# Chaos in The Duffing Oscillator Experiment

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**Abstract**

This study investigates the transition from regular to chaotic behavior in the duffing oscillator, a mechanical system composed of an oscillator coupled to two driven, elastic springs, with damping introduced via a magnetic field. We utilized Pasco Capstone for data acquisition and analysis, and manipulated the damping and frequency of the system to probe the boundary between regular and chaotic dynamics. Our experiment clearly demonstrated a threshold beyond which the system transitions from periodic oscillations to chaotic behavior. We computed various complexity quantifiers to characterize the chaotic dynamics of the system, and identified the underlying parameters and conditions conducive to chaos in the Duffing oscillator.

# 1 Introduction & Theoretical Framework

## 1.1 Theory

Chaos can be found in nature and in human-made systems. Among the various models that exhibit chaotic dynamics, the Duffing oscillator stands out due to its rich dynamical behaviors and practical relevance in fields such as mechanical engineering, seismology, and electronics.

The Duffing oscillator, a non-linear second-order differential equation, is defined by its characteristic equation:

$$\ddot{x} + \gamma \dot{x} + \beta x^3 + \alpha x = A \cos \omega t + \phi \tag{1}$$

where
$\ddot{x}$: is the second time derivative of $x$, which represents the acceleration
$\gamma$: is the damping coefficient, representing the resistance that opposes the motion of the oscillator, and $\dot{x}$ is the velocity term
$\beta$: is the non-linear stiffness coefficient, and $x^3$ is the nonlinear stiffness term
$\alpha$: is the linear stiffness coefficient, with $x$ being the linear stiffness term, which indicates the restoring force that is proportional to the displacement
$A$: is the amplitude of the periodic forcing function
$\omega$: is the driving angular frequency of the motor
$\phi$: is the phase of the forcing [1]

The nonlinear stiffness term, $\beta x^3$, is responsible for the diverse dynamics characteristic of the Duffing oscillator. Nonlinearity implies that the system's response is not directly proportional to the input, which can lead to multiple equilibrium points and the possibility of sudden jumps and bifurcations between different states of motion. These nonlinear effects can result in the coexistence of stable and unstable periodic orbits, which is, as we've already learned in class, a key feature of chaotic systems.

The damping term, $\gamma\dot{x}$, represents the rate at which energy is dissipated from the system. The balance between energy input from the driving force and energy loss due to damping determines whether the oscillator settles into a steady-state motion, exhibits sustained oscillations, or descends into chaos.

The term $A\cos\omega t + \phi$ models the time-dependent driving force. Here, $A$ dictates the strength of the forcing, while $\omega$ and $\phi$ determine its frequency and phase, respectively. As the driving frequency approaches the natural frequency of the oscillator, resonance can occur, leading to large amplitude oscillations. However, when the driving frequency or amplitude is varied, the system may transition through a series of bifurcations.

## 1.2 Complexity Quantifiers [2]

### 1.2.1 Lyapunov Exponent

The Lyapunov exponent ($\lambda$) serves as a measure of the average rate at which nearby trajectories in a dynamical system converge or diverge in phase space. A positive Lyapunov exponent is indicative of chaotic behavior.

Mathematically, it is defined as:

$$\lambda = \lim_{n\to\infty} (\frac{1}{n} \sum_{i=0}^{n-1} ln|f'(x_i)|) \tag{2}$$

We implemented an algorithm using Python to compute the Lyapunov exponent for the raw csv data files that we acquired. After reading the data and breaking it into components by column, we implemented the following algorithm using Python:

```
1. Initiate lambda = 0
2. Compute the derivative numerically (using numpy)
3. if derivative is > 0, compute lambda using eq. 2
```

#### 1.2.2 Shannon Entropy

Shannon entropy ($S$) is a statistical measure of randomness that quantifies the uncertainty associated with a random variable. In the context of chaotic systems, Shannon entropy measures the disorder and unpredictability of the system's states. It is defined as:

$$S = -\frac{\sum_{i=1}^{N} p_i ln(p_i)}{ln(N)} \tag{3}$$

where $0 \leq S \leq 1$.

#### 1.2.3 Permutation Entropy

Permutation entropy ($PE$) provides a measure of complexity in a time series by examining the ordering patterns among values. It converts the time series into a sequence of ordinal patterns. The diversity of these patterns, and their frequency distribution, are used to calculate the permutation entropy.

$$PE = -\sum_{i=1}^{D!} \frac{P_i ln(P_i)}{ln(D!)} \tag{4}$$

#### 1.2.4 Fisher Information

Fisher information measures the amount of information that an observable random variable carries about an unknown parameter upon which the probability depends. In dynamical systems, it quantifies the sensitivity of the system's state to changes in its parameters. Higher Fisher information suggests a system with significant order and structure, as opposed to randomness. Mathematically, it is defined as:

$$FIM = F_0 \sum_{i=1}^{N-1} ((p_{i+1})^{1/2} - (p_i)^{1/2})^2 \tag{5}$$

#### 1.2.5 Stability

Stability analysis in dynamical systems is important for understanding the long-term behavior of a system under small disturbances. It is categorized into local and global stability, with the former concerning the system's response near equilibrium points and the latter addressing the overall system behavior.

