Calorimeter moments analysis

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Abstract

These are some (more or less) random notes about the calorimeter moments analysis. The first part is a brief description of the basic code that I put together while trying to understand it. There's also some stuff dealing with possible improvements of the moments analysis, namely the measurement of the shower skeweness and the the error analysis aimed at the projection of the calorimeter clusters into the ACD.

1 Basics of the CAL moments analysis

The code for the moments analysis basically calculates the moment of inertia tensor (using energy as the weight instead of mass) and then diagonalizes this to get the three principal axes. The basic definitions can be found in [1, 2] and in our case they read:

$$\mathbb{I}_{xx} = \sum_{i=1}^{n} w_i(r_i^2 - x_i^2), \quad \mathbb{I}_{yy} = \sum_{i=1}^{n} w_i(r_i^2 - y_i^2), \quad \mathbb{I}_{zz} = \sum_{i=1}^{n} w_i(r_i^2 - z_i^2) \tag{1}$$

$$\mathbb{I}_{xy} = -\sum_{i=1}^{n} w_i x_i y_i, \quad \mathbb{I}_{xz} = -\sum_{i=1}^{n} w_i x_i z_i, \quad \mathbb{I}_{yz} = -\sum_{i=1}^{n} w_i y_i z_i \tag{2}$$

where the index i runs over the n hits in the calorimeter and the w_i are the weights associated with the hits (essentially the energy release). In addition to the moment of inertia, the sum of weights and the coordinates of the energy centroids are also used:

$$W = \sum_{i=1}^{n} w_i \tag{3}$$

$$\mathbf{r}_c = \frac{\sum_{i=1}^{n} w_i \mathbf{r}_i}{W} \tag{4}$$

In order to reduce the tensor of inertia to the principal axes we do have to solve the secular equation:

$$\det(\mathbb{I} - \lambda\mathbb{1}) = \det\begin{pmatrix} \mathbb{I}_{xx} - \lambda & \mathbb{I}_{xy} & \mathbb{I}_{xz} \\ \mathbb{I}_{xy} & \mathbb{I}_{yy} - \lambda & \mathbb{I}_{yz} \\ \mathbb{I}_{xz} & \mathbb{I}_{yz} & \mathbb{I}_{zz} - \lambda \end{pmatrix} = 0 \tag{5}$$

which is a cubic equation in λ yielding the three eigenvalues. By working out the tedious algebra we can write the equation as:

$$\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$$

where:

$$c_2 = -(\mathbb{I}_{xx} + \mathbb{I}_{yy} + \mathbb{I}_{zz}) \tag{6}$$

$$c_1 = \mathbb{I}_{xx}\mathbb{I}_{yy} + \mathbb{I}_{yy}\mathbb{I}_{zz} + \mathbb{I}_{xx}\mathbb{I}_{zz} - (\mathbb{I}_{xy}^2 + \mathbb{I}_{yz}^2 + \mathbb{I}_{xz}^2) \tag{7}$$

$$c_0 = -\mathbb{I}_{xx}\mathbb{I}_{yy}\mathbb{I}_{zz} - 2\mathbb{I}_{xy}\mathbb{I}_{yz}\mathbb{I}_{xz} + \mathbb{I}_{xx}\mathbb{I}_{yz}^2 + \mathbb{I}_{yy}\mathbb{I}_{xz}^2 + \mathbb{I}_{zz}\mathbb{I}_{xy}^2 \tag{8}$$

