Screening of a point charge by an anisotropic medium: Anamorphoses in the method of images

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(Received 21 August 2000; accepted 30 October 2000)

We extend the classical dielectric image problem for an external point charge interacting with a semi-infinite medium to the case of an anisotropic dielectric. We show that the exterior potential is a solution to an unconventional image problem. By contrast the interior potential is not a solution to any simple image system; instead it can be obtained through an anamorphic transformation of an image solution. We calculate the volume bound charge density that is induced within the anisotropic dielectric and distinguishes it from an isotropic medium where the volume density vanishes. This solution provides a very simple and physically relevant example in which the physics of the surface and volume bound charge densities in a polarized dielectric can be analyzed.

[DOI: 10.1119/1.1341252]

I. INTRODUCTION

In special geometries the electric potentials produced by charges interacting with conductors or dielectrics can be calculated using the method of images. Here one uses the idea that in certain situations the Laplace equation for the electrostatic potential V can be solved in a region space using a trick in which the required boundary conditions for V are simulated using one or more ''image'' charges placed in a different region. This approach to potential theory surprises most students of physics who first encounter it and dates back at least as far as the electrical researches of Thomson.[1]

When introducing students to this idea one stresses the idea that the fictitious charges in the image space simulate the effect of physical charges that typically are found at boundaries separating the ''physical'' and ''image'' spaces. Certainly, this is true for conductors where the classical theory requires that any free charge resides exactly at a boundary between a conducting and nonconducting medium. For interfaces involving dielectrics, however, the situation is much more subtle. For a linear homogeneous and *isotropic* (LHI) dielectric, the induced polarization P is related to the electric field strength E through a scalar susceptibility χ,

$$P=\chi E. \tag{1}$$

In general a dielectric can have a volume density of bound charge $\rho_b=-\nabla\cdot P$, but for a linear, homogeneous, and isotropic medium it is easy to demonstrate that

$$\rho_b=-(1-1/\epsilon)\rho_f, \tag{2}$$

where the dielectric constant $\epsilon=1+4\pi\chi$ and ρ_f denotes the volume density of free charges. Thus *within* a LHI medium where ρ_f is zero, ρ_b vanishes also, and the physical bound charges can be found only at boundaries where χ changes abruptly.[2] For example, recall that when a charge q is placed a perpendicular distance s above a planar boundary to a semifinite LHI dielectric, the exterior potential is the sum of the unscreened potential of the source q and the potential of its dielectric image which is a point charge of strength $q'=-q(\epsilon-1)/(\epsilon+1)$ located a distance s below the interface. This image charge simulates the effect of the physical surface bound charge density $\mathbf{P}\cdot\hat{n}$, which is induced exactly *on* the boundary to the dielectric with unit surface normal $\hat{n}$.

Because of the proportionality in Eq. (2) it is awkward to construct simple physical examples in electrostatics to illustrate the physics of the volume density ρ_b. Some of the simplest situations examine the charge distributions within domain walls of finite width separating different electrically poled phases of matter or the charge and field distributions found within a hypothetical linear medium where the susceptibility $\chi(r)$ is spatially nonuniform. In this paper we consider the physics of an alternative (and physically relevant) class of systems that involve linear, homogeneous, but *anisotropic* (LHA) dielectrics. Below we analyze several simple geometries in which free charges interacting with LHA dielectric media produce screened potentials that admit solutions using the method of images. Not surprisingly, these image solutions are *different* from the familiar dielectric image solutions appropriate to LHI media, and so these LHA media provide a family of simple but nontrivial image problems that may be of interest to intermediate students.

For completeness, in this paper we begin by studying the screening problem for an external point charge interacting with a LHA medium by explicitly solving the appropriate boundary value problem for the electrostatic potential. The solution will then be identified with that of a particular type of image problem. We then discuss the analogous results for the point charge embedded in a LHA medium near a planar interface to vacuum. Finally, we will see that these results can also be obtained by a different route: *anamorphosis*. In this last approach we solve for the potentials by a judicious nonuniform scale transformation applied to the spatial coordinates for these systems. This latter method is devious but powerful. It provides a transparent interpretation of the image solutions for LHA media and clearly distinguishes them from the more familiar dielectric image solutions for LHI systems.

