derivation for the potential of a point charge above thin layer of uniform dielectric above an anisotropic substrate bulk material

$$\boldsymbol{E} = -\nabla\phi$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\boldsymbol{D} = \varepsilon_{\parallel}\boldsymbol{E}_{\parallel} + \varepsilon_{\perp}\boldsymbol{E}_{\perp}$$

$$\boldsymbol{E}_{\parallel} = \frac{\partial\phi}{\partial R}$$

$$\boldsymbol{E}_{\perp} = \frac{\partial\phi}{\partial z}$$

$$\boldsymbol{D} = \varepsilon_{\parallel}\frac{\partial\phi}{\partial R} + \varepsilon_{\perp}\frac{\partial\phi}{\partial z}$$

$$\varepsilon_{\parallel}\frac{\partial^2\phi}{\partial R^2} + \varepsilon_{\perp}\frac{\partial^2\phi}{\partial z^2} = -\rho$$

$$\frac{\partial^2\phi_1}{\partial R^2} + \frac{\partial^2\phi_1}{\partial z^2} = -\frac{\rho}{\varepsilon_0\varepsilon}$$

$$\frac{\partial^2\phi_1}{\partial R^2} + \frac{\partial^2\phi_1}{\partial z^2} = -\frac{q}{\varepsilon_0\varepsilon}\left(\frac{\delta(R)\delta(z-z_0)}{2\pi R}\right)$$

$$\frac{\partial^2\phi_2}{\partial R^2} + \frac{\partial^2\phi_2}{\partial z^2} = 0$$

$$\varepsilon_{\parallel}\frac{\partial^2\phi_3}{\partial R^2} + \varepsilon_{\perp}\frac{\partial^2\phi_3}{\partial z^2} = 0$$

using the furrier transform

$$\phi(R,z) = \int e^{iKR}\hat{\phi}(K,z)dK$$

$$\frac{\partial^2\hat{\phi}_1}{\partial z^2} - K^2\hat{\phi}_1 = -\frac{q}{\varepsilon_0\varepsilon}\frac{\delta(z-z_0)}{2\pi}$$

$$\frac{\partial^2\hat{\phi}_2}{\partial z^2} - K^2\hat{\phi}_2 = 0$$

$$\frac{\partial^2\hat{\phi}_1}{\partial z^2} - \frac{\varepsilon_{\parallel}}{\varepsilon_{\perp}}K^2\hat{\phi}_3 = 0$$

apply boundary conditions $\hat{\phi}_1 \to 0$ as $z \to \infty$ and $\hat{\phi}_3 \to 0$ as $z \to -\infty$

$$\hat{\phi}_1 = De^{-Kz} + \frac{qe^{-K|z-z_0|}}{4\pi\varepsilon_0\varepsilon K}$$

$$\hat{\phi}_2 = Be^{Kz} + Ce^{-Kz}$$

$$\hat{\phi}_3 = Ae^{\sqrt{\frac{\varepsilon_{\parallel}}{\varepsilon_{\perp}}}Kz}$$

applying the conditions $\hat{\Phi}_{1(z=0)} = \hat{\Phi}_{2(z=0)}$, $\hat{\Phi}_{2(z=-h)} = \hat{\Phi}_{3(z=-h)}$

and $\varepsilon \frac{\partial\hat{\Phi}_1}{\partial z}_{(z=0)} = \varepsilon_L \frac{\partial\hat{\Phi}_2}{\partial z}_{(z=0)}$, $\varepsilon_{\perp}\frac{\partial\hat{\Phi}_3}{\partial z}_{(z=-h)} = \varepsilon_L \frac{\partial\hat{\Phi}_2}{\partial z}_{(z=-h)}$

$$Be^{K(-h)} + Ce^{-K(-h)} = Ae^{\sqrt{\frac{\varepsilon_{\parallel}}{\varepsilon_{\perp}}}K(-h)} \qquad (1)$$

$$\varepsilon_L BKe^{K(-h)} - \varepsilon_L KCe^{-K(-h)} = \varepsilon_\perp AK\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}e^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)} \qquad (2)$$

$$B + C = D + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K} \qquad (3)$$

$$\varepsilon_L BK - \varepsilon_L KC = -DK\varepsilon + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0} \qquad (4)$$

