Calorimeter moments analysis

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Abstract

These are some (more or less) random notes about the calorimeter moments analysis. The first part is a brief description of the basic code that I put together while trying to understand it. There's also some stuff dealing with possible improvements of the moments analysis, namely the measurement of the shower skeweness and the the error analysis aimed at the projection of the calorimeter clusters into the ACD.

1 Basics of the CAL moments analysis

The code for the moments analysis basically calculates the moment of inertia tensor (using energy as the weight instead of mass) and then diagonalizes this to get the three principal axes. The basic definitions can be found in [1, 2] and in our case they read:

$$\mathbb{I}_{xx} = \sum_{i=1}^n w_i(r_i^2 - x_i^2), \quad \mathbb{I}_{yy} = \sum_{i=1}^n w_i(r_i^2 - y_i^2), \quad \mathbb{I}_{zz} = \sum_{i=1}^n w_i(r_i^2 - z_i^2) \tag{1}$$

$$\mathbb{I}_{xy} = -\sum_{i=1}^n w_i x_i y_i, \quad \mathbb{I}_{xz} = -\sum_{i=1}^n w_i x_i z_i, \quad \mathbb{I}_{yz} = -\sum_{i=1}^n w_i y_i z_i \tag{2}$$

where the index i runs over the n hits in the calorimeter and the w_i are the weights associated with the hits (essentially the energy release). In addition to the moment of inertia, the sum of weights and the coordinates of the energy centroids are also used:

$$W = \sum_{i=1}^n w_i \tag{3}$$

$$\mathbf{r}_c = \frac{\sum_{i=1}^n w_i \mathbf{r}_i}{W} \tag{4}$$

In order to reduce the tensor of inertia to the principal axes we do have to solve the secular equation:

$$\det(\mathbb{I} - \lambda \mathbb{1}) = \det \begin{pmatrix} \mathbb{I}_{xx} - \lambda & \mathbb{I}_{xy} & \mathbb{I}_{xz} \\ \mathbb{I}_{xy} & \mathbb{I}_{yy} - \lambda & \mathbb{I}_{yz} \\ \mathbb{I}_{xz} & \mathbb{I}_{yz} & \mathbb{I}_{zz} - \lambda \end{pmatrix} = 0 \tag{5}$$

which is a cubic equation in λ yielding the three eigenvalues. By working out the tedious algebra we can write the equation as:

$$\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$$

where:

$$c_2 = -(\mathbb{I}_{xx} + \mathbb{I}_{yy} + \mathbb{I}_{zz}) \tag{6}$$

$$c_1 = \mathbb{I}_{xx}\mathbb{I}_{yy} + \mathbb{I}_{yy}\mathbb{I}_{zz} + \mathbb{I}_{xx}\mathbb{I}_{zz} - (\mathbb{I}_{xy}^2 + \mathbb{I}_{yz}^2 + \mathbb{I}_{xz}^2) \tag{7}$$

$$c_0 = -\mathbb{I}_{xx}\mathbb{I}_{yy}\mathbb{I}_{zz} - 2\mathbb{I}_{xy}\mathbb{I}_{yz}\mathbb{I}_{xz} + \mathbb{I}_{xx}\mathbb{I}_{yz}^2 + \mathbb{I}_{yy}\mathbb{I}_{xz}^2 + \mathbb{I}_{zz}\mathbb{I}_{xy}^2 \tag{8}$$

If we now define a new variable $\lambda' = \lambda + c_2/3$, the previous equation becomes:

$$\lambda'^3 + a\lambda' + b = 0$$

where:

$$a = \left(\frac{3c_1 - c_2^2}{3}\right) \tag{9}$$

$$b = \left(\frac{27c_0 + 2c_2^2 - 9c_1c_2}{27}\right) \tag{10}$$

(the algebra is fully worked out in [3]). We now set:

$$m = 2\sqrt{\frac{-a}{3}} \tag{11}$$

$$\psi = \frac{1}{3}\arccos\left(\frac{3b}{am}\right) \tag{12}$$

and, finally we get the three real solutions (guranteed by the fact that the tensor of inertia is symmetruc):

$$\lambda_0 = m\cos(\psi) - c_2/3 \tag{13}$$

$$\lambda_1 = m\cos(\psi + 2\pi/3) - c_2/3 \tag{14}$$

$$\lambda_2 = m\cos(\psi + 4\pi/3) - c_2/3 \tag{15}$$

Once we have the three eigenvalues we can work out the calcultaion of the eigenvectors $\mathbf{e}^i$ ($i = 1\ldots3$) defined by:

$$\mathbb{I}\mathbf{e}^i = \lambda_i\mathbf{e}^i$$

Following these conventions, λ_1 is the largest eigenvalue and, as a consequence, $\mathbf{e}^1$ is the principal axis of the cluster.

