

Chapter IV

Electrostatics

After these introductory chapters, we now return to the subject of electrostatics and treat it in some detail. There is quite a bit more to be said about the fields of singular sources, of which the point-charge and the dipole are only the first examples. It is especially important to understand the quadrupole and its interaction because of the application to nuclei in solids. There are some general techniques for solving electrostatic problems: images, inversion, complex variable, and separation of variables. Accounts and examples of each of these are given. It is perhaps an exaggeration to call them general techniques because the set of problems that they solve is, after all, quite small. However, every solved problem counts as progress along the way to understanding a subject. Finally in this chapter we give an account of polarization and dielectric phenomena, in static situations only.

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10. FIELDS NEAR SINGULAR SOURCES

Expansion of Potential outside Charges

For a point charge q at the origin

$$\begin{aligned}\rho &= q\delta(\mathbf{x}), \\ \varphi &= \frac{q}{r}, \\ \mathbf{E} &= q \frac{\mathbf{x}}{r^3},\end{aligned}\tag{10.1}$$

and for a dipole \mathbf{p} at the origin

$$\begin{aligned}\rho &= -\mathbf{p} \cdot \nabla \delta(\mathbf{x}), \\ \varphi &= -\mathbf{p} \cdot \nabla \frac{1}{r}, \\ \mathbf{E} &= -\mathbf{p} \cdot \nabla \frac{\mathbf{x}}{r^3}.\end{aligned}\tag{10.2}$$

There are, of course, other types of fields which can be obtained by higher-order differentiations. For example, the quadrupole field, of special importance in nuclear physics, is obtained by a similar differentiation process applied to the dipole field. Point sources of these types are interesting because particles and nuclei do exist in nature with charge, dipole moment, and so on. In fact any arbitrary distribution of charge $\rho(\mathbf{x})$ that is nonzero only in a finite region of space produces a field that looks like that of one of these singular sources at points sufficiently far away from the charges. Also, at remote field points, the field can be expanded in terms of the singular point sources. To see this, consider any distribution $\rho(\mathbf{x})$ that is zero external to a finite region of space. The potential is given by

$$\varphi(\mathbf{x}) = \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},\tag{10.3}$$

for any distribution $\rho(\mathbf{x}')$ and any field point \mathbf{x} . Now consider the potential in the region where ρ is zero and $|\mathbf{x}'|/|\mathbf{x}|$ is small throughout the integration. It is reasonable in this case to make an expansion for

small \mathbf{x}' . For any smooth function $f(\mathbf{x} - \mathbf{x}')$, a convenient way to write the expansion is

$$\begin{aligned} f(\mathbf{x} - \mathbf{x}') &= f(\mathbf{x}) - \mathbf{x}' \cdot \nabla f(\mathbf{x}) + \frac{1}{2} (\mathbf{x}' \cdot \nabla)^2 f(\mathbf{x}) - \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathbf{x}' \cdot \nabla)^n f(\mathbf{x}). \end{aligned} \quad (10.4)$$

That this expansion is valid for small \mathbf{x}' can be seen from its derivatives with respect to the primed variables evaluated at $\mathbf{x}' = 0$. Thus $\mathbf{a} \cdot \nabla' f(\mathbf{x} - \mathbf{x}')$ is given from Eq. (10.4) as the expansion

$$\mathbf{a} \cdot \nabla' f(\mathbf{x} - \mathbf{x}') = -\mathbf{a} \cdot \nabla f(\mathbf{x}) + (\mathbf{x}' \cdot \nabla)(\mathbf{a} \cdot \nabla)f(\mathbf{x}) - \cdots. \quad (10.5)$$

At $\mathbf{x}' = 0$, both sides simplify to $-\mathbf{a} \cdot \nabla f(\mathbf{x})$ since higher-order terms retain at least one power in \mathbf{x}' . Furthermore

$$(\mathbf{b} \cdot \nabla')(\mathbf{a} \cdot \nabla')f(\mathbf{x} - \mathbf{x}') = (\mathbf{b} \cdot \nabla)(\mathbf{a} \cdot \nabla)f(\mathbf{x}) - \cdots \quad (10.6)$$

is also valid when \mathbf{x}' is set equal to zero. All derivatives of $f(\mathbf{x} - \mathbf{x}')$ are the same on both sides of the equation when \mathbf{x}' is zero; this justifies Eq. (10.4). This formula can be applied on the right in Eq. (10.3); the expansion of the potential is then

$$\varphi(\mathbf{x}) = \left(\int d^3x' \rho(\mathbf{x}') \right) \left[\frac{1}{r} - \mathbf{x}' \cdot \nabla \frac{1}{r} + \frac{1}{2} (\mathbf{x}' \cdot \nabla)^2 \frac{1}{r} - \cdots \right], \quad (10.7)$$

and so

$$\begin{aligned} \varphi(\mathbf{x}) &= \left[\int d^3x' \rho(\mathbf{x}') \right] \frac{1}{r} + \left[- \int d^3x' \rho(\mathbf{x}') \mathbf{x}' \right] \cdot \nabla \frac{1}{r} \\ &\quad + \frac{1}{2} \left[\int d^3x' \rho(\mathbf{x}') x'_i x'_j \right] \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} + \cdots. \end{aligned} \quad (10.8)$$

Each successive term is smaller than the previous one by a factor of the order of $|\mathbf{x}'|/r$. For distant field points the first nonvanishing term dominates. Often $q' = \int d^3x' \rho(\mathbf{x}')$ does not vanish and in that case $\varphi(\mathbf{x}) \underset{r \rightarrow \infty}{\Rightarrow} q'/r$ is the potential of a point charge. However, if q' is zero, the next term is considered; it depends on

$$\mathbf{p}' = \int d^3x' \rho(\mathbf{x}') \mathbf{x}'. \quad (10.9)$$

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If \mathbf{p}' is not zero, the potential will tend to $-\mathbf{p}' \cdot \nabla(1/r)$ far away and the field will be that of a dipole. If \mathbf{p}' should also vanish, then the next term dominates and so on.

The charge q' is a property of the distribution $\rho(\mathbf{x}')$ which is independent of the origin. To demonstrate this, change to a new origin given by

$$\mathbf{x}' = \mathbf{a} + \mathbf{x}''. \quad (10.10)$$

Then since $d^3x' = d^3x''$,

$$\begin{aligned} q'' &= \int d^3x'' \rho''(\mathbf{x}'') \\ &= \int d^3x' \rho(\mathbf{x}') = q', \end{aligned} \quad (10.11)$$

where $\rho''(\mathbf{x}'') = \rho(\mathbf{x}')$. In general the dipole moment \mathbf{p}' depends on the choice of origin but if q' is zero, then \mathbf{p}' becomes independent of this choice and is a property of the distribution only; that is,

$$\begin{aligned} \mathbf{p}'' &= \int d^3x'' \rho''(\mathbf{x}'')\mathbf{x}'' \\ &= \int d^3x' \rho(\mathbf{x}')(\mathbf{x}' - \mathbf{a}) \\ &= \mathbf{p}' - \mathbf{a}q', \end{aligned} \quad (10.12)$$

so $\mathbf{p}'' = \mathbf{p}'$ if q' happens to vanish. In general, the first nonzero moment is a property of $\rho(\mathbf{x})$ independent of the origin and it dominates the field at large distances. If q' and \mathbf{p}' are both zero, the field far away looks like that of a quadrupole.

The Quadrupole Moment Tensor

The quadrupole term in Eq. (10.8) is

$$\varphi_{\text{quad}}(\mathbf{x}) = \frac{1}{2} \int d^3x' \rho(\mathbf{x}') x'_i x'_j \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right). \quad (10.13)$$

Since \mathbf{x} is a point outside the charge distribution (away from the origin),

$\partial^2 r^{-1}/\partial x_i \partial x_i = 0$. Hence only the type 2 ($l = 2$) part of $\partial^2 r^{-1}/\partial x_i \partial x_j$ contributes and so the potential for the quadrupole part of the field is

$$\begin{aligned}\varphi_{\text{quad}}(\mathbf{x}) &= \frac{1}{2} \int d^3x' \rho(\mathbf{x}') x'_i x'_j \left(\frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{3} \delta_{ij} \frac{\partial^2}{\partial x_k \partial x_k} \right) \frac{1}{r} \\ &= \frac{1}{2} \int d^3x' \rho(\mathbf{x}') \left(x'_i x'_j - \frac{1}{3} \delta_{ij} x'_k x'_k \right) \left(\frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{3} \delta_{ij} \frac{\partial^2}{\partial x_k \partial x_k} \right) \frac{1}{r}.\end{aligned}\quad (10.14)$$

Thus it is sensible to define the quadrupole moment tensor by

$$Q_{ij} = \int d^3x' \rho(\mathbf{x}') (3x'_i x'_j - \delta_{ij} r'^2). \quad (10.15)$$

That it is a tensor is demonstrated below. It is traceless and symmetric and therefore of type 2. Since $\partial^2 r^{-1}/\partial x_i \partial x_j$ is $-\delta_{ij}/r^3 + 3x_i x_j/r^5$ in the region away from $r = 0$, the potential for the quadrupole field can be written in the following ways:

$$\begin{aligned}\varphi_{\text{quad}}(\mathbf{x}) &= \frac{1}{6} Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right) \\ &= \frac{1}{2} Q_{ij} \frac{x_i x_j}{r^5} \\ &= \frac{1}{2} Q_{ij} \left(\frac{x_i x_j - \frac{1}{3} \delta_{ij} r^2}{r^5} \right).\end{aligned}\quad (10.16)$$

The higher-order terms in the expansion of the potential can also be organized into irreducible tensor moments contracted with irreducible tensors formed from x_i . This occurs because $\partial^l r^{-1}/\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}$ is symmetric and because contraction on any two indices results in a term like

$$\frac{\partial^{l-2}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_{l-2}}} \nabla^2 \left(\frac{1}{r} \right) = 0. \quad (10.17)$$

Thus the l th-order term in the expansion can be written in the form

$$\varphi^{(l)}(\mathbf{x}) = (\text{constant}) \int d^3x' \rho(\mathbf{x}') (x'_{i_1} x'_{i_2} \cdots x'_{i_l})^{(l)} \frac{(x_{i_1} x_{i_2} \cdots x_{i_l})^{(l)}}{r^l}, \quad (10.18)$$

where the functions $(x_{i_1} x_{i_2} \cdots x_{i_l})^{(l)}$ are l th-degree polynomials which

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are called *solid harmonics*. They are just the type l tensors built from products of x_i . The first four sets of these solid harmonics are

$$\begin{aligned} (1)^{(0)} &= 1, \\ (x_i)^{(1)} &= x_i, \\ (x_i x_j)^{(2)} &= x_i x_j - \frac{1}{3} \delta_{ij} r^2, \\ (x_i x_j x_k)^{(3)} &= x_i x_j x_k - \frac{1}{5} (\delta_{ij} x_k + \delta_{ki} x_j + \delta_{jk} x_i) r^2. \end{aligned} \quad (10.19)$$

The solid harmonics are special tensors which are said to be covariantly defined. The transformation rule of any tensor field is

$$T'_{i_1 i_2 \dots i_n}(\mathbf{x}') = a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n} T_{j_1 j_2 \dots j_n}(\mathbf{x}). \quad (10.20)$$

The covariantly defined tensors are special in that they are formed from the primed variables in the primed system in exactly the same way as they are formed from the unprimed variables in the unprimed system. That is, the function $T'_{i_1 i_2 \dots i_l}$ is the same as the function $T_{i_1 i_2 \dots i_l}$:

$$\begin{aligned} T'_{i_1 i_2 \dots i_l}(\mathbf{x}') &= T_{i_1 i_2 \dots i_l}(\mathbf{x}') \\ &= a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_l j_l} T_{j_1 j_2 \dots j_l}(\mathbf{x}). \end{aligned} \quad (10.21)$$

Later on it will be shown that the solid harmonics are simply related to the spherical harmonics. Thus the irreducible covariantly defined tensors have spherical harmonics for independent components.

It is instructive to consider at this point the extent to which a quadrupole moment can be visualized. It is known that the dipole moment defines a direction. The generalization of this fact is that the quadrupole moment defines a set of axes. The quadrupole moment is defined with respect to some set of axes as

$$Q_{ij} = \int d^3x \rho(\mathbf{x}) (3x_i x_j - \delta_{ij} r^2). \quad (10.22)$$

In a different set of axes given by $x'_i = a_{ij} x_j$

$$\begin{aligned} Q'_{ij} &= \int d^3x' \rho'(\mathbf{x}') (3x'_i x'_j - \delta_{ij} r'^2) \\ &= \int d^3x \rho(\mathbf{x}) (3a_{ik} a_{jl} x_k x_l - a_{ik} a_{jl} \delta_{kl} r^2) \\ &= a_{ik} a_{jl} Q_{kl}. \end{aligned} \quad (10.23)$$

In Eq. (10.23) the elementary facts $d^3x = d^3x'$ and $\rho(\mathbf{x}) = \rho'(\mathbf{x}')$ have been used. Thus the quadrupole moment tensor is a second rank tensor. The quadrupole moment tensor is not a field quantity, however. The components of the tensor may be organized in matrix form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}, \quad (10.24)$$

so that the transformation rule $Q'_{ij} = a_{ik}Q_{kl}(a^\top)_{lj}$ can be written

$$Q' = aQa^\top, \quad (10.25)$$

and computed by matrix multiplication. A standard result from the theory of matrices may be applied here: Given any real symmetric matrix Q , a real matrix a exists such that

$$\begin{aligned} aa^\top &= 1, \\ Q' &= aQa^\top \\ &= \begin{pmatrix} \bar{Q}^{(1)} & 0 & 0 \\ 0 & \bar{Q}^{(2)} & 0 \\ 0 & 0 & \bar{Q}^{(3)} \end{pmatrix}. \end{aligned} \quad (10.26)$$

That is, a real symmetric matrix can be diagonalized by a real orthogonal transformation. The proof of this theorem will not be reviewed here, but can be found in Chapter 3 of the standard work by Wigner [1]. Instead, the solution will be given in terms of the solutions of the eigenvalue problem,

$$QX = \bar{Q}X, \quad (10.27)$$

where X is a three component column matrix and \bar{Q} is a number. This is how the new axes may be found in a numerical application. Let the three solutions be $X^{(i)}$ and $\bar{Q}^{(i)}$, $i = 1, 2$, and 3 , all real. The $X^{(i)}$ must be chosen to be orthogonal and normalized

$$X^{(i)\top} X^{(j)} = \delta_{ij}. \quad (10.28)$$

Then the required matrix is $a_{ij} = X_j^{(i)}$; the rows of the matrix a are the eigenvectors of Q . This matrix is orthogonal since

$$\begin{aligned} (aa^\top)_{ij} &= a_{ik}(a^\top)_{kj} \\ &= a_{ik}a_{jk} \\ &= X_k^{(i)}X_k^{(j)} \\ &= (X^{(i)\top} X^{(j)}) \\ &= \delta_{ij}. \end{aligned} \quad (10.29)$$

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The components of the quadrupole tensor in the primed system are

$$\begin{aligned}
 Q'_{ij} &= (aQa^\top)_{ij} \\
 &= a_{ik}Q_{kl}a_{jl} \\
 &= X_k^{(i)}Q_{kl}X_l^{(j)} \\
 &= X_k^{(i)}\bar{Q}^{(j)}X_k^{(j)} \\
 &= \bar{Q}^{(j)}X^{(i)\top}X^{(j)} \\
 &= \bar{Q}^{(j)}\delta_{ij},
 \end{aligned} \tag{10.30}$$

in agreement with Eq. (10.26). Thus a charge cloud with a quadrupole moment selects a set of axes in which Q' is diagonal. Moreover the eigenvectors point out the new axes since $\mathbf{i}'_i = X_j^{(i)}\mathbf{i}_j$. The trace of Q' is zero so only two of the $\bar{Q}^{(i)}$ are independent. Of the five independent components of a quadrupole tensor, three may be thought of as specifying the orientation of the set of axes in which the tensor is diagonal. Notice that $-X^{(i)}$ is an eigenvector of Q_{ij} as well as $X^{(i)}$. The quadrupole tensor picks out three mutually perpendicular lines, but it does not prefer either direction along a line or the handedness of the coordinate frame in which it is diagonal.

Some of the components of the quadrupole moment tensor in the primed system are

$$\begin{aligned}
 Q'_{12} &= 3 \int d^3x' x'_1 x'_2 \rho'(\mathbf{x}') \\
 &= 0, \\
 Q'_{33} &= \int d^3x' \rho'(\mathbf{x}') (3x'^2_3 - r'^2) \\
 &= \int d^3x' \rho'(\mathbf{x}') (2x'^2_3 - x'^2_1 - x'^2_2).
 \end{aligned} \tag{10.31}$$

If the charge density is positive, positive (negative) contributions are made to Q'_{33} from points inside (outside) the cone $x'^2_1 + x'^2_2 = 2x'^2_3$. The most significant contributions to Q'_{33} come from points on the X'_3 -axis and points in the $X'_1X'_2$ -plane. For example, suppose there are two spheres of positive charge and a doughnut of negative charge halfway between the spheres such that there is axial symmetry and the net charge is zero. Figure 10.1 shows this charge distribution. An intelligent thing to do is to choose the X'_3 -axis along the line between

the centers of the spheres and choose the origin at the center of this line. This must be a principal axis because

$$Q'_{13} = \int d^3x' 3x'_1 x'_3 \rho'(\mathbf{x}') = 0. \quad (10.32)$$

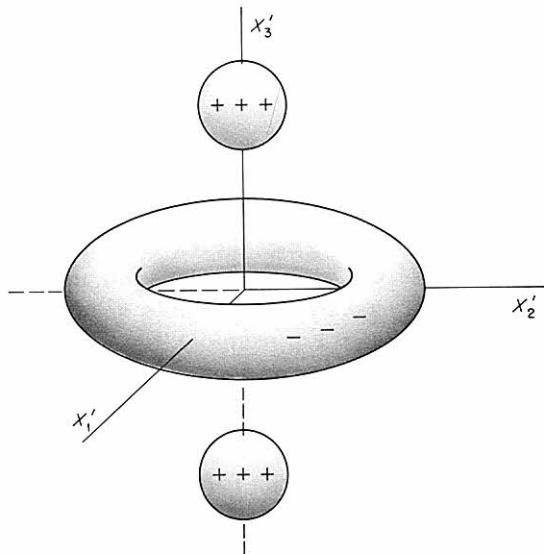


FIG. 10.1. Example of a charge distribution with a quadrupole moment.

At fixed x'_1 and x'_2 , there is equal charge at positive and negative values of x'_3 , so the contributions to Q'_{13} cancel. Similarly Q'_{23} is zero. The X'_1 - and X'_2 -axes can be chosen anywhere since the system is axially symmetric. The diagonal component

$$Q'_{33} = \int d^3x' \rho'(\mathbf{x}') (2x'^2_3 - x'^2_1 - x'^2_2) = \bar{Q}^{(3)} \quad (10.33)$$

is positive since there is positive charge near the X'_3 -axis and negative charge near the $X'_1 X'_2$ -plane. Also $Q'_{12} = 0$ and $Q'_{11} = Q'_{22}$ follow from the symmetry of the distribution about a plane through the X'_1 -axis. Since the trace of Q'_{ij} is zero, the tensor is completely specified by Q'_{33} ,

$$Q' = \begin{pmatrix} -\frac{1}{2}\bar{Q}^{(3)} & 0 & 0 \\ 0 & -\frac{1}{2}\bar{Q}^{(3)} & 0 \\ 0 & 0 & \bar{Q}^{(3)} \end{pmatrix}. \quad (10.34)$$

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Energy of Singular Sources in an External Electrostatic Field

The work done in moving a point charge q from a point where the potential φ is zero to the point \mathbf{x} in an external field $\mathbf{E} = -\nabla\varphi$ is

$$W = q\varphi(\mathbf{x}). \quad (10.35)$$

For any distribution of charge $\rho(\mathbf{x})$, one can sum the effects of all the charges to obtain

$$W = \int d^3x \rho(\mathbf{x})\varphi(\mathbf{x}), \quad (10.36)$$

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since this may be integrated to produce Eq. (10.39). Thus for a quadrupole distribution centered at \mathbf{x}_1

$$\rho_{\text{quad}}(\mathbf{x}) = \frac{1}{6} Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \delta(\mathbf{x} - \mathbf{x}_1), \quad (10.41)$$

and the energy in an external electrostatic field is

$$\begin{aligned} W &= \int d^3x \frac{1}{6} Q_{ij} \left[\frac{\partial^2}{\partial x_i \partial x_j} \delta(\mathbf{x} - \mathbf{x}_1) \right] \varphi(\mathbf{x}) \\ &= \int d^3x \frac{1}{6} Q_{ij} \delta(\mathbf{x} - \mathbf{x}_1) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(\mathbf{x}) \\ &= \frac{1}{6} Q_{ij} \underline{\frac{\partial^2 \varphi(\mathbf{x})}{\partial x_i \partial x_j}} \end{aligned}$$

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of the cloud in the external field is the sum of the energies of its various moments in the external field, or

$$W = qp(\mathbf{x}_0) - \mathbf{p} \cdot \mathbf{E}(\mathbf{x}_0) - \frac{1}{6} Q_{ij} \frac{\partial E_i(\mathbf{x}_0)}{\partial x_{0j}} + \dots \quad (10.46)$$

In many practical situations the series converges rapidly and is dominated by the first nonzero term. For example, suppose the net charge q is zero. A rough estimate of the relative importance of the dipole and quadrupole contributions is then given by

$$\begin{aligned} \frac{\text{quadrupole energy}}{\text{dipole energy}} &\sim \frac{Q \frac{\partial E}{\partial x}}{pE} \\ &\sim \frac{\alpha \frac{\partial E}{\partial x}}{E}, \end{aligned} \quad (10.47)$$

where $\alpha \sim Q/p$ is a measure of the size of the charge cloud. Thus

$$\frac{\text{quadrupole energy}}{\text{dipole energy}} \sim \frac{\text{change in } E \text{ across cloud}}{E}, \quad (10.48)$$

and this may very well be small.

Surface Singularities

Only fields which are singular at a point have been considered thus far. A complete development of line singularities could be made but there is not much application for this in electrostatics. However, surface singularities are important and will be considered in some detail. Any surface charge will be characterized by a charge density per unit area σ . Let the normal to the surface be \hat{n} , as in Fig. 10.2. The surface charge distribution produces a potential that is continuous but not differentiable at the surface, and correspondingly a field that is discontinuous at the surface. For example, consider the potential and field on the axis of a disk of radius R carrying charge per unit area σ . (For another related

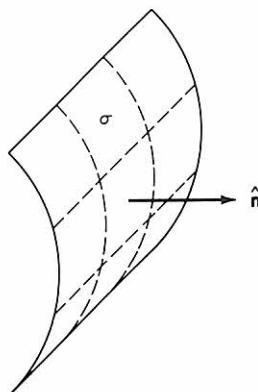
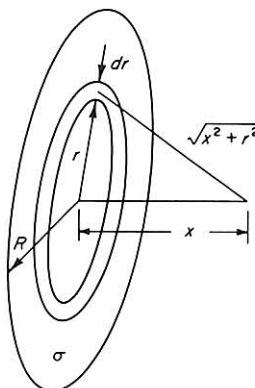


FIG. 10.2. Surface charge singularities.

example see Problem 2, Section 1.) The potential can be found by integration, Fig. 10.3 shows the geometry of the problem. The potential is

$$\begin{aligned}
 \varphi(x) &= \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\
 &= \int_0^R \frac{\sigma 2\pi r dr}{\sqrt{x^2 + r^2}} \\
 &= 2\pi\sigma \sqrt{x^2 + r^2} \Big|_0^R \\
 &= 2\pi\sigma (\sqrt{x^2 + R^2} - |x|). \tag{10.49}
 \end{aligned}$$

FIG. 10.3. The charged disk. The distance from the field point on the axis to an element of area at radius r is $\sqrt{x^2 + r^2}$.



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The potential is not differentiable at $x = 0$ because $|x|$ is not. The field is given by

$$\frac{d\phi}{dx} = \sigma \left[-\frac{x}{\sqrt{x^2 + R^2}} + \frac{|x|}{x} \right]. \quad (10.50)$$

Electric field at the surface is $2\pi\sigma[+1 - (-1)] = 4\pi\sigma$. This amount in the field must occur at a distance R from the center. This can be seen from Gauss's law. For any surface of radius R , the flux through side 1 can be distinguished from side 2 by letting \mathbf{E}_1 point outwards and \mathbf{E}_2 point inwards. Gauss's theorem can be applied inside the cylinder of radius R and thickness t , where the volume spans the entire cylinder. The volume charge density is ρ and the total charge is $4\pi\sigma A$. The theorem writes

$$\begin{aligned} \mathbf{E} \cdot d\mathbf{a} &= \int_{(*)} \nabla \cdot \mathbf{E} \, d^3x \\ &= 4\pi \int_{(*)} \rho \, d^3x, \\ \hat{\mathbf{n}} \cdot \mathbf{A}_1 + \mathbf{E}_2 \cdot \hat{\mathbf{n}} \cdot \mathbf{A}_2 &= 4\pi\sigma A. \end{aligned} \quad (10.51)$$

If the surface charge density is σ , then $(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}}$ is equal to $4\pi\sigma$. The question is whether such surface charges can really exist. In the case of a metal, the substance that permits charges to move about freely, if a uniform electric field \mathbf{E} exists in the metal, then charges will move until they stop, the field is zero and any point inside the metal has zero electric field.

$$=\frac{1}{4\pi} \nabla \cdot \mathbf{E} = 0. \quad (10.52)$$

Outside the metal, the electric field \mathbf{E}_{out} must be normal to the surface. Any component parallel to the surface would cause charges to flow parallel to the surface. The tangential component of \mathbf{E} across the surface is discontinuous. Then there must exist a surface charge σ on the surface.

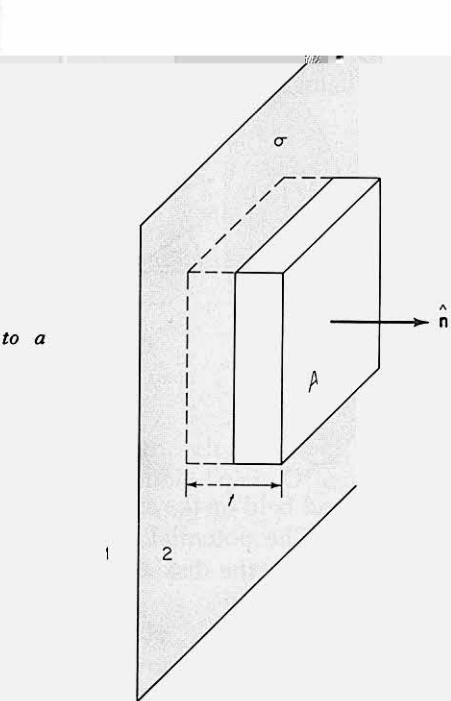
$$\sigma = \frac{1}{4\pi} E_{\text{out}}, \quad (10.53)$$

The discontinuity in the normal component of the electric field is given by

FIG. 10.4. Gauss's theorem applied to a cylindrical volume including some surface charge.

is equal to $4\pi\sigma$. Another property of the metal is that it is a constant throughout because the free electrons are distributed uniformly.

One can also consider a dipole layer on a metal surface. Such a layer is also called *displacement per unit area*. The dipole layer must be normal to the surface, the positive to the negative side. The surface charge density of the layer is given by $\sigma = D - \epsilon_0 E_{\text{out}}$. If the dipole distribution is zero except on the surface, it produces a singular dipole charge. The tangential component of the electric field is discontinuous. The smooth dependence of E in the interior is the same as f . Thus any component of E that is discontinuous just a distribution of charge on the surface. To understand this way, consider bringing together two negatively charged surfaces to form a new surface. The two surfaces, a new surface layer.



of the metal is that the potential is $= -\mathbf{E} = 0$.

moment per unit area τ on a surface, *area*. The dipole moment per unit area $= \tau \hat{n}$, where \hat{n} is the unit normal from the layer. The reason is that the charge $= (-\mathbf{p} \cdot \nabla) f$ where f is some charge surface. The normal component of \mathbf{p} distribution $(-\mathbf{p}_n \cdot \nabla) f$. However, in $\cdot \nabla) f$, the gradient acts only on the face and so this term is qualitatively the of \mathbf{p} that is parallel to the surface proper unit area. To put this in a different a positively charged surface and a small separation d . If d is parallel to distribution is produced, not a dipole

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A disk of radius R carrying constant dipole moment per unit area τ as in Fig. 10.5 will be considered as an example. It is easy to calculate the

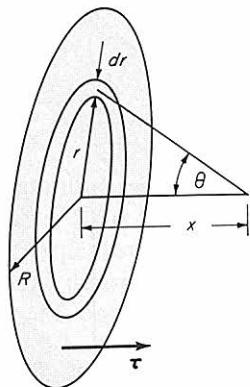


FIG. 10.5. The dipole layer disk.

potential and field on the axis of the disk and they illustrate what happens in general. The potential of a dipole \mathbf{p} at the origin is $\mathbf{p} \cdot \mathbf{x}/r^3$, so the potential due to the disk at any field point \mathbf{x} is

$$\varphi(\mathbf{x}) = \int d\mathbf{a}' \frac{\tau \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (10.54)$$

Evidently the potential may be interpreted as τ times the solid angle subtended by the surface at the field point. On the axis, the potential is

$$\begin{aligned} \varphi(x) &= \int_0^R \frac{2\pi r dr \tau \cos \theta}{x^2 + r^2} \\ &= 2\pi\tau x \int_0^R \frac{r dr}{[\sqrt{x^2 + r^2}]^3} \\ &= 2\pi\tau x \left[\frac{-1}{\sqrt{(x^2 + r^2)}} \right]_0^R \\ &= 2\pi\tau \left[\frac{x}{|x|} - \frac{x}{\sqrt{x^2 + R^2}} \right]. \end{aligned} \quad (10.55)$$

Thus there is a discontinuity of $4\pi\tau$ in the potential at the disk. The electric field is

$$\begin{aligned} E_x &= -\frac{d\varphi}{dx} \\ &= -2\pi\tau \frac{d}{dx} \left(\frac{x}{|x|} \right) + 2\pi\tau \left(\frac{1}{\sqrt{x^2 + R^2}} - \frac{x^2}{[\sqrt{x^2 + R^2}]^3} \right). \end{aligned} \quad (10.56)$$

A somewhat new mathematical problem appears in Eq. (10.56). It is the problem of differentiating the step function $x/|x|$. Obviously $(d/dx)(x/|x|)$ is zero so long as x is not zero, and if ϵ is some small positive number, one has

$$\int_{-\epsilon}^{+\epsilon} \left(\frac{d}{dx} \frac{x}{|x|} \right) dx = \frac{x}{|x|} \Big|_{-\epsilon}^{+\epsilon} = 2. \quad (10.57)$$

This meets the requirements of the delta function so $(d/dx)(x/|x|) = 2\delta(x)$, and the electric field is

$$E_x = -4\pi\tau\delta(x) + \frac{2\pi\tau R^2}{[\sqrt{x^2 + R^2}]^3}. \quad (10.58)$$

Heuristically, the delta function represents the electric field between the two oppositely charged surfaces in the dipole layer. Of course this field diverges as the two surfaces are brought together.

