Dynamical Analysis of the Nonlinear Chaotic Cusp Map

Feras Natsheh
PHYS-301: Complex Dynamics & Chaos
04/25/2024

This study explores the complex dynamics of the cusp map, defined by $x_{n+1} = 1 - r\sqrt{|x_n|}$, where $0 \leq r \leq 1$, via a thorough computational analysis. My objectives include generating the stability bifurcation diagram to study its dynamics, computing the Lyapunov exponent to verify these results, and exploring complexity quantifiers such as Shannon Entropy and Fisher Information Measure, relative to the control parameter. I aim to outline the various dynamical properties inherent to the cusp map and discuss their meanings.Through this analysis, I highlight the unique characteristics of the cusp map, providing insights into its behavior and the role of the control parameter in shaping its dynamics.

Introduction

Chaos theory is a branch of mathematics and physics focusing on the behavior of dynamical systems that are highly sensitive to initial conditions. It has has been a subject of profound interest since the pioneering work of Lorenz in the 1960s. This sensitivity, often referred to as the "butterfly effect," implies that small changes in the initial conditions of a system can lead to vastly different outcomes, rendering long-term predictions that are "impossible" in practical terms. Among the various models studied in chaos theory, one-dimensional discrete maps provide a framework for exploring chaotic dynamics due to their simple conceptual baseline and the complex behaviors they can exhibit.

The cusp map is one of the lesser-studied one-dimensional nonlinear dynamical systems. It is defined by the equation

$$x_{n+1} = 1 - r\sqrt{|x_n|} \tag{1}$$

where r is the control parameter influencing the system's evolution. The cusp map showcases a variety of dynamical behaviors from stable fixed points to chaotic regimes depending on the control parameter. Unlike the widely studied logistic or sine maps, the cusp map's absolute value inside the square root introduces a nonlinearity that is not commonly addressed in more traditional studies of logistic or tent maps, and as non-invertible map, contributes to a more complex bifurcation structure and chaos.[1]

This study aims to explore the cusp map by analyzing the impact of its control parameter on the system's dynamics, using both analytical and numerical methods. Through this exploration,

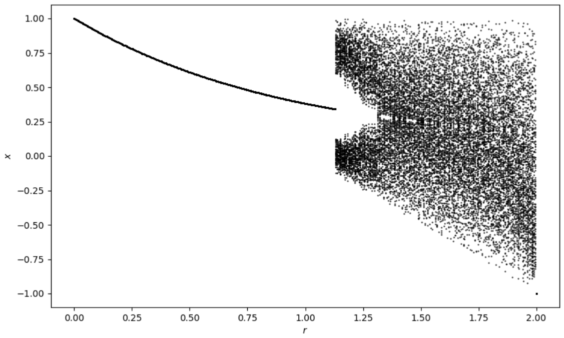

Figure 1: *Bifurcation Diagram for the cusp map. Initial condition is $x_0 = 0$*

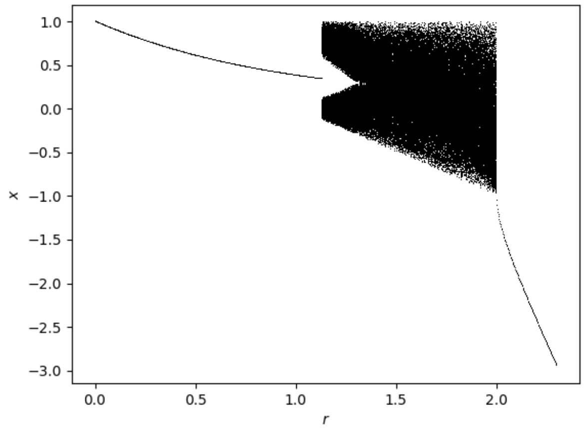

Figure 2: *Full range of the Bifurcation diagram*

I hope to elucidate the conditions under which the cusp map exhibits chaotic behavior, through generating its stability bifurcation diagram, and computing the Lyapunov exponent and other complexity quantifiers such as permutation entropy, Fisher Information Measure, and Shannon entropy, as functions of the control parameter. Finally, I aim to discuss the different dynamical regimes present in the system, thereby contributing to the broader discourse on stability analysis in nonlinear systems.

Stability & Bifurcations

As illustrated in Figure 1, when r is small, we see a single line which suggests the system settles into a stable fixed point. As r increases from zero, this line moves downwards, indicating that the fixed point's position changes smoothly with r. As r increases further, the line representing the stable fixed point splits into bifurcations. This suggests that the system transitions from a single stable state to a state where two different outcomes are possible depending on the initial conditions.

