mechanics_examples_notebook

April 20, 2024

[1]:
```
"""
_1_classical_mechanics.py

Installation in a Local Computer
===============================
sudo apt install sagemath
sudo pip3 install nbextensions, wolframclient

Intallation in a Cloud
======================
!pip install scipy, wolframclient

Functions:
==========
sympy.calculus.euler.euler_equations

References:
===========
    Books:
    ======
    Gerd Baumann, Mathematica for Theoretical Physics I, Classical Mechanics␣
↪and Nonlinear Dynamics, (Springer, 2nd Ed., 2005)(ISBN 0387016740)
    Armin Wachter, Henning Hoeber, Compendium of Theoretical Physics, Springer,␣
↪2006 (ISBN-10: 0-387-25799-3).
    Douglas Cline, Variational Principles In Classical Mechanics, https://
↪LibreTexts.org
    Christopher W. Kulp, Vasilis Pagonis, Classical Mechanics A Computational␣
↪Approach with Examples Using Mathematica and Python
    Gerald Jay Sussman, Jack Wisdom - Structure and Interpretation of Classical␣
↪Mechanics, MIT Press (2014)

    Python Books:
    =============
    Python Programming And Numerical Methods: A Guide For Engineers And␣
↪Scientists
        https://pythonnumericalmethods.berkeley.edu/notebooks/Index.html
```

```
    R. Johansson, Numerical Python A Practical Techniques Approach for␣
↪Industry, Berkeley, CA, APress, 2015.
        https://jrjohansson.github.io/numericalpython.html
        https://github.com/jrjohansson

    Problem Books:
    ==============
    Vladimir Pletser - Lagrangian and Hamiltonian Analytical Mechanics Forty␣
↪Exercises Resolved and Explained-Springer Singapore (2018)

    Web Sites:
    ==========
    1. The Full Python Tutorial, Luke Polson
       https://www.youtube.com/playlist?list=PLkdGijFCNuVnGxo-1fSNcdHh5gZc17oRM
       https://github.com/lukepolson/youtube_channel/tree/main/
↪Python%20Tutorial%20Series
    2. Physics Problems, Luke Polson
       https://www.youtube.com/playlist?list=PLkdGijFCNuVnMsuC4uFncWusSA9aUzzIp
       https://github.com/lukepolson/youtube_channel/tree/main/
↪Python%20Metaphysics%20Series

Homeworks
=========
    2341:"motion_on_a_helix"
    2342:"motion_of_a_projectile"

    p129- U ?, Eng,
    p130 2483 The Phase Diagram

    p156, 157 apply Laplace transform to driven oscillator ODE and obtain p157.

"""
import copy
import sys
import os
lstPaths = ["../../src"]
for ipath in lstPaths:
    if ipath not in sys.path:
        sys.path.append(ipath)
import scipy as sp
from libsympy import *
from mechanics import *
from sympy.physics import mechanics
mechanics.mechanics_printing()
# Mathematica Client
from wolframclient.evaluation import WolframLanguageSession
from wolframclient.language import wl, wlexpr
```

libsympy is loaded.

0.0.1 Settings

[2]:
```
### Settings
#----Settings
class sets:
    """
    Setttings class.

    Instead of settings class, settings nametuble might be used.
    Settings = namedtuple("Settings", "type dropinf delta")
    sets = Settings(type="symbolic", dropinf=True, delta=0.1)
    """
    global dictflow, test_all

    def __init__(self):
        pass

    # File settings
    input_dir  = "input/mechanics"
    output_dir = "output/mechanics"

    # Plotting settings
    plot_time_scale = {1:"xy", 2:"xz", 3:"yz"}[3]

    # Execution settings.
    test_all = {0:False, 1:True}[1]
    dictflow = {100:"get_formulary", 150:"get_subformulary",
                200:"simple_harmonic_oscillator_scalar", 201:
↪"simple_harmonic_oscillator_vectorial",
                2321:"Coordinate_Systems", 2322:"Moving_Particle",
                2341:"motion_on_a_helix", 2342:"motion_of_a_projectile",

                2484:"Damped_Harmonic_Oscillator",
                2485:"Driven_Oscillations",
                24861:"Driven_Oscillations_The_Laplace_Transform_Method",
                24862:"Driven_Oscillations_Greens_Function_Method",

                263:"Eulers_Equation", 2651:"Brachystochrone_Baumann",
                2652:"Brachystochrone_Wachter", 266:"Euler_Operator",
                267:"2.6.7 Euler Operator for q + p Dimensions",
                272:"2.7.2 Hamiltons Principle Historical Remarks",
                2731:"2.7.3.1 Example 1: Harmonic Oscillator",
                2732:"2.7.3.2 Example 2: Rolling Wheel on an Inclined Plane",
                2733:"2.7.3.3 Example 2: Sliding Mass Connected to a Pendulum",
                2820:"2.8.2.0 Motion in a uniform gravitational field",
```

```
                2821:"2.8.2.1 Example 1: Moving Beat on a String",
                2841:"2.8.4.1 Example 1: Motion on a Cylinder"}
    flow = [dictflow[i] for i in [2841]]
    if test_all: flow = [dictflow[i] for i in dictflow.keys()]
print(sets.flow)
```

```
['get_formulary', 'get_subformulary', 'simple_harmonic_oscillator_scalar',
'simple_harmonic_oscillator_vectorial', 'Coordinate_Systems', 'Moving_Particle',
'motion_on_a_helix', 'motion_of_a_projectile', 'Damped_Harmonic_Oscillator',
'Driven_Oscillations', 'Driven_Oscillations_The_Laplace_Transform_Method',
'Driven_Oscillations_Greens_Function_Method', 'Eulers_Equation',
'Brachystochrone_Baumann', 'Brachystochrone_Wachter', 'Euler_Operator', '2.6.7
Euler Operator for q + p Dimensions', '2.7.2 Hamiltons Principle Historical
Remarks', '2.7.3.1 Example 1: Harmonic Oscillator', '2.7.3.2 Example 2: Rolling
Wheel on an Inclined Plane', '2.7.3.3 Example 2: Sliding Mass Connected to a
Pendulum', '2.8.2.0 Motion in a uniform gravitational field', '2.8.2.1 Example
1: Moving Beat on a String', '2.8.4.1 Example 1: Motion on a Cylinder']
```

[ ]:
```
print(sys.version)
print(sys.path)
```

[ ]:
```
### Formulary
print("Test of the {0}.".format(sets.flow))
if "get_formulary" in sets.flow:
    omech.__init__("scalar")
    omech.get_formulary()
    omech.get_formulary(style="eq")

    omech.__init__("vectorial")
    omech.get_formulary()

    omech.__init__("EulerLagrange")
    omech.get_formulary()
```

[ ]:
```
if "get_subformulary" in sets.flow:
    omech.__init__()
    omech.get_subformulary()
```

0.1 2.4 Newtonian Mechanics

0.1.1 simple_harmonic_oscillator_scalar

[6]:
```
#----> simple_harmonic_oscillator_scalar
if "simple_harmonic_oscillator_scalar" in sets.flow: #␣
↪simple_harmonic_oscillator_scalar
    """
    Example: Solve a from F = ma
```

```
    """
#    omech = mechanics() # DO NOT create any instance.
    print("2.4.8.2 Harmonic Oscillator, p126.")
    omech.__init__("scalar")
    omech.verbose = True
    commands = ["solve", "NewtonsLaw2", omech.a.rhs]
    omech.process(commands)

    """
    Example: Solve position of a spring mass system.
    F = ma, F = -kx
    -kx = ma
    -kx = m d^2 x/dt^2
    w = sqrt(k/m)
    x(t) = C1*sin(wt) + C2*sin(wt)
    """
    # Scalar Way.
    omech.__init__("scalar")
    omech.verbose = True
    display("Newton's 2nd Law", omech.NewtonsLaw2,
            "Hooke's Law", omech.HookesLaw)

    commands = ["Eq", "NewtonsLaw2", "HookesLaw"]
    omech.process(commands)
#    commands = ["subs", "omech.result", [(a, diff(x, t, 2, evaluate=False))]]
    res = omech.process(commands)
    omech.result = Eq(simplify(res.lhs/m), simplify(res.rhs/m))
    commands = ["subs", "omech.result", [(k/m, w**2)]]
    omech.process(commands)
#    omech.result = Eq(omech.result.lhs.coeff(C.i), omech.result.rhs)
    commands = ["dsolve", "omech.result", omech.x]
    omech.process(commands)
    print("Codes:\n", *omech.get_codes())

    omech.x = omech.process(commands).rhs
    v = omech.v.evalf(subs={x:omech.x}).doit()
    a = omech.a.evalf(subs={x:omech.x}).doit()
    T = omech.T.evalf(subs={x:omech.x}).doit()
    U = omech.U.evalf(subs={x:omech.x}).doit()
    display(omech.result,v,a,T,U)

    # Numerical calculations
    [C1,C2] = symbols('C1 C2')
    numvals = {C1:1, C2:1, w:2}
#    commands = ["xreplace", "omech.x", numvals]
#    omech.process(commands)
    x = omech.x.evalf(subs=numvals).doit()
```

```python
    v = v.evalf(subs=numvals).rhs
    a = a.evalf(subs=numvals).rhs
    plot(x, (t,0,4*pi,200), xlabel="$t$", ylabel="$x(t)$")
    plot(v, (t,0,4*pi,200), xlabel="$t$", ylabel="$v(t)$")
    plot(a, (t,0,4*pi,200), xlabel="$t$", ylabel="$a(t)$")
    plot_sympfunc([x.subs({t:var('x')}),], (0, float(4*pi), 200),
                   xlabel="$t$", ylabel="$x(t)$")

    #--- 2.4.8.3 The Phase Diagram
    x = omech.result.rhs.evalf(subs=numvals).doit()
    plot_parametric((x,v),(t,1,5), xlabel="x", ylabel="x'")
```

2.4.8.2 Harmonic Oscillator, p126.

'solve NewtonsLaw2 Derivative(x(t), (t, 2))'

solve(Eq(F, m*Derivative(x(t), (t, 2))), Derivative(x(t), (t, 2)))

$$\left[\frac{F}{m}\right]$$

"Newton's 2nd Law"

$$F = m\frac{d^2}{dt^2}x(t)$$

"Hooke's Law"

$$F = -kx(t)$$

'Eq NewtonsLaw2 HookesLaw'

Equality(m*Derivative(x(t), (t, 2)), -k*x(t))

$$m\frac{d^2}{dt^2}x(t) = -kx(t)$$

'Eq NewtonsLaw2 HookesLaw'

Equality(m*Derivative(x(t), (t, 2)), -k*x(t))

$$m\frac{d^2}{dt^2}x(t) = -kx(t)$$

'subs omech.result [(k/m, w**2)]'

Eq(Derivative(x(t), (t, 2)), -k*x(t)/m)(subs, [(k/m, w**2)])

$$\frac{d^2}{dt^2}x(t) = -w^2x(t)$$

```
'dsolve omech.result x(t)'
```

```
dsolve(Eq(Derivative(x(t), (t, 2)), -w**2*x(t)), x(t))
```

$x(t) = C_1 \sin(tw) + C_2 \cos(tw)$

```
Codes:
 Equality(m*Derivative(x(t), (t, 2)), -k*x(t))
 Equality(m*Derivative(x(t), (t, 2)), -k*x(t))
 Eq(Derivative(x(t), (t, 2)), -k*x(t)/m)(subs, [(k/m, w**2)])
 dsolve(Eq(Derivative(x(t), (t, 2)), -w**2*x(t)), x(t))
```

```
'dsolve omech.result x(t)'
```

```
dsolve(Eq(x(t), C1*sin(t*w) + C2*cos(t*w)), x(t))
```

$x(t) = C_1 \sin(tw) + C_2 \cos(tw)$

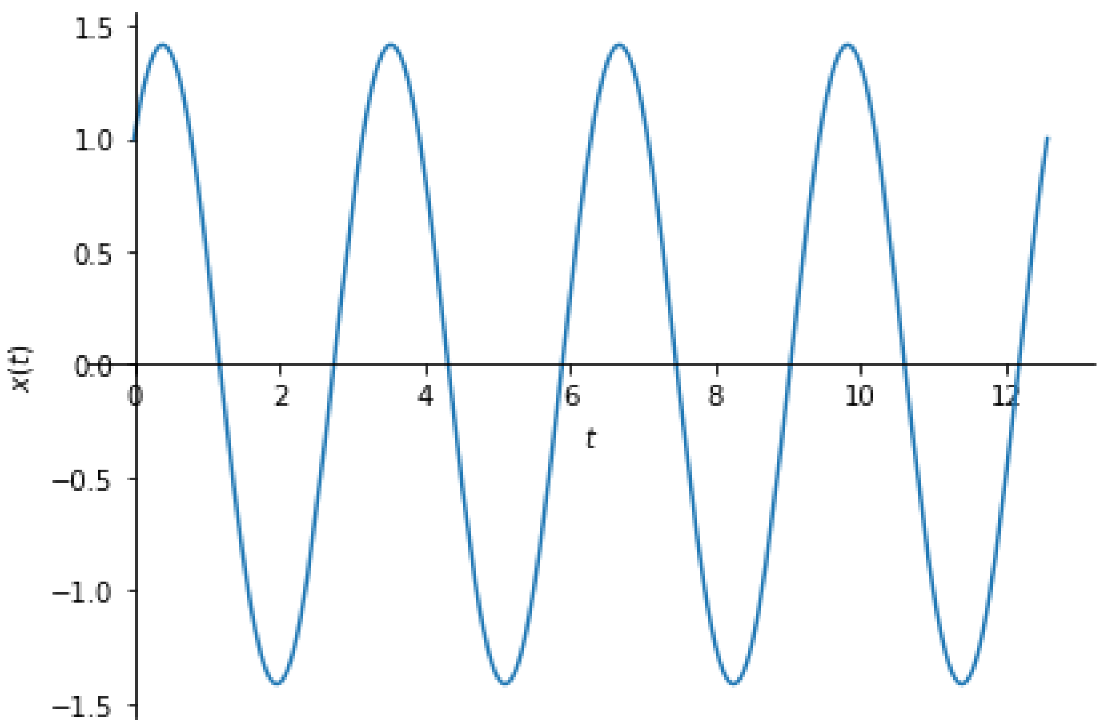
$x(t) = C_1 \sin(tw) + C_2 \cos(tw)$

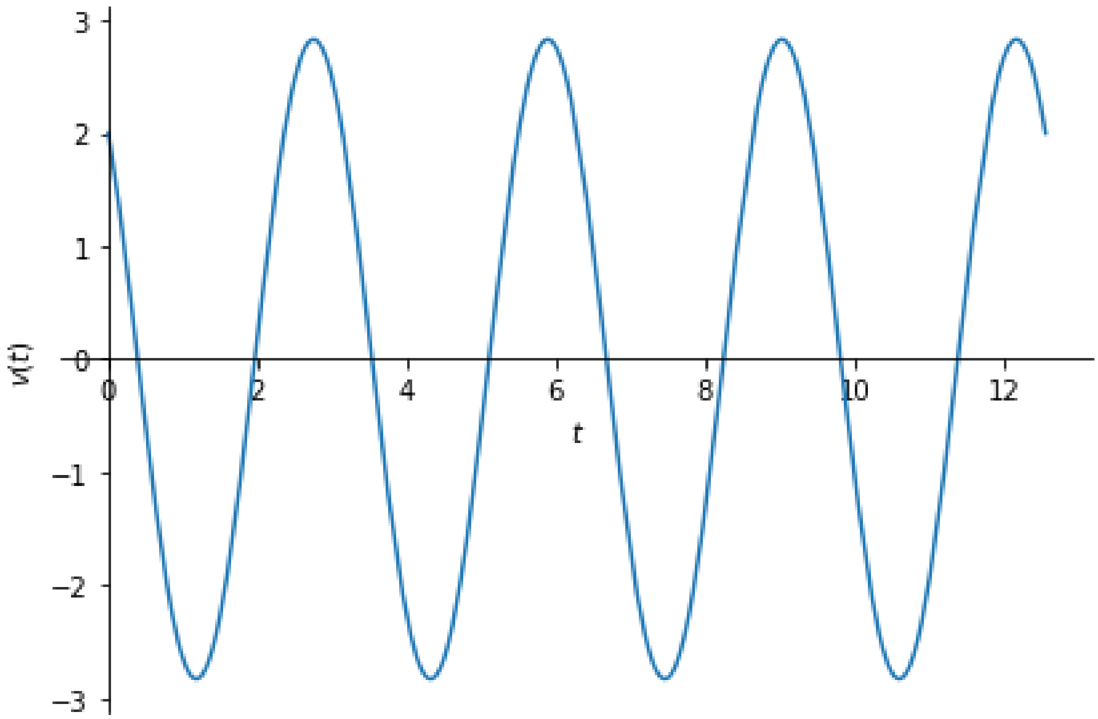
$v = C_1 w \cos(tw) - C_2 w \sin(tw)$

$a = -w^2 (C_1 \sin(tw) + C_2 \cos(tw))$

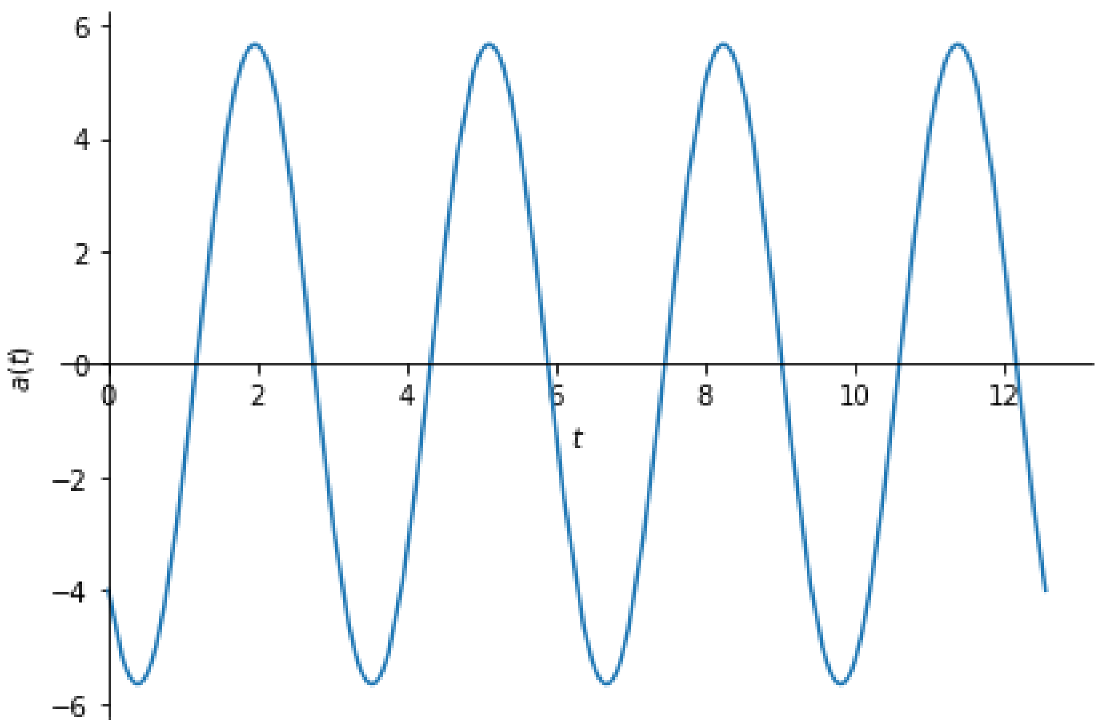
$T = 0.5m (C_1 w \cos(tw) - C_2 w \sin(tw))^2$

$U = k (C_1 \sin(tw) + C_2 \cos(tw))^2$

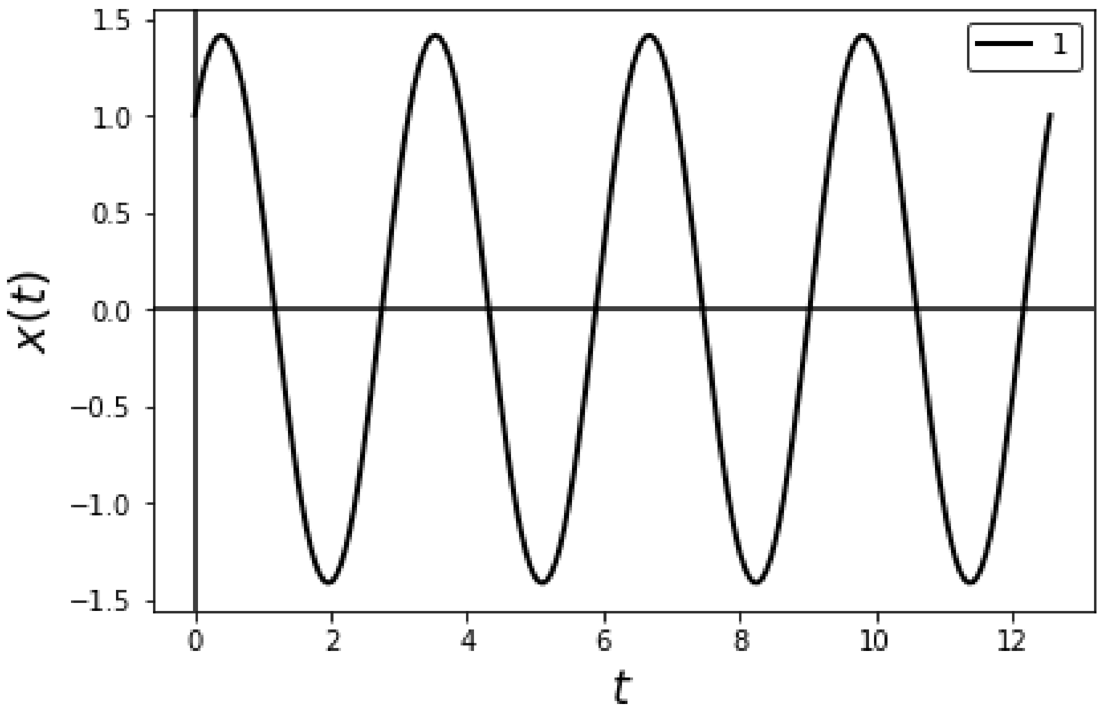




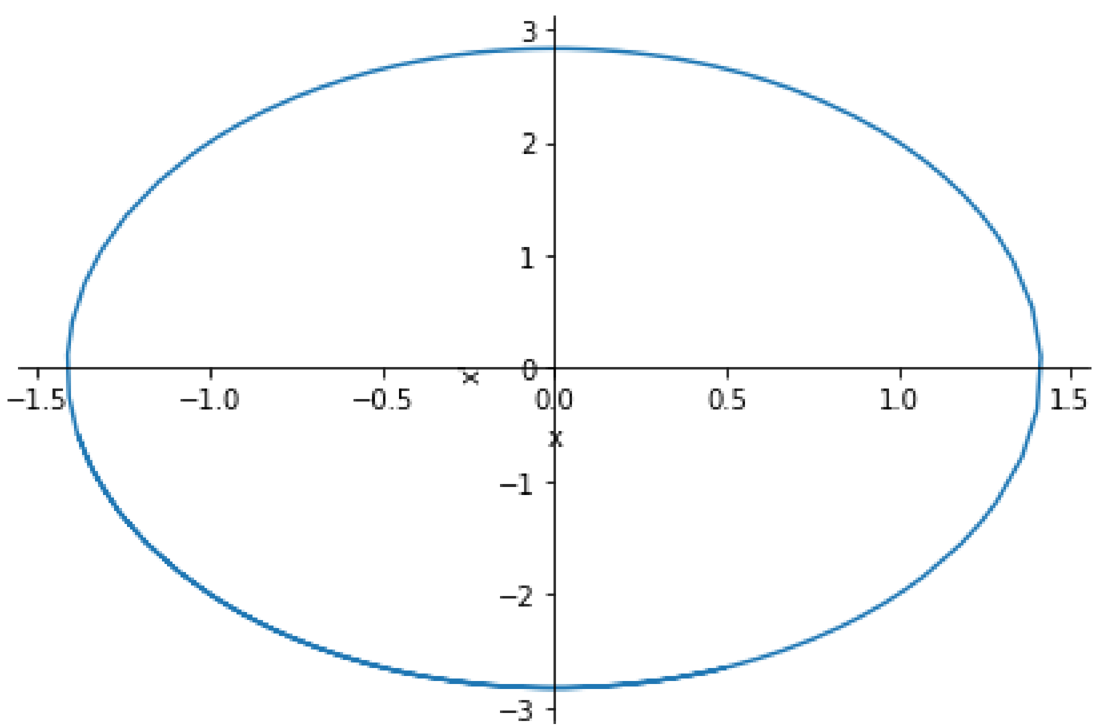
v(t)
t



a(t)
t



1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
$x(t)$
0
2
4
6
8
10
12
t
1



