Optimum Design of Prestressed Beams under Uncertainties in the Tendon Layout
Revision Document

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Abstract

In this study we develop a method that allows designing optimally prestressed beams with uniform cross section taking into account the effects that uncertainties in the construction process can produce in the real beam.

Optimal values of prestressing force, tendon configuration and cross sectional dimensions are obtained subject to constrains on design and stress limits. A practical example will be given at the end.

Keywords: Prestressed Beams , Uncertainties , Linear Optimization

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Notation

Different variables and constants used through the study:

A area of the cross section
C objective function parameter
d dimension of the cross section
P prestressing force
f reduction factor for the prestressing force
J number of configuration variables for tendon path
j subscript denoting configuration variable
K number of checked cross sections
k subscript denoting cross section
L length of the beam, superscript denoting upper bound
l subscript denoting load condition
M bending moment acting on beam
R number of checked points in a cross section
r subscript denoting point
U superscript denoting upper bound
I modulus of the section
X linear variable
e configuration variable
Z objective function
α constant parameter related with geometry of cross section
σ stress
Δe imposed maximum error

1 Introduction

When designing prestressed concrete beams a common approach is by trial an error which usually requires several computations and does not get the best possible answer. Trying to calculate also the effect of construction errors can prove to be a difficult task.

Mathematical optimization methods can facilitate greatly the design process. This problem can be solved with Linear Programming methods as long as a linear elastic behavior is assumed which is usual under service limits. A transformation of design variables will be performed in order to obtain a set of linear equations. Known algorithms [1] will give us the solution to our problem.

2 Basic Assumptions

Cross sectional variables

The design variables of the cross section will have to represent both the area and the modulus of the section. For that we define two design variables d_1 and d_2 and two constant parameters α_A, α_r shown in Figure 1 that will help us express the area

$$\left.\begin{aligned} A &= \alpha_A d_1 d_2 \\ I_r &= \alpha_r d_1 d_2^2 \end{aligned}\right\} \tag{1}$$

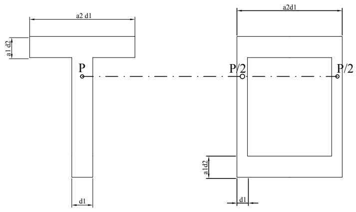

Figure 1: Cross section. - This image will be bigger in the final version

Constrains of the Desing Variables

For the design of the beam we have to consider a number of constraints:

1. Constrain on the Prestressing force P.
2. Constrains on the vertical coordinate of the tendon e_j.

$$\Big(\frac{(e_j \pm \Delta e_j)}{d_2}\Big)^L \leq \frac{(e_j \pm \Delta e_j)}{d_2} \leq \Big(\frac{(e_j \pm \Delta e_j)}{d_2}\Big)^U \tag{2}$$

   Where $(e_j \pm \Delta e_j/d_2)^L$ and $(e_j \pm \Delta e_j/d_2)^U$ are Lower and Upper bounds of $e_j \pm \Delta e_j/d_2$.

3. Constrains on cross sectional variables d_1 and d_2.

$$\left.\begin{aligned} d_1^L \leq d_1 \leq d_1^U \\ d_2^L \leq d_2 \leq d_2^U \end{aligned}\right\} \tag{3}$$

In order to help with the linear transformation to be performed later, the equations of constrains (3) will have to be replaced by:

$$\left.\begin{array}{l} (d_1 d_2)^L \leq d_1 d_2 \leq (d_1 d_2)^U \\ d_2^L \leq d_2 \leq d_2^U \end{array}\right\} \quad (4)$$

4. Constrains on the stresses defined by the service limits in which we will add the possible uncertainties. This will be formulated in the next section.