(2)

$$Be^{K(-h)} - Ce^{-K(-h)} = \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}Ae^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)} \qquad (5)$$

(1)+(5)

$$2Be^{K(-h)} = \left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)Ae^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)} \qquad (6)$$

(1)-(5)

$$2Ce^{-K(-h)} = \left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)Ae^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)} \qquad (7)$$

(4)

$$B - C = -\frac{\varepsilon}{\varepsilon_L}D + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon_L K} \qquad (8)$$

(8)+(3)

$$2B = \left(1 - \frac{\varepsilon}{\varepsilon_L}\right)D + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0 K}\left(\frac{1}{\varepsilon_L} + \frac{1}{\varepsilon}\right) \qquad (9)$$

(3)-(8)

$$2C = \left(1 + \frac{\varepsilon}{\varepsilon_L}\right)D + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0 K}\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_L}\right) \qquad (10)$$

sub 9 in to 6

$$\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)D + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0 K}\left(\frac{1}{\varepsilon_L} + \frac{1}{\varepsilon}\right) = \left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)Ae^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)}e^{Kh}$$

$$\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)D + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right) = \left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)Ae^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)}e^{Kh}$$

$$\frac{\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)D + \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)e^{Kh}} = Ae^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)} \qquad (11)$$

sub 10 in to 7

$$\left(1+\frac{\varepsilon}{\varepsilon_L}\right)D+\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)=\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)Ae^{\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}K(-h)}e^{-Kh} \qquad (12)$$

sub 11 in to 12

$$\left(1+\frac{\varepsilon}{\varepsilon_L}\right)D+\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)=\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)\frac{\left(1-\frac{\varepsilon}{\varepsilon_L}\right)D+\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)e^{Kh}}e^{-Kh}$$

$$\left(1+\frac{\varepsilon}{\varepsilon_L}\right)D+\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)=\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\left(1-\frac{\varepsilon}{\varepsilon_L}\right)D+\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)\right)e^{-2Kh}$$

$$\left(1+\frac{\varepsilon}{\varepsilon/_L}\right)D-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)De^{-2Kh}=\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)\right)e^{-2Kh}-\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)$$

$$D\left(\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}\right)=\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)\right)e^{-2Kh}-\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)$$

$$D=\frac{\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)\right)e^{-2Kh}-\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)}{\left(\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}\right)}$$

let h=0

$$D=\frac{\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)\right)-\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)}{\left(\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)\right)}$$

$$D=\frac{\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)-\left(1-\frac{\varepsilon}{\varepsilon_L}\right)}{\left(\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)\right)}\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}$$

$$D = \frac{\frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\frac{\varepsilon}{\varepsilon_L} + \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)} - 1 + \frac{\varepsilon}{\varepsilon_L}}{\left(\frac{\varepsilon}{\varepsilon_L} + 1 + \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\frac{\varepsilon}{\varepsilon_L} - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\right)}\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}$$

$$D = \frac{\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)}\frac{\varepsilon}{\varepsilon_L} + \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)} - 1 + \frac{\varepsilon}{\varepsilon_L}}{\left(\frac{\varepsilon}{\varepsilon_L} + 1 + \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)}\frac{\varepsilon}{\varepsilon_L} - \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\perp\varepsilon_\parallel}\right)}\right)}\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}$$

$$D = \frac{\varepsilon - \sqrt{\varepsilon_\perp\varepsilon_\parallel}}{\varepsilon + \sqrt{\varepsilon_\perp\varepsilon_\parallel}}\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}$$

which is the result which is expected when the layer is no longer present

let $h \to \infty$

$$D = \frac{\frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right)\right)e^{-2K(\infty)} - \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)}{\left(\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2K(\infty)}\right)}$$

$$D = \frac{-\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)}{\left(\left(1 + \frac{\varepsilon}{\varepsilon_L}\right)\right)}$$

$$D = \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\frac{\left(-1 + \frac{\varepsilon}{\varepsilon_L}\right)}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right)}$$