3
2
1
0
−1
−2
−3
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x
x

0.1.2 simple_harmonic_oscillator_vectorial

[3]:
```
#----> simple_harmonic_oscillator_vectorial
if "simple_harmonic_oscillator_vectorial" in sets.flow:
    # Vectorial Way.
    # omech.class_type = "vectorial"
    omech.__init__("vectorial")
    omech.verbose = True

    commands = ["Eq", "NewtonsLaw2", "HookesLaw"]
#    commands = ["subs", "omech.result", [(a, diff(x, t, 2, evaluate=False))]]
    res = omech.process(commands)
    omech.result = Eq(simplify(res.lhs/m), simplify(res.rhs/m))
    commands = ["subs", "omech.result", [(k/m, w**2)]]
    omech.process(commands)
    omech.result = Eq(omech.result.lhs.coeff(C.i), omech.result.rhs)
    commands = ["dsolve", "omech.result", omech.x]
    omech.process(commands)
    print("Codes:\n", *omech.get_codes())

    omech.x = omech.process(commands).rhs
    v = omech.v.evalf(subs={x:omech.x}).doit()
    a = omech.a.evalf(subs={x:omech.x}).doit()
    display(omech.result,v,a)

    # Numerical calculations
    [C1,C2] = symbols('C1 C2')
    numvals = {C1:1, C2:1, w:2}
#    commands = ["xreplace", "omech.x", numvals]
#    omech.process(commands)
    x = omech.x.evalf(subs=numvals).doit()
    v = v.evalf(subs=numvals).rhs.components[C.i]
    # a = a.evalf(subs=numvals).rhs.components[C.i]
    a = a.xreplace(numvals).rhs.components[C.i]

    plot(x, (t,0,4*pi,200), xlabel="$t$", ylabel="$x(t)$")
    plot(v, (t,0,4*pi,200), xlabel="$t$", ylabel="$v(t)$")
    plot(a, (t,0,4*pi,200), xlabel="$t$", ylabel="$a(t)$")
    plot_sympfunc([x.subs({t:var('x')}),], (0, float(4*pi), 200),
                  xlabel="$t$", ylabel="$x(t)$")

    # The Phase Diagram
    x = omech.result.rhs.evalf(subs=numvals).doit()
    plot_parametric((x,v),(t,1,5), xlabel="x", ylabel="x'")
```

```
'Eq NewtonsLaw2 HookesLaw'
```

```
Equality((m*Derivative(x(t), (t, 2)))*C.i + (m*Derivative(y(t), (t, 2)))*C.j +
(m*Derivative(z(t), (t, 2)))*C.k, -k*x(t))
```

$$\left(m\frac{d^2}{dt^2}x(t)\right)\hat{\mathbf{i}}_{\mathbf{C}} + \left(m\frac{d^2}{dt^2}y(t)\right)\hat{\mathbf{j}}_{\mathbf{C}} + \left(m\frac{d^2}{dt^2}z(t)\right)\hat{\mathbf{k}}_{\mathbf{C}} = -kx(t)$$

```
'subs omech.result [(k/m, w**2)]'
```

```
Eq((Derivative(x(t), (t, 2)))*C.i + (Derivative(y(t), (t, 2)))*C.j +
(Derivative(z(t), (t, 2)))*C.k, -k*x(t)/m)(subs, [(k/m, w**2)])
```

$$\left(\frac{d^2}{dt^2}x(t)\right)\hat{\mathbf{i}}_{\mathbf{C}} + \left(\frac{d^2}{dt^2}y(t)\right)\hat{\mathbf{j}}_{\mathbf{C}} + \left(\frac{d^2}{dt^2}z(t)\right)\hat{\mathbf{k}}_{\mathbf{C}} = -w^2x(t)$$

```
'dsolve omech.result x(t)'
```

```
dsolve(Eq(Derivative(x(t), (t, 2)), -w**2*x(t)), x(t))
```

$$x(t) = C_1\sin(tw) + C_2\cos(tw)$$

```
Codes:
 Equality((m*Derivative(x(t), (t, 2)))*C.i + (m*Derivative(y(t), (t, 2)))*C.j +
(m*Derivative(z(t), (t, 2)))*C.k, -k*x(t))
 Eq((Derivative(x(t), (t, 2)))*C.i + (Derivative(y(t), (t, 2)))*C.j +
(Derivative(z(t), (t, 2)))*C.k, -k*x(t)/m)(subs, [(k/m, w**2)])
 dsolve(Eq(Derivative(x(t), (t, 2)), -w**2*x(t)), x(t))
```

```
'dsolve omech.result x(t)'
```

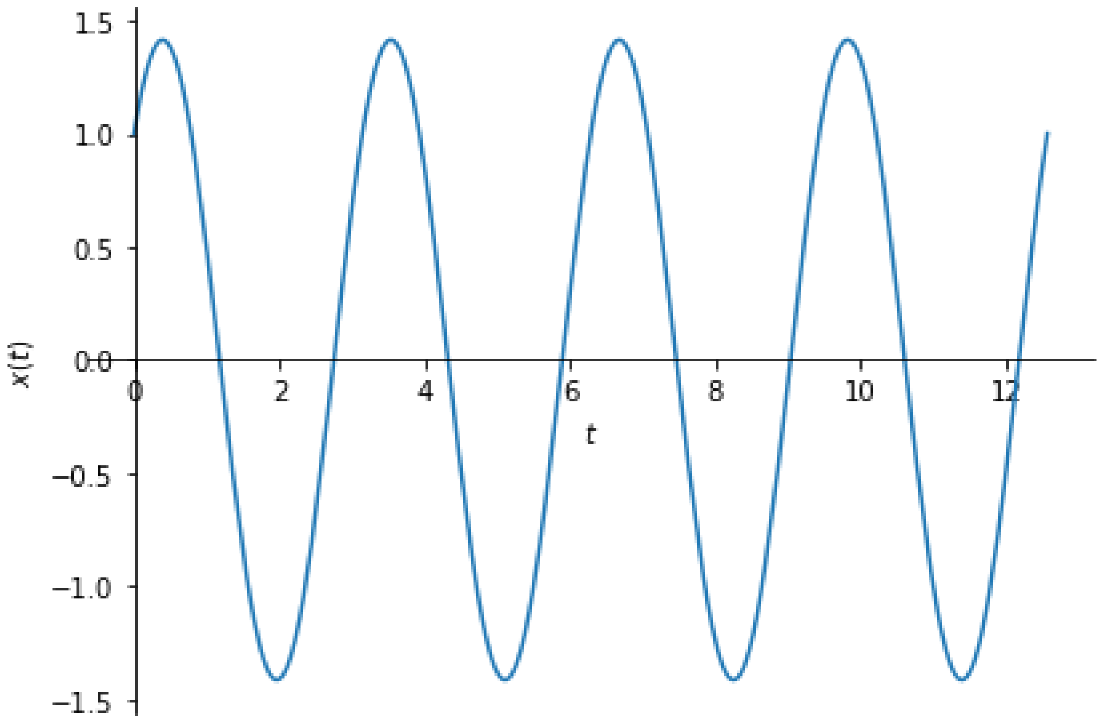
```
dsolve(Eq(x(t), C1*sin(t*w) + C2*cos(t*w)), x(t))
```

$$x(t) = C_1\sin(tw) + C_2\cos(tw)$$

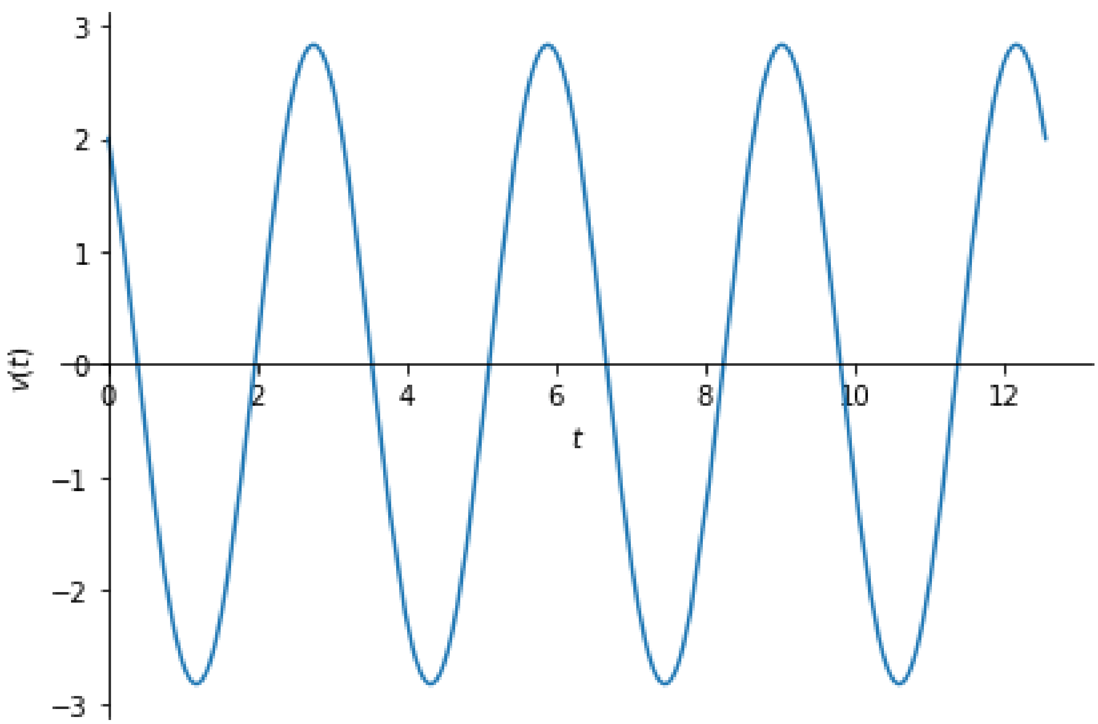
$$x(t) = C_1\sin(tw) + C_2\cos(tw)$$

$$(v_x)\hat{\mathbf{i}}_{\mathbf{C}} + (v_y)\hat{\mathbf{j}}_{\mathbf{C}} + (v_z)\hat{\mathbf{k}}_{\mathbf{C}} = (C_1w\cos(tw) - C_2w\sin(tw))\hat{\mathbf{i}}_{\mathbf{C}} + \left(\frac{d}{dt}y(t)\right)\hat{\mathbf{j}}_{\mathbf{C}} + \left(\frac{d}{dt}z(t)\right)\hat{\mathbf{k}}_{\mathbf{C}}$$

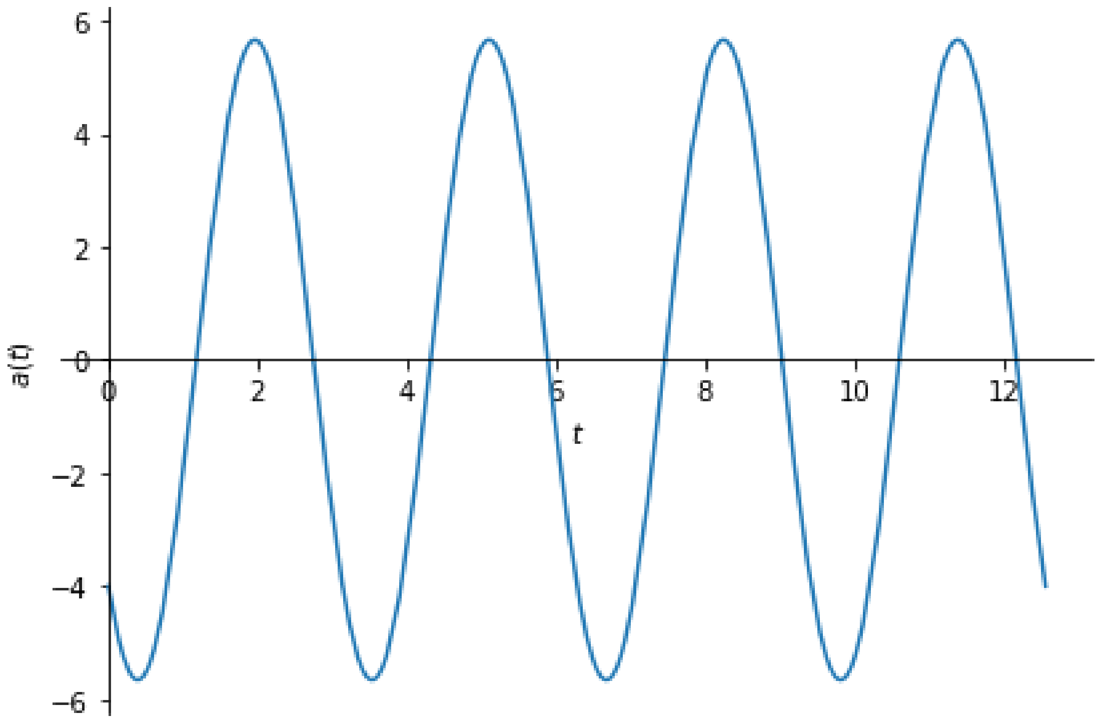
$$(a_x)\hat{\mathbf{i}}_{\mathbf{C}} + (a_y)\hat{\mathbf{j}}_{\mathbf{C}} + (a_z)\hat{\mathbf{k}}_{\mathbf{C}} = \left(-w^2\left(C_1\sin(tw) + C_2\cos(tw)\right)\right)\hat{\mathbf{i}}_{\mathbf{C}} + \left(\frac{d^2}{dt^2}y(t)\right)\hat{\mathbf{j}}_{\mathbf{C}} + \left(\frac{d^2}{dt^2}z(t)\right)\hat{\mathbf{k}}_{\mathbf{C}}$$



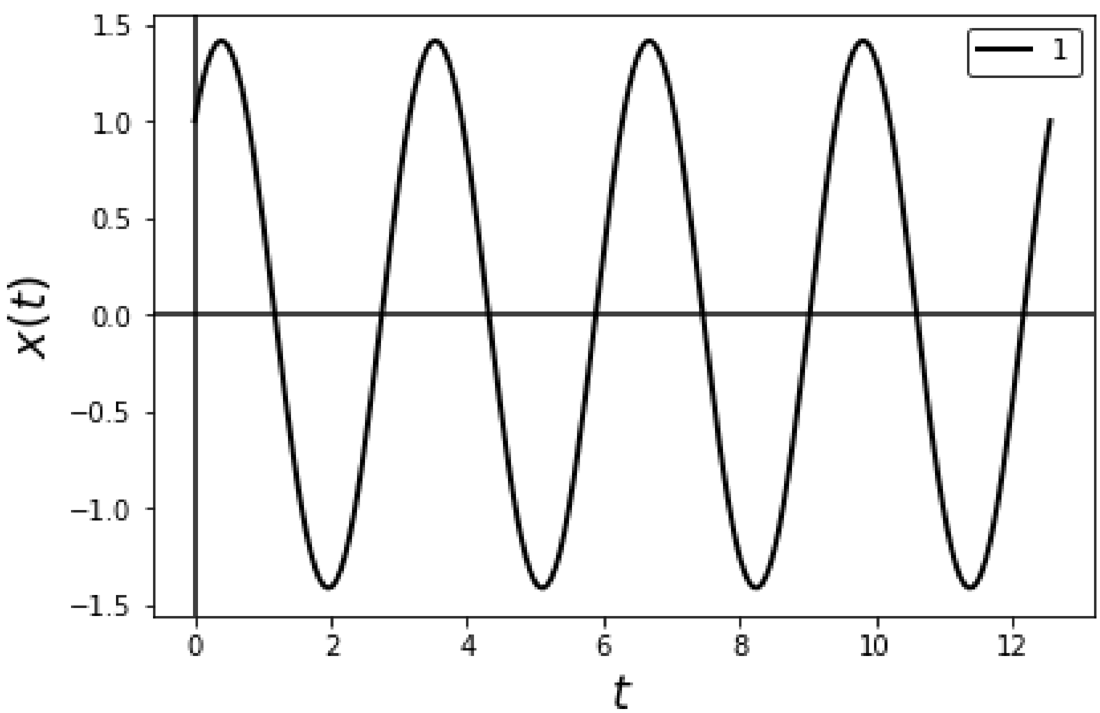
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
x(t)
0
2
4
6
8
10
12
t



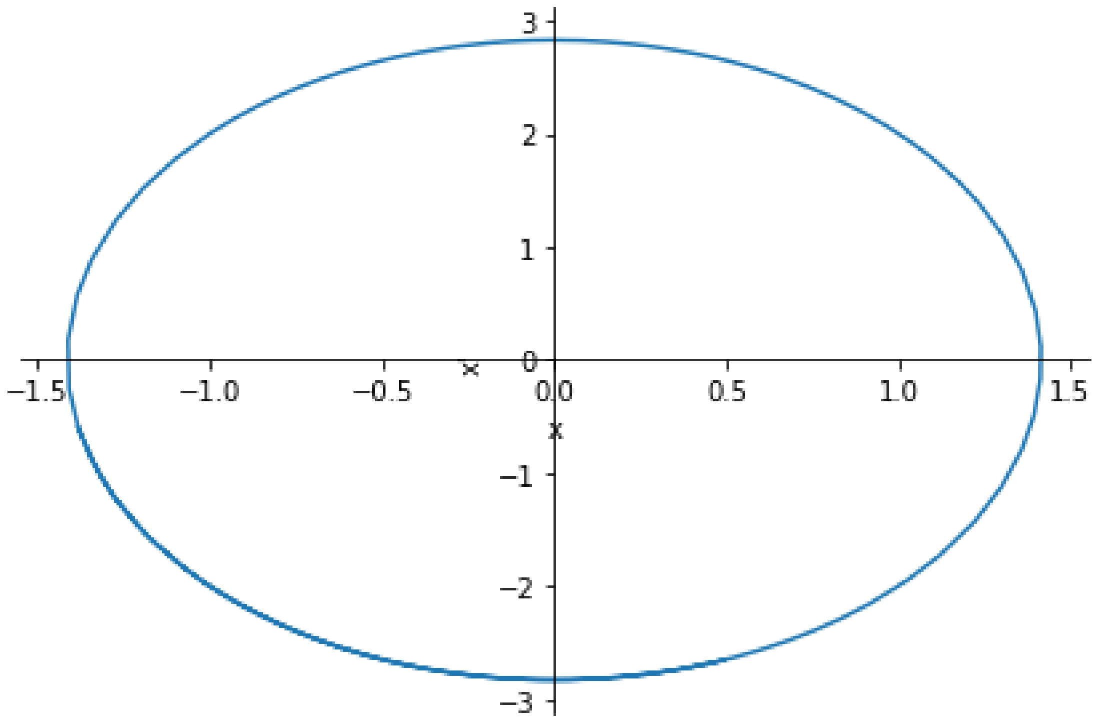
3
2
1
0
−1
−2
−3
v(t)
0
2
4
6
8
10
12
t



6
4
2
0
−2
−4
−6
a(t)
0
2
4
6
8
10
12
t



1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
x(t)
0
2
4
6
8
10
12
t
1



0.2 Coordinate_Systems

[5]:
```
#----> Coordinate_Systems
if "Coordinate_Systems" in sets.flow:
    print("Example 1. Coordinate Systems, p78.")
    print("Polar Coordinates")
    omech.__init__("vectorial")
    omech.verbose = False

    xreplaces = {x:r*cos(theta)*C.i,
                 y:r*sin(theta)*C.j,
                 z:0}
    xreplaces = {x:omech.subformulary.pol_to_cart_x,
                 y:omech.subformulary.pol_to_cart_y,
                 z:0} # C.k
    display(omech.r, omech.v, omech.a)
    display(xreplaces)

    commands = ["xreplace", "omech.r", xreplaces]
    r = omech.process(commands).doit()
    commands = ["xreplace", "omech.v", xreplaces]
    v = omech.process(commands).doit()
    commands = ["xreplace", "omech.a", xreplaces]
```

```python
    a = omech.process(commands).doit()
    display(x,y,z,r,v,a)

    print("Components of r")
    [display(r.rhs.args[i]) for i in range(2)]
    print("Components of v")
    [display(v.rhs.args[i]) for i in range(2)]
    print("Components of a")
    [display(a.rhs.args[i]) for i in range(2)]
```

```
Example 1. Coordinate Systems, p78.
Polar Coordinates
```

$$(r_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (r_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (r_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = (x(t))\,\widehat{\mathbf{i}}_{\mathbf{C}} + (y(t))\,\widehat{\mathbf{j}}_{\mathbf{C}} + (z(t))\,\widehat{\mathbf{k}}_{\mathbf{C}}$$

$$(v_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (v_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (v_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = \left(\frac{d}{dt}x(t)\right)\widehat{\mathbf{i}}_{\mathbf{C}} + \left(\frac{d}{dt}y(t)\right)\widehat{\mathbf{j}}_{\mathbf{C}} + \left(\frac{d}{dt}z(t)\right)\widehat{\mathbf{k}}_{\mathbf{C}}$$

$$(a_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (a_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (a_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = \left(\frac{d^2}{dt^2}x(t)\right)\widehat{\mathbf{i}}_{\mathbf{C}} + \left(\frac{d^2}{dt^2}y(t)\right)\widehat{\mathbf{j}}_{\mathbf{C}} + \left(\frac{d^2}{dt^2}z(t)\right)\widehat{\mathbf{k}}_{\mathbf{C}}$$

$$\{\sin(2t) + \cos(2t) : r(t)\cos(\theta(t)),\ y(t) : r(t)\sin(\theta(t)),\ z(t) : 0\}$$

$$(r_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (r_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (r_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = (x(t))\,\widehat{\mathbf{i}}_{\mathbf{C}} + (r(t)\sin(\theta(t)))\,\widehat{\mathbf{j}}_{\mathbf{C}}$$

