Constructing Equations of Motion for A Vehicle Rigid Body Model

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ABSTRACT

This paper focuses on generating equations of motion for a Vehicle model. The generated equations can be used for rigid body analysis such as vibration analysis, frequency domain analysis, and time domain analysis. The paper starts with the equations of motion of a 6-degree-of-freedom (DOF) system and demonstrates detailed approach in building the equations. Next, the equations of motion are built for a 12-DOF vehicle model, which can be used as an approximated Vehicle model in most cases. Finally, the rigid body of an accurate Vehicle model is created and the equations of motion are constructed following the presented method.

Nomenclature

Symbols

θ	Rotation along Y-axis(Pitch)
φ	Rotation along X-axis(Roll)
ψ	Rotation along Z-axis(Yaw)
Z	Displacement along Z-direction
K	Stiffness of R
C	Damping coefficient
I	Moment inertia of Vehicle component

Subscripts

w	Wheel
c	Cab
f	Frame
x	X-direction
y	Y-direction
z	Z-direction

INTRODUCTION

In vehicle design and analysis, NVH analysis (noise, vibration, and harshness) and rigid body analysis (RBA) have become important considerations for designers. In order to perform these analyses, a qualified rigid body model (RBM) for the vehicle and a set of equations of motion need to be generated.

Various rigid body models have been developed for different applications. Song et. al. [1] created simple and efficient rigid body models and used them for perturbation analysis and simulation of systems with frictional contacts. Hegazy et. al. [2] presented a complicated vehicle model for nonlinear rigid body analysis. This model had 94 degrees of freedom and comprised front/rear suspensions, steering system, wheels/tires, and vehicle inertia. The model was successfully used for the vehicle handling analysis. Zhang et. al. [3] used flexible and rigid beams to create a vehicle model which was used for road surface analyses in order to find dynamic properties of the vehicle chassis/suspension system. Pereira et. al.[4] presented a multibody dynamic model for train vehicles and constructed formulations based on it. The model and the formulations were used for crashworthiness analysis and design of the train vehicles. Mousseau et. al. [5] successfully developed a comprehensive vehicle dynamics model for simulating the dynamics response of ground vehicles on different road surfaces. In their research, the multibody system simulations (MSS) program was used to simulate the vehicle and a nonlinear finite element model was used for the tires. Lee et. al. [6] created a computer model for a vehicle system and used it for frequency response analyses. The computer model consisted of vehicle body, suspension systems, and tires. A coupled formulation is also obtained for the vehicle model by using the modal synthesis technique. Hunt [7] devised a method of creating a multi-axle vehicle model and used this model to represent all vehicles on the road for vibration analysis. The effects of vehicle mass, speed, and wheelbase on its ground vibration were discussed. Ibrahim et. al. [8] applied finite element method to create a truck frame model in order to analyze the vehicle dynamic responses. The effects of the frame flexibility on the ride vibration behavior of trucks were investigated through the FE model.

The purpose of this paper is to build a concept vehicle model with necessary components and develop the equations of motion by applying such model. The developed model and equations can be used for rigid body analysis and correctly reflect the vehicle's frequency and time responses.

6-DOF SYSTEM

EOF FOR EXAMPLE SYSTEM

This project starts with a simpler case: modeling and formulating a 6-DOF system, which is the basic component in a vehicle assembly. Figure1 shows a cab model, which is simplified to be a 6-DOF rigid body. From this figure, the cab body is connected to ground through 4 mounts, and each mount has 3 springs and dampers along X-, Y-, and Z-direction. Conventionally, the three pairs of spring and damper elements are defined as a 3-DOF viscoelastic element. In other words, in our model, each mount has a 3-DOF viscoelastic element. For simplification, it is assumed that the stiffness/damping along the same direction are equals at each mount.

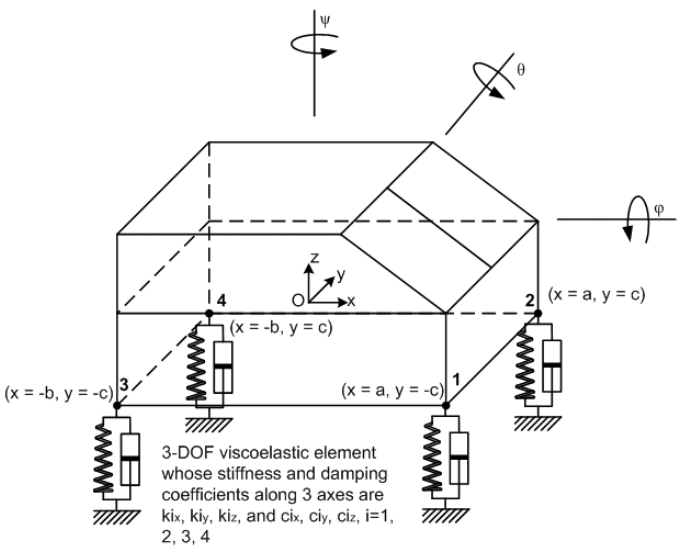

Fig. 1. 6-DOF cab model

In order to derive the EOM for such system, relationship among the degrees of freedom should be investigated at first. On analyzing figure 1, it is found that the cab's vertical displacement (Z), rolling (φ) and pitching (θ) are related to each other. In another words, the model's Z-displacement will change if it rotates about X- or Y-direction while stay unchanged if the cab only rotates about Z-direction. Similarly, the cab's lateral displacements (X and Y) and its yawing (ψ) are related and coupled together. Thus, the EOM of such 6-DOF rigid body can be represented as two uncoupled 3-DOF system. The EOM systems can be derived following Newton's law as

Forces along X-direction

$$m\ddot{x} = -k_{1x}(x - c\psi) - k_{2x}(x + c\psi) - k_{3x}(x - c\psi) - k_{4x}(x + c\psi)$$
$$- c_{1x}(\dot{x} - c\dot{\psi}) - c_{2x}(\dot{x} + c\dot{\psi}) - c_{3x}(\dot{x} - c\dot{\psi}) - c_{4x}(\dot{x} + c\dot{\psi})$$

(1-a)

Forces along Y-direction

$$m\ddot{y} = -k_{1y}(y + a\psi) - k_{2y}(y + a\psi) - k_{3y}(y - b\psi) - k_{4y}(y - b\psi)$$
$$- c_{1y}(\dot{y} + a\dot{\psi}) - c_{2y}(\dot{y} + a\dot{\psi}) - c_{3y}(\dot{y} - b\dot{\psi}) - c_{4y}(\dot{y} - b\dot{\psi})$$

