A Figure Eight
And other Interesting Solutions
to the N-Body Problem

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1 History of the n-Body Problem

The n-body problem was a famous unsolved problem in the 19$^{\text{th}}$ century. Newton had solved the two-body case, and certain solutions to the three-body version were known, but no general solution. In this case "solving" the problem requires finding a full general solution, which is, as it turns out, impossible. In particular, King Oscar II of Sweden offered a prize for a solution to the n-body problem:

> Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

This problem was not solved until much later, however Poincaré derived partial solutions for the $n = 3$ case that were sufficiently important that he was awarded the prize. The problem has since been solved for $n = 3$ and generalized to $n > 3$.

In particular the solution of interest that I will discuss forms a figure eight where three particles chase each other around the same path.

1.1 The Simple Case: n=2

Applying Newton's equations in order to explicitly find the differential equation to be solved, we get something of the form

$$m_i \frac{d^2 x_i}{dt^2} = -\sum_{j \neq i} \frac{m_i m_j (x_i - x_j)}{r_{ij}^3}, \qquad i = 1, \ldots, n$$

In the case of $n = 2$ the solution is fully solved, as proven by Newton the solutions take the form of conic sections in a generalized variable $x = x_1 - x_2$. The three-body problem, on the other hand

1.2 Some Solutions to n=3

It is impossible to describe fully a general solution to the three-body variant of the problem. So, we impose certain limitations on the set of solutions we wish to find. In particular, we assume that the solution is periodic with period T, that is, $x_i(t) = x_i(t+T)$. There are two very intuitive solutions in this case to the problem. The first was discovered by Euler and is where all three bodies are collinear and orbit the center of mass in such a way that this structure is maintained.

The second simple solution discovered by Lagrange is true more generally, in the case of different masses. This is the case where the three bodies form an equilateral triangle about the center of mass, and they each follow the same ellipse which is a solution to the equivalent two-body problem. Thus the equilateral triangle between the three is maintained.

The third obvious solution is most pertinent to astronomy, which is the case where two of the masses are close together orbiting one another, and that system orbits a third, much more massive particle. For example, a star-planet-moon system.

2 Basics of Lagrangian Mechanics

In order to derive the figure eight we assume some basic knowledge of Lagrangian mechanics: for the case of particles moving in space interacting only through gravitational forces, the Kinetic and Potential energies are given by

$$K = \frac{1}{2}\sum_i m_i ||\dot{x}_i||^2, \qquad -U = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$$

and the Lagrangian $\mathcal{L} = K + U$. Then the action of a path x is

$$A(x) \equiv \int_0^T \mathcal{L}(x(t), \dot{x}(t))\, dt.$$

The action of a path may be thought of as a generalized distance travelled (although not explicitly as such; we are integrating Energy along the length of the path). In the simple case of a single particle the energy is constant, and we literally get the length of the path.

The main principles to be used in derivation of the figure eight are that the Lagrangian satisfies the Euler-Reimann equation,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0.$$

And the second principle to be used is called the *Principle of Least Action*, which states that a solution x minimizes the action formula.

To give some physical grasp of why these conditions at least ought to be true, one can think of a simple example. For the case of a ball rolling down a slope, one would expect it to follow the path of the slope itself rather than some other path. As it turns out, the path of the slope does minimize the action given initial conditions.

3 Details on the Euler and Lagrange Solutions

3.1 Euler's Solution

As previously mentioned, for Euler's solution we start with three collinear bodies that will orbit each other in a plane containing all three. Furthermore one body is assumed to be much less massive than the other two, and we can therefore treat those as, to first order of approximation, nonmoving. The following proof is attributed to Euler and found in [3]. The two centers of force (namely the two more massive objects) lie in line with the origin, call these points $\pm a$. Thus the gravitational field at a point (x, y) is given by

$$U(x,y) = \frac{-\mu_1}{\sqrt{(x-a)^2+y^2}} + \frac{-\mu_2}{\sqrt{(x+a)^2+y^2}}$$

where μ_1 and μ_2 are proportionality constants; positive in the case of gravity but more general in the case of a coulomb force.