$$(v_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (v_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (v_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = \left(\frac{d}{dt}x(t)\right)\widehat{\mathbf{i}}_{\mathbf{C}} + \left(\left(r(t)\cos(\theta(t))\frac{d}{dt}\theta(t) + \sin(\theta(t))\frac{d}{dt}r(t)\right)\right)\widehat{\mathbf{j}}_{\mathbf{C}} + ((0))\,\widehat{\mathbf{k}}_{\mathbf{C}}$$

$$(a_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (a_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (a_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = \left(\frac{d^2}{dt^2}x(t)\right)\widehat{\mathbf{i}}_{\mathbf{C}} + \Bigg(\Bigg(-\left(\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)^2 - \cos(\theta(t))\frac{d^2}{dt^2}\theta(t)\right)r(t) + \sin(\theta(t))\frac{d^2}{dt^2}r(t)$$

$$((0))\,\widehat{\mathbf{k}}_{\mathbf{C}}$$

$$\sin(2t) + \cos(2t)$$

$$y(t)$$

$$z(t)$$

$$(r_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (r_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (r_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = (x(t))\,\widehat{\mathbf{i}}_{\mathbf{C}} + (r(t)\sin(\theta(t)))\,\widehat{\mathbf{j}}_{\mathbf{C}}$$

$$(v_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (v_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (v_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = \left(\frac{d}{dt}x(t)\right)\widehat{\mathbf{i}}_{\mathbf{C}} + \left(r(t)\cos(\theta(t))\frac{d}{dt}\theta(t) + \sin(\theta(t))\frac{d}{dt}r(t)\right)\widehat{\mathbf{j}}_{\mathbf{C}}$$

$$(a_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (a_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (a_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = \left(\frac{d^2}{dt^2}x(t)\right)\widehat{\mathbf{i}}_{\mathbf{C}} + \Bigg(-\left(\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)^2 - \cos(\theta(t))\frac{d^2}{dt^2}\theta(t)\right)r(t) + \sin(\theta(t))\frac{d^2}{dt^2}r(t) +$$

```
Components of r
```

$$(x(t))\,\widehat{\mathbf{i}}_{\mathbf{C}}$$

$$(r(t)\sin(\theta(t)))\,\widehat{\mathbf{j}}_{\mathbf{C}}$$

```
Components of v
```

$$\left(r(t)\cos(\theta(t))\frac{d}{dt}\theta(t)+\sin(\theta(t))\frac{d}{dt}r(t)\right)\hat{\mathbf{j}}_{\mathbf{C}}$$

$$\left(\frac{d}{dt}x(t)\right)\hat{\mathbf{i}}_{\mathbf{C}}$$

Components of a

$$\left(-\left(\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)^2-\cos(\theta(t))\frac{d^2}{dt^2}\theta(t)\right)r(t)+\sin(\theta(t))\frac{d^2}{dt^2}r(t)+2\cos(\theta(t))\frac{d}{dt}r(t)\frac{d}{dt}\theta(t)\right)\hat{\mathbf{j}}_{\mathbf{C}}$$

$$\left(\frac{d^2}{dt^2}x(t)\right)\hat{\mathbf{i}}_{\mathbf{C}}$$

0.3 Moving_Particle

[6]:
```
#----> Moving_Particle
if "Moving_Particle" in sets.flow:
    print("Example 2. Moving Particle, p80.")
    print("Spherical Coordinates")
    omech.class_type = "vectorial"
    omech.__init__()
    omech.verbose = False
    xreplaces = {x:omech.subformulary.sph_to_cart_x,
                 y:omech.subformulary.sph_to_cart_y,
                 z:omech.subformulary.sph_to_cart_z}
    x = omech.x.evalf(subs=xreplaces).doit()
    y = omech.y.evalf(subs=xreplaces).doit()
    z = omech.z.evalf(subs=xreplaces).doit()
    commands = ["xreplace", "omech.r", xreplaces]
    r = omech.process(commands).doit()
    commands = ["xreplace", "omech.v", xreplaces]
    v = omech.process(commands).doit()
    commands = ["xreplace", "omech.a", xreplaces]
    a = omech.process(commands).doit()
#    a = simplify( omech.a.rhs.evalf(subs=xreplaces).doit()) # Does not work.
    pprints("x=", x,
            "y=", y,
            "z=", z,
            "r=", r,
            "v=", v,
            "a=", a)
```

Example 2. Moving Particle, p80.
Spherical Coordinates

$$(r_x)\hat{\mathbf{i}}_{\mathbf{C}}+(r_y)\hat{\mathbf{j}}_{\mathbf{C}}+(r_z)\hat{\mathbf{k}}_{\mathbf{C}}=(x(t))\hat{\mathbf{i}}_{\mathbf{C}}+(r(t)\sin(\phi(t))\sin(\theta(t)))\hat{\mathbf{j}}_{\mathbf{C}}+(r(t)\cos(\theta(t)))\hat{\mathbf{k}}_{\mathbf{C}}$$

$$v=\frac{d}{dt}x(t)$$

$$a = \frac{d^2}{dt^2}x(t)$$

'x='

$$x(t)$$

'y='

$$r(t)\sin(\phi(t))\sin(\theta(t))$$

'z='

$$r(t)\cos(\theta(t))$$

'r='

$$(r_x)\,\widehat{\mathbf{i}}_{\mathbf{C}} + (r_y)\,\widehat{\mathbf{j}}_{\mathbf{C}} + (r_z)\,\widehat{\mathbf{k}}_{\mathbf{C}} = (x(t))\,\widehat{\mathbf{i}}_{\mathbf{C}} + (r(t)\sin(\phi(t))\sin(\theta(t)))\,\widehat{\mathbf{j}}_{\mathbf{C}} + (r(t)\cos(\theta(t)))\,\widehat{\mathbf{k}}_{\mathbf{C}}$$

'v='

$$v = \frac{d}{dt}x(t)$$

'a='

$$a = \frac{d^2}{dt^2}x(t)$$

0.4 2.4.8.4 Damped_Harmonic_Oscillator

[7]:
```
#----> Damped_Harmonic_Oscillator
if "Damped_Harmonic_Oscillator" in sets.flow:
    pprints("2.4.8.4 Damped Harmonic Oscillator, p133.",
            "General Solution")

    # General Solution.
    case = {1:"underdamped", 2:"critical_damped", 3:"overdamped"}[3]

    if case == "underdamped":
        omech.__init__("scalar")
        omech.verbose = True
        pprints("Underdamped Motion, p134.",
                omech.subformulary.underdamping_criteria)
        commands = ["dsolve", "damped_harmonic_oscillator1", omech.x]
        omech.process(commands)
        commands = ["dsolve", "damped_harmonic_oscillator2", omech.x]
        omech.x = omech.process(commands).rhs
        v = omech.v.evalf(subs={x:omech.x}).doit()
```

```
        T = omech.T.evalf(subs={x:omech.x}).doit()
        _U = Function('U')(t)          # Potential energy.
        _H = Function('H')(t)          # Total energy.
        U = Eq(_U, S(1)/2*k*(omech.x)**2)
        H = Eq(_H, T.rhs + U.rhs)
        display(v,T,U,H)

        # Numerical calculations.
        [C1,C2] = symbols('C1 C2')
        numvals = {C1:1, C2:1, beta:S(1)/7, w0:sqrt(1+(S(1)/7)**2), k:1, m:1} #␣
↪Exact solution's numerical values.
        envvals = {C1:1, C2:1, beta:S(1)/7, w0:S(1)/7} # Envelope function's␣
↪numerical values. w1->0.

    #    commands = ["xreplace", "omech.x", numvals]
    #    omech.process(commands)
        x = omech.x.evalf(subs=numvals)
        x_env = omech.x.evalf(subs=envvals)
        v = v.evalf(subs=numvals).rhs
        H = H.evalf(subs=numvals).rhs
        # Plot x(t) and envelope functions.
        plot(x, x_env, -x_env, (t,0,5*pi,200), xlabel="$t$", ylabel="$x(t)$")
        # Plot H and dH/dt.
        p = plot(H, diff(H,t), (t,0,5*pi,200), xlabel="$t$", ylabel="$H$, $dH/
↪dt$",
                   legend=True)
        p[0].label = 'H'
        p[1].label = 'dH/dt'
        p.show()
        # Plot phase diagram, x' versus x.
        plot_parametric((x,v), (t,0,25), xlabel="x", ylabel="x'")

    if case == "critical_damped":
        """
        Critical Damped Motion
        dsolve(omech.damped_harmonic_oscillator2.subs({w0:beta}), omech.x,␣
↪ics={omech.x.subs({t:0}):x0, diff(omech.x, t).subs({t:0}):v0})
        """
        omech.class_type = "scalar"
        omech.__init__()
        omech.verbose = True
        pprints("Critical Damped Motion",
                omech.subformulary.critical_damping_criteria)
        omech.x = dsolve(omech.damped_harmonic_oscillator2.subs(omech.
↪subformulary.critical_damping_criteria),
                          omech.x,
                          ics={omech.x.subs({t:0}):x0,
```

```
                                diff(omech.x, t).subs({t:0}):v0})
        display(omech.x)

        # Numerical calculations.
        numvals = {beta:S(1)/5, x0:1, v0:0}
        x_t = omech.x.evalf(subs=numvals).rhs
        # Plot x(t).
        plot(x_t, (t,0,25,200), xlabel="$t$", ylabel="$x(t)$")

    if case == "overdamped":
        """
        Overdamped Motion

        f = lambda i:x.rhs.subs(v0,i)
        list(map(f,[1,2]))
        """
        omech.class_type = "scalar"
        omech.__init__()
        omech.verbose = True
        pprints("Overdamped Motion",
                omech.subformulary.overdamping_criteria)
        omech.x = dsolve(omech.damped_harmonic_oscillator2.subs(omech.
↪subformulary.overdamping_criteria), omech.x, ics={omech.x.subs({t:0}):x0,␣
↪diff(omech.x, t).subs({t:0}):v0})
        v = diff(omech.x, t)
        display(omech.x,v)

        # Numerical calculations.
        # Plot x(t).
        numvals = {beta:S(1)/5, w2:S(1)/10, x0:1, v0:0}
        x_t = omech.x.evalf(subs=numvals).rhs
        plot(x_t, (t,0,25,200), xlabel="$t$", ylabel="$x(t)$", ylim=(-1,1))

        # Plot x(t) for various v0.
        fvals = {beta:S(1)/5, w2:S(1)/10, x0:1}
        x_t = omech.x.evalf(subs=fvals).rhs
        v_t = diff(x_t, t)
        fx = lambda i:x_t.subs(v0,i) # Lambda function
        fv = lambda i:v_t.subs(v0,i)
        x_funcs = list(map(fx, np.arange(-1,1,.25)))
        p = plot(*x_funcs, (t,0,25,200), xlabel="$t$", ylabel="$x(t)$",␣
↪legend=True)
        for i,ival in enumerate(np.arange(-1,1,.25)): p[i].label =␣
↪"v0="+str(ival) # Prepare legend texts.
        p.show()

        # Plot phase diagram, x' versus x.
```

```
        x_funcs = list(map(fx, np.arange(-2,2.25,.25)))
        v_funcs = list(map(fv, np.arange(-2,2.25,.25)))
        p = plot_parametric(*list(zip(x_funcs, v_funcs)), (t,0,25), xlabel="x",␣
↪ylabel="x'", legend=True)
        for i,ival in enumerate(np.arange(-2,2.25,.25)): p[i].label = ival #␣
↪Prepare legend texts.
        p.show()
```

```
'2.4.8.4 Damped Harmonic Oscillator, p133.'
```

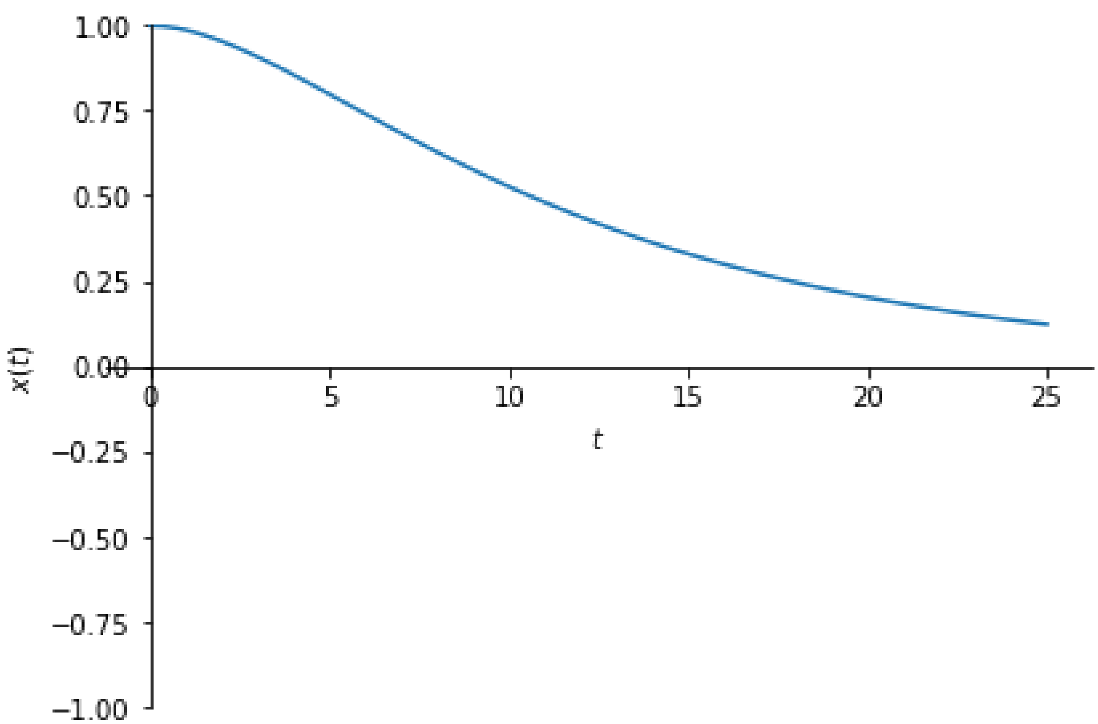
```
'General Solution'
```

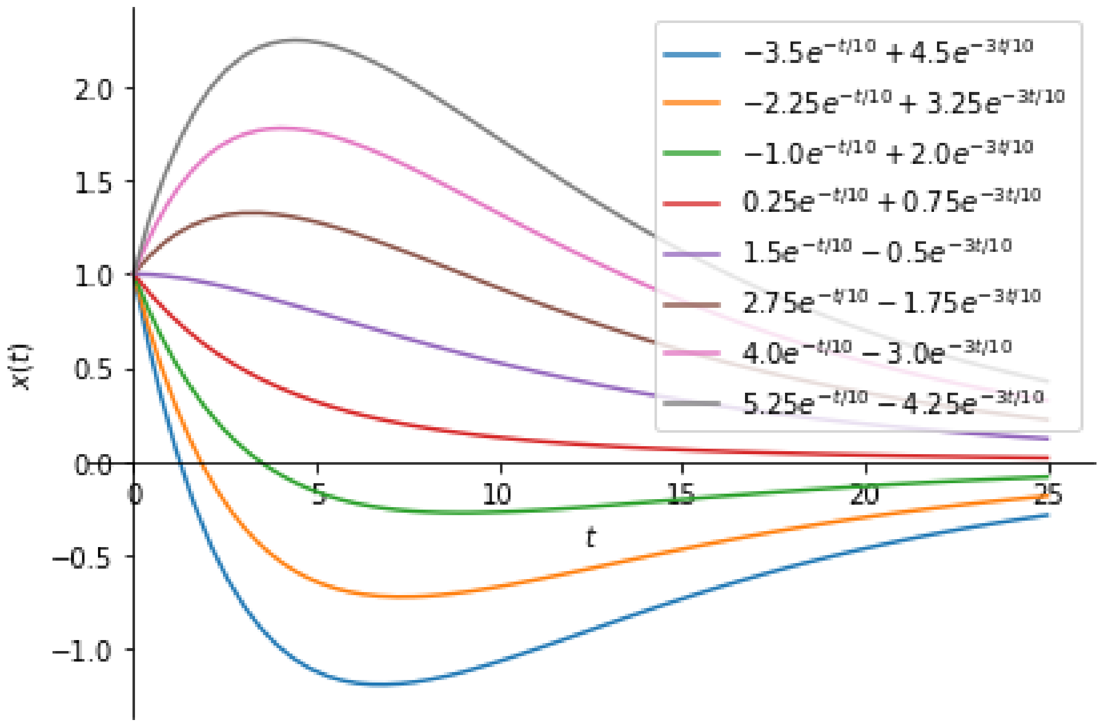
```
'Overdamped Motion'
```

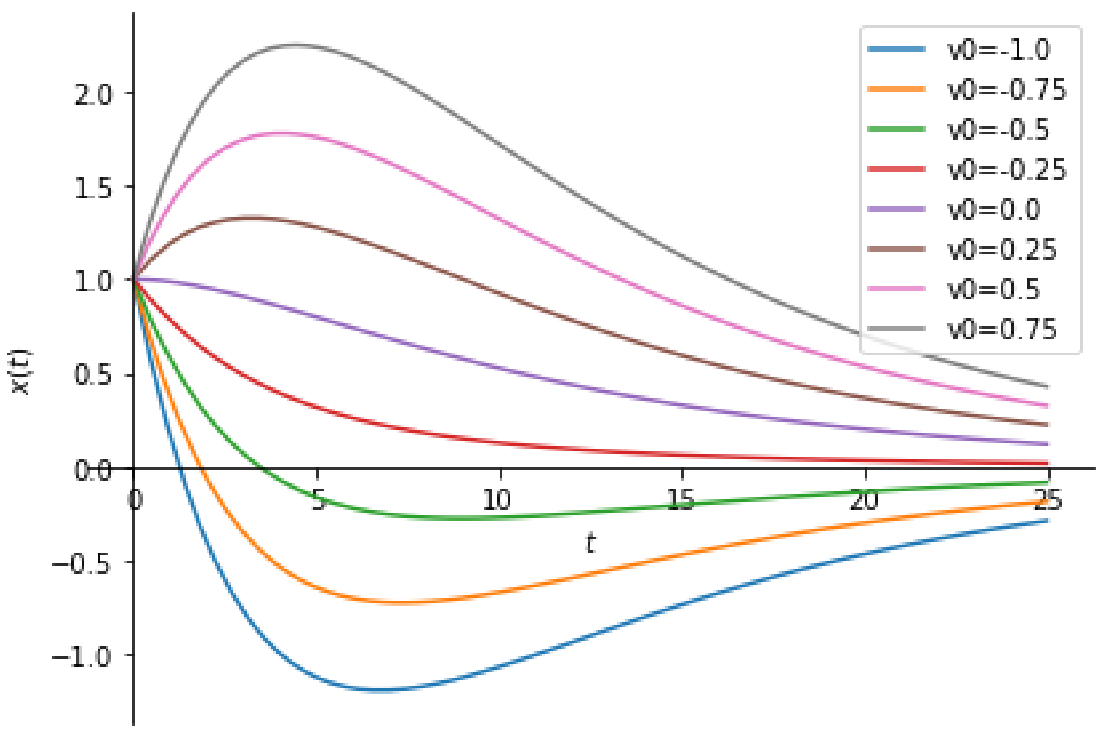
$$\left\{w_0 : \sqrt{\beta^2 - w_2^2}\right\}$$

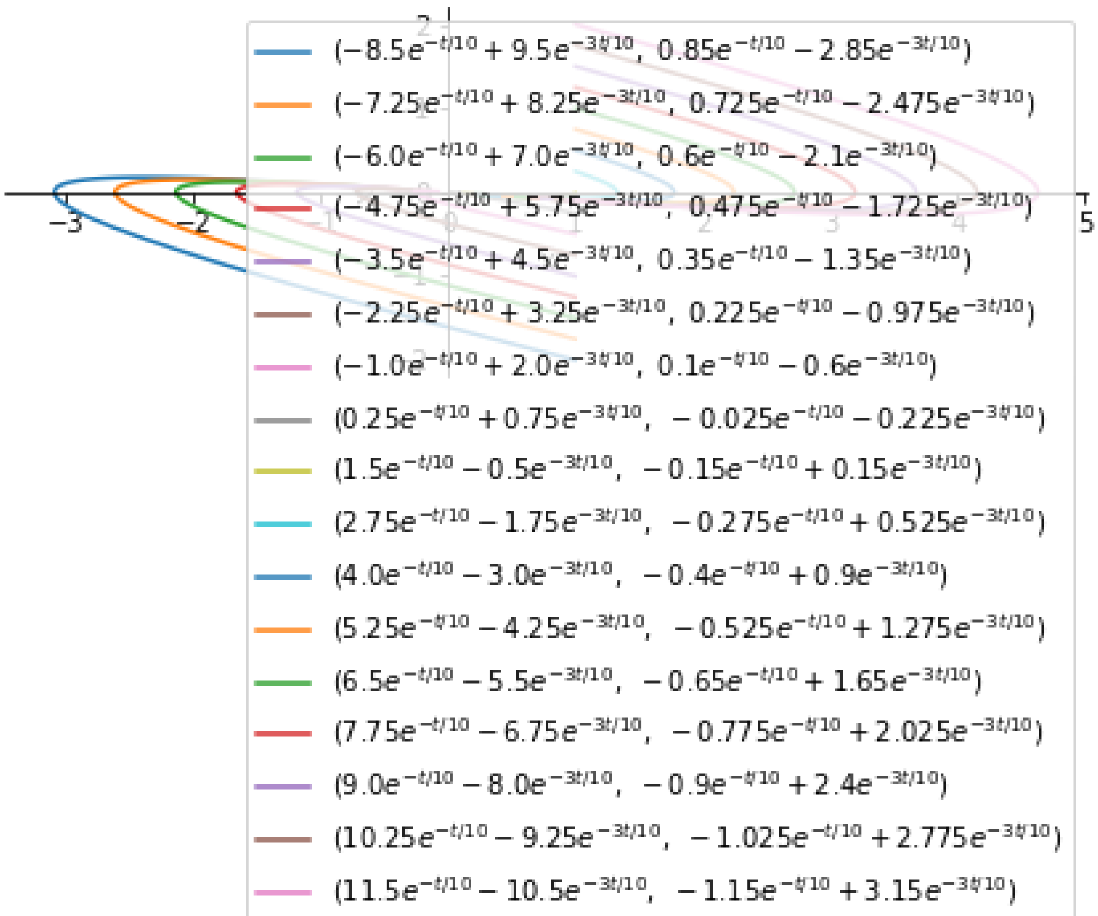
$$x(t) = \frac{\left(-\beta x_0 - v_0 + w_2 x_0\right) e^{-t(\beta + w_2)}}{2w_2} + \frac{\left(\beta x_0 + v_0 + w_2 x_0\right) e^{t(-\beta + w_2)}}{2w_2}$$

$$\frac{\partial}{\partial t} x(t) = \frac{\left(-\beta x_0 - v_0 + w_2 x_0\right) e^{-t(\beta + w_2)}}{2w_2} + \frac{\left(\beta x_0 + v_0 + w_2 x_0\right) e^{t(-\beta + w_2)}}{2w_2}$$