Local stability in the context of the Duffing oscillator, which is inherently nonlinear, can be characterized by linearizing the system around the equilibria to approximate the local behavior. An equilibrium point is considered stable if all perturbations decay over time, leading the system to return to equilibrium. An equilibrium point is considered unstable if any perturbation grows over time, causing the system to move away from equilibrium.

Global stability examines the system's ability to return to a particular state or behavior from a wide range of initial conditions. In chaotic systems, trajectories may diverge globally, but they can be confined to a bounded region of phase space known as an attractor. The nature of these attractors—whether they are point attractors, limit cycles, or strange attractors — provides insights into the system's global stability.

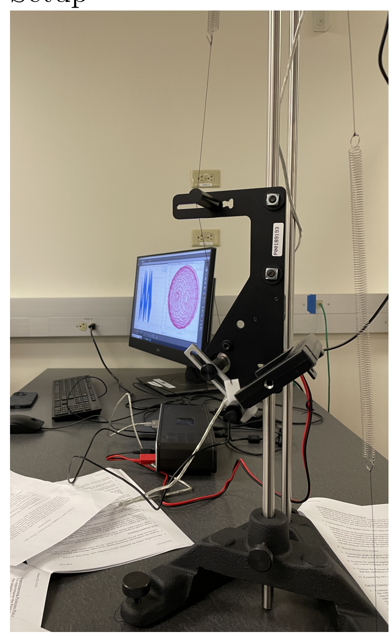
# 2 Experiment & Methodology

## 2.1 Experimental Setup

### Apparatus

1. The Duffing Oscillator
2. PASCO Capstone software
3. Driving motor
4. String angle and length adjuster

Figure 1: Experimental Setup



The experimental apparatus consisted of a rigid vertical frame from which a mechanical oscillator was suspended. The oscillator itself was a horizontal arm connected to two elastic springs, allowing for both linear and nonlinear motion. The springs were anchored to the frame at points equidistant from the central pivot of the arm. Additionally, a magnetic damping mechanism was positioned above the oscillator to introduce a controlled damping effect, essential for observing the system's transition to chaotic motion.

## 2.2 Computational Setup

The PASCO Capstone software was set to collect data. We collected plots of time series and phase space for each of the runs. We collected a total of 12 runs. The first 4 were test runs where we calibrated our measurements. The 8 remaining runs were split into two categories. The fist 4 runs we varied the frequency, $\omega$, while holding everything else, including $\gamma$, constant. The remaining 4 runs we varied the damping coefficient, $\gamma$, holding $\omega$ and all other parameters constant.

## 2.3 Procedure

On Day 1, we started by adjusting various parameters within the experiment in order to isolate which variables would be best in observing a transition from regular to chaotic behavior. We determined that individually adjusting the damping produced by the magnet on the disk, as well as the frequency of the driving force from the electric motor, would yield the best results.

On Day 2, we collected data while independently adjusting the position of the magnet and the frequency of the motor. We ran each trial for around 5 minutes, at a 20Hz polling rate, collecting data for a time series ($\theta$ vs t), and phase space ($\dot{\theta}$ vs $\theta$). Then, we stopped the motor and adjusted either parameter, cleared the data on PASCO, and resumed collecting with the new parameters.

# 3 Results & Analysis

## 3.1 Parameters

**Magnetic Damping Adjustment:** We adjusted the proximity of the magnet to the oscillating disk to vary the magnetic damping force. Starting with minimal damping, we incrementally moved the magnet closer to the disk, observing the effect on the oscillator's amplitude and frequency response. Each position was held constant for a series of cycles to ensure accuracy in the readings.

**Driving Force Frequency Adjustment:** The frequency of the electric motor driving the system was altered using the software controls. We started at a low frequency and gradually increased it, noting changes in the system's dynamics.

## 3.2 Quantitative Analysis

This section presents the results of all numerical and quantitative findings of our experiments. We present two cases, where we vary one parameter and hold the rest constant. For each case, we present the numerical values for the Lyapunov exponent, Shannon entropy, Permutation entropy, and Fisher Information Measure, and we indicate the meaning of their respective values.

### 3.2.1 Varying $\omega$

From the 8 datasets we're working with, 4 of them examine the angular frequency parameter $\omega$. By holding all other parameters constant and only varying $\omega$, we can explore whether the angular frequency has any impact on the system's chaotic behavior.