Since we know from [1] that the solutions to the Euler problem are superpositions of the known elliptical solutions to the 2-body problem, we do a coordinate transform into elliptical coordinates: $x = a\cosh\xi\cos\eta$ and $y = a\sinh\xi\sin\eta$. So we have the potential

$$\begin{aligned} U(\xi,\eta) &= \frac{-\mu_1}{a(\cosh\xi - \cos\eta)} + \frac{-\mu_2}{a(\cosh\xi + \cos\eta} \\ &= \frac{-\mu_1(\cosh\xi + \cos\eta) - \mu_2(\cosh\xi - \cos\eta)}{a(\cosh^2\xi - \cos^2\eta)} \end{aligned}$$

Then with the kinetic energy term defined exactly as expected,

$$K = \frac{ma^2}{2}\left(\cosh^2\xi - \cos^2\eta\right)\left(\dot{\xi}^2 + \dot{\eta}^2\right),$$

this become s Liouville system and we can take the general form of the solutions to Liouville integrals [4], which gives

$$\frac{1}{2}ma^2\left(\cosh^2\xi - \cos^2\eta\right)^2\dot{\xi}^2 = E\cosh^2\xi + \left(\frac{\mu_1+\mu_2}{a}\right)\cosh\xi - \gamma$$

$$\frac{1}{2}ma^2\left(\cosh^2\xi - \cos^2\eta\right)^2\dot{\eta}^2 = -E\cos^2\eta + \left(\frac{\mu_1-\mu_2}{a}\right)\cos\eta + \gamma$$

Finally if we introduce a generalized parameter u to these formulae and integrate, we have an explicit expression for parameterized paths that solve Euler's restriction of the three-body problem in terms of each elliptical variable separately.

$$u = \int \frac{d\xi}{\sqrt{E\cosh^2\xi + \left(\frac{\mu_1+\mu_2}{a}\right)\cosh\xi - \gamma}} = \int \frac{d\eta}{\sqrt{-E\cos^2\eta + \left(\frac{\mu_1-\mu_2}{a}\right)\cos\eta + \gamma}}.$$

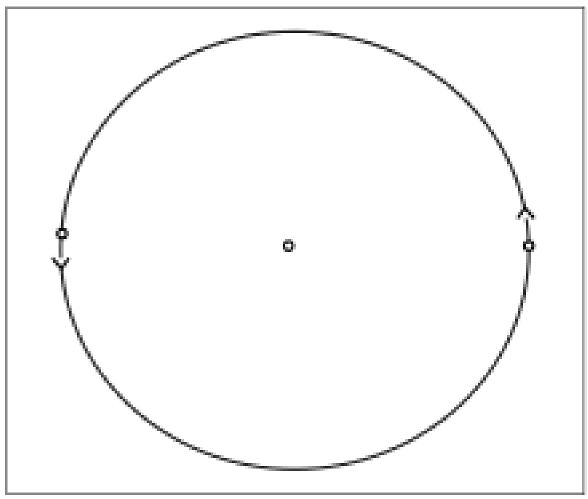

Figure 1: A depiction of the Euler solution.

3.2 Lagrange's Solution

The other known set of solutions is much easier to derive . We take three equal masses arranged in an equilateral triangle, and without loss of generality examining one mass in particular, approximate the other two by a single, larger particle located at their center of mass. We can then apply the two-body solutions. Where they are identical for all particles (i.e. the initial conditions are identical under rotation by $2\pi/3$ about the origin), then we have a stable solution and the equilateral triangle geometry is maintained.