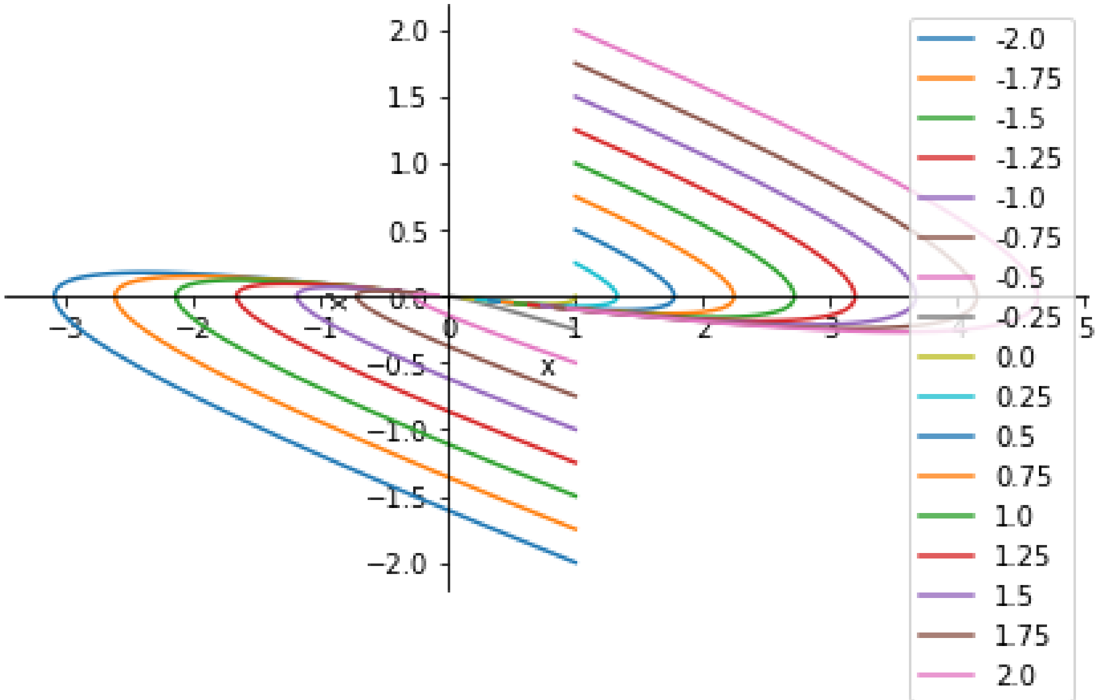



$-3.5e^{-t/10} + 4.5e^{-3t/10}$
$-2.25e^{-t/10} + 3.25e^{-3t/10}$
$-1.0e^{-t/10} + 2.0e^{-3t/10}$
$0.25e^{-t/10} + 0.75e^{-3t/10}$
$1.5e^{-t/10} - 0.5e^{-3t/10}$
$2.75e^{-t/10} - 1.75e^{-3t/10}$
$4.0e^{-t/10} - 3.0e^{-3t/10}$
$5.25e^{-t/10} - 4.25e^{-3t/10}$
$x(t)$
t
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
25


v0=-1.0
v0=-0.75
v0=-0.5
v0=-0.25
v0=0.0
v0=0.25
v0=0.5
v0=0.75
$x(t)$
t
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
25



$(-8.5e^{-t/10} + 9.5e^{-3t/10},\ 0.85e^{-t/10} - 2.85e^{-3t/10})$
$(-7.25e^{-t/10} + 8.25e^{-3t/10},\ 0.725e^{-t/10} - 2.475e^{-3t/10})$
$(-6.0e^{-t/10} + 7.0e^{-3t/10},\ 0.6e^{-t/10} - 2.1e^{-3t/10})$
$(-4.75e^{-t/10} + 5.75e^{-3t/10},\ 0.475e^{-t/10} - 1.725e^{-3t/10})$
$(-3.5e^{-t/10} + 4.5e^{-3t/10},\ 0.35e^{-t/10} - 1.35e^{-3t/10})$
$(-2.25e^{-t/10} + 3.25e^{-3t/10},\ 0.225e^{-t/10} - 0.975e^{-3t/10})$
$(-1.0e^{-t/10} + 2.0e^{-3t/10},\ 0.1e^{-t/10} - 0.6e^{-3t/10})$
$(0.25e^{-t/10} + 0.75e^{-3t/10},\ -0.025e^{-t/10} - 0.225e^{-3t/10})$
$(1.5e^{-t/10} - 0.5e^{-3t/10},\ -0.15e^{-t/10} + 0.15e^{-3t/10})$
$(2.75e^{-t/10} - 1.75e^{-3t/10},\ -0.275e^{-t/10} + 0.525e^{-3t/10})$
$(4.0e^{-t/10} - 3.0e^{-3t/10},\ -0.4e^{-t/10} + 0.9e^{-3t/10})$
$(5.25e^{-t/10} - 4.25e^{-3t/10},\ -0.525e^{-t/10} + 1.275e^{-3t/10})$
$(6.5e^{-t/10} - 5.5e^{-3t/10},\ -0.65e^{-t/10} + 1.65e^{-3t/10})$
$(7.75e^{-t/10} - 6.75e^{-3t/10},\ -0.775e^{-t/10} + 2.025e^{-3t/10})$
$(9.0e^{-t/10} - 8.0e^{-3t/10},\ -0.9e^{-t/10} + 2.4e^{-3t/10})$
$(10.25e^{-t/10} - 9.25e^{-3t/10},\ -1.025e^{-t/10} + 2.775e^{-3t/10})$
$(11.5e^{-t/10} - 10.5e^{-3t/10},\ -1.15e^{-t/10} + 3.15e^{-3t/10})$
−3
−2
5



0.5 Driven_Oscillations

[8]:
```
#----> Driven_Oscillations
if "Driven_Oscillations" in sets.flow:
    # simple_harmonic_oscillator_scalar
    # General Solution
    pprints("2.4.8.5 Driven Oscillations, p145",
            "General Solution")
    omech.__init__("scalar")
    omech.verbose = False
    pprints("Differential Equation",
             omech.driven_oscillator1,
             omech.driven_oscillator2)
    commands = ["dsolve", "driven_oscillator1", omech.x]
    omech.process(commands)
    commands = ["dsolve", "driven_oscillator2", omech.x]
    omech.x = omech.process(commands).rhs
    v = omech.v.evalf(subs={x:omech.x}).doit()
    display(omech.x, v)

    # General Solution
    sol_particular = simplify(omech.x.subs({C1:0,C2:0}))
    sol_complementary = together(simplify(omech.x-sol_particular))
```

```
    amplitude = sol_particular.subs({t:0})
    omech.scaled_amplitude = scaled_amplitude = sol_particular.subs({t:0})/A
    omech.phase = numer(omech.scaled_amplitude)/sqrt(denom(omech.
↪scaled_amplitude))
    omech.amplitude = 1/sqrt(denom(omech.scaled_amplitude))

    pprints(
            """The solution consists of two parts. The first part represents \
            the complementary solution containing initial conditions denoted by␣
↪the \
            constants of integration C1 and C2. The second part is the␣
↪particular \
            solution free of any constant of integration. This part is present␣
↪in any case \
            independent of the initial conditions.""",
            "General Solution",
            "x(t)=", omech.x,
            "Particular Solution",
            "C1->0, C2->0",
            "x_p(t)=", sol_particular,
            "Complementary Solution",
            "x_c(t)=", sol_complementary,
            "Amplitude", amplitude,
            "Scaled amplitude= delta = Delta/A", scaled_amplitude,
            "Phase=", omech.phase,
            "Amplitude=", omech.amplitude)

    # Numerical calculations.
    # Plot scaled amplitude versus w.
    fixed_vals = {A:1, w0:1}
    param_vals = np.arange(0.1,1.2,0.1)
    A_w_funcs = get_iterated_functions(omech.scaled_amplitude, fixed_vals,␣
↪beta, param_vals)
    p = plot(*A_w_funcs, (w,0,3,200), xlabel=r"$\omega$", ylabel=r"$\Delta /
↪A$", legend=True)
    for i,ival in enumerate(np.arange(0.1,1.2,0.1)): p[i].label =␣
↪"beta="+str(ival) # Prepare legend texts.
    p.show()

    # Plot amplitude versus w.
    A_w_funcs = get_iterated_functions(omech.amplitude, fixed_vals, beta,␣
↪param_vals)
    p = plot(*A_w_funcs, (w,0,4), xlabel=r"$\omega$", ylabel=r"$\Delta /A$",␣
↪legend=True)
    for i,ival in enumerate(np.arange(0.1,1.2,0.1)): p[i].label =␣
↪"beta="+f"{ival:.1f}" # Prepare legend texts.
```

```python
p.show()

# Solve Driven Oscillator Differential Equation
omech.class_type = "scalar"
omech.__init__()
omech.verbose = True
initial_conds = {omech.x.subs({t:0}):0,
                 diff(omech.x, t).subs({t:0}):0}

"""
OR todo fix erros.
commands = ["dsolve", "driven_oscillator2", omech.x, initial_conds]
omech.x = omech.process(commands).rhs
"""
omech.x = dsolve(omech.driven_oscillator2,
                 omech.x,
                 ics = initial_conds)

pprints("Solution of Driven Oscillator Differential Equation",
        "x(t)", omech.x,
        "simplified solution x(t)", simplify(omech.x),
        "with initial conditions",
        initial_conds)

# Plot x(t).
numvals = {A:1, beta:0.1, w0:2, w:1}
x_t = omech.x.rhs.evalf(subs=numvals) # x_t = omech.x.srhs.ubs(numvals)
plot(x_t, (t,0,25,300), xlabel="$t$", ylabel="$x(t)$", ylim=(-0.75,0.75))
```

'2.4.8.5 Driven Oscillations, p145'

'General Solution'

'Differential Equation'

$$\gamma\frac{d}{dt}x(t) + kx(t) + m\frac{d^2}{dt^2}x(t) = F_0\cos(tw)$$

$$2\beta\frac{d}{dt}x(t) + w_0^2x(t) + \frac{d^2}{dt^2}x(t) = A\cos(tw)$$

$$x(t) = C_1e^{\frac{t\left(-\gamma+\sqrt{\gamma^2-4km}\right)}{2m}} + C_2e^{-\frac{t\left(\gamma+\sqrt{\gamma^2-4km}\right)}{2m}} + \frac{F_0\gamma w\sin(tw)}{\gamma^2w^2+k^2-2kmw^2+m^2w^4} + \frac{F_0k\cos(tw)}{\gamma^2w^2+k^2-2kmw^2+m^2w^4} - \frac{F_0mw^2\cos(tw)}{\gamma^2w^2+k^2-2kmw^2+m^2w^4}$$

$$x(t) = \frac{2A\beta w\sin(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} - \frac{Aw^2\cos(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} + \frac{Aw_0^2\cos(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} +$$

$C_1 e^{t\left(-\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)} + C_2 e^{-t\left(\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)}$

$$\frac{2A\beta w \sin(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} - \frac{Aw^2\cos(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} + \frac{Aw_0^2\cos(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} + C_1 e^{t\left(-\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)} + C_2 e^{-t\left(\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)}$$

$$v = \frac{2A\beta w^2\cos(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} + \frac{Aw^3\sin(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} - \frac{Aww_0^2\sin(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} + C_1\left(-\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)e^{t\left(-\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)} + C_2\left(-\beta-\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)e^{-t\left(\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)}$$

'The solution consists of two parts. The first part represents the complementary so

'General Solution'

'x(t)='

$$\frac{2A\beta w \sin(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} - \frac{Aw^2\cos(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} + \frac{Aw_0^2\cos(tw)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4} + C_1 e^{t\left(-\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)} + C_2 e^{-t\left(\beta+\sqrt{\beta-w_0}\sqrt{\beta+w_0}\right)}$$

'Particular Solution'

'C1->0, C2->0'

'x_p(t)='

$$\frac{A\left(2\beta w\sin(tw) - w^2\cos(tw) + w_0^2\cos(tw)\right)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4}$$

'Complementary Solution'

'x_c(t)='

$$\left(C_1 e^{2t\sqrt{\beta-w_0}\sqrt{\beta+w_0}} + C_2\right)e^{-\beta t}e^{-t\sqrt{\beta-w_0}\sqrt{\beta+w_0}}$$

'Amplitude'

$$\frac{A\left(-w^2+w_0^2\right)}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4}$$

'Scaled amplitude= delta = Delta/A'

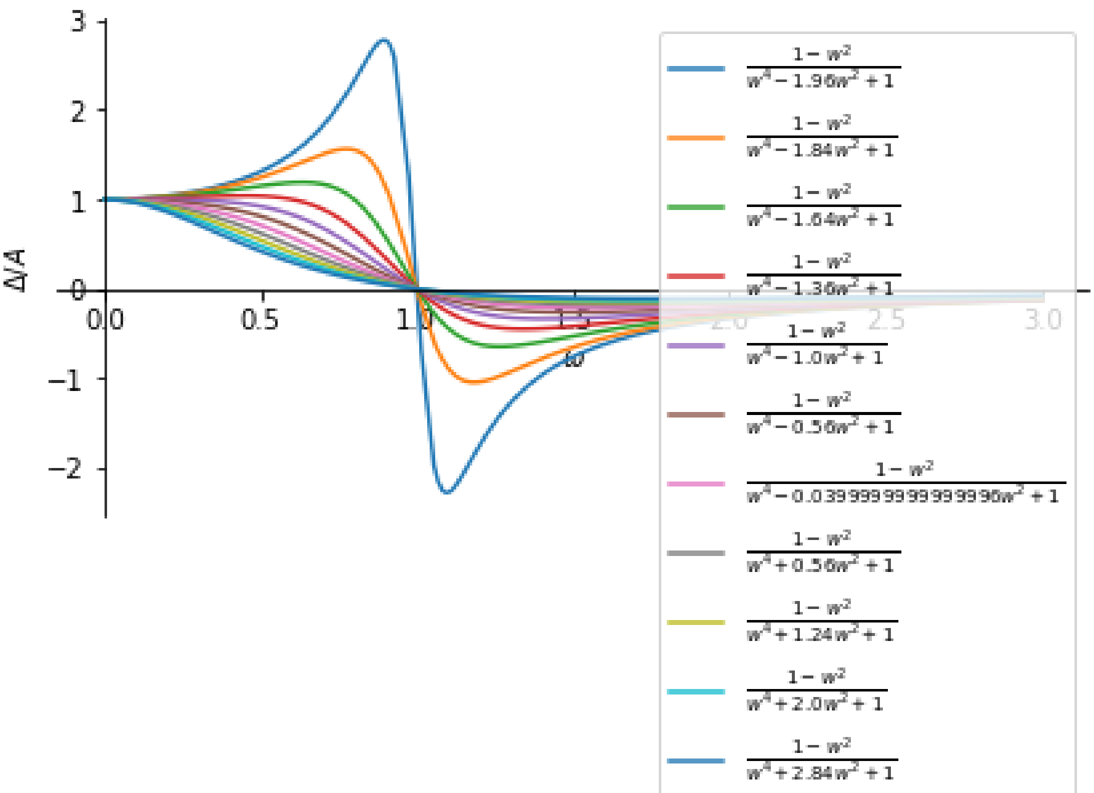
$$\frac{-w^2+w_0^2}{4\beta^2w^2+w^4-2w^2w_0^2+w_0^4}$$

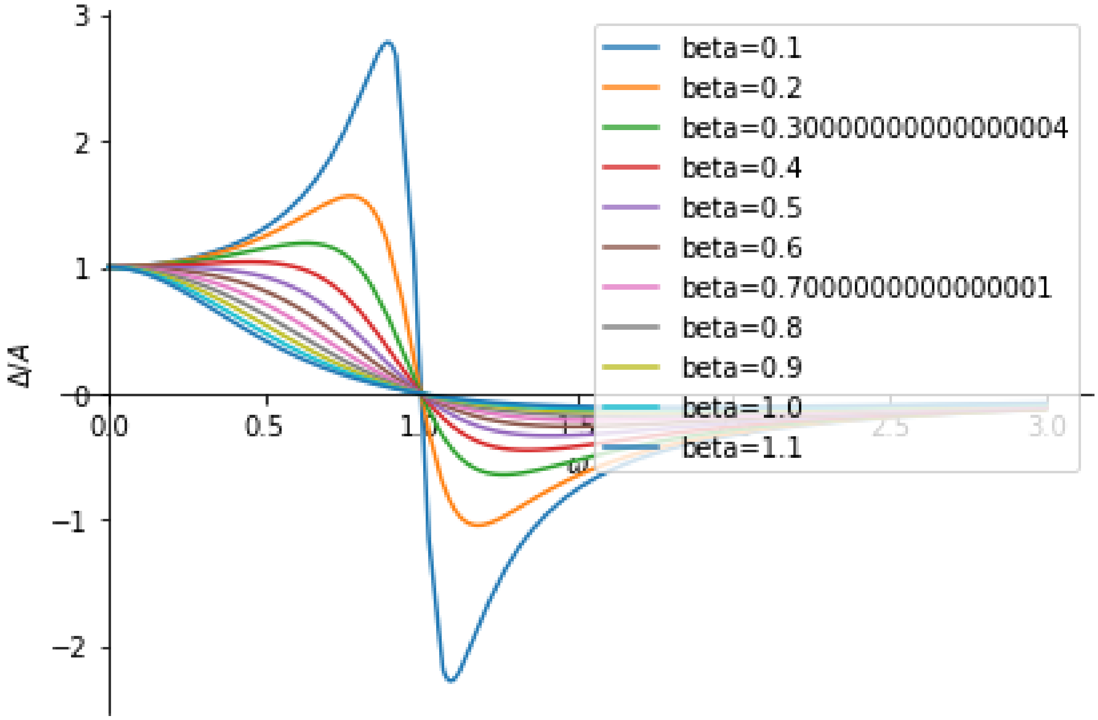
'Phase='

$$\frac{-w^2 + w_0^2}{\sqrt{4\beta^2 w^2 + w^4 - 2w^2 w_0^2 + w_0^4}}$$

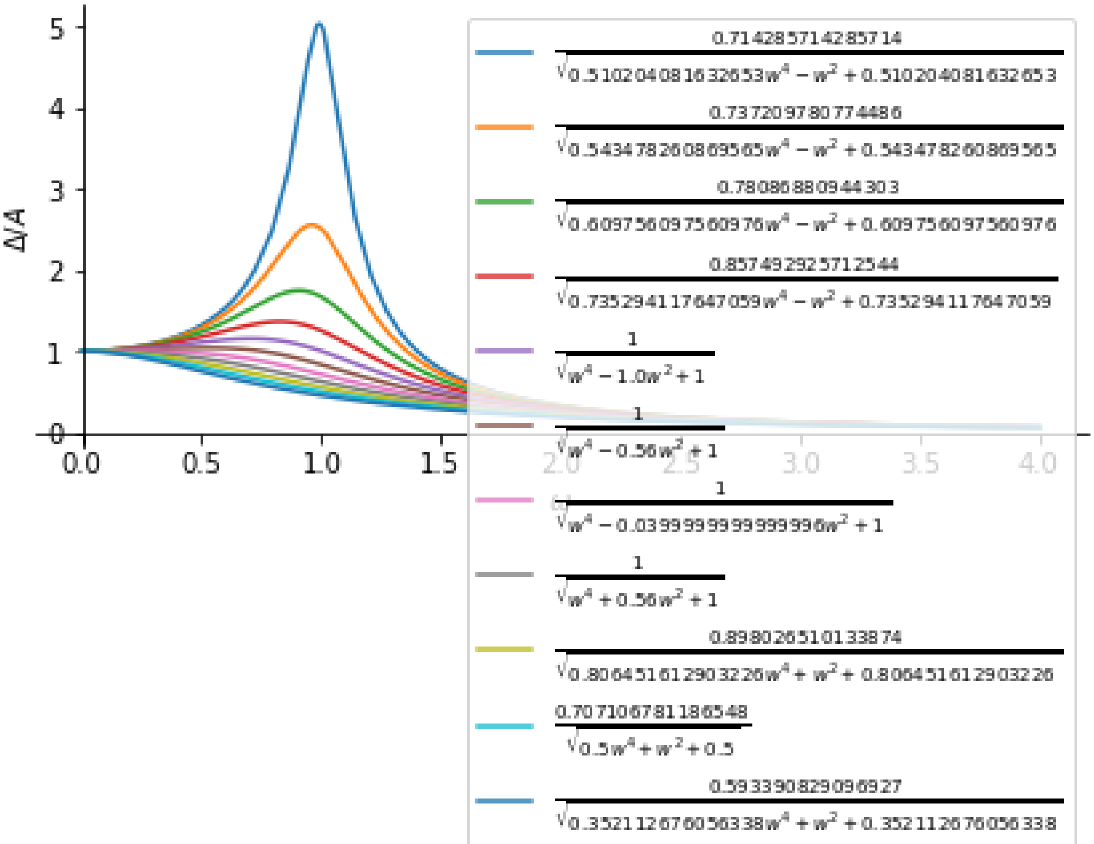
'Amplitude='

$$\frac{1}{\sqrt{4\beta^2 w^2 + w^4 - 2w^2 w_0^2 + w_0^4}}$$

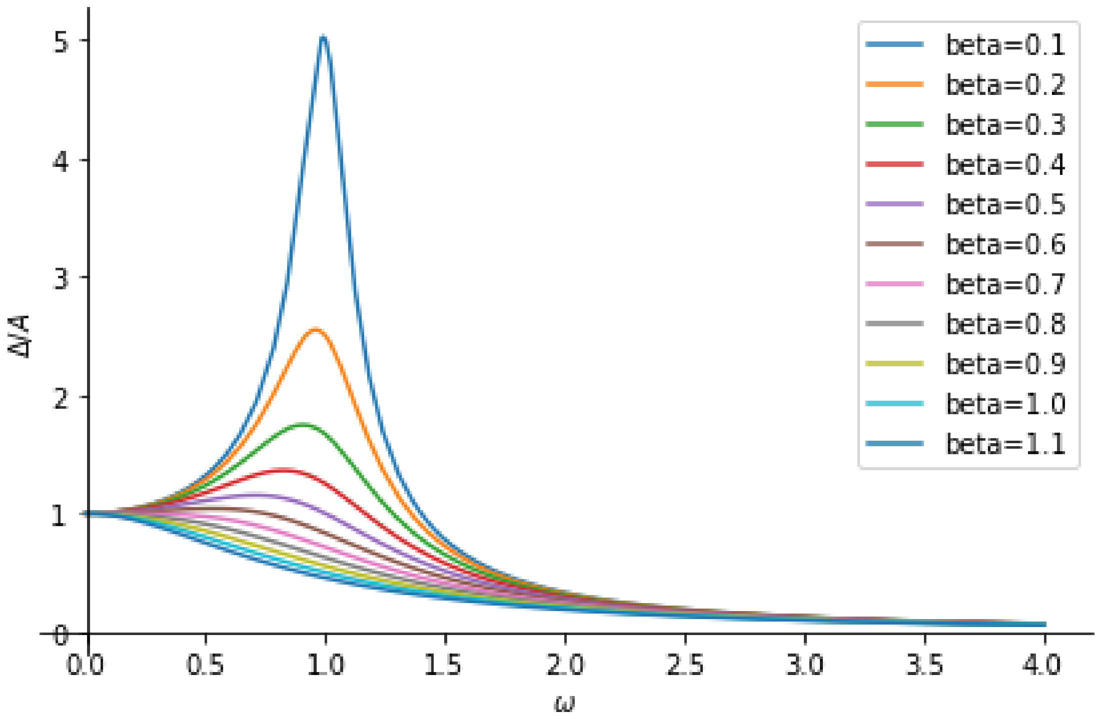




beta=0.1
beta=0.2
beta=0.30000000000000004
beta=0.4
beta=0.5
beta=0.6
beta=0.700000000000001
beta=0.8
beta=0.9
beta=1.0
beta=1.1
Δ/A
ω
3
2
1
0
−1
−2
0.0
0.5
1.0
1.5
2.5
3.0