Each of the 4 datasets has a different $\omega$ value, sorted in an ascending $\omega$ value order.

**Case 1:** $\omega = 0.25$ Hz. The numerical values of each complexity quantifier is as follows:

Lyapunov exponent: 0.2886

Shannon entropy: 0.9552

Permutation entropy: 0.5604

Fisher information: 1.1976

We can see that the Lyapunov exponent is greater than zero, which indicates chaotic behavior. A normalized Shannon entropy of 0.96 which is very close to 1, indicates a high degree of disorder or unpredictability in the system. A permutation entropy value of 0.6 indicates that the system exhibits a moderate level of complexity, which on its own is ambiguous and not indicative of chaotic behavior, but coupled with the other quantifiers, we can make better judgements. Finally, a Fisher Information Measure of 1.2 is fairly indicative of disorder. So, based on the quantitative analysis we have just conducted, we conclude that this case is chaotic.

**Case 2:** $\omega = 1.25$ Hz. The numerical values for each of the complexity quantifiers is as follows:

Lyapunov exponent: 0.9470

Shannon entropy: 0.9702

Permutation entropy: 0.5870

Fisher information: 1.1666

Given that the $\lambda$ value is much greater than zero, the $S$ value is very close to 1, the $PE$ and $FIM$ values are moderate, we conclude that, based on the quantitative analysis, this case exhibits levels of disorder and chaos.

**Case 3:** $\omega = 2.0$ Hz. The numerical values for each of the complexity quantifiers is as follows:

Lyapunov exponent: $-1.2312$

Shannon entropy: 0.8758

Permutation entropy: 0.6661

Fisher information: 2.4533

The $\lambda$ value is negative, which is not a sign of chaos. The $S$ value is close to 1, which is an indication of disorder. The $PE$ value is moderate, which does suggest chaos more than order, and the $FIM$ value is higher, which is a sign of predictability and order. Given the contradicting values of our quantifiers, we suspect that this is the transition phase from regular to chaotic.

**Case 4:** $\omega = 3.0$ Hz. The numerical values for each of the complexity quantifiers is as follows:

Lyapunov exponent: $-1.8133$

Shannon entropy: 0.4871

Permutation entropy: 0.7278

Fisher information: 2.9993

Given that the $\lambda$ value is much less than zero, the $S$ value is moderate and not indicative of chaos, the $PE$ value is on the chaotic end of the spectrum, and the $FIM$ value is indicative of order and predictability, we conclude that this case exhibits regular, non-chaotic behavior.

### 3.2.2 Varying $\gamma$

From the 8 datasets we're working with, 4 of them examine the dampening parameter $\gamma$. By holding all other parameters constant and only varying $\gamma$, we can explore whether the damping, or energy dissipated, has any impact on the system's chaotic behavior. Each of the 4 datasets has a different $\gamma$ value, sorted in an ascending $\gamma$ value order. Note that we did not record exact measurements of the distance of the magnet from the disc, instead estimating a minimum, maximum, and medium distance(s).

**Case 1:** $\gamma$ = Close. This is the highest level of damping. The numerical values of each complexity quantifier is as follows:

Lyapunov exponent: $-0.0783$

Shannon entropy: 0.9312

Permutation entropy: 0.5326

Fisher information: 1.9021

The Lyapunov exponent is negative, indicating that the system is likely stable and not sensitive to initial conditions. A negative Lyapunov exponent generally signifies that perturbations or deviations in the system's initial conditions decay exponentially over time. For Shannon Entropy, a value close to 1 suggests a high level of disorder or unpredictability within the system, but not maximal. This indicates a degree of complexity and unpredictability in the system's behavior. Permutation entropy is closer to 0.5, which indicates a moderate level of complexity in the time series. The FIM is 1.9, which is slightly on the higher side, considering this value of FIM is higher than the other 3 cases. This suggests that the system's outputs can be informed based off of its starting parameters, meaning the output has sensitivity to the initial conditions.

**Case 2:** $\gamma$ = Medium-Close. (In between the distances *Medium* and *Close*. The numerical values of each complexity quantifier is as follows:

Lyapunov exponent: 0.0266

Shannon entropy: 0.9404

Permutation entropy: 0.5519

Fisher information: 1.5441

Here, the lyapunov exponent is positive, which indicates that the system, with less dampening, has shifted to a more sensitive state, towards chaos. Shannon entropy is close to 1 here as well, which suggests a high level of disorder in the system. There is a slight increase in comparison to Case 1, which could be indicative of a marginal rise in complexity. Similarly, Permutation entropy has risen marginally, which could also indicate a complexity of the time series. However the FIM has decreased, which suggests that the output is less informative of the internal states of the system than when the magnet was close to the disc. This also means that the lower level of dampening could make ths system less sensitive to parameter changes that can be observed in the output.