Upon further observation, we are able to notice features of this map that raise our concern. First, the bifurcations start past the range we set for r. The first bifurcation starts around $r \approx 1.1$, which falls outside of $0 \leq r \leq 1$. This challenges our understanding of equation (1). It also does not match the cusp map bifurcation diagram given in Fig 3(b) in Spichak, Kupetsky & Aragoneses, which starts at $(0, 0)$ and starts the bifurcations a little before $r = 0.5$. The issue we see here is that the lowest value we can have for $|x_{n+1}|$ in eq (1) is 1 when x_n or r is zero. Therefore, eq (1) cannot physically have the point $(0, 0)$ mapped onto its bifrucation diagram.

By examining the map a little further, we notice, as shown by Figure 2, that the r range for which this form of the cusp map exhibits bifurcations is approximately $1.125 \leq r \leq 2$, while the initial value of x does not seem to have any visible impact of the behavior of the chaotic map.

Nevertheless, we can still pick out some of the map's characteristics and compare it to the well-studied logistic map. The cusp maps does not develop the same route to chaos that the logistic map does; there is no period-doubling, and no clear distinctions in most of their bifurcation map. [1] The logistic map typically has a smooth, predictable pattern of period-doubling bifurcations leading to chaos. The cusp map seems to have a more abrupt and less predictable transition.

Complexity Quantifiers

Lyapunov Exponent

The Lyapunov exponent (λ) serves as a measure of the average rate at which nearby trajectories in a dynamical system converge or diverge in phase space. A positive Lyapunov exponent is indicative of chaotic behavior. Mathematically, it is defined as:

$$\lambda = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=0}^{n-1} ln|f'(x_i)|\right) \tag{2}$$

The derivative of the cusp map is

$$f'(x) = \frac{-r}{2\sqrt{|x|}} \tag{3}$$

Using python, I developed an algorithm (see Appendix A) that iterates the Lyapunov exponent a 100,000 times for the initial conditions $r = 1.5$ and $x_0 = 0.1$. The resulting value for the exponent is $\lambda = 0.4815$ which is indicative of chaos since it's greater than zero.

Shannon Entropy

Shannon entropy (S) is a statistical measure of randomness that quantifies the uncertainty associated with a random variable. In the context of chaotic systems, Shannon entropy measures the disorder and unpredictability of the system's states. It is defined as:

$$S = -\frac{\sum_{i=1}^{N} p_i ln(p_i)}{ln(N)} \tag{4}$$

where $0 \leq S \leq 1$, p_i is the probability distribution, and N is the number of parts the system has been divided into. Generating a probability distribution for $N = 300$ bins and running an algorithm that computes the normalized Shannon entropy, I get a value of $S = 0.972$ which is very close to 1. The closer the S value to 1, the more complex and unpredictable the system is.

Fisher Information

Fisher information measures the amount of information that an observable random variable carries about an unknown parameter upon which the probability depends. In dynamical systems, it quantifies the sensitivity of the system's state to changes in its parameters. Higher Fisher information suggests a system with order and structure, as opposed to randomness. Mathematically, it is defined as:

$$FIM = F_0 \sum_{i=1}^{N-1} ((p_{i+1})^{1/2} - (p_i)^{1/2})^2 \tag{5}$$

where $F_0 = 1$ for the normalized version (if p_i adds up to 1). In this case, I am using the same probabilities I defined for the Shannon entropy, which are normalized. Thus, the value for F_0 is 1. Computing the FIM, we get the value of 0.0018 which is a low value, suggesting that the system is unordered and unstructured–random.

Knife Map

An interesting finding that I came across during my research is the knife map [2] variation of the cusp map, which takes the form

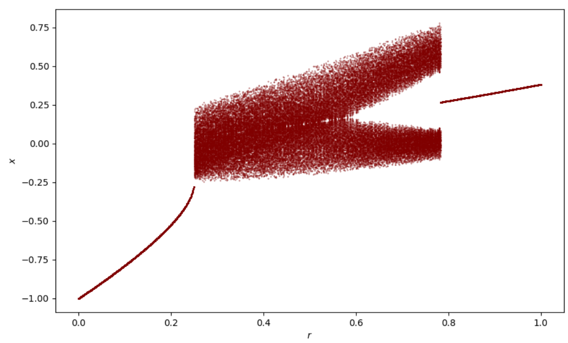

Figure 3: *Bifurcation Diagram of the Knife Map*

$$x_{n+1} = r - \sqrt{|x_n|} \tag{6}$$

This form of the cusp map solves the issues outlined in the Stability & Bifurcations section, as illustrated by Figure 3. We can see that the r values are confined to the range we hypothesized earlier, $0 \leq r \leq 1$, and that it is possible for the point $(0, 0)$ to be mapped on the bifurcation plane (if both r and x_n are zero).

Following the analysis conducted on the original cusp map, through some quick computations, I was able to obtain values for the Lyapunov exponent, the normalized Shannon entropy, and the Fisher Information Measure. The values are as follows:

$\lambda = 0.0760$

$S = 0.00028$

FIM= 1.987

Those complexity quantifiers are not as strongly indicative of chaos as the original cusp map. Although they do hint at a chaotic behavior, their values are still mild in comparison.