Objective Function

The cost function will be the sum of the area of concrete and the prestressing force with some weighting factors.

$$Z = C_1 P + C_2 A$$

Loads

The loads may be a function of d_1 and d_2 (e.g. self weight). In that case we should approximate the load with a polynomial equation

$$q(d_1, d_2) = c_1 + c_2 d_2 + c_3 d_1 d_2^2 \quad (5)$$

where c_1, c_2, c_3, are constant coefficients.

3 Problem Formulation

We have to find the prestressing force (P), the tendon configuration eccentricities (e_j) and the dimensions (d_1 ,d_2) of the cross section. That minimizes the cost function:

$$Z = C_1 P + C_2 \alpha_A d_1 d_2 \quad (6)$$

And also follow these constrains:

$$0 \leq P \tag{7a}$$
$$\left(\frac{e_j \pm \Delta e_j}{d_2}\right)^L \leq \frac{e_j \pm \Delta e_j}{d_2} \leq \left(\frac{e_j \pm \Delta e_j}{d_2}\right)^U \tag{7b}$$
$$(d_1 d_2)^L \leq d_1 d_2 \leq (d_1 d_2)^U \tag{7c}$$
$$d_2^L \leq d_2 \leq d_2^U \tag{7d}$$
$$\sigma_{k,r,l}^L \leq -\frac{P}{\alpha_A d_1 d_2} + \frac{P}{\alpha_r d_1 d_2^2} \sum_{j=1}^{J} a_{k,r,l}(e_j \pm \Delta e_j) + \frac{M_{k,l}}{\alpha_r d_1 d_2^2} \leq \sigma_{k,r,l}^U \tag{7e}$$
$$(j = 1, \ldots, J \quad k = 1, \ldots, K \quad r = 1, \ldots, R \quad l = 1, \ldots, L)$$

The constrain (7e) refers to the limits of stresses that a cross section can handle being $\sigma_{k,r,l}^L, \sigma_{k,r,l}^U$ the maximum compression and the maximum traction respectively, J is the number of sections in which we configure e_j, R is the number of positions in the cross section where we check the stress (top and bottom fiber for example), K is the number of sections in which we check the condition (usually coincides with J) and L is the number of load conditions.

Apart from this $a_{k,r,l}$ will give us in each position the bending moment associated with a certain eccentricity and the $M_{k,l}$ represents the bending moment generated by the different loads.

The error, $\pm\Delta e$, is a fixed value that will contemplate the difference between the beam designed and the actual beam that is constructed. This way we ensure that different construction tolerances will not affect the desired behavior of the beam.

4 Transformation of Variables

We have to define a set of linear equations for the linear programming problem. For that we will use the following transformations replacing P, y, d_1 and d_2.

$$\begin{aligned}
X_A &= P \\
X_B &= \alpha_A d_1 d_2 \\
X_C &= \frac{\alpha_A}{\alpha_r d_2} \\
X_j &= P \frac{\alpha_A}{\alpha_r d_2}(e_j \pm \Delta e_j)
\end{aligned} \tag{8}$$

With the formulas stated before (8) we can start transforming the equations of the constrains (7). Knowing that:

$$\frac{e_j \pm \Delta e_j}{d_2} = \frac{X_j}{X_A}\frac{\alpha_r}{\alpha_A} \tag{9}$$

And then substituting in (7b)

$$\Big(\frac{e_j \pm \Delta e_j}{d_2}\Big)^L \frac{\alpha_A}{\alpha_r} \leq \frac{X_j}{X_A} \leq \Big(\frac{e_j \pm \Delta e_j}{d_2}\Big)^U \frac{\alpha_A}{\alpha_r} \tag{10}$$

We know that

$$\frac{e_j \pm \Delta e_j}{d_2}\frac{\alpha_A}{\alpha_r} = \frac{X_j}{X_A} \tag{11}$$

With the equations (11) and (10) we arrive finally to the linear constrain:

$$\Big(\frac{X_j}{X_A}\Big)^L X_A \leq X_j \leq \Big(\frac{X_j}{X_A}\Big)^U X_A \tag{12}$$