If we now define a new variable $\lambda' = \lambda + c_2/3$, the previous equation becomes:

$$\lambda'^3 + a\lambda' + b = 0$$

where:

$$a = \left(\frac{3c_1 - c_2^2}{3}\right) \tag{9}$$

$$b = \left(\frac{27c_0 + 2c_2^3 - 9c_1c_2}{27}\right) \tag{10}$$

(the algebra is fully worked out in [3]). We now set:

$$m = 2\sqrt{\frac{-a}{3}} \tag{11}$$

$$\psi = \frac{1}{3}\arccos\left(\frac{3b}{am}\right) \tag{12}$$

and, finally we get the three real solutions (guranteed by the fact that the tensor of inertia is symmetruc):

$$\lambda_0 = m\cos(\psi) - c_2/3 \tag{13}$$

$$\lambda_1 = m\cos(\psi + 2\pi/3) - c_2/3 \tag{14}$$

$$\lambda_2 = m\cos(\psi + 4\pi/3) - c_2/3 \tag{15}$$

Once we have the three eigenvalues we can work out the calcultaion of the eigenvectors $\mathbf{e}^i$ ($i = 1 \ldots 3$) defined by:

$$\mathbb{I}\mathbf{e}^i = \lambda_i \mathbf{e}^i$$

Following these conventions, λ_1 is the largest eigenvalue and, as a consequence, $\mathbf{e}^1$ is the principal axis of the cluster.

Once the three principal axis of the cluster have been found, the cluster χ^2 (normalized to the number of *degree of freedom*) is calculated as:

$$\chi^2 = \frac{\sum_{i=1}^{n} w_i d_i^2}{nW} \tag{16}$$

where d_i are the distances from each of the calorimeter hits to the axis parallel to $\mathbf{e}^1$ and passing throught the cluster centroid. Finally the some well know Merit quantities are calculated:

$$\texttt{CalTransRms} = \sqrt{\frac{|\lambda_1|}{W}} \tag{17}$$

$$\texttt{CalLongRms} = \sqrt{\frac{|\lambda_0| + |\lambda_2|}{2W\log L}} \tag{18}$$

$$\texttt{CalLRmsAsym} = \sqrt{\frac{|\lambda_0| - |\lambda_2|}{|\lambda_0| + |\lambda_2|}} \tag{19}$$

where L is the number of radiation lengths transversed.

2 Shower development: basic formulæ

The longitudinal profile of an electromagnetic shower is described by:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = E_0 p(t) = E_0 k t^\alpha e^{-bt} \tag{20}$$

where

$$k = \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}$$

(with this definition $p(t)$ is normalized to 1 and is therefore a probability density) and the Euler Γ function, defined by:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \mathrm{d}t$$

satisfies the well know relation:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

The position of the shower maximum is given by the condition:

$$\left.\frac{\mathrm{d}p}{\mathrm{d}t}\right|_{t_{\mathrm{max}}} = k t_{\mathrm{max}}^{\alpha-1} e^{-bt_{\mathrm{max}}} (\alpha - bt_{\mathrm{max}}) = 0$$

and therefore:

$$t_{\mathrm{max}} = \frac{\alpha}{b} \tag{21}$$

The other two pieces of necessary information are the dependences of α and b on the energy. These are given by the relations:

$$b \approx 0.5 \tag{22}$$

and:

$$t_{\mathrm{max}} = \frac{\alpha}{b} = \ln\left(\frac{E_0}{E_c}\right) + C \tag{23}$$

where $C = 0.5$ for photons and $C = -0.5$ for electrons and E_c is the critical energy for the material.

3 Longitudinal moments

Let's start from the calculation of the lowest order moments of the shower longitudinal profile around $t = 0$. The first one is the mean:

$$\begin{aligned} \langle t \rangle = \mu = \int_0^\infty t p(t) \mathrm{d}t = k \int_0^\infty t^{\alpha+1} e^{-bt} \mathrm{d}t = \\ = \frac{b^{\alpha+1}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+2)}{b^{\alpha+2}} = \frac{(\alpha+1)}{b} \end{aligned} \tag{24}$$