(1-b)

Forces along Z-direction

$$m\ddot{z} = -k_{1z}(z - a\theta + c\varphi) - k_{2z}(z - a\theta - c\varphi) - k_{3z}(z + b\theta + c\varphi)$$
$$-k_{4z}(z + b\theta - c\varphi) - c_{1z}(\dot{z} - a\dot{\theta} + c\dot{\varphi}) - c_{2z}(\dot{z} - a\dot{\theta} - c\dot{\varphi})$$
$$-c_{3z}(\dot{z} + b\dot{\theta} + c\dot{\varphi}) - c_{4z}(\dot{z} + b\dot{\theta} - c\dot{\varphi})$$

(1-c)

Moments about X-axis

$$I_{xx}\ddot{\varphi} = -k_{1z}(z - a\theta + c\varphi)c + k_{2z}(z - a\theta - c\varphi)c - k_{3z}c(z + b\theta + c\varphi)$$
$$+k_{4z}c(z + b\theta - c\varphi) - c_{1z}(\dot{z} - a\dot{\theta} + c\dot{\varphi})c + c_{2z}(\dot{z} - a\dot{\theta} - c\dot{\varphi})c$$
$$-c_{3z}c(\dot{z} + b\dot{\theta} + c\dot{\varphi}) + c_{4z}c(\dot{z} + b\dot{\theta} - c\dot{\varphi})$$

(1-d)

Moments about Y-axis

$$I_{yy}\ddot{\theta} = k_{1z}a(z - a\theta + c\varphi) + k_{2z}a(z - a\theta - c\varphi) - k_{3z}b(z + b\theta + c\varphi)$$
$$-k_{4z}b(z + b\theta - c\varphi) + c_{1z}a(\dot{z} - a\dot{\theta} + c\dot{\varphi}) + c_{2z}z(\dot{z} - a\dot{\theta} - c\dot{\varphi})a$$
$$-c_{3z}b(\dot{z} + b\dot{\theta} + c\dot{\varphi}) - c_{4z}b(\dot{z} + b\dot{\theta} - c\dot{\varphi})$$

(1-e)

Moments about Z-axis

$$I_{zz}\ddot{\psi} = -k_{1y}a(y + a\psi) - k_{2y}a(y + a\psi) + k_{3y}b(y - b\psi) + k_{4y}b(y - b\psi)$$
$$-c_{1y}a(\dot{y} + a\dot{\psi}) - c_{2y}a(\dot{y} + a\dot{\psi}) + c_{3y}b(\dot{y} - b\dot{\psi}) + c_{4y}b(\dot{y} - b\dot{\psi})$$
$$+k_{1x}c(x - c\psi) - k_{2x}c(x + d\psi) + k_{3x}c(x - c\psi) - k_{4x}c(x + d\psi)$$
$$+c_{1x}c(\dot{x} - c\dot{\psi}) - c_{2x}c(\dot{x} + d\dot{\psi}) + c_{3x}c(\dot{x} - c\dot{\psi}) - c_{4x}c(\dot{x} + d\dot{\psi})$$

(1-f)

where a, b are distances from the gravity center of the cab model to its front and rear mounts; c, d are distances from the center to the left and right mounts. Here it is assumed that for a concept cab model, the distances to the two front mounts are equals, the same with the other mounts.

Rearranging the variables in the above formula and we have

$$m\ddot{x} + (k_{1x} + k_{2x} + k_{3x} + k_{4x})x + c(-k_{1x} - k_{3x} + k_{2x} + k_{4x})\psi$$
$$+ (c_{1x} + c_{2x} + c_{3x} + c_{4x})\dot{x} + c(-c_{1x} - c_{3x} + c_{2x} + c_{4x})\dot{\psi} = 0$$

(2-a)

$$m\ddot{y} - y(k_{1y} + k_{2y} + k_{3y} + k_{4y}) - \psi c(k_{1y} + k_{2y} - k_{3y} - k_{4y})$$
$$+ \dot{y}(c_{1y} + c_{2y} + c_{3y} + c_{4y}) + \dot{\psi} c(c_{1y} + c_{2y} - c_{3y} - c_{4y}) = 0$$

(2-b)

$$m\ddot{z} + (k_{1z} + k_{2z} + k_{3z} + k_{4z})z + (-ak_{1lz} - ak_{2z} + bk_{3z} + bk_{4z})\theta$$
$$+c(k_{1z} + k_{2z} - k_{3z} - k_{4z})\varphi + (c_{1z} + c_{2z} + c_{3z} + c_{4z})\dot{z}$$
$$+(-ac_{1z} - ac_{2z} + bc_{3z} + bc_{4z})\dot{\theta} + c(c_{1z} + c_{3z} - c_{2z} - c_{4z})\dot{\varphi} = 0$$

(2-c)

$$I_{xx}\ddot{\varphi}+(ck_{1z}+ck_{3z}-dk_{2z}-dk_{4z})z+c(-ak_{1z}+ak_{2z}+bk_{3z}-bk_{4z})\theta$$
$$+c^2(k_{1z}+k_{2z}+k_{3z}+k_{4z})\varphi+c(c_{1z}+cc_{3z}-c_{2z}-c_{4z})\dot{z}$$
$$+c(-ac_{1z}+ac_{2z}+bc_{3z}-bc_{4z})\dot{\theta}+c^2(c_{1z}+c_{2z}+c_{3z}+c_{4z})\dot{\varphi}=0 \tag{2-d}$$

$$I_{yy}\ddot{\theta}+(-ak_{1z}-ak_{2z}+bk_{3z}+bk_{4z})z+(a^2k_{1z}+a^2k_{2z}+b^2k_{3z}+b^2k_{4z})\theta$$
$$+c(-ak_{1z}+ak_{2z}+bk_{3z}-bk_{4z})\varphi+(-ac_{1z}-ac_{2z}+bc_{3z}+bc_{4z})\dot{z}$$
$$+(a^2c_{1z}+a^2c_{2z}+b^2c_{3z}+b^2c_{4z})\dot{\theta}+c(-ac_{1z}+ac_{2z}+bc_{3z}-bc_{4z})\dot{\varphi}=0 \tag{2-e}$$

$$I_{zz}\ddot{\psi}+(ak_{1y}+ak_{2y}-bk_{3y}-bk_{4y})y+(ac_{1y}+ac_{2y}-bc_{3y}-bc_{4y})\dot{y}$$
$$+(a^2k_{1y}+a^2k_{2y}+b^2k_{3y}+b^2k_{4y}+c^2k_{1x}+c^2k_{2x}+c^2k_{3x}+c^2k_{4x})\psi$$
$$+(a^2c_{1y}+a^2c_{2y}+b^2c_{3y}+b^2c_{4y}+c^2c_{1x}+c^2c_{2x}+c^2c_{3x}+c^2c_{4x})\dot{\psi}$$
$$+c(-k_{1x}-k_{3x}+k_{2x}+k_{4x})x+c(-c_{1x}-c_{3x}+c_{2x}+c_{4x})\dot{x}=0 \tag{2-f}$$

Eq. (1) and (2) explain how to create EOM for a 6-DOF model with 4 mounts connected to the ground. However, a more general EOM has to be developed for any 6-DOF model which connects to the ground through arbitrary number of mounts.

