Bending of Bernoulli beams and FEM

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1 Introduction

In this note we discuss how to solve the bending equation for an elastic Bernoulli beam via the finite element method (FEM). In order to keep things simple we do not use Sobolev spaces and work with piecewise differentiable functions instead.

2 Notation and facts from calculus

Basic notation. By $\mathcal{C}(a,b)$ we denote the set of continuous real valued functions on the closed interval $[a,b]\subset\mathbb{R}$. By $f'(x)$ we denote the first derivative at $x\in\mathbb{R}$ of the real valued function f. For notational convenience an integral $\int_a^b f(x)\,dx$ will be written as $\int_a^b f$. Sometimes we write $(fg)(x)$ instead of $f(x)g(x)$.

Piecewise continuity. By $\mathcal{C}(a,b)$ we denote the set of continuous real valued functions on the closed interval $[a,b]\subset\mathbb{R}$. A function $f:[a,b]\to\mathbb{R}$ is called piecewise continuous if there are finitely many points $a=x_1<x_2<\ldots<x_m=b$ and continuous functions $f_i\in\mathcal{C}(x_i,x_{i+1})$, $i=1,\ldots,m-1$ such that $f(x)=f_i(x)$ for $x_i<x<x_{i+1}$. At the points x_i the value $f(x_i)$ may not be defined. The set of piecewise continuous functions on $[a,b]$ is denoted by $\mathcal{C}^p(a,b)$. For $f,g\in\mathcal{C}^p(a,b)$ we write $f=g$, if $f(x)=g(x)$ for all $x\in[a,b]$ where f and g are both continuous.

Piecewise differentiability. A continuous function $f:[a,b]\to\mathbb{R}$ is called piecewise continuously differentiable if there are finitely many points $a=x_1<x_2<\ldots<x_m=b$ and continuous functions $g_i\in\mathcal{C}(x_i,x_{i+1})$, $i=1,\ldots,m-1$ such that $f'(x)=g_i(x)$ for $x_i<x<x_{i+1}$, where $f'(x)$ denotes the derivative of f at x. At the points x_i the derivative $f'(x_i)$ may not be defined. The set of piecewise continuously differentiable functions on $[a,b]$ is denoted by $\mathcal{C}^{1,p}(a,b)$. Hence, $f\in\mathcal{C}^{1,p}(a,b)$ if and only if $f\in\mathcal{C}(a,b)$ and $f'\in\mathcal{C}^p(a,b)$. The set of twice piecewise continuously differentiable functions is defined analogously,

$$\mathcal{C}^{2,p}(a,b) \quad=\quad \{\ f:[a,b]\to\mathbb{R}\mid f,f'\in\mathcal{C}(a,b),\ f''\in\mathcal{C}^p(a,b)\ \}.$$

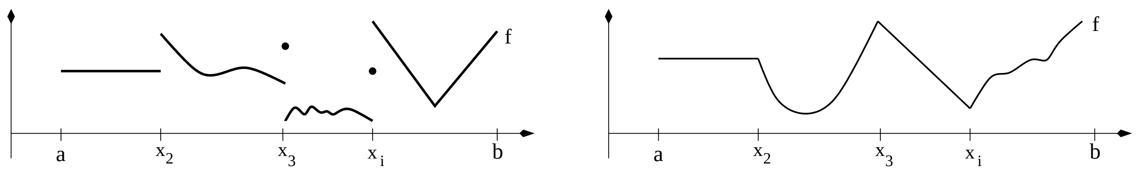


Figure 1: A piecewise cont. function (left) and a piecewise cont. differentiable function (right).

