodology adopted here, designing quantum circuits, simulating them on classical hardware, and analyzing measurement statistics. Follows the standard approach outlined in their textbook and adapted to a Qiskit-based implementation.

3 Project Methodology

Tools and Technologies

This project was developed using a set of software tools and libraries that support quantum circuit design, simulation, and visualization.

- **Python 3.11** – The primary programming language used for implementing the quantum circuits and executing the simulations.
- **Qiskit** – An open-source quantum computing framework used to build quantum circuits, apply quantum gates, simulate quantum behavior, and extract statevectors before measurement.

- **Qiskit Aer Simulator** – A high-performance backend used to run quantum circuits on a classical machine with support for multiple shots and statevector analysis.
- **Jupyter Notebook** – The development environment used to write, execute, and document all Python code in an interactive format.
- **Matplotlib** – A visualization library used to generate histograms and plots that illustrate measurement results and quantum state behavior.
- **NumPy** – Utilized for numerical operations when handling statevectors and probability distributions.

Methods

The project follows an experimental approach to understanding basic quantum behaviors such as superposition and entanglement within the standard circuit model of quantum computation. The methodology is divided into several structured steps:

1. **Define the Quantum Experiments**
   Three independent experiments were selected to illustrate key quantum concepts:
   - Superposition using the Hadamard gate.
   - Bit-flip behavior using the Pauli-X gate.
   - Entanglement using the Bell-state circuit (Hadamard + CNOT).
2. **Construct Quantum Circuits**
   Each experiment begins with the creation of a quantum circuit using Qiskit. The circuits specify:
   - the number of qubits and classical bits,
   - the quantum gates applied to each qubit,
   - measurement operations at the end of the circuit.
3. **Simulate Circuit Execution**
   The Aer simulator backend is used to execute the circuits with a predefined number of shots. This step captures the measurement results and provides statistical distributions of the outcomes.
4. **Analyze Statevectors Before Measurement**
   For experiments involving superposition and entanglement, the statevector is extracted before performing measurements. This allows examination of the underlying quantum state without collapsing it.

5. **Visualize and Interpret Results**

   Measurement results are displayed using histograms, and state behavior is illustrated using Bloch-sphere plots or printed statevectors. The visualizations help relate theoretical concepts to observable outcomes.

6. **Theoretical Separability Test for the Bell State**

   Before running the entanglement experiment, a theoretical analysis is performed to check whether the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

can be written as a separable product state of two single-qubit states, following the formalism of tensor-product spaces used in quantum information theory. [1] Motivated by foundational discussions on nonlocal correlations and entanglement by Einstein, Podolsky, and Rosen, and by Schrödinger, [2,3] we assume that

$$|\Phi^+\rangle = |\psi\rangle \otimes |\phi\rangle$$

where

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\phi\rangle = \gamma|0\rangle + \delta|1\rangle,$$

with complex amplitudes $\alpha, \beta, \gamma, \delta$ satisfying the normalization conditions $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$.

We then expand the tensor product explicitly:

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) \\ &= \alpha\gamma\,|00\rangle + \alpha\delta\,|01\rangle + \beta\gamma\,|10\rangle + \beta\delta\,|11\rangle. \end{aligned}$$

On the other hand, the Bell state $|\Phi^+\rangle$ has coefficients

$$\frac{1}{\sqrt{2}}|00\rangle \;+\; 0\cdot|01\rangle \;+\; 0\cdot|10\rangle \;+\; \frac{1}{\sqrt{2}}|11\rangle.$$

By matching the amplitudes of the corresponding basis states, we obtain the system of equations:

$$\begin{aligned}
\alpha\gamma &= \frac{1}{\sqrt{2}}, \\
\alpha\delta &= 0, \\
\beta\gamma &= 0, \\
\beta\delta &= \frac{1}{\sqrt{2}}.
\end{aligned}$$

We now solve these equations step by step:

- From $\alpha\delta = 0$, either $\alpha = 0$ or $\delta = 0$.
- From $\beta\gamma = 0$, either $\beta = 0$ or $\gamma = 0$.
- From $\alpha\gamma = \frac{1}{\sqrt{2}}$, both $\alpha \neq 0$ and $\gamma \neq 0$.
- From $\beta\delta = \frac{1}{\sqrt{2}}$, both $\beta \neq 0$ and $\delta \neq 0$.

However, this leads to a contradiction:

- Since $\alpha \neq 0$ and $\delta \neq 0$, the condition $\alpha\delta = 0$ cannot be satisfied.
- Since $\beta \neq 0$ and $\gamma \neq 0$, the condition $\beta\gamma = 0$ cannot be satisfied.