Once the three principal axis of the cluster have been found, the cluster χ^2 (normalized to the number of *degree of freedom*) is calculated as:

$$\chi^2 = \frac{\sum_{i=1}^n w_i d_i^2}{nW} \tag{16}$$

where d_i are the distances from each of the calorimeter hits to the axis parallel to $\mathbf{e}^1$ and passing throught the cluster centroid. Finally the some well know Merit quantities are calculated:

$$\texttt{CalTransRms} = \sqrt{\frac{|\lambda_1|}{W}} \tag{17}$$

$$\texttt{CalLongRms} = \sqrt{\frac{|\lambda_0| + |\lambda_2|}{2W\log L}} \tag{18}$$

$$\texttt{CalLRmsAsym} = \sqrt{\frac{|\lambda_0| - |\lambda_2|}{|\lambda_0| + |\lambda_2|}} \tag{19}$$

where L is the number of radiation lengths transversed.

1.1 Outline of the iterative moments analysis

Put here some details about the iteration scheme, as they are relevant for the calculation of the skewness (i.e. you get significantly different answers at different steps).

2 The (toy) problem in 2D

I thought the problem of the diagonalization of the inertia tensor in 2D could be useful for working out the error analysis, so here is a quick look at it. The basic definitions read as:

$$\mathbb{I}_{xx} = \sum_{i=1}^{n} w_i y_i^2, \quad \mathbb{I}_{yy} = \sum_{i=1}^{n} w_i x_i^2, \quad \mathbb{I}_{xy} = -\sum_{i=1}^{n} w_i x_i y_i \tag{20}$$

and the secular equation is:

$$\det(\mathbb{I} - \lambda \mathbb{1}) = \det \begin{pmatrix} \mathbb{I}_{xx} - \lambda & \mathbb{I}_{xy} \\ \mathbb{I}_{xy} & \mathbb{I}_{yy} - \lambda \end{pmatrix} = 0 \tag{21}$$

The eigenvalues are readily found:

$$\lambda_0 = \frac{(\mathbb{I}_{xx} + \mathbb{I}_{yy}) - \sqrt{(\mathbb{I}_{xx} - \mathbb{I}_{yy})^2 + 4\mathbb{I}_{xy}^2}}{2} \tag{22}$$

$$\lambda_1 = \frac{(\mathbb{I}_{xx} + \mathbb{I}_{yy}) + \sqrt{(\mathbb{I}_{xx} - \mathbb{I}_{yy})^2 + 4\mathbb{I}_{xy}^2}}{2} \tag{23}$$

$$\tag{24}$$

At this point, assuming that λ_0 is the smallest eigenvalue, we're left with the problem of calculating the corresponding eigenvector $\mathbf{e}^0$, obeying the equation:

$$\mathbb{I}\mathbf{e}^0 = \lambda_0 \mathbf{e}^0$$

Being the problem bi-dimensional, the two eigenvectors can be parametrized in terms of a single rotation angle ϕ (to be determined), representing the angle of the principal axis with respect to the original reference frame i.e.:

$$\begin{pmatrix} e_x^0 \\ e_y^0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \begin{pmatrix} e_x^1 \\ e_y^1 \end{pmatrix} = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \tag{25}$$

By putting everything together, from the definition of the principal axis we get:

$$\begin{pmatrix} \mathbb{I}_{xx} & \mathbb{I}_{xy} \\ \mathbb{I}_{xy} & \mathbb{I}_{yy} \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

The first of the two equations in the system can be squared, yielding:

$$\frac{(\mathbb{I}_{xx} - \mathbb{I}_{yy})}{\mathbb{I}_{xy}} \tan\phi = \tan^2\phi - 1$$

and eventually, through the trigonometric equation

$$\tan(2\phi) = \frac{2\tan\phi}{1 - \tan^2\phi}$$

we get:

$$\phi = -\frac{1}{2} \arctan\left(\frac{2\mathbb{I}_{xy}}{\mathbb{I}_{yy} - \mathbb{I}_{xx}} \right) \tag{26}$$

The rotation matrix between the original system and the one defined by the principal axis has the (transpose of) the two eigenvectors as its rows:

$$S = \begin{pmatrix} e_x^0 & e_y^0 \\ e_x^1 & e_y^1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{27}$$

and obviously we have:

$$\lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} = S\, \mathbb{I}\, S^{-1} = S\, \mathbb{I}\, S^{\mathrm{T}} \tag{28}$$

2.1 Error analysis in 2 dimensions

As a first ingredient we'll need the derivatives of the components of the tensor of inertia with respect to the coordinates and the weights:

$$\frac{\partial \mathbb{I}_{xx}}{\partial x_i} = 0, \quad \frac{\partial \mathbb{I}_{xx}}{\partial y_i} = 2w_i y_i, \quad \frac{\partial \mathbb{I}_{xx}}{\partial w_i} = y_i^2 \tag{29}$$

$$\frac{\partial \mathbb{I}_{yy}}{\partial x_i} = 2w_i x_i, \quad \frac{\partial \mathbb{I}_{yy}}{\partial y_i} = 0, \quad \frac{\partial \mathbb{I}_{yy}}{\partial w_i} = x_i^2 \tag{30}$$

$$\frac{\partial \mathbb{I}_{xy}}{\partial x_i} = -w_i y_i, \quad \frac{\partial \mathbb{I}_{xy}}{\partial y_i} = -w_i x_i, \quad \frac{\partial \mathbb{I}_{xy}}{\partial w_i} = -x_i y_i \tag{31}$$