Partial integration. For $f,g\in\mathcal{C}^{1,p}(a,b)$ we have

$$\int_a^b fg' = fg|_a^b - \int_a^b f'g, \qquad \text{where} \quad fg|_a^b = fg(b)-fg(a). \tag{1}$$

Proof. Let $a = x_1 < x_2 < \ldots < x_m = b$ be the points where f or g may not be differentiable. Then

$$\begin{aligned}
\int_a^b fg' + \int_a^b f'g &= \int_a^b (fg' + f'g) = \int_a^b (fg)' = \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} (fg)' = \sum_{i=1}^{m-1} fg|_{x_i}^{x_{i+1}} \\
&= (fg(b) - fg(x_{m-1})) + (fg(x_{m-1}) - fg(x_{m-2})) + \ldots \\
&\quad \ldots + (fg(x_3) - fg(x_2)) + (fg(x_2) - fg(a)) \\
&= fg|_a^b \qquad \square
\end{aligned}$$

Primitives. A continuous function $F : [a, b] \to \mathbb{R}$ is said to be a primitive of $f \in \mathcal{C}^s(a, b)$ if F is differentiable at all x where f is continuous and $F'(x) = f(x)$. Let $f^{[1]} : [a, b] \to \mathbb{R}$ be defined by

$$f^{[1]}(x) = \int_a^x f(t)\, dt, \qquad x \in [a, b].$$

Then by the main therorem of calculus $f^{[1]}$ is a primitive of f. Furthermore, all its primitives of f are of the form $F(x) = f^{[1]}(x) + c$, where $c \in \mathbb{R}$ is an arbitrary constant. Let $f^{[2]} = (f^{[1]})^{[1]}$. Then $(f^{[2]})'(x) = f^{[1]}(x)$ for all $x \in [a, b]$, and $(f^{[2]})''(x) = (x)$ for all $x \in [a, b]$ where f is continuous. We call $f^{[2]}$ a primitive of second order of f. We define primitves of higher order analogously: $f^{[3]} = (f^{[2]})^{[1]}$, $f^{[4]} = (f^{[3]})^{[1]} \ldots$.

Lemma of Du Bois-Reymond [1] for second derivatives. $f, g \in \mathcal{C}^p(a, b)$. Suppose that

$$\int_a^b f\,\psi'' = \int_a^b g\psi \tag{2}$$

for all $\psi \in \mathcal{C}^{2,p}(a, b)$ with $\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0$. Then $f \in \mathcal{C}^{2,p}(a, b)$ and $f'' = g$.

Proof. On applying partial integration twice to the right hand side of (2) we get

$$\int_a^b g\psi = \int_a^b (g^{[1]})'\psi = \underbrace{g^{[1]}\psi|_a^b}_{=0} - \int_a^b g^{[1]}\psi' = -\left(\underbrace{g^{[2]}\psi'|_a^b}_{=0} - \int_a^b g^{[2]}\psi''\right) = \int_a^b g^{[2]}\psi''. \tag{3}$$

This combined with (2) yields

$$\int_a^b (f - g^{[2]})\psi'' = 0 \tag{4}$$

for all $\psi \in \mathcal{C}^{2,p}(a, b)$ with $\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0$. We now define $\psi_0 = f^{[2]} - g^{[4]} - p$, where p is the Hermite interpolation polynomial of degree ≤ 3 satisfying the conditions

$$p(a) = (f^{[2]} - g^{[4]})(a), \quad p'(a) = (f^{[2]} - g^{[4]})'(a), \quad p(b) = (f^{[2]} - g^{[4]})(b), \quad p'(b) = (f^{[2]} - g^{[4]})'(b).$$

Then $\psi_0(a) = \psi_0(b) = \psi_0'(a) = \psi_0'(b) = 0$, and

$$f - g^{[2]} = \psi_0'' + p''. \tag{5}$$

Now, it follows from (4) that

$$\begin{aligned}
0 &= \int_a^b (f - g^{[2]})\psi_0'' = \int_a^b (\psi_0'' + p'')\psi_0'' \\
&= \int_a^b (\psi_0'')^2 + \int_a^b p''\psi_0'' \\
&= \int_a^b (\psi_0'')^2 + \underbrace{\int_a^b p^{(4)}\psi_0}_{=0} \qquad \text{(partial integration, and } p^{(4)} = 0\text{)} \\
&= \int_a^b (\psi_0'')^2
\end{aligned}$$

The latter implies $\psi_0'' = 0$, and hence by (4), $f = p'' + g^{[2]}$. Thus, f is twice differentaible and $f'' = (p'' + g^{[2]})'' = g \quad \square$

[1] David Paul Gustave Du Bois-Reymond (* 2. Dezember 1831 in Berlin; † 7. April 1889 in Freiburg im Breisgau) professor of mathematics at Freiburg, Tübingen and Technische Hochschule Berlin.