In all cases, the equations cannot be satisfied simultaneously. Therefore, there are no single-qubit states $|\psi\rangle$ and $|\phi\rangle$ such that

$$|\Phi^+\rangle = |\psi\rangle \otimes |\phi\rangle.$$

This proves that the Bell state $|\Phi^+\rangle$ is not separable and is therefore entangled from a purely theoretical point of view.

7. **Compare Results to Quantum Theory**
   After establishing the theoretical expectations, the simulation outputs of each experiment are compared with the predicted behaviors:

   - Hadamard gate → equal superposition of `|0>` and `|1>`.
   - Pauli-X gate → deterministic flip from `|0>` to `|1>`.
   - Bell-state circuit → correlated outputs `00` and `11`, consistent with the entangled Bell state derived in the theoretical separability test.

4 Project Implementation

System Design

The system is structured as a sequence of modular experiments that demonstrate the behavior of simple quantum circuits. The design follows a linear workflow consisting of four main components: circuit construction, simulation, visualization, and analysis.

1. **Circuit Construction**
   Each experiment begins by creating a quantum circuit with a specified number of qubits and classical bits. Basic quantum gates such as the Hadamard, Pauli-X, and CNOT are applied to prepare the desired quantum states.

2. **Simulation Engine**
   A Python-based quantum simulator is used to execute the circuits. The simulator runs each experiment multiple times (shots) to generate measurement statistics and extracts the statevector when required.

3. **Visualization Module**
   The system generates circuit diagrams, measurement histograms, and Bloch-sphere representations. These visual outputs allow clear interpretation of quantum behavior, including superposition and entanglement.

4. **Analysis Layer**
   The final stage evaluates the simulation results. This includes checking probability distributions, verifying gate behavior, and confirming the presence of entanglement in two-qubit circuits.

Implementation Details

The implementation consists of three primary experiments, each written in Python and executed in a notebook environment.

1. **Experiment 1: Superposition Using the Hadamard Gate**
   A single-qubit circuit is created with one classical bit. The Hadamard gate is applied to transform the initial state into a superposition, and the qubit is then measured. The statevector and Bloch-sphere visualizations are obtained before measurement, while the histogram of results confirms an equal probability of outcomes 0 and 1.

2. **Experiment 2: Bit Flip Using the Pauli-X Gate**
   A second single-qubit circuit demonstrates the effect of the Pauli-X gate. The qubit, initialized in the state `|0>`, is flipped to `|1>` using the X gate. Repeated simulation shows a consistent measurement outcome of 1, validating the behavior of the gate.

3. **Experiment 3: Bell-State Generation Using H and CNOT**
   A two-qubit circuit is constructed to generate an entangled Bell state. A Hadamard gate creates superposition on the first qubit, and a CNOT gate entangles it with the second. Measurement results show only the outcomes `00` and `11`, while the statevector confirms equal amplitudes for these states, demonstrating quantum entanglement.

5 Results and Analysis

Results

This section presents the outcomes of the quantum circuit simulations using the Qiskit Aer simulator. The results are shown in the form of plots, statevectors, and measurement distributions that correspond to the experiments described in the Methods section.

A. Single-Qubit Superposition (Hadamard Gate)

The Hadamard gate was applied to an initial $|0\rangle$ state. The pre-measurement statevector was visualized using a Bloch sphere, and the plot confirms that the resulting state lies exactly on the positive x-axis, corresponding to the $|+\rangle$ state.

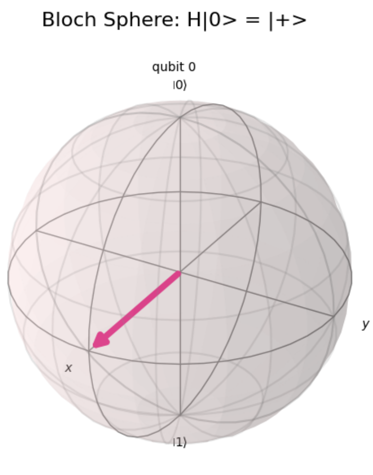

Figure 1: Bloch sphere representation of the state $H|0\rangle = |+\rangle$.

Before measuring the qubit, the experiment requires constructing the circuit shown below. The circuit consists of a single qubit initialized in $|0\rangle$, followed by a Hadamard gate, and finally a measurement mapped into the classical register.

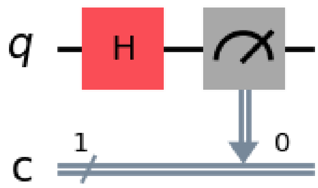

Figure 2: Quantum circuit used to generate the $|+\rangle$ superposition state.