We can then calculate the full covariance matrix of the errors using the usual formula (see [4], section 32.1.4 for instance). Assuming that the errors on the two positions and on the weights are mutually not correlated (i.e. their covariance matrix is diagonal), we have:

$$\Sigma_{k-l} = \sum_{i=1}^{n} \frac{\partial \mathbb{I}_k}{\partial x_i}\frac{\partial \mathbb{I}_l}{\partial x_i}(\Delta x_i)^2 + \frac{\partial \mathbb{I}_k}{\partial y_i}\frac{\partial \mathbb{I}_l}{\partial y_i}(\Delta y_i)^2 + \frac{\partial \mathbb{I}_k}{\partial w_i}\frac{\partial \mathbb{I}_l}{\partial w_i}(\Delta w_i)^2 \tag{32}$$

where k and l run over the three (double) indexes xx, yy and xy. We're ready to work out the details:

$$\Sigma_{xx-xx} = (\Delta \mathbb{I}_{xx})^2 = \sum_{i=1}^{n} \left[4w_i^2 y_i^2 (\Delta y_i)^2 + y_i^4 (\Delta w_i)^2\right] \tag{33}$$

$$\Sigma_{yy-yy} = (\Delta \mathbb{I}_{yy})^2 = \sum_{i=1}^{n} \left[4w_i^2 x_i^2 (\Delta x_i)^2 + x_i^4 (\Delta w_i)^2\right] \tag{34}$$

$$\Sigma_{xy-xy} = (\Delta \mathbb{I}_{xy})^2 = \sum_{i=1}^{n} \left[w_i^2 y_i^2 (\Delta x_i)^2 + w_i^2 x_i^2 (\Delta y_i)^2 + x_i^2 y_i^2 (\Delta w_i)^2\right] \tag{35}$$

$$\Sigma_{xx-yy} = \sum_{i=1}^{n} \left[x_i^2 y_i^2 (\Delta w_i)^2\right] \tag{36}$$

$$\Sigma_{xx-xy} = -\sum_{i=1}^{n} \left[2w_i^2 x_i y_i (\Delta y_i)^2 + x_i y_i^3 (\Delta w_i)^2\right] \tag{37}$$

$$\Sigma_{xy-yy} = -\sum_{i=1}^{n} \left[2w_i^2 x_i y_i (\Delta x_i)^2 + x_i^3 y_i (\Delta w_i)^2\right] \tag{38}$$

The rest of this section follows closely the prescription described in [5] for the error propagation. We can slice the 2×2 tensor of inertia and define a 4-component vector with the two columns one on top of the other (this is what we call the *vec* operator):

$$vec(\mathbb{I}) = \begin{pmatrix} \mathbb{I}_{xx} \\ \mathbb{I}_{xy} \\ \mathbb{I}_{xy} \\ \mathbb{I}_{yy} \end{pmatrix} \tag{39}$$

That said, we can rewite the equation (28) using the Kronecker product of the rotation matrix

$$T = S \otimes S$$

as:

$$vec(\lambda) = \begin{pmatrix} \lambda_0 \\ 0 \\ 0 \\ \lambda_1 \end{pmatrix} = T vec(\mathbb{I}) \tag{40}$$

It is useful to rearrange the the elements of the *vec* operator in such a way that the diagonal elements of the tensor come first, followed by the non-diagonal ones, getting rid of the duplicated terms. This is accomplished by introducing the matrix:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{41}$$

which allows to define a new operator v_d:

$$v_d(\mathbb{I}) = Dvec(\mathbb{I}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{I}_{xx} \\ \mathbb{I}_{xy} \\ \mathbb{I}_{xy} \\ \mathbb{I}_{yy} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_{xx} \\ \mathbb{I}_{yy} \\ \mathbb{I}_{xy} \end{pmatrix} \tag{42}$$

Talking about which, the covariance matrix of $v_d(\mathbb{I})$ reads:

$$\Sigma_{v_d(\mathbb{I})} = \begin{pmatrix} \Sigma_{xx-xx} & \Sigma_{xx-yy} & \Sigma_{xx-xy} \\ \Sigma_{xx-yy} & \Sigma_{yy-yy} & \Sigma_{xy-yy} \\ \Sigma_{xx-xy} & \Sigma_{xy-yy} & \Sigma_{xy-xy} \end{pmatrix} \tag{43}$$

in terms of the quantities we have calculated a few lines above. In order to go back from the v_d to the *vec* representation we need the so called pseudo-inverse of D:

$$D^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{44}$$

The paper [5] is wrong to this respect (see [6]). Using these definitions, equation (40) can be rewritten as:

$$v_d(\lambda) = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ 0 \end{pmatrix} = Dvec(\lambda) = DTvec(\mathbb{I}) = DTD^+v_d(\mathbb{I}) \tag{45}$$

