Determining Rotations Between Disc Axis and Line of Sight

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A disc photographed at an angle appears in the photo as an ellipse. From the shape and orientation of the ellipse, we can figure out the camera angle and the orientation of the disc.

The ellipse in the photo can be defined by the ratio of its axes and its tilt:

$$k \equiv \frac{b}{a} = \frac{\text{length of minor axis}}{\text{length of major axis}}$$

$\omega \equiv$ angle of tilt of the major axis (with respect to vertical in the photo) (0 = vertical, +ve clockwise)

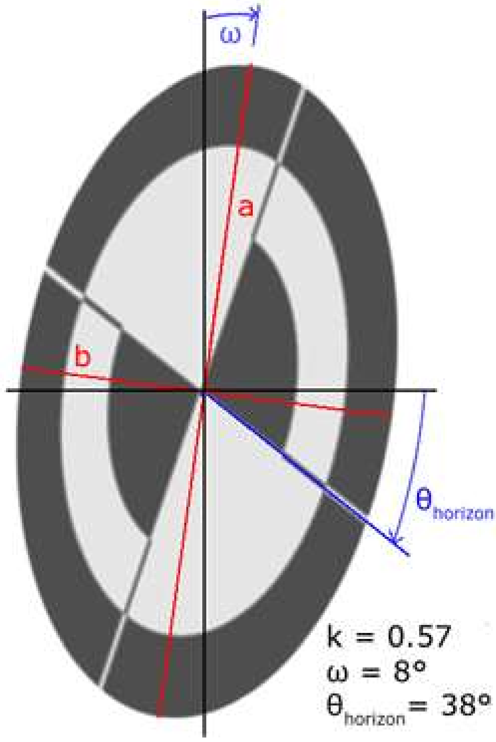


Figure 1. Circular perspective target on reference scale, photographed obliquely (from the right and above)

For the following analysis, we assume orthographic projection, ie., that the camera is viewing from an infinite distance. From an infinite distance, the center of the ellipse image in the photo will exactly match the center of the disc; in real life, there will be a small offset that depends on the focal length of the lens. Figure 1 is a photo taken with a digital camera with a moderate zoom lens (a focal length of about 105mm, in 35mm terms, and angle of view of 18 degrees); the center of the image ellipse is nearly coincident with the center of the disc.

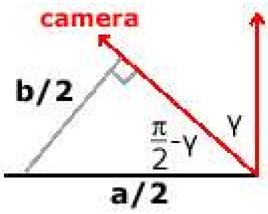
[A camera at a distance approaching infinity needs a telephoto lens with an angle of view approaching zero. Long telephoto lenses have angles of view of a few degrees, and a 'normal' lens (50mm on a 35mm camera) has an angle of view of about 40 degrees. The angle of view for a lens is given by $2 \tan^{-1} (d / 2f)$, where d is the width of the image and f is the focal length of the lens (both typically in millimeters). Thus a 200mm lens on a 35mm camera with 36mm-wide film frame has an angle of view of 10 degrees. For a given f, reducing d reduces the angle of view by roughly a proportional amount (so a 50mm lens on a digital SLR that has a sensor 50% the size of a 36mm film frame will have an angle of view ~50% smaller).]

(For the following derivations, Wikipedia's List of Trigonometric Identities is helpful.)

Our goal is to figure out the angle(s) between the camera's view and the face of the disc -- in other words, figure out what rotations would be necessary to make the disc face the camera face-on.

One Rotation

We know that no matter how a disc is positioned, its major axis will always be perpendicular to the camera's line of sight and thus its major axis will always be equal to its diameter (which is why a utility hole cover cannot fall into a utility hole). Thus the semiminor axis (b/2) is the projection of the disc radius (a/2) upon the line of sight vector, giving us the angle γ between the disc axis and the camera line of sight:



$$\sin(\pi/2 - \gamma) = \frac{b}{a} = k$$

$$\cos \gamma = k$$

$$\gamma = \cos^{-1} k$$

Thus one rotation of the ellipse of γ about the major axis will align the disc axis to our line of sight (and thus we'd see the disc face-on). Similarly, stretching the image perpendicular to the major axis by 1/k will restore the ellipse to a circular shape (and rectify the surrounding scene, to the extent that it is planar and parallel to the disc lying upon it).