The constrains (7c) and (7d) now become:

$$X_B^L \leq X_B \leq X_B^U \tag{13}$$

$$X_C^L \leq X_C \leq X_C^U \tag{14}$$

It should be noted that the upper and lower bounds in X_C are now reversed.

$$X_C^L = \frac{\alpha_A}{\alpha_r d_2^U}$$

$$X_C^U = \frac{\alpha_A}{\alpha_r d_2^L}$$

If we multiply the stress constrains equations (7e) by $\alpha_A d_1 d_2$ we get the following

$$\sigma_{k,r,l}^L \alpha_A d_1 d_2 \leq -P + \frac{P\alpha_A}{\alpha_r d_2}\sum_{j=1}^{J} a_{k,r,l}(e_j \pm \Delta e_j) + \frac{M_{k,l}\alpha_A}{\alpha_r d_2} \leq \sigma_{k,r,l}^U \alpha_A d_1 d_2$$
$$(j = 1, \ldots, J \quad k = 1, \ldots, K \quad r = 1, \ldots, R \quad l = 1, \ldots, L) \tag{15}$$

Substituting (8) into (15) we finally obtain

$$\sigma_{k,r,l}^L X_B \leq -P + \sum_{j=1}^{J} a_{k,r,l} X_j + M_{k,l} X_C \leq \sigma_{k,r,l}^U X_B$$
$$(j = 1, \ldots, J \quad k = 1, \ldots, K \quad r = 1, \ldots, R \quad l = 1, \ldots, L) \tag{16}$$

5 Linear Programming Problem

Finally we can reduce our non linear problem (7) to a linear one that, changing the new variables (8), will minimize the cost the function (6) which now will be:

$$Z = C_1 X_A + C_2 X_B \tag{17}$$

Subject to the constrains (12), (13) and (16):

$$\left.\begin{array}{l}
\left(\frac{X_j}{X_A}\right)^L X_A \leq X_j \leq \left(\frac{X_j}{X_A}\right)^U X_A \\
X_B^L \leq X_B \leq X_B^U \\
X_C^L \leq X_C \leq X_C^U \\
\sigma_{k,r,l}^L X_B \leq -P + \displaystyle\sum_{j=1}^{J} a_{k,r,l} X_j + M_{k,l} X_C \leq \sigma_{k,r,l}^U \\
(j = 1, \ldots, J \quad k = 1, \ldots, K \quad r = 1, \ldots, R \quad l = 1, \ldots, L)
\end{array}\right\} \tag{18}$$

6 Solving the Uncertainties Problem

When solving the linear programming problem for a given uncertainty $\pm\Delta e_j$ we will proceed solving the equations first for a positive value $(+\Delta e_j)$ and then for a negative one $(-\Delta e_j)$.

One may argue that when doing this we will only explore the solutions in the extremes of the interval and a different optimum value could be found in $e_j - \Delta e_j < e_j < e_j + \Delta e_j$.

However it can be demonstrated that the optimum will be always in the extremes.

$$\sigma_{k,r,l}^L \leq -\frac{P}{\alpha_A d_1 d_2} + \frac{P}{\alpha_r d_1 d_2^2} \sum_{j=1}^{J} a_{k,r,l}(e_j \pm \Delta e_j) + \frac{M_{k,l}}{\alpha_r d_1 d_2^2} \leq \sigma_{k,r,l}^U \tag{7e revisited}$$

If we take the stress formulation (7e) and change it in order to be graphically represented by $\frac{1}{P}$ and e_j in a Magnel diagram [2] first dividing by P and secondly ordering the terms, we can clearly see that a change in Δe_j will only affect the constant term on the equations resulting in a parallel to the original problem.