$$D = \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\frac{(\varepsilon - \varepsilon_L)}{(\varepsilon + \varepsilon_L)}$$

result as expected when the layer depth goes to infinity

let $\varepsilon_L = \varepsilon$

$$D = \frac{\frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon} + 1\right)\right)e^{-2Kh} - \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1 - \frac{\varepsilon}{\varepsilon}\right)}{\left(\left(1 + \frac{\varepsilon}{\varepsilon}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon}\right)e^{-2Kh}\right)}$$

$$D = \frac{2\dfrac{\left(1-\dfrac{\varepsilon_\perp}{\varepsilon}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\dfrac{\varepsilon_\perp}{\varepsilon}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\dfrac{qe^{-Kz_0-2Kh}}{4\pi\varepsilon_0\varepsilon K}}{2}$$

$$D = \frac{\left(1-\dfrac{\varepsilon_\perp}{\varepsilon}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\dfrac{\varepsilon_\perp}{\varepsilon}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\frac{qe^{-Kz_0-2Kh}}{4\pi\varepsilon_0\varepsilon K}$$

only issue here is that h is 2 times what is expected

let $\varepsilon' = \varepsilon_L, \varepsilon_\perp, \varepsilon_\parallel$

$$D = \frac{\dfrac{\left(1-\dfrac{\varepsilon'}{\varepsilon'}\sqrt{\dfrac{\varepsilon'}{\varepsilon'}}\right)}{\left(1+\dfrac{\varepsilon'}{\varepsilon'}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\dfrac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\dfrac{\varepsilon}{\varepsilon'}+1\right)\right)e^{-2Kh} - \dfrac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\dfrac{\varepsilon}{\varepsilon'}\right)}{\left(\left(1+\dfrac{\varepsilon}{\varepsilon'}\right) - \dfrac{\left(1-\dfrac{\varepsilon'}{\varepsilon'}\sqrt{\dfrac{\varepsilon'}{\varepsilon'}}\right)}{\left(1+\dfrac{\varepsilon'}{\varepsilon'}\sqrt{\dfrac{\varepsilon'}{\varepsilon'}}\right)}\left(1-\dfrac{\varepsilon}{\varepsilon'}\right)e^{-2Kh}\right)}$$

$$D = \frac{\dfrac{(1-1)}{(1+1)}\left(\dfrac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\dfrac{\varepsilon}{\varepsilon'}+1\right)\right)e^{-2Kh} - \dfrac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\dfrac{\varepsilon}{\varepsilon'}\right)}{\left(\left(1+\dfrac{\varepsilon}{\varepsilon'}\right) - \dfrac{(1-1)}{(1+1)}\left(1-\dfrac{\varepsilon}{\varepsilon'}\right)e^{-2Kh}\right)}$$

$$D = \frac{-\dfrac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\dfrac{\varepsilon}{\varepsilon'}\right)}{\left(1+\dfrac{\varepsilon}{\varepsilon'}\right)}$$

$$D = \frac{\varepsilon-\varepsilon'}{\varepsilon+\varepsilon'}\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}$$

all trivial cases return the expected results

$$D = \frac{\dfrac{\left(1-\dfrac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\dfrac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\dfrac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\dfrac{\varepsilon}{\varepsilon_L}+1\right)\right)e^{-2Kh} - \dfrac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\dfrac{\varepsilon}{\varepsilon_L}\right)}{\left(\left(1+\dfrac{\varepsilon}{\varepsilon_L}\right) - \dfrac{\left(1-\dfrac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\dfrac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\dfrac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\dfrac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}\right)} \tag{14}$$

sub 14 in to $\hat{\phi}_1$

$$\hat{\phi}_1 = \frac{\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)\right)e^{-2Kh} - \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)}{\left(\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}\right)} e^{-Kz} + \frac{qe^{-K|z-z_0|}}{4\pi\varepsilon_0\varepsilon K}$$