3 The static bending equation

In the sequel the set of piecewise twice differentiable functions on the interval $[0, L]$ is denoted by

$$V := \mathcal{C}^{2,p}(0, L) = \{\, \phi : [0, L] \to \mathbb{R} \mid \phi \in \mathcal{C}(0, L),\ \phi' \in \mathcal{C}(0, L),\ \ \phi'' \in \mathcal{C}^p(0, L)\}.$$

Static bending equation:

$$(EIw'')''(x) = q(x), \quad x \in [0, L], \quad \text{(piecewise, i.e. } q \in V,\ (EIw'') \in V) \tag{6}$$

where

$w(x)$	height of the neutral axis (bending curve) at x
$E = E(x)$	Young's module
$I = I(x)$	area moment of inertia: $I(x) = \int_{\text{cross section at } x} z^2 \ \ dy\, dz$
$q(x)$	load (force density) at x

Further notation:

$M^x(w) = EIw''(x)$	bending moment at x
$Q^x(w) = -(EIw'')'(x)$	shear force at x

The general solution of the bending equation is

$$w = \left(\frac{q^{[2]} + p_1}{EI}\right)^{[2]} + p_2,$$

where p_1, p_2 are arbitrary polynomials of degree ≤ 1. If EI is constant then the general solution of the bending equation is

$$w = \frac{q^{[4]}}{EI} + p$$

where p is a polynomial of degree ≤ 3. The general solution has 4 free parameters, the coefficients of p_1 and p_2. In order to specify a particular solution 4 boundary conditions are necessary. For a left sided clamped beam (cantilever, see figure 3) these boundary conditions are

$$w(0) = a, \qquad w'(0) = b, \qquad Q^L(w) = Q_L, \qquad M^L(w) = M_L,$$

where $a, b, Q_L, M_L \in \mathbb{R}$ are given numbers. The first two conditions, concerning $w(0)$ and $w'(0)$, are called essential or Dirichlet- bondary conditions. The remaining two conditions, concerning the shear force $Q^L(w)$ and the bending moment $M^L(w)$, are called physical or natural boundary conditions.

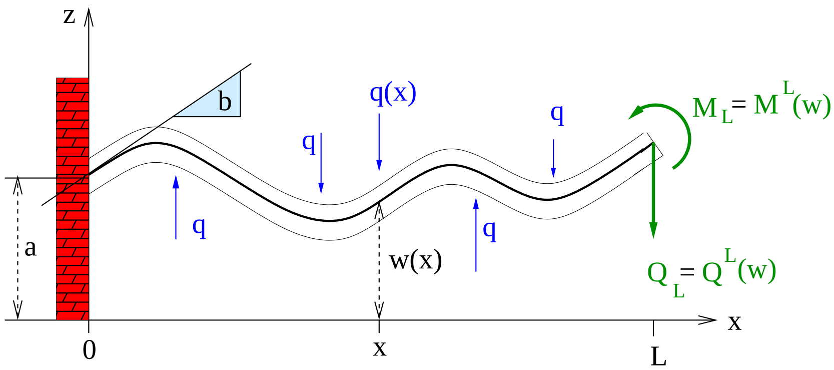


Figure 2: left sided clamped beam

4 Partial integration and the weak formulation of the bending equation

Lemma. If $(EIw'') \in V$ then for all $\psi \in V$,

$$\int_0^L (EIw'')''\psi = \int_0^L EIw''\psi'' - \beta(w,\psi), \tag{7}$$

where

$$\beta(w,\psi) = Q^L(w)\psi(L) - Q^0(w)\psi(0) + M^L(w)\psi'(L) - M^0(w)\psi'(0).$$

Proof. Use partial integration (twice) □

Weak formulation of the bending equation. Let $w \in V$ and $Q_0, Q_L, M_0, M_L \in \mathbb{R}$. Then the following statements are equivalent.