**Case 3:** $\gamma$ = Medium. The numerical values of each complexity quantifier is as follows:

Lyapunov exponent: 0.0037

Shannon entropy: 0.9334

Permutation entropy: 0.5520

Fisher information: 1.6301

As the dampening in the system decreases further, there appears to be a trend towards stabilization, where Chaotic behavior is becoming less extreme, and the output slightly more informative (as the FIM increased marginally). However, in comparison with Case 2, there is a slight reduction in Shannon Entropy, pointing to a decrease overall unpredictability.

**Case 4:** $\gamma$ = Far. This is the lowest amount of damping (Magnet is furthest as possible from disc). The numerical values of each complexity quantifier is as follows:

Lyapunov exponent: $-0.0065$

Shannon entropy: 0.9368

Permutation entropy: 0.5640

Fisher information: 1.6041

Here, the Lyapunov exponent has shifted back to a negative value, indicating a move from chaotic to a more stable regime. However, Shannon Entropy is slightly greater, which suggests that while the system is becoming more stable with its sensitivity to initial conditions, the overall complexity has still increased. This is similarly shown in the marginal rise in permutation entropy, showing a rise in the complexity of the time series. The FIM has decreased which points to a reduction in the clarity with which outputs reflect the system's states.

### 3.3 Qualitative Analysis

Our qualitative analysis started with the visual inspection of the oscillator's motion as parameters were varied. Through these observations, the sensitivity of the system to initial conditions was noted—slightly different initial states could lead to dramatically different trajectories over time, even within the same parameter set.

#### 3.3.1 Varying $\omega$

**Case 1:** $\omega = 0.25$ Hz.

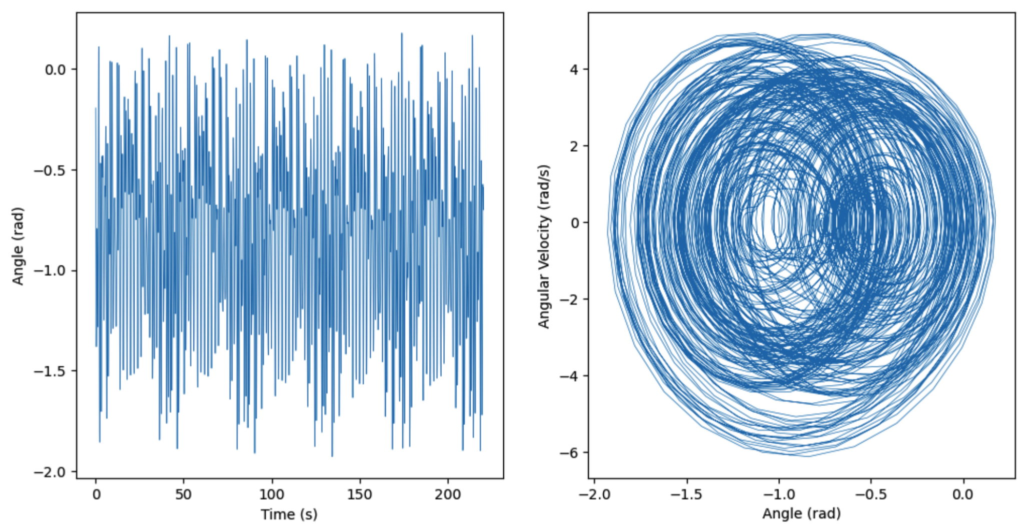



Figure 2: $\omega = 0.25$ Hz

We can see that at a driving frequency of 0.25 Hz, the system presents a time series with an aperiodic behavior, lacking any clear or repeating pattern over the interval observed. The system exhibits irregular fluctuations of varying amplitude, a behavior that suggests an underlying complexity. Furthermore, the phase space plot reinforces this hypothesis. In a non-chaotic system, we would expect to see closed loops representing periodic motion, or a fixed point indicating a steady state. However, the trajectory in phase space is neither closed nor fixed but fills a bounded region in a seemingly disordered manner. The presence of this phase space plot is reminiscent of the famous strange attractor, or "butterfly effect" phase space diagrams.