If we define:

$$V = DTD^+ \tag{46}$$

we can rewrite the previous equation as:

$$v_d(\lambda) = Vv_d(\mathbb{I}) \tag{47}$$

The infinitesimal change $\mathrm{d}S$ in the rotation matrix S when we change the rotation angle ϕ by an infinitesimal amount $\mathrm{d}\phi$ can be easily calculated by differentiating S itself:

$$\mathrm{d}S = \begin{pmatrix} -\sin\phi\,\mathrm{d}\phi & \cos\phi\,\mathrm{d}\phi \\ -\cos\phi\,\mathrm{d}\phi & -\sin\phi\,\mathrm{d}\phi \end{pmatrix}$$

If we introduce the antisimmetric tensor:

$$\Omega = \begin{pmatrix} 0 & -\mathrm{d}\phi \\ \mathrm{d}\phi & 0 \end{pmatrix} \tag{48}$$

we can rewrite the previous equation as:

$$\mathrm{d}S = -S\Omega \tag{49}$$

and, along the same lines, we have:

$$\mathrm{d}S^{\mathrm{T}} = \Omega S^{\mathrm{T}} \tag{50}$$

as can be verified by direct matrix multiplication. It seems a bit odd, here, to call Ω an infinitesimal quantity (I would have rather named it $\mathrm{d}\Omega$), but we'll bravely follow the conventions used in the paper to avoid confusion. We now define the quantity:

$$\Omega^p = S\Omega S^{\mathrm{T}} = \begin{pmatrix} 0 & -\mathrm{d}\phi \\ \mathrm{d}\phi & 0 \end{pmatrix} = \Omega \tag{51}$$

By putting all together we have:

$$S\mathrm{d}\mathbb{I}S^{\mathrm{T}} = \mathrm{d}\mu = \Omega^p\lambda - \lambda\Omega^p + \mathrm{d}\lambda = \begin{pmatrix} \mathrm{d}\lambda_0 & (\lambda_0-\lambda_1)\mathrm{d}\phi \\ (\lambda_0-\lambda_1)\mathrm{d}\phi & \mathrm{d}\lambda_1 \end{pmatrix} \tag{52}$$

and we can write:

$$vec(\mathrm{d}\mu) = G\beta \tag{53}$$

where:

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & (\lambda_1-\lambda_0) \\ 0 & 0 & (\lambda_1-\lambda_0) \\ 0 & 1 & 0 \end{pmatrix} \tag{54}$$

and

$$\beta = \begin{pmatrix} \mathrm{d}\lambda_0 \\ \mathrm{d}\lambda_1 \\ -\mathrm{d}\phi \end{pmatrix} \tag{55}$$

(again, β is a differential quantities, so it is a bit odd to name it without a d in front). Further on through the paper:

$$v_d(\mathrm{d}\mathbb{I}) = \begin{pmatrix} d\mathbb{I}_{xx} \\ d\mathbb{I}_{yy} \\ d\mathbb{I}_{xy} \end{pmatrix} = D(S^{\mathrm{T}} \otimes S^{\mathrm{T}})vec(d\mu) = D(S^{\mathrm{T}} \otimes S^{\mathrm{T}})G\beta = F\beta \tag{56}$$

where we have defined:

$$F = D(S^{\mathrm{T}} \otimes S^{\mathrm{T}})G \tag{57}$$

All we have to do is invert this equation, namely find F^{-1}. If we were dealing with square matrices, all we had to do would be:

$$F^{-1} = \left(D(S^{\mathrm{T}} \otimes S^{\mathrm{T}})G\right)^{-1} = G^{-1}\left(S^{\mathrm{T}} \otimes S^{\mathrm{T}}\right)^{-1} D^{-1} = G^{-1}\left(S \otimes S\right) D^{-1} = G^{-1}TD^{-1}$$

But in fact G and D are rectangular matrices, so we need again the pseudo inverses. We have already the one for D, while for G, since G^TG is not singular, the solution is even easier:

$$G^{+} = \left(G^{\mathrm{T}}G\right)^{-1} G^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2(\lambda_1-\lambda_0)} & \frac{1}{2(\lambda_1-\lambda_0)} & 0 \end{pmatrix} \tag{58}$$