II. CHARGE ABOVE A PLANAR BOUNDARY TO A LHA MEDIUM

We study the geometry illustrated in Fig. 1. A point charge of strength Q is located a perpendicular distance s above a semi-infinite linear dielectric medium where the sur-

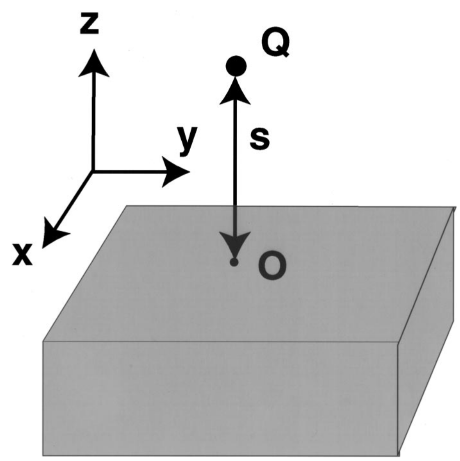

Fig. 1. Geometry of the image problem studied in this paper. A charge Q is placed a perpendicular distance s above a semi-infinite dielectric. The interface occupies the xy plane with the origin located directly below the charge Q. The susceptibility tensor χ in the dielectric is diagonal but anisotropic with tensor elements given by Eq. (5).

face of the dielectric lies in the xy plane. In the region $z>0$ the electric potential can be obtained by superposing the unscreened potential from the point charge with the potential produced by the bound charge induced in the dielectric. Both can be represented in cylindrical coordinates (r, ϕ, z) using the Fourier–Bessel expansion;[3] the unscreened charge produces an electric potential

$$V^{>}(r,z)=Q\int dq\, J_0(qr)e^{-q|z-s|}, \tag{3}$$

while the exterior potential produced by the bound charge in the dielectric is obtained by combining solutions to the Laplace equation

$$V_b^{>}(r,z)=\int dq\, F(q)J_0(qr)e^{-qz}, \tag{4}$$

where the function $F(q)$ ultimately will be determined by the boundary conditions.

In the region $z<0$ there is an induced polarization $\mathbf{P}=\chi\cdot\mathbf{E}$. We consider the case where the susceptibility tensor χ is *diagonal* but *anisotropic*,

$$\chi=\begin{pmatrix}\chi_r & 0 & 0\\ 0 & \chi_r & 0\\ 0 & 0 & \chi_z\end{pmatrix}. \tag{5}$$

In general a polarization of this medium produces the bound charge density[2]

$$\rho_b=-\boldsymbol{\nabla}\cdot\mathbf{P}=\boldsymbol{\nabla}\cdot\chi\cdot\boldsymbol{\nabla}V^{<}, \tag{6}$$

where $V^{<}$ is the interior potential which satisfies the Poisson equation

$$\boldsymbol{\nabla}^2V^{<}(r,z)=-4\pi\rho. \tag{7}$$

In the interior of the dielectric, the only sources appearing on the right-hand side of (7) are the bound charges (6), and by expressing all the parts of (7) in terms of the interior potential $V^{<}$ we obtain the useful result

$$(1+4\pi\chi_r)\frac{\partial}{r\partial r}\left(r\frac{\partial V}{\partial \tau}\right)+(1+4\pi\chi_z)\frac{\partial^2V^{<}}{\partial^2 z}=0, \tag{8}$$

which is a Laplace equation, but in the *scaled coordinates* $\xi_r=r/\sqrt{1+4\pi\chi_r}$ and $\xi_z=z/\sqrt{1+4\pi\chi_z}$. Here it is useful to identify the diagonal elements of the dielectric tensor, and an anisotropy parameter γ,

$$\begin{aligned}&\epsilon_z=1+4\pi\chi_z\,, \quad \epsilon_r=1+4\pi\chi_r\,,\\ &\gamma=\sqrt{\epsilon_z/\epsilon_r},\end{aligned} \tag{9}$$

and write the interior potential by superposing solutions to the Laplace equation in the scaled variables

$$V^{<}(r,z)=\int dq G(q)J_0(\gamma qr)e^{qz}, \tag{10}$$

where the function $G(q)$ will also be determined by the boundary conditions. Note that in the special case of an isotropic dielectric, $\gamma\equiv 1$ and in that case (10) degenerates to a solution to the Laplace equation in the *physical* variables r and z. Thus in the interior of the polarized but isotropic dielectric, the bound charge density $-\boldsymbol{\nabla}^2V/4\pi$ vanishes.

