| Prob. 1 | Prob. 2 | Prob. 3 | Prob. 4 |
|---|---|---|---|
| | | | |

Problem 1.

Problem 2.

First we need to prove that in any man-optimal stable matching, each woman has the worst partner that she can have in any stable matching (man-optimal = woman-pessimal).
Suppose the above assumption does not hold. Let $M_0$ be the man-optimal stable matching and suppose there is another stable matching $M_1$. Let the woman $w$ be matched with man $m_0$ in $M_0$ and with $m_1$ in $M_1$. Suppose that $w$ prefers $m_0$ to $m_1$. Then it is obvious that $(m, w)$ blocks $M_1$, since $m_0$ has no other better stable partner than his partner in $M_0$.
The same way it can be proved that a woman-optimal matching is man-pessimal.
Thus if a matching is both man-optimal and woman-optimal, it is implied that the man-optimal version of the stable matching and the man-pessimal version of the stable matching are equal and so are the woman-optimal and the woman-pessimal versions. But since the pessimal and the optimal matching are the same that implies that there can be only one matching.

Problem 3.

1. Strong instability
   Yes, there always exists a perfect matching without any strong instability.
   An algorithm in polynomial-time that can guarantee such a matching is actually the "man proposes, woman disposes" we've seen in class.
   This algorithm has a wort-case number of rounds of $n^2 - 2n + 2$ so it's running in polynomial-time.
   In the case of ties, we may end up with $m$ prefering $w$ than its final matching and $w$ having $m$ at the top of her ranking. But it means that the final matching of $w$ is a man she likes as much as $m$ not less otherwise it means than $m$ has never proposed to $w$ which contradicts the way the algorithm works.
   Thus with the "man proposes, woman disposes" algorithm we can never have a strong instability.

2. Weak instability
   First we can see that the algorithm used for avoiding strong instabilities doesn't work for weak instability. The explanation of the correctness used before actually shows a weak instability.
   But anyway there isn't any algorithm that can avoid weak instabilities for sure. For instance if you have $m_1$ and $m_2$ both prefering $w_1$ than $w_2$ and $w_1$ liking $m_1$ and $m_2$ equally (we don't even need to know the preferences of $w_2$). The two only possible matching without even taking rankings into account would be:

   (a) $m_1 \longleftrightarrow w_1$ and $m_2 \longleftrightarrow w_2$
   (b) $m_1 \longleftrightarrow w_2$ and $m_2 \longleftrightarrow w_1$

   In (a) we can see that there's a weak instability because $m_2$ prefers $w_1$ to his matching and $w_1$ likes $m_1$ and $m_2$ equally.
   In (b) there is also a weak instability because $m_1$ this time prefers $w_1$ to his matching and $w_1$ likes $m_1$ and $m_2$ equally.
   Thus there is no solution without a weak instability.

Problem 4.