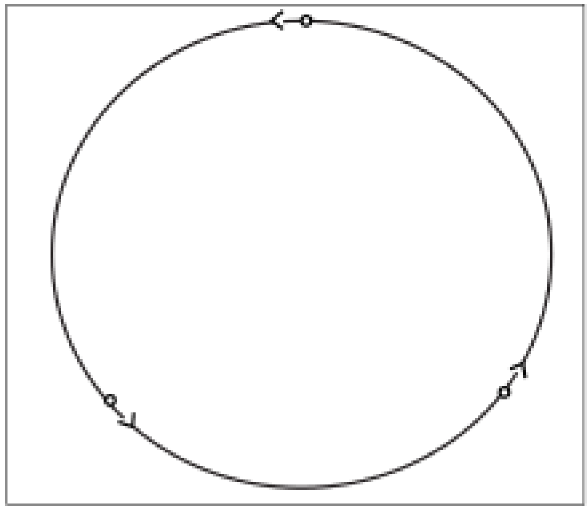

Figure 2: A depiction of the Lagrange solution.

4 The Figure Eight

An unexpected and interesting solution to the three-body problem, is when particles of equal masses all follow a figure eight orbit, each tracing out the same path about the center of mass.

The most surprising aspect of this solution is that it is stable under small perturbations. This is unexpected for several reasons, since very few stable solutions are actually known, and even fewer for the equal-mass case; the Euler solutions are never stable and the Lagrange solutions are only stable in certain cases where one mass is much greater than the other two. Furthermore, the method to derive an expression for the figure eight follows the Principle of Least Action, and involves minimizing the path the orbit takes. This is indeed a process that generally yields unstable orbits.

4.1 Stability

It turns out that the figure eight is stable, not quite in the normal sense, but a slightly different KAM stability [1]. However for our purposes, this effectively is the same, meaning that all parameters (masses, or more importantly mass ratios, and initial angular momenta) can be perturbed, resulting in an almost-periodic orbit that does not diverge.

The figure eight was proven to be a stable solution to the three-body problem through numerical methods by Simó [2] in 2000. This is the most fascinating aspect of the result because it means that we could, in theory, find a trinary star system orbiting in a figure eight. This is, however extremely improbable. Another calculation in [2] gives that the actual probability of finding such a star syste is somewhere between one per galaxy, and one per universe.

4.2 Mathematical Construction

We start with a few assumptions about the figure eight. In particular, we assume that it is periodic with period T and that the positions $x_1, x_2, x_3 \in \mathbb{R}^2$ are parameterized as functions of time such that $x_2(t) = x_1(t - T/3)$, and $x_3(t) = x_1(t - 2T/3)$. This is effectively is assuming that the system is stable, and not precessing.

Montgomery proceeds in [1] to project the solutions to the 3-body problem into a vector space he calls the *shape space*. All (normalized) solutions to the three-body problem lie on the surface of the unit sphere in shape space, and Montgomery proves the existence and uniqueness of the figure eight in terms of shape space [1].