(a) w satisfies the bending equation (6) and the boundary conditions

$$Q^0(w) = Q_0, \quad Q^L(w) = Q_L, \quad M^0(w) = M_0, \quad M^L(w) = M_L. \tag{8}$$

(b) For all $\psi \in V$,

$$\int_0^L EIw''\psi'' = \int_0^L q\psi + b(\psi), \tag{9}$$

where

$$b(\psi) = Q_L\psi(L) - Q_0\psi(0) + M_L\psi'(L) - M_0\psi'(0).$$

Proof. $(a) \Rightarrow (b)$. Multiply the bending equation with $\psi \in V$ and integrate to obtain

$$\int_0^L q\psi = \int_0^L (EIw'')''\psi \underset{\text{by (7)}}{=} \int_0^L (EIw'')\psi'' + \beta(w,\psi) \underset{\text{by (8)}}{=} \int_0^L EIw''\psi'' + b(\psi).$$

$(b) \Rightarrow (a)$. If $\psi(0) = \psi'(0) = \psi(L) = \psi'(L) = 0$ then $b(\psi) = 0$, and hence we (b),

$$\int_0^L EIw''\psi'' = \int_0^L q\psi.$$

Thus, $(EIw'')'' = q$ by the lemma of Du Bois-Reymond. Thus, by (b), we have for *all* $\psi \in V$,

$$\int_0^L EIw''\psi'' = \int_0^L q\psi + b(\psi) = \int_0^L (EIw'')''\psi + b(\psi) = \int_0^L EIw''\psi'' - \beta(w,\psi) + b(\psi)$$

Thus, $\beta(w,\psi) = b(\psi)$. Now, choose $\psi \in V$ such that $\psi(0) = 1$, $\psi'(0) = \psi(L) = \psi'(L) = 0$. It then follows that $Q^0(w) = \beta(w,\psi) = b(\psi) = Q_0$. Analogously one concludes the other identities in (8). □

5 Galerkin's method.

[2] In order to compute an approximative solution w_h of the bending equation choose a finite dimensional subspace V_h (Ansatzspace) of V with basis $\phi_1, \dots \phi_N : [0,L] \to \mathbb{R}$. Let

$$w_h(x) = \sum_{k=1}^N w_k\, \phi_k(x), \qquad w_k \in \mathbb{R}. \tag{10}$$

Insert this Ansatz in the weak formulation, and let $\psi = \phi_j$, $j = 1, \dots, N$. This yields the equations

$$\int_0^L EIw_h''\phi_j'' = \int_0^L q\phi_j + b(\phi_j), \qquad j = 1, \dots, N. \tag{11}$$

[2] Boris Grigorjewitsch Galjorkin, (* 1871 in Polozk, heute Weißrussland; † 1945 in Leningrad) sovjetischer Ingenieur und Mathematiker.

Observe that

$$\int_0^L EI w_h'' \phi_j'' = \sum_{k=1}^N \left(\int_0^L EI \phi_k'' \phi_j'' \right) w_k.$$

Hence, (11) is equivalent to

$$\sum_{k=1}^N \left(\int_0^L EI \phi_k'' \phi_j'' \right) w_k = \int_0^L q\phi_j + Q_L \phi_j(L) - Q_0 \phi_j(0) + M_L \phi_j'(L) - M_0 \phi_j'(0), \qquad j = 1, \dots, N.$$

These are N linear equations for the N unknowns $w_1, \dots, w_N$. The equations can be written in matrix-vector-form as follows.

$$S\underline{w} = \underline{q} + Q_L\, \underline{e}_L - Q_0\, \underline{e}_0 + M_L\, \underline{d}_L - M_0\, \underline{d}_0, \tag{12}$$

where

$$\underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \qquad \underline{q} = \begin{bmatrix} \int q\phi_1 \\ \vdots \\ \int q\phi_N \end{bmatrix}, \qquad \underline{e}_x = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{bmatrix}, \qquad \underline{d}_x = \begin{bmatrix} \phi_1'(x) \\ \vdots \\ \phi_N'(x) \end{bmatrix}, \tag{13}$$

and

$$S = \begin{bmatrix} s_{11} & \dots & s_{1N} \\ \vdots & & \vdots \\ s_{N1} & \dots & s_{NN} \end{bmatrix}, \qquad s_{jk} = \int_0^L EI \phi_k'' \phi_j''.$$