P R O B L E M S

1. From any vector \mathbf{v} one can make a symmetric tensor $T_{ij} = v_i v_j$. What are the eigenvectors and eigenvalues of T_{ij} ? Such a tensor is called *axial*; why is this? How does one specialize the general quadrupole to get an axial quadrupole?

Solution: Since

$$T_{ij}v_j = v_i v_j v_j = v^2 v_i,$$

one eigenvector is \mathbf{v} and it has the eigenvalue v^2 . For any vector \mathbf{a} that is perpendicular to \mathbf{v} ,

$$T_{ij}a_j = v_i v_j a_j = 0 a_i,$$

so \mathbf{a} is an eigenvector with the eigenvalue zero. There are, of course, two linearly independent vectors that are perpendicular to \mathbf{v} . The tensor $v_i v_j$ singles out only one axis in the $\pm \mathbf{v}$ -direction. All directions perpendicular to \mathbf{v} are equivalent, and so the tensor is said to be *axially symmetric*. The quadrupole potential in general is

$$\varphi_{\text{quad}}(\mathbf{x}) = \frac{1}{6} Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right),$$

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where Q_{ij} is traceless and symmetric. The general quadrupole simplifies to the axial case when it is of the form

$$\frac{1}{6}Q_{ii} = p_i p_i - \frac{1}{3}\delta_{ii}p^2.$$

One of the eigenvectors in this case is p_i since

$$\begin{aligned}[p_i p_j - \frac{1}{3}\delta_{ij}p^2]p_j &= p_i p^2 - \frac{1}{3}p_i p^2 \\ &= (\frac{2}{3}p^2)p_i,\end{aligned}$$

and the eigenvalue is $\frac{2}{3}p^2$. For any vector \mathbf{a} perpendicular to \mathbf{p} ,

$$[p_i p_j - \frac{1}{3}\delta_{ij}p^2]a_j = (-\frac{1}{3}p^2)a_i,$$

so the other two eigenvalues are both $-\frac{1}{3}p^2$. In general the quadrupole is axial if two of its eigenvalues are equal. Then the eigenvector for the third, distinct, eigenvalue points along the symmetry axis.

2. Derive formulas for the force and torque on a quadrupole in an external electrostatic field.
3. A system has the quadrupole moment tensor

$$Q = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

Find the principal moments and principal axes. Make a sketch showing the axes.

4. Show that the force and torque of an external electrostatic field on the charge distribution

$$\rho = (\mathbf{p} \cdot \nabla)^2 \delta(\mathbf{x})$$

are the same as those on the distribution

$$\rho = (p_i p_j - \frac{1}{3}\delta_{ij}p^2) \frac{\partial^2}{\partial x_i \partial x_j} \delta(\mathbf{x}).$$

5. A charge distribution consists of four lines, each of length l , arranged in a square as shown in Fig. 10.6. Two opposite lines have charge q uniformly distributed along them; the other two have charge $-q$ uniformly distributed. Explain why the lowest nonzero moment is the

quadrupole. Find the principal axes and the principal quadrupole moments.

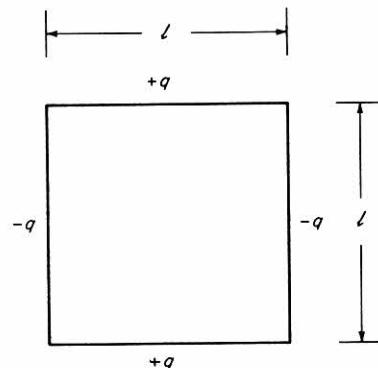


FIG. 10.6. A rectangular pattern of line charges which has a quadrupole moment.

6. Find the discontinuity in the tangential component of \mathbf{E} across a surface with surface charge per unit area and also across a surface with normal dipole moment per unit area.

Solution: Label the sides of the boundary 1 and 2 with the dipole layer in the direction pointing from side 1 to side 2. Consider a rectan-

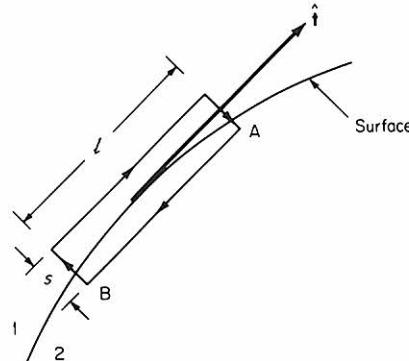


FIG. 10.7. Closed path at a surface for finding the discontinuity in E_{\tan} .

gular path of length l in the tangential direction $\hat{\mathbf{t}}$ and of width s in the normal direction, as indicated in Fig. 10.7, such that the path lies on both sides of the surface. Consider the limit as s and l go to zero. The

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integral around this path of $\mathbf{E} \cdot d\mathbf{l}$ is, with the help of Stokes's theorem, given by

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_{\text{loop}} \nabla \times \mathbf{E} \cdot d\mathbf{a} \\ = -\frac{1}{c} \frac{d}{dt} \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{a}.$$

In a static problem the right-hand side is zero. Even in a dynamic problem, if \mathbf{B} is finite everywhere the right-hand side is of order sl and is negligible compared to the left-hand side which is of order l . Therefore as s and l go to zero the equation becomes

$$(\mathbf{E}_1 - \mathbf{E}_2) \cdot \hat{\mathbf{t}}l + \varphi_{1A} - \varphi_{2A} + \varphi_{2B} - \varphi_{1B} = 0.$$

If there is only a surface charge per unit area the potential is continuous across the surface and so the tangential component of the field must be continuous also. If there is a displacement per unit area τ on the surface then $\varphi_2 - \varphi_1 = 4\pi\tau$ and so

$$(\mathbf{E}_1 - \mathbf{E}_2) \cdot \hat{\mathbf{t}} = 4\pi \left(\frac{\tau_A - \tau_B}{l} \right) = 4\pi \hat{\mathbf{t}} \cdot \nabla \tau,$$

where τ is thought of as a function of position on the surface. The equation holds for any direction $\hat{\mathbf{t}}$ in the surface so

$$(\mathbf{E}_1 - \mathbf{E}_2)_{\tan} = 4\pi \nabla \tau.$$

7. Show in several ways that the quadrupole moment of a spherically symmetric distribution of charge is zero.
8. Total charge q is uniformly distributed within an ellipsoid of revolution having semiaxes a , a , and c . Choose the origin at the center of the distribution and the Z -axis along the axis of length $2c$ and show that the quadrupole moment tensor is

$$Q = \begin{pmatrix} -\frac{q}{5}(c^2 - a^2) & 0 & 0 \\ 0 & -\frac{q}{5}(c^2 - a^2) & 0 \\ 0 & 0 & \frac{2q}{5}(c^2 - a^2) \end{pmatrix}.$$

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Uniqueness of Solution

When the charge density $\rho(\mathbf{x})$ is given inside a surface S , either $\varphi(\mathbf{x})$ or $\partial\varphi/\partial n$ (the normal component of the gradient) is given on the surface, and Poisson's equation $\nabla^2\varphi = -4\pi\rho$ holds, a standard mathematical problem has been posed. When φ (or $\partial\varphi/\partial n$) is given, the proposed problem is said to be a *Dirichlet* (or *Neumann*) boundary value problem for the potential $\varphi(\mathbf{x})$. It is possible to prove that a unique solution exists in either case. To prove uniqueness, assume contrarily that the problem possesses two solutions φ_1 and φ_2 . The standard trick is to consider the difference U

$$U = \varphi_2 - \varphi_1. \quad (11.1)$$

Then U or $\partial U/\partial n$ is zero on the boundary depending on whether Dirichlet or Neumann boundary conditions were given. Furthermore $\nabla^2 U = 0$ within the volume enclosed by S . Next one may consider integrating

$$\nabla \cdot (U \nabla U) = \nabla U \cdot \nabla U + U \nabla^2 U \quad (11.2)$$

over the volume inside S . The result of this is

$$\begin{aligned} \int_{\textcircled{O}} d\mathbf{a} \cdot U \nabla U &= \int_{\textcircled{*}} \nabla U \cdot \nabla U d^3x + \int_{\textcircled{*}} U \nabla^2 U d^3x, \\ \int_{\textcircled{*}} |\nabla U|^2 d^3x &= \int_{\textcircled{O}} U \frac{\partial U}{\partial n} d\mathbf{a} \\ &= 0. \end{aligned} \quad (11.3)$$

Since $|\nabla U|$ is necessarily positive, $\int_{\textcircled{*}} |\nabla U|^2 d^3x = 0$ means that inside the volume

$$\nabla U = 0. \quad (11.4)$$

Therefore U must be constant throughout the volume. For Dirichlet boundary conditions the constant is zero because $U = \varphi_2 - \varphi_1$ is zero on the surface S . For Neumann boundary conditions a solution φ is unique except for an additive constant.

The theorem is useful because, although various special ways of finding solutions can be developed, once any solution has been produced,

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it is then known to be the unique solution. For example, consider the problem of finding the field inside a cavity in a metal. It can be shown that the field is zero just by application of the theorem. The charge density ρ is zero inside and φ is constant on the boundary. One solution of $\nabla^2\varphi = 0$ for these boundary conditions is $\varphi = \varphi_0$, a constant everywhere inside the cavity. This solution is constant on the boundary and satisfies $\nabla^2\varphi = 0$ in the cavity. The solution is unique so this is the only correct answer. The field is

$$\mathbf{E} = -\nabla\varphi_0 = 0 \quad (11.5)$$

within the cavity. The uniqueness theorem applies within any volume, even multiply connected ones, since only Gauss' theorem was used in the proof. It also applies in an open volume, exterior to a surface S if $\int_{\text{O}\rightarrow} U(\partial U/\partial n) da$ is zero for a large sphere of radius R . This is true if $\varphi \xrightarrow[R \rightarrow \infty]{} 1/R$, which applies when the charges exist in finite space, because $U \sim 1/R$, $\partial U/\partial n \sim 1/R^2$, and

$$\int_{\text{O}\rightarrow} U \frac{\partial U}{\partial n} da \sim \frac{1}{R} \left(\frac{1}{R^2} \right) R^2 \xrightarrow[R \rightarrow \infty]{} 0. \quad (11.6)$$

Green's Function

Since Poisson's equation is linear, the potential for some charge distribution within a surface S , with Neumann or Dirichlet boundary conditions, can be found by adding up the contributions from the charges in each infinitesimal element of volume and the contributions from the boundary conditions in a certain way using a Green's function. A Green's function $G(\mathbf{x}, \mathbf{x}')$ for Poisson's equation is defined to be a solution of

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad (11.7)$$

inside the surface S . One possibility is that G is just the potential at \mathbf{x}' of a unit point charge at \mathbf{x}

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \quad (11.8)$$

However there are other solutions of the form

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}'), \quad (11.9)$$

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provided $F(\mathbf{x}, \mathbf{x}')$ satisfies Laplace's equation

$$\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0. \quad (11.10)$$

There are two choices of $F(\mathbf{x}, \mathbf{x}')$ that are used commonly, depending on the boundary conditions. For the Dirichlet problem, $F(\mathbf{x}, \mathbf{x}')$ is chosen such that the Green's function is zero when \mathbf{x}' lies on the boundary,

$$G_D(\mathbf{x}, \mathbf{x}') = 0, \quad \mathbf{x}' \text{ on } S. \quad (11.11)$$

This means that $F_D(\mathbf{x}, \mathbf{x}')$ must satisfy $\nabla'^2 F_D(\mathbf{x}, \mathbf{x}') = 0$ inside S and also $F_D(\mathbf{x}, \mathbf{x}') = -1/|\mathbf{x} - \mathbf{x}'|$, \mathbf{x}' on the surface S . Of course it is often a difficult problem to find $F_D(\mathbf{x}, \mathbf{x}')$, depending on the nature of the surface. However, as will be shown, once $F_D(\mathbf{x}, \mathbf{x}')$ is known, a solution of the original Dirichlet problem can be constructed. The Green's function separates out the part of the problem that is complicated by the geometry of the surface. It can be found once and for all, and then all Dirichlet boundary value problems are solved in terms of it. To see how this works, use the vector identities

$$\begin{aligned} \nabla \cdot (\varphi \nabla \psi) &= \nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi, \\ \nabla \cdot (\psi \nabla \varphi) &= \nabla \psi \cdot \nabla \varphi + \psi \nabla^2 \varphi, \end{aligned} \quad (11.12)$$

in the form

$$\nabla \cdot (\varphi \nabla \psi - \psi \nabla \varphi) = \varphi \nabla^2 \psi - \psi \nabla^2 \varphi. \quad (11.13)$$

When this expression is integrated over the volume within S , the result is

$$\int_{\bigcirclearrowright} d\mathbf{a} \cdot (\varphi \nabla \psi - \psi \nabla \varphi) = \int_{(*)} (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) d^3x, \quad (11.14)$$

which is often referred to as Green's theorem. If primed variables are used in the integration,

$$\int_{\bigcirclearrowright} d\mathbf{a}' \left[\varphi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \varphi}{\partial n'} \right] = \int_{(*)} (\varphi \nabla'^2 \psi - \psi \nabla'^2 \varphi) d^3x' \quad (11.15)$$

holds for the potential φ and any other function ψ . If $\psi(\mathbf{x}')$ is taken to be $G_D(\mathbf{x}, \mathbf{x}')$, satisfying Eqs. (11.7) and (11.11), then

$$\begin{aligned} \int_{\bigcirclearrowright} d\mathbf{a}' \left[\varphi(\mathbf{x}') \frac{\partial}{\partial n'} G_D(\mathbf{x}, \mathbf{x}') - G_D(\mathbf{x}, \mathbf{x}') \frac{\partial \varphi(\mathbf{x}')}{\partial n'} \right] \\ = \int_{(*)} d^3x' [\varphi(\mathbf{x}') \nabla'^2 G_D(\mathbf{x}, \mathbf{x}') - G_D(\mathbf{x}, \mathbf{x}') \nabla'^2 \varphi(\mathbf{x}')]. \end{aligned} \quad (11.16)$$

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Now $G_D(\mathbf{x}, \mathbf{x}')$ is zero on the surface, so

$$\begin{aligned} & \int_{\bigcirclearrowright} da' \varphi(\mathbf{x}') \frac{\partial}{\partial n'} G_D(\mathbf{x}, \mathbf{x}') \\ &= \int_{(*)} d^3x' \{ \varphi(\mathbf{x}') [-4\pi\delta(\mathbf{x} - \mathbf{x}')] - G_D(\mathbf{x}, \mathbf{x}') [-4\pi\rho(\mathbf{x}')] \} \\ &= -4\pi\varphi(\mathbf{x}) + 4\pi \int d^3x' G_D(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}'). \end{aligned} \quad (11.17)$$

Thus a formal solution to the boundary value problem is

$$\varphi(\mathbf{x}) = \int d^3x' G_D(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi} \int_{\bigcirclearrowright} da' \varphi(\mathbf{x}') \frac{\partial}{\partial n'} G_D(\mathbf{x}, \mathbf{x}'). \quad (11.18)$$

That is, if $G_D(\mathbf{x}, \mathbf{x}')$ is known, the Dirichlet boundary value problem can be solved by integration for any given $\rho(\mathbf{x}')$ within the volume and for any $\varphi(\mathbf{x}')$ given on the surface S .

The natural thing to do for the Neumann boundary-value problem is to choose the Neumann Green's function $G_N(\mathbf{x}, \mathbf{x}')$ such that $\partial G_N(\mathbf{x}, \mathbf{x}')/\partial n'$ is zero on the surface. This would parallel the successful treatment of the Dirichlet problem. However, this approach leads to a contradiction. To see this, note that

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \quad (11.19)$$

so, integrating over the volume inside S ,

$$\int_{(*)} d^3x' \nabla' \cdot \nabla' G(\mathbf{x}, \mathbf{x}') = -4\pi \int_{(*)} d^3x' \delta(\mathbf{x} - \mathbf{x}').$$

This means that

$$\int_{\bigcirclearrowright} da' \frac{\partial}{\partial n'} G(\mathbf{x}, \mathbf{x}') = -4\pi. \quad (11.20)$$

It would not make sense to require $\partial G(\mathbf{x}, \mathbf{x}')/\partial n'$ to be zero. This result suggests, however, that the boundary condition on the Neumann Green's function should be

$$\frac{\partial}{\partial n'} G_N(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{A}, \quad \mathbf{x}' \text{ on } S \quad (11.21)$$

where A is the area of the surface, and where only closed surfaces are

considered. This then is consistent with Eq. (11.20). Green's theorem may now be applied with $\psi(\mathbf{x}') = G_N(\mathbf{x}, \mathbf{x}')$, where G_N satisfies Eqs. (11.19) and (11.21),

$$\begin{aligned} & \int_{\bigcirclearrowright} da' [\varphi(\mathbf{x}') \frac{\partial}{\partial n'} G_N(\mathbf{x}, \mathbf{x}') - G_N(\mathbf{x}, \mathbf{x}') \frac{\partial}{\partial n'} \varphi(\mathbf{x}')] \\ &= \int_{(*)} d^3x' [\varphi(\mathbf{x}') \nabla'^2 G_N(\mathbf{x}, \mathbf{x}') - G_N(\mathbf{x}, \mathbf{x}') \nabla'^2 \varphi(\mathbf{x}')], \\ & - \frac{4\pi}{A} \int_{\bigcirclearrowright} da' \varphi(\mathbf{x}') - \int_{\bigcirclearrowright} da' G_N(\mathbf{x}, \mathbf{x}') \frac{\partial}{\partial n'} \varphi(\mathbf{x}') \\ &= -4\pi\varphi(\mathbf{x}) + 4\pi \int_{(*)} d^3x' G_N(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}'). \end{aligned} \quad (11.22)$$

As a shorthand notation, $\langle \varphi \rangle_S = (1/A) \int_{\bigcirclearrowright} da' \varphi(\mathbf{x}')$ will be used, so the solution of the Neumann boundary value problem is

$$\varphi(\mathbf{x}) - \langle \varphi \rangle_S = \int_{(*)} d^3x' \rho(\mathbf{x}') G_N(\mathbf{x}, \mathbf{x}') + \frac{1}{4\pi} \int_{\bigcirclearrowright} da' \frac{\partial \varphi(\mathbf{x}')}{\partial n'} G_N(\mathbf{x}, \mathbf{x}'). \quad (11.23)$$

Thus the Neumann problem is solved, to within a constant, if $G_N(\mathbf{x}, \mathbf{x}')$ can be found. The boundary conditions do not specify the constant and the Neumann problem possesses a unique solution only to within a constant, so this aspect of the solution might have been anticipated. However, for the one surface S , each charge density $\rho(\mathbf{x})$ and boundary function $\partial\varphi/\partial n$ determines a potential which can be found with the one Green's function.

P R O B L E M S

1. Given a source distribution $\rho(\mathbf{x})$ inside a surface S , the potential φ and its normal derivative on S , and the equation

$$\nabla^2 \varphi - \kappa^2 \varphi = -4\pi\rho,$$

find $\varphi(\mathbf{x})$ inside S .

2. Show that in free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on the point.

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3. Show that it is impossible for a solution of Laplace's equation to have a local minimum. A local minimum would be a point such that φ increases in every direction away from the point.

4. Calculate the capacitance of two large flat conducting sheets of area A separated by distance d . The capacitance is defined as the ratio of the charge on one conductor to the potential difference between the two when equal and opposite charges are placed on them.

Solution: Suppose that charge q is placed on one plate and charge $-q$ on the other. In the statement of the problem, the word "large" means that fringe effects at the edges of the plates are negligible. Consequently the field is taken to be uniform and perpendicular from one plate to the other. The charge per unit area on the plates is then constant at $\pm q/A$. The discontinuity in the normal component of the field at each plate is $4\pi\sigma$ so the field must be

$$E = 4\pi \frac{q}{A},$$

pointing from the positively charged plate. The potential difference between the plates is then

$$\begin{aligned}\varphi_+ - \varphi_- &= - \int_{-}^{+} \mathbf{E} \cdot d\mathbf{l} \\ &= 4\pi \frac{qd}{A}.\end{aligned}$$

The capacitance is therefore $C = A/4\pi d$.

5. Find the capacitance of two concentric conducting spheres of radii a and b , where $b > a$. What is the result as $b \rightarrow \infty$?

6. The potential outside a surface S , due to charges inside S , is the same as the potential due to a certain distribution of charge and dipole moment on S . Prove that, if S is an equipotential, only a charge distribution is required.

Solution: From Eq. (11.15) with $\psi = 1/|\mathbf{x} - \mathbf{x}'|$ it is known that for any volume

$$\varphi(\mathbf{x}) = \int_{(*)} d^3x' \frac{\rho(\mathbf{x}')}{R} + \frac{1}{4\pi} \int_{\bigcirclearrowright} da' \left[\frac{1}{R} \frac{\partial \varphi}{\partial n'} - \varphi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right],$$

where R is $|\mathbf{x} - \mathbf{x}'|$. In this problem the potential $\varphi(\mathbf{x})$ is supposed to be

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produced by a charge distribution ρ located inside the surface S . Therefore, for a point \mathbf{x} outside S , the potential can be expressed in the form

$$\varphi(\mathbf{x}) = \frac{1}{4\pi} \int_{\bigcirc \leftarrow S} \left[\frac{1}{R} \frac{\partial \varphi}{\partial n'} - \varphi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right] da',$$

where, since the theorem is applied to the volume outside S , the normal is now inward to S . The contribution from the other boundary (a large sphere) is zero since φ is produced by ρ inside S and so goes like $1/R$ for large R . This expression for the potential is the same as that which would be produced by the surface charge per unit area

$$\sigma = \frac{1}{4\pi} \frac{\partial \varphi}{\partial n'}$$

plus a normal dipole moment per unit area [see Eq. (10.54)]

$$\tau = \frac{1}{4\pi} \varphi$$

directed out of S . Now suppose that S is an equipotential, that is, $\varphi = \varphi_0$ on the surface. In this case the dipole moment term simplifies to

$$\begin{aligned} -\frac{1}{4\pi} \int_{\bigcirc \leftarrow S} \varphi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) da' &= \frac{\varphi_0}{4\pi} \int_{\bigcirc \rightarrow} d\mathbf{a}' \cdot \nabla \left(\frac{1}{R} \right) \\ &= \frac{\varphi_0}{4\pi} \int_{(*)} d^3x' \nabla'^2 \left(\frac{1}{R} \right) \\ &= -\varphi_0 \int_{(*)} d^3x' \delta(\mathbf{x} - \mathbf{x}'), \end{aligned}$$

which is zero because \mathbf{x} is outside the surface S .

7. Prove that the Dirichlet Green's function is symmetric in the sense that

$$G_D(\mathbf{x}, \mathbf{x}') = G_D(\mathbf{x}', \mathbf{x}).$$

Solution: Start from

$$\int_{(*)} (\varphi \nabla_y^2 \psi - \psi \nabla_y^2 \varphi) d^3y = \int_{\bigcirc \rightarrow} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) da_y,$$

and let $\varphi(\mathbf{y})$ be $G_D(\mathbf{x}, \mathbf{y})$ and $\psi(\mathbf{y})$ be $G_D(\mathbf{x}', \mathbf{y})$. Then since the Dirichlet

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Green's function is zero on the boundary and obeys $\nabla_y^2 G_D(\mathbf{x}, \mathbf{y}) = -4\pi\delta(\mathbf{x} - \mathbf{y})$ one has

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of a charge $-q$ at $(-d, 0, 0)$ with the boundary disregarded. This is the solution because the potential at any point in the YZ -plane receives equal and opposite contributions from the two charges, and for $x_1 > 0$.

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physical sense are induced charges on the surfaces of the conductors. For example, in the XZ -plane the induced charge per unit area is

$$\begin{aligned}\sigma &= \frac{1}{4\pi} (E_y)_{y=0} \\ &= -\frac{1}{4\pi} \left(\frac{\partial \varphi}{\partial y} \right)_{y=0} \\ &= -\frac{q}{4\pi} \left(\frac{2d}{[\sqrt{(x-c)^2 + d^2 + z^2}]^3} - \frac{2d}{[\sqrt{(x+c)^2 + d^2 + z^2}]^3} \right).\end{aligned}\quad (12.5)$$

It will be noticed that the induced charge density goes to zero at $x = 0$.

Conducting Spheres

Another problem that can be solved by images is that of a point charge near a conducting sphere. Figure 12.1 shows the charge, the sphere,

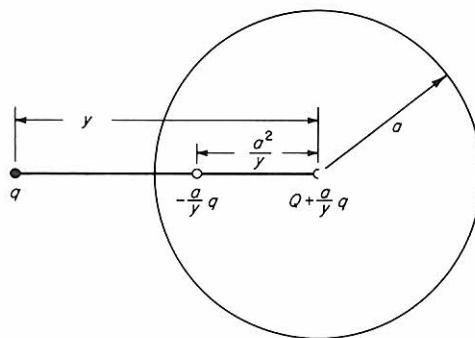


FIG. 12.1. Solution by the image technique for the potential of a point charge near a conducting sphere. The sphere is ungrounded and carries total charge Q .

and the image charges derived below. In any electrostatic problem for which the solution is known, one can replace an equipotential surface by a conductor at that potential, and thus obtain another solved problem. It happens that one of the equipotential surfaces due to two unequal oppositely signed charges is a sphere. Hence the problem posed by a point charge q outside of a conducting sphere can be solved by images. To see this in detail, consider a charge q at the origin and a charge q'

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at $(d, 0, 0)$, where $|q| > |q'|$ and the two charges are oppositely signed. The potential is

$$\varphi(x, y, z) = \frac{q}{\sqrt{x^2 + y^2 + z^2}} + \frac{q'}{\sqrt{(x - d)^2 + y^2 + z^2}}. \quad (12.6)$$

In general the equipotential surfaces ($\varphi = \text{constant}$) are of fourth degree and so are not spheres. However, when $\varphi = 0$ the surface degenerates to the second degree:

$$0 = \frac{q}{\sqrt{x^2 + y^2 + z^2}} + \frac{q'}{\sqrt{(x - d)^2 + y^2 + z^2}},$$

$$(x - d)^2 + y^2 + z^2 = \left(\frac{q'}{q}\right)^2 (x^2 + y^2 + z^2).$$
(12.7)

Let q' be $-nq$ so that $0 < n < 1$. Then the equation for this special equipotential surface is

$$(x - d)^2 + y^2 + z^2 = n^2(x^2 + y^2 + z^2),$$

$$(1 - n^2)x^2 - 2dx + (1 - n^2)(y^2 + z^2) = -d^2,$$

$$x^2 - \frac{2d}{1 - n^2}x + \left(\frac{d}{1 - n^2}\right)^2 + y^2 + z^2 = -\frac{d^2}{1 - n^2} + \left(\frac{d}{1 - n^2}\right)^2, \quad (12.8)$$

$$\left(x - \frac{d}{1 - n^2}\right)^2 + y^2 + z^2 = \left(\frac{dn}{1 - n^2}\right)^2.$$

Thus the potential is zero on the sphere of radius $dn/(1 - n^2)$ centered at $(d/(1 - n^2), 0, 0)$. Since $d/(1 - n^2)$ is greater than d , the center of the sphere is on the side of q' away from q . Furthermore since $dn/(1 - n^2)$ is less than $d/(1 - n^2)$ the surface does not enclose the charge q . Since the distance from the center of the sphere to q' is

$$\frac{d}{1 - n^2} - d = \frac{dn^2}{1 - n^2}$$

$$< \frac{dn}{1 - n^2}, \quad (12.9)$$

the sphere does include the charge q' . These last two points are essential for the solution of the point charge near a conducting sphere by images. Thus the field is the same as that of the original charge and an appropriate image charge which is guaranteed to fall inside the sphere. Ordinarily the

This problem can be shown to be unique. The charges bring the

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But this is precisely the problem that has been solved. That is, the potential at \mathbf{x}' due to a unit point charge at \mathbf{x} in the presence of a grounded sphere has been found by the method of images. The image charge is $-a/x$ located at $a^2\hat{\mathbf{x}}/x$ (for a unit charge at \mathbf{x} , a sphere of radius a , and the center of sphere at the origin). The Green's function is therefore

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}' - \mathbf{x}|} - \frac{a}{x \left| \mathbf{x}' - \frac{a^2}{x^2} \mathbf{x} \right|}. \quad (12.15)$$

The potential produced by charge density ρ in a region subject to a given value of φ on the boundary was found in Section 11 to be

$$\varphi(\mathbf{x}) = \int d^3x' \rho(\mathbf{x}') G_D(\mathbf{x}, \mathbf{x}') - \frac{1}{4\pi} \int_{\text{O}_2} da' \varphi(\mathbf{x}') \frac{\partial}{\partial n'} G_D(\mathbf{x}, \mathbf{x}'). \quad (12.16)$$

The outward normal in the present problem means the inward normal to the sphere so $\partial/\partial n'$ means $-\partial/\partial x'$, the derivative taken at constant $\hat{\mathbf{x}}'$. To evaluate this derivative, one may write

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{x'^2 - 2x'\hat{\mathbf{x}}' \cdot \mathbf{x} + x^2}} - \frac{1}{\sqrt{x'^2 \frac{x^2}{a^2} - 2x'\hat{\mathbf{x}}' \cdot \mathbf{x} + a^2}}. \quad (12.17)$$

The result of the differentiation is then

$$\begin{aligned} & -\frac{\partial}{\partial x'} G_D(\mathbf{x}, \mathbf{x}') \\ &= \frac{x' - \hat{\mathbf{x}}' \cdot \mathbf{x}}{(\sqrt{x'^2 - 2x'\hat{\mathbf{x}}' \cdot \mathbf{x} + x^2})^3} - \frac{x' \frac{x^2}{a^2} - \hat{\mathbf{x}}' \cdot \mathbf{x}}{\left(\sqrt{x'^2 \frac{x^2}{a^2} - 2x'\hat{\mathbf{x}}' \cdot \mathbf{x} + a^2} \right)^3}. \end{aligned} \quad (12.18)$$

This derivative of the Green's function is needed on the surface, namely at $x' = a$, so the result simplifies to

$$\left[\frac{\partial}{\partial n'} G_D(\mathbf{x}, \mathbf{x}') \right]_{x'=a} = \frac{a - (x^2/a)}{[\sqrt{a^2 - 2ax'\hat{\mathbf{x}}' \cdot \mathbf{x} + x^2}]^3}, \quad (12.19)$$

and the complete solution of the Dirichlet problem outside a sphere is given by

$$\begin{aligned}\varphi(\mathbf{x}) = & \int_{\text{out sph}} d^3x' \rho(\mathbf{x}') \left(\frac{1}{|\mathbf{x}' - \mathbf{x}|} - \frac{a}{x' \left| \mathbf{x}' - \frac{a^2}{x'^2} \mathbf{x} \right|} \right) \\ & + \frac{1}{4\pi} \int_{\text{O}'} da' \varphi(\mathbf{x}') \frac{x^2 - a^2}{a[\sqrt{a^2 - 2a\hat{\mathbf{x}}' \cdot \mathbf{x}} + x^2]^3}. \quad (12.20)\end{aligned}$$

PROBLEMS

1. A point charge q is near a conducting plane which has a hemispherical boss of radius a on it. Find the potential everywhere.