5
4
3
2
1
0
Δ/A
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$\frac{0.714285714285714}{\sqrt{0.510204081632653w^4 - w^2 + 0.510204081632653}}$
$\frac{0.737209780774486}{\sqrt{0.543478260869565w^4 - w^2 + 0.543478260869565}}$
$\frac{0.78086880944303}{\sqrt{0.609756097560976w^4 - w^2 + 0.609756097560976}}$
$\frac{0.857492925712544}{\sqrt{0.735294117647059w^4 - w^2 + 0.735294117647059}}$
$\frac{1}{\sqrt{w^4 - 1.0w^2 + 1}}$
$\frac{1}{\sqrt{w^4 - 0.36w^2 + 1}}$
$\frac{1}{\sqrt{w^4 - 0.0399999999999996w^2 + 1}}$
$\frac{1}{\sqrt{w^4 + 0.36w^2 + 1}}$
$\frac{0.898026510133874}{\sqrt{0.806451612903226w^4 + w^2 + 0.806451612903226}}$
$\frac{0.707106781186548}{\sqrt{0.5w^4 + w^2 + 0.5}}$
$\frac{0.593390829096927}{\sqrt{0.352112676056338w^4 + w^2 + 0.352112676056338}}$



'Solution of Driven Oscillator Differential Equation'

'x(t)'

$$x(t) = \frac{2A\beta w \sin(tw)}{4\beta^2 w^2 + w^4 - 2w^2 w_0^2 + w_0^4} - \frac{Aw^2 \cos(tw)}{4\beta^2 w^2 + w^4 - 2w^2 w_0^2 + w_0^4} + \frac{Aw_0^2 \cos(tw)}{4\beta^2 w^2 + w^4 - 2w^2 w_0^2 + w_0^4} +$$

$$\left(-\frac{A\beta w^2}{8\beta^2 w^2\sqrt{\beta - w_0}\sqrt{\beta + w_0} + 2w^4\sqrt{\beta - w_0}\sqrt{\beta + w_0} - 4w^2 w_0^2\sqrt{\beta - w_0}\sqrt{\beta + w_0} + 2w_0^4\sqrt{\beta - w_0}\sqrt{\beta + w_0}} - \frac{}{8\beta^2 w^2\sqrt{}}\right.$$

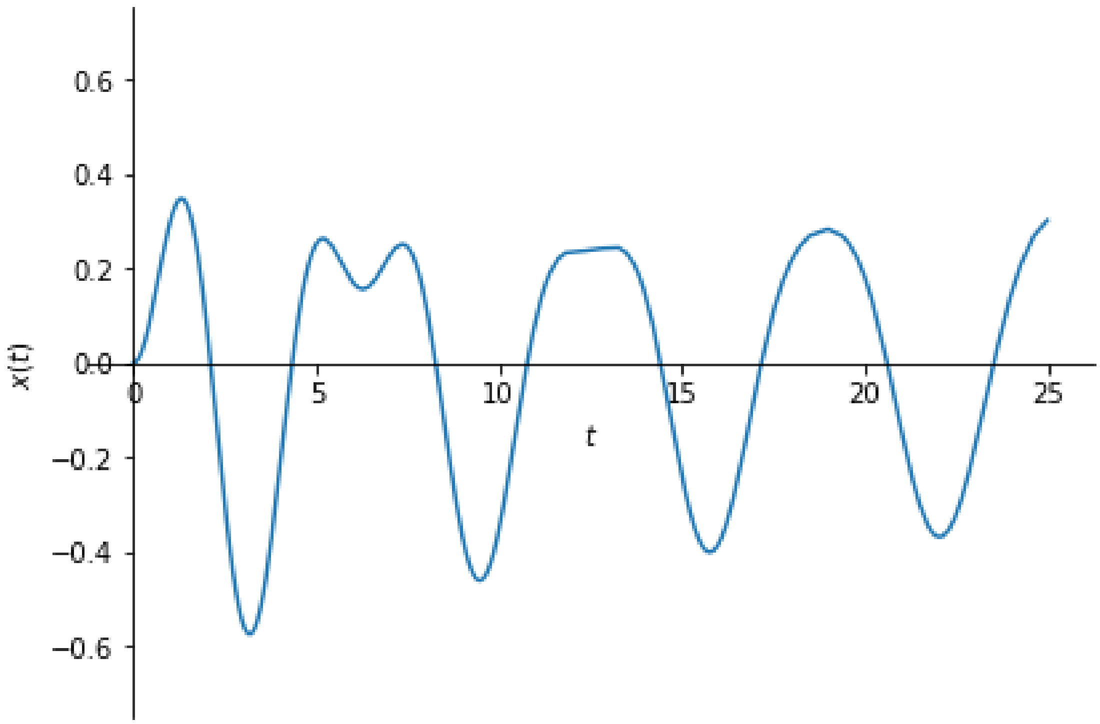
$$\left(\frac{A\beta w^2}{8\beta^2 w^2\sqrt{\beta - w_0}\sqrt{\beta + w_0} + 2w^4\sqrt{\beta - w_0}\sqrt{\beta + w_0} - 4w^2 w_0^2\sqrt{\beta - w_0}\sqrt{\beta + w_0} + 2w_0^4\sqrt{\beta - w_0}\sqrt{\beta + w_0}} + \frac{}{8\beta^2 w^2\sqrt{\beta}}\right.$$

'simplified solution x(t)'

$$x(t) = \frac{A\left(2(\beta - w_0)(\beta + w_0)\left(2\beta w \sin(tw) - w^2\cos(tw) + w_0^2\cos(tw)\right)e^{2\beta t} + \sqrt{\beta^2 - w_0^2}\left(-\beta w^2 - \beta w_0^2 + w^2\sqrt{\beta^2}\right.\right.}{2(\beta - w_0)(\beta +}$$

'with initial conditions'

$$\left\{x(0) : 0,\ \left.\frac{d}{dt}x(t)\right|_{t=0} : 0\right\}$$



0.6 Driven_Oscillations_The_Laplace_Transform_Method

[4]:
```
#----> Driven_Oscillations_The_Laplace_Transform_Method
if "Driven_Oscillations_The_Laplace_Transform_Method" in sets.flow:
    pprints("2.4.8.6a Solution Procedures of Linear Differential Equations,␣
↪p154",
            "The Laplace Transform Method")

    """
    sudo pip3 install wolframclient
    from wolframclient.evaluation import WolframLanguageSession
    from wolframclient.language import wl, wlexpr
    session = WolframLanguageSession()

    math_expr = "InverseLaplaceTransform[{0}, p, t]".
↪format(mathematica_code(sol_IC_lap_trans))
    math_result = session.evaluate(wlexpr(math_expr))
    convert_str = 'ExportString[{0}, "PythonExpression"]'.format(math_result)
    session.evaluate(wlexpr(convert_str))
    """
    omech.__init__("scalar")
    omech.verbose = True
```

```
    pprints("Differential Equation of The Driven Oscillator",
            omech.driven_oscillator1,
            omech.driven_oscillator2,
            "laplace_transform(exp(-alpha*t), t, p)=",
            laplace_transform(exp(-alpha*t), t, p, noconds=True))

    # Laplace Transform Method
    # 1. Way: By using sympy.
    lap_trans = Eq(laplace_transform(omech.driven_oscillator2.lhs, t, p),
                   laplace_transform(omech.driven_oscillator2.rhs, t, p,␣
↪noconds=True))
    IC_lap_trans = lap_trans.subs({omech.x.subs({t:0}):0, diff(omech.x, t).
↪subs({t:0}):0})  # Set initial conditions.
    sol_IC_lap_trans = factor(solve(IC_lap_trans, LaplaceTransform(omech.x, t,␣
↪p)))[0]      # Solve for L(x(t))

    pprints("Laplace transform of the differential equation",
            lap_trans,
            "Apply initial conditions to Laplace transform",
            IC_lap_trans,
            "Solve algebraic equation for L(x(t))",
            sol_IC_lap_trans)

    # 2. Way: By using libphysics.
    substitutions = {omech.x.subs({t:0}):0, diff(omech.x, t).subs({t:0}):0}
    commands = ["laplace_transform", "driven_oscillator2", (t,p)]
    omech.process(commands)
    commands = ["subs", "omech.result", substitutions]
    omech.process(commands)
    commands = ["solve", "omech.result", LaplaceTransform(omech.x, t, p)]
    display(factor(omech.process(commands)))

    # sol_diffeq = inverse_laplace_transform(sol_IC_lap_trans, p, t)

    # Plot x(t) graph.
    # fixed_vals = {A:1, w0:2, beta:4, w:1}
    # x_t = simplify(sol_diffeq.subs(fixed_vals))
    # plot(x_t, (t,0,25,500), xlabel="$t$", ylabel="$x(t)$")

    # Calling Mathematica for evaluating the inverse Laplace transformation.
    # 1. Way sympy -> latex -> evaluate ERROR PRONE!!! in multiplications.
    """
    # Ap != A*p becomes after latex conversion.

    import re
    from sympy.parsing.mathematica import parse_mathematica
    from sympy.parsing.latex import parse_latex
```

```
    session = WolframLanguageSession()

    inputTex = latex(sol_IC_lap_trans)
    inputMath = f'ToExpression["{inputTex}", TeXForm]'
    math_expr = f"InverseLaplaceTransform[{inputMath}, p, t]"
    math_expr = re.sub(r'\\', r'\\\\', math_expr)
    math_result = session.evaluate(wlexpr(math_expr))
    math_result = str(math_result).replace("<<","").replace(">>", "")
    pprint(parse_mathematica(math_result))
    """

    # Call Mathematica for evaluating the inverse Laplace transformation.
    # 2. Way sympy -> evaluate
    from sympy.parsing.mathematica import parse_mathematica
    session = WolframLanguageSession()
    math_expr = wlexpr(mathematica_code(sol_IC_lap_trans))
    math_expr = str(math_expr).replace("w_0","w0")
    math_expr = f"InverseLaplaceTransform[{math_expr}, p, t]"
    math_result = session.evaluate(session.normalize_input(math_expr))
    math_result = str(math_result).replace("<<","").replace(">>", "")
    # parse_mathematica(math_result)
    print(math_result)
```

'2.4.8.6a Solution Procedures of Linear Differential Equations, p154'

'The Laplace Transform Method'

'Differential Equation of The Driven Oscillator'

$$\gamma \frac{d}{dt} x(t) + kx(t) + m \frac{d^2}{dt^2} x(t) = F_0 \cos(tw)$$

$$2\beta \frac{d}{dt} x(t) + w_0^2 x(t) + \frac{d^2}{dt^2} x(t) = A \cos(tw)$$

'laplace_transform(exp(-alpha*t), t, p)='

$$\frac{1}{\alpha + p}$$

'Laplace transform of the differential equation'

$$2\beta \left(p\mathcal{L}_t\left[x(t)\right](p) - x(0)\right) + p^2 \mathcal{L}_t\left[x(t)\right](p) - px(0) + w_0^2 \mathcal{L}_t\left[x(t)\right](p) - \left.\frac{d}{dt} x(t)\right|_{t=0} = \frac{Ap}{p^2 + w^2}$$

'Apply initial conditions to Laplace transform'

$$2\beta p\mathcal{L}_t\left[x(t)\right](p) + p^2\mathcal{L}_t\left[x(t)\right](p) + w_0^2\mathcal{L}_t\left[x(t)\right](p) = \frac{Ap}{p^2+w^2}$$

```
'Solve algebraic equation for L(x(t))'
```

$$\frac{Ap}{(p^2+w^2)\left(2\beta p + p^2 + w_0^2\right)}$$

```
'laplace_transform driven_oscillator2 (t, p)'


Laplace transform of the driven_oscillator2 equation.
Eq(laplace_transform(2*beta*Derivative(x(t), t) + w0**2*x(t) + Derivative(x(t),
(t, 2)), t, p), laplace_transform(A*cos(t*w), t, p, noconds=True))
```

$$2\beta\left(p\mathcal{L}_t\left[x(t)\right](p) - x(0)\right) + p^2\mathcal{L}_t\left[x(t)\right](p) - px(0) + w_0^2\mathcal{L}_t\left[x(t)\right](p) - \left.\frac{d}{dt}x(t)\right|_{t=0} = \frac{Ap}{p^2+w^2}$$

```
'subs omech.result {x(0): 0, Subs(Derivative(x(t), t), t, 0): 0}'


Eq(2*beta*(p*LaplaceTransform(x(t), t, p) - x(0)) + p**2*LaplaceTransform(x(t),
t, p) - p*x(0) + w0**2*LaplaceTransform(x(t), t, p) - Subs(Derivative(x(t), t),
t, 0), A*p/(p**2 + w**2))(subs, {x(0): 0, Subs(Derivative(x(t), t), t, 0): 0})
```

$$2\beta p\mathcal{L}_t\left[x(t)\right](p) + p^2\mathcal{L}_t\left[x(t)\right](p) + w_0^2\mathcal{L}_t\left[x(t)\right](p) = \frac{Ap}{p^2+w^2}$$

```
'solve omech.result LaplaceTransform(x(t), t, p)'


solve(Eq(2*beta*p*LaplaceTransform(x(t), t, p) + p**2*LaplaceTransform(x(t), t,
p) + w0**2*LaplaceTransform(x(t), t, p), A*p/(p**2 + w**2)),
LaplaceTransform(x(t), t, p))
```

$$\left[\frac{Ap}{2\beta p^3 + 2\beta p w^2 + p^4 + p^2w^2 + p^2w_0^2 + w^2w_0^2}\right]$$

$$\left[\frac{Ap}{(p^2+w^2)\left(2\beta p + p^2 + w_0^2\right)}\right]$$

```
Times[Global`A, Plus[Times[Rational[-1, 2], Power[Plus[Power[Global`beta, 2],
Times[-1, Power[Global`w0, 2]]], Rational[-1, 2]], Power[Plus[Times[4,
Power[Global`beta, 2], Power[Global`w, 2]], Power[Global`w, 4], Times[-2,
Power[Global`w, 2], Power[Global`w0, 2]], Power[Global`w0, 4]], -1],
Plus[Times[-1, Global`beta, Power[E, Times[Global`t, Plus[Times[-1,
Global`beta], Times[-1, Power[Plus[Power[Global`beta, 2], Times[-1,
Power[Global`w0, 2]]], Rational[1, 2]]]]]], Power[Global`w, 2]],
Times[Global`beta, Power[E, Times[Global`t, Plus[Times[-1, Global`beta],
Power[Plus[Power[Global`beta, 2], Times[-1, Power[Global`w0, 2]]], Rational[1,
2]]]]], Power[Global`w, 2]],  4 , Times[Power[E, Times[Global`t, Plus[Times[-1,
Global`beta], Times[-1, Power[Plus[Power[Global`beta, 2], Times[-1,
Power[Global`w0, 2]]], Rational[1, 2]]]]]], Power[Global`w0, 2],
```

```
Power[Plus[Power[Global`beta, 2], Times[-1, Power[Global`w0, 2]]], Rational[1,
2]]], Times[Power[E, Times[Global`t, Plus[Times[-1, Global`beta],
Power[Plus[Power[Global`beta, 2], Times[-1, Power[Global`w0, 2]]], Rational[1,
2]]]]], Power[Global`w0, 2], Power[Plus[Power[Global`beta, 2], Times[-1,
Power[Global`w0, 2]]], Rational[1, 2]]]]], Times[Power[Plus[Times[4,
Power[Global`beta, 2], Power[Global`w, 2]], Power[Global`w, 4], Times[-2,
Power[Global`w, 2], Power[Global`w0, 2]], Power[Global`w0, 4]], -1],
Plus[Times[Plus[Times[-1, Power[Global`w, 2]], Power[Global`w0, 2]],
Cos[Times[Global`t, Global`w]]], Times[2, Global`beta, Global`w,
Sin[Times[Global`t, Global`w]]]]]]]
```

0.7 Driven_Oscillations_Greens_Function_Method

[5]:
```
#----> Driven_Oscillations_Greens_Function_Method
if "Driven_Oscillations_Greens_Function_Method" in sets.flow:
    pprints("2.4.8.6b Solution Procedures of Linear Differential Equations,␣
↪p158",
            "Green's Function Method",
            "FAILED at Green's Function Implementation !!!")
    """
    G_conds = {omech.G.subs({t:0}):omech.G_t_tau.subs({t:tau, tau:tau}),
               diff(omech.G, t).subs({t:0}):diff(omech.G_t_tau.subs({t:tau, tau:
↪tau}),t)}

    References:
        Dean G. Duffy, Greens Functions with Applications, 2nd Edition, CRC␣
↪Press, 2015.
    """
    omech.__init__("scalar")
    omech.verbose = True

    substitutions = {omech.x:omech.G, omech.driven_oscillator3.rhs:
↪DiracDelta(t-tau)}
    commands = ["subs", "driven_oscillator3", substitutions]
    omech.process(commands)
    eq_green_func = omech.result

    method = {1:"Laplace_transform", 2:"Fourier_transform"}[2]
    if method == "Laplace_transform":
        # Laplace Transform Method
        """
        Laplace transform in sympy cannot handle functions with more than 1␣
↪variable.
        """
    #    lap_trans = Eq(laplace_transform(green_func_eq.lhs, t, p,␣
↪noconds=True),
```

```
    #                   laplace_transform(green_func_eq.rhs, t, p,␣
↪noconds=True))
    #    IC_lap_trans = lap_trans.subs({omech.G(t,tau):omech.G(0,tau),␣
↪diff(omech.G(t,tau), t):diff(omech.G(0,tau), t)})  # Set initial conditions.
        # sol_IC_lap_trans = factor(solve(IC_lap_trans, LaplaceTransform(omech.
↪G(t,tau), t, p)))[0]      # Solve for L(x(t))

        # Intial conditions
        # IC1 = Eq(G, 0)
        # IC2 = Eq(diff(G, t), 1)
        # sol1 = solve([IC1, IC2], [omech.G.subs({t:0}), diff(omech.G, t).
↪subs({t:0})])
        # display(sol1)

    elif method == "Fourier_transform":
        # Fourier Transform Method
        """
        fourier_trans = Eq(fourier_transform(eq_green_func.lhs, t, k,␣
↪noconds=True),
                           fourier_transform(DiracDelta(t-tau), t, k,␣
↪noconds=True))
        sol_fourier_trans = solve(expand(fourier_trans),␣
↪fourier_transform(omech.G, t, k))[0]
        sol_G = inverse_fourier_transform(sol_fourier_trans, k, t)
        """
        substitutions = {omech.G:omech.IFT_Gw.rhs}
        commands = ["subs", "omech.result", substitutions]
        omech.process(commands)
        omech.result = omech.result.doit()
        substitutions = {DiracDelta(t-tau):omech.IFT_Dirac_delta.rhs}
        commands = ["subs", "omech.result", substitutions]
        omech.process(commands)
        display(omech.result)
        eq_IFT_green_func1 = Eq(diff(omech.result.lhs, w), diff(omech.result.
↪rhs, w))
        eq_IFT_green_func2 = simplify(eq_IFT_green_func1)
        display(eq_IFT_green_func1, eq_IFT_green_func2)

        # eq_diff_green_func = Eq(diff(eq_green_func.lhs, w),␣
↪diff(eq_green_func.rhs, w))
        sol_Gw = solve(eq_IFT_green_func2, omech.Gw)[0]
        sol_Gt = omech.IFT_Gw.subs(omech.Gw, sol_Gw)

        # Sympy cannot solve the integral.
        # sol = integrate(sol_Gt.args[-1].args[0], w)
        # sol = integrate(sol_Gt.args[-1].args[0], (w, -inf, inf))
```

```
    # todo check below.
    # x(t) = x_homogeneous(t) + integrate(f(t)*G(t,tau), (tau,0,t))
    sol_complementary = simplify(integrate(omech.
↪G_driven_oscillator_critical_damping*omech.driven_oscillator3.rhs, (tau, 0,␣
↪t)))
    sol_complementary = simplify(integrate(omech.
↪G_driven_oscillator_weak_damping*omech.driven_oscillator3.rhs, (tau, 0, t)))
    sol_complementary = simplify(integrate(omech.
↪G_driven_oscillator_strong_damping*omech.driven_oscillator3.rhs, (tau, 0,␣
↪t)))

    omech.x = dsolve(Eq(omech.driven_oscillator2.lhs, 0), omech.x)
    omech.x = omech.x.rhs + sol_complementary

    # Plot x(t).
    numvals = {A:1, beta:0.1, w0:2, w:1, m:1, F0:1, gamma:0.2, C1:1, C2:1}
    x_t = omech.x.evalf(subs=numvals) # x_t = omech.x.srhs.ubs(numvals)
    plot(x_t, (t,0,25,300), xlabel="$t$", ylabel="$x(t)$", ylim=(-1,1))

    pprints("G(t,tau)=", sol_Gt,
            "Take integral and get G(t,tau).",
            "x(t) = x_homogeneous + integrate(G(t,tau)*f(t), t)",
            "Complementary Solution",
            "x_c(t)=", sol_complementary
            )
```

```
'2.4.8.6b Solution Procedures of Linear Differential Equations, p158'

"Green's Function Method"

"FAILED at Green's Function Implementation !!!"

'subs driven_oscillator3 {x(t): G(t, tau), F0*cos(t*w): DiracDelta(t - tau)}'

Eq(2*gamma*m*Derivative(x(t), t) + m*w0**2*x(t) + m*Derivative(x(t), (t, 2)),
F0*cos(t*w))(subs, {x(t): G(t, tau), F0*cos(t*w): DiracDelta(t - tau)})
```

$$2\gamma m\frac{\partial}{\partial t}G(t,\tau) + mw_0^2G(t,\tau) + m\frac{\partial^2}{\partial t^2}G(t,\tau) = \delta(t-\tau)$$

```
'subs omech.result {G(t, tau): sqrt(2)*Integral(Gtilde(w)*exp(I*w*(t - tau)), w)/(2*sqrt(pi))}

Eq(2*gamma*m*Derivative(G(t, tau), t) + m*w0**2*G(t, tau) + m*Derivative(G(t,
tau), (t, 2)), DiracDelta(t - tau))(subs, {G(t, tau):
sqrt(2)*Integral(Gtilde(w)*exp(I*w*(t - tau)), w)/(2*sqrt(pi))})
```

$$2\gamma m\left(\frac{\sqrt{2}\int iw\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{2\sqrt{\pi}}\right) + \frac{\sqrt{2}mw_0^2\int\tilde{G}(w)e^{iw(t-\tau)}\,dw}{2\sqrt{\pi}} + m\left(-\frac{\sqrt{2}\int w^2\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{2\sqrt{\pi}}\right) = \delta\left(t-\tau\right)$$

```
'subs omech.result {DiracDelta(t - tau): Integral(exp(I*w*(t - tau)), w)/(2*pi)}'
```