(i.e. the mean of the profile is exactly $1/b$ radiation lengths to the right of the shower maximum). Along the same lines:

$$\langle t^2 \rangle = \frac{b^{\alpha+1}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+3)}{b^{\alpha+3}} = \frac{(\alpha+2)(\alpha+1)}{b^2} \tag{25}$$

and:

$$\langle t^3 \rangle = \frac{b^{\alpha+1}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+4)}{b^{\alpha+4}} = \frac{(\alpha+3)(\alpha+2)(\alpha+1)}{b^3} \tag{26}$$

We can apply the usual formulæ for the moments M_n centered around the mean (as opposed to the ones centered around 0):

$$M_2 = \sigma^2 = \langle t^2 \rangle - \mu^2 = \frac{(\alpha+1)}{b^2} \tag{27}$$

and

$$M_3 = \langle t^3 \rangle - 3\mu\sigma^2 - \mu^3 = \frac{2(\alpha+1)}{b^3} \tag{28}$$

The skewness γ is given by:

$$\gamma = \frac{M_3}{\sigma^3} = \frac{2}{\sqrt{\alpha+1}} \tag{29}$$

Let's look at the problem from a different perspective, which will hopefully turn out to be handy in the following. Integrating by parts, we get:

$$\begin{aligned}
\langle t^n \rangle &= k \int_0^\infty t^n \cdot t^\alpha e^{-bt} \mathrm{d}t = k \int_0^\infty t^\alpha e^{-bt} \mathrm{d}\left(\frac{t^{n+1}}{n+1}\right) = \\
&= k\, \frac{t^{n+1}}{n+1} t^\alpha e^{-bt} \Big|_0^\infty - k \int_0^\infty \frac{t^{n+1}}{n+1} \left(\alpha t^{\alpha-1} e^{-bt} - b t^\alpha e^{-bt}\right) \mathrm{d}t = \\
&= \frac{kb}{n+1} \int_0^\infty t^{\alpha+n+1} e^{-bt} \mathrm{d}t - \frac{k\alpha}{n+1} \int_0^\infty t^{\alpha+n} e^{-bt} \mathrm{d}t = \frac{b \langle t^{n+1} \rangle - \alpha \langle t^n \rangle}{n+1}
\end{aligned}$$

from which it follows that:

$$\langle t^{n+1} \rangle = \frac{(\alpha+n+1)}{b} \langle t^n \rangle \tag{30}$$

For $n = 1$ we get:

$$\langle t^2 \rangle = \frac{(\alpha+2)}{b} \langle t \rangle$$

or:

$$\sigma^2 = \frac{(\alpha+2)}{b} \mu - \mu^2 \tag{31}$$

Whereas for $n = 2$:

$$\langle t^3 \rangle = \frac{(\alpha+3)}{b} \langle t^2 \rangle$$

which translates into:

$$\gamma = \frac{\mu}{\sigma^3} \left[\frac{(\alpha+3)(\alpha+2)}{b^2} - 3\sigma^2 - \mu^2 \right] \tag{32}$$

All this equations can be directly verified by plugging in the expressions for μ, σ and γ explicitly obtained before, but the hope is to generalize them to the case in which we don't sample the entire shower (see the following section).

4 Longitudinal moments over a finite interval

We can generalize the previous relations to the case in which we only sample a finite fraction of the longitudinal shower development, say between t_1 and t_2. The formalism is essentially identical, except for the fact that now we're dealing with a probability density function over a finite interval:

$$p_\mathrm{f}(t) = k_\mathrm{f} t^\alpha e^{-bt}$$

with k_f being:

$$k_\mathrm{f} = \frac{1}{\int_{t_1}^{t_2} t^\alpha e^{-bt} \mathrm{d}t}$$

(physically k_f is the ratio between the raw energy deposited in the calorimeter and the true energy of the particle). So now we have:

$$\langle t^{n+1} \rangle = \frac{(\alpha+n+1)}{b} \langle t^n \rangle - \frac{k_\mathrm{f}}{b} t^{(\alpha+n+1)} e^{-bt} \Big|_{t_1}^{t_2} \tag{33}$$