The symmetric matrix $S \in \mathbb{R}^{N \times N}$ is called the stiffness matrix. Note that for all x,

$$w_h(x) = \underline{e}_x^\top \underline{w}, \qquad w_h'(x) = \underline{d}_x^\top \underline{w},$$

Remark. The stiffness matrix is positive semidefinite since for any $\underline{w} \in \mathbb{R}^n$,

$$\underline{w}^\top S \underline{w} = \sum_{j,k} \left(\int_0^L EI \phi_k'' \phi_j'' \right) w_j w_k = \int_0^L EI \left(\sum_j w_j \phi_j'' \right) \left(\sum_k w_j \phi_k'' \right) = \int_0^L EI \underbrace{\left(\sum_j w_j \phi_j'' \right)^2}_{\geq 0} \geq 0.$$

However, in general S is not positive definite since there might be nonzero vectors $\underline{w}$ such that $\sum_j w_j \phi_j'' = (\sum_j w_j \phi_j)'' = 0$.

Remark. The elastic energy stored in the bended beam is given by the formula

$$E(w) = \frac{1}{2} \int_0^L EI (w'')^2.$$

If w is of the form $w = \sum_k w_k \phi_k$ then

$$E(w) = \frac{1}{2} \underline{w}^\top S \underline{w}.$$

6 Fixing the beam.

In order to get a unique solution of the beam equation one has to add at least two Dirichlet boundary conditions. For a left sided clamped beam (cantilever) these conditions are $w(0) = a$, $w'(L) = b$, where $a, b \in \mathbb{R}$ are given. The Galerkin approximation should satisfy these conditions, too, i.e.

$$a = w_h(0) = \underline{e}_0^\top \underline{w}, \qquad b = w_h'(0) = \underline{d}_0^\top \underline{w}.$$

These equations combined with (12) are equivalent to

$$\begin{bmatrix} S & \underline{e}_0 & \underline{d}_0 \\ \underline{e}_0^\top & 0 & 0 \\ \underline{d}_0^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{w} \\ Q_0 \\ M_0 \end{bmatrix} = \begin{bmatrix} \underline{q} + Q_L\, \underline{e}_L + M_L\, \underline{d}_L \\ a \\ b \end{bmatrix}. \tag{14}$$

If the quantities on the right hand side of this linear equation are given then $\underline{w}$, Q_0 and M_0 can be computed. If the beam is supported (but not not clamped) at both ends, then the corresponding equation is

$$\begin{bmatrix} S & \underline{e}_0 & -\underline{e}_L \\ \underline{e}_0^\top & 0 & 0 \\ -\underline{e}_L^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{w} \\ Q_0 \\ Q_L \end{bmatrix} = \begin{bmatrix} \underline{q} - M_0\, \underline{d}_0 + M_L\, \underline{d}_L \\ a_0 \\ -a_L \end{bmatrix} \tag{15}$$

The equations (14) and (15) are both of the form

$$\underbrace{\begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix}}_{S_e} \begin{bmatrix} \underline{x} \\ \underline{\mu} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{a} \end{bmatrix} \tag{16}$$

where S is symmetric and positive semidefinite. The matrix S_e is called the extended stiffness matrix. The matrix C must be such that S_e is nonsingular. Otherwise (16) has not a unique solution.

