APPENDIX C

The Bessel function of the second kind, for integral order, may be defined by the relation

$$Y_n(v) = \lim_{\nu \to n} \frac{\cos \nu\pi J_\nu(v) - J_{-\nu}(v)}{\sin \nu\pi} \tag{C.5}$$

and application of L'Hospital's rule to (C.5) leads to the series (3.73). It therefore follows, since Y_n is definable in terms of Bessel functions of the first kind, which obey (C.4), that

$$Y_{-n}(v) = (-1)^n Y_n(v) \tag{C.6}$$

Because of the nature of the defining relations (3.76) and (3.77) the Hankel functions also obey this law. However, the manner in which the modified Bessel functions are defined leads to the result that

$$I_{-n}(w) = I_n(w) \qquad K_{-n}(w) = K_n(w) \tag{C.7}$$

with $w = \mathfrak{k}r$. These formulas are useful when working with the orthogonal representations (3.85) and (3.87).

When both sides of (C.2) are differentiated with respect to t, one obtains

$$\begin{aligned}\sum_{n=-\infty}^{\infty} nt^{n-1}J_n(v) &= \frac{1}{2}v\left(1 + \frac{1}{t^2}\right)\exp\left[\frac{1}{2}v\left(t - \frac{1}{t}\right)\right] \\ &= \frac{1}{2}v\left(1 + \frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} t^n J_n(v)\end{aligned}$$

If the expression on the right is arranged in powers of t and the coefficients of t^{n-1} are equated, it is evident that

$$nJ_n(v) = \frac{v}{2}[J_{n-1}(v) + J_{n+1}(v)] \tag{C.8}$$

If any two successive Bessel functions are known, the third in sequence can be deduced from (C.8) and then this process may be repeated indefinitely.

Alternatively, if both sides of (C.2) are differentiated with respect to v, one obtains

$$\begin{aligned}\sum_{n=-\infty}^{\infty} t^n J_n'(v) &= \frac{1}{2}\left(t - \frac{1}{t}\right)\exp\left[\frac{1}{2}v\left(t - \frac{1}{t}\right)\right] \\ &= \frac{1}{2}\left(t - \frac{1}{t}\right)\sum_{n=-\infty}^{\infty} t^n J_n(v)\end{aligned}$$

Upon equating coefficients of t^n on the two sides of this identity, one finds that

$$J_n'(v) = \tfrac{1}{2}[J_{n-1}(v) - J_{n+1}(v)] \tag{C.9}$$

Equations (C.8) and (C.9) are known as recurrence relations and are also satisfied by Bessel functions of the second and third kind. However, because of the nature of the definitions (3.80) and (3.81), the modified Bessel functions satisfy

$$nI_n(w) = \frac{w}{2}[I_{n-1}(w) - I_{n+1}(w)] \tag{C.10}$$

$$I_n'(w) = \tfrac{1}{2}[I_{n-1}(w) + I_{n+1}(w)] \tag{C.11}$$

and

$$nK_n(w) = -\frac{w}{2}[K_{n-1}(w) - K_{n+1}(w)] \tag{C.12}$$

$$K_n'(w) = -\tfrac{1}{2}[K_{n-1}(w) + K_{n+1}(w)] \tag{C.13}$$

Upon eliminating either J_{n-1} or J_{n+1} from (C.8) and (C.9) one obtains

$$vJ_n'(v) + nJ_n(v) = vJ_{n-1}(v)$$
$$vJ_n'(v) - nJ_n(v) = -vJ_{n+1}(v)$$

which are equivalent to

$$\frac{d}{dv}[v^n J_n(v)] = v^n J_{n-1}(v) \tag{C.14}$$

$$\frac{d}{dv}[v^{-n} J_n(v)] = -v^{-n} J_{n+1}(v) \tag{C.15}$$

These differentiation formulas are also obeyed by the Bessel functions of the second and third kind. For $n = 0$ the result is simply

$$J_0'(v) = -J_1(v) \tag{C.16}$$

Because of the difference in the recurrence relations, the modified Bessel functions satisfy