$$\hat{\phi}_1 = \frac{\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)\right)e^{-2Kh}}{\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} e^{-Kz} - \frac{qe^{-Kz_0}}{4\pi\varepsilon_0\varepsilon K}\frac{\left(1-\frac{\varepsilon}{\varepsilon_L}\right)}{\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} e^{-Kz} + \frac{qe^{-K|z-z_0|}}{4\pi\varepsilon_0\varepsilon K}$$

let $k_0 = \frac{q}{4\pi\varepsilon_0\varepsilon}$

$$\hat{\phi}_1 = \frac{\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)}{\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} k_0\frac{e^{-K(z_0+z+2h)}}{K} - k_0\frac{e^{-K(z+z_0)}}{K}\frac{\left(1-\frac{\varepsilon}{\varepsilon_L}\right)}{\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} + k_0\frac{e^{-K|z-z_0|}}{K}$$

test cases

let $h = 0$

$$\hat{\phi}_1 = \frac{\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L}+1\right)}{\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)} k_0\frac{e^{-K(z_0+z)}}{K} - k_0\frac{e^{-K(z+z_0)}}{K}\frac{\left(1-\frac{\varepsilon}{\varepsilon_L}\right)}{\left(1+\frac{\varepsilon}{\varepsilon_L}\right)-\frac{\left(1-\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1+\frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1-\frac{\varepsilon}{\varepsilon_L}\right)} + k_0\frac{e^{-K|z-z_0|}}{K}$$

$$\hat{\phi}_1 = \frac{\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)}(\varepsilon + \varepsilon_L) - (\varepsilon_L - \varepsilon)}{(\varepsilon_L + \varepsilon) - \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)}(\varepsilon_L - \varepsilon)} k_0 \frac{e^{-K(z_0+z)}}{K} + k_0 \frac{e^{-K|z-z_0|}}{K}$$

$$\hat{\phi}_1 = \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)(\varepsilon + \varepsilon_L) - (\varepsilon_L - \varepsilon)\left(\varepsilon_L + \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)}{(\varepsilon_L + \varepsilon)\left(\varepsilon_L + \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right) - \left(\varepsilon_L - \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)(\varepsilon_L - \varepsilon)} k_0 \frac{e^{-K(z_0+z)}}{K} + k_0 \frac{e^{-K|z-z_0|}}{K}$$

$$\hat{\phi}_1 = \frac{\varepsilon\varepsilon_L + \varepsilon_L^2 - \varepsilon\sqrt{\varepsilon_\perp \varepsilon_\parallel} - \varepsilon_L\sqrt{\varepsilon_\perp \varepsilon_\parallel} + \varepsilon\varepsilon_L - \varepsilon_L^2 + \varepsilon\sqrt{\varepsilon_\perp \varepsilon_\parallel} - \varepsilon_L\sqrt{\varepsilon_\perp \varepsilon_\parallel}}{\varepsilon\varepsilon_L + \varepsilon_L^2 + \varepsilon\sqrt{\varepsilon_\perp \varepsilon_\parallel} + \varepsilon_L\sqrt{\varepsilon_\perp \varepsilon_\parallel} + \varepsilon\varepsilon_L - \varepsilon_L^2 - \varepsilon\sqrt{\varepsilon_\perp \varepsilon_\parallel} + \varepsilon_L\sqrt{\varepsilon_\perp \varepsilon_\parallel}} k_0 \frac{e^{-K(z_0+z)}}{K} + k_0 \frac{e^{-K|z-z_0|}}{K}$$

$$\hat{\phi}_1 = \frac{2\varepsilon_L\left(\varepsilon - \sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)}{2\varepsilon_L\left(\varepsilon\varepsilon_L + \varepsilon_L\sqrt{\varepsilon_\perp \varepsilon_\parallel}\right)} k_0 \frac{e^{-K(z_0+z)}}{K} + k_0 \frac{e^{-K|z-z_0|}}{K}$$

let $h = \infty$

$$\hat{\phi}_1 = \frac{\frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right)}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} k_0 \frac{e^{-K(z_0+z+2h)}}{K}$$

$$- k_0 \frac{e^{-K(z+z_0)}}{K} \frac{\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} + k_0 \frac{e^{-K|z-z_0|}}{K}$$