**Case 2:** $\omega = 1.25$ Hz. The system exhibits the following time series and phase space plots

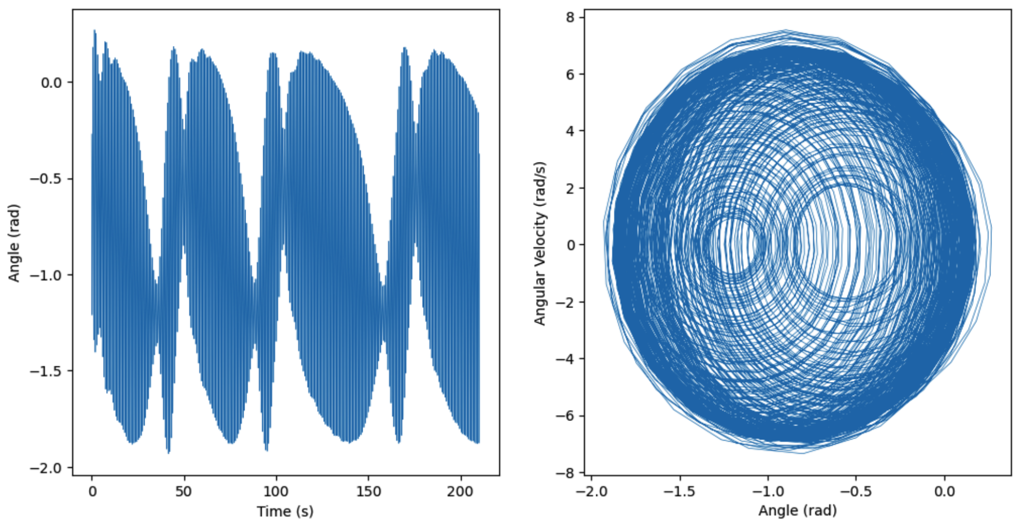



Figure 3: $\omega = 1.25$ Hz

For this case, we can see a quasi-periodic behavior, with the time series exhibiting a somewhat periodic behavior, and the phase space polarizing into one big loop. This observation typically indicates a resonant behavior associated with predictable and orderly dynamics. However, the quantitative analysis for this case indicates otherwise: despite the apparent regularity, the system exhibits chaotic and complex behavior.

**Case 3:** $\omega = 2.0$ Hz. The system exhibits the following time series and phase space plots

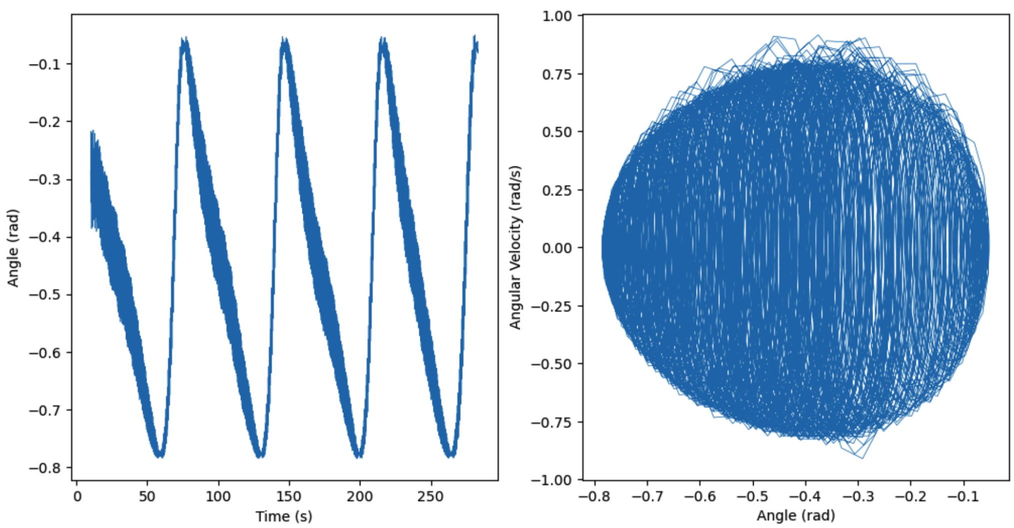



Figure 4: $\omega = 2.0$ Hz

We can see that the time series shows a pattern of oscillations with a consistent amplitude over time, which might suggest a periodic behavior. The phase space plot, however, presents a more complex structure, showing a dense, cloud-like formation of points without clear, distinct loops or a single converging point, which could imply a more complicated dynamical state. Given the contradicting quantitative analysis too, we further hypothesize that this case is in a transition state between regular to chaotic.