The functions $F(q)$ and $G(q)$ need to be chosen to enforce the boundary conditions at the matching plane $z=0$: continuity of the parallel component of $\mathbf{E}$ (equivalently, continuity of the potential) and continuity of the normal component of $\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}$. Explicitly, these two conditions require that

$$\int dq(Qe^{-qs}+F(q))J_0(qr)=\int dq\, G(q)J_0(\gamma qr) \tag{11}$$

and

$$\int dq\, q(Qe^{-qs}-F(q))J_0(qr)=\int dq\, qG(q)J_0(\gamma qr). \tag{12}$$

The functions $F(q)$ and $G(q)$ are extracted from the integrals (11) and (12) by using the orthogonality relations for the Bessel functions,[4]

$$\int_0^\infty J_0(qr)J_0(q'r)r\,dr=\delta(q-q')/q', \tag{13}$$

from which we obtain

$$F(q)=-Q\left(\frac{\sqrt{\epsilon_r\epsilon_z}-1}{\sqrt{\epsilon_r\epsilon_z}+1}\right)e^{-qs} \tag{14}$$

and

$$G(q)=Q\left(\frac{2}{\sqrt{\epsilon_r\epsilon_z}+1}\right)e^{-qs}. \tag{15}$$

Using (14) and (15) we calculate the exterior and interior potentials

$$V^{>}(r,z)=Q\int dq\, J_0(qr)\left(e^{-q|z-s|}-\left(\frac{\sqrt{\epsilon_r\epsilon_z}-1}{\sqrt{\epsilon_r\epsilon_z}+1}\right)e^{-q(s+z)}\right) \tag{16}$$

and

$$V^{<}(r,z)=Q\int dq\, J_0(qr)\left(\frac{2}{\sqrt{\epsilon_r\epsilon_z}+1}\right)e^{q(z/\gamma-s)}. \tag{17}$$

Equations (16) and (17) give a complete solution for the electrostatic potentials.

III. IMAGE SOLUTIONS

For an isotropic dielectric $\gamma = 1$, and the dielectric constant $\epsilon = 1 + 4\pi\chi$. For this case the solution in (16) and (17) degenerates to the well-known results for dielectric images at a planar interface.[2] In the outside space, an observer sees the potential (16) produced by the surface bound charge as though it were the potential produced by an image charge of strength

$$Q_i = -Q\,\frac{\epsilon - 1}{\epsilon + 1} \tag{18}$$

situated a distance s below the matching plane. An observer inside the dielectric sees the potential (17) of a screened point charge

$$Q_i' = \frac{2Q}{\epsilon + 1} = Q + Q_i\,. \tag{19}$$

Equation (19) emphasizes that both the exterior and interior observers ''see'' the same screening charge at the boundary.

For the anisotropic problem, the outside observer sees the potential produced by bound charge in the dielectric as the potential of an image point charge Q_i located a perpendicular distance s below the interface, and with a magnitude

$$Q_i = -Q\,\frac{\sqrt{\epsilon_r \epsilon_z} - 1}{\sqrt{\epsilon_r \epsilon_z} + 1}. \tag{20}$$

Thus the *effective* dielectric constant of the medium is given by the geometric mean $\epsilon_{\mathrm{eff}} = \sqrt{\epsilon_r \epsilon_z}$. In particular in the extreme anisotropic limit where $\epsilon_z = 1$, the dielectric image depends on the *square root* of the dielectric constant $\sqrt{\epsilon_r}$ in the strongly polarizable direction.

However, the most important distinction between the anisotropic and isotropic solutions occurs for the interior potentials. For the anisotropic problem, the potential in Eq. (17) does not admit a simple image solution. Instead, by carrying out the integral in (17) one finds that

$$V^{<}(r,z) = \left(\frac{2Q}{\sqrt{\epsilon_r \epsilon_z} + 1}\right) \Big/ \sqrt{r^2 + (z/\gamma - s)^2}. \tag{21}$$

This is the potential of a charge of strength $Q_i' = (2Q/(\sqrt{\epsilon_r \epsilon_z} + 1))$, but seen in a distorted space in which the spatial coordinates *within* the dielectric have their z components rescaled by the anisotropy parameter γ. The effects of the rescaling can be seen clearly in the contour plots of the potential displayed in Fig. 2, which show the equipotential surfaces calculated for three situations with $\gamma = 5$ (strongly polarizable along the z direction), 1 (isotropic), and 1/5 (strongly polarizable in the radial direction). For the isotropic solution, one sees that the equipotential surfaces inside the dielectric are concentric spheres centered on the position of the physical charge in the exterior space. For the anisotropic solutions, the equipotentials are stretched ($\gamma > 1$) or compressed ($\gamma < 1$) along the z direction.