Remark. The vector $\underline{x}$ is the solution of a constrained energy minimization problem:

$$\mathcal{E}(\underline{x}) = \min\{\, \mathcal{E}(\underline{\xi}),\ |\ \underline{\xi} \in \mathbb{R}^N,\ C^\top \underline{\xi} = \underline{a}\, \}, \qquad \mathcal{E}(\underline{\xi}) := \frac{1}{2}\, \underline{\xi}^\top S \underline{\xi} - \underline{f}^\top \underline{\xi}.$$

The vector $\underline{\mu}$ is the associated Lagrangian multiplier (consisting of constraint forces and constraint moments). The multiplier can be eliminated from the equation in the follwing way. Let

$$\hat{P} = C(C^\top C)^{-1} C^\top, \qquad P = I - \hat{P}.$$

Then P and $\hat{P}$ are complementary orthogonal projectors. In particular,

$$\underline{x} = P\underline{x} + \hat{P}\underline{x} \qquad \text{for all } \underline{x} \in \mathbb{R}^N. \tag{17}$$

Furthermore, $PC = 0$, $\hat{P}C = C$. On multiplying the equations

$$S\underline{x} + C\underline{\mu} = \underline{f}, \qquad C^\top \underline{x} = \underline{a}$$

with P and $C(C^\top C)^{-1}$, respectively, one obtains

$$PS\underline{x} = P\underline{f}, \qquad \hat{P}\underline{x} = \underline{\widetilde{a}}, \qquad \text{where } \underline{\widetilde{a}} := C(C^\top C)^{-1}\underline{a}.$$

Inserting (17) in the first of these equations yields

$$PSP\underline{x} = P(\underline{f} - S\underline{\widetilde{a}}).$$

In order to obtain an equation for $\underline{x}$ with an invertible matrix one can add a multiple of $\hat{P}\underline{x}$:

$$\underbrace{(\gamma\, \hat{P} + PSP)}_{=:S_{mod}}\, \underline{x} = \underbrace{P(\underline{f} - S\underline{\widetilde{a}}) + \gamma\, \underline{\widetilde{a}}}_{=:f_{mod}}, \qquad \gamma \in \mathbb{R} \setminus \{0\}.$$

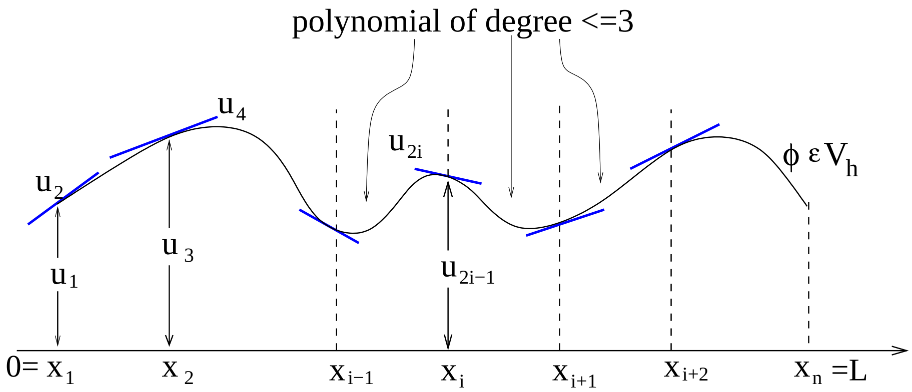
7 Choice of the Ansatz space V_h.

In principle the Galerkin ansatz (10) works with an arbitrary ansatz space V_h (for instance one could take a space of trigonometric functions). However, not every ansatz space contains a close approximation w_h to the exact solution w. A good choice for V_h is a space of piecewise cubic polynomials as described below.

Choose $n \geq 2$. Let $h = L/(n-1)$, $x_i = h(i-1)$, $i = 1, \ldots, n$ and

$$V_h \quad = \quad \{\, \phi \in V \mid \phi|_{]x_i, x_{i+1}[} \text{ is a polynomial of degree } \leq 3,\ i = 1, \ldots, n-1\, \}.$$

The points x_i are called the nodes. Each function $\phi \in V_h$ is uniquely determined by the values $u_{2i-1} := \phi(x_i)$ and $u_{2i} := \phi'(x_i)$.