(this satisfies $G^{+}G = I_{3\times3}$, as can be easily verified by direct multiplication). Summarizing:

$$\beta = F^{-1}v_d(\mathrm{d}\mathbb{I}) \tag{59}$$

and, more important, we get the full covariance matrix of the eigenvalues and rotation angle:

$$\Sigma_\beta = F^{-1}\Sigma_{v_d(\mathbb{I})}(F^{-1})^T \tag{60}$$

Once we have the error on the rotation angle:

$$(\Delta\phi)^2 = (\Sigma_\beta)_{33} \tag{61}$$

the covariance matrix of the two components of the principal axis is:

$$\Sigma_{\mathbf{e}^0} = \begin{pmatrix} \frac{\partial e_x^0}{\partial\phi}\frac{\partial e_x^0}{\partial\phi}(\Delta\phi)^2 & \frac{\partial e_x^0}{\partial\phi}\frac{\partial e_y^0}{\partial\phi}(\Delta\phi)^2 \\ \frac{\partial e_x^0}{\partial\phi}\frac{\partial e_y^0}{\partial\phi}(\Delta\phi)^2 & \frac{\partial e_y^0}{\partial\phi}\frac{\partial e_y^0}{\partial\phi}(\Delta\phi)^2 \end{pmatrix} = \begin{pmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{pmatrix}(\Delta\phi)^2 \tag{62}$$

2.2 Error analysis in 3 dimensions

Along the same lines of the previous subsection the derivatives of the components of the inertia tensor are:

$$\frac{\partial \mathbb{I}_{xx}}{\partial x_i} = 0, \quad \frac{\partial \mathbb{I}_{xx}}{\partial y_i} = 2w_i y_i, \quad \frac{\partial \mathbb{I}_{xx}}{\partial z_i} = 2w_i z_i, \quad \frac{\partial \mathbb{I}_{xx}}{\partial w_i} = (y_i^2 + z_i^2) \tag{63}$$

$$\frac{\partial \mathbb{I}_{yy}}{\partial x_i} = 2w_i x_i, \quad \frac{\partial \mathbb{I}_{yy}}{\partial y_i} = 0, \quad \frac{\partial \mathbb{I}_{yy}}{\partial z_i} = 2w_i z_i, \quad \frac{\partial \mathbb{I}_{yy}}{\partial w_i} = (x_i^2 + z_i^2) \tag{64}$$

$$\frac{\partial \mathbb{I}_{zz}}{\partial x_i} = 2w_i x_i, \quad \frac{\partial \mathbb{I}_{zz}}{\partial y_i} = 2w_i y_i, \quad \frac{\partial \mathbb{I}_{zz}}{\partial z_i} = 0, \quad \frac{\partial \mathbb{I}_{zz}}{\partial w_i} = (x_i^2 + y_i^2) \tag{65}$$

$$\frac{\partial \mathbb{I}_{xy}}{\partial x_i} = -w_i y_i, \quad \frac{\partial \mathbb{I}_{xy}}{\partial y_i} = -w_i x_i, \quad \frac{\partial \mathbb{I}_{xy}}{\partial z_i} = 0, \quad \frac{\partial \mathbb{I}_{xy}}{\partial w_i} = -x_i y_i \tag{66}$$

$$\frac{\partial \mathbb{I}_{xz}}{\partial x_i} = -w_i z_i, \quad \frac{\partial \mathbb{I}_{xz}}{\partial y_i} = 0, \quad \frac{\partial \mathbb{I}_{xz}}{\partial z_i} = -w_i x_i, \quad \frac{\partial \mathbb{I}_{xz}}{\partial w_i} = -x_i z_i \tag{67}$$

$$\frac{\partial \mathbb{I}_{yz}}{\partial x_i} = 0, \quad \frac{\partial \mathbb{I}_{yz}}{\partial y_i} = -w_i z_i, \quad \frac{\partial \mathbb{I}_{yz}}{\partial z_i} = -w_i y_i, \quad \frac{\partial \mathbb{I}_{yz}}{\partial w_i} = -y_i z_i \tag{68}$$

The elements of the covariance matrix are:

$$\Sigma_{k-l} = \sum_{i=1}^{n} \frac{\partial \mathbb{I}_k}{\partial x_i}\frac{\partial \mathbb{I}_l}{\partial x_i}(\Delta x_i)^2 + \frac{\partial \mathbb{I}_k}{\partial y_i}\frac{\partial \mathbb{I}_l}{\partial y_i}(\Delta y_i)^2 + \frac{\partial \mathbb{I}_k}{\partial z_i}\frac{\partial \mathbb{I}_l}{\partial z_i}(\Delta z_i)^2 + \frac{\partial \mathbb{I}_k}{\partial w_i}\frac{\partial \mathbb{I}_l}{\partial w_i}(\Delta w_i)^2 \tag{69}$$

where now k and l run over the 6 independent (double) indexes of the indertia tensor xx, yy, zz, xy, xz and yz. Therefore the 6×6 covariancewe will have $6(6+1)/2 = 21$ components:

$$\Sigma_{xx-xx} = \sum_{i=1}^{n} \left\{ 4w_i^2 \left[y_i^2(\Delta y_i)^2 + z_i^2(\Delta z_i)^2 \right] + \left(y_i^2 + z_i^2 \right)^2 (\Delta w_i)^2 \right\} \tag{70}$$