**Case 4:** $\omega = 3.0$ Hz. The system exhibits the following time series and phase space plots

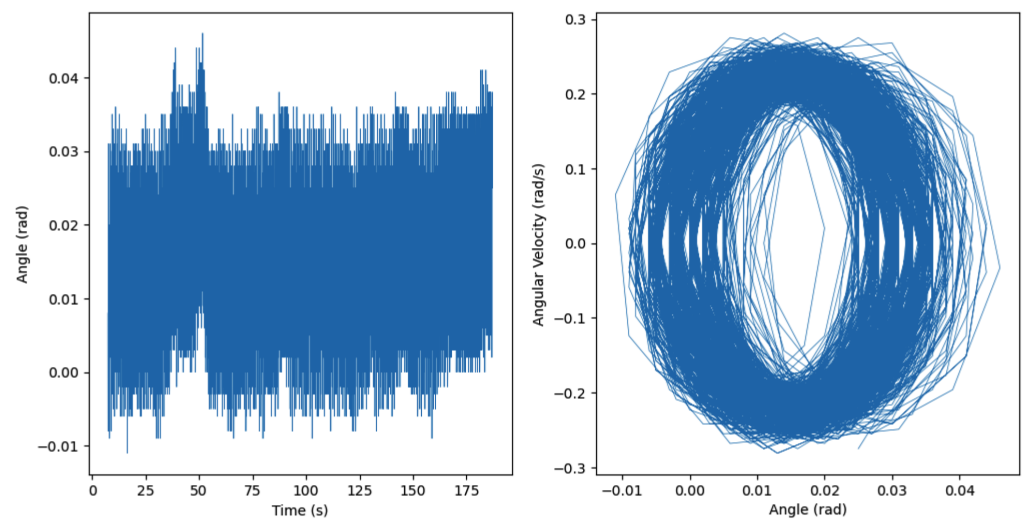



Figure 5: $\omega = 3.0$ Hz

For this case, the time series data exhibit a noisy, fluctuating pattern without a clear periodic structure. The lack of repeating peaks and troughs may suggest potentially chaotic behavior. However, the irregularity alone does not confirm chaos; it could also result from measurement noise. The phase space plot does not show the thin, stretched, and folded structures that are typically indicative of a strange attractor in a low-dimensional chaotic system. Instead, the phase space is filled in a manner that could be consistent with noise-driven dynamics. In line with our quantitative analysis, we hypothesize that this system is non-chaotic, and the disorder we observe is due to measurement noise or other sources of error.

### 3.3.2 Varying $\gamma$

**Case 1:** $\gamma =$ High. The system exhibits the following time series and phase space plots

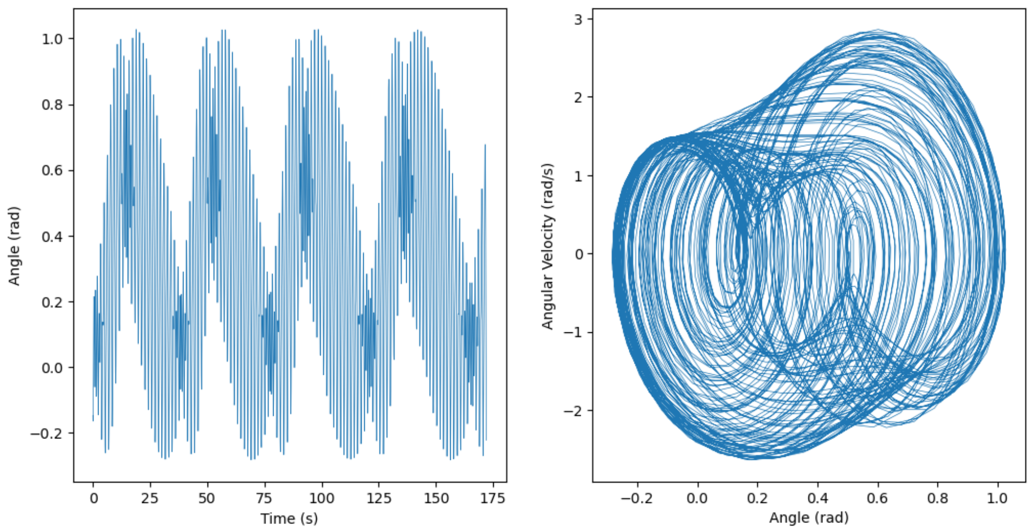



Figure 6: $\gamma =$ High

We can see that this figure is the odd one out in comparison to the 3 other cases of varying $\gamma$. The amplitude of the oscillations appears to decay over time, but as the time series progresses, the amplitude does not continue to decay to zero but rather maintains a certain fluctuating level. This behavior could be the result of a balance between the energy supplied by the driving force and the energy dissipated due to damping. The phase space plot presents a complex trajectory that covers a significant area but seems to be confined within a bounded region. This alone might suggest a chaotic attractor due to its familiar structure; however, the high damping effect typically suppresses chaotic dynamics by smoothing out the trajectories and preventing the system from exploring all possible states within the attractor.

