Kalman Filter

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Kalman filter is an estimator which estimates the state of a linear system. It uses measurements that are generated as a linear function of the system state but corrupted by additive Gaussian noise. Kalman filter employs two procedures called time update(prediction) and measurement update(correction). Let x_k be the state of the system at time step k, then state transition at each time step can be written as a linear combination of the previous state, a control signal and noise:

$$x_k = Ax_{k-1} + Bu_k + w_k \tag{1}$$

where A is a matrix called state transition matrix, u_k is a control signal assuming the system is controlled by another system, B is a matrix representing the control model and w_k is the process noise. Measurements from the system can be modelled as a linear transform of the state with additive Gaussian noise:

$$z_k = Hx_k + v_k \tag{2}$$

where z_k is the measurement, H is the measurement model and v_k is the measurement noise. A, B and H matrices of the system are constant and must be known or correctly modelled for Kalman filter to work correctly.

Time Update (Prediction)

Kalman filter makes a prediction about the state of the system in next iteration. The prediction is simply made by using the state transition model:

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k \tag{3}$$

where $\hat{x}_k^-$ is the state prediction or the state prior. Since it is an estimator, Kalman filter has a state estimation covariance matrix denoted as P_k. Prediction step also makes a prediction about what estimation covariance will be at next iteration, in other words it projects ahead the error covariance for one step:

$$P_k^- = AP_{k-1}A^T + Q \tag{4}$$

where Q is the covariance matrix of the process noise w_k.

Measurement Update (Correction)

In correction equations, new measurement about the system is used to update previously predicted state estimate. Firstly, Kalman gain, denoted as K_k is calculated. Kalman gain is a weighting factor that determines how much the new measurement affects the new estimate of the state. Kalman gain is calculated by using the state estimation covariance matrix P_k and measurement covariance matrix R:

$$K_k = P_k^- H^T(HP_k^- H^T + R)^{-1} \tag{5}$$

Note that R is a constant user defined value, so one must correctly define how noisy the measurements would be. State prediction is updated with a value called the measurement prediction error or the residual.

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-) \tag{6}$$

$H\hat{x}_k^-$ is what would the measurement be if the state prediction is correct. Therefore $(z_k - H\hat{x}_k^-)$ is the difference between the measurement z_k and predicted measurement $H\hat{x}_k^-$, so it is the measurement residual. State estimate is corrected by the measurement residual weighted by Kalman gain. Correction step also corrects the state estimation covariance prediction P_k^- using calculated Kalman gain:

$$P_k = (I - K_k H)P_k^- \tag{7}$$

Prediction and correction procedures run at every iteration respectively. Outputs of every iteration are used for inputs to next iteration.