$$\sigma_{k,r,l}\left(\frac{1}{P}\right) = -\frac{1}{A} + \frac{\sum_{j=1}^{J} a_{k,r,l}(e_j)}{I} \pm \frac{\sum_{j=1}^{J} a_{k,r,l}(\pm\Delta e_j)}{I} + \frac{M_{k,l}}{I}\left(\frac{1}{P}\right) \tag{19}$$

$$\left(\sigma_{k,r,l} - \frac{M_{k,l}}{I}\right)\left(\frac{1}{P}\right) = \frac{\sum_{j=1}^{J} a_{k,r,l}}{I} e_j + \left(\frac{\sum_{j=1}^{J} a_{k,r,l}(\pm\Delta e_j)}{I} - \frac{1}{A}\right) \tag{20}$$

In the Figure 2 we can see an example of how the magnel diagram is modified with a positive and negative uncertainties in the path of the cable in a given section. In this case we can observe that the feasible domain is reduced and the maximum efficiency point corresponds with the diagram of $-\Delta e_j$

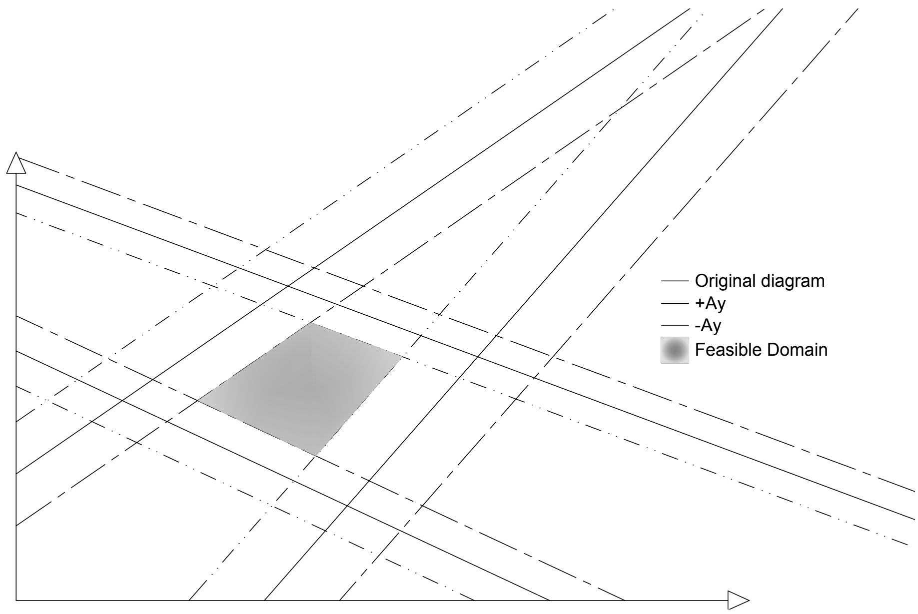

Figure 2: Magnel diagram showing the difference in uncertainties (legend to be changed)

7 Examples

An example based on the one by U. Kirsch [3]. Given a continuous beam with uniformly distributed live and dead load of $DL = 3\ t/m$ and $LL = 4\ t/m$ and a prestressing cable consisting of two parabolas as shown in Figure 3. The cross section will be a double T with $\alpha_A = 2.2$ and $\alpha_R = 0.6047$, with the geometry shown in Figure 4.

In the first load pattern we will have only the dead load, no prestressing losses and a maximum compression of 1500 t/m. On the second load pattern we consider both the $DL + LL$ with a loss in prestressing of the 20% the maximum value of compression will be 1200 t/m. In any of the cases can't be traction.