**Case 2:** $\gamma$ = Medium-High. The system exhibits the following time series and phase space plots

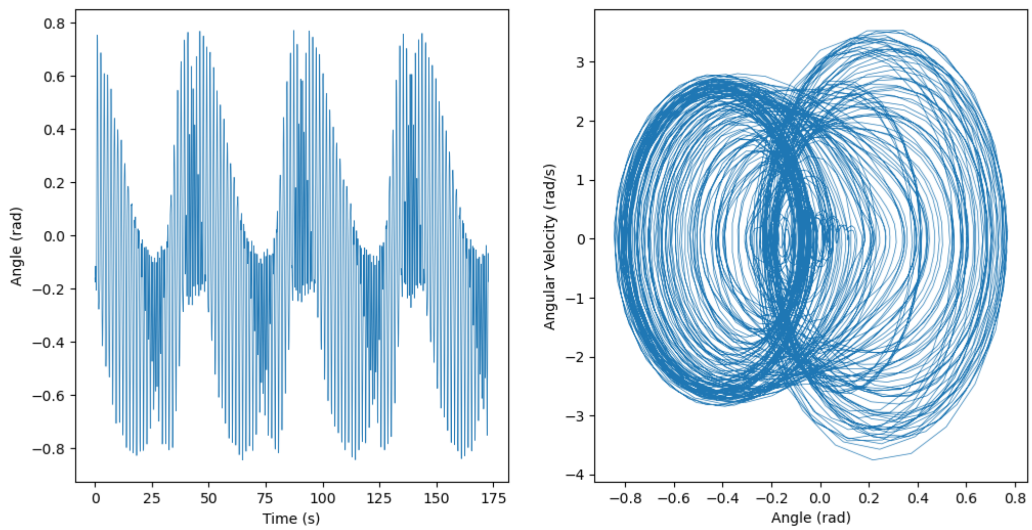



Figure 7: $\gamma$ = Medium-High

Here, the time series exhibits a distinct pattern of peaks and troughs with regular intervals, but with noticeable variations in amplitude that do not decay over time. The phase space shows a dense structure that is more complex than a simple periodic one but lacks the sensitive dependence on initial conditions often seen in chaotic attractors. The loops appear to be well-defined and organized, and seem to explore the phase space over time. In line with our quantitative analysis, this case seems to be the one exhiting the most chaos in the $\gamma$ cases.

**Case 3:** $\gamma$ = Medium. The system exhibits the following time series and phase space plots

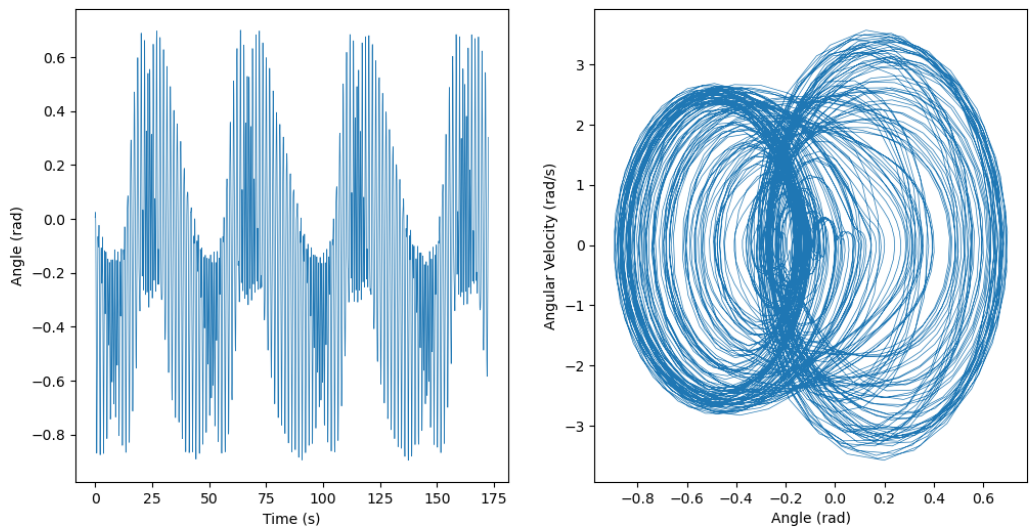



Figure 8: $\gamma$ = Medium

Similar to Figure 7, this case also presents a quasi-periodic behavior, which might suggest a transition phase from regular to chaotic.