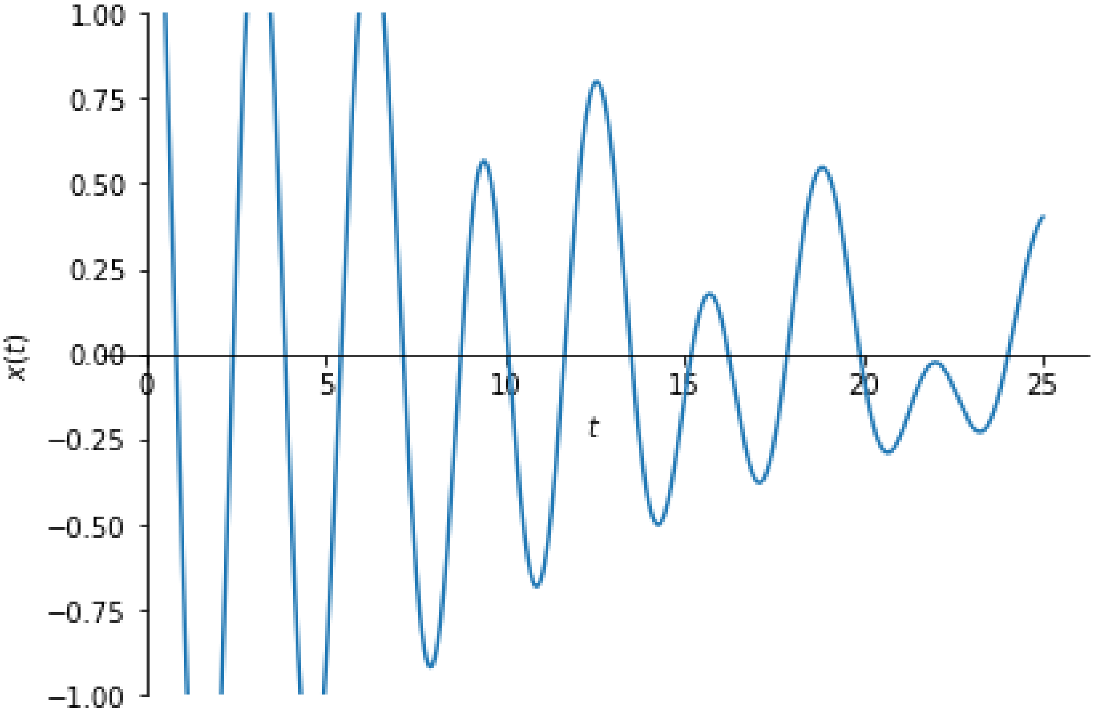
```
Eq(sqrt(2)*gamma*m*Integral(I*w*Gtilde(w)*exp(I*t*w)*exp(-I*tau*w), w)/sqrt(pi)
+ sqrt(2)*m*w0**2*Integral(Gtilde(w)*exp(I*t*w)*exp(-I*tau*w), w)/(2*sqrt(pi)) -
sqrt(2)*m*Integral(w**2*Gtilde(w)*exp(I*t*w)*exp(-I*tau*w), w)/(2*sqrt(pi)),
DiracDelta(t - tau))(subs, {DiracDelta(t - tau): Integral(exp(I*w*(t - tau)),
w)/(2*pi)})
```

$$\frac{\sqrt{2}\gamma m\int iw\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{\sqrt{\pi}} + \frac{\sqrt{2}mw_0^2\int\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{2\sqrt{\pi}} - \frac{\sqrt{2}m\int w^2\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{2\sqrt{\pi}} = \frac{\int e^{iw(t-\tau)}\,dw}{2\pi}$$

$$\frac{\sqrt{2}\gamma m\int iw\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{\sqrt{\pi}} + \frac{\sqrt{2}mw_0^2\int\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{2\sqrt{\pi}} - \frac{\sqrt{2}m\int w^2\tilde{G}(w)e^{itw}e^{-i\tau w}\,dw}{2\sqrt{\pi}} = \frac{\int e^{iw(t-\tau)}\,dw}{2\pi}$$

$$\frac{\sqrt{2}i\gamma mw\tilde{G}(w)e^{itw}e^{-i\tau w}}{\sqrt{\pi}} - \frac{\sqrt{2}mw^2\tilde{G}(w)e^{itw}e^{-i\tau w}}{2\sqrt{\pi}} + \frac{\sqrt{2}mw_0^2\tilde{G}(w)e^{itw}e^{-i\tau w}}{2\sqrt{\pi}} = \frac{e^{iw(t-\tau)}}{2\pi}$$

$$\frac{e^{iw(t-\tau)}}{2\pi} = \frac{\sqrt{2}m\left(2i\gamma w - w^2 + w_0^2\right)\tilde{G}(w)e^{iw(t-\tau)}}{2\sqrt{\pi}}$$



'G(t,tau)='

$$G(t,\tau) = \frac{\sqrt{2}\int \frac{\sqrt{2}e^{iw(t-\tau)}}{2\sqrt{\pi}m\left(2i\gamma w-w^2+w_0^2\right)}\, dw}{2\sqrt{\pi}}$$

'Take integral and get G(t,tau).'

'x(t) = x_homogeneous + integrate(G(t,tau)*f(t), t)'

'Complementary Solution'

'x_c(t)='

$$\frac{F_0\left(\gamma-\left(\gamma+\sqrt{\mathrm{polar\_lift}\left(\gamma^2-w_0^2\right)}\right)e^{2t\sqrt{\mathrm{polar\_lift}\left(\gamma^2-w_0^2\right)}}+2e^{t\left(\gamma+\sqrt{\mathrm{polar\_lift}\left(\gamma^2-w_0^2\right)}\right)}\sqrt{\mathrm{polar\_lift}\left(\gamma^2-w_0^2\right)}-\sqrt{\mathrm{p}}\right.}{2mw_0^2\sqrt{\gamma^2-w_0^2}}$$

0.8 2.6 Calculus of Variations

0.8.1 2.6.3 Euler's Equation

[ ]:
```
#----> Euler's Equation
if "Eulers_Equation" in sets.flow:
    pprints("2.6.3 Euler's Equation")
    """
    References:
        https://github.com/cnkndmr/brachistochrone-problem/blob/master/
↪shortest_path.ipynb
    """
    omech.__init__("EulerLagrange")
    omech.verbose = True
    pprints("Euler Equation",
            omech.Eulers_equation)
```

0.8.2 2.6.5 Algorithm Used in the Calculus of Variations

0.8.3 2.6.5.1 Brachystochrone

[4]:
```
#----> 2.6.5.1 Brachystochrone_Baumann
if "Brachystochrone_Baumann" in sets.flow:
    pprints("2.6.5.1 Brachystochrone")
    pprints("Baumann's Approach")
    pprints("Includes ERROR !!!. Check It !!!")
    omech.__init__("EulerLagrange")
    omech.verbose = True

    u,v = [Function('u')(t), Function('v')(t)]
    a, g, theta = symbols('a g theta', real=True, positive=True)
    f = Eq(omech.f, sqrt((1 + diff(omech.u, t, evaluate=False)**2)/(2*g*t)))
    brachystochrone_eq = simplify(omech.Eulers_equation_1D(f.rhs, [u, diff(u,␣
↪t, evaluate=False)], t)[0].doit())
    steps = omech.Eulers_equation_1D(f.rhs, [u, diff(u, t, evaluate=False)],␣
↪t)[1]

    num_brachystochrone_eq = Eq(numer(brachystochrone_eq.lhs), 0)
    diffeq_inner = simplify(simplify( Eq(steps[1].rhs**2, (1/(2*sqrt(a*g)))**2)␣
↪))
    du_dx = solve(diffeq_inner, u.diff(t))[1]
    subs_int = {t:a*(1-cos(theta))}
    dtheta = diff(a*(1-cos(theta)), theta)
    du_dx_transformed = simplify(du_dx.xreplace(subs_int))
    sol_int = du_dx_transformed*dtheta
#    sol_int = simplify(dsolve(diffeq_inner, u, ics={u.subs({t:a}):0})[0].
↪subs(C1,0))
```

```python
    # Do integration with sage
    import sage.all as sg # this is mandatory to initialize Sage
    a,theta = sg.var('a theta')
    sol_x = simplify(sympify(sg.integrate(sol_int, theta)))

    pprints(f,
        "Euler equation calculation steps",
        *steps,
        "Brachystochrone Equation",
        brachystochrone_eq,
        "Numerator of the Brachystochrone equation",
        num_brachystochrone_eq,
        "A simple differential equation obtained from Euler equation␣
↪calculation",
        diffeq_inner,
        du_dx_transformed,
        "u'(x)=", du_dx,
        "u(x)=", sol_x,
        )

    # Plot u(x).
    numvals = {a:2}
    u_x = sol_x.evalf(subs=numvals)
    x_funcs  = [-a.subs(numvals)*(1-cos(theta)), -a.
↪subs(numvals)*(1-cos(theta))]
    ux_funcs = [-u_x, u_x]
    p = plot_parametric(*list(zip(x_funcs, ux_funcs)),
                          (theta,0,float(2*pi),200),
                          xlabel="x", ylabel="u(x)")
```

'2.6.5.1 Brachystochrone'

"Baumann's Approach"

'Includes ERROR !!!. Check It !!!'

$$f(u(t), t) = \frac{\sqrt{2}\sqrt{\left(\frac{d}{dt}u(t)\right)^2 + 1}}{2\sqrt{g}\sqrt{t}}$$

'Euler equation calculation steps'

$$\left(\frac{\partial}{\partial \xi_1} L\left(\xi_1, \frac{d}{dt}u(t)\right)\Big|_{\xi_1=u(t)}\right) = 0$$

$$\frac{d}{d\frac{d}{dt}u(t)}L\left(u(t),\frac{d}{dt}u(t)\right)=\frac{\sqrt{2}\frac{d}{dt}u(t)}{2\sqrt{g}\sqrt{t}\sqrt{\left(\frac{d}{dt}u(t)\right)^2+1}}$$

'Brachystochrone Equation'

$$\frac{\sqrt{2}\left(-2t\frac{d^2}{dt^2}u(t)+\left(\frac{d}{dt}u(t)\right)^3+\frac{d}{dt}u(t)\right)}{4\sqrt{g}t^{\frac{3}{2}}\left(\left(\frac{d}{dt}u(t)\right)^2+1\right)^{\frac{3}{2}}}=0$$

'Numerator of the Brachystochrone equation'

$$\sqrt{2}\left(-2t\frac{d^2}{dt^2}u(t)+\left(\frac{d}{dt}u(t)\right)^3+\frac{d}{dt}u(t)\right)=0$$

'A simple differential equation obtained from Euler equation calculation'

$$\frac{1}{4ag}=\frac{\left(\frac{d}{dt}u(t)\right)^2}{2gt\left(\left(\frac{d}{dt}u(t)\right)^2+1\right)}$$

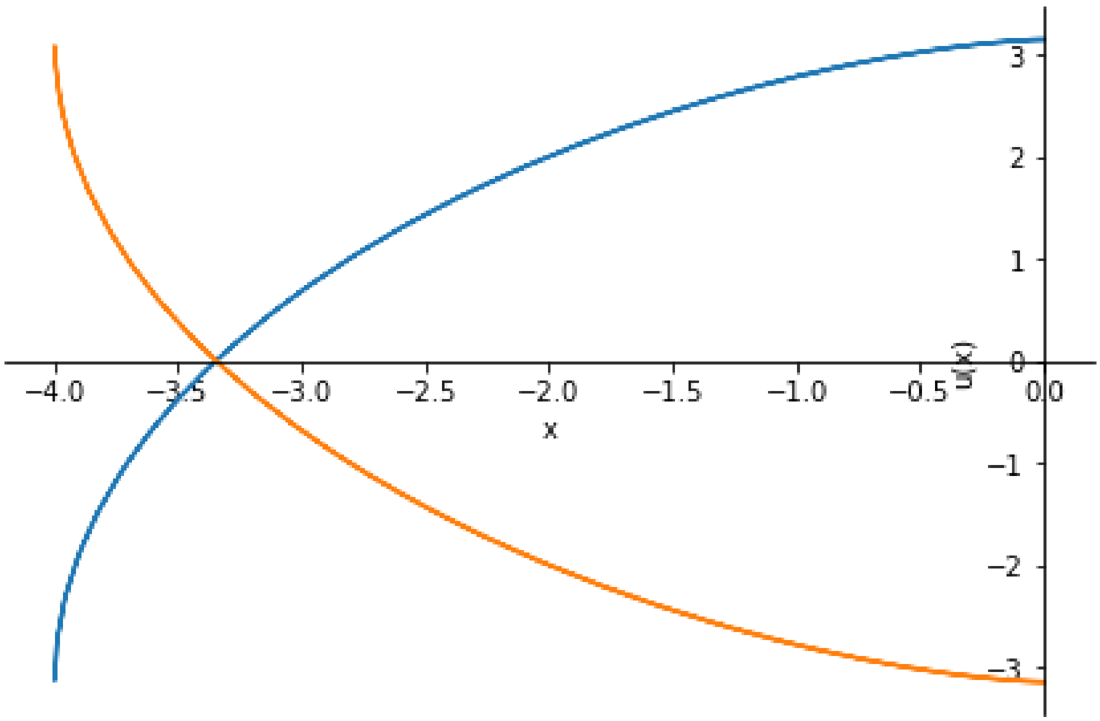
$$\sqrt{1-\cos(\theta)}\sqrt{\frac{1}{\cos(\theta)+1}}$$

"u'(x)="

$$\sqrt{t}\sqrt{\frac{1}{2a-t}}$$

'u(x)='

$$-a\left(\sqrt{\sin^2(\theta)}+\operatorname{asin}(\cos(\theta))\right)$$



0.8.4 2.6.5.1 Brachystochrone_Wachter

[5]:
```
#----> 2.6.5.1 Brachystochrone_Wachter
if "Brachystochrone_Wachter" in sets.flow:
    pprints("2.6.5.1 Brachystochrone")
    pprints("Wachter's Approach")
    pprints("Includes ERROR !!!. Check It !!!")
    omech.__init__("EulerLagrange")
    omech.verbose = True

    x = Symbol('x')
    y = Function('y')(x)
    r = Matrix([[x], [y]])
    h, g = symbols('h g', real=True, positive=True)
    f = Eq( omech.f, 1/sqrt(2*g)*sqrt((1+y.diff(x)**2)/(h-y)) )
    brachystochrone_eq = simplify(omech.Eulers_equation_1D(f.rhs, [y, diff(y,␣
↪x, evaluate=False)], x)[0].doit())
    num_brachystochrone_eq = Eq(numer(brachystochrone_eq.lhs), 0)
#    sol = dsolve(num_brachystochrone_eq, y) # cannot be solved by the␣
↪factorable group method
    sol = solve(num_brachystochrone_eq, y.diff(x,2))[0]

    import sage.all as sg # this is mandatory to initialize Sage
```

```
    x,h = sg.var('x h')
    y = sg.function('y')(x)
    sol = sg.desolve(diff(y,x,2) - sol == 0, dvar=y, ivar=x)
    pprints("Solution", sol)
```

```
'2.6.5.1 Brachystochrone'

"Wachter's Approach"

'Includes ERROR !!!. Check It !!!'

'Solution'

[-((h*e^_K1 - e^_K1*y(x))*sqrt(-(h*e^_K1 - e^_K1*y(x) + 1)/(h*e^_K1 - e^_K1*y(x))) - arctan(sq
 ((h*e^_K1 - e^_K1*y(x))*sqrt(-(h*e^_K1 - e^_K1*y(x) + 1)/(h*e^_K1 - e^_K1*y(x))) - arctan(sqr
```

0.8.5 2.6.6 Euler Operator for q Dependent Variables

[6]:
```
#----> 2.6.6 Euler Operator for q Dependent Variables
if "Euler_Operator" in sets.flow:
    pprints("2.6.6 Euler Operator for q Dependent Variables")
    omech.__init__("EulerLagrange")
    omech.verbose = True
    q,u,v = [Function('q')(t), Function('u')(t), Function('v')(t)]

    #----> Lagrangian Density
    """
    l = t + q + q.diff(t)
    eu_eq_q = simplify(omech.Eulers_equation_1D(l, [q, q.diff(t)], [t])[0])
    pprints("2.6.6 Euler Operator for q Dependent Variables",
            "l=", l,
            eu_eq_q)
    """



    #----> Two-Dimensional Oscillator System
    l = u*v + (u.diff(t))**2 + (v.diff(t))**2 - u**2 - v**2
    eu_eq_u,steps_u = simplify(omech.Eulers_equation_1D(l, [u, u.diff(t)], t))
    eu_eq_v,steps_v = simplify(omech.Eulers_equation_1D(l, [v, v.diff(t)], t))
    pprints("2.6.6.1 Two-Dimensional Oscillator System by libphysics",
            "l=", l,
            "The corresponding system of second-order equations follows by",
            eu_eq_u,
            eu_eq_v)
```

```python
# Correct Way.
eu_eqs = euler_equations(l, [u,v], t)
pprints("2.6.6.1 Two-Dimensional Oscillator System by SymPy",
        "l=", l,
        "The corresponding system of second-order equations follows by",
        *eu_eqs)


#----> Two-Dimensional Lagrangian
l = u*v + (u.diff(t))**2 + (v.diff(t))**2 + 2*u.diff(t)*v.diff(t)
eu_eq_u, steps_u = simplify(omech.Eulers_equation_1D(l, [u, u.diff(t)], t))
eu_eq_v, steps_v = simplify(omech.Eulers_equation_1D(l, [v, v.diff(t)], t))
pprints("2.6.6.2 Two-Dimensional Lagrangian by libphysics",
        "l=", l,
        "The corresponding Euler-Lagrange equations read",
        eu_eq_u,
        eu_eq_v)
```

'2.6.6 Euler Operator for q Dependent Variables'

'2.6.6.1 Two-Dimensional Oscillator System by libphysics'

'l='

$$-u^2(t) + u(t)v(t) - v^2(t) + \left(\frac{d}{dt}u(t)\right)^2 + \left(\frac{d}{dt}v(t)\right)^2$$

'The corresponding system of second-order equations follows by'

$$-2u(t) + v(t) - 2\frac{d^2}{dt^2}u(t) = 0$$

$$u(t) - 2v(t) - 2\frac{d^2}{dt^2}v(t) = 0$$

'2.6.6.1 Two-Dimensional Oscillator System by SymPy'

'l='

$$-u^2(t) + u(t)v(t) - v^2(t) + \left(\frac{d}{dt}u(t)\right)^2 + \left(\frac{d}{dt}v(t)\right)^2$$

'The corresponding system of second-order equations follows by'

$$-2u(t) + v(t) - 2\frac{d^2}{dt^2}u(t) = 0$$

$$u(t) - 2v(t) - 2\frac{d^2}{dt^2}v(t) = 0$$

'2.6.6.2 Two-Dimensional Lagrangian by libphysics'

'l='

$$u(t)v(t) + \left(\frac{d}{dt}u(t)\right)^2 + 2\frac{d}{dt}u(t)\frac{d}{dt}v(t) + \left(\frac{d}{dt}v(t)\right)^2$$

'The corresponding Euler-Lagrange equations read'

$$v(t) - 2\frac{d^2}{dt^2}u(t) - 2\frac{d^2}{dt^2}v(t) = 0$$

$$u(t) - 2\frac{d^2}{dt^2}u(t) - 2\frac{d^2}{dt^2}v(t) = 0$$

0.8.6 2.6.7 Euler Operator for q + p Dimensions

[7]:
```
#----> 2.6.7 Euler Operator for q + p Dimensions
if "2.6.7 Euler Operator for q + p Dimensions" in sets.flow:
    pprints("2.6.7 Euler Operator for q + p Dimensions",
            "Example1: Quadratic Density",
            "Euler Operator for q + p Dimensions is Not Impelemented in␣
 ↪mechanics.py")
    omech.__init__("EulerLagrange")
    omech.verbose = True

    """
    # IndexedBased Functions with Function
    x = IndexedBase('x', shape=(3))
    u = Function('u')(x[1], x[2], x[3])
    du_dx1 = u.diff(x[1])
    pprints("x=", x,
            "u=", u,
            "du_dx1=", du_dx1)

    # IndexedBased Functions with Lambda Function
    # ValueError:
    #    Can not calculate derivative wrt Lambda((x[1], x[2], x[3]), u(x[1],
    #    x[2], x[3])).
    x = IndexedBase('x', shape=(3))
    u = Lambda((x[1], x[2], x[3]), Function('u')(x[1],x[2],x[3]))
    du_dx1 = u(x[1],x[2],x[3]).diff(x[1])
    pprints("x=", x,
            "u=", u,
            "du_dx1=", du_dx1)
    """
```

```python
#    Example 1: Quadratic Density
    x = IndexedBase('x', shape=(3))
    u = Function('u')(x[1], x[2], x[3])
    f = S(1)/2*(u.diff(x[1])**2 - u.diff(x[2])**2 - u.diff(x[3])**2)
    eu_eq_ux1 = simplify(omech.Eulers_equation_1D(f, [u, u.diff(x[1])],␣
↪x[1])[0])
    eu_eq_ux2 = simplify(omech.Eulers_equation_1D(f, [u, u.diff(x[2])],␣
↪x[2])[0])
    eu_eq_ux3 = simplify(omech.Eulers_equation_1D(f, [u, u.diff(x[3])],␣
↪x[3])[0])
    res1 = simplify( eu_eq_ux1.lhs + eu_eq_ux2.lhs + eu_eq_ux3.lhs )
    pprints("Example 1: Quadratic Density",
            "f=", f,
            "eu_eq_ux1=", eu_eq_ux1,
            "eu_eq_ux2=", eu_eq_ux2,
            "eu_eq_ux3=", eu_eq_ux3,
            "res=", res1)
```

'2.6.7 Euler Operator for q + p Dimensions'

'Example1: Quadratic Density'

'Euler Operator for q + p Dimensions is Not Impelemented in mechanics.py'

'Example 1: Quadratic Density'

'f='

$$\frac{\left(\frac{\partial}{\partial x_1}u(x_1,x_2,x_3)\right)^2}{2}-\frac{\left(\frac{\partial}{\partial x_2}u(x_1,x_2,x_3)\right)^2}{2}-\frac{\left(\frac{\partial}{\partial x_3}u(x_1,x_2,x_3)\right)^2}{2}$$

'eu_eq_ux1='

$$\frac{\partial^2}{\partial x_1^2}u(x_1,x_2,x_3)=0$$

'eu_eq_ux2='

$$\frac{\partial^2}{\partial x_2^2}u(x_1,x_2,x_3)=0$$

'eu_eq_ux3='

$$\frac{\partial^2}{\partial x_3^2}u(x_1,x_2,x_3)=0$$

'res='

$$\frac{\partial^2}{\partial x_1^2}u(x_1,x_2,x_3) + \frac{\partial^2}{\partial x_2^2}u(x_1,x_2,x_3) + \frac{\partial^2}{\partial x_3^2}u(x_1,x_2,x_3)$$