Each function $\phi \in V_h$ can be uniquely written as

$$\phi = \sum_{k=1}^{2n} u_k\,\phi_k = \sum_{i=1}^{n}(u_{2i-1}\,\phi_{2i-1} + u_{2i}\,\phi_{2i})$$

with the basis functions $\phi_1, \phi_2, \ldots, \phi_{2n-1}, \phi_{2n} \in V_h$ which are defined as follows. For $i = 2, \ldots, n-1$,

$$\phi_{2i-1}(x) = \begin{cases} \bar\phi_3\left(\frac{x-x_{i-1}}{h}\right) & x \in [x_{i-1}, x_i] \\ \bar\phi_1\left(\frac{x-x_i}{h}\right) & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise.} \end{cases} \qquad \phi_{2i}(x) = \begin{cases} h\,\bar\phi_4\left(\frac{x-x_{i-1}}{h}\right) & x \in [x_{i-1}, x_i] \\ h\,\bar\phi_2\left(\frac{x-x_i}{h}\right) & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise.} \end{cases}$$

Additionally:

$$\phi_1(x) = \begin{cases} \bar\phi_1\left(\frac{x}{h}\right) & x \in [0, h] \\ 0 & \text{otherwise.} \end{cases} \qquad \phi_2(x) = \begin{cases} h\,\bar\phi_2\left(\frac{x}{h}\right) & x \in [0, h] \\ 0 & \text{otherwise.} \end{cases}$$

und

$$\phi_{2n-1}(x) = \begin{cases} \bar\phi_3\left(\frac{x-x_{n-1}}{h}\right) & x \in [x_{n-1}, L] \\ 0 & \text{otherwise.} \end{cases} \qquad \phi_{2n}(x) = \begin{cases} h\,\bar\phi_4\left(\frac{x-x_{n-1}}{h}\right) & x \in [x_{n-1}, L] \\ 0 & \text{otherwise,} \end{cases}$$

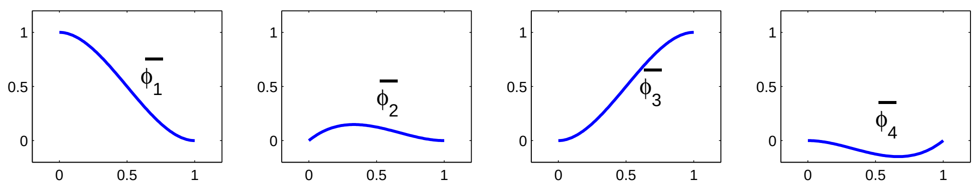
where

$$\begin{aligned} \bar\phi_1(\xi) &= 1 - 3\xi^2 + 2\xi^3, & \bar\phi_3(\xi) &= 3\xi^2 - 2\xi^3, \\ \bar\phi_2(\xi) &= \xi(\xi-1)^2, & \bar\phi_4(\xi) &= \xi^2(\xi-1). \end{aligned}$$

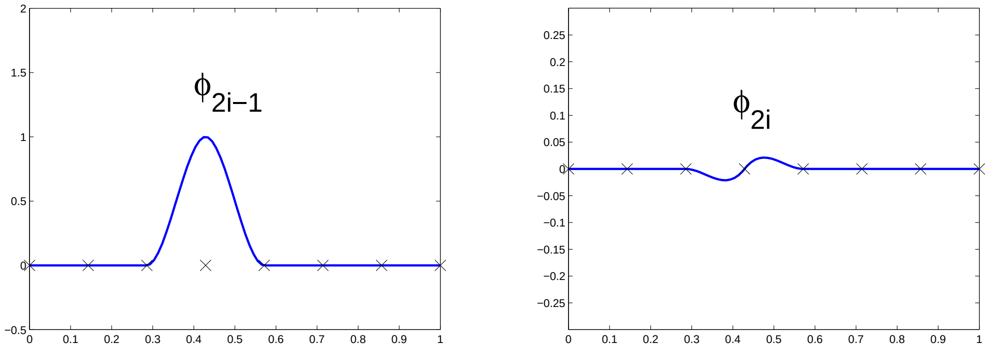
The functions $\bar\phi_i$ are called form functions. They are polynomials of degree 3 which take the following values at 0 and 1.