**Case 4:** $\gamma$ = Low. The system exhibits the following time series and phase space plots

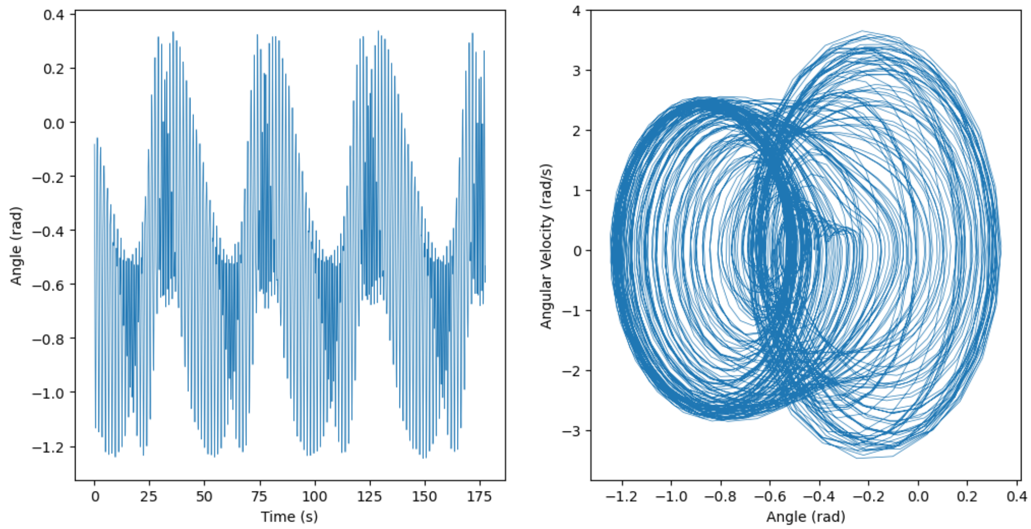



Figure 9: $\gamma$ = Low

Similarly, the time series shows repeated patterns with fluctuations in amplitude. Such clear oscillations could indicate a periodic or quasi-periodic system, but the variability in amplitude could also be a sign of disorder in the system's response. The phase space plot does not settle into a single loop or point, but instead shows strange-attractor-like behavior. This case could be classified as either chaotic or a quasi-periodic transition phase.

# 4 Conclusion & Discussion

## 4.1 Regular vs Chaotic Behavior

In the cases where we varied $\omega$, the inconsistency between the qualitative appearances and the complexity quantifiers hints at the possibility that the system in such cases is in a quasi-periodic state, and could be part of a more extensive chaotic attractor or a transitional behavior near a bifurcation point. For example, in case 3 where $\omega = 2.0$ Hz, the phase space shows a complex diagram, but nevertheless, it is not indicative of chaos. Because in cases of chaos, we expect our phase space diagram to exhibit a behavior similar to the strange attractor or butterfly effect, where the system explores new paths in the phase space (or phase volume in a 3D plot) without crossing or repeating a previously explored path. This is not apparent in this case, where we see the phase space sphere denser in regions and less dense in other regions, which might suggest that the system is neither chaotic nor regular, but somewhere in between. Furthermore, we see a lot of noise around the sphere, which we believe is due to measurement noise and other sources of error, which we will discuss in the next section

For the $\gamma$ variation cases, we found Case 2, with a magnet distance of Medium-Close, to be the most chaotic, as well as represent the transition from regular to chaotic behavior as damping decreased. A positive Lyapunov exponent indicates chaos, with larger values signifying more chaotic behavior. Case 2 has the largest positive Lyapunov exponent. For Shannon Entropy, higher values indicate more disorder, which case 2 again had the highest Shannon Entropy. For permutation entropy, Case 4 had the highest, however, the differences among the cases are marginal. Lastly, lower values inn the FIM potentially indicate a system with more chaos, as it would be harder to predicate the system's internal state based off of its output. Case 2 had the lowest FIM value. Based off of these numerical values, we can conclude that Case 2, or a Medium-to-High level of damping, represented the transition from regular (in case 1) to chaotic behavior, as well as being the most chaotic state overall when adjusting $\gamma$ values. Since the chaotic behavior diminished as damping further decreased (the magnet moved further away from the disc), this leads us to believe that a drastically high or low level of damping will diminish the potential for chaos in this system.

## 4.2 Sources of Error

The experimental setup presented a few potential sources of error. One of them is the magnet's distance to the disc, which was not quantifiable and presented a challenge in attempting to characterize it. We thought it's best to just indicate whether the magnet was close or far from the disc and what that implies about the damping coefficient, $\gamma$.

Another source of error, as briefly mentioned in section 4.1, is the measurement noise. We believe that some of the irregularities that we've seen in the data is due to perturbations caused by either a machine imprecision or a software inaccuracy. This is because such kinds of irregularities did not fit into the characteristics of the chaotic behavior we expected, but rather hinted at noise-driven chaos. This could also be a mere speculation and misjudgement on our end, because chaos is chaos, and chaos doesn't necessarily have to fit into the "strange attractor's" characteristics of chaos. In this case, we fail to characterize such chaos, due to our lack of knowledge about other way to quantify "unorthodox" chaos.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## References


[1] https://www.cfm.brown.edu/people/dobrush/am34/Mathematica/ch3/duffing.html

[2] PHYS-301 Lecture 10 and Lecture 11 notes