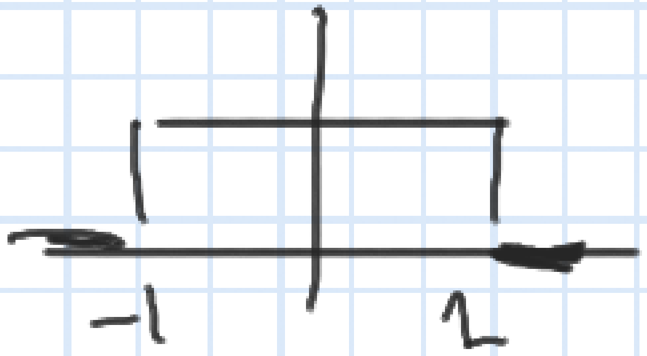
X e Y son <u>indep</u> Si:

- $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ (v.a. c)
- $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$ (v.a. d)
- $F_{X,Y}(x,y) = F_X(x)\, F_Y(y)$ (general)

Si X e Y son indep $\Rightarrow \operatorname{cov}(X,Y) = 0$

$\not\Leftarrow$ contra ej:



$X \sim \mathcal{U}(-1, 1)$ $Y = X^2$ $\operatorname{cov}(X,Y) = 0$

pero no son indep.

Si X e son indep

$$f_{Y|X=x}(y) = f_Y(y)$$

$$= \frac{f_{X,Y}(x,y)}{f_X(x)} \overset{\text{indep}}{=} \frac{\cancel{f_X(x)}\, f_Y(y)}{\cancel{f_X(x)}}$$

$$f_{X,Y}(x,y) = f_X(x)\, f_{Y|X=x}(y)$$

$$= f_Y(y)\, f_{X|Y=y}(x)$$

1. La probabilidad de acertar a un blanco es ⅕. Se realizan 10 tiros independientes y se cuenta la cantidad de aciertos. Sean X la cantidad de aciertos en los 10 tiros, e Y la cantidad de aciertos en el primer tiro. Hallar la distribución de X|Y=y y Y|X=x.

A _ A _ _ A _ _ _ _ _ _

$Sop_X = \{0, 1, \dots, 10\}$

$X|Y=1 = 1 + V$

$X|Y=0 = 0 + V$

A _ _ _ _ _ _ _ _ _ _

$X \sim Bin(10, 1/5)$

$Y \sim Be(1/5)$

X= "# de aciertos en 10 tiros"

Y= "# " " en el 1º tiro"

$Y = \begin{cases} 1 & \text{si acierto el 1º tiro} \\ 0 & \text{si no} \end{cases}$

V= "# de aciertos en los tiros 2 al 10"

$V \sim Bin(9, 1/5)$

$$p_{X|Y=1}(x) = \mathbb{P}(X=x \mid Y=1) = \frac{\mathbb{P}(X=x,\ Y=1)}{\mathbb{P}(Y=1)}$$

$$\underline{A}\ \underbrace{\_\ \_\ \_\ A\ \_\ \_\ \_\ \_\ \_\ A\ \_\ \_\ \_}_{x-1}$$

$$= \frac{\binom{9}{x-1}\left(\frac{4}{5}\right)^{9-(x-1)}\left(\frac{1}{5}\right)^{\cancel{x}\ x-1}}{}$$

$$= \binom{9}{x-1}\left(1/5\right)^{x-1}\left(\frac{4}{5}\right)^{9-(x-1)}$$

$$\cancel{1/5}$$

$$\left|\ \binom{n}{x} p^{x} (1-p)^{n-x}\right.$$

$$p_{X|Y=1}(3) = \frac{\mathbb{P}(X=3,\ Y=1)}{\mathbb{P}(Y=1)}$$

$$Y \mid X = x \in \{0, 1\}$$

$$Y \mid X = x \sim \text{Ber}(p_x)$$

$$\mathbb{P}(Y = 1 \mid X = x) = \frac{\mathbb{P}(Y = 1, X = x)}{\mathbb{P}(X = x)} = \frac{\binom{9}{x-1} \cancel{(4/5)}^{\overbrace{9-(x-1)}^{10-x}} \cancel{1/5}^{x}}{\binom{10}{x} \cancel{4/5}^{10-x} \cancel{1/5}^{x}}$$

$$= \frac{\dfrac{\cancel{9!}}{\cancel{(x-1)!}\,\cancel{(9-(x-1))!}}}{\dfrac{\cancel{10!}\ 10}{x\ \cancel{x!}\,\cancel{(10-x)!}}} = \frac{x}{10}$$

$$Y \mid X = x \sim \text{Ber}\left(\frac{x}{10}\right)$$

Func. de reg.

$X | Y = y = y + V \quad , \quad V \sim Bin(9, 1/5)$

$\varphi(y) = \mathbb{E}[X | Y = y] = \mathbb{E}[y + V] = \mathbb{E}[y] + \mathbb{E}[V]$