$$\Sigma_{xx-yy} = \sum_{i=1}^{n} \left\{ 4w_i^2 z_i^2 (\Delta z_i)^2 + (x_i^2 + z_i^2)(y_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{71}$$

$$\Sigma_{xx-zz} = \sum_{i=1}^{n} \left\{ 4w_i^2 y_i^2 (\Delta y_i)^2 + (x_i^2 + y_i^2)(y_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{72}$$

$$\Sigma_{xx-xy} = -\sum_{i=1}^{n} x_i y_i \left\{ 2w_i^2 (\Delta y_i)^2 + (y_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{73}$$

$$\Sigma_{xx-xz} = -\sum_{i=1}^{n} x_i z_i \left\{ 2w_i^2 (\Delta z_i)^2 + (y_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{74}$$

$$\Sigma_{xx-yz} = -\sum_{i=1}^{n} y_i z_i \left\{ 2w_i^2 \left[(\Delta y_i)^2 + (\Delta z_i)^2 \right] + (y_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{75}$$

$$\Sigma_{yy-yy} = \sum_{i=1}^{n} \left\{ 4w_i^2 \left[x_i^2(\Delta x_i)^2 + z_i^2(\Delta z_i)^2 \right] + \left(x_i^2 + z_i^2 \right)^2 (\Delta w_i)^2 \right\} \tag{76}$$

$$\Sigma_{yy-zz} = \sum_{i=1}^{n} \left\{ 4w_i^2 x_i^2 (\Delta x_i)^2 + (x_i^2 + y_i^2)(x_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{77}$$

$$\Sigma_{yy-xy} = -\sum_{i=1}^{n} x_i y_i \left\{ 2w_i^2 (\Delta x_i)^2 + (x_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{78}$$

$$\Sigma_{yy-xz} = -\sum_{i=1}^{n} x_i z_i \left\{ 2w_i^2 \left[(\Delta x_i)^2 + (\Delta z_i)^2 \right] + (x_i^2 + z_i^2)(\Delta w_i)^2 \right\} \tag{79}$$

$$\Sigma_{yy-yz} = -\sum_{i=1}^{n} y_i z_i \left\{2w_i^2(\Delta z_i)^2 + (x_i^2 + z_i^2)(\Delta w_i)^2\right\} \tag{80}$$

$$\Sigma_{zz-zz} = \sum_{i=1}^{n} \left\{4w_i^2 \left[x_i^2(\Delta x_i)^2 + y_i^2(\Delta y_i)^2\right] + \left(x_i^2 + y_i^2\right)^2 (\Delta w_i)^2\right\} \tag{81}$$

$$\Sigma_{zz-xy} = -\sum_{i=1}^{n} x_i y_i \left\{2w_i^2 \left[(\Delta x_i)^2 + (\Delta y_i)^2\right] + (x_i^2 + y_i^2)(\Delta w_i)^2\right\} \tag{82}$$

$$\Sigma_{zz-xz} = -\sum_{i=1}^{n} x_i z_i \left\{2w_i^2(\Delta x_i)^2 + (x_i^2 + y_i^2)(\Delta w_i)^2\right\} \tag{83}$$

$$\Sigma_{zz-yz} = -\sum_{i=1}^{n} y_i z_i \left\{2w_i^2(\Delta y_i)^2 + (x_i^2 + y_i^2)(\Delta w_i)^2\right\} \tag{84}$$

$$\Sigma_{xy-xy} = \sum_{i=1}^{n} \left\{w_i^2 \left[y_i^2(\Delta x_i)^2 + x_i^2(\Delta y_i)^2\right] + x_i^2 y_i^2(\Delta w_i)^2\right\} \tag{85}$$

$$\Sigma_{xy-xz} = \sum_{i=1}^{n} y_i z_i \left\{w_i^2(\Delta x_i)^2 + x_i^2(\Delta w_i)^2\right\} \tag{86}$$

$$\Sigma_{xy-yz} = \sum_{i=1}^{n} x_i z_i \left\{w_i^2(\Delta y_i)^2 + y_i^2(\Delta w_i)^2\right\} \tag{87}$$

$$\Sigma_{yz-yz} = \sum_{i=1}^{n} \left\{w_i^2 \left[z_i^2(\Delta y_i)^2 + y_i^2(\Delta z_i)^2\right] + y_i^2 z_i^2(\Delta w_i)^2\right\} \tag{88}$$

$$\Sigma_{yz-xz} = \sum_{i=1}^{n} x_i y_i \left\{w_i^2(\Delta z_i)^2 + z_i^2(\Delta w_i)^2\right\} \tag{89}$$

$$\Sigma_{xz-xz} = \sum_{i=1}^{n} \left\{w_i^2 \left[z_i^2(\Delta x_i)^2 + x_i^2(\Delta z_i)^2\right] + x_i^2 z_i^2(\Delta w_i)^2\right\} \tag{90}$$