Three Rotations

We can obtain different information by decomposing the single rotation above into two rotations, one horizontal and one vertical, by using the tilt of the ellipse in the image. These two rotations will be with respect to the frame of reference defined by the camera, and will only match the real world if: 1) the camera was horizontal, and 2) the camera was at the same vertical elevation as the disc.

Furthermore, if we know the disc was aligned with real-world horizontal (eg., the disc in Figure 1 aligned such that its horizontal line is level with the horizon, even if the disc is leaning back) and if the camera was level, then we can learn even more information from the image -- we can determine the real world horizontal and vertical rotations of the disc, and the angle by which the camera was looking down, all with respect to real-world horizontal.

Let's assume the camera was level but possibly looking down (or up) at an angle δ. Further, let's assume the disc is possibly lying back at an angle β and possibly rotated away from the camera at angle α. In other words:

- **α** Angle of rotation about the real-world vertical axis between the disc axis and the horizontal direction of view of the camera (0 = no rotation; 90 degrees = viewed left edge-on, disc facing right)
- **β** Angle of rotation about the real-world horizontal axis ('lie back') between the disc axis and the real world horizontal plane (0 = no rotation, disc is vertical; 90 degrees = disc is flat, facing up)
- **δ** Angle that the camera is looking down, relative to real world horizontal (0 = level; 90 degrees = looking straight down)

Assume orthogonal unit vectors **x**,**y**,**z**, with the disc in the **x**,**y** plane with its axis aligned with the **z** axis.

Consider the following rotations:

1. Rotate the disc about the horizontal axis (**x**) by β (positive β moves the top of the disc (+**y**) back, toward **-z**)
2. Then rotate the disc about the vertical axis (**y**) by α (positive α moves the right of the disc (+**x**) back, toward **-z**)

   The above-two rotations put the disc into any position. Now let's consider a final rotation to account for the possibility that the camera may be looking down or up upon the disc:
3. Rotate the disc about the horizontal axis (**x**) by δ (positive δ moves the top of the disc (+**y**) forward, toward +**z**)

These rotations are given by the following standard transformation matrices for rotation about an axis, with adjustment of the signs for the rotation directions defined above:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{vmatrix} \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & \sin\delta \\ 0 & -\sin\delta & \cos\delta \end{vmatrix} \begin{vmatrix} x \\ y \\ 0 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & -\sin\alpha\sin\beta \\ \sin\alpha\sin\delta & \cos\beta\cos\delta + \cos\alpha\sin\beta\sin\delta \end{vmatrix} \qquad (1)$$

Let **i**,**j** be axes for the 2-dimensional photo plane; the transformation above maps points (x,y,0) of the disc to (i,j) in the photo (recall we assumed the disc is in the x,y plane, so all disc points have z=0).

[i j] = T [x y], where T is the transformation above, at (1), giving:

$$i = x\cos\alpha - y\sin\alpha\sin\beta \qquad (2)$$
$$j = x\sin\alpha\sin\delta + y(\cos\beta\cos\delta + \cos\alpha\sin\beta\sin\delta) \qquad (3)$$

The parametric equation of an ellipse at the origin of the **i**,**j** plane with its major axis the j axis is:

$i = k \sin t$ (4) (an unrotated ellipse, aligned with the j axis)
$j = \cos t$ (5)

('k' is the ellipse ratio defined above, not a third coordinate)

This ellipse can be matched to the one in the photo by rotating by ω (clockwise from the j axis) using the following rotation transformation:

$$i' = i \cos \omega + j \sin \omega \qquad (6)$$
$$j' = -i \sin \omega + j \cos \omega \qquad (7)$$

Substituting i and j from (4),(5) into (6),(7):

$$i' = k \sin t \cos \omega + \cos t \sin \omega$$

$$= k \cos \omega \sin t + \sin \omega \cos t$$

Applying the identity:

$$a \sin t + b \cos t = \sqrt{a^2 + b^2} \sin (t + \tan^{-1} b/a)$$

$$= \sqrt{k^2 \cos^2 \omega + \sin^2 \omega} \sin (t + ...) \qquad (8)$$

and:

$$j' = -k \sin t \sin \omega + \cos t \cos \omega$$

$$= -k \sin \omega \sin t + \cos \omega \cos t$$

$$a \sin t + b \cos t = \sqrt{a^2 + b^2} \sin (t + \tan^{-1} b/a)$$

$$= \sqrt{k^2 \sin^2 \omega + \cos^2 \omega} \sin (t + ...) \qquad (9)$$

This rotated ellipse, in terms of k and ω, matches the photo ellipse. We can also obtain an ellipse matching the photo ellipse, but in terms of α β, and δ, by applying transformation (1) to a circle in the **x**,**y** plane, as follows:

$$x = \cos \alpha \cos t + \sin \alpha \sin \beta \sin t$$
$$= \sqrt{\cos^2 \alpha + \sin^2 \alpha \sin^2 \beta} \sin (t + ...) \qquad (10)$$

$$y = \sin \alpha \sin \delta \cos t + (\cos \beta \cos \delta + \cos \alpha \sin \beta \sin \delta) \sin t$$
$$= \sqrt{\sin^2 \alpha \sin^2 \delta + (\cos \beta \cos \delta + \cos \alpha \sin \beta \sin \delta)^2} \sin (t + ...) \qquad (11)$$

Now we have two sets of ellipse equations that match the photo ellipse. The maximum extent of i′ and j′ in ellipse equations (6) and (7) must match the maximum extent of x and y in ellipse equations (10) and (11). Thus:

$$\sqrt{\cos^2 \alpha + \sin^2 \alpha \sin^2 \beta} = \sqrt{k^2 \cos^2 \omega + \sin^2 \omega} \qquad \text{(12), from (10) \& (8)}$$

and:

$$\sqrt{\sin^2 \alpha \sin^2 \delta + (\cos \beta \cos \delta + \cos \alpha \sin \beta \sin \delta)^2} = \sqrt{k^2 \sin^2 \omega + \cos^2 \omega} \qquad \text{(13), from (11) \& (8)}$$

It'd be nice to solve for α and β in terms of δ, ω, and k, but I don't think it reduces to anything simple. Let's try a different tack:

Cross-hair markings on the disc provide orientation information for the disc; let's assume the disc is positioned with the markings aligned with horizontal/vertical, such that the horizontal cross-hair line is level (aligned with the real-world horizon).

Moving the camera out of the horizontal plane that contains the disc causes distortion of cross-hair print on the disc that is dependent on δ. In the photo to the right, vertical cross-hair line appears to lean to the right in the photo; the vertical cross-hair line has been rotated clockwise by $\theta_{vertical}$ degrees. Similarly, the horizontal cross-hair line appears rotated clockwise by $\theta_{horizon}$ degrees.

We can use this information to determine δ. If the disc was oriented with the horizontal cross-hair line level, the vertical cross-hair line was originally aligned with the y-axis and the horizontal cross-hair line was originally aligned with the x-axis. (We could maybe remove the dependence on the disc's original orientation by using the difference between the two -- something for further investigation.)

The disc circumference point in the photo at $\theta_{horizon} + \pi/2$ was at (1,0) in the **x**,**y** plane that contains the disc before rotations, and it is mapped to (i, j) in the photo by these equations copied from above:

$$i = x \cos \alpha - y \sin \alpha \sin \beta \qquad \text{Copy of (2)}$$
$$j = x \sin \alpha \sin \delta + y (\cos \beta \cos \delta + \cos \alpha \sin \beta \sin \delta) \qquad \text{Copy of (3)}$$

For point (x=1, y=0), plugging x=1 and y=0 into (2) and (3), we get:

$$i_{horizon} = \cos a \qquad (14)$$
$$j_{horizon} = \sin a \sin \delta \qquad (15)$$

Also, we can find ($i_{horizon}$, $j_{horizon}$) for that point in terms of k and ω because we know the ellipse in the photo is given by:

$$i' = k \cos \omega \sin t + \sin \omega \cos t = i_{horizon} \qquad (16)$$
$$j' = -k \sin \omega \sin t + \cos \omega \cos t = j_{horizon} \qquad (17)$$

tan = y/x, so:

$$\tan -\theta_{horizon} = -\tan \theta_{horizon} = \frac{j_{horizon}}{i_{horizon}} = \frac{-k \sin \omega \sin t_{horizon} + \cos \omega \cos t_{horizon}}{k \cos \omega \sin t_{horizon} + \sin \omega \cos t_{horizon}}$$