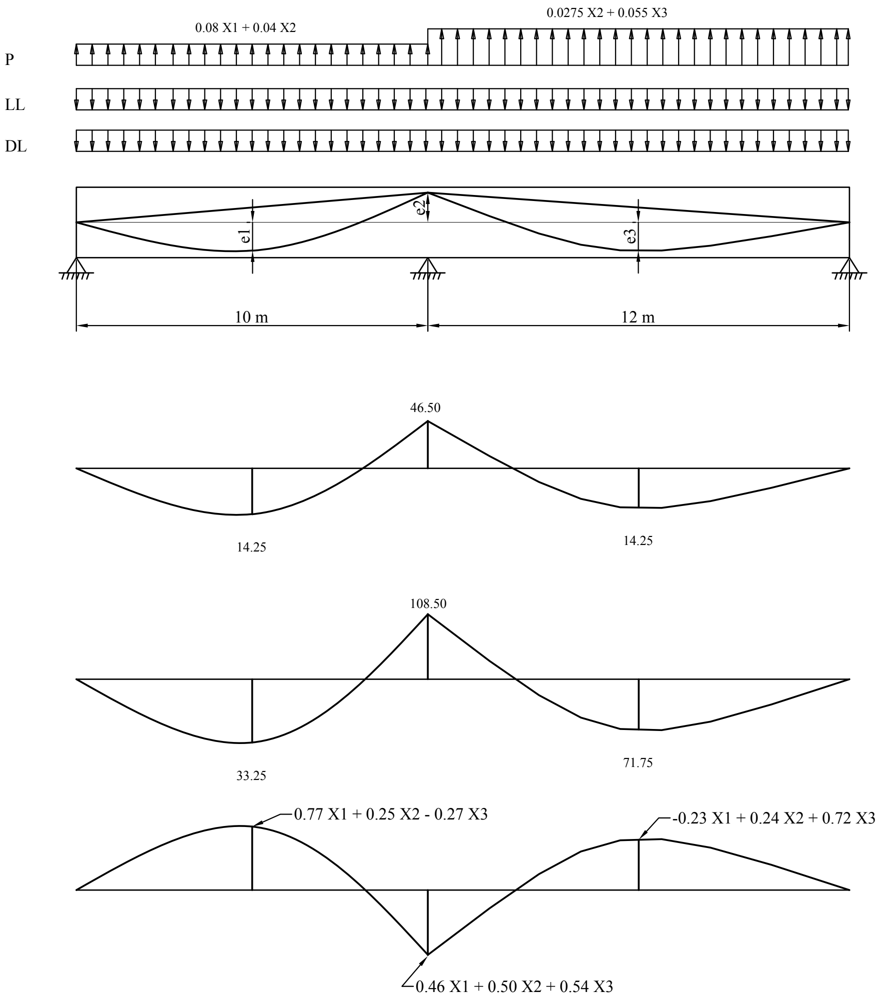

Figure 3: Continuous beam with different loads detailed.

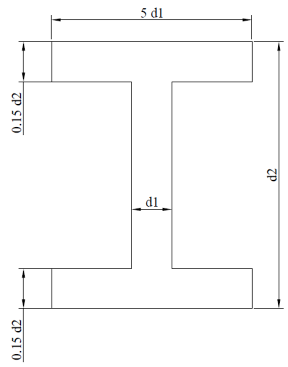

Figure 4: Continuous beam's cross section.

The stress constrains are therefore:

$$
\begin{aligned}
&-1500X_B \leq -X_A \pm (0.77X_1 + 0.25X_2 - 0.27X_3) \mp 14.25X_C \leq 0\\
&\pm 1500X_B \leq -X_A \pm (-0.46X_1 - 0.5X_2 - 0.54X_3) \pm 46.50X_C \leq 0\\
&-1500X_B \leq -X_A \pm (-0.23X_1 + 0.24X_2 - 0.72X_3) \mp 30.75X_C \leq 0\\
&-1200X_B \leq -0.8X_A \pm 0.8(0.77X_1 + 0.25X_2 - 0.27X_3) \mp 33.25X_C \leq 0\\
&\pm 1200X_B \leq -0.8X_A \pm 0.8(-0.46X_1 - 0.5X_2 - 0.54X_3) \pm 108.50X_C \leq 0\\
&-1200X_B \leq -0.8X_A \pm 0.8(-0.23X_1 + 0.24X_2 - 0.72X_3) \mp 71.75X_C \leq 0
\end{aligned}
$$

We will calculate this beam for different geometric boundaries and cost weights (C_1, C_2) inputs for 15 examples will be given in Table 1 and the outputs on Table 2.