MATRICES FOR GENERAL 6-DOF SYSTEM

Assume the coordinate of the i_{th} mount is (x_i, y_i, z_i), and k_{ix}, k_{iy}, k_{iz}, c_{ix}, c_{iy}, c_{iz} are its spring stiffness and damping coefficients along X-, Y-, and Z- direction. From Eq. (2), the system stiffness and damper matrices caused by the i_{th} mount can be written as

$$K_i=\begin{bmatrix} k_{ix} & 0 & 0 & 0 & 0 & -k_{ix}y_i \\ 0 & k_{iy} & 0 & 0 & 0 & k_{iy}x_i \\ 0 & 0 & k_{iz} & -k_{iz}x_i & k_{iz}y_i & 0 \\ 0 & 0 & -k_{iz}x_i & k_{iz}x_i^2 & -k_{iz}x_iy_i & 0 \\ 0 & 0 & k_{iz}y_i & -k_{iz}x_iy_i & k_{iz}y_i^2 & 0 \\ -k_{ix}y_i & k_{iy}x_i & 0 & 0 & 0 & k_{ix}y_i^2+k_{iy}x_i^2 \end{bmatrix} \tag{3}$$

$$C_i=\begin{bmatrix} c_{ix} & 0 & 0 & 0 & 0 & -c_{ix}y_i \\ 0 & c_{iy} & 0 & 0 & 0 & c_{iy}x_i \\ 0 & 0 & c_{iz} & -c_{iz}x_i & c_{iz}y_i & 0 \\ 0 & 0 & -c_{iz}x_i & c_{iz}x_i^2 & -c_{iz}x_iy_i & 0 \\ 0 & 0 & c_{iz}y_i & -c_{iz}x_iy_i & c_{iz}y_i^2 & 0 \\ -c_{ix}y_i & c_{iy}x_i & 0 & 0 & 0 & c_{ix}y_i^2+c_{iy}x_i^2 \end{bmatrix} \tag{3}$$

Eq. (3) and (4) are global stiffness and damping matrices. To simplify those matrices, local moment arm matrix, local stiffness matrix and local damping matrix are introduced.

$$K_i^{'}=\begin{bmatrix} k_{ix} & 0 & 0 \\ 0 & k_{iy} & 0 \\ 0 & 0 & k_{iz} \end{bmatrix} \tag{4}$$

$$C_i^{'}=\begin{bmatrix} c_{ix} & 0 & 0 \\ 0 & c_{iy} & 0 \\ 0 & 0 & c_{iz} \end{bmatrix} \tag{5}$$

$$L_i=\begin{bmatrix} 0 & 0 & -y_i \\ 0 & 0 & x_i \\ -x_i & y_i & 0 \end{bmatrix} \tag{6}$$

where K_i‘ is the system's local stiffness matrix due to the i_{th} mount, C_i‘ is the local damping matrix, and L_i is the moment arm matrix related to K_i‘ and C_i‘.

Therefore, the global matrices (eq. (3, 4)) can be simplified by the local matrices and moment arm matrix as

$$K_i=\begin{bmatrix} K_i^{'} & K_i^{'}L_i \\ L_i^{T}K_i^{'T} & K_i^{'}L_i^{T}L_i \end{bmatrix} \tag{8}$$

$$C_i=\begin{bmatrix} C_i^{'} & C_i^{'}L_i \\ L_i^{T}C_i^{'T} & C_i^{'}L_i^{T}L_i \end{bmatrix} \tag{9}$$

In eq. (8, 9), the upper left sub-matrices, K_i‘ and C_i‘, represent global translational stiffness and damping due to pure translation motions appears on the i^{th} mount. The upper right sub-matrices, K_i‘L_i and C_i‘L_i are the global translational stiffness and damping caused by pure rotations. The lower left sub-matrices, $L_i^TK_i^{'T}$ and $L_i^TC_i^{'T}$ represent the global rotational stiffness and damping due to the pure translational motions, which are transposed from the upper right sub-matrices. The lower right terms, K_i‘$L_i^TL_i$ and C_i‘$L_i^TL_i$ represent the global rotational stiffness and damping caused by the pure rotations. From eq. (3) and (5), it can also be concluded that the lower right terms K_i‘$L_i^TL_i$ and C_i‘$L_i^TL_i$ are symmetric matrices and the upper left terms K_i‘ and C_i‘ are diagonal matrices.

The whole system's stiffness and damping matrices then can be easily obtained by assembling all the sub stiffness and damping matrices together. These sub stiffness and damping matrices are brought by every single mount in the system (eq. (8, 9)).

Correspondingly, the mass matrix of the system is

$$[M]=\begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx} & I_{xy} & I_{xz} \\ 0 & 0 & 0 & I_{yx} & I_{yy} & I_{yz} \\ 0 & 0 & 0 & I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{10}$$

After obtaining M, K, and C, the EOM for the entire system can be easily expressed as

$$M\ddot{X}+C\dot{X}+KX=F \tag{11}$$

where X is the system's displacement vector, and F is applied force vector.

12-DOF SYSTEM

In this section, the EOM and rigid body model for a 12-DOF system are created, which can be considered as a combination of two 6-DOF system. Fig. 2 shows a cab-frame assembly model, which is a typical 12-DOF system. In figure 2, the cab and frame are considered as 6-DOF rigid bodies and the cab is connected to the frame through 4 mounts, which are the viscoelastic elements along Y-direction. We also assume that these viscoelastic elements have the same stiffness and damping coefficient denoted as k_{cf} and c_{cf}.