3 Shower development: basic formulæ

The longitudinal profile of an electromagnetic shower is described by:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = E_0 p(t) = E_0 k t^{\alpha} e^{-bt} \tag{91}$$

where

$$k = \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}$$

(with this definition $p(t)$ is normalized to 1 and is therefore a probability density) and the Euler Γ function, defined by:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} \mathrm{d}t$$

satisfies the well know relation:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

The position of the shower maximum is given by the condition:

$$\left.\frac{\mathrm{d}p}{\mathrm{d}t}\right|_{t_{\mathrm{max}}} = k t_{\mathrm{max}}^{\alpha-1} e^{-bt_{\mathrm{max}}} (\alpha - bt_{\mathrm{max}}) = 0$$

and therefore:

$$t_{\mathrm{max}} = \frac{\alpha}{b} \tag{92}$$

The other two pieces of necessary information are the dependences of α and b on the energy. These are given by the relations:

$$b \approx 0.5 \tag{93}$$

and:

$$t_{\max} = \frac{\alpha}{b} = \ln\left(\frac{E_0}{E_c}\right) + C \tag{94}$$

where $C = 0.5$ for photons and $C = -0.5$ for electrons and E_c is the critical energy for the material.

4 Longitudinal moments

Let's start from the calculation of the lowest order moments of the shower longitudinal profile around $t = 0$. The first one is the mean:

$$\begin{aligned}\langle t\rangle = \mu &= \int_0^\infty tp(t)]\mathrm{d}t = k\int_0^\infty t^{\alpha+1}e^{-bt}\mathrm{d}t = \\ &= \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+2)}{b^{\alpha+2}} = \frac{(\alpha+1)}{b}\end{aligned} \tag{95}$$

(i.e. the mean of the profile is exactly $1/b$ radiation lengths to the right of the shower maximum). Along the same lines:

$$\left\langle t^2\right\rangle = \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+3)}{b^{\alpha+3}} = \frac{(\alpha+2)(\alpha+1)}{b^2} \tag{96}$$

and:

$$\left\langle t^3\right\rangle = \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+4)}{b^{\alpha+4}} = \frac{(\alpha+3)(\alpha+2)(\alpha+1)}{b^3} \tag{97}$$

We can apply the usual formulæ for the moments M_n centered around the mean (as opposed to the ones centered around 0):

$$M_2 = \sigma^2 = \left\langle t^2\right\rangle - \mu^2 = \frac{(\alpha+1)}{b^2} \tag{98}$$

and

$$M_3 = \left\langle t^3\right\rangle - 3\mu\sigma^2 - \mu^3 = \frac{2(\alpha+1)}{b^3} \tag{99}$$

The skewness γ is given by:

$$\gamma = \frac{M_3}{\sigma^3} = \frac{2}{\sqrt{\alpha+1}} \tag{100}$$

Let's look at the problem from a different perspective, which will hopefully turn out to be handy in the following. Integrating by parts, we get:

$$\begin{aligned}\langle t^n\rangle &= k\int_0^\infty t^n\cdot t^\alpha e^{-bt}\mathrm{d}t = k\int_0^\infty t^\alpha e^{-bt}\mathrm{d}\left(\frac{t^{n+1}}{n+1}\right) = \\ &= k\left.\frac{t^{n+1}}{n+1}t^\alpha e^{-bt}\right|_0^\infty - k\int_0^\infty \frac{t^{n+1}}{n+1}\left(\alpha t^{\alpha-1}e^{-bt} - bt^\alpha e^{-bt}\right)\mathrm{d}t = \\ &= \frac{kb}{n+1}\int_0^\infty t^{\alpha+n+1}e^{-bt}\mathrm{d}t - \frac{k\alpha}{n+1}\int_0^\infty t^{\alpha+n}e^{-bt}\mathrm{d}t = \frac{b\left\langle t^{n+1}\right\rangle - \alpha\left\langle t^n\right\rangle}{n+1}\end{aligned}$$

from which it follows that:

$$\left\langle t^{n+1}\right\rangle = \frac{(\alpha+n+1)}{b}\left\langle t^n\right\rangle \tag{101}$$

For $n = 1$ we get:

$$\langle t^2 \rangle = \frac{(\alpha + 2)}{b} \langle t \rangle$$

or:

$$\sigma^2 = \frac{(\alpha + 2)}{b}\mu - \mu^2 \tag{102}$$

Whereas for $n = 2$:

$$\langle t^3 \rangle = \frac{(\alpha + 3)}{b} \langle t^2 \rangle$$

which translates into:

$$\gamma = \frac{\mu}{\sigma^3} \left[\frac{(\alpha + 3)(\alpha + 2)}{b^2} - 3\sigma^2 - \mu^2 \right] \tag{103}$$

All this equations can be directly verified by plugging in the expressions for μ, σ and γ explicitly obtained before, but the hope is to generalize them to the case in which we don't sample the entire shower (see the following section).