The measurement histogram obtained from 4096 shots shows an approximately equal distribution between the outcomes `0` and `1`, indicating that the qubit was placed into the expected superposition state:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

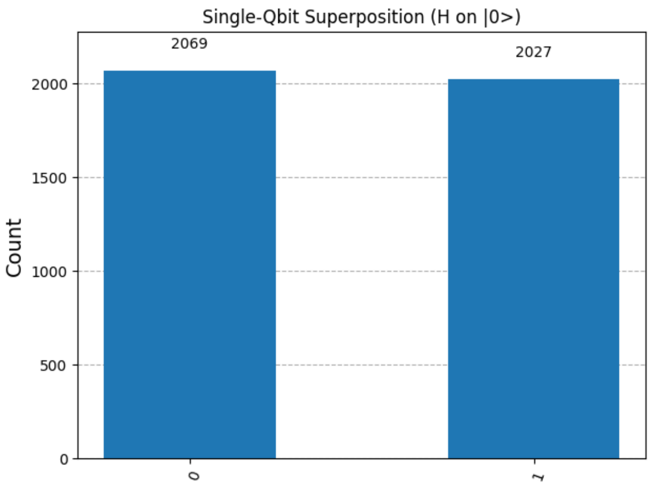

Figure 3: Measurement histogram for the Hadamard superposition experiment.

B. Single-Qubit Bit Flip (Pauli-X Gate)

The second experiment examines the Pauli-X gate, which performs a quantum bit flip analogous to the classical NOT operation. The qubit begins in the state $|0\rangle$, and applying the Pauli-X gate flips it deterministically to $|1\rangle$.

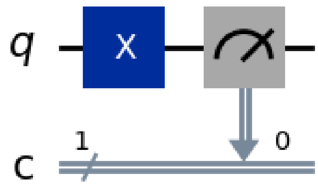

Figure 4: Quantum circuit demonstrating the Pauli-X bit-flip operation.

Given that the Pauli-X gate satisfies

$$X|0\rangle = |1\rangle,$$

the measurement results are expected to be entirely `1`. Executing the circuit for 4096 shots confirms this, as shown in the histogram below.

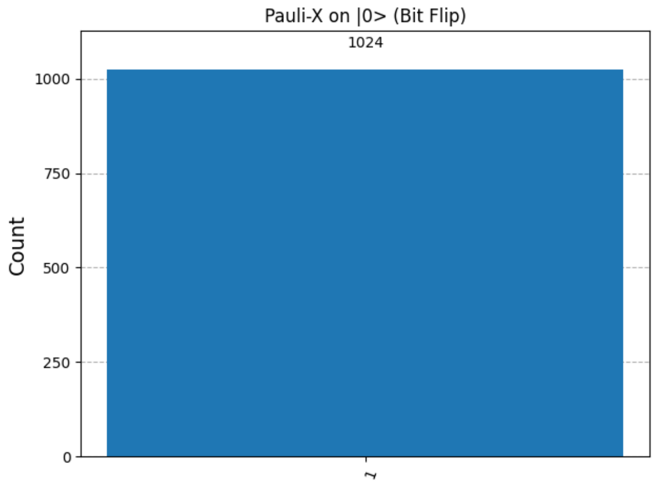

Figure 5: Measurement histogram for the Pauli-X experiment. All shots result in the state `1`.

C. Bell-State Circuit (Hadamard + CNOT)

To generate the Bell state $|\Phi^+\rangle$, a two-qubit circuit was constructed. A Hadamard gate is first applied to the initial qubit to create a superposition, and a CNOT gate then entangles it with the second qubit. The full circuit used in the experiment is shown below.

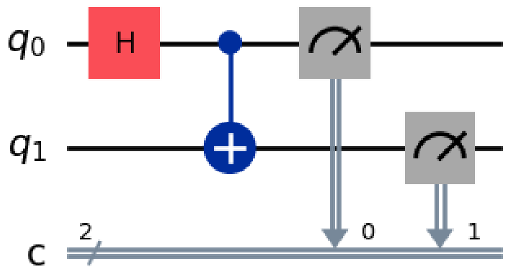

Figure 6: Quantum circuit used to generate the Bell state $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

Since both qubits are initialized in the state $|00\rangle$, the theoretical state of the circuit prior to measurement can be derived directly from the action of the gates. Applying a Hadamard gate to the first qubit creates a superposition, and the subsequent CNOT gate entangles it with the second qubit. This sequence deterministically prepares the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

which is the expected entangled superposition produced by this configuration.