$$\frac{d}{dw}[w^n I_n(w)] = w^n I_{n-1}(w) \tag{C.17}$$

$$\frac{d}{dw}[w^{-n} I_n(w)] = w^{-n} I_{n+1}(w) \tag{C.18}$$

and

$$\frac{d}{dw}[w^n K_n(w)] = -w^n K_{n-1}(w) \tag{C.19}$$

$$\frac{d}{dw}[w^{-n} K_n(w)] = -w^{-n} K_{n+1}(w) \tag{C.20}$$

If $v = kr$ is real, the $J_n(v)$ functions oscillate and each has a sequence of roots which may be designated by $\gamma_{n1}, \gamma_{n2}, \ldots, \gamma_{nm}, \ldots$, such that $J_n(\gamma_{nm}) = 0$,

$$m = 1, 2, 3, \ldots$$

A family of functions

$$\sqrt{v}\, J_n\left(\gamma_{nm}\frac{v}{v_0}\right) \tag{C.21}$$

can be generated with the property that each of these functions has a null at $v = v_0$; for the mth function there are m nulls in the interval $0 \le v \le v_0$. That the individual members of the family (C.21) are orthogonal to each other may be seen by the following argument:

Let

$$f_m(v) = \sqrt{v}\, J_n\left(\gamma_{nm}\frac{v}{v_0}\right)$$

$$f_p(v) = \sqrt{v}\, J_n\left(\gamma_{np}\frac{v}{v_0}\right)$$

be any two members of the family. By direct substitution they are seen to satisfy the differential equations

$$4v^2\frac{d^2f_m}{dv^2} + \left(4\frac{\gamma_{nm}^2}{v_0^2}v^2 - 4n^2 + 1\right)f_m = 0$$

$$4v^2\frac{d^2f_p}{dv^2} + \left(4\frac{\gamma_{np}^2}{v_0^2}v^2 - 4n^2 + 1\right)f_p = 0$$

Multiplying the first of these by f_p and the second by f_m and subtracting furnishes the identity

$$-\frac{\gamma_{nm}^2 - \gamma_{np}^2}{v_0^2}f_mf_p = f_m''f_p - f_p''f_m$$

Integration of both sides of this identity from 0 to $\mathfrak{v}$ yields

$$\begin{aligned}-\frac{\gamma_{nm}^2 - \gamma_{np}^2}{v_0^2}\int_0^{\mathfrak{v}} f_mf_p\,dv &= \int_0^{\mathfrak{v}} (f_m''f_p - f_p''f_m)\,dv\\ &= \left[f_m'f_p\Big|_0^{\mathfrak{v}} - \int_0^{\mathfrak{v}} f_m'f_p'\,dv\right] - \left[f_p'f_m\Big|_0^{\mathfrak{v}} - \int_0^{\mathfrak{v}} f_p'f_m'\,dv\right]\\ &= [f_m'f_p - f_p'f_m]_0^{\mathfrak{v}}\end{aligned}$$

which can be written

$$\begin{aligned}&\frac{\gamma_{nm}^2 - \gamma_{np}^2}{v_0^2}\int_0^{\mathfrak{v}} vJ_n\left(\gamma_{nm}\frac{v}{v_0}\right)J_n\left(\gamma_{np}\frac{v}{v_0}\right)dv\\ &\qquad = \frac{\mathfrak{v}}{v_0}\left[\gamma_{np}J_n\left(\gamma_{nm}\frac{\mathfrak{v}}{v_0}\right)J_n'\left(\gamma_{np}\frac{\mathfrak{v}}{v_0}\right) - \gamma_{nm}J_n\left(\gamma_{np}\frac{\mathfrak{v}}{v_0}\right)J_n'\left(\gamma_{nm}\frac{\mathfrak{v}}{v_0}\right)\right]\end{aligned} \tag{C.22}$$

If $m \neq p$ and $\mathfrak{v} = v_0$, since $J_n(\gamma_{nm}) = J_n(\gamma_{np}) = 0$, the above formula reduces to

$$\int_0^{v_0} vJ_n\left(\gamma_{nm}\frac{v}{v_0}\right)J_n\left(\gamma_{np}\frac{v}{v_0}\right)dv = 0 \tag{C.23}$$

and thus (C.21) is an orthogonal family of functions.

Upon differentiating (C.22) with respect to γ_{nm} and then letting $m = p$ and $\mathfrak{v} = v_0$ one obtains

$$\int_0^{r_0} vJ_n^2\left(\gamma_{nm}\frac{v}{v_0}\right)dv = \frac{v_0^2}{2}J_n'^2(\gamma_{nm}) \tag{C.24}$$