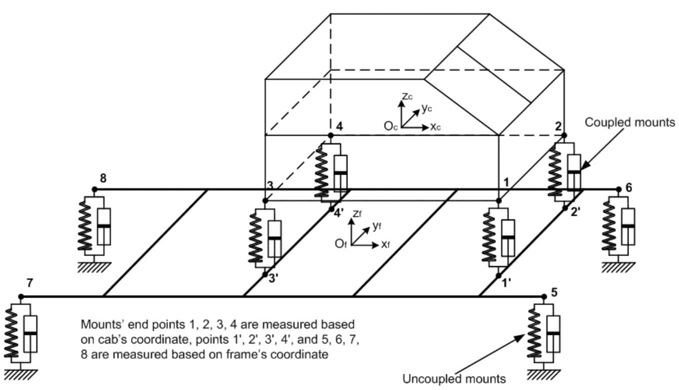

Fig. 2. 12-DOF cab-frame model

MATRICES FOR EXAMPLE SYSTEM

To find EOM for the 12-DOF system, the relationship between the cab and frame models has to be determined. As addressed before, for a 6-DOF rigid body, its vertical displacement, rolling, and pitching are correlated to each other; its lateral displacements and its yawing (ψ) are coupled together. However, in this 12-DOF system, the cab's vertical displacement is also affected by the frame's rolling and pitching, which is because the frame's rolling and pitching will input to the cab model and cause the rolling and pitching of cab therefore affect the cab's vertical displacement. For the same reason, the cab's lateral displacements are also influenced by the frame's yawing. Similarly, the cab's translation and rotation also affect the frame's motion, and vice versa.

Thus, the coupled effects between cab and frame can be described as: the cab and the frame's vertical displacements, rolling, and pitching are coupled together; the cab and the frame's lateral displacements and yawing are correlated to each other. Based on the coupling characteristics, the 12-DOF system's equations of motion can be represented as two uncoupled differential equation systems by Newton's law. Following the same way for the 6-DOF system, the EOM of the 12-DOF system can be derived and represented as matrix form.

MATRICES FOR GENERAL 12-DOF SYSTEM

So far, the EOM for a particular 12-DOF system was developed. However, a more general EOM needs to be constructed which are applicable to any 12-DOF system that has mounts with arbitrary numbers and dynamic properties, and at any locations.

As shown in fig. 2, the 12-DOF system has two local coordinate systems centered at the gravity centers of cab and frame. Meanwhile there're two types of viscoelastic elements within this model: one type viscoelastic elements only connect the frame to the ground, which doesn't contribute to the coupling effects (uncoupled mount). Another type of viscoelastic elements connect the two rigid bodies together and cause the coupling effects (coupled mount). Thus, the positions of ends of such type of viscoelastic elements have to be represented by different coordinate systems.

Assume that the coordinates of the i_{th} coupled mount's end on the cab is at point (x_i^c, y_i^c, z_i^c), where the superscript "c" denotes the cab's coordinate system. On the other hand, the coordinates of the other end on the frame model is (x_i^f, y_i^f, z_i^f), where the superscript "f" denotes the frame's coordinate system. Also, the mount's stiffness and damping coefficient along three directions are k_{ix}^c, k_{iy}^c, k_{iz}^c and c_{ix}^c, c_{iy}^c, c_{iz}^c respectively, where the superscript "c" means "coupled". For the uncoupled mounts, assuming that the coordinates of the i_{th} uncoupled mount's end on the frame model is (x_i^u, y_i^u, z_i^u), which is on the frame's coordinate system, and the superscript "u" means "uncoupled". Accordingly, the mount's stiffness and damping coefficient are k_{ix}^u, k_{iy}^u, k_{iz}^u and c_{ix}^u, c_{iy}^u, c_{iz}^u.

<u>Diagonal sub matrices</u>

Similar to the 6-DOF system, the stiffness matrix caused by the i_{th}, which is a 12 by 12 square matrix, is divided into 4 parts and each part is a 6 by 6 square matrix. The 4 sub-matrices are: the upper left matrix K_i^{11}, the upper right matrix K_i^{12}, the lower left matrix K_i^{21}, and the lower right matrix K_i^{22}. Following the same way, the damping matrix is also divided into 4 sub-matrices: C_i^{11}, C_i^{12}, C_i^{21}, and C_i^{22}. As from fig. 3, the sub stiffness matrix K_i^{11} can be written as

$$K_i^{11} = \begin{bmatrix} k_{ix} & 0 & 0 & 0 & 0 & -k_{ix} y_i \\ 0 & k_{iy} & 0 & 0 & 0 & k_{ix} x_i \\ 0 & 0 & k_{iz} & -k_{iz} x_i & k_{iz} y_i & 0 \\ 0 & 0 & -k_{iz} x_i & k_{iz} x_i^2 & -k_{iz} x_i\, y_i & 0 \\ 0 & 0 & k_{iz} y_i & -k_{iz} x_i\, y_i & k_{iz} y_i^2 & 0 \\ -k_{ix} y_i & k_{iy} x_i & 0 & 0 & 0 & k_{ix} y_i^2 + k_{iy} x_i^2 \end{bmatrix} \tag{72}$$

where x_i, y_i and z_i are the arms of the i_{th} mounts based on appropriate coordinate system. Please note that eq. (16) is applicable to both coupled and uncoupled mounts. Similarly, the sub damping matrix C_i^{11} is

$$C_i^{11} = \begin{bmatrix} c_{ix} & 0 & 0 & 0 & 0 & -c_{ix} y_i \\ 0 & c_{iy} & 0 & 0 & 0 & c_{ix} x_i \\ 0 & 0 & c_{iz} & -c_{iz} c_i & c_{iz} y_i & 0 \\ 0 & 0 & -c_{iz} x_i & c_{iz} c_i^2 & -c_{iz} x_i\, y_i & 0 \\ 0 & 0 & c_{iz} y_i & -c_{iz} x_i\, y_i & c_{iz} y_i^2 & 0 \\ -c_{ix} y_i & c_{iy} x_i & 0 & 0 & 0 & c_{ix} y_i^2 + c_{iy} x_i^2 \end{bmatrix} \tag{13}$$

The lower right sub matrices K_i^{22} and C_i^{22} have the same form as eq. (16) and (17). In our model, we let K_i^{11} and C_i^{11} are stiffness and damping caused by the i_{th} mount connects to the frame model and K_i^{22} and C_i^{22} are stiffness and damping matrices due to the mount connects to the cab model.