In the examples from 1 to 6 we can observe the change in the cost function in relation to an increase in the uncertainty of the path of the cable. If we analyze this data we will see almost a linear relationship (see Figure 5) even though the cost function grows exponentially which is shown thanks to the addition of example 7. An error in the construction of such magnitude (a radius of 100 mm) is non realistic with normal construction methods therefore in this case a linear relationship could be used in order to estimate the increase of the cost in a beam. Also case 15 shows the effect of different uncertainties along the beam.

In examples 8, 9 and 10 we can observe how the different boundaries change the configuration of the beam and the prestressing force. It can be observed that the optimum solution is always the one with the most efficient section values, which is using the maximum d_2 possible. The simplifications used at the beginning of this study in section 4 transform or desired constrains:

$$
\begin{aligned}
&0.15 \leq d_1 \leq 0.50\\
&0.70 \leq d_2 \leq 1.00 \qquad \text{(3 revisited)}
\end{aligned}
$$

Ex	C_1	C_2	$(d_1d_2)^L$	$(d_1d_2)^U$	d_2^L	d_2^U	$\frac{e_j}{d_2}^L$	$\frac{e_j}{d_2}^U$	Δe_1	Δe_2	Δe_3
1	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
2	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 1	$\pm$ 1	$\pm$ 1
3	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 3	$\pm$ 3	$\pm$ 3
4	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 5	$\pm$ 5	$\pm$ 5
5	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 7	$\pm$ 7	$\pm$ 7
6	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 9	$\pm$ 9	$\pm$ 9
7	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 100	$\pm$ 100	$\pm$ 100
8	1	1	0.135	0.35	0.7	0.90	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
9	1	1	0.120	0.35	0.7	0.80	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
10	1	1	0.105	0.35	0.7	0.70	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
11	1	2000	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
12	1	2000	0.135	0.35	0.7	0.90	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
13	1	2000	0.120	0.35	0.7	0.80	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
14	1	2000	0.105	0.35	0.7	0.70	0	0.4	$\pm$ 0	$\pm$ 0	$\pm$ 0
15	1	1	0.15	0.35	0.7	1.00	0	0.4	$\pm$ 5	$\pm$ 7	$\pm$ 9

Table 1: Input data. (Δe_j in mm)

Ex	P	d_1	d_2	e_1	e_2	e_3	Z
1	162.13	0.15	1	0.316	0.400	0.400	163.35
2	163.02	0.15	1	0.400	0.325	0.393	165.15
3	164.82	0.15	1	0.400	0.305	0.400	165.15
4	166.67	0.15	1	0.354	0.400	0.340	167.00
5	168.57	0.15	1	0.268	0.400	0.400	168.90
6	170.50	0.15	1	0.255	0.400	0.400	170.83
7	356.90	0.22	1	0.000	0.400	0.101	357.37
8	180.14	0.15	0.9	0.284	0.360	0.360	180.44
9	202.66	0.154	0.8	0.253	0.320	0.320	202.93
10	231.61	0.201	0.7	0.221	0.280	0.280	231.92
11	162.13	0.15	1	0.316	0.400	0.400	822.13
12	180.14	0.15	0.9	0.284	0.360	0.360	774.14
13	202.66	0.154	0.8	0.253	0.320	0.320	743.08
14	231.61	0.201	0.7	0.221	0.400	0.400	849.23
15	168.68	0.15	1	0.268	0.280	0.280	169.01

Table 2: Output data.