The circuit was executed using 4096 shots. The resulting histogram reveals that only the outcomes `00` and `11` appear, each with nearly equal probability, while `01` and `10` never occur. This strong correlation is a signature of entanglement.

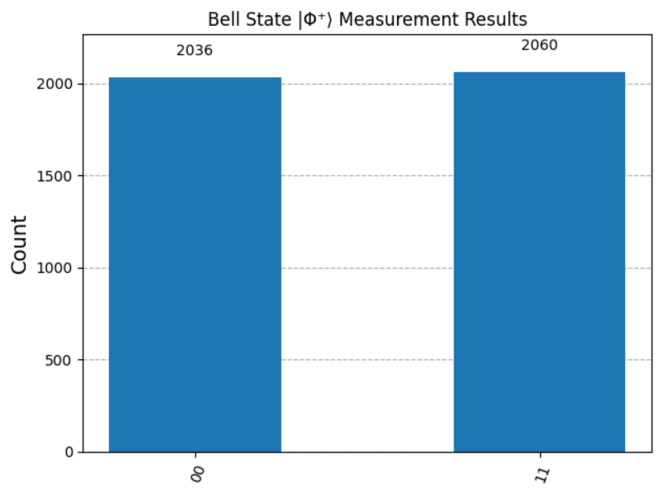

Figure 7: Measurement results for the Bell-state experiment. Only `00` and `11` occur, confirming entanglement.

Analysis

The experimental results align closely with the theoretical expectations established in the Methods section.

A. Superposition Behavior

The Hadamard experiment demonstrated a balanced distribution of measurement outcomes between `0` and `1`, confirming that the qubit was successfully placed into an equal superposition. The Bloch-sphere representation further validates this by showing the state located exactly on the x-axis, which corresponds to the ideal $|+\rangle$ state.

B. Pauli-X Bit Flip Behavior

The Pauli-X experiment exhibited fully deterministic behavior. All measurement shots resulted in the outcome `1`, confirming that the operator acts as expected on the basis state $|0\rangle$. This verifies correct implementation of the bit-flip operation in the simulated environment.

C. Verification of Bell-State Entanglement

The Bell-state experiment provides strong evidence of entanglement. The histogram reveals that measurements collapse exclusively to `00` or `11`, which is the defining characteristic of the $|\Phi^+\rangle$ Bell state. These correlated outcomes indicate that the two qubits do not behave independently.

This directly supports the theoretical separability test performed in the Methods section. The system of equations derived from assuming that the Bell state can be factored into a product state leads to contradictions. The simulation results validate this conclusion: the absence of `01` and `10` outcomes confirms that the Bell state cannot be expressed as $|\psi\rangle \otimes |\phi\rangle$.

D. Alignment with Objectives

The experiments successfully demonstrate:

- the creation of superposition using the Hadamard gate,
- deterministic bit flipping using the Pauli-X gate,
- and entanglement generation through the Bell-state circuit.

Overall, the results clearly meet the project's objectives and experimentally verify the core behaviors of simple quantum gates.

6 Challenges Encountered

A key challenge in this project was ensuring an accurate interpretation of entanglement. Demonstrating that the Bell state is not separable required a complete theoretical derivation, careful manipulation of the algebraic expressions, and verification that no choice of single-qubit states could reproduce the Bell state. Aligning this mathematical proof with the simulated measurement outcomes also required attention, since the results had to show the same correlations predicted by theory. Overcoming this challenge involved performing a detailed separability analysis and confirming its conclusions through the strong measurement correlations observed in the Bell-state experiment. This process strengthened the overall understanding of both the theoretical and practical aspects of entanglement.

7 Conclusion

This project demonstrated three fundamental quantum behaviors: superposition, bit flipping, and entanglement, through the construction and simulation of simple quantum circuits. The experiments confirmed that the Hadamard gate creates an equal superposition, the Pauli-X gate performs a deterministic bit flip, and the combination of Hadamard and CNOT gates generates the entangled Bell state $|\Phi^+\rangle$.

A theoretical separability test was carried out to show that the Bell state cannot be written as a product of two single-qubit states. This analysis established the presence of entanglement from first principles. The simulation results further validated this conclusion by producing perfectly correlated measurement outcomes.

By combining theoretical reasoning with practical simulation, the project provided a clear and structured investigation into the behavior of basic quantum gates and their role in generating key quantum phenomena. These results contribute to a foundational understanding of quantum computation and serve as a basis for exploring more advanced quantum algorithms and multi-qubit systems in future work.

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