The equipotentials in the interior space are nonspherical precisely because the bound charge induced in this system is *not* confined to the planar boundary between the vacuum and the dielectric. The interior potential is given by a solution to Eq. (8), but in general a solution to this *scaled* Laplace equation $\boldsymbol{\nabla}_\xi^2 V^{<} = 0$ requires that the physical bound charge density $\rho_b = -\boldsymbol{\nabla}^2 V^{<}/4\pi$ is nonzero, and instead is given by

$$\rho_b = \left(\frac{1 - \gamma^2}{4\pi}\right) \frac{\partial^2 V^{<}(r,z)}{\partial z^2}. \tag{22}$$

In addition, if $\epsilon_z \neq 1$ there is a true surface contribution to the bound charge $\sigma_b = \mathbf{P} \cdot \hat{z}$, which can be expressed explicitly

$$\sigma_b = \frac{Q}{2\pi\gamma}\,\frac{s(1 - \epsilon_z)}{(1 + \sqrt{\epsilon}}$$

$$(1+4\pi\chi_r)\frac{\partial}{r\partial r}\left(r\frac{\partial V^<}{\partial r}\right)+(1+4\pi\chi_z)\frac{\partial^2 V^<}{\partial^2 z}$$
$$=-4\pi Q\,\delta(z+s)\,\delta(r)/r. \qquad (24)$$

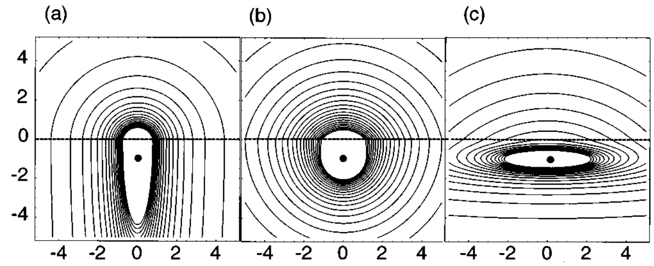

Fig. 4. Equipotentials for a point charge embedded in a semi-infinite anisotropic dielectric near a planar interface to vacuum. (a)–(c) $(\sqrt{\epsilon_z}, \sqrt{\epsilon_r}) = (5,1)$, (5, 5), and (1, 5). The dashed line denotes the boundary between vacuum and the dielectric medium.

induces a ''plume'' of charge (of the same sign!) which extends along the highly polarizable direction. This is accompanied by a ''ring'' of bound charge of opposite sign which surrounds the free charge in its weakly polarizable direction. In addition to the volume contributions, a singular bound charge

$$Q_s = -\left(1 - \frac{1}{\sqrt{\epsilon_r \epsilon_z}}\right) Q \tag{34}$$

accumulates exactly at the site of the free charge and partially compensates the free charge. The total bound charge (which consists of volume, singular, *and* surface contributions) integrates to zero.