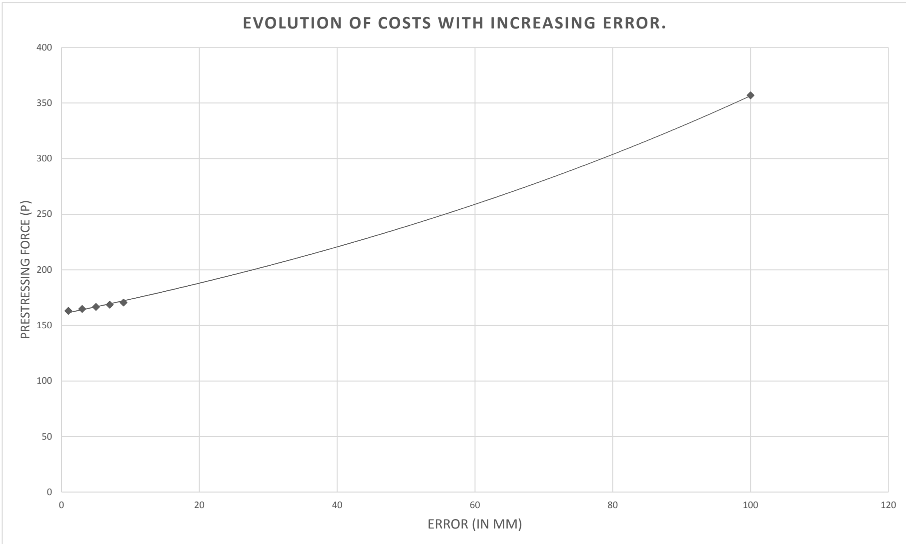

Figure 5: Relation between Prestressing and Uncertainty level.

Were replaced by new constrains:

$$\begin{aligned} &0.15 \leq d_1 d_2 \leq 0.35 \\ &0.70 \leq d_2 \leq 1.00 \end{aligned} \tag{4 revisited}$$

In examples 11 to 14 cost of the sectional area of the concrete has been increased substantially, two the point that a m^2 of concrete costs two thousand times the t of prestressing force. We can see that example 13 which is itself contained inside example 11 has a lower cost (743 versus 822 respectively). This happens because of the simplification stated before and, for an optimum value, we should solve the optimization problem for several cases of d_2 in order to find a suitable solution.

When doing said simplifations there was a change in the range of possible answers in the geometrical variables d_1 and d_2 as shown in the table 6

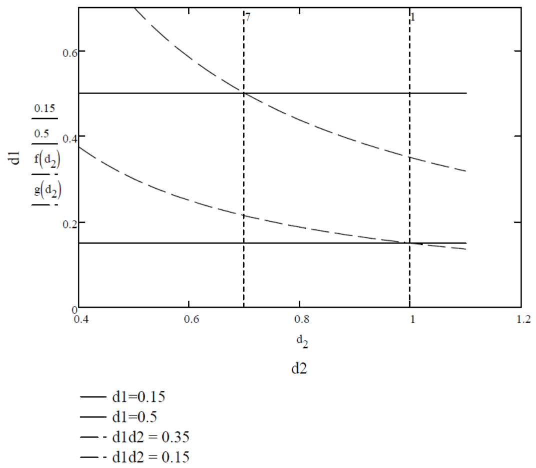

Figure 6: Constrains on cross-sectional dimensions.

8 Conclusions

In the length of this study we've learned that prestressed beams can be solved as a linear programming problem in order to get optimal solutions. In cases where the assumptions made in Section 2 are not sufficient we can obtain the optimal values analyzing the problem in for different heights in the beam (d_2) and the best sections is calculated with a precision equal to the iterations made (different values of d_2 tested).

Also we have calculated how construction uncertaties in the path of the tendons affects the final cost of the structure taking into account different variations in each section of control. The points with the most cost are the ones located in the extremes of the possible range of the eccentricity (either $e + \Delta_e$ or $e - \Delta_e$) where the need of prestressing P is maximum or minimum as demonstrated in Section 6. The cost on the design has proved to increase exponentially with the uncertainties in the structure although in the scale of the example given the growth in cost can be approximated accurately by a linear relation.

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