let $\varepsilon_L = \varepsilon$

$$\hat{\phi}_1 = \frac{\frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right)}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} k_0 \frac{e^{-K(z_0+z+2h)}}{K}$$

$$- k_0 \frac{e^{-K(z+z_0)}}{K} \frac{\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} + k_0 \frac{e^{-K|z-z_0|}}{K}$$

let $\varepsilon' = \varepsilon_L, \varepsilon_\perp, \varepsilon_\parallel$

$$\hat{\phi}_1 = \frac{\frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right)}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} k_0 \frac{e^{-K(z_0+z+2h)}}{K}$$

$$- k_0 \frac{e^{-K(z+z_0)}}{K} \frac{\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} + k_0 \frac{e^{-K|z-z_0|}}{K}$$

therefore integral is

$$\phi_1 = k_0 \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right) \int_0^\infty \frac{1}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} \frac{e^{-K(z_0+z+2h)}}{K} J_0(KR)K\,dK$$

$$- k_0 \left(1 - \frac{\varepsilon}{\varepsilon_L}\right) \int_0^\infty \frac{e^{-K(z+z_0)}}{K} \frac{1}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} J_0(KR)K\,dK$$

$$+ k_0 \int_0^\infty \frac{e^{-K(z-z_0)}}{K} J_0(KR)\,K dK$$

$$\phi_1 = k_0 \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(\frac{\varepsilon}{\varepsilon_L} + 1\right) \int_0^\infty \frac{e^{-K(z_0+z+2h-iR)}}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}}\,dK$$

$$- k_0 \left(1 - \frac{\varepsilon}{\varepsilon_L}\right) \int_0^\infty \frac{e^{-K(z+z_0-iR)}}{\left(1 + \frac{\varepsilon}{\varepsilon_L}\right) - \frac{\left(1 - \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}{\left(1 + \frac{\varepsilon_\perp}{\varepsilon_L}\sqrt{\frac{\varepsilon_\parallel}{\varepsilon_\perp}}\right)}\left(1 - \frac{\varepsilon}{\varepsilon_L}\right)e^{-2Kh}} dK + k_0 \frac{1}{\sqrt{R^2 + (z - z_0)^2}}$$

$$\phi_1 = k_0 \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel \varepsilon_\perp}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel \varepsilon_\perp}\right)} \int_0^\infty \frac{e^{-K(z_0+z+2h-iR)}}{1 - \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel \varepsilon_\perp}\right)(\varepsilon_L - \varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel \varepsilon_\perp}\right)(\varepsilon_L + \varepsilon)} e^{-2Kh}}\,dK$$

$$- k_0 \frac{(\varepsilon_L - \varepsilon)}{(\varepsilon_L + \varepsilon)} \int_0^\infty \frac{e^{-K(z+z_0-iR)}}{1 - \frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel \varepsilon_\perp}\right)(\varepsilon_L - \varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel \varepsilon_\perp}\right)(\varepsilon_L + \varepsilon)} e^{-2Kh}} dK + k_0 \frac{1}{\sqrt{R^2 + (z - z_0)^2}}$$

using result for table of intergrals

$$\int_0^\infty \frac{e^{-qx}}{1-ae^{-px}}dx = \Sigma_{k=0}^{\infty}\frac{a^k}{q+kp}, \qquad [0<a<1]$$

$$\phi_1 = k_0\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k}{z_0+z+2h(1+n)-iR}\right) - k_0\frac{(\varepsilon_L-\varepsilon)}{(\varepsilon_L+\varepsilon)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k}{z_0+z+2nh-iR}\right) + k_0\frac{1}{\sqrt{R^2+(z-z_0)^2}}$$

$$\phi_1 = k_0\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k(z_0+z+2h(1+n)+iR)}{\left(z_0+z+2h(1+n)\right)^2+R^2}\right) - k_0\frac{(\varepsilon_L-\varepsilon)}{(\varepsilon_L+\varepsilon)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k(z_0+z+2nh+iR)}{(z_0+z+2nh)^2+R^2}\right) + k_0\frac{1}{\sqrt{R^2+(z-z_0)^2}}$$