Since K_i^{11}, C_i^{11} and K_i^{22}, C_i^{22} share the same form, then we can have the general form for the main diagonal sub matrices

$$K_i^{jj} = \begin{bmatrix} k_{ix} & 0 & 0 & 0 & 0 & -k_{ix} y_i \\ 0 & k_{iy} & 0 & 0 & 0 & k_{ix} x_i \\ 0 & 0 & k_{iz} & -k_{iz} x_i & k_{iz} y_i & 0 \\ 0 & 0 & -k_{iz} x_i & k_{iz} x_i^2 & -k_{iz} x_i\, y_i & 0 \\ 0 & 0 & k_{iz} y_i & -k_{iz} x_i\, y_i & k_{iz} y_i^2 & 0 \\ -k_{ix} y_i & k_{iy} x_i & 0 & 0 & 0 & k_{ix} y_i^2 + k_{iy} x_i^2 \end{bmatrix} \tag{14}$$

where the elements k, x, y, and z are measured from the same rigid body and within the same coordinate system. Hence, there's no coupling effects existing in the diagonal sub matrices K_i^{jj}, this conclusion and the eq. (18) can be applied to any multi-DOF system.

The introduced local stiffness and damping matrices can also be used to simplify eq. (18) along with the moment arm matrix. (eq. (5-7)). The sub matrices K_i^{jj} and C_i^{jj} can also be represented as eq.(8) and (9) with K_i and C_i being replaced by K_i^{jj} and C_i^{jj}.

The main diagonal sub matrices K^{jj} and C^{jj} of a 12-DOF system with any number of mounts then can be easily obtained by assembling all the matrices K_i^{jj} or C_i^{jj} together, which are caused by the i_{th} mount. Also, from above equations it can be seen that for any multi-DOF systems, the main diagonal sub matrices K^{jj} and C^{jj} are symmetric matrices, as well as any K_i^{jj} and C_i^{jj}.

<u>Off-diagonal sub matrices</u>

Compared to the diagonal sub stiffness matrices, the off-diagonal sub stiffness matrices K_i^{12} and K_i^{21} are affected by the coupled mounts connected the cab to the frame. The sub stiffness matrix K_i^{12} due to the i_{th} coupled mount can be written as

$$K_i^{12} = \begin{bmatrix} k_{ix}^c & 0 & 0 & 0 & 0 & -k_{ix}^c y_i^c \\ 0 & k_{iy}^c & 0 & 0 & 0 & k_{ix}^c x_i^c \\ 0 & 0 & k_{iz}^c & -k_{iz}^c x_i^c & k_{iz}^c y_i^c & 0 \\ 0 & 0 & -k_{iz}^c x_i^f & k_{iz}^c x_i^c x_i^f & -k_{iz}^c x_i^f y_i^c & 0 \\ 0 & 0 & k_{iz}^c y_i^f & -k_{iz}^c x_i^c y_i^f & k_{iz}^c y_i^c y_i^f & 0 \\ -k_{ix}^c y_i^f & k_{iy}^c x_i^f & 0 & 0 & 0 & k_{ix}^c y_i^c y_i^f + k_{iy}^c x_i^c x_i^f \end{bmatrix} \tag{15}$$

In this problem, K_i^{12} reflects the influences of the coupled effects on the cab's motion. The coupled effects are caused by the relative motion between the cab and the frame and include the coupled forces and moments. In eq. (19), the coupled effects are represented through the points measured on both local coordinate systems (x_i^c, y_i^c, z_i^c) and (x_i^f, y_i^f, z_i^f), which are both ends of the coupled mount.

Similarly, for the off diagonal sub damping matrix C_i^{12} we have

$$C_i^{12} = \begin{bmatrix} c_{ix}^c & 0 & 0 & 0 & 0 & -c_{ix}^c y_i^c \\ 0 & c_{iy}^c & 0 & 0 & 0 & c_{ix}^c x_i^c \\ 0 & 0 & c_{iz}^c & -c_{iz}^c x_i^c & c_{iz}^c y_i^c & 0 \\ 0 & 0 & -c_{iz}^c x_i^f & c_{iz}^c x_i^c x_i^f & -c_{iz}^c x_i^f y_i^c & 0 \\ 0 & 0 & c_{iz}^c y_i^f & -c_{iz}^c x_i^c y_i^f & c_{iz}^c y_i^c y_i^f & 0 \\ -c_{ix}^c y_i^f & c_{iy}^c x_i^f & 0 & 0 & 0 & c_{ix}^c y_i^c y_i^f + c_{iy}^c x_i^c x_i^f \end{bmatrix} \tag{16}$$

K_i^{12} and C_i^{12} can also be simplified using the local stiffness and damping matrices, and the moment arm matrix. For the off-diagonal sub matrices, the local stiffness and damping matrices are the same as eq. (5) and (6). The moment arm matrix has the same form as eq. (7) but needs to be separated into two matrices, whose arms are measured from the two local coordinate systems.

$$L_i^f = \begin{bmatrix} 0 & 0 & -y_i^f \\ 0 & 0 & x_i^f \\ -x_i^f & y_i^f & 0 \end{bmatrix} \tag{17}$$

$$L_i^c = \begin{bmatrix} 0 & 0 & -y_i^c \\ 0 & 0 & x_i^c \\ -x_i^c & y_i^c & 0 \end{bmatrix} \tag{18}$$

Where L_i^f is the moment arm matrix of the i_{th} coupled mount whose end connects to the frame (therefore measured in frame coordinate system), and L_i^c is the moment arm matrix of the coupled mount whose end connects to the cab.

The off-diagonal sub stiffness and damping matrices of the 12-DOF system's due to the i_{th} coupled mount can be written as

$$K_i^{12} = \begin{bmatrix} K_i^{c'} & K_i^{c'} L_i^{c} \\ L_i^{f^T} K_i^{c'T} & K_i^{c'} L_i^{fT} L_i^{c} \end{bmatrix} \tag{19}$$

$$C_i^{12} = \begin{bmatrix} C_i^{c'} & C_i^{c'} L_i^{c} \\ L_i^{fT} C_i^{c'T} & C_i^{c'} L_i^{fT} L_i^{c} \end{bmatrix} \tag{20}$$

The upper left sub matrices $K_i^{c'}$ and $C_i^{c'}$ represent one rigid body's global translational stiffness and damping coefficient affected by another rigid body's pure translations; the upper right sub matrices $K_i^{c'}L_i^{c}$ and $C_i^{c'}L_i^{c}$ represent one rigid body's global translational stiffness and damping coefficient affected by another rigid body's pure rotations; the lower left sub matrices $L_i^{f}K_i^{c'T}$ and $L_i^{f}C_i^{c'T}$ represent the rigid body's global rotational stiffness and damping coefficient caused by another body's pure translations; the lower right sub matrices $K_i^{c'}L_i^{fT}L_i^{c}$ and $C_i^{c'}L_i^{fT}L_i^{c}$ represent the body's global rotational stiffness and damping coefficient due to the other body's pure rotations.