[8]:
```
# Example 1: Quadratic Density
x1,x2,x3 = symbols('x_1,x_2,x_3', real=True)
u = Function('u')(x1, x2, x3)
f = S(1)/2*(u.diff(x1)**2 - u.diff(x2)**2 - u.diff(x3)**2)
eu_eqs, steps = simplify(omech.Eulers_equation_1D(f, [u,u.diff(x1),u.diff(x2)],␣
↪x1))
pprints("Example 1: Quadratic Density, todo last sign is wrong",
        "f=", f,
        "eu_eqs=", eu_eqs)

# Correct Way.
eu_eqs = euler_equations(f, [u], [x1,x2,x3])
pprints("Example 1: Quadratic Density",
        "f=", f,
        "The corresponding system of second-order equations follows by",
        *eu_eqs)
```

'Example 1: Quadratic Density, todo last sign is wrong'

'f='

$$\frac{\left(\frac{\partial}{\partial x_1}u(x_1,x_2,x_3)\right)^2}{2} - \frac{\left(\frac{\partial}{\partial x_2}u(x_1,x_2,x_3)\right)^2}{2} - \frac{\left(\frac{\partial}{\partial x_3}u(x_1,x_2,x_3)\right)^2}{2}$$

'eu_eqs='

$$-\frac{\partial^2}{\partial x_1^2}u(x_1,x_2,x_3) - \frac{\partial^3}{\partial x_2 \partial x_1^2}u(x_1,x_2,x_3) = 0$$

'Example 1: Quadratic Density'

'f='

$$\frac{\left(\frac{\partial}{\partial x_1}u(x_1,x_2,x_3)\right)^2}{2} - \frac{\left(\frac{\partial}{\partial x_2}u(x_1,x_2,x_3)\right)^2}{2} - \frac{\left(\frac{\partial}{\partial x_3}u(x_1,x_2,x_3)\right)^2}{2}$$

'The corresponding system of second-order equations follows by'

$$-\frac{\partial^2}{\partial x_1^2}u(x_1,x_2,x_3) + \frac{\partial^2}{\partial x_2^2}u(x_1,x_2,x_3) + \frac{\partial^2}{\partial x_3^2}u(x_1,x_2,x_3) = 0$$

[9]:
```
# Example 2: Diffusion of Two Components
t,x = symbols('t x', real=True)
u,v = [Function('u')(x,t), Function('v')(x,t)]
l = v*u.diff(t) + u.diff(x)*v.diff(x) + u**2*v**2
eu_eq_ux = simplify(omech.Eulers_equation_1D(l, [u,u.diff(x)], x)[0])
eu_eq_ut = simplify(omech.Eulers_equation_1D(l, [u,u.diff(t)], t)[0])
eu_eq_vx = simplify(omech.Eulers_equation_1D(l, [v,v.diff(x)], x)[0])
eu_eq_vt = simplify(omech.Eulers_equation_1D(l, [v,v.diff(t)], t)[0])
res_u = eu_eq_ux.lhs + eu_eq_ut.lhs
res_v = eu_eq_vx.lhs + eu_eq_vt.lhs
pprints("Example 2: Diffusion of Two Components",
    "Lagrangian density = l=", l,
    "eu_eq_ux=", eu_eq_ux,
    "eu_eq_ut=", eu_eq_ut,
    "eu_eq_vx=", eu_eq_vx,
    "eu_eq_vt=", eu_eq_vt,
    "res_u=", res_u,
    "res_v=", res_v,
    "Baumann found  = 2*u(x, t)*v(x, t)**2 - Derivative(v(x, t), t) -␣
 ↪Derivative(v(x, t), (x, 2))",
    "We found WRONG = 4*u(x, t)*v(x, t)**2 - Derivative(v(x, t), t) -␣
 ↪Derivative(v(x, t), (x, 2))"
    )

# Correct Way.
#    eu_eqs = euler_equations(l, [u,v], [x,t])
eu_eqs,steps = omech.Eulers_equation_sympy(l, [u,v], [x,t])
pprints("Example 2: Diffusion of Two Components",
        "Lagrangian density = l=", l,
        "Steps=", *steps,
        "The corresponding system of differential equations follows by",
        *eu_eqs)
```

'Example 2: Diffusion of Two Components'

'Lagrangian density = l='

$$u^2(x,t)v^2(x,t) + v(x,t)\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}v(x,t)$$

'eu_eq_ux='

$$2u(x,t)v^2(x,t) - \frac{\partial^2}{\partial x^2}v(x,t) = 0$$

'eu_eq_ut='

$$2u(x,t)v^2(x,t) - \frac{\partial}{\partial t}v(x,t) = 0$$

'eu_eq_vx='

$$2u^2(x,t)v(x,t) + \frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0$$

'eu_eq_vt='

$$2u^2(x,t)v(x,t) + \frac{\partial}{\partial t}u(x,t) = 0$$

'res_u='

$$4u(x,t)v^2(x,t) - \frac{\partial}{\partial t}v(x,t) - \frac{\partial^2}{\partial x^2}v(x,t)$$

'res_v='

$$4u^2(x,t)v(x,t) + 2\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t)$$

```
'Baumann found  = 2*u(x, t)*v(x, t)**2 - Derivative(v(x, t), t) - Derivative(v(x, t), (x, 2))'

'We found WRONG = 4*u(x, t)*v(x, t)**2 - Derivative(v(x, t), t) - Derivative(v(x, t), (x, 2))'

'Example 2: Diffusion of Two Components'

'Lagrangian density = l='
```

$$u^2(x,t)v^2(x,t) + v(x,t)\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}v(x,t)$$

'Steps='

$$\frac{\partial}{\partial u(x,t)}L(u(x,t),v(x,t)) = 2u(x,t)v^2(x,t)$$

$$\frac{\partial}{\partial v(x,t)}L(u(x,t),v(x,t)) = 2u^2(x,t)v(x,t) + \frac{\partial}{\partial t}u(x,t)$$

'The corresponding system of differential equations follows by'

$$2u(x,t)v^2(x,t) - \frac{\partial}{\partial t}v(x,t) - \frac{\partial^2}{\partial x^2}v(x,t) = 0$$

$$2u^2(x,t)v(x,t) + \frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0$$

0.8.7 2.7.2 Hamiltons Principle Historical Remarks

[3]:
```
#----> 2.7.2 Hamiltons Principle Historical Remarks
if "2.7.2 Hamiltons Principle Historical Remarks" in sets.flow:
    pprints("2.7.2 Hamilton's Principle Historical Remarks")
    omech.__init__("EulerLagrange")
    omech.verbose = True

    # Lagrangian L = T - V
    L = S(1)/2*m*q.diff()**2-V
    eu_eqs,steps = omech.Eulers_equation_sympy(L, [q], [t])
    pprints("Lagrangian L = T - V",
            "For velocity-independent potentials, Lagrange's equations become",
            "Lagrangian= L=", L,
            "Steps=", *steps,
            "The corresponding differential equation follows by",
            *eu_eqs,
            "which, in the case of cartesian coordinates, are just Newton's␣
 ↪equations.")
```

"2.7.2 Hamilton's Principle Historical Remarks"

'Lagrangian L = T - V'

"For velocity-independent potentials, Lagrange's equations become"

'Lagrangian= L='

$$\frac{m\left(\frac{d}{dt}q(t)\right)^2}{2} - V(q_i(t))$$

'Steps='

$$\frac{d}{dq(t)}L(q(t)) = 0$$

'The corresponding differential equation follows by'

$$-m\frac{d^2}{dt^2}q(t) = 0$$

"which, in the case of cartesian coordinates, are just Newton's equations."

0.8.8 2.7.3 Hamiltons Principle

[4]:
```
#----> 2.7.3 Hamiltons Principle
if "2.7.3.1 Example 1: Harmonic Oscillator" in sets.flow:
    pprints("2.7.3 Hamiltons Principle")
    pprints("2.7.3.1 Example 1: Harmonic Oscillator")
    omech.__init__("EulerLagrange")
    omech.verbose = True

    # Example 1: Harmonic Oscillator
    V = S(1)/2*k*q**2
    L = omech.T.rhs - V
#    eu_eqs,steps = omech.Eulers_equation_sympy(L, [q], [t]); eq_SHO = eu_eqs[0]
    eu_eqs,steps = omech.Eulers_equation_1D(L, [q,D(q)], t); eq_SHO = eu_eqs
    omech.result = eq_SHO
    commands = ["dsolve", "omech.result", q]
    omech.q = omech.process(commands)
    pprints("Example 1: Harmonic Oscillator",
            "T=", T, "V=", V,
            "Lagrangian= L=", L,
            "Steps=", *steps,
            "The corresponding differential equation follows by", eu_eqs,
            "Solution of differential equation", omech.q)
```

'2.7.3 Hamiltons Principle'

'2.7.3.1 Example 1: Harmonic Oscillator'

'dsolve omech.result q(t)'

dsolve(Eq(-k*q(t) - Derivative(0, t), 0), q(t))

$q(t) = 0$

'Example 1: Harmonic Oscillator'

'T='

$T(q_i(t), \dot{q}_i(t), t)$

'V='

$$\frac{kq^2(t)}{2}$$

'Lagrangian= L='

$$-\frac{kq^2(t)}{2} + T(q_i(t), \dot{q}_i(t), t)$$

'Steps='

$$\left(\frac{\partial}{\partial \xi_1} L\left(\xi_1, \frac{d}{dt}q(t)\right)\bigg|_{\xi_1=q(t)}\right) = -kq(t)$$

$$\frac{d}{d\frac{d}{dt}q(t)} L\left(q(t), \frac{d}{dt}q(t)\right) = 0$$

'The corresponding differential equation follows by'

$$-kq(t) - (0) = 0$$

'Solution of differential equation'

$$q(t) = 0$$

0.8.9 2.7.3.2 Example 2: Rolling Wheel on an Inclined Plane

[5]:
```
#----> 2.7.3.2 Example 2: Rolling Wheel on an Inclined Plane
if "2.7.3.2 Example 2: Rolling Wheel on an Inclined Plane" in sets.flow:
    # Prepare Lagrangian
    pprints("2.7.3.2 Example 2: Rolling Wheel on an Inclined Plane")
    pprints("1. Way: Eulers_equation_1D")
    omech.__init__("EulerLagrange")
    omech.verbose = True
    calc_type = {1:"1. Way: Eulers_equation_1D",
                 2:"2. Way: Eulers_equation_sympy",
                 3:"3. Way: euler_equations",
                 4:"4. Way: Lagrange_equations_I"}[1]

    l,R = symbols('l R', real=True, positive=True)
    # fg  = Function('g')(t)
    Icm = Eq(var('I'), S(1)/2*m*R**2)
    T   = Eq(S('T'), S(1)/2*m*D(y,t)**2 + S(1)/2*var('I')*D(theta,t)**2)
    T   = T.xreplace({var('I'):Icm.rhs})
    V   = Eq(S('V'), m*g*(l-y)*sin(alpha))
    L   = Eq(S('L'), T.rhs-V.rhs)
    const_g = Eq(y-R*theta, 0)

    sol_theta = solve(const_g,theta)[0]
    Ly = ratsimp(simplify(L.rhs.subs({theta:sol_theta})))
    Ltheta = ratsimp(simplify(L.rhs.subs({y:R*theta})))

    eu_eq_y     = euler_equations(Ly, y, t)[0]
    eu_eq_theta = euler_equations(Ltheta, theta, t)[0]
```

```
    pprints(T,V,
            "Lagrangian=", L,
            "Constraint equation=", const_g,
            Ly, Ltheta, eu_eq_y, eu_eq_theta)
```

'2.7.3.2 Example 2: Rolling Wheel on an Inclined Plane'

'1. Way: Eulers_equation_1D'

$$T = \frac{R^2 m \left(\frac{d}{dt}\theta(t)\right)^2}{4} + \frac{m \left(\frac{d}{dt}y(t)\right)^2}{2}$$

$$V = gm\left(l - y(t)\right)\sin\left(\alpha\right)$$

'Lagrangian='

$$L = \frac{R^2 m \left(\frac{d}{dt}\theta(t)\right)^2}{4} - gm\left(l - y(t)\right)\sin\left(\alpha\right) + \frac{m \left(\frac{d}{dt}y(t)\right)^2}{2}$$

'Constraint equation='

$$-R\theta(t) + y(t) = 0$$

$$-glm\sin\left(\alpha\right) + gmy(t)\sin\left(\alpha\right) + \frac{3m\left(\frac{d}{dt}y(t)\right)^2}{4}$$

$$\frac{3R^2 m\left(\frac{d}{dt}\theta(t)\right)^2}{4} + Rgm\theta(t)\sin\left(\alpha\right) - glm\sin\left(\alpha\right)$$

$$gm\sin\left(\alpha\right) - \frac{3m\frac{d^2}{dt^2}y(t)}{2} = 0$$

$$-\frac{3R^2 m\frac{d^2}{dt^2}\theta(t)}{2} + Rgm\sin\left(\alpha\right) = 0$$

0.8.10 2.7.3.3 Example 3: Sliding Mass Connected to a Pendulum

[6]:
```
#----> 2.7.3.3 Example 3: Sliding Mass Connected to a Pendulum
if "2.7.3.3 Example 2: Sliding Mass Connected to a Pendulum" in sets.flow:
    # Prepare Lagrangian
    pprints("Example 3: Sliding Mass Connected to a Pendulum")
    pprints("1. Way: Eulers_equation_1D")
    omech.__init__("EulerLagrange")
    omech.verbose = True
    calc_type = {1:"1. Way: Eulers_equation_1D", # todo solve bug
                 2:"2. Way: Eulers_equation_sympy",
                 3:"3. Way: euler_equations",
                 4:"4. Way: Lagrange_equations_I"}[2]
```

```python
l = symbols('l')
T1 = Eq(symbols('T1'), S(1)/2*m1*(D(x1)**2 + D(z1)**2))
T2 = Eq(symbols('T2'), S(1)/2*m2*(D(x2)**2 + D(z2)**2))
V1 = Eq(symbols('V1'), 0)
V2 = Eq(symbols('V2'), m2*g*z2)
omech.T = Eq(symbols('T'), T1.rhs + T2.rhs)
omech.V = Eq(symbols('V'), V1.rhs + V2.rhs)
L = omech.L = Eq(symbols('L'), omech.T.rhs - omech.V.rhs)
display(T1,T2,V1,V2,omech.T,omech.V,omech.L)

# Transform to generalized coordinates
generalized_coordinates = {x1:x, z1:0,
                           x2:x+l*sin(phi), z2:-l*cos(phi)}
T = omech.T = omech.T.xreplace(generalized_coordinates)
V = omech.V = omech.V.xreplace(generalized_coordinates)
L = omech.L = Eq(symbols('L'), omech.T.rhs - omech.V.rhs)
Lag = simplify(omech.L.doit())

# Apply Euler-Lagrange operator
if calc_type == "1. Way: Eulers_equation_1D":
    eu_eq_x, steps = omech.Eulers_equation_1D(Lag.rhs, [x,D(x)], t)
    sim_eu_eq_x = expand(simplify(eu_eq_x))
    eu_eq_phi, steps = omech.Eulers_equation_1D(Lag.rhs, [phi,D(phi)], t)
    sim_eu_eq_phi = expand(simplify(eu_eq_phi))
    pprints("1. Way: Eulers_equation_1D",
            "generalized_coordinates=", generalized_coordinates,
            T, T.doit(), V, V.doit(), L, L.doit(),
            "Lagrangian= L=", Lag,
            "Steps=", *steps,
            "Differential equation for x(t)", eu_eq_x, sim_eu_eq_x,
            "Differential equation for phi(t)", eu_eq_phi, sim_eu_eq_phi
            )
if calc_type == "2. Way: Eulers_equation_sympy":
    eu_eqs,steps = omech.Eulers_equation_sympy(Lag.rhs, [x,phi], t)
    eu_eq_x, eu_eq_phi = [eu_eqs[0], eu_eqs[1]]
    pprints("2. Way: Eulers_equation_sympy",
            "Steps=", *steps,
            "Differential equation for x(t)", expand(simplify(eu_eqs[0])),
            "Differential equation for phi(t)", expand(simplify(eu_eqs[1]))
            )
if calc_type == "3. Way: euler_equations":
    eu_eqs = euler_equations(Lag.rhs, [x,phi], t)
    eu_eq_x, eu_eq_phi = [eu_eqs[0], eu_eqs[1]]
    pprints("3. Way: euler_equations",
            "Differential equation for x(t)", expand(simplify(eu_eqs[0])),
            "Differential equation for phi(t)", expand(simplify(eu_eqs[1]))
```

```
                )
    if calc_type == "4. Way: Lagrange_equations_I":
        substitutions  = {q_i:x, q_idot:D(x,t), omech.L.lhs:Lag.rhs}
        eu_eq_x = expand(simplify(omech.Lagrange_equations_I.
↪xreplace(substitutions).doit()))
        substitutions  = {q_i:phi, q_idot:D(phi,t), omech.L.lhs:Lag.rhs}
        eu_eq_phi = expand(simplify(omech.Lagrange_equations_I.
↪xreplace(substitutions).doit()))
        pprints("4. Way: Lagrange_equations_I",
                "Differential equation for x(t)",   eu_eq_x,
                "Differential equation for phi(t)", eu_eq_phi)

    if calc_type == "SymPy: A rolling disc using Lagrange's Method":
        print("todo")

    if calc_type == "SymPy: A rolling disc, with Kane’s method":
        print("todo")

    # Solution of ODEs
    sol_ode = {0:False, 1:True}[1]
    if sol_ode:
        # Reduce 2nd order derivatives to 1st order derivatives.
        y1, y2, y3, y4 = symbols("y_1, y_2, y_3, y_4", cls=Function)
        varchange = {x.diff(t,t):y2(t).diff(t),
                     x:y1(t),
                     phi.diff(t,t):y4(t).diff(t),
                     phi:y3(t)}
        ode1, ode2 = [eu_eq_x.lhs.subs(varchange),
                      eu_eq_phi.lhs.subs(varchange)]
        ode3 = y1(t).diff(t) - y2(t)
        ode4 = y3(t).diff(t) - y4(t)

        y = Matrix([y1(t), y2(t), y3(t), y4(t)])
        vcsol = solve((ode1, ode2, ode3, ode4), y.diff(t), dict=True)
        f = y.diff(t).subs(vcsol[0])
        eq_S = Eq(y.diff(t), f)
        jac = Matrix([[fj.diff(yi) for yi in y] for fj in f])

        # Numerical calculations
        params = {m1:1, m2:0.5, l:0.7, g:9.81}
        f_np = lambdify((t, y), f.subs(params), 'numpy')
        jac_np = lambdify((t, y), jac.subs(params), 'numpy')

        # y0 = [x(0), x'(0), phi(0), phi'(0)]
        y0 = [0.1, 0.01, 0.1, 0.01]
        # y0 = [0.1, 0.1, 0.5, 0.9]
        t = np.linspace(0, 20, 1000)
```

```
    r = sp.integrate.ode(f_np, jac_np).set_initial_value(y0, t[0]);
    dt = t[1] - t[0]
    y = np.zeros((len(t), len(y0)))
    idx = 0
    while r.successful() and r.t < t[-1]:
        y[idx, :] = r.y
        r.integrate(r.t + dt)
        idx += 1

    fig = plt.figure(figsize=(10, 4))
    ax1 = plt.subplot2grid((2, 5), (0, 0), colspan=3)
    ax2 = plt.subplot2grid((2, 5), (1, 0), colspan=3)
    ax3 = plt.subplot2grid((2, 5), (0, 3), colspan=2, rowspan=2)

    ax1.plot(t, y[:, 0], 'r')
    ax1.set_ylabel(r'$x(t)$', fontsize=18)

    ax2.plot(t, y[:, 2], 'b')
    ax2.set_xlabel('$t$', fontsize=18)
    ax2.set_ylabel(r'$\phi(t)$', fontsize=18)

    ax3.plot(y[:, 0], y[:, 2], 'k')
    ax3.set_xlabel(r'$x(t)$', fontsize=18)
    ax3.set_ylabel(r'$\phi(t)$', fontsize=18)

    fig.tight_layout()

    pprints("Solution of ODEs:",
            "Reduction of derivatives:", varchange,
            "ODEs:", *[ode1,ode2,ode3,ode4],
            "New ODEs:", eq_S,
            "Jacobian Matrix of the System:", jac)
```

```
'Example 3: Sliding Mass Connected to a Pendulum'
```

```
'1. Way: Eulers_equation_1D'
```

$$T_1 = \frac{m_1 \left(\left(\frac{d}{dt} x_1(t)\right)^2 + \left(\frac{d}{dt} z_1(t)\right)^2\right)}{2}$$

$$T_2 = \frac{m_2 \left(\left(\frac{d}{dt} x_2(t)\right)^2 + \left(\frac{d}{dt} z_2(t)\right)^2\right)}{2}$$

$$V_1 = 0$$

$$V_2 = g m_2 z_2(t)$$

$$T = \frac{m_1 \left(\left(\frac{d}{dt} x_1(t) \right)^2 + \left(\frac{d}{dt} z_1(t) \right)^2 \right)}{2} + \frac{m_2 \left(\left(\frac{d}{dt} x_2(t) \right)^2 + \left(\frac{d}{dt} z_2(t) \right)^2 \right)}{2}$$

$$V = g m_2 z_2(t)$$

$$L = -g m_2 z_2(t) + \frac{m_1 \left(\left(\frac{d}{dt} x_1(t) \right)^2 + \left(\frac{d}{dt} z_1(t) \right)^2 \right)}{2} + \frac{m_2 \left(\left(\frac{d}{dt} x_2(t) \right)^2 + \left(\frac{d}{dt} z_2(t) \right)^2 \right)}{2}$$

```
'2. Way: Eulers_equation_sympy'
```

```
'Steps='
```

$$\frac{d}{dx(t)} L(x(t), \phi(t)) = 0$$

$$\frac{d}{d\phi(t)} L(x(t), \phi(t)) = -g l m_2 \sin(\phi(t)) - l m_2 \sin(\phi(t)) \frac{d}{dt}\phi(t) \frac{d}{dt} x(t)$$

```
'Differential equation for x(t)'
```

$$-l m_2 \sin(\phi(t)) \left(\frac{d}{dt} \phi(t) \right)^2 + l m_2 \cos(\phi(t)) \frac{d^2}{dt^2}\phi(t) + m_1 \frac{d^2}{dt^2} x(t) + m_2 \frac{d^2}{dt^2} x(t) = 0$$

```
'Differential equation for phi(t)'
```

$$g l m_2 \sin(\phi(t)) + l^2 m_2 \frac{d^2}{dt^2}\phi(t) + l m_2 \cos(\phi(t)) \frac{d^2}{dt^2} x(t) = 0$$

```
'Solution of ODEs:'
```

```
'Reduction of derivatives:'
```

$$\left\{ \phi(t) : y_3(t),\ x(t) : y_1(t),\ \frac{d^2}{dt^2}\phi(t) : \frac{d}{dt} y_4(t),\ \frac{d^2}{dt^2} x(t) : \frac{d}{dt} y_2(t) \right\}$$

```
'ODEs:'
```

$$-m_1 \frac{d}{dt} y_2(t) - m_2 \left(-l \sin(y_3(t)) \left(\frac{d}{dt} y_3(t) \right)^2 + l \cos(y_3(t)) \frac{d}{dt} y_4(t) + \frac{d}{dt} y_2(t) \right)$$

$$-g l m_2 \sin(y_3(t)) - l m_2 \left(l \frac{d}{dt} y_4(t) - \sin(y_3(t)) \frac{d}{dt} y_1(t) \frac{d}{dt} y_3(t) + \cos(y_3(t)) \frac{d}{dt} y_2(t) \right) - l m_2 \sin(y_3(t)) \frac{d}{dt} y_1(t) \frac{d}{dt} y_3(t)$$

$$-y_2(t) + \frac{d}{dt} y_1(t)$$

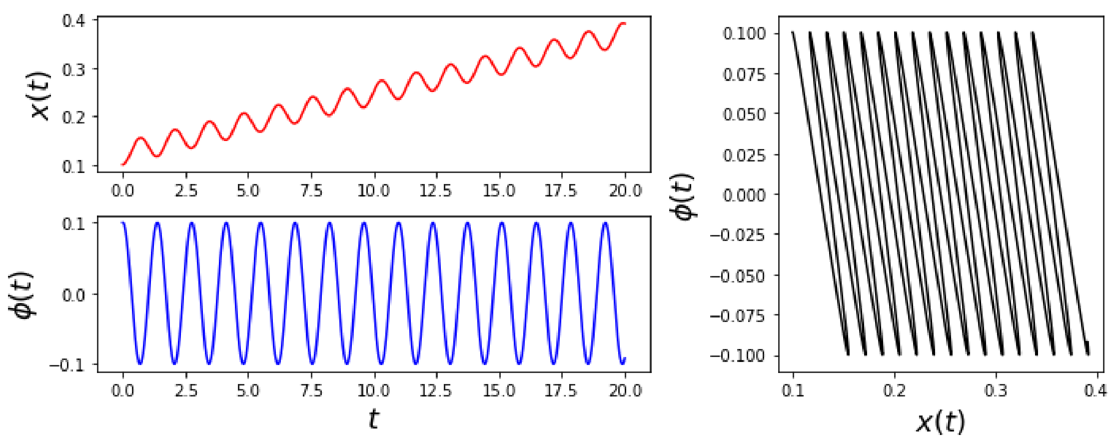
$$-y_4(t) + \frac{d}{dt} y_3(t)$$

'New ODEs:'

$$\begin{bmatrix}\frac{d}{dt}y_1(t)\\\frac{d}{dt}y_2(t)\\\frac{d}{dt}y_3(t)\\\frac{d}{dt}y_4(t)\end{bmatrix}=\begin{bmatrix}y_2(t)\\\frac{m_2\left(g\cos\left(y_3(t)\right)+ly_4^2(t)\right)\sin\left(y_3(t)\right)}{m_1+m_2\sin^2\left(y_3(t)\right)}\\y_4(t)\\-\frac{\left(gm_1+gm_2+lm_2y_4^2(t)\cos\left(y_3(t)\right)\right)\sin\left(y_3(t)\right)}{l\left(m_1+m_2\sin^2\left(y_3(t)\right)\right)}\end{bmatrix}$$