Solution: Choose the origin at the center of the boss, the Y - and Z -axes in the plane, and choose the XY -plane through the charge q , such that the charge has coordinates $(c, d, 0)$. The problem is solved with image charges for q through the plane and through the sphere plus a third image charge chosen so that each charge possesses an image through each surface. Figure 12.2 shows the ensemble of images that

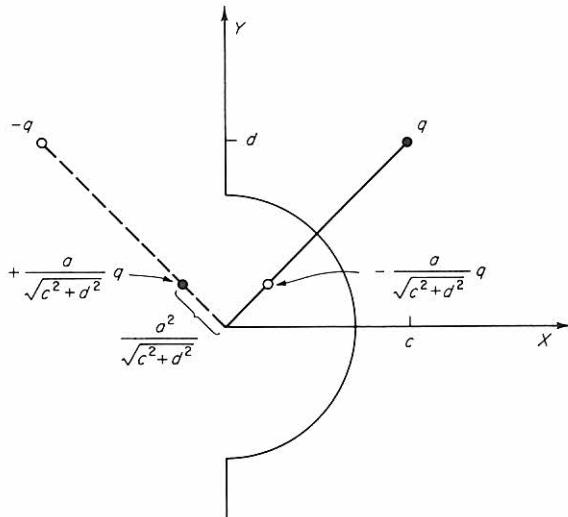


FIG. 12.2. Images for finding the potential produced by a point charge near a grounded plane with a hemispherical boss.

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solves the problem. For this purpose the sphere may be regarded as completed inside the metal. The potential is

$$\begin{aligned}\varphi(\mathbf{x}) = & \frac{q}{\sqrt{(x-c)^2 + (y-d)^2 + z^2}} - \frac{q}{\sqrt{(x+c)^2 + (y-d)^2 + z^2}} \\ & - \frac{aq}{\sqrt{c^2 + d^2} \sqrt{\left(x - \frac{a^2 c}{c^2 + d^2}\right)^2 + \left(y - \frac{a^2 d}{c^2 + d^2}\right)^2 + z^2}} \\ & + \frac{aq}{\sqrt{c^2 + d^2} \sqrt{\left(x + \frac{a^2 c}{c^2 + d^2}\right)^2 + \left(y - \frac{a^2 d}{c^2 + d^2}\right)^2 + z^2}}\end{aligned}$$

outside the conductor, and φ vanishes inside it.

2. An infinite line charge of density q/l runs parallel to an insulated conducting cylinder that carries charge per unit length $-q/l$. Show that the induced field is that of the given charge and a properly located image. Find the position of the image in terms of the cylinder radius a and the distance d between the line charge and the cylinder axis.

Solution: The potential of a line of charge may be found by applying Gauss's theorem to a cylinder of length L and radius r and with axis along the line of charge

$$\int_{\textcircled{O}} \mathbf{E} \cdot d\mathbf{a} = \int_{(*)} 4\pi\rho d^3x,$$

$$2\pi r L E = 4\pi \frac{q}{l} L,$$

$$E = 2 \left(\frac{q}{l} \right) r^{-1}.$$

The field points radially out from the line of charge. The potential is

$$\begin{aligned}\varphi &= - \int E dr \\ &= -2 \frac{q}{l} \int \frac{dr}{r} \\ &= -2 \frac{q}{l} \log r + \text{constant.}\end{aligned}$$

Since the charges are not confined to finite space, the constant in the

potential cannot be chosen to make the potential vanish infinitely far from the charge. To investigate the possibility of solutions by images, consider the potential produced by two lines of charge. Let the two lines pass through the X -axis and point in the Z -direction. Suppose there is line charge q/l at distance d from the origin and line charge q'/l at distance d' from the origin, where q'/l is to be the image in a cylinder of radius a centered at the origin. At any point the potential due to these charges, disregarding the constants, is

$$\varphi = -\log\{[(x-d)^2 + y^2]^{q/l}[(x-d')^2 + y^2]^{q'/l}\}.$$

The question now is whether q' and d' can be chosen such that φ is constant on the cylinder. If $q' = -q$, then the equipotentials will be second degree surfaces governed by

$$(x-d')^2 + y^2 = K[(x-d)^2 + y^2],$$

$$x^2 - 2 \left[\frac{d' - Kd}{1 - K} \right] x + y^2 = \frac{Kd^2 - d'^2}{1 - K},$$

where K is just a constant. This equipotential will be a circle centered at $x = y = 0$ provided $d' - Kd$ vanishes. In that case the radius a of the circle will be given by $a^2 = dd'$. Thus $q' = -q$ and $dd' = a^2$ determine the image of a line charge in a parallel conducting cylinder, the charge and image again being at inverse points.

3. An insulated conducting cylinder of radius a with charge per unit length q/l on it has its axis parallel to and a distance d away from a conducting plane. What is the potential of the cylinder relative to the plane?

4. A dipole \mathbf{p} is at a distance d from a metal plane. How much work is required to rotate it from normal (toward the plane) to parallel orientation?

Solution: Suppose the dipole is at an angle α from the normal orientation. The solution of the electrostatic problem is provided by an image dipole. One can see this by bringing together two equal and oppositely signed charges and watching the image of the process. Take the origin at the image dipole, and the Y -axis such that \mathbf{p} is in the XY -plane, the X -axis along the line between the two sources as shown in Fig. 12.3. By visualizing the construction of the dipole and its image,

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one sees that the image dipole is given by $p'_x = p_x$, $p'_y = -p_y$. The field of the image is

$$\mathbf{E} = \frac{3\mathbf{x}(\mathbf{p}' \cdot \mathbf{x}) - \mathbf{p}'r^2}{r^5},$$

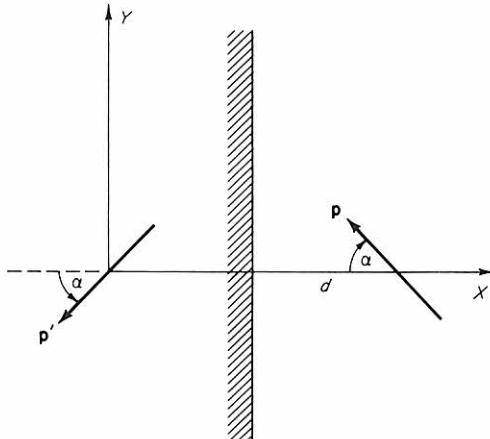


FIG. 12.3. The image of a dipole in a conducting plane.

and, at the location of the real dipole, specializes to

$$\mathbf{E} = -\frac{p \cos \alpha}{4d^3} \mathbf{i} + \frac{p \sin \alpha}{8d^3} \mathbf{j}.$$

The torque of the image on the dipole is therefore

$$\begin{aligned}\mathbf{T} &= \mathbf{p} \times \mathbf{E} \\ &= \frac{p^2}{8d^3} \sin \alpha \cos \alpha \mathbf{k}.\end{aligned}$$

The work that has to be done to rotate the dipole from the normal to the parallel orientation against this torque is

$$\begin{aligned}W &= \int_0^{\pi/2} T d\alpha \\ &= \frac{p^2}{16d^3} \int_0^{\pi/2} \sin 2\alpha d\alpha \\ &= \frac{p^2}{16d^3}.\end{aligned}$$

5. A positive point charge q is near a spherical conductor with positive charge Q on it. Derive the formula for the force between the charge and the conductor. When q/Q is small, how far from the sphere will the charge be when the force changes from attractive to repulsive?
6. Suppose there is a given arrangement of charges in space together with some fixed conductors either grounded or insulated. To what extent is the potential determined?

13. INVERSION OF SOLUTIONS

Basic Theorem

From any solved problem in electrostatics, others can be generated by a process known as *inversion*. If $\varphi(r, \theta, \phi)$ is a solution in all space of Poisson's equation

$$\nabla^2\varphi(r, \theta, \phi) = -4\pi\rho(r, \theta, \phi) \quad (13.1)$$

for some charge density ρ , then the functions

$$\begin{aligned}\varphi'(r, \theta, \phi) &= \frac{a}{r} \varphi\left(\frac{a^2}{r}, \theta, \phi\right), \\ \rho'(r, \theta, \phi) &= \left(\frac{a}{r}\right)^5 \rho\left(\frac{a^2}{r}, \theta, \phi\right)\end{aligned} \quad (13.2)$$

also satisfy Poisson's equation, as will be demonstrated below. It will be noticed that the values of the new potential and charge density are given at \mathbf{x} by the values of the old ones at the inverse point of \mathbf{x} in a sphere of radius a centered at the origin.

When φ and ρ are continuous, one can demonstrate that φ' and ρ' satisfy Poisson's equation directly. The Laplacian can be written in spherical coordinates as (see Problem 8, Section 15)

$$\nabla^2 = \frac{1}{r} \frac{\partial^2}{\partial r^2} r - \frac{L^2}{r^2}, \quad (13.3)$$

where

$$L^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \quad (13.4)$$

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Acting on the new potential with the Laplacian results in

$$\nabla^2 \varphi'(r, \theta, \phi) = \frac{a}{r} \frac{\partial^2}{\partial r^2} \varphi\left(\frac{a^2}{r}, \theta, \phi\right) - \frac{a^5}{r^5} \left(\frac{r}{a^2}\right)^2 L^2 \varphi\left(\frac{a^2}{r}, \theta, \phi\right). \quad (13.5)$$

Letting $r' = a^2/r$, one finds that

$$\begin{aligned} \frac{\partial}{\partial r} &= -\frac{a^2}{r^2} \frac{\partial}{\partial r'}, \\ \frac{\partial^2}{\partial r^2} &= \frac{2a^2}{r^3} \frac{\partial}{\partial r'} + \frac{a^4}{r^4} \frac{\partial^2}{\partial r'^2} \\ &= \frac{a^4}{r^4} \frac{1}{r'} \frac{\partial^2}{\partial r'^2} r'. \end{aligned} \quad (13.6)$$

Thus Eq. (13.5) simplifies to

$$\begin{aligned} \nabla^2 \varphi'(r, \theta, \phi) &= \frac{a^5}{r^5} \left[\frac{1}{r'} \frac{\partial^2}{\partial r'^2} r' - \frac{L^2}{r'^2} \right] \varphi(r', \theta, \phi) \\ &= -4\pi \frac{a^5}{r^5} \rho(r', \theta, \phi) \\ &= -4\pi \rho(r, \theta, \phi), \end{aligned} \quad (13.7)$$

as required.

Now for a point charge q_1 at \mathbf{x}_1 , on the other hand, the charge density in spherical coordinates would be given by

$$\rho(r, \theta, \phi) = \frac{q_1}{r^2 \sin \theta} \delta(r - r_1) \delta(\theta - \theta_1) \delta(\phi - \phi_1), \quad (13.8)$$

since this is zero everywhere except at $\mathbf{x} = \mathbf{x}_1$ and integrates to

$$\begin{aligned} \int \rho d^3x &= q_1 \int_0^\infty r^2 dr \frac{\delta(r - r_1)}{r^2} \int_0^\pi \sin \theta d\theta \frac{\delta(\theta - \theta_1)}{\sin \theta} \int_0^{2\pi} d\phi \delta(\phi - \phi_1) \\ &= q_1. \end{aligned} \quad (13.9)$$

We will verify that Eqs. (13.2) apply to give a new solution of Poisson's equation in this case also. The potential due to the point charge is

$$\varphi(r, \theta, \phi) = \frac{q_1}{\sqrt{r^2 + r_1^2 - 2rr_1 \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}_1}}, \quad (13.10)$$

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so the potential in the inverted problem should be

$$\begin{aligned}
 \varphi'(r, \theta, \phi) &= \frac{a}{r} \varphi\left(\frac{a^2}{r}, \theta, \phi\right) \\
 &= \frac{\frac{aq_1}{r_1}}{\sqrt{\left(\frac{a^2}{r}\right)^2 + r_1^2 - 2\left(\frac{a^2}{r}\right)r_1 \hat{x} \cdot \hat{x}_1}} \\
 &= \frac{\frac{aq_1}{r_1}}{\sqrt{\left(\frac{a^2}{r_1}\right)^2 + r^2 - 2r\left(\frac{a^2}{r_1}\right)\hat{x} \cdot \hat{x}_1}}. \quad (13.11)
 \end{aligned}$$

This is the potential of a point charge aq_1/r_1 located at $(a^2/r_1, \theta_1, \phi_1)$. The charge density that produces it is therefore

$$\rho'(r, \theta, \varphi) = \frac{\frac{aq_1}{r_1}}{r^2 \sin \theta} \delta\left(r - \frac{a^2}{r_1}\right) \delta(\theta - \theta_1) \delta(\varphi - \phi_1), \quad (13.12)$$

by comparison with Eq. (13.8). However, in the statement of the theorem, the charge density for the inversion problem is said to be

$$\begin{aligned}
 \rho'(r, \theta, \phi) &= \left(\frac{a}{r}\right)^5 \rho\left(\frac{a^2}{r}, \theta, \phi\right) \\
 &= \frac{a^5}{r^5} \frac{q_1}{\left(\frac{a^2}{r}\right)^2 \sin \theta} \delta\left(\frac{a^2}{r} - r_1\right) \delta(\theta - \theta_1) \delta(\phi - \phi_1). \quad (13.13)
 \end{aligned}$$

To reconcile the two expressions it will be necessary to unravel the delta function of $(a^2/r) - r_1$. Consider integration with any function $f(r)$,

$$\begin{aligned}
 \int_0^\infty \delta\left(\frac{a^2}{r} - r_1\right) f(r) dr &= \int_0^\infty \delta(x - r_1) f\left(\frac{a^2}{x}\right) \frac{a^2}{x^2} dx \\
 &= f\left(\frac{a^2}{r_1}\right) \frac{a^2}{r_1^2} \\
 &= \int_0^\infty \delta\left(r - \frac{a^2}{r_1}\right) \frac{r^2}{a^2} f(r) dr, \quad (13.14)
 \end{aligned}$$

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where the integration variable was changed to $x = a^2/r$ in the first step. This equation implies that

$$\delta\left(\frac{a^2}{r} - r_1\right) = \frac{r^2}{a^2} \delta\left(r - \frac{a^2}{r_1}\right). \quad (13.15)$$

The charge density of Eq. (13.13) becomes

$$\begin{aligned} \rho'(r, \theta, \phi) &= \frac{a}{r^3} \frac{q_1}{\sin \theta} \frac{\delta\left(r - \frac{a^2}{r_1}\right)}{\left(\frac{a^2}{r^2}\right)} \delta(\theta - \theta_1) \delta(\phi - \phi_1) \\ &= \frac{aq_1}{r^2 \sin \theta} \delta\left(r - \frac{a^2}{r_1}\right) \delta(\theta - \theta_1) \delta(\phi - \phi_1). \end{aligned} \quad (13.16)$$

In the final step of Eq. (13.16), a^2/r has been replaced by r_1 to show that the charge densities of Eqs. (13.12) and (13.13) are equal. Thus it is established that Eqs. (13.2) relate solutions of Poisson's equation, for either continuous or discrete distributions of charge.

Geometry of Inversion

A solved problem in electrostatics can be inverted to produce another solved problem. Next one studies what sort of a mapping this inversion process is, in order to see what solutions it relates. Consider points P at (r, θ, φ) and P' at (r', θ, φ) which are inverse to one another with respect to a sphere of radius a , centered at 0. Figure 13.1 shows the geometry, r' and r are related by $rr' = a^2$. Obviously if the point P is outside (or inside or on) the sphere the point P' will be inside (or outside or on) it. The inverse of the inverse of a point is the original point so inversion in a sphere is said to be of *period two*.

If a given plane passes through the point 0, each point in that plane will lie on a straight line containing the center of the sphere. Since the inverse of the point falls on that line, the inverse of a plane through 0 is itself. However, if the plane does not pass through 0, it is generated by points that satisfy

$$lx + my + nz + q = 0, \quad (13.17)$$

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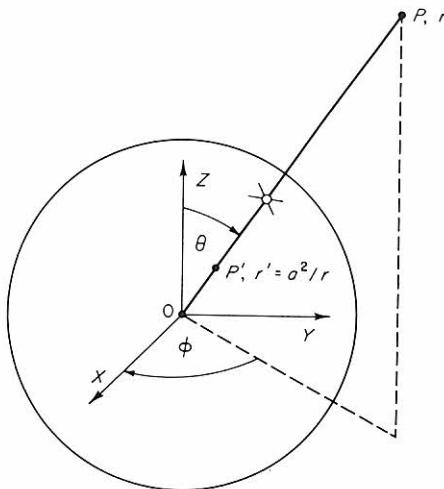


FIG. 13.1. Inverse points in a sphere.

where q is not zero. Then since

$$\begin{aligned}\frac{x}{r} &= \frac{x'}{r'}, \\ \frac{y}{r} &= \frac{y'}{r'}, \\ \frac{z}{r} &= \frac{z'}{r'},\end{aligned}\tag{13.18}$$

and $rr' = a^2$, the inverse of the plane is the surface described by

$$\begin{aligned}l \frac{a^2}{r'^2} x' + m \frac{a^2}{r'^2} y' + n \frac{a^2}{r'^2} z' + q &= 0, \\ x'^2 + y'^2 + z'^2 + \frac{a^2}{q} (lx' + my' + nz') &= 0,\end{aligned}\tag{13.19}$$

which is the equation of a sphere that passes through the origin. A sphere that does not pass through O is given by

$$x^2 + y^2 + z^2 + lx + my + nz + q = 0,\tag{13.20}$$

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where q is not zero. The inverse points of this surface satisfy

$$\frac{a^4}{r'^4} (x'^2 + y'^2 + z'^2) + \frac{a^2}{r'^2} (lx' + my' + nz') + q = 0, \quad (13.21)$$

$$x'^2 + y'^2 + z'^2 + \frac{a^2}{q} (lx' + my' + nz') + \frac{a^4}{q} = 0,$$

which is another sphere that does not pass through the origin. A plane can be regarded as a sphere of infinitely large radius. With this understanding, the outstanding feature of the inversion process is that it transforms spheres into spheres.

Another geometrical feature of the inversion process is that angles are preserved. To confirm this, consider a small triangle in a plane containing the origin O as in Fig. 13.2. Let A and C be at the same radius r and let B

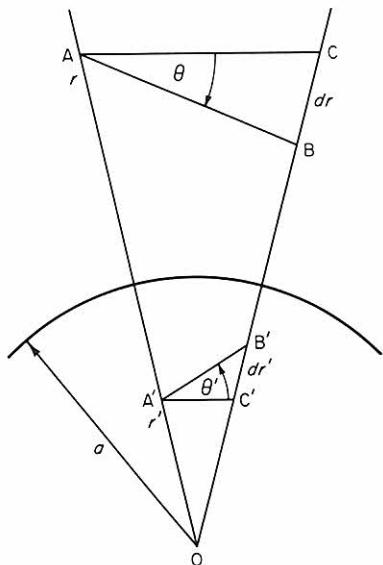


FIG. 13.2. Transformation of angles under inversion.

lie on the line connecting O and C at radius $r - dr$. As the sides of the triangle ABC are shrunk to zero, the image triangle A'B'C' becomes similar to ABC and the angles θ and θ' are equal. To see this note that

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the angles ACB and $A'C'B'$ tend to 90 degrees, and $A'C' = (r'/r)AC$. Since $r' = a^2/r$,

$$\begin{aligned} dr' &= -\frac{a^2}{r^2} dr \\ &= -\frac{r'}{r} dr, \\ B'C' &= \frac{r'}{r} BC. \end{aligned} \tag{13.22}$$

Thus the triangles are similar and the angles θ and θ' are equal. For any two lines in a plane containing O , the angles of intersection will be the same for the original lines as for the inverted lines since the intersection angle is a sum of two angles to which the above proof applies. It is also clear that angles on the surfaces of spheres about O are unchanged by inversion. Thus any small geometrical shape inverts into a similar shape with all sides in proportion and angles equal, though the inverted shape is a reflection of the original.

The rule for the transformation of an infinitesimal area is particularly simple. If the area $d\mathbf{a}$ is located at radius r , it subtends the solid angle

$$d\Omega = \frac{|d\mathbf{a} \cdot \hat{\mathbf{r}}|}{r^2} \tag{13.23}$$

at the origin. But under inversion each point in the area is transformed into a new point such that both points lie on the same line through the origin. The solid angle subtended by the inverse area element is therefore the same

$$\begin{aligned} d\Omega' &= \frac{|d\mathbf{a}' \cdot \hat{\mathbf{r}}|}{r'^2} \\ &= \frac{|d\mathbf{a} \cdot \hat{\mathbf{r}}|}{r^2}. \end{aligned} \tag{13.24}$$

Keeping in mind that the angle is unchanged by the inversion, one sees that the transformation rule for an infinitesimal element of area is given by

$$\frac{da'}{r'^2} = \frac{da}{r^2}. \tag{13.25}$$

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Surface Charges

The transformation law for charge under inversion is given by

$$q' = \frac{aq}{r} \quad (13.26)$$

for discrete charge q at radius r . This rule can be applied to the charge on a small area of a charged surface in the form

$$\sigma'(r', \theta, \phi) da' = \frac{a}{r} \sigma(r, \theta, \phi) da, \quad (13.27)$$

and with the help of the transformation rule for an element of area, the inverted charge per unit area is found to be

$$\begin{aligned} \sigma'(r', \theta, \phi) &= \frac{a}{r} \sigma(r, \theta, \phi) \frac{da}{da'} \\ &= \frac{a}{r} \frac{r^2}{r'^2} \sigma(r, \theta, \phi). \end{aligned} \quad (13.28)$$

Thus since r' is a^2/r

$$\begin{aligned} \sigma' \left(\frac{a^2}{r}, \theta, \phi \right) &= \left(\frac{r}{a} \right)^3 \sigma(r, \theta, \phi), \\ \sigma'(r, \theta, \phi) &= \left(\frac{a}{r} \right)^3 \sigma \left(\frac{a^2}{r}, \theta, \phi \right). \end{aligned} \quad (13.29)$$

Example of Inversion

Consider a sphere of radius R centered at iR which has charge q uniformly distributed on its surface. Let us declare its surface to be at zero potential. Then the potential is

$$\varphi(\mathbf{x}) = \begin{cases} 0, & |\mathbf{x} - iR| < R \\ \frac{q}{|\mathbf{x} - iR|} - \frac{q}{R}, & |\mathbf{x} - iR| > R. \end{cases} \quad (13.30)$$

Notice that the potential is $-q/R$ at infinity. Now if an inversion is performed with $a > 2R$, the sphere is transformed into a plane as shown

in Fig. 13.3. The inside of the charged sphere is transformed to the right of the plane $x = d$, where $d(2R) = a^2$. The potential there is a/r times the potential at the corresponding points inside the sphere, namely

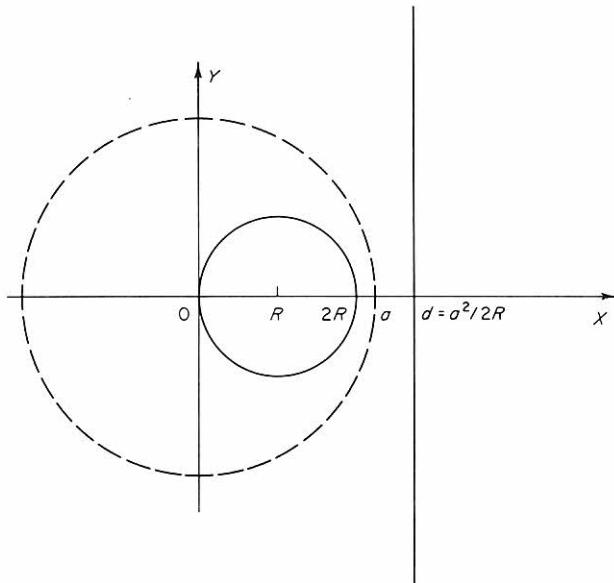


FIG. 13.3. Inversion for charged sphere at zero potential.

zero. To the left of the plane the potential consists of two parts. The contribution from the inversion of $-q/R$ is

$$\varphi' = -\frac{aq}{rR}. \quad (13.31)$$

Thus the constant part of the potential inverts to the potential of a point charge $-aq/R$ at the center of inversion. The other term in the potential is that of a point charge q located at $x = R$. The inverse of this, for $x < d$, is the potential of a point charge aq/R located at $a^2/R = 2d$. Thus, we recognize the inverted problem as that of a point charge $-aq/R$ located a distance $d = a^2/2R$ from an infinite grounded plane.

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PROBLEMS

1. Charge q is spread uniformly on a sphere of radius R . Write down the solution of Poisson's equation that is zero inside the sphere. Find the inverse solution, choosing the center of inversion outside the sphere.
2. Find the capacitance of a conductor formed by two spheres of radii R_1 and R_2 intersecting orthogonally.

Solution: Choose the origin at a point on the line of intersection of the two spheres and consider the inverse problem in a sphere of radius $2R_1$, where R_1 is the larger radius. Each sphere inverts into a plane when this is done, and the planes intersect at 90 degrees since the spheres were orthogonal. Figure 13.4 shows the two spheres and planes. In the inverted problem suppose there is a charge q at the origin along with three image charges arranged so the potential vanishes on both planes. The corresponding solution in the original problem has $\varphi = 0$ on the conductor and

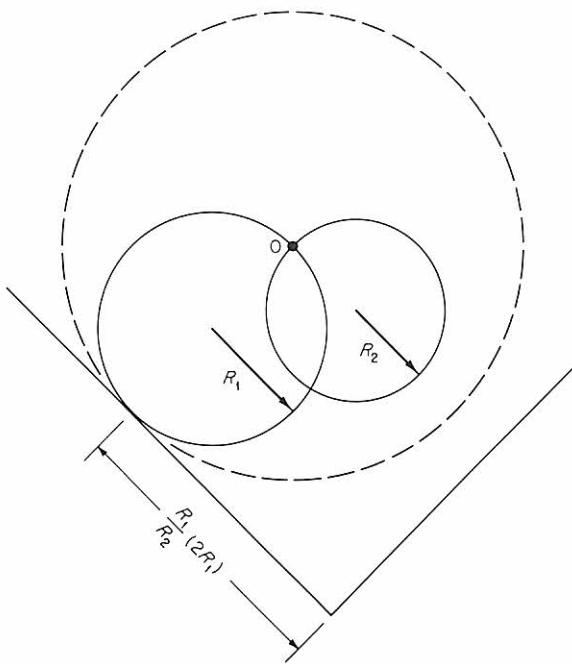


FIG. 13.4. Inversion process leading to the capacitance of a conductor formed from orthogonally intersecting spheres.

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$\varphi = q/2R_1$ at infinity. The other three charges have inverses inside the conductor which can be found from the rule

inverse charge = (inversion radius)(charge)(distance from inversion center) $^{-1}$.