Following the same way illustrated before, the off-diagonal sub matrices K^{12} and C^{12} of a 12-DOF system with any number of mounts can be easily derived by adding all sub stiffness and damper matrices together. These sub matrices are caused by coupled mounts that connect both rigid bodies.

The off-diagonal sub matrices K_i^{21} and C_i^{21} reflect the influences of the coupled effects on the frame's motion. Again, the coupled effects include the coupled forces and moments which are caused by the relative motion between the cab and frame. Both off-diagonal sub matrices can be expressed as

$$K_i^{21} = \begin{bmatrix} k_{ix}^c & 0 & 0 & 0 & 0 & -k_{ix}^c y_i^f \\ 0 & k_{iy}^c & 0 & 0 & 0 & k_{ix}^c x_i^f \\ 0 & 0 & k_{iz}^c & -k_{iz}^c x_i^f & k_{iz}^c y_i^f & 0 \\ 0 & 0 & -k_{iz}^c x_i^c & k_{iz}^c x_i^c x_i^f & -k_{iz}^c x_i^c y_i^f & 0 \\ 0 & 0 & k_{iz}^c y_i^c & -k_{iz}^c x_i^f y_i^c & k_{iz}^c y_i^c y_i^f & 0 \\ -k_{ix}^c y_i^c & k_{iy}^c x_i^c & 0 & 0 & 0 & k_{ix}^c y_i^c y_i^f + k_{iy}^c x_i^c x_i^f \end{bmatrix} \tag{21}$$

$$C_i^{21} = \begin{bmatrix} c_{ix}^c & 0 & 0 & 0 & 0 & -c_{ix}^c y_i^f \\ 0 & c_{iy}^c & 0 & 0 & 0 & c_{ix}^c x_i^f \\ 0 & 0 & c_{iz}^c & -c_{iz}^c x_i^f & c_{iz}^c y_i^f & 0 \\ 0 & 0 & -c_{iz}^c x_i^c & c_{iz}^c x_i^c x_i^f & -c_{iz}^c x_i^c y_i^f & 0 \\ 0 & 0 & c_{iz}^c y_i^c & -c_{iz}^c x_i^f y_i^c & c_{iz}^c y_i^c y_i^f & 0 \\ -c_{ix}^c y_i^c & c_{iy}^c x_i^c & 0 & 0 & 0 & c_{ix}^c y_i^c y_i^f + c_{iy}^c x_i^c x_i^f \end{bmatrix} \tag{22}$$

Similarly, the K_i^{21} and C_i^{21} can be simplified with the local sub matrices and the moment arm matrices. K^{21} and C^{21} of any 12-DOF system can also be obtained based on the constructed sub matrices K_i^{21} and C_i^{21}.

Final stiffness, damping, and mass matrices

By comparing K_i^{12}, C_i^{12} and K_i^{21}, C_i^{21}, it is found that K_i^{12} and K_i^{21} are transpose matrices as well as C_i^{12} and C_i^{21}. Therefore, the matrix K^{12} is transposed to K^{21} and C^{12} is transposed to C^{21}. Also because the matrices K^{jj} and C^{jj} are symmetric, the 12-DOF system's stiffness and damping matrices K and C are symmetric matrices which are obtained by assembling K^{11}, K^{12}, K^{21} and K^{22} together. This conclusion is valid for any multi-DOF system.

In summary, the 12-DOF system's stiffness matrix can be obtained by combining K^{11}, K^{12}, K^{21} and K^{22} into one matrix. K^{11}, K^{12}, K^{21} and K^{22} can be calculated using the demonstrated methods. The 12-DOF system's damping matrix can be evaluated in the same way. Methods of generating the general EOM have been illustrated, which are applicable to any 12-DOF system with arbitrary number of coupled or uncoupled mounts.

The overall mass matrix for this 12-DOF system can be easily obtained as

$$M = \begin{bmatrix} M_f & [0] \\ [0] & M_c \end{bmatrix} \tag{23}$$

where

$$M_f = \begin{bmatrix} m_f & 0 & 0 & 0 & 0 & 0 \\ 0 & m_f & 0 & 0 & 0 & 0 \\ 0 & 0 & m_f & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx}^f & I_{xy}^f & I_{xz}^f \\ 0 & 0 & 0 & I_{yx}^f & I_{yy}^f & I_{yz}^f \\ 0 & 0 & 0 & I_{zx}^f & I_{zy}^f & I_{zz}^f \end{bmatrix} \tag{24}$$

and

$$M_c = \begin{bmatrix} m_c & 0 & 0 & 0 & 0 & 0 \\ 0 & m_c & 0 & 0 & 0 & 0 \\ 0 & 0 & m_c & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx}^c & I_{xy}^c & I_{xz}^c \\ 0 & 0 & 0 & I_{yx}^c & I_{yy}^c & I_{yz}^c \\ 0 & 0 & 0 & I_{zx}^c & I_{zy}^c & I_{zz}^c \end{bmatrix} \tag{25}$$

EOM for the 12-DOF system can be easily obtained based on the derived M, K, and C matrices, as introduced for the 6-DOF system.

FULL VEHICLE MODEL

In this section, the EOM for a full vehicle model is derived. In this paper, a concept vehicle model is presented, which correctly reflects the vehicle's transient responses and dynamic properties while only contains basic vehicle components. Fig. 3 shows this model, which includes one cab, one frame, two suspensions, and four wheels. In this model, the cab is connected to the frame with 4 3-DOF viscoelastic elements. The frame is connected to the two suspensions through 4 viscoelastic elements and each wheel is assumed as a mass node and connected to the ground through a viscoelastic element. Since the most popular input to a running vehicle is excited by road surfaces along vertical (Z-) direction, the viscoelastic elements of suspensions and wheels are approximated as 1-DOF elements. In this vehicle model, all the major components are assumed as 6-DOF rigid bodies. Here, the EOM is created for this 48-DOF system.