'Jacobian Matrix of the System:'

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & -\frac{gm_2\sin^2\left(y_3(t)\right)}{m_1+m_2\sin^2\left(y_3(t)\right)}-\frac{2m_2^2\left(g\cos\left(y_3(t)\right)+ly_4^2(t)\right)\sin^2\left(y_3(t)\right)\cos\left(y_3(t)\right)}{\left(m_1+m_2\sin^2\left(y_3(t)\right)\right)^2}+\frac{m_2\left(g\cos\left(y_3(t)\right)+ly_4^2(t)\right)\cos\left(y_3(t)\right)}{m_1+m_2\sin^2\left(y_3(t)\right)}\\0 & 0 & 0\\0 & 0 & \frac{m_2y_4^2(t)\sin^2\left(y_3(t)\right)}{m_1+m_2\sin^2\left(y_3(t)\right)}+\frac{2m_2\left(gm_1+gm_2+lm_2y_4^2(t)\cos\left(y_3(t)\right)\right)\sin^2\left(y_3(t)\right)\cos\left(y_3(t)\right)}{l\left(m_1+m_2\sin^2\left(y_3(t)\right)\right)^2}-\frac{\left(gm_1+gm_2+lm_2y_4^2(t)\cos\left(y_3(t)\right)\right)\cos\left(y_3(t)\right)}{l\left(m_1+m_2\sin^2\left(y_3(t)\right)\right)}\end{bmatrix}$$



0.8.11 2.8 Hamiltonian Dynamics

[3]:
```
#----> 2.8.2.0 Motion in a uniform gravitational field
if "2.8.2.0 Motion in a uniform gravitational field" in sets.flow:
    # Prepare Lagrangian
    pprints("2.8.2.0 Motion in a uniform gravitational field")
    omech.__init__("EulerLagrange")
    omech.verbose = True
    omech.T = Eq(S('T'), S(1)/2*m*(D(x)**2 + D(y)**2 + D(z)**2))
    omech.T = Eq(S('T'), S(1)/2*m*(xdot**2 + ydot**2 + zdot**2))
    omech.V = Eq(S('V'), m*g*z)
    omech.L = Eq(S('L'), omech.T.rhs - omech.V.rhs)

    calc_type = {1:"1. Way",
                 2:"2. Way"}[2]
```

```
    if calc_type == "1. Way":
        # 1. Way: Implementation step by step.
        """
        1. Calculate generalize momenta by taking derivative of Lagrangian with␣
↪respect to q_idot.
        2. Solve q_idots from generalize momenta equations.
        3. Replace q_idots in Lagrangian with corresponding generalize momenta.
        4. Replace pi*qidot in Hamiltonian with expressions written in terms of␣
↪generalize momenta.
        5. Calculate qidot, p_idot, p_idot by Hamilton's equations.
        """
    #    1. Calculate generalize momenta by taking derivative of Lagrangian␣
↪with respect to q_idot.
        eq_px = omech.p_i.xreplace({L:omech.L.rhs, q_idot:xdot, p_i:px}).doit()
        eq_py = omech.p_i.xreplace({L:omech.L.rhs, q_idot:ydot, p_i:py}).doit()
        eq_pz = omech.p_i.xreplace({L:omech.L.rhs, q_idot:zdot, p_i:pz}).doit()
    #    2. Solve q_idots from generalize momenta equations.
        sol_xdot = solve(eq_px, xdot)[0]
        sol_ydot = solve(eq_py, ydot)[0]
        sol_zdot = solve(eq_pz, zdot)[0]
    #    3. Replace q_idots in Lagrangian with corresponding generalize momenta.
        sub_qidots = {xdot:sol_xdot, ydot:sol_ydot, zdot:sol_zdot}
        omech.L = omech.L.subs(sub_qidots)
    #    4. Replace pi*qidot in Hamiltonian with expressions written in terms␣
↪of generalize momenta.
        piqidot = Matrix([[px,py,pz]]).
↪dot(Matrix([[sol_xdot,sol_ydot,sol_zdot]]))
        substitutions = {n:1, L:omech.L.rhs, p_i*q_idot:piqidot}
        omech.H = simplify(omech.H.xreplace(substitutions).doit())
    #    5. Calculate qidot, p_idot by Hamilton's equations.
        xdot = omech.q_idot.xreplace({H:omech.H.rhs, p_i:px, q_idot:xdot})
        ydot = omech.q_idot.xreplace({H:omech.H.rhs, p_i:py, q_idot:ydot})
        zdot = omech.q_idot.xreplace({H:omech.H.rhs, p_i:pz, q_idot:zdot})
    #    zdot = omech.Hamiltons_equations_I.xreplace({H:omech.H.rhs, p_i:pz,␣
↪q_idot:zdot})
        pxdot = omech.p_idot.xreplace({H:omech.H.rhs, q_i:x, p_idot:pxdot})
        pydot = omech.p_idot.xreplace({H:omech.H.rhs, q_i:y, p_idot:pydot})
        pzdot = omech.p_idot.xreplace({H:omech.H.rhs, q_i:z, p_idot:pzdot})
    elif calc_type == "2. Way":
        # 2. Way: Implementation step by step.
        lst_qi    = [x,y,z]
        lst_qidot = [xdot, ydot, zdot]
        lst_pi    = [px,py,pz]
        lst_pidot = [pxdot, pydot, pzdot]
```

```
    [[xdot,ydot,xdot], [pxdot,pydot,pzdot]] = omech.
↪Hamiltons_equations(omech.L, [x,y,z], [xdot, ydot, zdot],
                                                              ␣
↪  [px,py,pz], [pxdot, pydot, pzdot])

  pprints("Example 8.5.1 : Motion in a uniform gravitational field [Cline]")
  display(f"calc_type={calc_type}",
          omech.L,
          omech.H,
          omech.Hamiltons_equations_I,
          omech.Hamiltons_equations_II,
          xdot,ydot,xdot,
          pxdot,pydot,pzdot)
```

'2.8.2.0 Motion in a uniform gravitational field'

$$L = -gmz(t) + \frac{m\left(\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)\right)}{2}$$

$$[x(t),\ y(t),\ z(t)]$$

$$[\dot{x}(t),\ \dot{y}(t),\ \dot{z}(t)]$$

$$[p_x(t),\ p_y(t),\ p_z(t)]$$

$$[\dot{p}_x(t),\ \dot{p}_y(t),\ \dot{p}_z(t)]$$

$$p_i(t) = \frac{d}{d\dot{q}_i(t)} L(q_i(t), \dot{q}_i(t), t)$$

$$[p_x(t) = m\dot{x}(t),\ p_y(t) = m\dot{y}(t),\ p_z(t) = m\dot{z}(t)]$$

$$\left\{\dot{x}(t) : \frac{p_x(t)}{m},\ \dot{y}(t) : \frac{p_y(t)}{m},\ \dot{z}(t) : \frac{p_z(t)}{m}\right\}$$

$$L = -gmz(t) + \frac{m\left(\frac{p_x^2(t)}{m^2} + \frac{p_y^2(t)}{m^2} + \frac{p_z^2(t)}{m^2}\right)}{2}$$

$$\left\{n : 1,\ p_i(t)\dot{q}_i(t) : \frac{p_x^2(t)}{m} + \frac{p_y^2(t)}{m} + \frac{p_z^2(t)}{m},\ L(q_i(t), \dot{q}_i(t), t) : -gmz(t) + \frac{m\left(\frac{p_x^2(t)}{m^2} + \frac{p_y^2(t)}{m^2} + \frac{p_z^2(t)}{m^2}\right)}{2}\right\}$$

$$H = gmz(t) + \frac{p_x^2(t) + p_y^2(t) + p_z^2(t)}{2m}$$

$$\dot{q}_i(t) = \frac{d}{dp_i(t)} H(q_i(t), p_i(t), t)$$

$$\dot{p}_i(t) = -\frac{d}{dq_i(t)} H(q_i(t), p_i(t), t)$$

$$\dot{x}(t) = \frac{p_x(t)}{m}$$

$\dot{y}(t) = \frac{p_y(t)}{m}$

$\dot{z}(t) = \frac{p_z(t)}{m}$

$\dot{p}_x(t) = 0$

$\dot{p}_y(t) = 0$

$\dot{p}_z(t) = -gm$

'Example 8.5.1 : Motion in a uniform gravitational field [Cline]'

'calc_type=2. Way'

$$L = -gmz(t) + \frac{m\left(\frac{p_x^2(t)}{m^2} + \frac{p_y^2(t)}{m^2} + \frac{p_z^2(t)}{m^2}\right)}{2}$$

$$H = gmz(t) + \frac{p_x^2(t) + p_y^2(t) + p_z^2(t)}{2m}$$

$$\dot{q}_i(t) = \frac{d}{dp_i(t)} H(q_i(t), p_i(t), t)$$

$$\dot{p}_i(t) = -\frac{d}{dq_i(t)} H(q_i(t), p_i(t), t)$$

$\dot{z}(t) = \frac{p_z(t)}{m}$

$\dot{y}(t) = \frac{p_y(t)}{m}$

$\dot{z}(t) = \frac{p_z(t)}{m}$

$\dot{p}_x(t) = 0$

$\dot{p}_y(t) = 0$

$\dot{p}_z(t) = -gm$

0.8.12 2.8.2.1 Example 1: Moving Beat on a String

[3]:
```
#----> 2.8.2.1 Example 1: Moving Beat on a String
if "2.8.2.1 Example 1: Moving Beat on a String" in sets.flow:
    # todo check
    pprints("2.8.2.1 Example 1: Moving Beat on a String")
    omech.__init__("EulerLagrange")
    omech.verbose = True
    omech.T = Eq(S('T'), S(1)/2*m*(D(x)**2 + D(y)**2))
    omech.T = Eq(S('T'), S(1)/2*m*(xdot**2 + ydot**2))
    omech.V = Eq(S('V'), m*g*y)
    omech.L = Eq(S('L'), omech.T.rhs - omech.V.rhs)
```

```
    f = Function('f')(x)
    omech.L = omech.L.xreplace({y:f, ydot:D(f)})
    [lst_qidot, lst_pidot] = omech.Hamiltons_equations(omech.L, [x], [xdot],
                              [px], [pxdot])
    display(lst_qidot, lst_pidot)
```

'2.8.2.1 Example 1: Moving Beat on a String'

$$L = -gmf(x(t)) + \frac{m\left(\dot{x}^2(t) + \left(\left(\frac{d}{dx(t)}f(x(t))\frac{d}{dt}x(t)\right)\right)^2\right)}{2}$$

$[x(t)]$

$[\dot{x}(t)]$

$[p_x(t)]$

$[\dot{p}_x(t)]$

$$p_i(t) = \frac{d}{d\dot{q}_i(t)}L(q_i(t), \dot{q}_i(t), t)$$

$[p_x(t) = m\dot{x}(t)]$

$$\left\{\dot{x}(t) : \frac{p_x(t)}{m}\right\}$$

$$L = -gmf(x(t)) + \frac{m\left(\left(\left(\frac{d}{dx(t)}f(x(t))\frac{d}{dt}x(t)\right)\right)^2 + \frac{p_x^2(t)}{m^2}\right)}{2}$$

$$\left\{n : 1,\ p_i(t)\dot{q}_i(t) : \frac{p_x^2(t)}{m},\ L(q_i(t), \dot{q}_i(t), t) : -gmf(x(t)) + \frac{m\left(\left(\left(\frac{d}{dx(t)}f(x(t))\frac{d}{dt}x(t)\right)\right)^2 + \frac{p_x^2(t)}{m^2}\right)}{2}\right\}$$

$$H = gmf(x(t)) - \frac{m\left(\frac{d}{dx(t)}f(x(t))\right)^2\left(\frac{d}{dt}x(t)\right)^2}{2} + \frac{p_x^2(t)}{2m}$$

$$\dot{q}_i(t) = \frac{d}{dp_i(t)}H(q_i(t), p_i(t), t)$$

$$\dot{p}_i(t) = -\frac{d}{dq_i(t)}H(q_i(t), p_i(t), t)$$

$$\dot{x}(t) = \frac{p_x(t)}{m}$$

$$\dot{p}_x(t) = -gm\frac{d}{dx(t)}f(x(t)) + m\frac{d}{dx(t)}f(x(t))\frac{d^2}{dx(t)^2}f(x(t))\left(\frac{d}{dt}x(t)\right)^2$$

$$\left[\dot{x}(t) = \frac{p_x(t)}{m}\right]$$

$$\left[\dot{p}_x(t) = -gm\frac{d}{dx(t)}f(x(t)) + m\frac{d}{dx(t)}f(x(t))\frac{d^2}{dx(t)^2}f(x(t))\left(\frac{d}{dt}x(t)\right)^2 \right]$$

0.8.13 2.8.4.1 Example 1: Motion on a Cylinder

[3]:
```
#----> 2.8.4.1 Example 1: Motion on a Cylinder
if "2.8.4.1 Example 1: Motion on a Cylinder" in sets.flow:
    pprints("2.8.4.1 Example 1: Motion on a Cylinder")
    omech.class_type = "EulerLagrange"
    omech.__init__()
    omech.verbose = True
    omech.output_style = {1:"latex", 2:"display"}[2]
    R,kappa = symbols('R kappa', real=True)
    theta = Function('theta')(t)
    thetadot = Function('thetadot')(t)
    p_theta, p_thetadot = symbols('p_theta pdot_theta', real=True)

    omech.T = Eq(S('T'), S(1)/2*m*(zdot**2 + R**2*thetadot**2))
    omech.V = Eq(S('V'), k/2*(R**2 + z**2))
    omech.L = Eq(S('L'), omech.T.rhs - omech.V.rhs)
    lst_qi    = [z,theta]
    lst_pi    = [pz, p_theta]
    lst_qidot = [zdot, thetadot]
    lst_pidot = [pzdot, p_thetadot]
    [res_qidot, res_pidot] = omech.Hamiltons_equations(omech.L, [z,theta],
                            [zdot, thetadot], [pz, p_theta], [pzdot,␣
↪p_thetadot])
    # pprints(lst_qidot, lst_pidot)

    eq1 = Eq(diff(res_qidot[0].lhs,t), diff(res_qidot[0].rhs,t))
    eq1 = eq1.subs({lst_qidot[0]:diff(lst_qi[0]),
                diff(lst_pi[0]):res_pidot[0].rhs})
    omech.z = dsolve(eq1, lst_qi[0])
    omech.theta = dsolve(Eq(m*R**2*diff(theta), kappa), theta)


    # Numerical calculations 1. Way, sympy
    [C1,C2] = symbols('C1 C2')
    numvals = {C1:0, C2:1, R:1, m:1, k:0.1, kappa:2}
    z = omech.z.rhs
    theta = omech.theta.rhs
    x = (R*sin(theta)).xreplace(numvals)
    y = (R*cos(theta)).xreplace(numvals)
    z = z.xreplace(numvals)
    plot3d_parametric_line(x, y, z, (t, 0, 6*pi))

    # Numerical calculations 2. Way, matplotlib
```

```
# https://stackoverflow.com/questions/45627187/plot-a-curve-in-3d-with-sympy
t = symbols('t')
alpha = [x,y,z]
f = lambdify(t, alpha)
# T = [6*math.pi/1000*n for n in range(1000)]
T = np.linspace(0, 6*np.pi, 200)
F = [f(t) for t in T]

fig1, ax1 = plt.subplots(subplot_kw=dict(projection='3d'))
ax1.plot(*zip(*F))
ax1.set_aspect('auto')
plt.show()

# todo: matplotlib animate
```

'2.8.4.1 Example 1: Motion on a Cylinder'

$$L = -\frac{k\left(R^2 + z^2(t)\right)}{2} + \frac{m\left(R^2\dot{\theta}^2(t) + \dot{z}^2(t)\right)}{2}$$

$$[z(t),\ \theta(t)]$$

$$\left[\dot{z}(t),\ \dot{\theta}(t)\right]$$

$$[p_z(t),\ p_\theta]$$

$$[\dot{p}_z(t),\ \dot{p}_\theta]$$

$$p_i(t) = \frac{d}{d\dot{q}_i(t)}L(q_i(t), \dot{q}_i(t), t)$$

$$\left[p_z(t) = m\dot{z}(t),\ p_\theta = R^2 m\dot{\theta}(t)\right]$$

$$\left\{\dot{\theta}(t) : \frac{p_\theta}{R^2 m},\ \dot{z}(t) : \frac{p_z(t)}{m}\right\}$$

$$L = -\frac{k\left(R^2 + z^2(t)\right)}{2} + \frac{m\left(\frac{p_z^2(t)}{m^2} + \frac{p_\theta^2}{R^2 m^2}\right)}{2}$$

$$\left\{n : 1,\ p_i(t)\dot{q}_i(t) : \frac{p_z^2(t)}{m} + \frac{p_\theta^2}{R^2 m},\ L(q_i(t), \dot{q}_i(t), t) : -\frac{k\left(R^2 + z^2(t)\right)}{2} + \frac{m\left(\frac{p_z^2(t)}{m^2} + \frac{p_\theta^2}{R^2 m^2}\right)}{2}\right\}$$

$$H = \frac{kx^2(t)}{2} + \frac{m\left(\frac{d}{dt}x(t)\right)^2}{2}$$

$$\dot{q}_i(t) = \frac{d}{dp_i(t)}H(q_i(t), p_i(t), t)$$

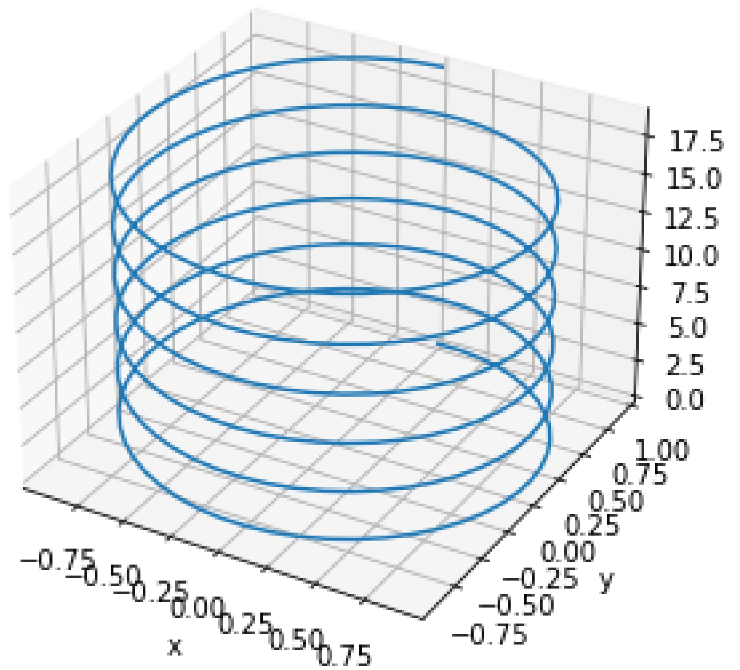
$$\dot{p}_i(t) = -\frac{d}{dq_i(t)}H(q_i(t), p_i(t), t)$$

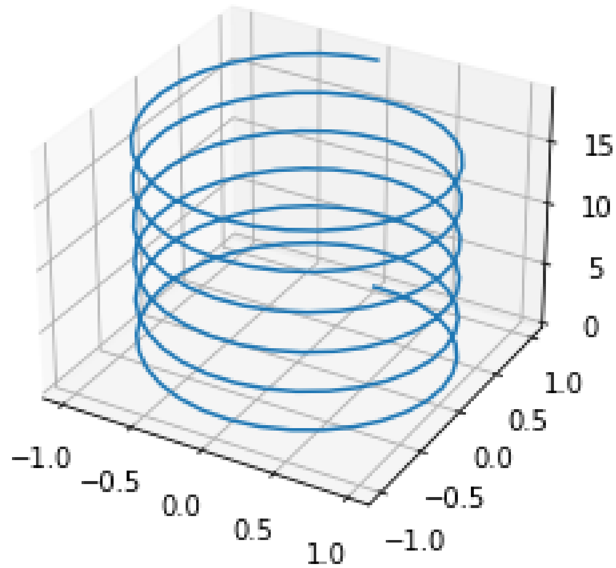
$\dot{z}(t) = 0$

$\dot{\theta}(t) = 0$

$\dot{p}_z(t) = 0$

$\dot{p}_\theta = 0$





[ ]:
```
# HW todo:
    # Example 4: Sliding Mass on a Curve p335 (Baumann)

    """
    # todo: Future Work,
    2.8.6 Poisson Brackets
    2.8.7 Manifolds and Classes
    2.8.8 Canonical Transformations
    2.8.9 Generating Functions
    2.8.10 Action Variables

    FINAL
    """
```