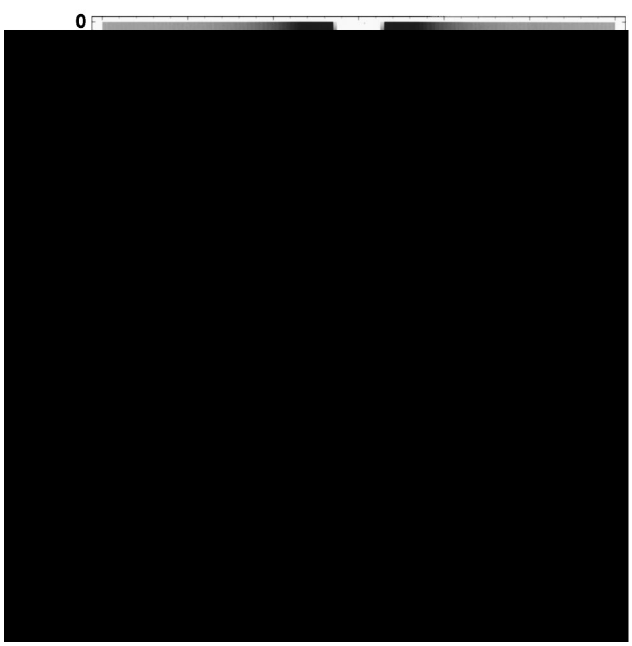

Fig. 5. Grayscale plots of the volume bound charge density induced in an anisotropic medium that contains an embedded charge a perpendicular distance s from an interface with vacuum interface. The vacuum–dielectric interface lies in the xy plane, and results are plotted inside the dielectric in the xz plane with $z<0$. In the top panel the medium is strongly polarizable along the z direction, and in the lower panel it is strongly polarizable in the radial direction. Bright regions denote bound charge with the same sign as the free charge and the dark regions denote regions where the bound charge has the opposite sign. For clarity, a singular contribution to the bound charge which accumulates at the location of the free charge is not plotted.

V. ANAMORPHOSIS

The results obtained in the previous three sections can be obtained directly by using anamorphosis. We have been making repeated use of this idea in our previous discussion, but in this section we will make the connection explicit. The anamorphic approach uses geometrical perspective to solve for the potentials. *Anamorphosis* is derived from the Greek *morphe* (shape) and *ana* (again), suggesting that the observer can ''reshape'' the problem into a simpler form.[5,6]

The central mathematical result is contained in Eq. (8), which demonstrates that inside the anisotropic but linear medium the electrostatic potential is a solution to a Laplace equation when expressed in the scaled coordinates ξ_r and ξ_z rather than in the physical spatial variables r and z. Therefore the solutions for the electric potential in the LHA medium are the ordinary solutions for the electric potential in an empty, albeit scaled space. The only complication is that the scale factors Γ_i change discontinuously at the vacuum dielectric interface, and therefore we need to properly connect the exterior unscaled solutions for $z>0$ with their interior scaled counterparts for $z<0$. This is accomplished by decomposing the potential into orthogonal radial and isotropic modes and representing the full potential by superposing the separable solutions of the form $J_0(qr)Z_q(z)$.

In the exterior space the function $Z(z)$ is chosen to ensure that the potential has no net curvature; this in turn requires that

$$Z_q^{>}(z) = A_q e^{qz} + B_q e^{-qz}. \tag{35}$$

In the interior space, the potential has no net curvature in the *scaled* variables and the solution which is regular when $z \to -\infty$ is

$$Z^{<}(z) = C_q e^{qz/\gamma}. \tag{36}$$

Our matching conditions at the surface of the dielectric require continuity of the potential and continuity of the normal component $D_z = -\epsilon_z^{<(>)} \partial_z V^{<(>)}$ (note that this is a z derivative and not a ξ_z derivative inside the medium) which are solved to give

$$B_q = -A_q \left(\frac{\sqrt{\epsilon_r \epsilon_z} - 1}{\sqrt{\epsilon_r \epsilon_z} + 1} \right) \tag{37}$$

and

$$C_q = A_q \left(\frac{2}{\sqrt{\epsilon_r \epsilon_z} + 1} \right). \tag{38}$$

Finally, the unknown coefficients A_q are obtained by applying this construction at the plane $z=s$. For $(z>s)$ only the factor $Z(z) \propto e^{-qz}$ satisfies the boundary conditions at $z \to \infty$ and the z component of the electric field contains a discontinuity at $z=s$ due to the external free charge Q; thus $(E_z(r,s^+) - E_z(r,s^-))/4\pi = Q\delta(r)/2\pi r$. From this we conclude that $A_q = Qe^{-qs}$ and the solutions (37) and (38) recover the solution for the electric potential which were obtained previously in Sec. II.

In fact the steps leading to this solution are essentially identical to those of Sec. II, although the anamorphic construction offers an additional insight. Here we find that our problem describes a free charge Q in an empty region of space near a boundary to a medium which can be represented as an empty region of a space that is ''stretched'' or ''compressed'' along the z direction. In appropriately scaled variables, the potentials everywhere are given by the curvature-free solutions to the Laplace equation. However, the *scaling function* changes discontinuously at the boundary between these media; the physical electrostatic boundary conditions at this interface thus generate the ''reflected'' ($Z \sim e^{-qz}$) and the ''transmitted'' ($Z \sim e^{q\xi_z}$) parts of the full potential. Of course the distortion of the equipotentials in the LHA medium is a direct consequence of the scaling transformations applied to the spatial coordinates.