and therefore:

$$\langle t^2 \rangle = \frac{(\alpha+2)}{b} \langle t \rangle - \frac{k_\mathrm{f}}{b} \left[t_2^{(\alpha+2)} e^{-bt_2} - t_1^{(\alpha+2)} e^{-bt_1} \right] \tag{34}$$

and:

$$\langle t^3 \rangle = \frac{(\alpha+3)}{b} \langle t^2 \rangle - \frac{k_\mathrm{f}}{b} \left[t_2^{(\alpha+3)} e^{-bt_2} - t_1^{(\alpha+3)} e^{-bt_1} \right] \tag{35}$$

Some more formula that might turn out to be useful for the normalization of the skewness to the expected value for electronmagnetic showers. The moments of the longitudinal distribution can be written as a function of the incomplete gamma function, defined as:

$$\gamma(\alpha, t) = \frac{1}{\Gamma(\alpha)} \int_0^t t^{\alpha-1} e^{-t} \mathrm{d}t \tag{36}$$

from which it follows that:

$$\int_{t_1}^{t_2} t^\alpha e^{-bt} \mathrm{d}t = \frac{\Gamma(\alpha+1)}{b^{\alpha+1}} \left(\gamma(\alpha+1, bt_2) - \gamma(\alpha+1, bt_1) \right) \tag{37}$$

If we define:

$$\mathcal{G}(\alpha, b, t_1, t_2) = \frac{\Gamma(\alpha)}{b^\alpha} \left(\gamma(\alpha, bt_2) - \gamma(\alpha, bt_1) \right)$$

we have:

$$\langle t^n \rangle = \frac{\mathcal{G}(\alpha+n+1, b, t_1, t_2)}{\mathcal{G}(\alpha+1, b, t_1, t_2)} \tag{38}$$

Caution: the stuff in the appendix is mostly crap, at this time. I'll move it into appropriate sections as soon as it's in a reasonable shape (and, of course, this does not mean that people should not take a look).

A Tentative error analysis

This is an attempt to work out the error analysis for the calorimeter moments analysis. Let's start out with with the derivatives of the components of the inertia tensor:

$$\frac{\partial \mathbb{I}_{xx}}{\partial x_i} = 0, \quad \frac{\partial \mathbb{I}_{xx}}{\partial y_i} = 2w_i y_i, \quad \frac{\partial \mathbb{I}_{xx}}{\partial z_i} = 2w_i z_i, \quad \frac{\partial \mathbb{I}_{xx}}{\partial w_i} = (y_i^2 + z_i^2) \tag{39}$$

$$\frac{\partial \mathbb{I}_{yy}}{\partial x_i} = 2w_i x_i, \quad \frac{\partial \mathbb{I}_{yy}}{\partial y_i} = 0, \quad \frac{\partial \mathbb{I}_{yy}}{\partial z_i} = 2w_i z_i, \quad \frac{\partial \mathbb{I}_{yy}}{\partial w_i} = (x_i^2 + z_i^2) \tag{40}$$

$$\frac{\partial \mathbb{I}_{zz}}{\partial x_i} = 2w_i x_i, \quad \frac{\partial \mathbb{I}_{zz}}{\partial y_i} = 2w_i y_i, \quad \frac{\partial \mathbb{I}_{zz}}{\partial z_i} = 0, \quad \frac{\partial \mathbb{I}_{zz}}{\partial w_i} = (x_i^2 + y_i^2) \tag{41}$$

$$\frac{\partial \mathbb{I}_{xy}}{\partial x_i} = -w_i y_i, \quad \frac{\partial \mathbb{I}_{xy}}{\partial y_i} = -w_i x_i, \quad \frac{\partial \mathbb{I}_{xy}}{\partial z_i} = 0, \quad \frac{\partial \mathbb{I}_{xy}}{\partial w_i} = -x_i y_i \tag{42}$$