	$\bar\phi_j(0)$	$\bar\phi'_j(0)$	$\bar\phi_j(1)$	$\bar\phi'_j(1)$
$j=1$	1	0	0	0
$j=2$	0	1	0	0
$j=3$	0	0	1	0
$j=4$	0	0	0	1

Here are the graphs of the form functions:



The basis functions with odd index $2i-1$ have the form of a hump and attain the maximal value 1 at the node x_i. The basis functions with even index $2i$ have the form of a wave with slope 1 a the node x_i:



Remark. A one dimensional finite element is a finite interval with a set (vector space) of polynomials defined on it. So our ansatz space V_h is built from finite elements.

Remark. It can be shown that for our choice of the ansatz functions (piecewise cubic polynomials) each exact solution w of the bending equation (+ boundary conditions) coincides with its Galerkin approximation w_h at the nodes: $w_h(x_i) = w(x_i)$, $w_h'(x_i) = w'(x_i)$, $i = 1, \dots, N$.

8 The dynamic case

In the dynamic case the bending curve as well as all forces and boundary conditions depend on time. In particular the bending curve at time t is given by $w(x,t)$, $x \in [0, L]$. The bending equation is

$$\mu\, \ddot{w} + (EIw'')'' = q, \tag{18}$$

where $\mu = \mu(x)$ is the mass density (more precisely: the mass per unit length) and $\ddot{w}$ denotes the second derivative of w with respect to t. The same procedure as in the static case (multiplication with ψ and partial integration) yields the weak formulation

$$\int_0^L \ddot{w}\psi + \int_0^L EIw''\psi'' = \int_0^L q(\cdot,t)\psi + b(\psi,t), \tag{19}$$

where

$$b(\psi,t) = Q_L(t)\psi(L) - Q_0(t)\psi(0) + M_L(t)\psi'(L) - M_0(t)\psi'(0)$$

and $\psi \in V$ ($\psi = \psi(x)$ does not depend on time). Inserting in (19) for w the (time depentent) Galerkin Ansatz

$$w_h(x,t) \sum_{k=1}^{N} w_k(t)\phi_k(x)$$

and for ψ the basis functions ϕ_j, $\quad j = 1, \dots, N$ we obtain the ordinary differential equation of second order

$$M\underline{\ddot{w}}(t) + S\underline{w}(t) = \underline{q}(t) + Q_L(t)\,\underline{e}_L - Q_0(t)\,\underline{e}_0 + M_L(t)\,\underline{d}_L - M_0(t)\,\underline{d}_0,$$

where $\underline{w}(t) = [w_1(t)\ \ w_2(t)\ \ \dots w_N(t)]^\top$, $\underline{e}_x, \underline{d}_x$ and $\underline{q}$ are defined as in (13) and

$$M = \begin{bmatrix} m_{11} & \dots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \dots & m_{NN} \end{bmatrix}, \qquad m_{jk} = \int_0^L \mu\phi_k\phi_j$$

is the mass matrix. The latter is symmetric and positive definite. By adding the Dirichlet boundary conditions for the left sided clamped beam one obtains the differential algebraic equation (DAE)

$$\begin{bmatrix} M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{\ddot{w}} \\ \ddot{Q}_0 \\ \ddot{M}_0 \end{bmatrix} + \begin{bmatrix} S & \underline{e}_0 & \underline{d}_0 \\ \underline{e}_0^\top & 0 & 0 \\ \underline{d}_0^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{w} \\ Q_0 \\ M_0 \end{bmatrix} = \begin{bmatrix} \underline{q} + Q_L\,\underline{e}_L + M_L\,\underline{d}_L \\ a \\ b \end{bmatrix}.$$

This is a DAE of the form

$$\underbrace{\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}}_{M_e} \begin{bmatrix} \ddot{\underline{x}} \\ \ddot{\underline{\mu}} \end{bmatrix} + \underbrace{\begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix}}_{S_e} \begin{bmatrix} \underline{x} \\ \underline{\mu} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{a} \end{bmatrix} \tag{20}$$

with extended mass matrix M_e and extended stiffness matrix S_e (Note: $\underline{\mu}$ is the vector of contraint forces and moments and not a vector of mass densities.)