Part 2) $\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$

$$x=x'\cos(\theta)-y'\sin(\theta)$$
$$y=x'\sin(\theta)+y'\cos(\theta)$$

$$\frac{\partial x}{\partial x'}=\cos(\theta) \qquad \frac{\partial y}{\partial x'}=\sin(\theta) \qquad \frac{\partial^2 x}{\partial x'^2}=0 \qquad \frac{\partial^2 y}{\partial x'^2}=0$$

$$\frac{\partial x}{\partial y'}=-\sin(\theta) \qquad \frac{\partial y}{\partial y'}=\cos(\theta) \qquad \frac{\partial^2 x}{\partial y'^2}=0 \qquad \frac{\partial^2 y}{\partial y'^2}=0$$

$$\frac{\partial f}{\partial x'}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial x'}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \qquad \frac{\partial f}{\partial y'}=\frac{\partial f}{\partial y}\frac{\partial y}{\partial y'}+\frac{\partial f}{\partial x}\frac{\partial x}{\partial y'}$$

$$\frac{\partial^2 f}{\partial x'^2}=\frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial x'^2}+\frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial x'^2}+\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial x'}\right)^2+\frac{\partial x}{\partial x'}\frac{\partial y}{\partial x'}\left(\frac{\partial^2 f}{\partial y\partial x}\ \frac{\partial^2 f}{\partial x\partial y}\right)+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial x'}\right)^2$$

$$\frac{\partial^2 f}{\partial x'^2}=\frac{\partial f}{\partial x}\cdot 0+\frac{\partial f}{\partial y}\cdot 0+\frac{\partial^2 f}{\partial x^2}\cdot\cos(\theta)^2+\cos(\theta)\cdot\sin(\theta)\cdot\left(\frac{\partial^2 f}{\partial y\partial x}\ \frac{\partial^2 f}{\partial x\partial y}\right)+\frac{\partial^2 f}{\partial y^2}\sin(\theta)^2$$

$$\frac{\partial^2 f}{\partial x'^2}=\cos(\theta)^2\frac{\partial^2 f}{\partial x^2}+\cos(\theta)\sin(\theta)\left(\frac{\partial^2 f}{\partial y\partial x}\ \frac{\partial^2 f}{\partial x\partial y}\right)+\sin(\theta)^2\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial y'^2}=\frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial y'^2}+\frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial y'^2}+\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial y'}\right)^2+\frac{\partial x}{\partial y'}\frac{\partial y}{\partial y'}\left(\frac{\partial^2 f}{\partial y\partial x}\ \frac{\partial^2 f}{\partial x\partial y}\right)+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial y'}\right)^2$$

$$\frac{\partial^2 f}{\partial y'^2}=\frac{\partial f}{\partial x}\cdot 0+\frac{\partial f}{\partial y}\cdot 0+\frac{\partial^2 f}{\partial x^2}\cdot\sin(\theta)^2-\sin(\theta)\cdot\cos(\theta)\cdot\left(\frac{\partial^2 f}{\partial y\partial x}\ \frac{\partial^2 f}{\partial x\partial y}\right)+\frac{\partial^2 f}{\partial y^2}\cos(\theta)^2$$

$$\frac{\partial^2 f}{\partial y'^2}=\sin(\theta)^2\frac{\partial^2 f}{\partial x^2}-\cos(\theta)\sin(\theta)\left(\frac{\partial^2 f}{\partial y\partial x}\ \frac{\partial^2 f}{\partial x\partial y}\right)+\cos(\theta)^2\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2}=\left(\cos(\theta)^2+\sin(\theta)^2\right)\frac{\partial^2 f}{\partial x^2}+\left(\cos(\theta)\sin(\theta)-\cos(\theta)\sin(\theta)\right)\left(\frac{\partial^2 f}{\partial y\partial x}\ \frac{\partial^2 f}{\partial x\partial y}\right)+\left(\cos(\theta)^2+\sin(\theta)^2\right)\frac{\partial^2 f}{\partial y^2}$$

$$\cos(\theta)^2+\sin(\theta)^2=1$$

$$\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\nabla^2 f$$

Part 3)

h(x , y)=3 f (x , y)+2 f (x −1, y)+2 f (x +1, y)−17 f (x , y−1)+ 99 f (x , y+1)

a) Is h linear? Prove your answer.

For linearity we need $O(af_1+bf_2) = aO(f_1) + bO(f_2)$, f1 and f2 are images, a and b constants

$$
\begin{aligned}
& h(af_1+bf_2) = 3(af_1+bf_2)(x,y)+2(af_1+bf_2)(x-1,y)+2(af_1+bf_2)(x+1,y)-17(af_1+bf_2)(x,y-1) \\
& 99(af_1+bf_2)(x,y+1) \\
& = 3af_1(x,y)+3bf_2(x,y)+2af_1(x-1,y)+2bf_2(x-1,y)+2af_1(x+1,y)+2bf_2(x+1,y) \\
& -17af_1(x,y-1)-17bf_2(x,y-1)+99af_1(x,y+1)+99bf_2(x,y+1) \\
& = a(3f_1(x,y)+2f_1(x-1,y)+2f_1(x+1,y)-17f_1(x,y-1)+99f_1(x,y+1)) \\
& \quad +b(3f_2(x,y)+2f_2(x-1,y)+2f_2(x+1,y)-17f_2(x,y-1)+99f_2(x,y+1)) \\
& = ah(f_1)+ah(f_2)
\end{aligned}
$$

h is linear.

b) Provide the convolution mask corresponding to h.

$$
\begin{bmatrix} f(x-1,y-1) & f(x,y-1) & f(x+1,y-1) \\ f(x-1,y) & f(x,y) & f(x+1,y) \\ f(x-1,y+1) & f(x,y+1) & f(x+1,y+1) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -17 & 0 \\ 2 & 3 & 2 \\ 0 & 99 & 0 \end{bmatrix}
$$

After flipping h = $\begin{bmatrix} 0 & 99 & 0 \\ 2 & 3 & 2 \\ 0 & -17 & 0 \end{bmatrix}$