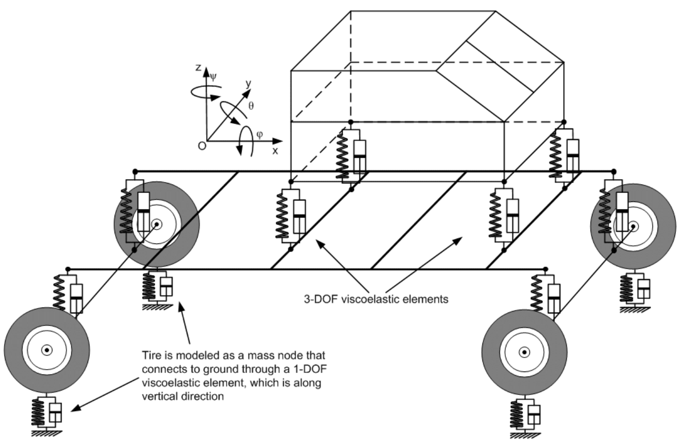

Fig. 3. Full vehicle model

As demonstrated above, the connected components' vertical displacement, rolling, and pitching are coupled together, and their lateral displacements and yawing are coupled together. However, for unconnected components, there's no direct relationship among their motions. For example, in this model, the cab is only related to the frame. Thus, the EOM of the cab model in the full vehicle is exactly the same as the EOM derived for the 12-DOF system.

Following the similar way, the whole vehicle model can be considered as a group of 6-DOF system and coupled 12-DOF system. The final EOM can be obtained by adding all sub matrices together. Detailed steps of generating EOM for a full vehicle model and any multi-DOF model with arbitrary number of mounts are described as

1. Build mass matrix for the whole vehicle model.
2. Build stiffness and damping matrices for a vehicle component due to one mount, including main diagonal and off-diagonal sub matrices.
3. Build entire stiffness and damping matrices for this component by adding together all the matrices derived in step 2.
4. Develop stiffness and damping matrices for all vehicle components following the same way.
5. Construct EOM for the vehicle model based on the derived mass, stiffness, and damping matrices using Newton's law.

Part of dynamic equations of motion for this vehicle model is listed in appendix.

CONCLUSION

This paper analyzes the 6-DOF system, 12-DOF system and the concept full vehicle model. The properties of viscoelastic mounts are investigated and the mass, stiffness, and damping matrices are created for each system. In multi-body system, the coupled effects between rigid bodies are discussed and the influences of the coupled effects on the body's stiffness and damping characteristics are illustrated. Rigid body models and equations of motion are constructed for these systems that can have any number of coupled and uncoupled mounts. A general method is finally presented for how to create EOM for a multi-DOF system such as a full vehicle model. The proposed method can be embedded into an algorithm so that the matrices and EOM for a given model can be automatically created. The created EOM and rigid body models can be used for vehicle analyses, such as NVH (noise, vibration, harshness) analysis, dynamic analysis, and transient analysis in order to find its dynamic properties and frequency/time responses.

REFERENCES

1. P. Song, P. Kraus, V. Kumar, P. Dupont, "Analysis of rigid-body dynamic models for simulation of systems with frictional contacts", ASME-Journal of Applied Mechanics, 68(1) (2001) 118-128.
2. S. Hegazy, H. Rahnejat, K. Hussain, "Multi-body dynamics in full-vehicle handling analysis under transient manoeuvre", Vehicle System Dynamics-International Journal of Vehicle Mechanics and Mobility, 34(1) (2000) 1-24.
3. Y. Zhang, P. Xiao, T. Palmer, A. Farahani, "Vehicle chassis/suspension dynamics analysis - finite element model vs. rigid body model", SAE Transactions, Paper no. 980900.
4. M.S. Pereia, J.A.C. Ambrosio, J.P. Dias, "Crashworthiness analysis and design using rigid-flexible multibody dynamics with application to train vehicles", International Journal for Numerical Methods in Engineering, 40(4) (1998) 655-687.
5. C.W. Moussau, T.A. Laursen, M. Lidberg, R.L. Taylor, "Vehicle dynamics simulations with coupled multibody and finite element models", Finite Element in Analysis and Design, 31(4) (1999) 295-315.
6. J. Lee, D.J. Thompson, H.-H. Yoo, J.-M. Lee, "Vibration analysis of a vehicle body and suspension system using a substructure synthesis method", International Journal of Vehicle Design, 24(4) (2000) 360-371.
7. H.E.M. Hunt, "Modelling of road vehicles for calculation of traffic-induced ground vibration as a random process", Journal of Sound and Vibration, 144(1) (1991), 41-51.
8. I.M. Ibrahim, D.A. Crolla, D.C. Barton, "Effect of frame flexibility on the ride vibration of trucks", Computers & Structures, 58(4) (1996) 709-713.

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APPENDIX

Four tires in X-translation

$$m_t\ddot{x}_{ti}+k_{fx}x_{ti}-k_{fx}x_f+c_{fx}\dot{x}_{ti}-c_{fx}\dot{x}_f-\frac{k_{fx}a_3}{2}\theta_f-\frac{c_{fx}a_3}{2}\dot{\theta}_f=0$$

i = 1, …, 4

(A-1)

Frame in X-translation

$$m_f\ddot{x}_f+4(k_{cx}+k_{fx})x_f-4k_{cx}x_c+4(c_{cx}+c_{fx})\dot{x}_f-4c_{cx}\dot{x}_c+2k_{cx}(a_3-2a_1-a_2)\theta_f+2c_{cx}(a_3-2a_1-a_2)\dot{\theta}_f-k_{fx}(x_{t1}+x_{t2}+x_{t3}+x_{t4})-c_{fx}(\dot{x}_{t1}+\dot{x}_{t2}+\dot{x}_{t3}+\dot{x}_{t4})=0$$

(8)

Cab in X-translation

$$m_c\ddot{x}_c+4k_{cx}x_c-4k_{cx}x_f+4c_{cx}\dot{x}_c-4c_{cx}\dot{x}_f-2k_{cx}(a_3-2a_1-a_2)\theta_f-2c_{cx}(a_3-2a_1-a_2)\dot{\theta}_f=0$$

(A-3)

Four tires in Y-translation

$$m_t\ddot{y}_{ti}+(k_{fy}+k_t)y_{ti}-k_{fy}y_f+(c_{fy}+c_t)\dot{y}_{ti}-c_{fy}\dot{y}_f-\frac{k_{fy}a_3}{2}\varphi_f-\frac{k_{fy}d_3}{2}\psi_f,\quad -\frac{c_{fy}a_3}{2}\dot{\varphi}_f-\frac{c_{fy}d_3}{2}\dot{\psi}_f=0$$

i = 1, …, 4

(9)