Thus the charge on the conductor in the original problem is

$$\begin{aligned} Q &= \frac{2R_1}{4R_1}(-q) + \frac{2R_1}{\left(\frac{4R_1^2}{R_2}\right)}(-q) + \frac{2R_1}{\sqrt{(4R_1)^2 + \left(\frac{4R_1^2}{R_2}\right)^2}}(q) \\ &= -\frac{1}{2}q - \frac{R_2}{2R_1}q + \frac{q}{2\sqrt{1 + \left(\frac{R_1}{R_2}\right)^2}} \\ &= \frac{q}{2} \left[-\left(1 + \frac{R_2}{R_1}\right) + \frac{1}{\sqrt{1 + \left(\frac{R_1}{R_2}\right)^2}} \right]. \end{aligned}$$

With this charge and potential zero at infinity, the conductor would have potential $-q/2R_1$ so the capacitance is

$$\begin{aligned} C &= \frac{Q}{\varphi} = R_1 \left[\left(1 + \frac{R_2}{R_1}\right) - \frac{1}{\sqrt{1 + \left(\frac{R_1}{R_2}\right)^2}} \right] \\ &= R_1 + R_2 - \frac{R_1 R_2}{\sqrt{R_1^2 + R_2^2}}. \end{aligned}$$

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Some two-dimensional problems in electrostatics can be given elegant solutions based on the properties of functions $w(z)$ of a complex variable $z = x + iy$. We will review here some of the basic ideas in the theory of functions of a complex variable on a heuristic level and then develop the electrostatic applications. For a more thorough and logical development of the mathematics, which the subject deserves, see, for example, Churchill [2].

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Differentiation of a Function of a Complex Variable

If for every value of $z = x + iy$ a value of $w = \varphi + i\psi$ is prescribed, one says w is a *function* of z . Thus a complex function corresponds to two real functions

$$\begin{aligned}\varphi &= \varphi(x, y), \\ \psi &= \psi(x, y).\end{aligned}\tag{14.1}$$

The derivative of a function of a complex variable is defined by

$$\frac{dw(z)}{dz} = \lim_{\Delta z \rightarrow 0} \left[\frac{w(z + \Delta z) - w(z)}{\Delta z} \right].\tag{14.2}$$

In general the limit depends on the direction of Δz or, in other words, on the ratio of its real to its imaginary part. The derivative is said to *exist*, however, when the limit does not depend on this direction. For some arbitrary direction one has

$$\begin{aligned}\frac{dw}{dz} &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left[\frac{\varphi(x + \Delta x, y + \Delta y) - \varphi(x, y)}{\Delta x + i \Delta y} + i \frac{\psi(x + \Delta x, y + \Delta y) - \psi(x, y)}{\Delta x + i \Delta y} \right] \\ &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left[\frac{\left[\Delta x \frac{\partial \varphi}{\partial x} + i \Delta y \frac{\partial \psi}{\partial y} \right] + i \left[\Delta x \frac{\partial \psi}{\partial x} - i \Delta y \frac{\partial \varphi}{\partial y} \right]}{\Delta x + i \Delta y} \right].\end{aligned}\tag{14.3}$$

The limit is independent of the particular path along which Δx and Δy are taken to zero if the functions $\varphi(x, y)$ and $\psi(x, y)$ satisfy

$$\begin{aligned}\frac{\partial \psi}{\partial y} &= \frac{\partial \varphi}{\partial x}, \\ \frac{\partial \varphi}{\partial y} &= -\frac{\partial \psi}{\partial x},\end{aligned}\tag{14.4}$$

since then the $\Delta x + i \Delta y$ cancels out. These equations are called the *Cauchy-Riemann conditions*. They are seen here to be sufficient conditions for the existence of the derivative of $w(z)$ at the point $z = x + iy$. (Actually the partial derivatives $\partial \varphi / \partial x$, $\partial \varphi / \partial y$, $\partial \psi / \partial x$, $\partial \psi / \partial y$ must be continuous as well. The manipulations that lead to Eq. (14.3) do not hold without this continuity.) The conditions are also necessary. To see

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this suppose that dw/dz exists and evaluate it from Eq. (14.2) in case $\Delta y = 0$,

$$\frac{dw}{dz} = \frac{\partial \varphi}{\partial x} + i \frac{\partial \psi}{\partial x}, \quad (14.5)$$

and in case $\Delta x = 0$,

$$\frac{dw}{dz} = -i \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial y}. \quad (14.6)$$

Comparison of these two results gives Eqs. (14.4).

A function $w(z)$ is said to be *analytic at the point z* if the derivative exists throughout some neighborhood of z . A function is called *analytic in a region* if it is analytic at each point of the region. For example, polynomials in z are analytic except at infinity. If f is an analytic function of w and w is an analytic function of z , then $f(w(z))$ is analytic in z .

Often functions of a single real variable generalize into analytic functions of a complex variable. For example suppose $f(x)$ has a power series expansion

$$f(x) = \sum_n a_n x^n. \quad (14.7)$$

Then a function $f(z)$ can be defined by

$$f(z) = \sum_n a_n z^n. \quad (14.8)$$

It follows directly from the definition, Eq. (14.2), that the derivative of z^n is $n z^{n-1}$ so

$$\frac{df}{dz} = \sum_n n a_n z^{n-1}. \quad (14.9)$$

This depends only on z so the derivative exists and the function is analytic. Of course the important question is where these series converge, but anyway the function of a real variable gives a function of a complex variable that is analytic in some region.

The function $\log z$ is many valued since, writing z as $r e^{i(\theta+2\pi n)}$, where n is any integer, one obtains

$$\log z = \log r + i\theta + i2\pi n. \quad (14.10)$$

The sensible thing to do in differentiating this according to the definition

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given by Eq. (14.2) would be to choose values of $w(z)$ and $w(z + \Delta z)$ that are contiguous. It is evident then that the derivative is independent of the particular value of n chosen in the process. The derivative of $\log z$ defined in this way exists everywhere except at the origin and at infinity. In many problems one avoids ambiguities by defining the functions in a single-valued way in the first place. For example, as part of the definition of $\log z$, one may impose the requirement

$$2\pi > \operatorname{Im}(\log z) \geq 0. \quad (14.11)$$

The function is then single valued and the derivative exists everywhere except at the origin, at infinity, and along the line $\theta = 0$. The positive real axis is, in this case, referred to as the *cut*, because the function is analytic in the finite plane away from the origin if the plane is regarded as cut apart along $\theta = 0$.

Integration on Closed Paths in the Complex Plane

Next we will review Cauchy's theorem and Cauchy's integral formula. These ideas are used later in evaluating some integrals and are essential in the development of dispersion relations (Section 25).

To begin with, consider specializing Stokes's theorem

$$\oint dl = \int_{\text{closed path}} da \times \nabla \quad (14.12)$$

to act on functions of x and y only where the closed path lies in the XY -plane. The direction in which the circuit is traversed and the Z -axis are related by the right-hand rule if da is taken to be $+dx dy$. By taking components and dropping terms involving $\partial/\partial z$ one finds

$$\begin{aligned} \oint dx &= - \iint dx dy \frac{\partial}{\partial y}, \\ \oint dy &= + \iint dx dy \frac{\partial}{\partial x}, \end{aligned} \quad (14.13)$$

where the double integrations extend over the area in the XY -plane enclosed by the path. This special form of Stokes's theorem is known as *Green's theorem*. Now the integral of a function of the complex variable

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$z = x + iy$ over a closed path \mathcal{C} in the complex plane can be expressed in the form

$$\begin{aligned}\int_{\mathcal{C}} w(z) dz &= \int_{\mathcal{C}} (\varphi + i\psi)(dx + i dy) \\ &= \int_{\mathcal{C}} (\varphi dx - \psi dy) + i \int_{\mathcal{C}} (\psi dx + \varphi dy),\end{aligned}\quad (14.14)$$

just by separating real and imaginary parts. However one may now use Green's theorem in both integrals to obtain

$$\int_{\mathcal{C}} w(z) dz = \iint dx dy \left(-\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x} \right) + i \iint dx dy \left(-\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right).\quad (14.15)$$

Thus if $w(z)$ is an analytic function over the region of the boundary and its interior one may employ the Cauchy–Riemann equations $-\partial\varphi/\partial y = \partial\psi/\partial x$ and $\partial\varphi/\partial x = \partial\psi/\partial y$ to show that the integral vanishes,

$$\int_{\mathcal{C}} w(z) dz = 0.\quad (14.16)$$

This result is known as *Cauchy's theorem*.

Next consider integrating the function $w(z)/(z - z_0)$ over some closed path. It will be supposed that $w(z)$ is analytic over the path \mathcal{C}_1 and its interior and that the point z_0 lies in the interior. The function $w(z)/(z - z_0)$ is therefore analytic on the boundary and in the enclosed region except at the point z_0 . Now the value of the integral is unaltered if the path \mathcal{C}_1 is shrunk about the point z_0 . To see this, consider the contours shown in Fig. 14.1. The contour $\mathcal{C}_1 + \mathcal{C}_2 + (-\mathcal{C}_3) + \mathcal{C}_4$ is closed and $w(z)/(z - z_0)$ is analytic over it and its interior so

$$\int_{\mathcal{C}_1} \frac{w(z)}{z - z_0} dz + \int_{\mathcal{C}_2 + \mathcal{C}_4} \frac{w(z)}{z - z_0} dz = \int_{\mathcal{C}_3} \frac{w(z)}{z - z_0} dz,\quad (14.17)$$

by Cauchy's theorem. However the integrals over \mathcal{C}_2 and \mathcal{C}_4 cancel because these paths in the complex plane are the same except for direction. Therefore the value of the integral over \mathcal{C}_1 is unaltered by changing to the inner contour \mathcal{C}_3 . Moreover \mathcal{C}_3 is unspecified except that it must lie wholly within \mathcal{C}_1 and must also surround the point z_0 . Therefore

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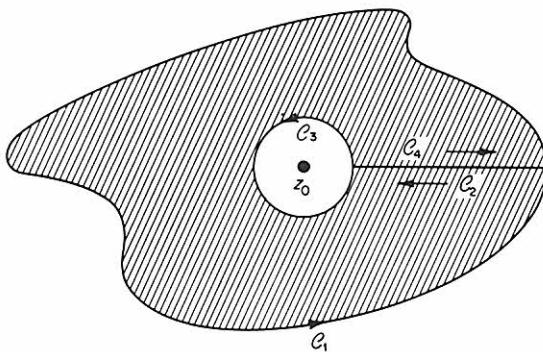


FIG. 14.1. Contours in the complex plane.

one can choose the inner path to be a circle of radius ϵ about the point z_0 and consider the limit as ϵ tends to zero. The result of this is

$$\begin{aligned} \int_{\mathcal{C}_3} \frac{w(z)}{z - z_0} dz &= w(z_0) \int_{\mathcal{C}_3} \frac{dz}{z - z_0} \\ &= w(z_0) \int_0^{2\pi} \frac{i\epsilon e^{i\phi} d\phi}{\epsilon e^{i\phi}} \\ &= w(z_0)i2\pi, \end{aligned} \quad (14.18)$$

where the substitution $z = z_0 + \epsilon e^{i\phi}$ defines the contour \mathcal{C}_3 . The result

$$w(z_0) = \frac{1}{i2\pi} \int_{\mathcal{C}} \frac{w(z)}{z - z_0} dz \quad (14.19)$$

for $w(z)$ analytic throughout \mathcal{C} and its interior, with z_0 a point in the enclosed region, is known as *Cauchy's integral formula*. The integral is taken counterclockwise around the contour.

Cauchy's integral formula illustrates the spectacular properties of analytic functions. Equation (14.19) relates the value of $w(z)$ at z_0 to its values on a path in the complex plane which does not pass through the point z_0 . The denominator $z - z_0$ is therefore bounded from below in magnitude throughout the integration and intuitively one sees nothing wrong in differentiating the Cauchy integral formula to arbitrary order. In fact where $w(z)$ is analytic on and within the contour \mathcal{C} the formula

$$w^{(n)}(z_0) = \frac{n!}{i2\pi} \int_{\mathcal{C}} dz \frac{w(z)}{(z - z_0)^{n+1}} \quad (14.20)$$

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for the n th derivative of $w(z)$ at z_0 can be made rigorous. This means that if a function of a complex variable is analytic at a point, it is differentiable indefinitely many times at that point.

Equation (14.20) can be converted into a Taylor series for $w(z)$ about z_0 very easily. All one needs to do is sum the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n = \frac{z' - z_0}{z' - z} \quad (14.21)$$

and substitute into Cauchy's integral formula, taking note of Eq. (14.20):

$$\begin{aligned} w(z) &= \frac{1}{i2\pi} \int_{\mathcal{C}} dz' \frac{w(z')}{z' - z} \\ &= \frac{1}{i2\pi} \int_{\infty} dz' \frac{w(z')}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n \\ &= \frac{1}{i2\pi} \sum_{n=0}^{\infty} (z - z_0)^n \int_{\mathcal{C}} dz' \frac{w(z')}{(z' - z_0)^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n w^{(n)}(z_0)}{n!}. \end{aligned} \quad (14.22)$$

From this manipulation one can understand the domain of applicability of the Taylor series expansion. The geometric series summation, Eq. (14.21), applies as long as $|(z - z_0)/(z' - z_0)| < 1$ or, in other words, as long as the distance from z' to z_0 is greater than the distance from z to z_0 . In order to apply the expansion at all points z' on the contour \mathcal{C} it is necessary that \mathcal{C} lie outside a circle drawn about z_0 through z . However, one can apply Eq. (14.20) for the derivative only if $w(z)$ is analytic on and inside the contour. The conclusion is that the Taylor series expansion for $w(z)$ about z_0 converges within a circle drawn about z_0 and extending out to the first singularity of $w(z)$.

Analytic Functions as Solutions of Laplace's Equation

Every analytic function satisfies Laplace's equation in two dimensions because, as a consequence of the Cauchy-Riemann conditions,

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial \psi}{\partial y} \\ &= \frac{\partial}{\partial y} \left(-\frac{\partial \varphi}{\partial y} \right), \end{aligned} \quad (14.23)$$

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and

$$\begin{aligned}\frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial}{\partial x} \left(-\frac{\partial \varphi}{\partial y} \right) \\ &= -\frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right).\end{aligned}\quad (14.24)$$

Thus φ and ψ obey Laplace's equation separately. Alternatively suppose that $f(x)$ is some real function of a real variable corresponding to an analytic function $f(z)$. Then

$$\begin{aligned}\frac{\partial^2 f(z)}{\partial x^2} &= \frac{d^2 f}{dz^2}, \\ \frac{\partial^2 f(z)}{\partial y^2} &= -\frac{d^2 f}{dz^2}.\end{aligned}\quad (14.25)$$

Thus $f(z)$ is a complex solution of Laplace's equation and so is $f^*(x + iy)$. The asterisk denotes complex conjugation, the operation of replacing i by $-i$. Since we have assumed $f(x)$ is real, we have $f^*(x + iy) = f(x - iy)$. Hence the real part of $f(z)$

$$\varphi = \frac{1}{2}[f(x + iy) + f(x - iy)] \quad (14.26)$$

is a real solution of Laplace's equation (and so is the imaginary part $(1/2i)[f(x + iy) - f(x - iy)]$).

Application to Electrostatics

Every analytic function corresponds to a solution of a two-dimensional boundary value problem in electrostatics. To see this consider two lines at constant φ , $\varphi = c_1$ and $\varphi = c_2$ as in Fig. 14.2. The electrostatic boundary value problem reduces to two dimensions when there is translational symmetry in the boundary and sources in one direction. The two boundaries $\varphi(x, y) = c_1$ and $\varphi(x, y) = c_2$ can therefore be interpreted as the cross sections of the plates of a condenser that is infinite along the Z -axis and $\varphi(x, y)$ as the electrostatic potential in the region between them.

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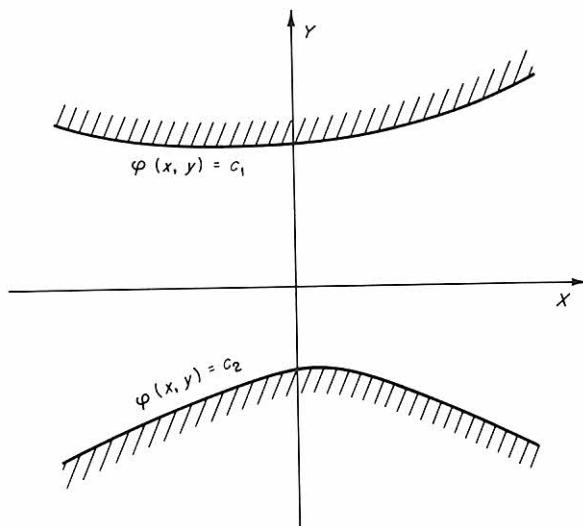


FIG. 14.2. *Solution of condenser problem by an analytic function.*

Now lines at constant ψ are always orthogonal to lines at constant φ when $\varphi + i\psi$ is an analytic function of $z = x + iy$ since

$$\begin{aligned}\nabla\varphi \cdot \nabla\psi &= \frac{\partial\varphi}{\partial x} \frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y} \frac{\partial\psi}{\partial y} \\ &= -\frac{\partial\varphi}{\partial x} \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial y} \frac{\partial\varphi}{\partial x} \\ &= 0.\end{aligned}\tag{14.27}$$

The orthogonality follows from this and the fact that $\nabla\varphi$ (or $\nabla\psi$) is orthogonal to lines of constant φ (or ψ). To see this one has only to consider two nearby points \mathbf{x} and $\mathbf{x} + d\mathbf{x}$

$$\begin{aligned}\varphi(\mathbf{x} + d\mathbf{x}) - \varphi(\mathbf{x}) &= \frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy + \frac{\partial\varphi}{\partial z} dz \\ &= \nabla\varphi \cdot d\mathbf{x}.\end{aligned}\tag{14.28}$$

Thus if $d\mathbf{x}$ points along a direction of constant φ

$$0 = \nabla\varphi \cdot d\mathbf{x}.\tag{14.29}$$

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Now, supposing that $\varphi(x, y)$ is the electrostatic potential, the field can be obtained from

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial\varphi}{\partial x} + i \frac{\partial\psi}{\partial x} \\ &= \frac{\partial\varphi}{\partial x} - i \frac{\partial\varphi}{\partial y} \\ &= -E_x + iE_y.\end{aligned}\tag{14.30}$$

Thus E_x and E_y are the real and imaginary components of $(-dw/dz)^*$. The standard terminology here is

$$\begin{aligned}w &= \varphi + i\psi \\ &= \text{complex potential}, \\ \varphi &= \text{potential}, \\ \psi &= \text{stream function}.\end{aligned}\tag{14.31}$$

The electric field $\mathbf{E} = -\nabla\varphi$ is everywhere in the direction of the lines $\psi = \text{constant}$; they are the field lines of the problem. The function ψ is called the *stream function* because of the hydrodynamic application of potential theory. In it \mathbf{E} is an actual flow velocity and $\psi = \text{constant}$ are the stream lines. The physical significance of the stream function in the present context is that it is a measure of the charge density at the boundary, considering the boundary to be a conducting sheet. On the boundary φ is constant while lines at constant ψ are normal to the boundary. Choose the normal $\hat{\mathbf{n}}$ to be outward from the boundary and choose the tangential direction $\hat{\mathbf{t}}$ so that $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ correspond to a rotation of \mathbf{i} and \mathbf{j} as in Fig. 14.3. Then the charge per unit area on the boundary is

$$\begin{aligned}\sigma &= \frac{1}{4\pi} \mathbf{E} \cdot \hat{\mathbf{n}} \\ &= -\frac{1}{4\pi} \left[\frac{\partial\varphi}{\partial x} \hat{n}_x + \frac{\partial\varphi}{\partial y} \hat{n}_y \right] \\ &= -\frac{1}{4\pi} \left[\frac{\partial\psi}{\partial y} \hat{n}_x - \frac{\partial\psi}{\partial x} \hat{n}_y \right].\end{aligned}\tag{14.32}$$

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But since $\hat{t}_x = \hat{n}_y$ and $\hat{t}_y = -\hat{n}_x$, the charge density is

$$\sigma = + \frac{1}{4\pi} \nabla\psi \cdot \hat{t}, \quad (14.33)$$

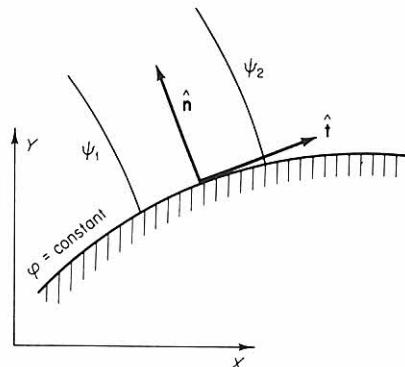


FIG. 14.3. Physical significance of the stream function.

and therefore the charge per unit length between ψ_1 and ψ_2 on the boundary is

$$\begin{aligned} \frac{dQ}{dl} &= \frac{1}{4\pi} \int_1^2 \frac{\partial\psi}{\partial t} dt \\ &= \frac{1}{4\pi} (\psi_2 - \psi_1). \end{aligned} \quad (14.34)$$

As an example, consider the complex function

$$w(z) = -\frac{2q}{l} \log z. \quad (14.35)$$

If one switches to cylindrical polar coordinates with $z = re^{i\theta}$, using the logarithm as defined by Eq. (14.11), then

$$w(z) = -2 \frac{q}{l} (\log r + i\theta). \quad (14.36)$$

The electrostatic potential at r is

$$\varphi = -2 \frac{q}{l} \log r, \quad (14.37)$$

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that is, the potential of an infinite line of charge q/l . (Since there is infinite charge, the potential cannot be made to go to zero as r goes to infinity.) The stream function is

$$\psi(\theta) = -2 \frac{q}{l} \theta. \quad (14.38)$$

Since θ is conventionally measured counterclockwise, opposite to $\hat{\mathbf{t}}$, the total charge per unit length inside radius r is

$$\begin{aligned} \frac{dQ}{dl} &= \frac{1}{4\pi} [\psi(0) - \psi(2\pi)] \\ &= \frac{q}{l}, \end{aligned} \quad (14.39)$$

as expected. The directions $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ are shown in Fig. 14.4.

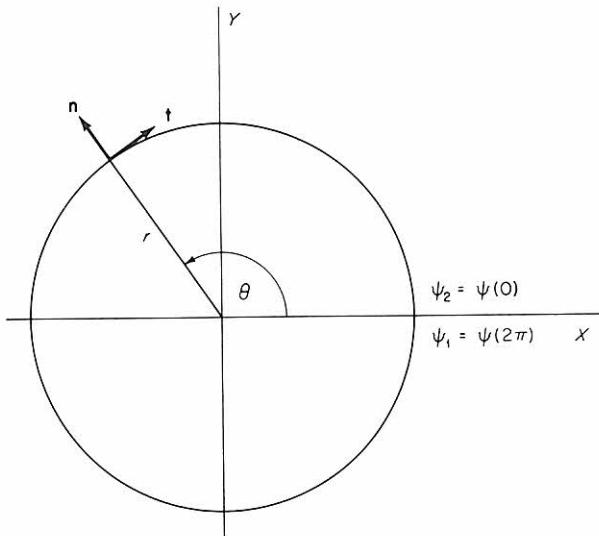


FIG. 14.4. Potential and stream function for a line of charge.

Another example is a treatment of the edge effects of a condenser by the complex variable technique. Suppose that $w(z)$ is implicitly defined by

$$\begin{aligned} iz &= iw + e^{iw}, \\ ix - y &= i\varphi - \psi + e^{i\varphi-\psi}, \\ x &= \varphi + e^{-\psi} \sin \varphi, \\ y &= \psi - e^{-\psi} \cos \varphi. \end{aligned} \quad (14.40)$$

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Now consider the region bounded by the equipotential surfaces $\varphi = +\pi$ and $\varphi = -\pi$. When $\varphi = \pm\pi$ one has

$$\begin{aligned} x &= \pm\pi, \\ y &= \psi + e^{-\psi}. \end{aligned} \tag{14.41}$$

The function $\psi + e^{-\psi}$ has an extremum when

$$\frac{d}{d\psi} (\psi + e^{-\psi}) = 1 - e^{-\psi} = 0. \tag{14.42}$$

This occurs at $\psi = 0$. It is therefore clear that $y = \psi + e^{-\psi}$ is always positive and achieves its minimum $y = 1$ when ψ is zero. Therefore the boundaries $\varphi = \pm\pi$ locating the plates of the condenser are as shown in Fig. 14.5. As ψ (and therefore y) tends to plus infinity, one has

$$x = \varphi \quad \text{and} \quad y = \psi, \tag{14.43}$$

and the equipotential surfaces are straight vertical lines. Their spacing in the X -direction is proportional to φ . This is, of course, the situation inside the condenser far from any edges. As ψ tends to minus infinity one has

$$\begin{aligned} x &= e^{-\psi} \sin \varphi, \\ y &= -e^{-\psi} \cos \varphi, \end{aligned} \tag{14.44}$$

provided that neither $\sin \varphi$ nor $\cos \varphi$ happens to be zero. If this is the case one has

$$\frac{x}{y} = -\tan \varphi, \tag{14.45}$$

and thus the equipotential lines tend to straight lines. It is clear from Eq. (14.40) that the equipotential $\varphi = 0$ is the entire Y -axis. When $\varphi = \pi/2$ one has

$$x = \frac{\pi}{2} + e^{-\psi} \quad \text{and} \quad y = \psi, \tag{14.46}$$

so the equation for this equipotential line is

$$x = \frac{\pi}{2} + e^{-y}. \tag{14.47}$$

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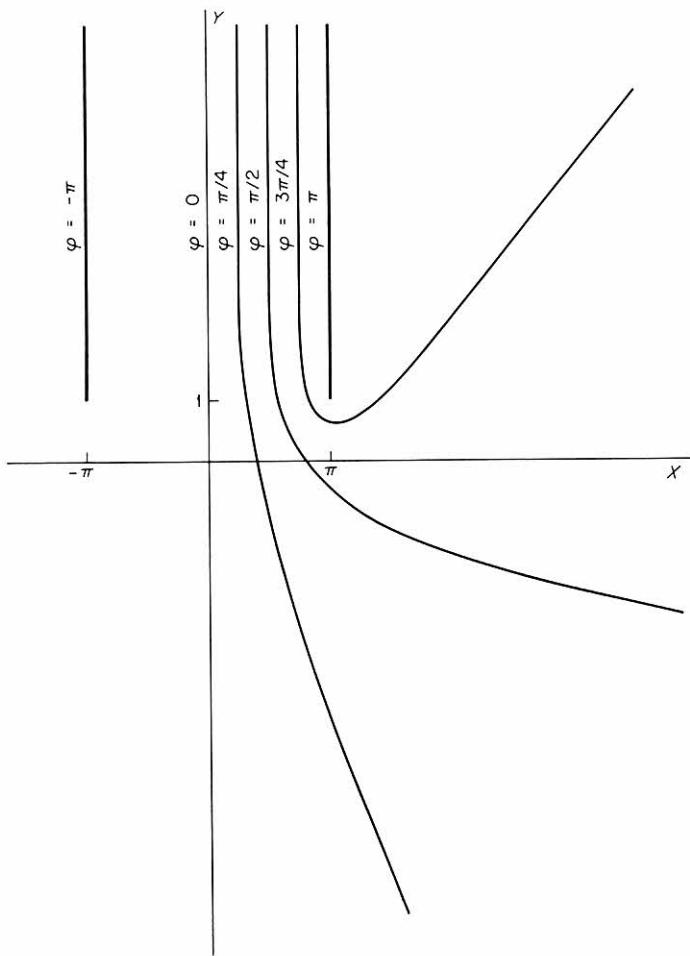


FIG. 14.5. Edge effects of a semi-infinite condenser.

It will be noticed that this never tends to a straight line as do the equipotentials for φ on either side of $\pi/2$. Some of the $\varphi = \text{constant}$ lines are shown in the figure.

As an example of a problem of a line charge near a conducting surface, consider finding the potential for the arrangement shown in Fig. 14.6. The charge per unit length q/l is located at $(2c, 2c)$ and the equation for

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the surface is $xy = c^2$. Suppose the surface is grounded. An efficient way to proceed here is to consider the new variable z' where

$$\begin{aligned} z' &= z^2 \\ &= x^2 - y^2 + i2xy, \\ x' &= x^2 - y^2, \\ y' &= 2xy. \end{aligned} \tag{14.48}$$

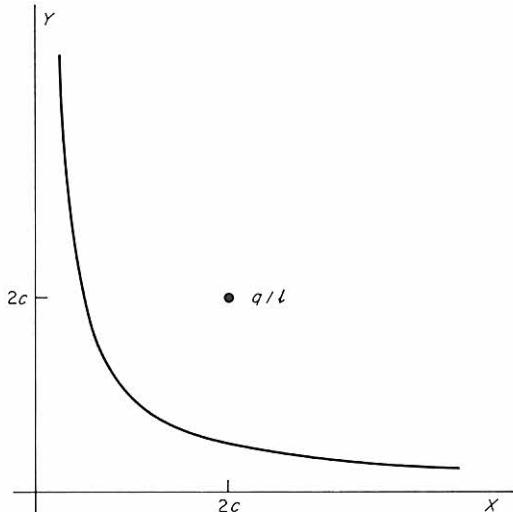


FIG. 14.6. Line charge with hyperbolic grounded surface.