5 Longitudinal moments over a finite interval

We can generalize the previous relations to the case in which we only sample a finite fraction of the longitudinal shower development, say between t_1 and t_2. The formalism is essentially identical, except for the fact that now we're dealing with a probability density function over a finite interval:

$$p_{\rm f}(t) = k_{\rm f} t^\alpha e^{-bt}$$

with $k_{\rm f}$ being:

$$k_{\rm f} = \frac{1}{\int_{t_1}^{t_2} t^\alpha e^{-bt} {\rm d}t}$$

(physically $k_{\rm f}$ is the ratio between the raw energy deposited in the calorimeter and the true energy of the particle). So now we have:

$$\langle t^{n+1} \rangle = \frac{(\alpha + n + 1)}{b} \langle t^n \rangle - \frac{k_{\rm f}}{b} t^{(\alpha+n+1)} e^{-bt} \Big|_{t_1}^{t_2} \tag{104}$$

and therefore:

$$\langle t^2 \rangle = \frac{(\alpha + 2)}{b} \langle t \rangle - \frac{k_{\rm f}}{b} \left[t_2^{(\alpha+2)} e^{-bt_2} - t_1^{(\alpha+2)} e^{-bt_1} \right] \tag{105}$$

and:

$$\langle t^3 \rangle = \frac{(\alpha + 3)}{b} \langle t^2 \rangle - \frac{k_{\rm f}}{b} \left[t_2^{(\alpha+3)} e^{-bt_2} - t_1^{(\alpha+3)} e^{-bt_1} \right] \tag{106}$$

Some more formula that might turn out to be useful for the normalization of the skewness to the expected value for electronmagnetic showers. The moments of the longitudinal distribution can be written as a function of the incomplete gamma function, defined as:

$$\gamma(\alpha, t) = \frac{1}{\Gamma(\alpha)} \int_0^t t^{\alpha-1} e^{-t} {\rm d}t \tag{107}$$

from which it follows that:

$$\int_{t_1}^{t_2} t^\alpha e^{-bt} {\rm d}t = \frac{\Gamma(\alpha + 1)}{b^{\alpha+1}} \left(\gamma(\alpha + 1, bt_2) - \gamma(\alpha + 1, bt_1) \right) \tag{108}$$

If we define:

$$\mathcal{G}(\alpha, b, t_1, t_2) = \frac{\Gamma(\alpha)}{b^\alpha} \left(\gamma(\alpha, bt_2) - \gamma(\alpha, bt_1) \right)$$

we have:

$$\langle t^n \rangle = \frac{\mathcal{G}(\alpha + n + 1, b, t_1, t_2)}{\mathcal{G}(\alpha + 1, b, t_1, t_2)} \tag{109}$$

Caution: the stuff in the appendix is mostly crap, at this time. I'll move it into appropriate sections as soon as it's in a reasonable shape (and, of course, this does not mean that people should not take a look).

Let's go back to the basic equation for the principal eigenvector:

$$\mathbb{I}\mathbf{e}^1 = \lambda_1 \mathbf{e}^1$$

Doing a full error propagation is not easy, since in this equation we do have error on the six independent components of the inertia tensor, as well as on the eigenvalue λ_1 we've just calculated. The errors on the $\mathbb{I}_{ij}$ are reasonably easy to calculate, starting from the errors associated with the finite dimensions of the crystals. On the other side the propagation of the errors to λ_1 is not trivial, as the expression is complicated. On top of that, these different error are not indipendent from each other, as λ_1 is calculated starting from the component of the inertia tensor.

The solution to this equation is:

$$e^1_x = \frac{1}{\sqrt{1 + \frac{A^2}{B^2} + \frac{A^2}{C^2}}} \tag{110}$$

$$e^1_y = \frac{1}{\sqrt{1 + \frac{B^2}{A^2} + \frac{B^2}{C^2}}} \tag{111}$$

$$e^1_z = \frac{1}{\sqrt{1 + \frac{C^2}{A^2} + \frac{C^2}{B^2}}} \tag{112}$$

where:

$$A = \mathbb{I}_{yz}(\mathbb{I}_{xx} - \lambda_1) - \mathbb{I}_{xy}\mathbb{I}_{xz} \tag{113}$$

$$B = \mathbb{I}_{xz}(\mathbb{I}_{yy} - \lambda_1) - \mathbb{I}_{xy}\mathbb{I}_{yz} \tag{114}$$

$$C = \mathbb{I}_{xy}(\mathbb{I}_{zz} - \lambda_1) - \mathbb{I}_{xz}\mathbb{I}_{yz} \tag{115}$$

References

[1] H. Goldstein, *Classical mechanics.*

[2] L. D. Landau, E. M. Lifŝic, *Mechanics.*

[3] `http://mathworld.wolfram.com/CubicFormula.html`

[4] PDG Review of Particle Physics, *Physics Letters B.* (2004) **592**

[5] T. Soler and B. H. W. van Gelder, *Geophys. J. Int.* (1991) **105**, 537–546.

[6] T. Soler and B. H. W. van Gelder, *Geophys. J. Int.* (2006) **165**, 382.