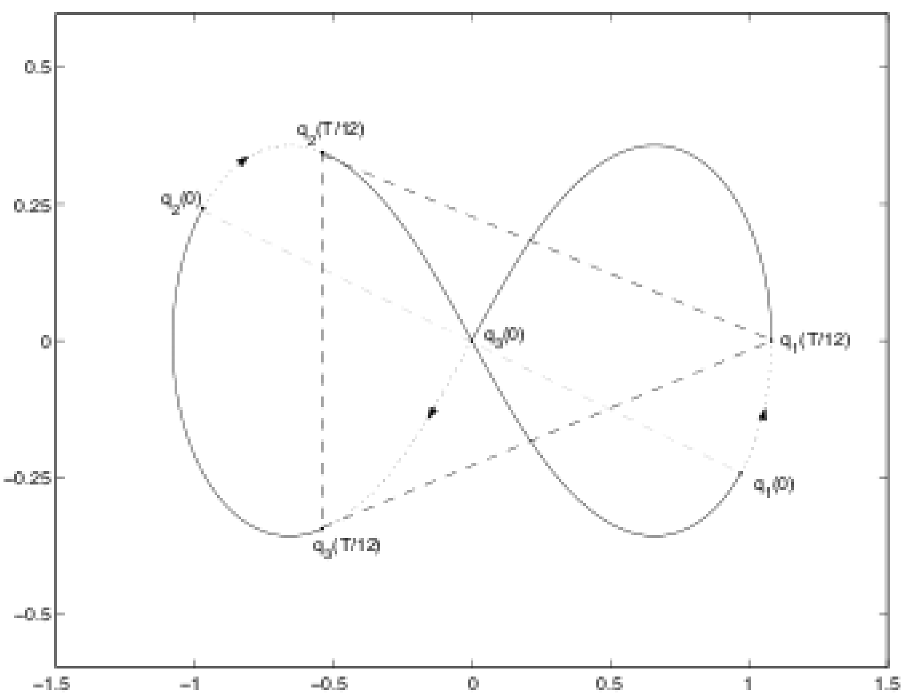

Figure 3: One twelfth of the path of the orbit, with the isoscles geometry of the later state traced out.

4.3 Phase Space

Finally in order to check the legitimacy of previous calculations, [1] transforms the derived equations of motion into a phase-space variant, and from that traces out the path that the particles follow as seen above in Figure 3 (which does indeed create an eight). It is important to note that the particle in the center traces out a much greater distance in the same amount of time ($T/12$), which is the expected result by approximations with the *Virial Theorem* (particles toward the center have greater kinetic energy, while outlying particles have greater potential energy). This result also checks by comparison to harmonic oscillators, where we expect the center to have the greatest instantaneous velocity.

5 Generalizing to N bodies

There will be another class of solutions to the n-body problem of very similar form to the figure eight solution, in which case each of the N particles will trace out the same periodic path about the center of mass. This can result in some interesting orbital shapes in the plane (Figure 4). In [1] these are referred to choreographies, which can take very interesting and beautiful shapes. These are also stable solutions for the same reason why the figure eight is, but are even less likely to occur in reality.

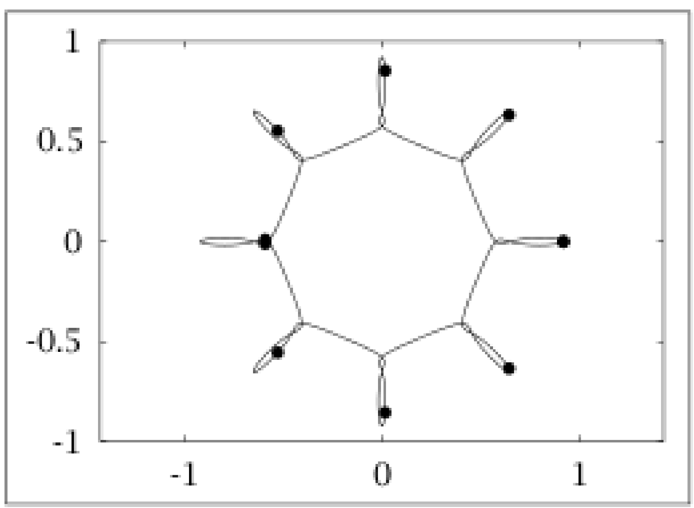

Figure 4: An eleven-body choreography from Montgomery.

It is important to note, however, that these solutions will only apply for the equal mass case because in any other case the differing masses will result in different paths, which results in chaotic motion rather than each particle following the same path.

5.1 Physics is Cool!

I have explored just a few of the many aspects of the n-body problem. Some of the greatest mathemeticians have worked on this problem: Newton, Euler, Lagrange, Poincar to name just a few. There are many potential branches of future study, from the search for a real figure eight, to the mathematical proof of other more complicated shapes, possibly not in the plane!

It is quite amazing that such a complicated process can yield such elegant and simple results in both mathematics and physics!

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