The line $xy = c^2$ maps into the line $y' = 2c^2$, while the point $x = y = 2c$ maps into the point $x' = 0, y' = 8c^2$. Now the problem of a line charge q/l at $z' = i8c^2$ and a grounded plane at $y' = 2c^2$ can be solved with the image $-q/l$ at $z' = -i4c^2$. Thus the complex potential to consider is

$$\begin{aligned} w(z) &= -2 \frac{q}{l} \log(z' - i8c^2) + 2 \frac{q}{l} \log(z' + i4c^2) \\ &= -2 \frac{q}{l} \log \frac{z' - i8c^2}{z' + i4c^2}, \end{aligned}$$

and therefore

$$w(z) = -2 \frac{q}{l} \log \frac{z^2 - i8c^2}{z^2 + i4c^2}. \tag{14.49}$$

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Notice that $w(z)$ is analytic except at the points

$$\begin{aligned} z &= \pm 2(1 + i)c, \\ z &= \pm \sqrt{2}(1 - i)c, \end{aligned} \tag{14.50}$$

and that only the point $z = 2(1 + i)c$ lies within the physical region. The complex potential can be reexpressed as

$$\begin{aligned} w(z) &= -2 \frac{q}{l} \log[z + 2(1 + i)c] - 2 \frac{q}{l} \log[z - 2(1 + i)c] \\ &\quad + 2 \frac{q}{l} \log[z + \sqrt{2}(1 - i)c] + 2 \frac{q}{l} \log[z - \sqrt{2}(1 - i)c]. \end{aligned} \tag{14.51}$$

Thus the singularities of $w(z)$ can be interpreted as arising from the original source in the physical region and three images in the unphysical region. The potential itself is given by

$$\begin{aligned} \varphi(x, y) &= \frac{1}{2} [w + w^*] \\ &= -\frac{q}{l} \log \left[\frac{x^2 - y^2 + i2(xy - 4c^2)}{x^2 - y^2 + i2(xy + 2c^2)} \right] - \frac{q}{l} \log \left[\frac{x^2 - y^2 - i2(xy - 4c^2)}{x^2 - y^2 - i2(xy + 2c^2)} \right] \\ &= -\frac{q}{l} \log \left[\frac{(x^2 - y^2)^2 + 4(xy - 4c^2)^2}{(x^2 - y^2)^2 + 4(xy + 2c^2)^2} \right]. \end{aligned} \tag{14.52}$$

There is no doubt that this is the solution of the problem; it approaches the potential of the line charge near the charge, it is evidently constant on the surface $xy = c^2$, and it satisfies Laplace's equation everywhere in the allowed region except at the source.

PROBLEMS

- Find the field and the surface charge density in the example of a line charge q/l at $(2c, 2c)$ bounded by a conducting surface at zero potential given in the first quadrant by $xy = c^2$.
- A horizontal half-plane at zero potential has its edge parallel to and at a distance c from an infinite vertical plane at potential $\pi/2$. Find the charge densities on both planes. *Suggestion:* Study w defined implicitly by $c \cos w = z$.

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3. Find the complex potential due to a line of charge per unit length q/l at $x = 0, y = a$ and another $-q/l$ at $w = 0, y = -a$. Prove that the equipotentials are circular cylinders.

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Separation of Laplace's Equation in Spherical Coordinates

In the space outside the charge distribution one must have $\nabla^2\varphi = 0$. If the boundary conditions are most naturally expressed in spherical coordinates, then it is sensible to look for solutions of Laplace's equation in the form

$$\varphi(r, \theta, \phi) = R(r)P(\theta)Q(\phi). \quad (15.1)$$

Laplace's equation in spherical coordinates takes the form [Problem 8]

$$\frac{1}{r} \left(\frac{\partial^2}{\partial r^2} \right) r\varphi + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2} = 0, \quad (15.2)$$

and so, with the substitution $R(r) = U(r)/r$, one has

$$\left(\frac{1}{r} \frac{d^2 U}{dr^2} \right) PQ + \frac{U}{r^3} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) \right] Q + \frac{U}{r^3} \frac{P}{\sin^2 \theta} \frac{d^2 Q}{d\phi^2} = 0, \quad (15.3)$$

where the partial derivatives of Eq. (15.2) have simplified to total derivatives. If one divides by UPQ and multiplies by $r^3 \sin^2 \theta$ in Eq. (15.3), the result is

$$r^2 \sin^2 \theta \frac{U''}{U} + \frac{\sin \theta}{P} (\sin \theta P')' + \frac{Q''}{Q} = 0. \quad (15.4)$$

Now Q depends only on ϕ , so Q''/Q does not depend on r or θ . But from Eq. (15.4), Q''/Q is equal to an expression that depends only on r and θ . Therefore both expressions must be equal to some constant number, say

$$\begin{aligned} \frac{Q''}{Q} &= -m^2, \\ Q'' + m^2 Q &= 0. \end{aligned} \quad (15.5)$$

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Thus the ϕ -dependence of φ has been isolated and determined. This is what is meant by separation of variables. The solution for Q is of the form

$$Q = \begin{cases} A \cos(|m|\phi + \eta), & |m| \neq 0, \\ C + \phi D, & |m| = 0, \end{cases} \quad (15.6)$$

where two undetermined constants are included in the general solution of this homogeneous, second-order, differential equation. The solution of interest for electrostatics are single valued since $\nabla \times \mathbf{E}$ is zero. In most problems ϕ ranges over all values and it must be that

$$\begin{aligned} Q(\phi + 2\pi) &= Q(\phi), \\ \cos(|m|\phi + 2\pi|m| + \eta) &= \cos(|m|\phi + \eta), \quad |m| \neq 0, \\ (\phi + 2\pi)D &= \phi D. \quad |m| = 0. \end{aligned} \quad (15.7)$$

Therefore $|m|$ must be an integer 0, 1, 2, ... and, if $|m| = 0$, D must be equal to zero. Note that if the separation constant is taken to be positive, say $Q''/Q = \alpha^2$, then the solutions are of the form $Ae^\alpha + Be^{-\alpha}$ and will not be single valued unless $\alpha = 0$. Although the potential must be a real function it is a mathematical convenience to consider complex solutions. Equation (15.6) can be expanded in the form

$$\begin{aligned} Q &= \frac{A}{2} [e^{i(|m|\phi+\eta)} + e^{-i(|m|\phi+\eta)}] \\ &= \left(\frac{1}{2} A e^{i\eta}\right) e^{i|m|\phi} + \left(\frac{1}{2} A e^{-i\eta}\right) e^{-i|m|\phi}. \end{aligned} \quad (15.8)$$

The complex single-valued solutions can therefore be taken to be of the form

$$Q = (\text{constant}) e^{im\phi}, \quad (15.9)$$

where m is merely an integer, $m = 0, \pm 1, \pm 2, \dots$. For any electrostatic potential, the solutions for $\pm m$ will be combined to make Q a real function, $Q = A \cos(|m|\phi + \eta)$.

Now to return to Eq. (15.4), the remaining differential equation in r and θ is

$$\begin{aligned} r^2 \sin^2 \theta \frac{U''}{U} + \frac{\sin \theta}{P} (\sin \theta P')' - m^2 &= 0, \\ r^2 \frac{U''}{U} + \frac{1}{\sin \theta P} (\sin \theta P')' - \frac{m^2}{\sin^2 \theta} &= 0. \end{aligned} \quad (15.10)$$

Since $U = U(r)$, $r^2 U''/U$ does not depend on θ so the differential equations in r and θ separate. The separation constant is chosen deviously to be $l(l + 1)$ so that

$$\frac{d^2U}{dr^2} = \frac{l(l + 1)}{r^2} U,$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + l(l + 1)P - \frac{m^2}{\sin^2 \theta} P = 0. \quad (15.11)$$

The reader has probably studied the properties of the solutions of this P -equation elsewhere so the necessary results will be stated without proof here. A more complete set of formulas is given by Edmonds [3], and Pauling and Wilson [4] have given a very clear derivation of the solutions from first principles. The equation for P has solutions that are everywhere finite only if l is a nonnegative integer such that

$$-l \leq m \leq l. \quad (15.12)$$

The solutions are called the *associated Legendre polynomials*, $P_l^m(\cos \theta)$, and are defined by

$$P_l^m(x) = \frac{1}{2^l l!} (1 - x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l. \quad (15.13)$$

Solutions of the equation for U can be found in the form

$$U(r) = Ar^\alpha. \quad (15.14)$$

One sees that

$$r^2 \frac{d^2U(r)}{dr^2} = \alpha(\alpha - 1)U,$$

$$[\alpha(\alpha - 1) - l(l + 1)]U = 0, \quad (15.15)$$

so α can be $l + 1$ or $-l$. Cauchy's equation, $\sum_n C_n r^n d^n U(r)/dr^n = 0$, can always be solved this way since an algebraic equation for α results from substitution of Eq. (15.14). Thus the function $U(r)$ is of the form

$$U(r) = Ar^{l+1} + Br^{-l}. \quad (15.16)$$

The sum of two solutions of Laplace's equation is again a solution,

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and so the most general solution in spherical coordinates, found by this process, is of the form

$$\varphi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} [A_{lm} r^l + B_{lm} r^{-l-1}] Y_{lm}(\theta, \phi). \quad (15.17)$$

Here the θ - and ϕ -dependence has been collected in the form of the spherical harmonics,

$$Y_{lm}(\theta, \phi) = (-)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}, \quad (15.18)$$

and A_{lm} and B_{lm} are constants.

Summary of Properties of the Spherical Harmonics

We will quote here some of the results needed later mostly without proof. The spherical harmonics are constructed to satisfy the eigenvalue equations

$$\begin{aligned} L^2 Y_{lm} &= l(l+1) Y_{lm}, \\ L_z Y_{lm} &= m Y_{lm}, \end{aligned} \quad (15.19)$$

where $\mathbf{L} = -i\mathbf{x} \times \nabla$ and $L^2 = L_x^2 + L_y^2 + L_z^2$. In spherical coordinates the operators are given by

$$\begin{aligned} L_x &= -\sin \phi \frac{1}{i} \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{1}{i} \frac{\partial}{\partial \phi}, \\ L_y &= +\cos \phi \frac{1}{i} \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{1}{i} \frac{\partial}{\partial \phi}, \\ L_z &= \frac{1}{i} \frac{\partial}{\partial \phi}, \\ L^2 &= -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \end{aligned} \quad (15.20)$$

The harmonics are orthogonal with respect to integration over the solid angle

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'}, \quad (15.21)$$

and have the property under complex conjugation

$$Y_{lm}^*(\theta, \phi) = (-)^m Y_{l,-m}(\theta, \phi). \quad (15.22)$$

There is also a formula for the sum over a product of the harmonics in two different sets of angles. If θ, ϕ and θ', ϕ' are the polar angles that describe the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$, then

$$P_l(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi). \quad (15.23)$$

This is called the *spherical harmonic addition theorem*. A useful expansion for $1/|\mathbf{x} - \mathbf{x}'|$ can be obtained from Eq. (15.23) and the generating function for the Legendre polynomials $P_l(x)$ [defined to be $P_l^0(x)$]. The generating function is

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l, \quad |t| < 1, \quad (15.24)$$

where t is real. First observe that

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{x}'|} &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}'}} \\ &= \frac{1}{r \sqrt{1 - 2 \frac{r'}{r} \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}' + \left(\frac{r'}{r}\right)^2}} \\ &= \frac{1}{r' \sqrt{1 - 2 \frac{r}{r'} \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}' + \left(\frac{r}{r'}\right)^2}}. \end{aligned} \quad (15.25)$$

Now let $r_>$ and $r_<$ be the larger and smaller of $|\mathbf{x}|$ and $|\mathbf{x}'|$. Then

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{x}'|} &= \frac{1}{r_> \sqrt{1 - 2 \frac{r_<}{r_>} \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}' + \left(\frac{r_<}{r_>}\right)^2}} \\ &= \frac{1}{r_>} \sum_{l=0}^{\infty} P_l(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}') \left(\frac{r_<}{r_>}\right)^l. \end{aligned} \quad (15.26)$$

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Finally from Eq. (15.24) one obtains

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi). \quad (15.27)$$

Multipole Expansion of the Potential of a Distribution of Charges

The multipole expansion, when the charges are confined to a finite region of space, was developed in Section 10. It is constructive to reconsider this subject now in the light of the properties of the spherical harmonics $Y_{lm}(\theta, \phi)$. The former result, Eq. (10.7), for the potential was

$$\varphi(\mathbf{x}) = \int d^3x' \rho(\mathbf{x}') \left\{ 1 - (\mathbf{x}' \cdot \nabla) + \frac{1}{2} (\mathbf{x}' \cdot \nabla)^2 - \dots \right\} \frac{1}{r}, \quad (15.28)$$

where r is $|\mathbf{x}|$ and it is assumed that $|\mathbf{x}'| < r$ for any point \mathbf{x}' where the charge density $\rho(\mathbf{x}')$ is nonzero. It was learned that the potential is a sum over l of the l th order contributions $\varphi^{(l)}(\mathbf{x})$ given by

$$\varphi^{(l)}(\mathbf{x}) = \left[\text{constant} \int d^3x' \rho(\mathbf{x}') (x'_{i_1} x'_{i_2} \cdots x'_{i_l})^{(l)} \right] \frac{(x_{i_1} x_{i_2} \cdots x_{i_l})^{(l)}}{r^{2l+1}}. \quad (15.29)$$

The solid harmonics $(x_{i_1} x_{i_2} \cdots x_{i_l})^{(l)}$ are covariantly defined irreducible tensors. They have $2l+1$ independent components. The first three sets of solid harmonics, for example, are

$$\begin{aligned} (1)^{(0)} &= 1, \\ (x_i)^{(1)} &= x_i, \\ (x_i x_j)^{(2)} &= x_i x_j - \frac{1}{3} \delta_{ij} r^2. \end{aligned} \quad (15.30)$$

The problem of finding the potential outside the distribution of charges can also be solved by the method of separation of variables; the result is of the form of Eq. (15.17). The potential will vanish at infinity so the constants A_{lm} are evidently zero and, in terms of some different constants q_{lm} ,

$$\varphi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}. \quad (15.31)$$

Another way to see this result, which also determines the q_{lm} , is to use Eq. (15.27). The point is that

$$\begin{aligned}\varphi(\mathbf{x}) &= \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &= \int d^3x' \rho(\mathbf{x}') \sum_{lm} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi).\end{aligned}\quad (15.32)$$

By comparison with Eq. (15.31), then,

$$q_{lm} = \int d^3x' \rho(\mathbf{x}') r'^l Y_{lm}^*(\theta', \phi'). \quad (15.33)$$

However, in Eq. (15.29) the \mathbf{x} -dependence of the l th order contribution to the potential was

$$\frac{(x_{i_1} x_{i_2} \cdots x_{i_l})^{(l)}}{r^{2l+1}} = \frac{(\hat{\mathbf{x}}_{i_1} \hat{\mathbf{x}}_{i_2} \cdots \hat{\mathbf{x}}_{i_l})^{(l)}}{r^{l+1}}, \quad (15.34)$$

and so the irreducible covariantly defined tensors $(\hat{\mathbf{x}}_{i_1} \hat{\mathbf{x}}_{i_2} \cdots \hat{\mathbf{x}}_{i_l})^{(l)}$ must be linearly related to the spherical harmonics Y_{lm} , where m ranges by integer steps from $-l$ up to $+l$. For example the spherical harmonics for $l = 0$ and $l = 1$ are

$$\begin{aligned}Y_{00}(\theta, \phi) &= \sqrt{\frac{1}{4\pi}}, \\ Y_{11}(\theta, \phi) &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} = -\sqrt{\frac{3}{8\pi}} \left(\frac{x + iy}{r} \right), \\ Y_{10}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}, \\ Y_{-11}(\theta, \phi) &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi} = \sqrt{\frac{3}{8\pi}} \left(\frac{x - iy}{r} \right),\end{aligned}\quad (15.35)$$

and these are just linear combinations of the components of the covariantly defined irreducible tensors $(\hat{\mathbf{x}}_{i_1} \hat{\mathbf{x}}_{i_2} \cdots \hat{\mathbf{x}}_{i_l})^{(l)}$ for $l = 0$ and 1.

The coordinate vector is a covariantly defined tensor because

$$x'_i = a_{ij} x_j \quad (15.36)$$

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is the transformation rule under an orthogonal transformation a . Since

$$\frac{\partial}{\partial x'_i} = a_{ij} \frac{\partial}{\partial x_j}, \quad (15.37)$$

differentiation with respect to the coordinate vector is also a covariantly defined operation. In quantum mechanics, a set of spin operators s_i are introduced, and these are also covariantly defined. With the extension of covariantly defined tensors to include operators, one can combine \mathbf{x} , ∇ , and \mathbf{s} to construct more such covariantly defined tensors; for example

$$\begin{aligned} \mathbf{L} &= -i\mathbf{x} \times \nabla, \\ \mathbf{J} &= \mathbf{L} + \mathbf{s}. \end{aligned} \quad (15.38)$$

The spherical components of any covariantly defined tensor are defined from its Cartesian components by the same linear combinations that relate Y_{lm} to $(\hat{\mathbf{x}}_{i_1} \hat{\mathbf{x}}_{i_2} \cdots \hat{\mathbf{x}}_{i_l})^{(l)}$. Thus, for example, if \mathbf{v} is a covariantly defined vector with Cartesian components v_x, v_y, v_z , then the spherical components v_{+1}, v_0, v_{-1} are defined, by comparison with Eq. (15.35), to be

$$\begin{aligned} v_{1\pm 1} &= \mp \sqrt{\frac{3}{8\pi}} (v_x \pm iv_y), \\ v_0 &= \sqrt{\frac{3}{4\pi}} v_z. \end{aligned} \quad (15.39)$$

Sometimes, however, a constant factor such as $\sqrt{3/4\pi}$ in Eq. (15.39) may be dropped.

The spherical basis has a wide range of application in quantum mechanics because of the Wigner–Eckart theorem. Suppose that $\psi(JM)$ are a set of wave functions with total angular momentum quantum number J and with $J_z = M$. That is, $\psi(JM)$ is an eigenfunction for the operators $\mathbf{J} \cdot \mathbf{J}$ and J_z

$$\begin{aligned} \mathbf{J} \cdot \mathbf{J} \psi(JM) &= J(J+1)\psi(JM), \\ J_z \psi(JM) &= M\psi(JM). \end{aligned} \quad (15.40)$$

The phases of the wavefunctions $\psi(JM)$ are chosen as usual such that

$$(J_x \pm iJ_y)\psi(JM) = [(J \mp M)(J \pm M+1)]^{1/2}\psi(J, M \pm 1). \quad (15.41)$$

Now if $\mathcal{O}(J'M')$ stands for one of the $2J' + 1$ spherical components of any covariantly defined type J' operator, then the theorem states that

$$\int \psi^*(J_f M_f) \mathcal{O}(J'M') \psi(J_i M_i) = C(J_i J' J_f; M_i M' M_f) R. \quad (15.42)$$

where the C 's depend only on the J 's and M 's and R is independent of the M 's. In Eq. (15.42) the integral sign indicates in a formal way that integration over the continuous spatial coordinates, and summations over the discrete spin coordinates are to be performed. The numbers $C(J_i J' J_f; M_i M' M_f)$ are known as *Clebsch-Gordan or Wigner or vector-coupling coefficients*. They are zero unless $|J_i - J'| \leq J_f \leq |J_i + J'|$ and $M_i + M' = M_f$. The Clebsch-Gordan coefficients have been studied and tabulated thoroughly since they only depend on the M 's and J 's; an extensive table of values was made by Rotenberg and co-workers [5]. The factor R is called the *reduced matrix element*; an essential part of the Wigner-Eckart theorem is that R is independent of M_f , M' , and M_i . A proof of the theorem is given by Edmonds [3].

The Quadrupole Moment in Nuclear Physics

Many nuclei possess a measurable quadrupole moment. The effect of an external electrostatic field is to split the nuclear energy levels in a way that can be understood with the aid of the Wigner-Eckart theorem. To see this in detail, choose the origin at the center of mass of the nucleus having N neutrons and Z protons, where $A = N + Z$ is the mass number. In the absence of an external field the wavefunction is $\psi_m^I(\mathbf{x}_1 s_1, \mathbf{x}_2 s_2, \dots, \mathbf{x}_A s_A)$, where \mathbf{x}_α is the spatial coordinate and $s_\alpha = \pm \frac{1}{2}$ is the spin coordinate of the α th nucleon. In the nonrelativistic theory the wavefunction satisfies the eigenvalue equations

$$\begin{aligned} \mathcal{H}_0 \psi_m^I &= W \psi_m^I, \\ \mathbf{J} \cdot \mathbf{J} \psi_m^I &= I(I+1) \psi_m^I, \\ J_z \psi_m^I &= m \psi_m^I, \\ P \psi_m^I &= \pm \psi_m^I. \end{aligned} \quad (15.43)$$

That is, ψ_m^I describes a $(2I+1)$ -fold degenerate nuclear level which has

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energy W and (spin)^{parity} equal to I^\pm . The parity operator is defined by

$$P\psi_m^I(\mathbf{x}_1 s_1, \mathbf{x}_2 s_2, \dots, \mathbf{x}_A s_A) = \psi_m^I(-\mathbf{x}_1 s_1, -\mathbf{x}_2 s_2, \dots, -\mathbf{x}_A s_A). \quad (15.44)$$

Now suppose that an external electrostatic field $\mathbf{E}(\mathbf{x})$ is applied to the nucleus. Then the Hamiltonian becomes

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}, \quad (15.45)$$

where \mathcal{H}_{int} incorporates the effect of the field interacting with the nucleons

$$\mathcal{H}_{\text{int}} = \sum_{\alpha=1}^A e_\alpha \varphi(\mathbf{x}_\alpha). \quad (15.46)$$

The electric charge e_α is zero for each neutron and $e = 4.8(10)^{-10}$ esu for each proton. Since the external field is likely to be slowly varying over the small dimensions of the nucleus, an expansion of the field about the origin is appropriate. In fact the physical situation is clarified using only the first two terms of the expansion:

$$E_i(\mathbf{x}) = (E_i)_0 + x_j \left(\frac{\partial E_i}{\partial x_j} \right)_0. \quad (15.47)$$

A potential for this field is

$$\varphi(\mathbf{x}) = \varphi_0 - x_i (E_i)_0 - \frac{1}{2} x_i x_j \left(\frac{\partial E_i}{\partial x_j} \right)_0 \quad (15.48)$$

since

$$-\frac{\partial \varphi(\mathbf{x})}{\partial x_k} = (E_k)_0 + \frac{1}{2} x_j \left(\frac{\partial E_k}{\partial x_j} \right)_0 + \frac{1}{2} x_i \left(\frac{\partial E_i}{\partial x_k} \right)_0. \quad (15.49)$$

Here the last two terms are the same since $\nabla \times \mathbf{E} = 0$ so $\partial E_i / \partial x_k = \partial E_k / \partial x_i$. The interaction part of the Hamiltonian is therefore

$$\mathcal{H}_{\text{int}} = \varphi_0 \sum_{\alpha=1}^A e_\alpha - (E_i)_0 \sum_{\alpha=1}^A e_\alpha x_{\alpha i} - \frac{1}{2} \left(\frac{\partial E_i}{\partial x_j} \right)_0 \sum_{\alpha=1}^A e_\alpha x_{\alpha i} x_{\alpha j}. \quad (15.50)$$

Equation (15.50) is just the expansion of the energy into charge, dipole, and quadrupole contributions in operator form.

Ordinarily the effect of the field is small. For nuclear spin I , the field

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splits the $2I+1$ levels apart in energy by amounts that are small compared to the separation in energy of the zero-field levels. In this situation the perturbation technique is appropriate and the energies of the $2I+1$ levels are given by the eigenvalues of the $(2I+1) \times (2I+1)$ matrix

$$\begin{aligned} \mathcal{H}_{mm'} &= \int \psi_m^{I*} \mathcal{H}_{\text{int}} \psi_m^I \\ &= \varphi_0 \sum_{\alpha=1}^A e_\alpha \int \psi_m^{I*} \psi_{m'}^I - (E_i)_0 \int \psi_m^{I*} \sum_{\alpha=1}^A e_\alpha x_{\alpha i} \psi_{m'}^I \\ &\quad - \frac{1}{2} \left(\frac{\partial E_i}{\partial x_j} \right)_0 \int \psi_m^{I*} \sum_{\alpha=1}^A e_\alpha x_{\alpha i} x_{\alpha j} \psi_{m'}^I, \end{aligned} \quad (15.51)$$

neglecting the constant term $W\delta_{mm'}$. The eigenfunctions ψ_m^I of Eq. (15.43) are chosen to be orthonormal

$$\int \psi_m^{I*} \psi_{m'}^I = \delta_{mm'}. \quad (15.52)$$

The first term in Eq. (15.51) merely shifts all the levels by the same amount and is therefore neglected in the subsequent discussion. The second term (dipole) is zero because of the definite parity of the nuclear levels. That is, consider the change of variables $\mathbf{x}_{\alpha i} \rightarrow -\mathbf{x}_{\alpha i}$; it leads to

$$\begin{aligned} \int \psi_m^{I*} \sum_{\alpha=1}^A e_\alpha x_{\alpha i} \psi_{m'}^I &= \int (P\psi_m^I)^* \left[-\sum_{\alpha=1}^A e_\alpha x_{\alpha i} \right] (P\psi_{m'}^I) \\ &= - \int \psi_m^{I*} \sum_{\alpha=1}^A e_\alpha x_{\alpha i} \psi_{m'}^I \\ &= 0. \end{aligned} \quad (15.53)$$

Thus the interesting splitting comes from the quadrupole term in \mathcal{H}_{int} in Eq. (15.50) and, neglecting terms that are constant for fixed nuclear spin I , the energies are the eigenvalues of the matrix

$$\mathcal{H}_{mm'} = -\frac{1}{2} \left(\frac{\partial E_i}{\partial x_j} \right)_0 \int \psi_m^{I*} \sum_{\alpha=1}^A e_\alpha \left[x_{\alpha i} x_{\alpha j} - \frac{1}{3} \delta_{ij} r_\alpha^2 \right] \psi_{m'}^I, \quad (15.54)$$

where the term $-\frac{1}{3}\delta_{ij}r_\alpha^2$ can be inserted for free since $(\partial E_i / \partial x_i)_0$ is zero

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(the external field is produced by charges external to the nucleus). However $x_{\alpha i}x_{\alpha j} - \frac{1}{3}\delta_{ij}r_{\alpha}^2$ is a covariantly defined tensor operator of type 2 so, apart from a constant factor, its effect is the same as that of any other covariantly defined tensor operator of the same type. This is the application of the Wigner–Eckart theorem. In particular, one may write

$$\int \psi_m^{I*} \sum_{\alpha=1}^A e_{\alpha} [x_{\alpha i}x_{\alpha j} - \frac{1}{3}\delta_{ij}r_{\alpha}^2] \psi_{m'}^I \\ = \text{constant} \int \psi_m^{I*} [J_i J_j + J_j J_i - \frac{2}{3}\delta_{ij} \mathbf{J} \cdot \mathbf{J}] \psi_{m'}^I, \quad (15.55)$$

where the constant factor is independent of i, j, m , and m' . Substituting this result back into Eq. (15.54) leads to

$$\mathcal{H}_{mm'} = -\frac{1}{2} \left(\frac{\partial E_i}{\partial x_j} \right)_0 \frac{Qe}{2I(2I-1)} \int \psi_m^{I*} (J_i J_j + J_j J_i) \psi_{m'}^I, \quad (15.56)$$

where the $\delta_{ij} \mathbf{J} \cdot \mathbf{J}$ term has been discarded and the constant factor is chosen by convention to be $Qe/2I(2I-1)$. Here Q is called the *quadrupole moment of the level*; the whole effect is absent for nuclear spin less than one so the factor $2I(2I-1)$ causes no trouble. An interpretation of the quadrupole moment Q can be obtained by setting $m = m' = I$ and $i = j = 3$ in Eq. (15.55):

$$\int \psi_I^{I*} \sum_{\alpha=1}^A e_{\alpha} \left(z_{\alpha}^2 - \frac{1}{3} r_{\alpha}^2 \right) \psi_I^I = \frac{Qe}{2I(2I-1)} \int \psi_I^{I*} \left(2J_z^2 - \frac{2}{3} \mathbf{J} \cdot \mathbf{J} \right) \psi_I^I \\ = \frac{Qe}{3}, \quad (15.57)$$

$$Q = \frac{1}{e} \int \psi_I^{I*} \sum_{\alpha=1}^A e_{\alpha} (3z_{\alpha}^2 - r_{\alpha}^2) \psi_I^I.$$

The wave functions ψ_m^I are eigenfunctions of $\mathbf{J} \cdot \mathbf{J}$ and J_z and are related for different values of m by Eq. (15.41). Thus in the standard representation of the angular momentum matrices (rows and columns are labeled with m and m' , respectively, each starting from I and going down to $-I$) the splitting is determined by the eigenvalues of the matrix

$$\mathcal{H} = -\frac{Qe}{4I(2I-1)} \left(\frac{\partial E_i}{\partial x_j} \right)_0 (J_i J_j + J_j J_i), \quad (15.58)$$

where here J_i denote the $(2I+1)$ -square angular momentum matrices.

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This eigenvalue problem is completely defined by the value of the gradient of the field and the single number Q for the nuclear spin I .

The field gradient tensor is symmetric, $\partial E_i / \partial x_j = \partial E_j / \partial x_i$, and therefore diagonalizable by appropriate choice of coordinate axes. By convention the principal axes are chosen such that

$$\left| \frac{\partial E_z}{\partial z} \right| \geq \left| \frac{\partial E_y}{\partial y} \right| \geq \left| \frac{\partial E_x}{\partial x} \right| \quad (15.59)$$

at the origin of the nucleus. Since $\nabla \cdot \mathbf{E}$ is zero, the sum of the diagonal elements of the field gradient tensor is zero and only two numbers are needed to specify $\partial E_i / \partial x_j$ in the principal axis system. Let these two numbers be

$$q = -\frac{1}{e} \frac{\partial E_z}{\partial z}, \quad (15.60)$$

$$\eta = \frac{\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y}}{\frac{\partial E_z}{\partial z}},$$

where η is known as the *asymmetry parameter*. Now since \mathbf{E} is an external field

$$\nabla \cdot \mathbf{E} = 0, \quad (15.61)$$

$$\frac{\partial E_z}{\partial z} = -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y},$$

and this imposes a definite limitation on η . That is, using Eqs. (15.59) and (15.61) one sees that $\partial E_x / \partial x$ and $\partial E_y / \partial y$ must be of the same sign so

$$\begin{aligned} \eta &= \frac{\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y}}{\frac{\partial E_z}{\partial z}} \\ &= \frac{\frac{\partial E_y}{\partial y} - \frac{\partial E_x}{\partial x}}{\frac{\partial E_y}{\partial y} + \frac{\partial E_x}{\partial x}} \\ &= \frac{\left| \frac{\partial E_y}{\partial y} \right| - \left| \frac{\partial E_x}{\partial x} \right|}{\left| \frac{\partial E_y}{\partial y} \right| + \left| \frac{\partial E_x}{\partial x} \right|}. \end{aligned} \quad (15.62)$$

Thus η is restricted to lie in the interval $0 \leq \eta \leq 1$.