Conclusion

This study has presented a comprehensive analysis of the nonlinear dynamics of the cusp map. Through computational methods and the generation of various complexity quantifiers, we have explored the complex behavior of this dynamical system under the influence of the control parameter r. The results revealed that the given cusp map exhibits bifurcations in the range $1.125 \leq r \leq 2$ instead of the expected $0 \leq r \leq 1$. The Lyapunov exponent, Shannon entropy, and Fisher Information Measure provided strong indications of chaotic behavior, confirming the theoretical predictions associated with the map.

A different form of the cusp map, namely the Knife map, shows the chaotic behavior that we expect in the region $0 \leq r \leq 1$, comprised of bifurcations. Both the cusp and the knife map illustrate their unique chaotic nature which differs from more commonly studied models like the logistic map. The main differences are that there are no bifurcation period-doubling, and no clear distinction between the bifurcations. Future work could focus on expanding the analysis of the cusp map in both of its forms. One of the challenges faced in this paper is the lack of published research surrounding the cusp map and its bifurcations.

References

[1] Spichak, D., Kupetsky, A., Aragoneses, A. (2021). Characterizing complexity of non-invertible chaotic maps in the Shannon–Fisher information plane with ordinal patterns. Chaos, Solitons Fractals/Chaos, Solitons and Fractals, 142, 110492. https://doi.org/10.1016/j.chaos.2020.110492

[2] Gilmore, R. (2014, June 3). A tale of two maps - physics - drexel university. https://www.physics.drexel.edu bob/ Presentations/LeHavre$_draft.pdf$

Appendix A

```
def cusp_map(x, r):
    return 1 - r * np.sqrt((np.abs(x)))

r_values = np.linspace(0, 2.2, 1000)
iterations = 1000
last = 100

x_values = []
r_points = []

for r in r_values:
    x = 1
    for i in range(iterations):
        x = cusp_map(x, r)
        if i >= (iterations - last):
            x_values.append(x)
            r_points.append(r)

plt.figure(figsize=(10, 6))
plt.scatter(r_points, x_values, s=0.1)
plt.xlabel('$r$')
plt.savefig('full range bifrucation')
plt.ylabel('$x$')
plt.show()

#####################

def cusp_map(x, r):
    return r - np.sqrt((np.abs(x)))

r_values = np.linspace(0, 1, 1000)
iterations = 1000
last = 100

x_values = []
r_points = []

for r in r_values:
    x = 0.1
    for i in range(iterations):
        x = cusp_map(x, r)
        if i >= (iterations - last):
            x_values.append(x)
            r_points.append(r)

plt.figure(figsize=(10, 6))
plt.scatter(r_points, x_values, s=0.1)
plt.xlabel('$r$')
plt.ylabel('$x$')
#plt.xlim(1,0)
plt.savefig('knifemap')
plt.show()


######################

n_bins = 300

x_values = [x0]
x = x0
for i in range(n_iter - 1):
    x = cusp_map(x, r)
    x_values.append(x)

hist, bin_edges =
np.histogram(x_values, bins=n_bins, density=True)

probabilities = hist * np.diff(bin_edges)
```

```
shannon_entropy = -np.sum(probabilities *
    np.log(probabilities,
        where=(probabilities > 0)))

normalized_shannon_entropy =
shannon_entropy / np.log(n_bins)
print(normalized_shannon_entropy)
```

```
print(normalized_shannon_entropy)

F_0 = 1
FIM = F_0 * np.sum((np.sqrt(probabilities[1:])
    - np.sqrt(probabilities[:-1]))**2)
print(FIM)
```

```
######################

F_0 = 1
FIM = F_0 * np.sum((np.sqrt(probabilities[1:])
    - np.sqrt(probabilities[:-1]))**2)
print(FIM)

#####################

### KNIFE MAP ###
def knife_map(x, r):
    return r - np.sqrt((np.abs(x)))
### Lyapunov exponent ###
r = 1.5
x0 = 0.1
n_iter = 100000

lyapunov_sum = 0
x = x0

for i in range(n_iter):
    derivative = 1 / (2 * np.sqrt(np.abs(x)))
    if derivative != 0:
        lyapunov_sum += np.log(np.abs(derivative))
    x = 1 - r * np.sqrt(np.abs(x))

lyapunov_exponent = lyapunov_sum / n_iter
print(lyapunov_exponent)


n_bins = int(np.sqrt(n_iter))

x_values = [x0]
x = x0
for i in range(n_iter - 1):
    x = knife_map(x, r)
    x_values.append(x)

hist, bin_edges = np.histogram(x_values, bins=n_bins, density=True)

probabilities = hist * np.diff(bin_edges)

shannon_entropy = -np.sum(probabilities * np.log(probabilities, where=(probabilities > 0)))

normalized_shannon_entropy = shannon_entropy / np.log(n_bins)
```