$$\phi_1 = k_0\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k}{\sqrt{\left(z_0+z+2h(1+n)\right)^2+R^2}}\right) - k_0\frac{(\varepsilon_L-\varepsilon)}{(\varepsilon_L+\varepsilon)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k}{\sqrt{(z_0+z+2nh)^2+R^2}}\right) + k_0\frac{1}{\sqrt{R^2+(z-z_0)^2}}$$

check sigh and derivation again first term looks right along with last term

let h=0

$$\phi_1 = k_0\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k}{\sqrt{(z_0+z)^2+R^2}}\right) - k_0\frac{(\varepsilon_L-\varepsilon)}{(\varepsilon_L+\varepsilon)}\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k}{\sqrt{(z_0+z)^2+R^2}}\right) + k_0\frac{1}{\sqrt{R^2+(z-z_0)^2}}$$

$$\phi_1 = \left(k_0\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)} - k_0\frac{(\varepsilon_L-\varepsilon)}{(\varepsilon_L+\varepsilon)}\right)\Sigma_{n=0}^{\infty}\left(\frac{\left(\frac{\left(\varepsilon_L - \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L-\varepsilon)}{\left(\varepsilon_L + \sqrt{\varepsilon_\parallel\varepsilon_\perp}\right)(\varepsilon_L+\varepsilon)}\right)^k}{\sqrt{(z_0+z)^2+R^2}}\right) + k_0\frac{1}{\sqrt{R^2+(z-z_0)^2}}$$

let $h = \infty$

$$\phi_1 = k_0 \frac{(\varepsilon_L - \sqrt{\varepsilon_\parallel \varepsilon_\perp})}{(\varepsilon_L + \sqrt{\varepsilon_\parallel \varepsilon_\perp})} \Sigma_{n=0}^{\infty} \left(\frac{\left(\frac{(\varepsilon_L - \sqrt{\varepsilon_\parallel \varepsilon_\perp})(\varepsilon_L - \varepsilon)}{(\varepsilon_L + \sqrt{\varepsilon_\parallel \varepsilon_\perp})(\varepsilon_L + \varepsilon)} \right)^k}{\sqrt{\left(z_0 + z + 2h(1+n)\right)^2 + R^2}} \right) - k_0 \frac{(\varepsilon_L - \varepsilon)}{(\varepsilon_L + \varepsilon)} \Sigma_{n=0}^{\infty} \left(\frac{\left(\frac{(\varepsilon_L - \sqrt{\varepsilon_\parallel \varepsilon_\perp})(\varepsilon_L - \varepsilon)}{(\varepsilon_L + \sqrt{\varepsilon_\parallel \varepsilon_\perp})(\varepsilon_L + \varepsilon)} \right)^k}{\sqrt{(z_0 + z + 2nh)^2 + R^2}} \right) + k_0 \frac{1}{\sqrt{R^2 + (z - z_0)^2}}$$

let $\varepsilon' = \varepsilon_L, \varepsilon_\parallel, \varepsilon_\perp$

$$\phi_1 = k_0 \frac{(\varepsilon' - \varepsilon')}{(\varepsilon' + \varepsilon')} \Sigma_{n=0}^{\infty} \left(\frac{\left(\frac{(\varepsilon' - \varepsilon')(\varepsilon' - \varepsilon)}{(\varepsilon' + \varepsilon')(\varepsilon' + \varepsilon)} \right)^k}{\sqrt{\left(z_0 + z + 2h(1+n)\right)^2 + R^2}} \right) - k_0 \frac{(\varepsilon' - \varepsilon)}{(\varepsilon' + \varepsilon)} \Sigma_{n=0}^{\infty} \left(\frac{\left(\frac{(\varepsilon' - \varepsilon')(\varepsilon' - \varepsilon)}{(\varepsilon' + \varepsilon')(\varepsilon' + \varepsilon)} \right)^k}{\sqrt{(z_0 + z + 2nh)^2 + R^2}} \right) + k_0 \frac{1}{\sqrt{R^2 + (z - z_0)^2}}$$