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In principal axes the field gradient tensor takes the form

$$\frac{\partial E_i}{\partial x_j} = -\frac{eq}{2} \begin{pmatrix} \eta - 1 & 0 & 0 \\ 0 & -\eta - 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (15.63)$$

Finally, substituting this principal axis gradient into the expression for the matrix that is to be diagonalized for the energy levels, one obtains the result

$$\begin{aligned} \mathcal{H} &= +\frac{Qe^2q}{8I(2I-1)} [2(\eta-1)J_x^2 - 2(\eta+1)J_y^2 + 4J_z^2] \\ &= \frac{e^2qQ}{4I(2I-1)} [3J_z^2 - \mathbf{J} \cdot \mathbf{J} + \eta J_x^2 - \eta J_y^2]. \end{aligned} \quad (15.64)$$

An especially simple case occurs when $\partial E_x/\partial x = \partial E_y/\partial y$. Then the field gradient tensor has axial symmetry, η is zero, and the energy levels for various values of m are equal to some overall constant plus

$$\mathcal{H}_{mm} = \frac{e^2Qq}{4I(2I-1)} [3m^2 - I(I+1)]. \quad (15.65)$$

Otherwise the problem of finding the energy levels has been reduced to finding the eigenvalues of the matrix in Eq. (15.64).

Separation of Laplace's Equation in Cylindrical Coordinates

If the boundary conditions may be expressed most naturally in cylindrical coordinates, then solutions of $\nabla^2\varphi = 0$ should be sought in the form

$$\varphi(r, z, \phi) = R(r)Z(z)Q(\phi), \quad (15.66)$$

where

$$\begin{aligned} x &= r \cos \phi, \\ y &= r \sin \phi \end{aligned} \quad (15.67)$$

define the coordinates. Laplace's equation takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \quad (15.68)$$

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Substituting from Eq. (15.66) and then multiplying by r^2/RZQ , one finds

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{Q} \frac{d^2 Q}{d\phi^2} + \frac{r^2}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (15.69)$$

The dependence in the variable ϕ does separate from that in r and z . If all values of ϕ are in the allowed region, the potential will be single valued when $Q^{-1} d^2 Q / d\phi^2 = -m^2$ with m an integer and Q of the form $Ae^{im\phi}$. Next, by dividing by r^2 , the equation becomes

$$\frac{1}{rR} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \frac{m^2}{r^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (15.70)$$

The dependence in z separates from that in r . The separation constant is be taken as a positive number to obtain real exponential solutions

$$\begin{aligned} \frac{d^2 Z}{dz^2} &= \kappa^2 Z, \\ Z(z) &= Ae^{\kappa z} + Be^{-\kappa z}, \end{aligned} \quad (15.71)$$

although oscillatory solutions are possible also. If the sources are limited to a finite region of space and an exterior solution is desired, it is necessary to choose a solution decreasing away from the sources.

Finally the dependence of the potential on r is determined by Bessel's differential equation

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) + (\kappa^2 r^2 - m^2) R = 0. \quad (15.72)$$

With the change of variable $\rho = \kappa r$ the equation becomes

$$\rho \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + (\rho^2 - m^2) R = 0, \quad (15.73)$$

and a particular solution exists in the form

$$R = \sum_{n=0}^{\infty} a_n \rho^{n+p}, \quad (15.74)$$

where the coefficients a_n and the index p are to be determined. The equation then becomes

$$\sum_{n=0}^{\infty} [(n+p)^2 - m^2 + \rho^2] a_n \rho^{n+p} = 0. \quad (15.75)$$

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Thus a recursion relation results for the coefficients a_n

$$\begin{aligned} (p^2 - m^2)a_0 &= 0, \\ [(1 + p)^2 - m^2]a_1 &= 0, \\ [(n + p)^2 - m^2]a_n + a_{n-2} &= 0, \quad n \geq 2. \end{aligned} \quad (15.76)$$

A solution with $a_0 \neq 0$ results with $a_1 = 0$ and $p = \pm |m|$. Consider first the case that $p \geq 0$ and return to the case of p negative later. The general coefficient is in this case to be found from

$$\begin{aligned} a_{2n+1} &= 0, & n = 0, 1, 2, \dots, \\ a_{2n} &= -\frac{a_{2n-2}}{2^{2n}(n+p)}, & n = 1, 2, \dots. \end{aligned}$$

By writing the similar equations with n replaced by $n - 1, n - 2, \dots, 1$ and multiplying them all together, one finds

$$a_{2n} = \frac{(-)^n a_0}{2^{2n} n! (p+1)(p+2) \cdots (p+n)}, \quad n = 1, 2, \dots. \quad (15.77)$$

If the coefficient a_0 is chosen to be $1/(2^p p!)$ the solution is known as the *Bessel function of the first kind* $J_p(\rho)$,

$$J_p(\rho) = \sum_{n=0}^{\infty} \frac{(-)^n (\rho)^{2n+p}}{2^{2n+p} n! (n+p)!}. \quad (15.78)$$

Nothing new results from this procedure if $a_0 = 0$ is considered. The function $J_p(\rho)$ is regular at $\rho = 0$ and oscillates with decreasing amplitude as ρ increases. We will take for granted here this and other properties of the Bessel functions, as summarized by Jahnke and co-workers [6]. Since Eq. (15.73) is a second order differential equation, there is another solution $N_p(\rho)$ and the general solution of Eq. (15.73) is written as (p is still $|m|$)

$$R(r) = A J_p(\kappa r) + B N_p(\kappa r). \quad (15.79)$$

$N_p(\rho)$ is called the *Bessel function of the second kind*. Like $J_p(\rho)$, it is oscillatory and well behaved at infinity. However the function of the second kind is singular at the origin and will not be present in problems where the origin is in the physical region. If the boundary conditions require the potential to oscillate as a function of z , then a negative

separation constant would be chosen in Eq. (15.70). Instead of κ being real, one would have $\kappa = ik$, where k is real. In this case also, the properties of the solutions are well known [6].

The parameter κ is not limited to discrete values. The general solution must therefore be expressed as an integral over the allowed range of κ . The general solution for the dependence of the potential on z was of the form $Z(z) = Ae^{\kappa z} + Be^{-\kappa z}$ so that $Z(z)$ can be taken to be of the form $Ae^{\kappa z}$, both positive and negative values of κ then being allowed. Thus the solution of Laplace's equation in cylindrical coordinates (that this method leads to) is of the form

$$\varphi(r, z, \phi) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\kappa e^{\kappa z} [A_m(\kappa) J_{|m|}(\kappa r) + B_m(\kappa) N_{|m|}(\kappa r)] e^{im\phi}. \quad (15.80)$$

Solutions which are oscillatory in z (for which κ is imaginary) would be written differently. As an example, the potential of a point charge q at the origin is axially symmetric, and therefore only solutions with $m = 0$ contribute to it. The potential finds expression in the form

$$\frac{q}{\sqrt{r^2 + z^2}} = q \int_0^\infty e^{-\kappa|z|} J_0(\kappa r) d\kappa, \quad (15.81)$$

as can be shown readily by a term-by-term integration of the power series for $J_0(\kappa r)$ weighted by $e^{-\kappa|z|}$.

A generating function for the Bessel functions of the first kind and of integer order can be developed easily. To accomplish this, one may consider the function

$$e^{(\rho/2)[t - (1/t)]}$$

as a product of two series

$$\begin{aligned} e^{(\rho/2)[t - (1/t)]} &= \sum_{n=0}^{\infty} \frac{\left(\frac{\rho}{2}\right)^n t^n}{n!} \sum_{m=0}^{\infty} \frac{\left(-\frac{\rho}{2}\right)^m t^{-m}}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^m \left(\frac{\rho}{2}\right)^{n+m} t^{n-m}}{n! m!}. \end{aligned} \quad (15.82)$$

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With the change of summation indices from m and n to m and $p = n - m$, this becomes

$$e^{(\rho/2)[t-(1/t)]} = \sum_{p=-\infty}^{+\infty} t^p \sum_{m=0}^{\infty} \frac{(-)^m \left(\frac{\rho}{2}\right)^{2m+p}}{m! (m+p)!}. \quad (15.83)$$

The sum over m has been extended from $m = -p$ to $m = 0$ when $p < 0$ in the lower limit. This is permitted because $1/(m+p)!$ is zero at the extra values. With the understanding that J_p for negative p is also defined by Eq. (15.78), the result is

$$e^{(\rho/2)[t-(1/t)]} = \sum_{p=-\infty}^{+\infty} J_p(\rho) t^p. \quad (15.84)$$

Finally, the arbitrary choice of positive p that was made following Eqs. (15.76) will be reconsidered: choose $p = -|m|$, a negative integer, and look for a solution of the recursion relation. Again, the a_n with odd n must all be zero. The a_n for $n = 0, 2, \dots, (-2p-2)$ must vanish because the coefficient of a_n in Eqs. (15.76) is zero for $n = -2p$. Therefore one chooses a_{-2p} arbitrarily and obtains the other coefficients from

$$[(n+p)^2 - m^2]a_n + a_{n-2} = 0, \quad n > -2p. \quad (15.85)$$

Equation (15.78) holds again, even with p negative, since the $(n+p)!$ term in the denominator is infinite for $n < -p$, making the excess terms vanish. However with p negative one also obtains

$$\begin{aligned} J_p(\rho) &= \sum_{n=-p}^{\infty} \frac{(-1)^n \rho^{2n+p}}{2^{2n+p} n! (n+p)!} \\ &= \sum_{l=0}^{\infty} \frac{(-1)^{l-p} \rho^{2l-p}}{2^{2l-p} (l-p)! l!} \\ &= (-1)^{-p} J_{-p}(\rho), \end{aligned} \quad (15.86)$$

where n was replaced by $l - p$ in the first step. Thus the effect of choosing $p = -|m|$ is at most a sign change. The situation for Bessel functions with p nonintegral is different; in that case J_p and J_{-p} are the two independent solutions.

Expansions in Complete Sets of Functions

One aspect of a problem in electrostatics is that of finding the special functions that emerge in the general solution of Laplace's equation in the variables that have been chosen. Another aspect of the problem then arises, and that is to express whatever is given on the boundaries as an expansion in the functions of the problem. When this is accomplished, the problem is usually solved. For example, if the potential is given on the boundary, one specializes the general solution to the boundary surface and then determines the coefficients in the general solution by matching them to those in the expansion of the given potential.

The primary example of an expansion in terms of a complete set of functions is the representation of a function by a Fourier series. An important mathematical result is that, if $f(x)$ is a function defined in the interval $-a/2 \leq x \leq a/2$ and if $f(x)$ is continuous except for a finite number of finite discontinuities, then there is an expansion for $f(x)$ in the form

$$f(x) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[A_m \cos 2\pi m \frac{x}{a} + B_m \sin 2\pi m \frac{x}{a} \right]. \quad (15.87)$$

The equality here means that if the partial sums $S_n(x)$ are defined by

$$S_0(x) = \frac{1}{2} A_0, \quad (15.88)$$

$$S_n(x) = \frac{1}{2} A_0 + \sum_{m=1}^n \left[A_m \cos 2\pi m \frac{x}{a} + B_m \sin 2\pi m \frac{x}{a} \right],$$

then the series converges to $f(x)$,

$$\lim_{n \rightarrow \infty} S_n(x) = f(x), \quad (15.89)$$

under suitably restricted conditions. A theorem which gives conditions under which the Fourier series representation of a function actually converges to that function is known as a *Fourier theorem*. Remarkably, a set of such conditions that are both necessary and sufficient for convergence are not known. However a useful set of sufficient conditions is discussed by Churchill [7]. There is no question about the definition

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of the coefficients A_n and B_n in the series. These are found by the Fourier trick; that is, by operation with

$$\int_{-a/2}^{+a/2} dx \cos 2\pi n \frac{x}{a} \quad \text{and} \quad \int_{-a/2}^{+a/2} dx \sin 2\pi n \frac{x}{a}$$

on both sides of Eq. (15.87). For example, if m' and m are positive integers, then

$$\begin{aligned} & \int_{-a/2}^{+a/2} dx \cos 2\pi m' \frac{x}{a} \cos 2\pi m \frac{x}{a} \\ &= \frac{1}{2} \int_{-a/2}^{+a/2} dx \left[\cos 2\pi(m' + m) \frac{x}{a} + \cos 2\pi(m' - m) \frac{x}{a} \right] \\ &= \begin{cases} \frac{1}{2} \left[\frac{\sin 2\pi(m' + m) \frac{x}{a}}{2\pi(m' + m) \frac{1}{a}} + x \right]_{-a/2}^{+a/2}, & m' = m, \\ \frac{1}{2} \left[\frac{\sin 2\pi(m' + m) \frac{x}{a}}{2\pi(m' + m) \frac{1}{a}} + \frac{\sin 2\pi(m' - m) \frac{x}{a}}{2\pi(m' - m) \frac{1}{a}} \right]_{-a/2}^{+a/2}, & m' \neq m, \end{cases} \\ &= \frac{a}{2} \delta_{mm'}. \end{aligned} \tag{15.90}$$

In a similar way one finds that

$$\begin{aligned} & \int_{-a/2}^{+a/2} dx \cos 2\pi m' \frac{x}{a} \sin 2\pi m \frac{x}{a} = 0, \\ & \int_{-a/2}^{+a/2} dx \cos 2\pi m' \frac{x}{a} = 0, \quad m' \neq 0. \end{aligned} \tag{15.91}$$

Thus by applying these results to the Fourier series one finds, for $m' > 0$,

$$\begin{aligned} & \int_{-a/2}^{+a/2} dx \cos 2\pi m' \frac{x}{a} f(x) = \sum_{m=1}^{\infty} A_m \frac{a}{2} \delta_{mm'} \\ &= \frac{a}{2} A_{m'}. \end{aligned} \tag{15.92}$$

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The coefficient A_m is therefore given by

$$A_m = \frac{2}{a} \int_{-a/2}^{+a/2} dx \cos 2\pi m \frac{x}{a} f(x), \quad (15.93)$$

and this is also found to hold for $m = 0$. The similar formula for the coefficients B_m is

$$B_m = \frac{2}{a} \int_{-a/2}^{+a/2} dx \sin 2\pi m \frac{x}{a} f(x). \quad (15.94)$$

The coefficients A_m and B_m are found from $f(x)$ by an integration process. Therefore, they do not depend upon a particular value that may be assigned to the function f at a discontinuity. If the function $f(x)$ is discontinuous at $x = x_1$, then under certain mild conditions its Fourier series converges to the average of f over the discontinuity

$$\lim_{n \rightarrow \infty} S_n(x_1) = \lim_{\epsilon \rightarrow 0} \frac{1}{2}[f(x_1 + \epsilon) + f(x_1 - \epsilon)]. \quad (15.95)$$

A formal simplification in the process of representing a function by its Fourier series results when the whole procedure is recast into complex form. Since the trigonometric functions involved are

$$\begin{aligned} \cos 2\pi m \frac{x}{a} &= \frac{1}{2} [e^{i2\pi mx/a} + e^{-i2\pi mx/a}], \\ \sin 2\pi m \frac{x}{a} &= \frac{1}{2i} [e^{i2\pi mx/a} - e^{-i2\pi mx/a}], \end{aligned} \quad (15.96)$$

the Fourier series representation of f becomes

$$\begin{aligned} f(x) &= \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[\left(\frac{A_m}{2} + \frac{B_m}{2i} \right) e^{i2\pi mx/a} + \left(\frac{A_m}{2} - \frac{B_m}{2i} \right) e^{-i2\pi mx/a} \right] \\ &= \sum_{m=-\infty}^{+\infty} \alpha_m e^{i2\pi mx/a}. \end{aligned} \quad (15.97)$$

The coefficients α_m which have been introduced are related to the coefficients A_m and B_m by

$$\begin{aligned} \alpha_{-m} &= \frac{A_m}{2} - \frac{B_m}{2i}, \quad m = 1, 2, \dots, \\ \alpha_0 &= \frac{A_0}{2}, \\ \alpha_m &= \frac{A_m}{2} + \frac{B_m}{2i}, \quad m = 1, 2, \dots. \end{aligned} \quad (15.98)$$

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One could substitute back into the definitions of A_m and B_m to derive the coefficients α_m in terms of $f(x)$. However it is just as easy to start all over using the property

$$\begin{aligned}
 & \int_{-a/2}^{+a/2} dx \left(\frac{e^{i2\pi m'x/a}}{\sqrt{a}} \right)^* \left(\frac{e^{i2\pi mx/a}}{\sqrt{a}} \right) \\
 &= \frac{1}{a} \int_{-a/2}^{+a/2} dx e^{i2\pi(m-m')x/a} \\
 &= \begin{cases} 1, & m' = m, \\ \frac{1}{a} \left[\frac{e^{i2\pi(m-m')x/a}}{i2\pi(m-m')\frac{1}{a}} \right]_{-a/2}^{+a/2}, & m' \neq m \end{cases} \\
 &= \delta_{mm'}.
 \end{aligned} \tag{15.99}$$

Thus the coefficients α_m are readily found from the integration process and

$$\begin{aligned}
 \frac{1}{a} \int_{-a/2}^{+a/2} dx (e^{i2\pi m'x/a})^* f(x) &= \sum_{m=-\infty}^{+\infty} \alpha_m \frac{1}{a} \int_{-a/2}^{+a/2} dx (e^{i2\pi m'x/a})^* e^{i2\pi mx/a} \\
 &= \sum_{m=-\infty}^{+\infty} \alpha_m \delta_{mm'} \\
 &= \alpha_{m'}.
 \end{aligned} \tag{15.100}$$

It will be observed that $\alpha_{-m} = \alpha_m^*$ when the function f is real. However one can also expand a complex function. Thus if $f(x)$ and $g(x)$ are two real functions of x with Fourier coefficients α_m and β_m , then

$$\begin{aligned}
 f(x) + ig(x) &= \sum_{m=-\infty}^{+\infty} a_m e^{i2\pi mx/a} + i \sum_{m=-\infty}^{+\infty} \beta_m e^{i2\pi mx/a} \\
 &= \sum_{m=-\infty}^{+\infty} (\alpha_m + i\beta_m) e^{i2\pi mx/a}.
 \end{aligned} \tag{15.101}$$

Thus the complex function $\psi(x) = f(x) + ig(x)$ has as Fourier coefficients $\gamma_m = \alpha_m + i\beta_m$

$$\psi(x) = \sum_{m=-\infty}^{+\infty} \gamma_m e^{i2\pi mx/a}. \tag{15.102}$$

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We still have

$$\begin{aligned} \frac{1}{a} \int_{-a/2}^{+a/2} dx (e^{i2\pi mx/a})^* \psi(x) &= \alpha_m + i\beta_m \\ &= \gamma_m, \end{aligned} \quad (15.103)$$

but γ_{-m} is not generally equal to γ_m^* .

As a further example of the expansion process, we will consider the Fourier integral. This can also be thought of as the extension of the Fourier series to the case when the function is defined over an infinite interval. Substituting for the expansion coefficient γ_m in the complex Fourier expansion of the function $\psi(x)$ leads to

$$\psi(x) = \sum_{m=-\infty}^{+\infty} \left[\frac{1}{a} \int_{-a/2}^{+a/2} dy (e^{i2\pi my/a})^* \psi(y) \right] e^{i2\pi mx/a}. \quad (15.104)$$

One can insert a factor of $\Delta m = m - (m - 1)$ in the sum since Δm is just unity. Then, in the limit as a tends to infinity, one has

$$\psi(x) = \frac{1}{2\pi} \lim_{a \rightarrow \infty} \sum_{m=-\infty}^{+\infty} \Delta \left(\frac{2\pi m}{a} \right) \int_{-a/2}^{+a/2} dy e^{i(2\pi m/a)(x-y)} \psi(y). \quad (15.105)$$

The sum over m , in the limit a tends to infinity and consequently $\Delta(2\pi m/a)$ tends to zero, requires further investigation. It does not coincide with the definition of the integral on $2\pi m/a$. However, there is some similarity to an infinite integral on the variable k

$$k = \frac{2\pi m}{a}, \quad (15.106)$$

in which case the equation reads

$$\psi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dy e^{ik(x-y)} \psi(y). \quad (15.107)$$

This result, when decorated with a set of conditions which $\psi(x)$ must satisfy, is known as the Fourier integral theorem. In a thorough treatment of the subject, as for example given by Churchill [7], it is shown that the principal limitation on $\psi(x)$ is square-integrability:

$$\int_{-\infty}^{+\infty} dx |\psi(x)|^2 < \infty. \quad (15.108)$$

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The Fourier integral theorem gives a useful representation of the δ function. That is, since

$$\psi(x) = \int_{-\infty}^{+\infty} dy \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik(x-y)} \right) \psi(y), \quad (15.109)$$

one has

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik(x-y)} = \delta(x-y). \quad (15.110)$$

The spherical harmonics furnish another example of expansion in terms of a complete set of functions. They are orthonormal with respect to integration over the solid angle,

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'}, \quad (15.111)$$

so it is easy to find the coefficients in the expansion of some function $f(\theta, \phi)$

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{lm} Y_{lm}(\theta, \phi) \quad (15.112)$$

with the Fourier trick. That is, the coefficients A_{lm} are found by the process

$$\begin{aligned} & \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{l'm'}^*(\theta, \phi) f(\theta, \phi) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{lm} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{lm} \delta_{ll'} \delta_{mm'} \\ &= A_{l'm'}. \end{aligned} \quad (15.113)$$

Some of the features of the expansion process are independent of the set of functions involved and can be discussed in a general way. To this end, let $U_n(\xi)$ denote some set of functions such as $\sqrt{1/a} e^{i2\pi mx/a}$ or $Y_{lm}(\theta, \phi)$ and in the integration process let $\int d\xi$ stand for an integral such as $\int_{-a/2}^{+a/2} dx$ or $\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta$. The index n stands for a set of labels such

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as m or l, m in the examples cited. The functions $U_n(\xi)$ are chosen to be orthonormal with respect to the integration,

$$\int d\xi U_n^*(\xi) U_m(\xi) = \delta_{nm}. \quad (15.114)$$

The set of functions is said to be *complete* if an arbitrary function $f(\xi)$ can be expanded in terms of them,

$$f(\xi) = \sum_n a_n U_n(\xi). \quad (15.115)$$

Then, for complete sets of functions, the coefficients a_n are found by the usual process,

$$\begin{aligned} \int d\xi U_m^* f(\xi) &= \sum_n a_n \int d\xi U_m^*(\xi) U_n(\xi) \\ &= \sum_n a_n \delta_{nm} \\ &= a_m. \end{aligned} \quad (15.116)$$

When this result is substituted back into the expansion of the arbitrary function $f(\xi)$,

$$\begin{aligned} f(\xi) &= \sum_n \left[\int d\xi' U_n^*(\xi') f(\xi') \right] U_n(\xi) \\ &= \int d\xi' \left[\sum_n U_n^*(\xi') U_n(\xi) \right] f(\xi'), \end{aligned} \quad (15.117)$$

a representation of the δ function emerges, that is

$$\sum_n U_n^*(\xi') U_n(\xi) = \delta(\xi' - \xi). \quad (15.118)$$

This last result is known as the *closure relation*. The form of the δ function naturally depends on that of the integral $\int d\xi$. For example, for the spherical harmonics, $\int d\xi$ is interpreted as

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta = \int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos \theta), \quad (15.119)$$

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so the δ function $\delta(\xi' - \xi)$ is to be interpreted as

$$\frac{1}{\sin \theta} \delta(\theta - \theta') \delta(\phi - \phi') = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi'). \quad (15.120)$$

Thus the closure relation for the spherical harmonics is expressed by

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi'). \quad (15.121)$$

Eigenfunction Expansion of the Dirichlet Green's Function

The geometry of the boundary in a problem leads naturally to an investigation of a set of related functions. On the other hand, the geometry of the boundary determines the Green's function from which the electrostatic potential may be found, given any charge density $\rho(\mathbf{x})$ and any potential on the boundary. There is a direct relation between the Green's function and the set of functions. To see the relation, consider finding a Green's function, for Dirichlet boundary conditions, for some boundary and the general equation

$$\nabla^2 \psi(\mathbf{x}) + [f(\mathbf{x}) + \lambda] \psi(\mathbf{x}) = -4\pi\rho(\mathbf{x}). \quad (15.122)$$

Many problems fall into this general class. Complex solutions $\psi(\mathbf{x})$ will be considered, but $f(\mathbf{x})$ is to be real; in the electrostatic problem $f(\mathbf{x})$ and λ are zero. A Dirichlet Green's function $G_D(\mathbf{x}', \mathbf{x})$ for Eq. (15.122) satisfies

$$\begin{aligned} \nabla^2 G_D(\mathbf{x}', \mathbf{x}) + [f(\mathbf{x}) + \lambda] G_D(\mathbf{x}', \mathbf{x}) &= -4\pi\delta(\mathbf{x} - \mathbf{x}'), \\ G_D(\mathbf{x}', \mathbf{x}) &= 0, \quad \text{for } \mathbf{x} \text{ on the boundary.} \end{aligned} \quad (15.123)$$

To find the Green's function, one may first solve the eigenvalue problem for the operator $-\nabla^2 - f(\mathbf{x})$. That is, functions $\psi_n(\mathbf{x})$ and numbers λ_n will be found such that

$$\begin{aligned} [-\nabla^2 - f(\mathbf{x})]\psi_n(\mathbf{x}) &= \lambda_n \psi_n(\mathbf{x}), \\ \psi_n(\mathbf{x}) &= 0 \quad \text{for } \mathbf{x} \text{ on the boundary.} \end{aligned} \quad (15.124)$$

This eigenvalue problem is homogeneous in $\psi_n(\mathbf{x})$ so that a constant multiple of a solution is not recognized as a different solution. The trivial solution $\psi_n(\mathbf{x}) = 0$, λ_n arbitrary, is also excluded. It may be

noted, for example, that the Schroedinger equation and attendant boundary conditions in nonrelativistic quantum mechanics is an eigenvalue problem of this form. There functions $\psi_n(\mathbf{x})$ and numbers E_n are to be found that satisfy

$$\begin{aligned} -\nabla^2\psi_n(\mathbf{x}) + \frac{2m}{\hbar^2} V(\mathbf{x})\psi_n(\mathbf{x}) &= \frac{2m}{\hbar^2} E_n\psi_n(\mathbf{x}), \\ \psi_n(\mathbf{x}) &\stackrel{\rightarrow}{|_{|\mathbf{x}| \rightarrow \infty}} 0. \end{aligned} \quad (15.125)$$

This equation describes a particle of mass m moving in a potential $V(\mathbf{x})$, and $\psi_n(\mathbf{x})$ and E_n are the eigenfunctions and total energy eigenvalues of the problem.

The eigenvalues λ_n have to be real because $f(\mathbf{x})$ is real. This can be seen by operating on Eqs. (15.124) with $\int_{(*)} d^3x \psi_n^*(\mathbf{x})$:

$$-\int_{(*)} d^3x \psi_n^*(\mathbf{x}) \nabla^2 \psi_n(\mathbf{x}) - \int_{(*)} d^3x \psi_n^*(\mathbf{x}) f(\mathbf{x}) \psi_n(\mathbf{x}) = \lambda_n \int_{(*)} d^3x \psi_n^*(\mathbf{x}) \psi_n(\mathbf{x}). \quad (15.126)$$

The first term on the left-hand side is real because

$$\begin{aligned} \int_{(*)} d^3x \psi_n^* \nabla^2 \psi_n &= \int_{(*)} d^3x \nabla \cdot (\psi_n^* \nabla \psi_n) - \int_{(*)} d^3x (\nabla \psi_n^*) \cdot (\nabla \psi_n) \\ &= \int_{\text{O}_r} d\mathbf{a} \cdot \psi_n^* \nabla \psi_n - \int_{(*)} d^3x (\nabla \psi_n^*) \cdot (\nabla \psi_n) \\ &= -\int_{(*)} d^3x (\nabla \psi_n^*) \cdot (\nabla \psi_n). \end{aligned} \quad (15.127)$$

The surface integral is discarded because it is to be taken over the boundary surface, where $\psi_n(\mathbf{x})$ vanishes. The second term on the left-hand side of Eq. (15.126) is real because $f(x)$ is real. Since λ_n multiplied by the real number $\int d^3x \psi_n^* \psi_n$ is real, λ_n must be real also. One can also show that eigenfunctions that possess different eigenvalues must be orthogonal. For this purpose consider two distinct eigenvalues λ_n and λ_m , $\lambda_n \neq \lambda_m$. Then

$$\begin{aligned} (-\nabla^2 - f)\psi_n &= \lambda_n\psi_n, \\ (-\nabla^2 - f)\psi_m^* &= \lambda_m\psi_m^*, \end{aligned} \quad (15.128)$$

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where the second equation has been complex conjugated. Operating with $\int_{\textcircled{*}} d^3x \psi_m^*$ on the first, $\int_{\textcircled{*}} d^3x \psi_n$ on the second, and forming the difference leads to

$$\begin{aligned}
 & (\lambda_n - \lambda_m) \int_{\textcircled{*}} d^3x \psi_m^* \psi_n \\
 &= - \int_{\textcircled{*}} d^3x \psi_m^* \nabla^2 \psi_n - \int_{\textcircled{*}} d^3x \psi_m^* f \psi_n + \int_{\textcircled{*}} d^3x \psi_n \nabla^2 \psi_m^* + \int_{\textcircled{*}} d^3x \psi_n f \psi_m^* \\
 &= \int_{\textcircled{*}} d^3x \nabla \cdot [(\nabla \psi_m^*) \psi_n - \psi_m^* (\nabla \psi_n)] \\
 &= \int_{\textcircled{*}} d\mathbf{a} \cdot [(\nabla \psi_m^*) \psi_n - \psi_m^* (\nabla \psi_n)] \\
 &= 0.
 \end{aligned} \tag{15.129}$$

The last step depends on the fact that the eigenfunctions ψ_n and ψ_m both vanish on the boundary. Therefore if $\lambda_n \neq \lambda_m$, the orthogonality relation results,

$$\int_{\textcircled{*}} d^3x \psi_m^* \psi_n = 0. \tag{15.130}$$

It sometimes occurs that there may be several linearly independent eigenfunctions that have the same eigenvalue. It can be shown by construction that even in this case an orthogonal set of such functions exists, although the proof of this will not be reviewed here. [See, for example Merzbacher [8], p. 145.] Since ψ_n and $a\psi_n$ satisfy the eigenvalue problem, it is convenient to choose the normalization of ψ_n such that

$$\int_{\textcircled{*}} d^3x \psi_n^* \psi_n = 1. \tag{15.131}$$

Since different eigenfunctions are orthogonal, the general orthonormality relation,

$$\int_{\textcircled{*}} d^3x \psi_m^* \psi_n = \delta_{mn}, \tag{15.132}$$

then holds. In the many cases that have been investigated, the set of eigenfunctions $\psi_n(\mathbf{x})$ is complete with respect to the set of functions $f(\mathbf{x})$

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that vanish on the boundary. Then an expansion for the Green's function may be found in the form

$$G_D(\mathbf{x}', \mathbf{x}) = \sum_n a_n(\mathbf{x}') \psi_n(\mathbf{x}). \quad (15.133)$$

This must satisfy

$$\begin{aligned} [\nabla^2 + f(\mathbf{x}) + \lambda] \sum_n a_n(\mathbf{x}') \psi_n(\mathbf{x}) &= \sum_n a_n(\mathbf{x}') (\lambda - \lambda_n) \psi_n(\mathbf{x}) \\ &= -4\pi \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (15.134)$$

The usual Fourier trick, operation with $\int_{\mathbb{R}} d^3x \psi_m^*$, yields

$$\begin{aligned} \sum_n a_n(\mathbf{x}') (\lambda - \lambda_n) \delta_{mn} &= -4\pi \int_{(*)} d^3x \psi_m^*(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}'), \\ a_m(\mathbf{x}') &= 4\pi \frac{\psi_m^*(\mathbf{x}')}{\lambda_m - \lambda}. \end{aligned} \quad (15.135)$$

Thus the Dirichlet Green's function is

$$G_D(\mathbf{x}', \mathbf{x}) = 4\pi \sum_m \frac{\psi_m^*(\mathbf{x}') \psi_m(\mathbf{x})}{\lambda_m - \lambda}, \quad (15.136)$$

and so the solution of the eigenvalue problem, Eq. (15.124), neatly solves the geometry implicit in the original problem, Eq. (15.122).