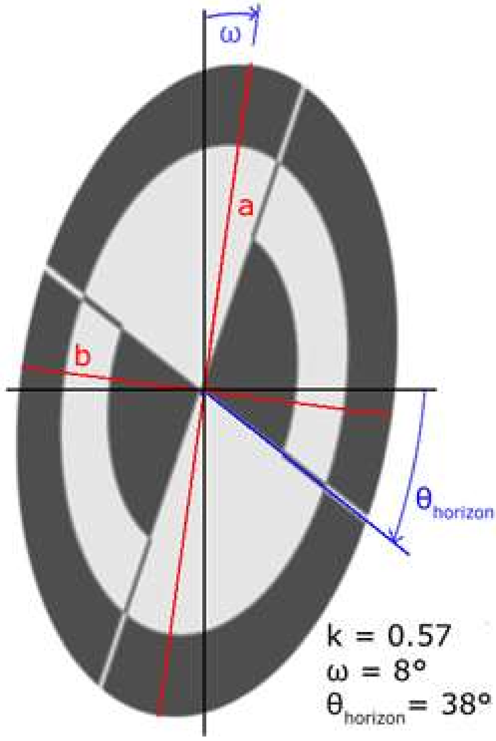
Solving for $t_{horizon}$:

$$t_{horizon} = \tan^{-1} \frac{1}{k \tan (\omega - \theta_{horizon})} \qquad (+ \pi \text{ when } \pi/2 < |w - \theta_{horizon}| < 3\pi/2)$$

which allows us to solve (16) and (17) at $t = t_{horizon}$, and thus obtain a and β from (14) and (15):

$$a = \cos^{-1} i_{horizon}$$

$$\delta = \sin^{-1} \frac{j_{horizon}}{\sin a} = \sin^{-1} \frac{j_{horizon}}{\sqrt{1 - i_{horizon}^2}}$$



β is obtained from (10):

$$\beta = \cos^{-1} \pm\sqrt{\frac{\cos^2 \omega\,(1 - k^2)}{\sin^2 a}} \qquad \text{(using equation (13) to resolve the } \pm)$$

These results are used in the preceding ellipse analysis web page.

Two Rotations

The one rotation about the major axis that aligns the disc's axis with the line of sight can be decomposed into two rotations about the two axes (vertical and horizontal) defined by the camera's frame of reference (which may or may not correspond to the real world, depending on whether the camera was level and at the same elevation as the disc). This might be useful if the disc or camera weren't aligned with real-world vertical, or if a disc with no markings were used, etc.

a_2 Angle of rotation about the ***camera frame's*** vertical axis between the disc axis and the horizontal direction of view of the camera (0 = no rotation; 90 degrees = viewed left edge-on, disc facing right)

β_2 Angle of rotation about the ***camera frame's*** horizontal axis ('lie back') between the disc axis and the real world horizontal plane (0 = no rotation, disc is vertical; 90 degrees = disc is flat, facing up)

From the camera's frame of reference, a_2 and β_2 are relative to the camera and δ is always 0. Thus we can obtain the two rotation solution as a special case of the three rotation solution.

When δ = 0, equation (13) reduces to:

$$\cos \beta_2 = \pm\sqrt{k^2 \sin^2 \omega + \cos^2 \omega} \qquad (18)$$

Solving (12) and (18) for a and β:

$$a_2 = \sin^{-1} \pm\sqrt{\frac{\cos^2 \omega\,(k^2 - 1)}{\sin^2 \omega\,(1 - k^2) - 1}} \qquad \text{(- if omega > 0)}$$

$$\beta_2 = \sin^{-1} \pm\sqrt{\sin^2 \omega \, (1 - k^2)}$$

Four and Five Rotations?

A fourth rotation, for cases when the disc was not positioned with the cross-hair horizontal, could probably be deduced given the tilt of the vertical cross-hair line as additional information. The angle between the horizontal and vertical cross-hair lines varies, I believe, as they rotate -- if true, that could be exploited. The fourth rotation would be about the z-axis and occur first.

Real-world information (eg., the horizon), if available in the photo, could be used in a fifth rotation to compensate for the camera not being level.

Conceivably an elliptical image of a real-world disc could be isolated from a photo and measured using computer vision techniques, thus obtaining information about orientation.