$$\frac{\partial \mathbb{I}_{xz}}{\partial x_i} = -w_i z_i, \quad \frac{\partial \mathbb{I}_{xz}}{\partial y_i} = 0, \quad \frac{\partial \mathbb{I}_{xz}}{\partial z_i} = -w_i x_i, \quad \frac{\partial \mathbb{I}_{xz}}{\partial w_i} = -x_i z_i \tag{43}$$

$$\frac{\partial \mathbb{I}_{yz}}{\partial x_i} = 0, \quad \frac{\partial \mathbb{I}_{yz}}{\partial y_i} = -w_i z_i, \quad \frac{\partial \mathbb{I}_{yz}}{\partial z_i} = -w_i y_i, \quad \frac{\partial \mathbb{I}_{yz}}{\partial w_i} = -y_i z_i \tag{44}$$

Assuming that the errors on the spatial coordinates and the energy are independent from each other, and dropping (for the time being) the term depending on the derivative with respect to w_i we get:

$$(\Delta\mathbb{I}_{xx})^2 = \sum_{i=1}^{n} \left(\frac{\partial \mathbb{I}_{xx}}{\partial x_i}\right)^2 (\Delta x_i)^2 + \sum_{i=1}^{n} \left(\frac{\partial \mathbb{I}_{xx}}{\partial y_i}\right)^2 (\Delta y_i)^2 + \sum_{i=1}^{n} \left(\frac{\partial \mathbb{I}_{xx}}{\partial z_i}\right)^2 (\Delta z_i)^2 =$$
$$= 4\sum_{i=1}^{n} w_i^2 \left[y_i^2 (\Delta y_i)^2 + z_i^2 (\Delta z_i)^2\right]$$

And the same thing for all the others:

$$\Delta\mathbb{I}_{xx} = \sqrt{\sum_{i=1}^{n} 4w_i^2 \left[y_i^2 (\Delta y_i)^2 + z_i^2 (\Delta z_i)^2\right]} \tag{45}$$

$$\Delta\mathbb{I}_{yy} = \sqrt{\sum_{i=1}^{n} 4w_i^2 \left[x_i^2 (\Delta x_i)^2 + z_i^2 (\Delta z_i)^2\right]} \tag{46}$$

$$\Delta\mathbb{I}_{zz} = \sqrt{\sum_{i=1}^{n} 4w_i^2 \left[x_i^2 (\Delta x_i)^2 + y_i^2 (\Delta y_i)^2\right]} \tag{47}$$

$$\Delta\mathbb{I}_{xy} = \sqrt{\sum_{i=1}^{n} w_i^2 \left[y_i^2 (\Delta x_i)^2 + x_i^2 (\Delta y_i)^2\right]} \tag{48}$$

$$\Delta\mathbb{I}_{xz} = \sqrt{\sum_{i=1}^{n} w_i^2 \left[z_i^2 (\Delta x_i)^2 + x_i^2 (\Delta z_i)^2\right]} \tag{49}$$

$$\Delta\mathbb{I}_{yz} = \sqrt{\sum_{i=1}^{n} w_i^2 \left[z_i^2 (\Delta y_i)^2 + y_i^2 (\Delta z_i)^2\right]} \tag{50}$$

We now bravely carry on to c_2, c_1 and c_0 defined earlier:

$$\frac{\partial c_2}{\partial x_i} = -4 w_i x_i \tag{51}$$

$$\frac{\partial c_2}{\partial y_i} = -4 w_i y_i \tag{52}$$

$$\frac{\partial c_2}{\partial z_i} = -4 w_i z_i \tag{53}$$

$$\tag{54}$$

and therefore it follows that:

$$(\Delta c_2)^2 = \sum_{i=1}^{n} \left(\frac{\partial c_2}{\partial x_i}\right)^2 (\Delta x_i)^2 + \sum_{i=1}^{n} \left(\frac{\partial c_2}{\partial y_i}\right)^2 (\Delta y_i)^2 + \sum_{i=1}^{n} \left(\frac{\partial c_2}{\partial z_i}\right)^2 (\Delta z_i)^2 = \tag{55}$$

$$= \sum_{i=1}^{n} 16 w_i^2 \left[x_i^2 (\Delta x_i)^2 + y_i^2 (\Delta y_i)^2 + z_i^2 (\Delta z_i)^2\right] \tag{56}$$