In the specialization of this mathematical machinery to electrostatics, one has $\lambda = 0$ and $f(\mathbf{x}) = 0$. The eigenvalue problem that must be solved is therefore

$$\begin{aligned} -\nabla^2 \psi_n(\mathbf{x}) &= \lambda_n \psi_n(\mathbf{x}), \\ \psi_n(\mathbf{x}) &= 0 \quad \text{for } \mathbf{x} \text{ on the boundary.} \end{aligned} \quad (15.137)$$

To further illustrate the procedure, the Dirichlet Green's function for a rectangular box of sides a , b , and c will be found. In rectangular coordinates one looks for solutions of Eqs. (15.137) in the form

$$\psi = X(x)Y(y)Z(z). \quad (15.138)$$

Substituting this into the eigenvalue equation and dividing by XYZ , one has

$$-\frac{X''}{X} - \frac{Y''}{Y} - \frac{Z''}{Z} = \lambda. \quad (15.139)$$

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Obviously each term must be equal to a constant. That is, the equation is separable in Cartesian coordinates. For example,

$$\begin{aligned}\frac{X''}{X} &= -\alpha^2, \\ X'' + \alpha^2 X &= 0, \\ X &= A \sin \alpha x + B \cos \alpha x.\end{aligned}\tag{15.140}$$

Since X must be zero at $x = 0$ (the coordinate axes will be taken to coincide with one corner of the box) B must be zero. Since $X(a)$ is zero, αa must be an integer multiple of π , say $l\pi$, where $l = 1, 2, 3, \dots$. The functions $Y(y)$ and $Z(z)$ are found in the same way, so the eigenfunctions and eigenvalues are

$$\begin{aligned}\psi_{lmn} &= A \sin l\pi \frac{x}{a} \sin m\pi \frac{y}{b} \sin n\pi \frac{z}{c}, \\ \lambda_{lmn} &= \pi^2 \left[\left(\frac{l}{a} \right)^2 + \left(\frac{m}{b} \right)^2 + \left(\frac{n}{c} \right)^2 \right],\end{aligned}\tag{15.141}$$

where l , m , and n range over the positive integers. The eigenfunctions are orthogonal since

$$\begin{aligned}\int_0^a dx \sin l'\pi \frac{x}{a} \sin l\pi \frac{x}{a} &= \frac{1}{2} \int_0^a dx \left[\cos(l' - l)\pi \frac{x}{a} - \cos(l' + l)\pi \frac{x}{a} \right] \\ &= \begin{cases} \frac{a}{2} - \frac{1}{2} \left[\frac{\sin(l' + l)\pi \frac{x}{a}}{(l' + l)\pi \frac{1}{a}} \right]_0^a, & l' = l, \\ 0, & l' \neq l \end{cases} \\ &= \frac{a}{2} \delta_{l'l}.\end{aligned}\tag{15.142}$$

Therefore,

$$\int_{(*)} d^3x \psi_{l'm'n'}^* \psi_{lmn} = \frac{abc}{8} |A|^2 \delta_{l'l} \delta_{m'm} \delta_{n'n}.\tag{15.143}$$

Thus the eigenfunctions

$$\psi_{lmn} = \sqrt{\frac{8}{abc}} \sin l\pi \frac{x}{a} \sin m\pi \frac{y}{b} \sin n\pi \frac{z}{c}\tag{15.144}$$

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are orthonormal and the Dirichlet Green's function for the box is

$$G_D(\mathbf{x}', \mathbf{x}) = 4\pi \sum_{lmn} \frac{8}{abc} \frac{\sin l\pi \frac{x'}{a} \sin l\pi \frac{x}{a} \sin m\pi \frac{y'}{b} \sin m\pi \frac{y}{b} \sin n\pi \frac{z'}{c} \sin n\pi \frac{z}{c}}{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2}. \quad (15.145)$$

PROBLEMS

1. A circular line of radius c and charge per unit length $q/2\pi c$ is in the equatorial plane of a grounded sphere of radius $a < c$. Find the induced charge density on the sphere correct to order $(a/c)^2$.

Solution: The image of a charge q at radius c is charge $-aq/c$ at radius a^2/c so the field outside the sphere is the same as that of two circular lines of charge in the equatorial plane with centers at the center of the sphere. These are shown on Fig. 15.1. One has charge per

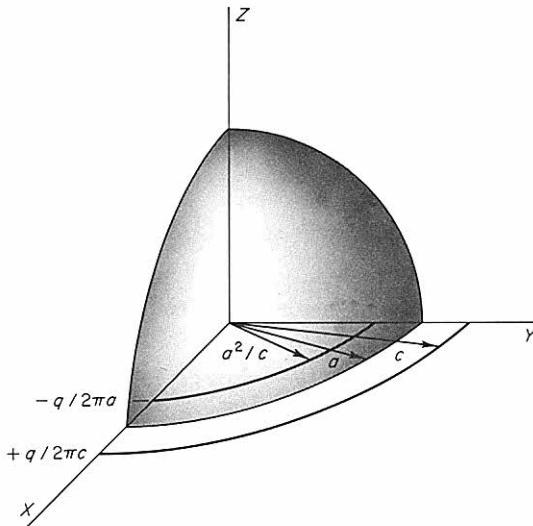


FIG. 15.1. A ring of charge outside a conducting sphere.

unit length $q/2\pi c$ and radius c ; the other has charge per unit length

$$-\frac{a}{c} q \cdot \frac{1}{2\pi \frac{a^2}{c}} = -\frac{q}{2\pi a}$$

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and radius a^2/c . The next step is to determine the potential due to a ring of charge q at radius c both for $r > c$ and for $r < c$. The general solution in spherical coordinates is

$$\varphi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} [A_{lm} r^l + B_{lm} r^{-l-1}] Y_{lm}(\theta, \phi).$$

Because of the axial symmetry of this problem, only $m = 0$ contributes to the potential. The first three spherical harmonics with $m = 0$ are

$$Y_{00} = \sqrt{\frac{1}{4\pi}},$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta,$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1).$$

However the potential at θ must be the same as the potential at $\pi - \theta$ and since $\cos(\pi - \theta) = -\cos \theta$, the term Y_{10} cannot contribute. Thus for $r > c$, where an expansion in $1/r$ is appropriate,

$$\varphi = \frac{B_0}{r} \sqrt{\frac{1}{4\pi}} + \frac{B_2}{r^3} \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) + \dots,$$

while for $r < c$ where an expansion in r is required,

$$\varphi = A_0 \sqrt{\frac{1}{4\pi}} + A_2 r^2 \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) + \dots.$$

The constants B_0, B_2 and A_0, A_2 can be determined by specialization to $\theta = 0$ where the potential is known to be just $q/\sqrt{r^2 + c^2}$. For $r > c$ this is

$$\begin{aligned} \frac{q}{\sqrt{r^2 + c^2}} &= \frac{q}{r} \frac{1}{\sqrt{1 + \left(\frac{c}{r}\right)^2}}, \\ &= \frac{q}{r} - \frac{1}{2} \frac{qc^2}{r^3} + \dots, \end{aligned}$$

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whereas for $r < c$ one has

$$\begin{aligned}\frac{q}{\sqrt{r^2 + c^2}} &= \frac{q}{c} \frac{1}{\sqrt{1 + \left(\frac{r}{c}\right)^2}}, \\ &= \frac{q}{c} - \frac{qr^2}{2c^3} + \cdots.\end{aligned}$$

Thus the expansion for the potential comes to

$$\varphi = \begin{cases} \frac{q}{r} - \frac{1}{4} \frac{qc^2}{r^3} (3 \cos^2 \theta - 1) + \cdots, & r > c, \\ \frac{q}{c} - \frac{1}{4} \frac{qr^2}{c^3} (3 \cos^2 \theta - 1) + \cdots, & r < c. \end{cases}$$

Now for the problem at hand. To obtain the potential near the surface of the sphere, the large r expansion is used for the inner ring of charge $-qa/c$ at a^2/c and the small r expansion is used for the outer ring of charge q at c , so

$$\begin{aligned}\varphi &= -\frac{aq}{c} \left[\frac{1}{r} - \frac{1}{4} \left(\frac{a^2}{c} \right)^2 \frac{1}{r^3} (3 \cos^2 \theta - 1) + \cdots \right] \\ &\quad + \frac{q}{c} - \frac{1}{4} \frac{qr^2}{c^3} (3 \cos^2 \theta - 1) + \cdots.\end{aligned}$$

The field in the radial direction is

$$\begin{aligned}E_r &= -\frac{\partial \varphi}{\partial r} = +\frac{aq}{c} \left[-\frac{1}{r^2} + \frac{3}{4} \left(\frac{a^2}{c} \right)^2 \frac{1}{r^4} (3 \cos^2 \theta - 1) + \cdots \right] \\ &\quad + \frac{1}{2} \frac{q}{c^3} r (3 \cos^2 \theta - 1) + \cdots,\end{aligned}$$

and so the charge density on the surface of the conductor at radius a is

$$\begin{aligned}\sigma &= \frac{1}{4\pi} E_r \Big|_{r=a} \\ &= \left(\frac{1}{4\pi} \right) \left(\frac{aq}{c} \right) \left[-\frac{1}{a^2} + \frac{3}{4c^2} (3 \cos^2 \theta - 1) + \cdots \right] \\ &\quad + \frac{1}{4\pi} \cdot \frac{q}{2c^3} a (3 \cos^2 \theta - 1) + \cdots \\ &= -\frac{1}{4\pi a^2} \left(\frac{aq}{c} \right) \left[1 - \frac{5}{4} \left(\frac{a}{c} \right)^2 (3 \cos^2 \theta - 1) + \cdots \right].\end{aligned}$$

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2. A line charge along the Z -axis has charge per unit length $\tau(z)$ and is confined to $-R < z < +R$. Show that the multipole expansion of the potential $\varphi(\mathbf{x})$ for $|\mathbf{x}| > R$ is

$$\varphi(\mathbf{x}) = \sum_l \left\{ \int_{-R}^{+R} dz' \tau(z') z'^l \right\} \frac{P_l(\cos \theta)}{r^{l+1}}.$$

3. Prove that

$$\int_0^\infty dx e^{-\kappa|z|} J_0(\kappa r) = \frac{1}{\sqrt{r^2 + z^2}}$$

for $|r| < |z|$ by substituting in the series for $J_0(\kappa r)$ and performing the integration term-by-term. The result actually holds in general.

4. Suppose the potential on the surface of a circular cylinder of radius $r = a$ depends only on the azimuthal angle so that $\varphi(a, z, \phi) = f(\phi)$, where f is a certain given function. Suppose also that there are no sources inside the cylinder. Find the potential inside using the method of separation of variables.

Solution: The boundary conditions are independent of z , and since there are no sources in the region of interest, the function $Z(z)$ is merely a constant. Thus the potential is a sum of terms like

$$\varphi(r, z, \phi) = R(r)Q(\phi),$$

where $Q(\phi)$ is of the form $Ae^{im\phi}$, m an integer, and $R(r)$ is determined by the equation

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) - m^2 R = 0.$$

This is just Cauchy's equation, so one substitutes

$$R(r) = Br^p,$$

with the result

$$(p^2 - m^2)Br^p = 0.$$

For the interior solution, $p = |m|$ is appropriate so the general solution with translational symmetry along the Z -axis is

$$\varphi(r, z, \phi) = \sum_{m=-\infty}^{+\infty} A_m \left(\frac{r}{a} \right)^{|m|} e^{im\phi}.$$

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A factor of $(a)^{|m|}$ has been incorporated into the expansion coefficients A_m for convenience in the next step. The coefficients A_m may be found by specializing the general form to the surface $r = a$ where one requires

$$f(\phi) = \sum_{m=-\infty}^{+\infty} A_m e^{im\phi}.$$

Now applying the Fourier trick, one finds

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi (e^{im'\phi})^* f(\phi) &= \sum_{m=-\infty}^{+\infty} A_m \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi e^{i(m-m')\phi} \\ &= \sum_{m=-\infty}^{+\infty} A_m \delta_{mm'} = A_{m'}. \end{aligned}$$

Furthermore, since $f(\phi)$ is real, one has $A_{-m} = A_m^*$. The desired expression for the potential is therefore

$$\varphi(r, z, \phi) = \sum_{m=-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi' (e^{im\phi'})^* f(\phi') \right] \left(\frac{r}{a} \right)^{|m|} e^{im\phi}.$$

By combining the sums over positive and negative integers m one obtains

$$\begin{aligned} \varphi(r, z, \phi) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi' f(\phi') + \frac{1}{2\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^m \int_{-\pi}^{+\pi} d\phi' f(\phi') \left[\begin{array}{c} e^{-im\phi'} e^{im\phi} \\ + e^{im\phi'} e^{-im\phi} \end{array} \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi' f(\phi') + \frac{1}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^m \int_{-\pi}^{+\pi} d\phi' f(\phi') \cos m(\phi' - \phi). \end{aligned}$$

5. Show that the Green's function for the Dirichlet problem in the volume between two concentric spheres of radii a and b , where $a < b$, is

$$G_D(\mathbf{x}', \mathbf{x}) = \sum_{lm} g_l(r, r') Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi),$$

with $r_>$ and $r_<$ denoting the greater and lesser of r and r' . Here $g_l(r, r')$ is

$$g_l(r, r') = \frac{4\pi}{(2l+1) \left[1 - \left(\frac{a}{b} \right)^{2l+1} \right]} \left[r_{<}^l - \frac{a^{2l+1}}{r_{<}^{l+1}} \right] \left[\frac{1}{r_{>}^{l+1}} - \frac{r_{>}^l}{b^{2l+1}} \right].$$

(See Jackson [9], pp. 78–81.)

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6. Suppose a function $f(\xi)$ is to be approximated by a linear combination of a finite number of orthonormal functions $U_n(\xi)$

$$f(\xi) \sim \sum_{n=1}^N a_n U_n(\xi).$$

Show that the coefficients a_n which minimize the mean square error

$$M = \int \left| f(\xi) - \sum_{n=1}^N a_n U_n(\xi) \right|^2 d\xi$$

are given by

$$a_n = \int U_n^*(\xi) f(\xi) d\xi.$$

Solution: The mean square error M is a real function of the $2N$ real variables a_{1n}, a_{2n} where

$$a_n = a_{1n} + ia_{2n}$$

is complex. Then M will be extremal when

$$\frac{\partial M}{\partial a_{1n}} = \frac{\partial M}{\partial a_{2n}} = 0.$$

Since the mean square error is

$$M = \int \left[f^*(\xi) - \sum_{m=1}^{\infty} (a_{1m} - ia_{2m}) U_m^*(\xi) \right] \left[f(\xi) - \sum_{m'=1}^{\infty} (a_{1m'} + ia_{2m'}) U_{m'}(\xi) \right] d\xi,$$

the conditions are

$$\begin{aligned} \frac{\partial M}{\partial a_{1n}} &= - \int U_n^*(\xi) \left[f(\xi) - \sum_{m'=1}^{\infty} (a_{1m'} + ia_{2m'}) U_{m'}(\xi) \right] d\xi \\ &\quad - \int \left[f^*(\xi) - \sum_{m=1}^{\infty} (a_{1m} - ia_{2m}) U_m^*(\xi) \right] U_n(\xi) d\xi \\ &= 0, \\ \frac{\partial M}{\partial a_{2n}} &= i \int U_n^*(\xi) \left[f(\xi) - \sum_{m'=1}^{\infty} (a_{1m'} + ia_{2m'}) U_{m'}(\xi) \right] d\xi \\ &\quad - i \int \left[f^*(\xi) - \sum_{m=1}^{\infty} (a_{1m} - ia_{2m}) U_m^*(\xi) \right] U_n(\xi) d\xi \\ &= 0. \end{aligned}$$

With the orthonormality of the functions $U_n(\xi)$, these relations lead to

$$-\int U_n^*(\xi) f(\xi) d\xi + a_n - \int U_n(\xi) f^*(\xi) d\xi + a_n^* = 0,$$

$$\int U_n^*(\xi) f(\xi) d\xi - a_n - \int U_n(\xi) f^*(\xi) d\xi + a_n^* = 0,$$

or

$$a_{1n} = \operatorname{Re} \left[\int U_n^*(\xi) f(\xi) d\xi \right],$$

$$a_{2n} = \operatorname{Im} \left[\int U_n^*(\xi) f(\xi) d\xi \right],$$

$$a_n = \int U_n^*(\xi) f(\xi) d\xi.$$

The result is the same when a formal differentiation with respect to the complex variable a_n is performed with a_n^* held constant. That procedure amounts to regarding a_n and a_n^* as independent variables in place of a_{1n} and a_{2n} . Then, because M itself is a real quantity, $\partial M / \partial a_n = 0$ implies $\partial M / \partial a_n^* = 0$.

7. A line carrying charge per unit length q/l lies along the Y -axis between two grounded conducting planes located at $z = h$ and $z = -h$.

(a) Show that the potential between the planes is given by

$$\varphi(x, z) = \frac{2q}{l} \int_0^\infty \frac{\sinh \mu(h - |z|) \cos \mu x}{\mu \cosh \mu h} d\mu.$$

(b) Show that the potential between the planes is also given by

$$\varphi(x, z) = \frac{4q}{l} \sum_{\text{odd } n} \frac{1}{n} e^{-n\pi|x|/2h} \cos \frac{n\pi z}{2h}$$

where the sum ranges over the odd integers, $n = 1, 3, 5, \dots$.

(c) Prove directly that the two formulas are equivalent.

Outline of Solution: (a) Proceed by separation of variables in the dependence of the potential in x and z . To obtain the first form of the solution, take the separation constant in such a way that the x -dependence is expressed in trigonometric functions. Then for $z > 0$, say region I, one has

$$\varphi_I(x, z) = \int_{-\infty}^{+\infty} A_1(\mu) \sinh \mu(h - z) \cos \mu x d\mu$$

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and for $z < 0$, denoted as region II,

$$\varphi_{\text{II}}(x, z) = \int_{-\infty}^{+\infty} A_{\text{II}}(\mu) \sinh \mu(h+z) \cos \mu x \, d\mu.$$

The boundary conditions needed to match up these two solutions are

$$\begin{aligned} [\varphi_{\text{I}} - \varphi_{\text{II}}]_{z=0} &= 0, \\ \left[-\frac{\partial \varphi_{\text{I}}}{\partial z} + \frac{\partial \varphi_{\text{II}}}{\partial z} \right]_{z=0} &= 4\pi \frac{q}{l} \delta(x) \\ &= \frac{2q}{l} \int_{-\infty}^{+\infty} \cos \mu x \, d\mu. \end{aligned}$$

(b) Choose the separation constant so as to express the z -dependence in trigonometric functions.

(c) The two expressions are shown to be equivalent by performing what is called a *Sommerfeld–Watson transform*. The result in (b) is rewritten in the form

$$\varphi = \frac{4q}{l} \left(\frac{\pi}{2h} \right) \frac{1}{i2\pi} \int_{\mathcal{C}} \frac{1}{\nu} e^{-\nu|z|} \cos \nu z \left(\frac{-h \sin \nu h}{\cos \nu h} \right) d\nu,$$

where the contour comes in from infinity just above the real axis, crosses it just to the left of $\nu h = \pi/2$, and goes out to infinity just below the real axis. An alternate expression is

$$\varphi = -\frac{q}{l} \left(\frac{1}{i^2} \right) \int_{\mathcal{C}} \frac{1}{\nu} \frac{e^{-\nu|z|}}{\cos \nu h} [\sin \nu(h+z) + \sin \nu(h-z)] d\nu.$$

The two terms in the integrand contribute equally so we may condense the integral to

$$\varphi = -\frac{q}{l} \left(\frac{1}{i} \right) \int_{\mathcal{C}} \frac{1}{\nu} \frac{e^{-\nu|z|}}{\cos \nu h} \sin \nu(h - |z|) d\nu.$$

The final step is to deform the contour into an integral along the imaginary axis.

Solutions to (a) and (b) are derived in detail in Panofsky and Phillips [10] in the more general case when the wire is not midway between the plates. The relation between the two formulas for the potential was pointed out by Cooper [11].

8. (a) In spherical coordinates

$$x = r \sin \theta \cos \phi,$$

$$y = r \sin \theta \sin \phi,$$

$$z = r \cos \theta,$$

one can derive

$$\frac{\partial}{\partial x} = \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi},$$

and analogous formulas for $\partial/\partial y$ and $\partial/\partial z$. For this, it helps to recognize that dx , dy , and dz are related to dr , $r d\theta$ and $r \sin \theta d\phi$ by an orthogonal matrix. Show by direct substitution that

$$\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi},$$

where

$$\hat{\mathbf{r}} = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k},$$

$$\hat{\theta} = \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k},$$

$$\hat{\phi} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}.$$

Make a sketch showing the way that these unit vectors point in the directions of increasing r , θ , and ϕ .

(b) Show that

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

Intermediate results needed are

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\theta}, \quad \frac{\partial \hat{\mathbf{r}}}{\partial \phi} = \sin \theta \hat{\phi},$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{\mathbf{r}}, \quad \frac{\partial \hat{\theta}}{\partial \phi} = \cos \theta \hat{\phi},$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\sin \theta \hat{\mathbf{r}} - \cos \theta \hat{\theta}.$$

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(c) Any vector \mathbf{v} may be resolved into components in the $\hat{\mathbf{r}}$, $\hat{\theta}$, $\hat{\phi}$ directions, say

$$\mathbf{v} = v_r \hat{\mathbf{r}} + v_\theta \hat{\theta} + v_\phi \hat{\phi}.$$

Show that

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi},$$

$$\begin{aligned} \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ &\quad + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} \\ &\quad + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi}. \end{aligned}$$

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Polarization

When, in an electrostatic problem, some of the volume is occupied by insulating material, the fields induce polarization, meaning dipole moment per unit volume, in the material. The potential of a single dipole \mathbf{p} at the origin is

$$\begin{aligned} \varphi(\mathbf{x}) &= -\mathbf{p} \cdot \nabla \left(\frac{1}{r} \right) \\ &= \frac{\mathbf{p} \cdot \mathbf{x}}{r^3}, \end{aligned} \tag{16.1}$$

so the potential produced by a polarized substance carrying dipole moment per unit volume $\mathbf{P}(\mathbf{x})$ is

$$\begin{aligned} \varphi(\mathbf{x}) &= - \int_{(*)} d^3x' \mathbf{P}(\mathbf{x}') \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= + \int_{(*)} d^3x' \mathbf{P}(\mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \int_{(*)} d^3x' \nabla' \cdot \left[\frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right] - \int_{(*)} d^3x' \frac{\nabla' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &= \int_{\bigcirclearrowright} \frac{d\mathbf{a}' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \int_{(*)} d^3x' \frac{\nabla' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \tag{16.2}$$

Thus the potential of the polarized material is equivalent to that produced by a surface charge per unit area P_n and a charge per unit volume $-\nabla \cdot \mathbf{P}$ inside the material.

Usually there are other charges besides the effective ones of the polarized material. As a matter of notation, one writes the total charge density as ρ_t , where

$$\rho_t = \rho + \rho_P, \quad (16.3)$$

where $\rho(\mathbf{x})$ is the density of nonpolarization charges, also called *true* charges, and the polarization charge density ρ_P is

$$\rho_P(\mathbf{x}) = -\nabla \cdot \mathbf{P}(\mathbf{x}). \quad (16.4)$$

Then, for electrostatic problems, the field is determined by

$$\begin{aligned} \nabla \times \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{E} &= 4\pi\rho_t. \end{aligned} \quad (16.5)$$

The effect of the polarization is equivalent to that of a charge per unit volume $-\nabla \cdot \mathbf{P}$ and a surface charge per unit area P_n as in Eq. (16.2). One only needs to consider ρ_P as $-\nabla \cdot \mathbf{P}$ because it includes the effect at the surface. To see this, an integration over a small box of area A at the surface of the polarized material may be performed as illustrated in Fig. 16.1.

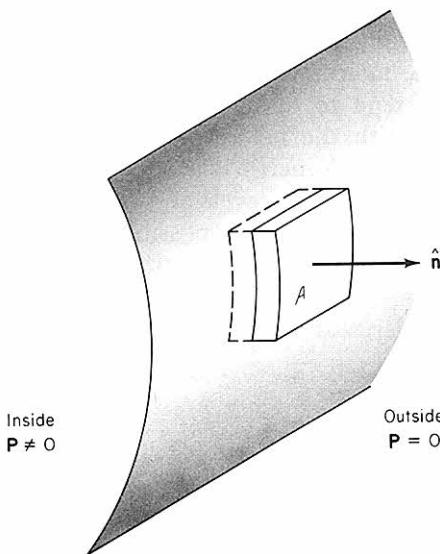


FIG. 16.1. Surface effect of polarization charge.

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One finds that

$$\begin{aligned} \int_{(*)} d^3x [-\nabla \cdot \mathbf{P}(\mathbf{x})] &= -\int_{\bigcirclearrowright} \mathbf{P} \cdot d\mathbf{a}, \\ &= -(-\mathbf{P} \cdot \hat{\mathbf{n}})A, \\ &= P_n A, \end{aligned} \quad (16.6)$$

where $\hat{\mathbf{n}}$ is the outward normal to the surface. Thus a charge per unit area P_n on the surface is included. Further clarification of the situation arises from the observation that the sum total of all polarization charges must be zero. This can be seen by applying Gauss's law to the integral over a volume that extends beyond the material of the polarization charge $\rho_P = -\nabla \cdot \mathbf{P}$.

The origin of polarization in materials lies ultimately in their atomic structure. One can begin to understand the polarizability of an atom on the basis of a very simple model in which the atom is thought of as a small sphere of mobile charges. When a field is applied to such an atom, the positive and negative charges are pulled in opposite directions, and the atom acquires a dipole moment that is lined up with the external field. Because the separation in energy of different electron energy levels is large, the atomic polarization is temperature independent. Molecules, on the other hand, sometimes have permanent electric dipole moments. In the material, the molecules may be randomly oriented but with an external field they tend to orient themselves to minimize the energy $-\mathbf{p} \cdot \mathbf{E}$, lining up with the field, and so the material acquires a net polarization. Orientational energy levels are closely spaced compared to the spacing of electron energy levels and so this type of polarization is temperature dependent. It is also possible, in the solid phase, for a material to have a spontaneous polarization over certain ranges of temperature.

In general the polarization induced in a material is a function of the electric field

$$\mathbf{P} = \mathbf{P}(\mathbf{E}). \quad (16.7)$$

Except for certain materials in special temperature ranges, a low order expansion for the polarization is adequate, say

$$P_i = P_{0i} + a_{ij}E_j. \quad (16.8)$$

Furthermore \mathbf{P}_0 may be nonzero. In this case the material is called a *ferroelectric*. Most substances have no spontaneous polarization. For isotropic materials

$$\mathbf{P} = \chi_e \mathbf{E}, \quad (16.9)$$

that is: all directions are equivalent and the polarization is proportional to the field. Then χ_e is called the *electric susceptibility*. This is the simplest kind of material and the one that we will treat especially.