VI. DISCUSSION

In this paper we have discussed a simple physical model for a dielectric in which one can investigate the screening effects of both the volume and surface bound charge densities. The traditional discussion of bound charges in a linear dielectric focuses on the role of screening by surface or interface charges, since in the simplest situation of a linear homogeneous and isotropic system the volume bound charge density vanishes [see Eq. (2)]. However, dielectric anisotropy is a generic property of most condensed phases of matter where the isotropy of free space is broken. This gives an opportunity to study the physics of the bound charge density in a physically relevant family of problems. As a bonus, one discovers that the solution contains an interesting geometrical wrinkle: The self-consistently screened potential in a LHA medium can be obtained by a suitable scaling of the potentials in empty space. This method has a straightforward extension to a system with a general susceptibility tensor where a generalized coordinate transformation can be introduced.[7]

The anamorphic solution provides a direct method for obtaining the electrostatic potentials in yet more complicated geometries. For example, for a point charge a perpendicular distance s above a dielectric film of thickness d, the exterior electrostatic potentials have an image solution requiring an *infinite series* of image point charges (images of images of images, etc.) separated by distance $2d$ which is the round-trip distance through the dielectric film.[3] For the anisotropic system one can show that the spacing between images is scaled to $2d/\gamma$, reflecting the effective rescaling of the ''z coordinate'' in the anisotropic medium.

Some closely related results have been reported previously in the technical literature. In Ref. 8, Wait analyzes the electrostatic potential in an anisotropic conductor that carries a steady current. This requires a solution to a Laplace equation for the electrostatic potential in the interior of the anisotropic current carrying conductor, which is solved using the same scaling methods that are presented in this paper. Interestingly, this solution demonstrates that an anisotropic current carrying conductor can maintain a nonvanishing *free charge* density in its interior in the steady state. In Ref. 9 Lindell *et al.* analyze the electrostatic potential for a point charge interacting with a semi-infinite medium characterized by a general symmetric positive definite susceptibility tensor. Remarkably, the image solutions in this problem consist of an image point augmented by a continuous charge distribution on a conical surface that terminates at the image charge.

Three-dimensional phases composed of polymers or bundles of nanotubes present physical situations where the dielectric properties are locally strongly anisotropic. This has a profound effect on the effective interaction between charged species embedded in such a medium. In fact, for special geometries in low dimensions, one can engineer a dielectric environment that modifies the analytic form of the effective electrostatic interactions between free charges. This has an important application in controlling band bending and designing electrostatic gates in molecular electronics.[10]

ACKNOWLEDGMENTS

This work was supported in part by the Department of Energy under Grant No. DE-FG-02-84ER45118 and by the Laboratory for Research on the Structure of Matter. I am happy to thank Mike Cohen and Andrew Zangwill for particularly helpful discussions on this topic.

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[1]W. Thomson, *Reprints of Papers on Electrostatics and Magnetism* (MacMillian, London, 1872). In his *Treatise on Electricity and Magnetism*, Maxwell attributes the discovery of the method of images to Thomson's work published in 1848 in *The Cambridge and Dublin Mathematical Journal*.

[2]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice–Hall, Englewood Cliffs, NJ, 1999), p. 160f.

[3]W. R. Smythe, *Static and Dynamic Electricity* (McGraw–Hill, New York, 1968), Sec. 5.303.

[4]G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge U.P., London, 1952).

[5]M. Schuyt and J. Elffers (organizing artists), *Anamorphoses* (Abrams, New York, 1976).

[6]A nice web site with a history and illustrations of anamorphoses is http://www.SkillTeam.com/SKTWeb.ns4/all/Anamor_anamor.html.

[7]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Elsevier Science, New York, 1960), Sec. 14.

[8]J. P. Wait, ''Current Flow into a Three Dimensional Anisotropic Conductor,'' Radio Sci. **25**, 689–694 (1990).

[9]I. V. Lindell, K. I. Nikoskinen, and A. Viljanen, ''Electrostatic Image Method for the Anisotropic Half Space,'' IEE Proc.: Sci., Meas. Technol. **144**, 156–162 (1997).

[10]C. L. Kane and E. J. Mele, ''Dielectric Control of Electrostatic Barriers in Molecular Electronics,'' Appl. Phys. Lett. (in press).