Frame in Y-translation

$$m_f\ddot{y}_f+4(k_{cy}+k_{fy})y_f-4k_{cy}y_c+4(c_{cy}+c_{fy})\dot{y}_f-4c_{cy}y_c-2k_{cy}(d_3-2d_1-d_2)\psi_f-2c_{cy}(d_3-2d_1-d_2)\dot{\psi}_f-k_{fy}(y_{t1}+y_{t2}+y_{t3}+y_{t4})-c_{fy}(\dot{y}_{t1}+\dot{y}_{t2}+\dot{y}_{t3}+\dot{y}_{t4})=0$$

(A-5)

Cab in Y-translation

$$m_c\ddot{y}_c+4k_{cy}y_c-4k_{cy}y_f+4c_{cy}\dot{y}_c-4c_{cy}\dot{y}_f-2k_{cy}(d_3-2d_1-d_2)\psi_f-2c_{cy}(d_3-2d_1-d_2)\dot{\psi}_f=0$$

(A-6)

Four tires in Z-translation

$$m_t\ddot{z}_{ti}+k_{fz}z_{ti}-k_{fz}z_f+c_{fz}\dot{z}_{ti}-c_{fz}\dot{z}_f-\frac{k_{fz}d_3}{2}\theta_f-\frac{c_{fz}d_3}{2}\dot{\theta}_f=0$$

i = 1, …, 4

(A-7)

Frame in Z-translation

$$m_f\ddot{z}_f+4(k_{cz}+k_{fz})z_f-4k_{cz}z_c+4(c_{cz}+c_{fz})\dot{z}_f-4c_{cz}\dot{z}_c+2k_{cz}(d_3-2d_1-d_2)\theta_f+2c_{cz}(d_3-2d_1-d_2)\dot{\theta}_f-k_{fz}(z_{t1}+z_{t2}+z_{t3}+z_{t4})-c_{fz}(z_{t1}+z_{t2}+z_{t3}+z_{t4})=0$$

(A-8)

Cab in Z-translation

$$m_c\ddot{z}_c+4k_{cz}z_c-4k_{cz}z_f+4c_{cz}\dot{z}_c-4c_{cz}\dot{z}_f-2k_{cz}(d_3-2d_1-d_2)\theta_f-2c_{cz}(d_3-2d_1-d_2)\dot{\theta}_f=0$$

(A-9)

Frame in X-rotation

$$I_{xxf}\ddot{\varphi}_f+(a_2^2k_{cy}+a_3^2k_{fy})\varphi_f-a_2^2k_{cy}\varphi_c+\frac{a_3}{2}k_{fy}(y_{t1}+y_{t3}-y_{t2}-y_{t4})+(a_2^2c_{cy}+a_3^2c_{fy})\dot{\varphi}_f-a_2^2c_{cy}\dot{\varphi}_c+\frac{a_3}{2}c_{fy}(\dot{y}_{t1}+\dot{y}_{t3}-\dot{y}_{t2}-\dot{y}_{t4})$$

(A-10)

Cab in X-rotation

$$I_{xxc}\ddot{\varphi}_c+a_2^2k_{cy}\varphi_c-a_2^2k_{cy}\varphi_f+a_2^2c_{cy}\dot{\varphi}_c-a_2^2c_{cy}\dot{\varphi}_f=0$$

(A-11)

Frame in Y-rotation

$$I_{xxf}\ddot{\theta}_f+2k_{cz}(d_3-2d_1-d_2)z_f-2k_{cz}(d_3-2d_1-d_2)z_c-(a_2^2k_{cx}+d_2^2k_{cz})\theta_c+(a_2^2k_{cx}+a_3^2k_{fx}+d_3^2k_{fz}-2k_{cz}d_2(d_3+2d_1-d_2))\theta_f-\frac{d_3}{2}k_{fz}(z_{t1}+z_{t2}-z_{t3}-z_{t4})-\frac{a_3}{2}c_{fy}(z_{t1}+z_{t3}-z_{t2}-z_{t4})+2c_{cz}(d_3-2d_1-d_2)\dot{z}_f-2c_{cz}(d_3-2d_1-d_2)\dot{z}_c-(a_2^2c_{cx}+d_2^2c_{cz})\dot{\theta}_c+(a_2^2c_{cx}+a_3^2c_{fx}+d_3^2k_{fz}-2c_{cz}d_2(d_3+2d_1-d_2))\dot{\theta}_f-\frac{d_3}{2}c_{fz}(\dot{z}_{t1}+\dot{z}_{t2}-\dot{z}_{t3}-\dot{z}_{t4})-\frac{a_3}{2}c_{fy}(\dot{z}_{t1}+\dot{z}_{t3}-\dot{z}_{t2}-\dot{z}_{t4})=0$$

(A-12)

Cab in X-rotation

$$I_{yyc}\ddot{\theta}_c+\left(d_2^2k_{cz}-a_2^2k_{cx}\right)\theta_c+\left(a_2^2k_{cx}-d_2^2k_{cz}\right)\theta_f+\left(d_2^2c_{cz}-a_2^2c_{cx}\right)\dot{\theta}_c+\left(a_2^2c_{cx}-d_2^2c_{cz}\right)\dot{\theta}_f=0$$

(A-13)

Frame in Z-rotation

$$
\begin{aligned}
&I_{zzf}\ddot{\psi}_f + 2k_{cy}(d_3 - 2d_1 - d_2)y_c - 2k_{cy}(d_3 - 2d_1 - d_2)y_f \\
&- d_2^2 k_{cy}\psi_c - [2k_{cy}d_2(d_3 + 2d_1 - d_2) - d_3^2 k_{fy}]\psi_f \\
&+ \frac{d_3}{2}k_{fy}(y_{t1} + y_{t2} - y_{t3} - y_{t4}) + 2c_{cy}(d_3 - 2d_1 - d_2)\dot{y}_c \\
&- 2c_{cy}(d_3 - 2d_1 - d_2)\dot{y}_f - d_2^2 c_{cy}\dot{\psi}_c - [2c_{cy}d_2(d_3 + 2d_1 - d_2) \\
&- d_3^2 c_{fy}]\dot{\psi}_f + \frac{d_3}{2}c_{fy}(\dot{y}_{t1} + \dot{y}_{t2} - \dot{y}_{t3} - \dot{y}_{t4}) = 0
\end{aligned}
\tag{A-14}
$$

Cab in Z-rotation

$$
I_{zzc}\ddot{\psi}_c + k_{cy}a_2d_2\psi_c - k_{cy}a_2d_2\psi_f + k_{cy}a_2d_2\dot{\psi}_c - k_{cy}a_2d_2\dot{\psi}_f = 0
\tag{A-15}
$$

Note: t-tire, c-cab, f-frame, a_i and d_i are distances measured from the vehicle model.