Since

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 4\pi\rho_t, \\ \nabla \cdot \mathbf{P} &= -\rho_P,\end{aligned}\quad (16.10)$$

if one defines the displacement vector \mathbf{D} by

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad (16.11)$$

then

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 4\pi\rho_t - 4\pi\rho_P, \\ &= 4\pi\rho.\end{aligned}\quad (16.12)$$

The important property of \mathbf{D} is that its divergence is proportional to the true nonpolarization charge. This means that D_n is continuous across a surface that carries no true surface charges. In contrast, E_n will ordinarily be discontinuous. However,

$$\begin{aligned}\nabla \times \mathbf{D} &= \nabla \times \mathbf{E} + 4\pi\nabla \times \mathbf{P}, \\ &= 4\pi\nabla \times \mathbf{P},\end{aligned}\quad (16.13)$$

and so the tangential component of \mathbf{D} may be discontinuous at a surface, whereas \mathbf{E}_t will be continuous as before.

For a simple material which can be characterized by an electric susceptibility,

$$\begin{aligned}\mathbf{D} &= \mathbf{E} + 4\pi\mathbf{P} \\ &= \mathbf{E} + 4\pi\chi_e \mathbf{E} \\ &= \epsilon \mathbf{E},\end{aligned}\quad (16.14)$$

where $\epsilon = 1 + 4\pi\chi_e$ is called the *dielectric constant* of the material. Within a region where ϵ is constant, one has

$$\begin{aligned}\nabla \times \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{D} &= 4\pi\rho, \\ \nabla \cdot \mathbf{E} &= \frac{4\pi}{\epsilon} \rho.\end{aligned}\quad (16.15)$$

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That is, the field \mathbf{E} satisfies the electrostatic field equations in this region except that the source is modified to $1/\epsilon$ times the true charge density.

The Problem of a Condenser Filled with a Dielectric

Suppose that the plates of a condenser have area A , that they are parallel at a distance d apart, and that the intervening material can be characterized by a dielectric constant ϵ . Figure 16.2 shows the elements

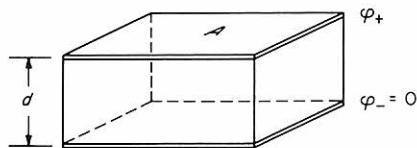


FIG. 16.2. *The dielectric filled condenser.*

of this problem. It will be assumed that one of the plates is grounded and the other is held at positive potential φ_+ by a battery. Neglecting edge effects, the field strength E is φ_+/d independent of ϵ . The magnitude of \mathbf{D} is therefore $\epsilon\varphi_+/d$. Now the true charge on the plates can be found when \mathbf{D} is known since $\nabla \cdot \mathbf{D}$ is $4\pi\rho$. That is, if σ is the true charge per unit area, then

$$\begin{aligned}\sigma &= \frac{1}{4\pi} D_n \\ &= \frac{\epsilon\varphi_+}{4\pi d}\end{aligned}\quad (16.16)$$

on the positive plate. The total charge on the plate is $A\sigma = A\epsilon\varphi_+/4\pi d$ so the capacitance C of the condenser is

$$\begin{aligned}C &= \frac{\text{charge on one plate}}{\text{potential difference}} \\ &= \frac{\epsilon A}{4\pi d}.\end{aligned}\quad (16.17)$$

It will be observed that the capacitance is increased by a factor of ϵ by the presence of the dielectric.

The total charge per unit area existing at the interface between the positive conductor and the dielectric is

$$\begin{aligned}\sigma_t &= \frac{E_n}{4\pi} \\ &= \frac{\varphi_+}{4\pi d},\end{aligned}\quad (16.18)$$

since $\nabla \cdot \mathbf{E}$ is $4\pi\rho_t$. The polarization charge per unit area is therefore

$$\begin{aligned}\sigma_p &= \sigma_t - \sigma \\ &= \frac{\varphi_+}{4\pi d} - \frac{\epsilon\varphi_+}{4\pi d} \\ &= -(\epsilon - 1) \frac{\varphi_+}{4\pi d}.\end{aligned}\quad (16.19)$$

That is, some of the true charge density is balanced by polarization charges. In effect, one has to introduce more true charge in order to produce the same potential drop between the plates, and therefore C increases due to the dielectric.

The Problem of a Charge near a Dielectric Boundary

Suppose now a charge q is located at $x = y = 0, z = d$ and the dielectric constant is ϵ_1 for $z > 0$ and ϵ_2 for $z < 0$ as in Fig. 16.3. What

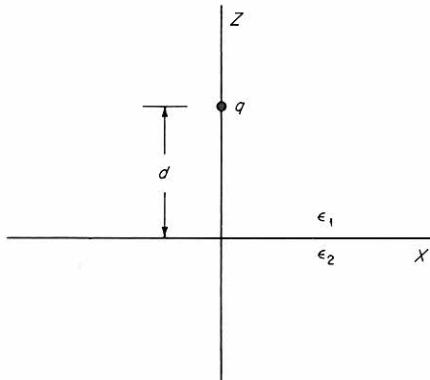


FIG. 16.3. Charge near a dielectric boundary.

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will be the fields \mathbf{D} and \mathbf{E} in this case? The fields are determined by the equations

$$\nabla \times \mathbf{E} = 0,$$

$$\nabla \cdot \mathbf{D} = 4\pi q \delta(z - d) \delta(x) \delta(y), \quad (16.20)$$

$$\mathbf{D} = \begin{cases} \epsilon_1 \mathbf{E}, & z > 0, \\ \epsilon_2 \mathbf{E}, & z < 0. \end{cases}$$

The electric field \mathbf{E} will have vanishing divergence except at the location of the charge q and at the boundary $z = 0$. The curl of \mathbf{E} is zero everywhere and in particular at the interface. Therefore the tangential component of \mathbf{E} at the interface is continuous. The displacement \mathbf{D} has zero divergence except at the charge. The normal component of \mathbf{D} is therefore continuous at the interface.

The problem can be solved with images. For the potential for $z > 0$, one considers two charges q and q' in a space filled with the dielectric ϵ_1 and located on the Z -axis at $\pm d$. In cylindrical coordinates the potential is

$$\varphi_{>}(r, z, \theta) = \frac{q}{\epsilon_1 \sqrt{r^2 + (z - d)^2}} + \frac{q'}{\epsilon_1 \sqrt{r^2 + (z + d)^2}}. \quad (16.21)$$

For $z < 0$ the potential in the original problem is supposed to be the same as that of an effective charge q'' located on the Z -axis at $z = d$ in a space filled with the dielectric ϵ_2 . That is

$$\varphi_{<}(r, z, \theta) = \frac{q''}{\epsilon_2 \sqrt{r^2 + (z - d)^2}}. \quad (16.22)$$

To complete the solution by images, the charges q' and q'' have to be found that lead to agreement with Eqs. (16.20). First of all the normal component of \mathbf{D} must be continuous at $z = 0$. This condition is

$$\begin{aligned} \epsilon_2 \left[\frac{\partial \varphi_{<}}{\partial z} \right]_{z=0} &= \epsilon_1 \left[\frac{\partial \varphi_{>}}{\partial z} \right]_{z=0}, \\ -\epsilon_2 \left[\frac{q''(z - d)}{\epsilon_2 [\sqrt{r^2 + (z - d)^2}]^3} \right]_{z=0} &= -\epsilon_1 \left[\frac{q(z - d)}{\epsilon_1 [\sqrt{r^2 + (z - d)^2}]^3} + \frac{q'(z + d)}{\epsilon_1 [\sqrt{r^2 + (z + d)^2}]^3} \right]_{z=0}, \quad (16.23) \\ q'' &= q - q'. \end{aligned}$$

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Also the tangential component of \mathbf{E} must be continuous at $z = 0$. This condition leads to

$$\begin{aligned} \left. \frac{\partial \varphi_{<}}{\partial r} \right|_{z=0} &= \left. \frac{\partial \varphi_{>}}{\partial r} \right|_{z=0}, \\ - \left[\frac{q''r}{\epsilon_2[\sqrt{r^2 + (z-d)^2}]^3} \right]_{z=0} &= - \left[\frac{qr}{\epsilon_1[\sqrt{r^2 + (z-d)^2}]^3} + \frac{q'r}{\epsilon_1[\sqrt{r^2 + (z+d)^2}]^3} \right]_{z=0}, \quad (16.24) \\ \frac{q''}{\epsilon_2} &= \frac{q + q'}{\epsilon_1}. \end{aligned}$$

It is easy to solve for q' and q'' from Eqs. (16.23) and (16.24). Eliminating q'' gives

$$\begin{aligned} \frac{q - q'}{\epsilon_2} &= \frac{q + q'}{\epsilon_1}, \\ q' &= \left[\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \right] q \quad \text{and} \quad q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} q. \end{aligned} \quad (16.25)$$

It is interesting to see what polarization charges are induced in this problem. Away from the interface one has $\nabla \cdot \mathbf{E} = 0$ for $z < 0$ and

$$\nabla \cdot \mathbf{E} = \frac{4\pi q}{\epsilon_1} \delta(z-d) \delta(x) \delta(y), \quad z > 0. \quad (16.26)$$

Since $\nabla \cdot \mathbf{E}$ is $4\pi\rho_t$, where ρ_t is the total charge density, and the polarization charge density ρ_p is $\rho_t - \rho$, the polarization charge density is

$$\begin{aligned} \rho_p &= 4\pi q \delta(z-d) \delta(x) \delta(y) \left[\frac{1}{\epsilon_1} - 1 \right] \\ &= - \frac{(\epsilon_1 - 1)}{\epsilon_1} 4\pi q \delta(z-d) \delta(x) \delta(y), \quad z \neq 0. \end{aligned} \quad (16.27)$$

On the interface there are no true charges so the polarization and total

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charge densities are equal, $\rho_P = \rho_t$, and the polarization charge per unit area is

$$\begin{aligned}
\sigma_P &= \frac{1}{4\pi} [E_z(z > 0) - E_z(z < 0)]_{z=0} \\
&= -\frac{1}{4\pi} \left[\frac{\partial \varphi_>}{\partial z} - \frac{\partial \varphi_<}{\partial z} \right]_{z=0} \\
&= +\frac{1}{4\pi} \left[\frac{q(z-d)}{\epsilon_1[\sqrt{r^2+(z-d)^2}]^3} + \frac{q'(z+d)}{\epsilon_1[\sqrt{r^2+(z+d)^2}]^3} \right]_{z=0} \\
&\quad - \frac{1}{4\pi} \left[\frac{q''(z-d)}{\epsilon_2[\sqrt{r^2+(z-d)^2}]^3} \right]_{z=0} \\
&= \frac{d}{4\pi[\sqrt{r^2+d^2}]^3} \left[-\frac{q}{\epsilon_1} + \frac{q'}{\epsilon_1} + \frac{q''}{\epsilon_2} \right] \\
&= \frac{qd}{4\pi\epsilon_1[\sqrt{r^2+d^2}]^3} \left[-1 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} + \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2} \right] \\
&= \frac{qd}{2\pi\epsilon_1[\sqrt{r^2+d^2}]^3} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}. \tag{16.28}
\end{aligned}$$

The polarization charge on the interface vanishes when $\epsilon_1 = \epsilon_2$, as expected.

The Clausius–Mossotti Relation

Suppose that a material is composed of polarizable molecules so that in a field \mathbf{E}_0 each acquires a dipole moment

$$\mathbf{p} = \alpha \mathbf{E}_0. \tag{16.29}$$

It is beyond the scope of E & M to calculate the polarizability α except for some simplified models. However, if the material has no spontaneous polarization and has at least cubic symmetry, it is possible to relate the dielectric constant ϵ to α directly. In fact one finds

$$\alpha = \frac{3}{4\pi n} \left(\frac{\epsilon - 1}{\epsilon + 2} \right), \tag{16.30}$$

where n is the number of molecules per unit volume. Equation (16.30) is known as the *Clausius–Mossotti relation*. To see how this rule can be

derived, let \mathbf{E} be the average electric field in the material. For simplicity, it will be assumed that the polarization is constant throughout the material. Then \mathbf{E} is produced by the true charges and by the effective surface charge P_n on the boundary of the material with free space. The field acting on an individual molecule, tending to polarize it, is then

$$\mathbf{E}_0 = \mathbf{E} + \mathbf{E}_i, \quad (16.31)$$

where \mathbf{E}_i is obtained by summing the electric fields produced by all the other molecules. They do produce fields, of course, because they are polarized. Although \mathbf{E}_i would be rapidly varying, on a scale commensurate with the spacing between molecules, we only have to evaluate it at a very special point. The problem here is really to calculate \mathbf{E}_i . One proceeds by choosing origin at the observed molecule and by drawing a sphere of radius R about the chosen molecule such that R is large compared to the spacing between molecules. For convenience, let the Z -axis be chosen in the direction of the average field \mathbf{E} and polarization \mathbf{P} as in Fig. 16.4. Assuming that \mathbf{P} is constant, the only effect of the material

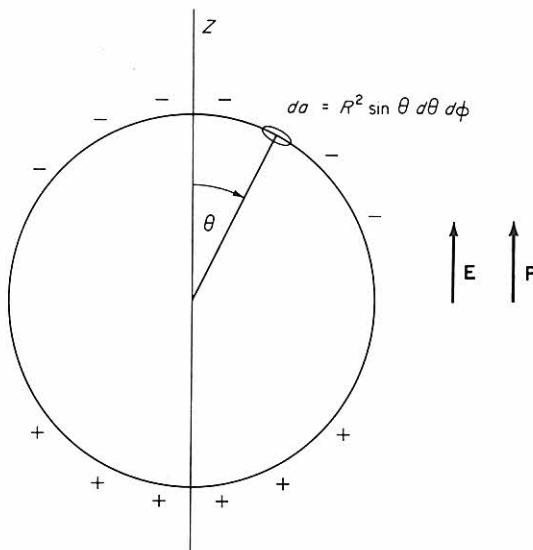


FIG. 16.4. Geometry for finding the field at a molecule. A section through the Z -axis and the area element da is shown.

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outside of the sphere is that produced by the effective surface charge P_n . Of course this is not the surface charge that contributes to \mathbf{E} . At an angle θ from the Z -axis, this effective surface charge is just

$$P_n = -P \cos \theta. \quad (16.32)$$

The surface charge produces a field at the center of the sphere that is in the Z -direction and is given by

$$\begin{aligned} (E_i)_{\text{mat out}} &= - \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta \frac{R^2 P_n \cos \theta}{R^2} \\ &= +2\pi P \int_0^{\pi} d\theta \sin \theta \cos^2 \theta \\ &= 2\pi P \left[-\frac{\cos^3 \theta}{3} \right]_0^{\pi} \\ &= \frac{4\pi}{3} P. \end{aligned} \quad (16.33)$$

For the contribution of the molecules inside the sphere, one considers a detailed sum. Each of these molecules possesses a dipole \mathbf{p} which, by the symmetry of the problem, is in the Z -direction. At the origin the field of these molecules, located at points \mathbf{x}_α , is

$$(\mathbf{E}_i)_{\text{mat in}} = \sum_{\alpha} \left[\frac{3(\mathbf{p} \cdot \mathbf{x}_{\alpha}) \mathbf{x}_{\alpha}}{r_{\alpha}^5} - \frac{\mathbf{p}}{r_{\alpha}^3} \right], \quad (16.34)$$

where α is a serial number for the molecules inside the sphere, the molecule at the origin not included.

Now it is assumed that the material has at least cubic symmetry. In the coordinate system defined by the crystal axes one has

$$\begin{aligned} \sum_{\alpha} \frac{x_{\alpha 1}^2}{r_{\alpha}^5} &= \sum_{\alpha} \frac{x_{\alpha 2}^2}{r_{\alpha}^5} = \sum_{\alpha} \frac{x_{\alpha 3}^2}{r_{\alpha}^5} \\ &= \frac{1}{3} \sum_{\alpha} \frac{x_{\alpha 1}^2 + x_{\alpha 2}^2 + x_{\alpha 3}^2}{r_{\alpha}^5} \\ &= \frac{1}{3} \sum_{\alpha} \frac{1}{r_{\alpha}^3}. \end{aligned} \quad (16.35)$$

Furthermore $\sum_{\alpha} x_{\alpha 1} x_{\alpha 2} / r_{\alpha}^5$ is zero because there are equal numbers of molecules at $\overset{\alpha}{+} x_{\alpha 1}$ and $-x_{\alpha 1}$. Similarly, $\sum_{\alpha} x_{\alpha 1} x_{\alpha 3} / r_{\alpha}^5$ and $\sum_{\alpha} x_{\alpha 2} x_{\alpha 3} / r_{\alpha}^5$ are zero so

$$\sum_{\alpha} \frac{x_{\alpha i} x_{\alpha j}}{r_{\alpha}^5} = \frac{1}{3} \delta_{ij} \sum_{\alpha} \frac{1}{r_{\alpha}^3}. \quad (16.36)$$

In any other set of Cartesian axes given by

$$\begin{aligned} x'_{\alpha i} &= a_{ij} x_{\alpha j}, \\ a_{ij} a_{kj} &= \delta_{ik}, \end{aligned} \quad (16.37)$$

one finds that the result is the same

$$\begin{aligned} \sum_{\alpha} \frac{x'_{\alpha i} x'_{\alpha j}}{r_{\alpha}^5} &= \sum_{\alpha} \frac{a_{ik} x_{\alpha k} a_{jl} x_{\alpha l}}{r_{\alpha}^5} \\ &= a_{ik} a_{jl} \frac{1}{3} \delta_{kl} \sum_{\alpha} \frac{1}{r_{\alpha}^3} \\ &= a_{ik} a_{jk} \frac{1}{3} \sum_{\alpha} \frac{1}{r_{\alpha}^3} \\ &= \frac{1}{3} \delta_{ij} \sum_{\alpha} \frac{1}{r_{\alpha}^3}. \end{aligned} \quad (16.38)$$

Therefore the molecules inside the sphere produce, at the origin, the field

$$\begin{aligned} [(E_i)_{\text{mat in}}]_j &= 3p_k \sum_{\alpha} \frac{x_{\alpha k} x_{\alpha j}}{r_{\alpha}^5} - p_j \sum_{\alpha} \frac{1}{r_{\alpha}^3} \\ &= 3p_k \frac{1}{3} \delta_{kj} \sum_{\alpha} \frac{1}{r_{\alpha}^3} - p_j \sum_{\alpha} \frac{1}{r_{\alpha}^3} \\ &= 0. \end{aligned} \quad (16.39)$$

Therefore the field on each molecule is

$$\mathbf{E}_0 = \mathbf{E} + \frac{4\pi}{3} \mathbf{P}. \quad (16.40)$$

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This field \mathbf{E}_0 is known as the *Lorentz effective field*. It does not apply if the molecules have permanent dipole moments or if the material is anisotropic.

Now the polarization density \mathbf{P} is just $n\mathbf{p}$, that is,

$$\begin{aligned}\mathbf{P} &= n\alpha\mathbf{E}_0 \\ &= n\alpha \left(\mathbf{E} + \frac{4\pi}{3} \mathbf{P} \right).\end{aligned}\quad (16.41)$$

And since \mathbf{P} is $\chi_e \mathbf{E}$ and ϵ is $(1 + 4\pi\chi_e)$,

$$\begin{aligned}\chi_e &= \frac{\epsilon - 1}{4\pi}, \\ \frac{\epsilon - 1}{4\pi} \mathbf{E} &= n\alpha \left(\mathbf{E} + \frac{4\pi}{3} \frac{\epsilon - 1}{4\pi} \mathbf{E} \right), \\ \frac{\epsilon - 1}{4\pi} &= n\alpha \left(\frac{2 + \epsilon}{3} \right).\end{aligned}\quad (16.42)$$

Thus the Clausius–Mossotti relation,

$$\alpha = \frac{3}{4\pi n} \left(\frac{\epsilon - 1}{\epsilon + 2} \right), \quad (16.43)$$

is established.

PROBLEMS

1. What collection of image charges solves the problem pictured in Fig. 16.5?
2. A circular cylinder of dielectric is placed in a uniform electric field, the axis of the cylinder being perpendicular to the field. Find the potential everywhere.
3. A spherical shell of dielectric with inner radius a and outer radius b is placed in a uniform electric field. Find the potential everywhere. Make a sketch showing how the ratio of the field inside to the field outside varies with ϵ for various values of a/b . This is the problem of *dielectric shielding*.

Solution: At large distances from the sphere let the field be \mathbf{E}_0 taken, for convenience, to be in the Z -direction. Choose the origin of coordinates at the center of the sphere (see Fig. 16.6). The general solution of Laplace's equation in spherical coordinates r, θ, ϕ is

$$\varphi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} [A_{lm} r^l + B_{lm} r^{-l-1}] Y_{lm}(\theta, \phi).$$

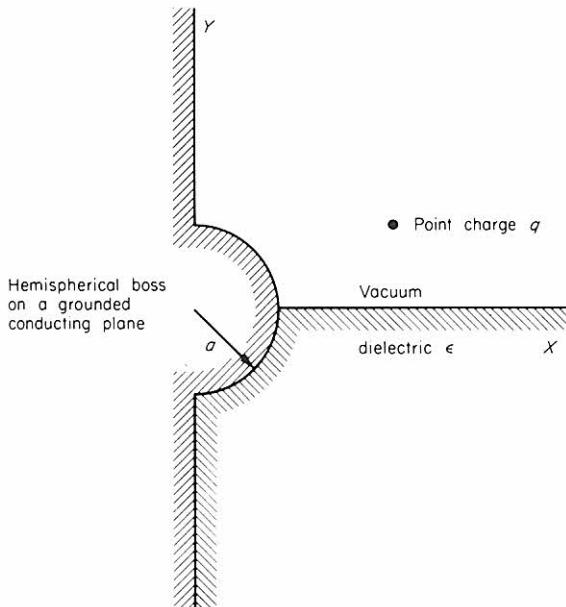


FIG. 16.5. A point charge near a conducting plane with a hemispherical boss. The space outside the conductor is half filled with dielectric.

However, the problem has azimuthal symmetry so only $m = 0$ contributes. Let the three regions be labeled 1, 2, 3 starting from the interior of the shell. In region 3 at large distances the potential becomes $-E_0 r \cos \theta$. Since Y_{10} is just $\sqrt{3/4\pi} P_1(\cos \theta) = \sqrt{3/4\pi} \cos \theta$ it is suggested that only $l = 1, m = 0$ in the sum will contribute in each of the regions. Since the potential will have to be finite at the origin and an

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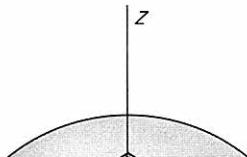
additive constant in the potential in region 3 is of no consequence, a solution may be sought in the form

$$\varphi_1 = A \left(\frac{r}{a} \right) \cos \theta,$$

$$\varphi_2 = B \sqrt{\frac{b}{a}} \left(\frac{r}{b} \right) \cos \theta + C \left(\frac{a}{b} \right) \left(\frac{b}{r} \right)^2 \cos^3 \theta$$

$$= B \sqrt{\frac{a}{b}} \left(\frac{r}{a} \right) \cos \theta + C \left(\frac{b}{a} \right) \left(\frac{a}{r} \right)^2 \cos \theta,$$

$$\varphi_3 = -E_0 b \left(\frac{r}{b} \right) \cos \theta + D \left(\frac{b}{r} \right)^2 \cos \theta.$$



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no true charges on the boundaries and $\nabla \cdot \mathbf{D} = 4\pi\rho$. The first of these conditions leads to

$$B \sqrt{\frac{a}{b}} + C \frac{b}{a} = A,$$

$$B \sqrt{\frac{b}{a}} + C \frac{a}{b} = D - E_0 b.$$

If one lets $\xi = a/b$, then the solution of these equations for B and C takes the form

$$B = \frac{\sqrt{\xi}}{1 - \xi^3} [(D - E_0 b) - A\xi^2],$$

$$C = \frac{1}{1 - \xi^3} [A\xi - (D - E_0 b)\xi^2].$$

Thus the potential in the three regions must be

$$\varphi_1 = A \left(\frac{r}{a} \right) \cos \theta,$$

$$\xi = \frac{r}{a}, \quad r < a, \quad \text{and} \quad \varphi_1(r) \rightarrow 0$$

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which leads to

$$A = \epsilon \left[\frac{\xi[(D - E_0 b) - A\xi^2]}{1 - \xi^3} - \frac{2[A - (D - E_0 b)\xi]}{1 - \xi^3} \right],$$

$$\epsilon \left[\frac{(D - E_0 b) - A\xi^2}{1 - \xi^3} - \frac{2\xi^2[A - (D - E_0 b)\xi]}{1 - \xi^3} \right] = -E_0 b - 2D.$$

By multiplying the first equation by ξ^2 and adding to the second, and multiplying the second by ξ and adding to the first, one finds

$$\epsilon[(D - E_0 b) - A\xi^2] = -3E_0 b - 2(D - E_0 b) - A\xi^2,$$

$$2\epsilon[A - (D - E_0 b)\xi] = [-3E_0 b - 2(D - E_0 b)]\xi - A.$$

Then, after solving for $(D - E_0 b)$ in the first equation,

$$D - E_0 b = \frac{-3E_0 b + A\xi^2(\epsilon - 1)}{\epsilon + 2},$$

one can solve for A in the second,

$$A = -(E_0 b) \frac{9\epsilon\xi}{(2\epsilon + 1)(\epsilon + 2) - 2\xi^3(\epsilon - 1)^2}.$$

Finally D is given by

$$D = E_0 b + (D - E_0 b)$$

$$= E_0 b \frac{(\epsilon - 1)(2\epsilon + 1)(1 - \xi^3)}{(2\epsilon + 1)(\epsilon + 2) - 2\xi^3(\epsilon - 1)^2}.$$

The potential is determined everywhere by these values of A and D .

over, the function is positive for positive ϵ since ξ must lie in the range $0 < \xi < 1$. The ratio, as a function of ϵ , has an extremum when

$$\begin{aligned} \frac{d}{d\epsilon} \left[\frac{9\epsilon}{(2\epsilon + 1)(\epsilon + 2) - 2\xi^3(\epsilon - 1)^2} \right] \\ = \frac{9}{(2\epsilon + 1)(\epsilon + 2) - 2\xi^3(\epsilon - 1)^2} - \frac{9\epsilon[4\epsilon + 5 - 4\xi^3(\epsilon - 1)]}{[(2\epsilon + 1)(\epsilon + 2) - 2\xi^3(\epsilon - 1)^2]^2} \\ = 0, \end{aligned}$$

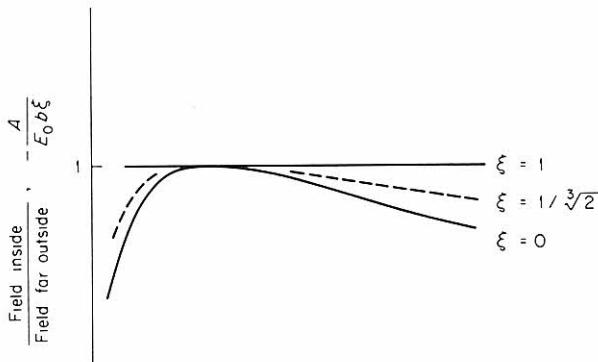
that is, when

$$9[2\epsilon^2 + 5\epsilon + 2 - 2\xi^3(\epsilon - 1)^2] = 9\epsilon[4\epsilon + 5 - 4\xi^3(\epsilon - 1)],$$

or

$$(\epsilon^2 - 1)(1 - \xi^3) = 0.$$

The extremum, at $\epsilon = 1$, is obviously a maximum. The function is still positive as ϵ tends to zero since the denominator tends to $2(1 - \xi^3)$. (Note the curves plotted in Fig. 16.7.)



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Find N for a sphere, a thin rod parallel to the field, and a thin disk normal to the field.

Solution: The potential inside a dielectric sphere of radius b in an external field \mathbf{E}_0 that is directed in the positive Z -direction is

$$\varphi = -\frac{3E_0}{\epsilon + 2} r \cos \theta,$$

in spherical coordinates. This result can be found by applying the method used in Problem 3 or by simply specializing φ_2 found there to the case $\xi = 0$. The field inside is therefore $3\mathbf{E}_0/(\epsilon + 2)$. The electric displacement \mathbf{D} and polarization \mathbf{P} are therefore given by

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} \\ &= \frac{3\epsilon}{\epsilon + 2} \mathbf{E}_0, \\ \mathbf{P} &= \frac{\mathbf{D} - \mathbf{E}}{4\pi} \\ &= \frac{3}{4\pi} \frac{\epsilon - 1}{\epsilon + 2} \mathbf{E}_0.\end{aligned}$$

However $\mathbf{E}_0 - \mathbf{E}$ is just $[1 - 3/(\epsilon + 2)]\mathbf{E}_0$, so that

$$\begin{aligned}\mathbf{E}_0 - \mathbf{E} &= \frac{\epsilon - 1}{\epsilon + 2} \mathbf{E}_0 \\ &\quad \left. \right\} \\ &= \frac{4\pi}{3} \mathbf{P},\end{aligned}$$

Therefore, the depolarization factor for the dielectric sphere is $4\pi/3$.

For a thin rod parallel to the field, the field inside is the same as that outside since the tangential component of \mathbf{E} is continuous across the boundary. For an infinitely long cylinder parallel to \mathbf{E}_0 , the field would

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The polarization is $(\mathbf{D} - \mathbf{E})/4\pi$ or $(\epsilon - 1)\mathbf{E}/4\pi$ or just $(\epsilon - 1)\mathbf{E}_0/4\pi\epsilon$. Therefore

$$\mathbf{E}_0 - \mathbf{E} = \frac{\epsilon - 1}{\epsilon} \mathbf{E}_0 = 4\pi\mathbf{P},$$

and the depolarization factor is 4π .

5. A conducting sphere of radius a carries charge q . The dielectric constant outside the sphere varies with the radial distance from the center of the sphere according to

$$\epsilon = 1 + \frac{b}{r}.$$

- (a) Find the potential in the region outside the sphere.
- (b) What will the polarization surface charge density on the dielectric surface at $r = a$ be?

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