Along the same lines:

$$\frac{\partial c_1}{\partial x_i} = 2 w_i \left[(2\mathbb{I}_{xx} + \mathbb{I}_{yy} + \mathbb{I}_{zz}) x_i + \mathbb{I}_{xy} y_i + \mathbb{I}_{xz} z_i\right] \tag{57}$$

$$\frac{\partial c_1}{\partial y_i} = 2 w_i \left[\mathbb{I}_{xy} x_i + (\mathbb{I}_{xx} + 2\mathbb{I}_{yy} + \mathbb{I}_{zz}) y_i + \mathbb{I}_{yz} z_i\right] \tag{58}$$

$$\frac{\partial c_1}{\partial z_i} = 2 w_i \left[\mathbb{I}_{xz} x_i + \mathbb{I}_{yz} y_i + (\mathbb{I}_{xx} + \mathbb{I}_{yy} + 2\mathbb{I}_{zz}) z_i\right] \tag{59}$$

and just like before:

$$(\Delta c_1)^2 = \sum_{i=1}^{n} \left(\frac{\partial c_1}{\partial x_i}\right)^2 (\Delta x_i)^2 + \sum_{i=1}^{n} \left(\frac{\partial c_1}{\partial y_i}\right)^2 (\Delta y_i)^2 + \sum_{i=1}^{n} \left(\frac{\partial c_1}{\partial z_i}\right)^2 (\Delta z_i)^2 \tag{60}$$

which is now too long to fit in one row.

Let's go back to the basic equation for the principal eigenvector:

$$\mathbb{I}\mathbf{e}^1 = \lambda_1 \mathbf{e}^1$$

Doing a full error propagation is not easy, since in this equation we do have error on the six independent components of the inertia tensor, as well as on the eigenvalue λ_1 we've just calculated. The errors on the $\mathbb{I}_{ij}$ are reasonably easy to calculate, starting from the errors associated with the finite dimensions of the crystals. On the other side the propagation of the errors to λ_1 is not trivial, as the expression is complicated. On top of that, these different error are not indipendent from each other, as λ_1 is calculated starting from the component of the inertia tensor.

The solution to this equation is:

$$e_x^1 = \frac{1}{\sqrt{1 + \frac{A^2}{B^2} + \frac{A^2}{C^2}}} \tag{61}$$

$$e_y^1 = \frac{1}{\sqrt{1 + \frac{B^2}{A^2} + \frac{B^2}{C^2}}} \tag{62}$$

$$e_z^1 = \frac{1}{\sqrt{1 + \frac{C^2}{A^2} + \frac{C^2}{B^2}}} \tag{63}$$

where:

$$A = \mathbb{I}_{yz}(\mathbb{I}_{xx} - \lambda_1) - \mathbb{I}_{xy}\mathbb{I}_{xz} \tag{64}$$

$$B = \mathbb{I}_{xz}(\mathbb{I}_{yy} - \lambda_1) - \mathbb{I}_{xy}\mathbb{I}_{yz} \tag{65}$$

$$C = \mathbb{I}_{xy}(\mathbb{I}_{zz} - \lambda_1) - \mathbb{I}_{xz}\mathbb{I}_{yz} \tag{66}$$

References

[1] H. Goldstein, *Classical mechanics.*

[2] L. D. Landau, E. M. Lifŝic, *Mechanics.*

[3] http://mathworld.wolfram.com/CubicFormula.html