$\to \mathbb{E}[X|Y] = \varphi(Y) = Y + 9/5$ linear $\quad = y + 9\,1/5,$

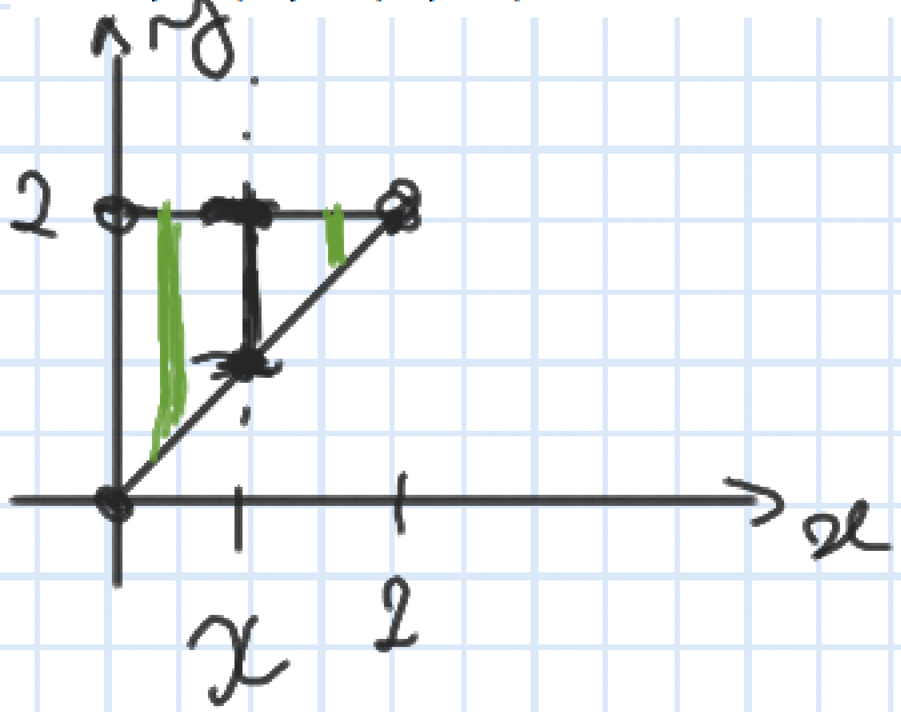
$y = \{0, 1\}$

$Y | X = x \sim Ber\left(\frac{x}{10}\right)$

$\varphi(x) = \mathbb{E}[Y | X = x] = \frac{x}{10}, \quad x \in \{0, 1, \dots, 10\}.$

$\to \mathbb{E}[Y | X] = \varphi(X) = \frac{X}{10}$

2. Sean X,Y dos v.a. conjuntamente uniformes en el triángulo de vértices (0,0), (2,2), (0,2). Hallar la función de densidad de Y|X=x.



$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{1}{2}\mathbb{1}\{0<x<y<2\}}{1-x/2\,\mathbb{1}\{0<x<2\}}$$

$$f_X(x) = \int_{\mathbb{R}} \frac{1}{2}\,\mathbb{1}\{0<x<y<2\}\,dy = \int_x^2 1/2\,dy\,\mathbb{1}\{0<x<2\}$$

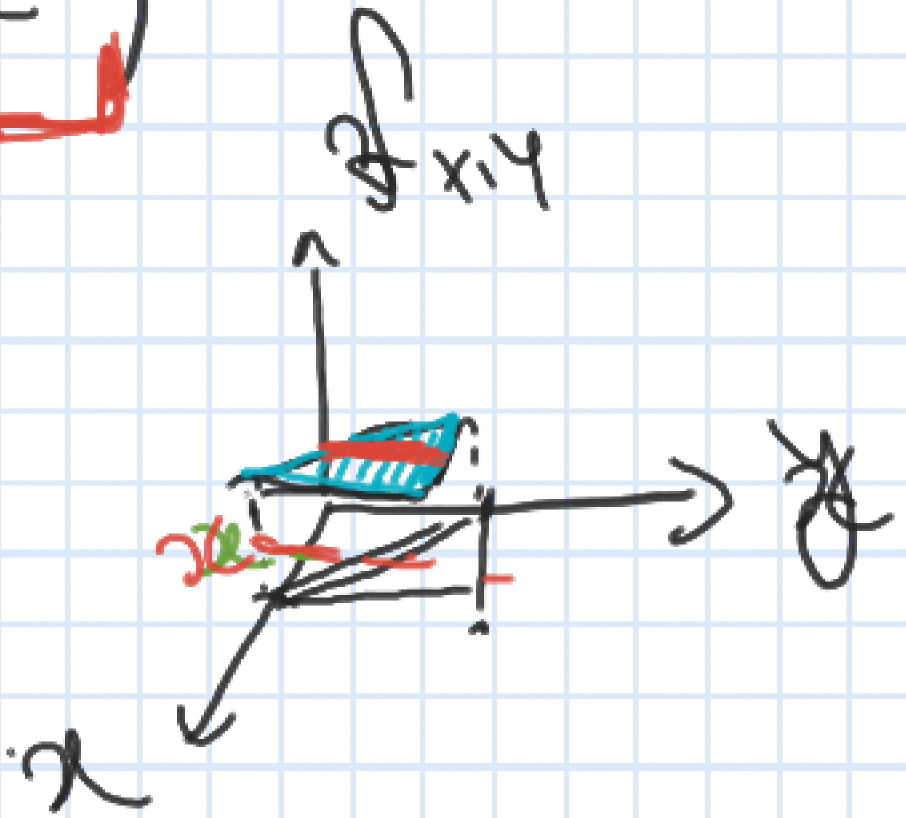
$$= \frac{1}{2}(2-x)\,\mathbb{1}\{0<x<2\}$$

$$= 1-x/2\,\mathbb{1}\{0<x<2\}.$$

$$f_{Y|X=x}(y) = \frac{\cancel{1/2} \ \mathbb{1}\{0<x<y<2\}}{\cancel{\frac{1}{2}}(2-x) \cdot \mathbb{1}\{0<x<2\}} = \boxed{\frac{1}{2-x}} \mathbb{1}\{x<y<2\}, \ 0<x<2$$

cte

$$Y|X=x \sim \mathcal{U}(\underline{x,2})$$



func. de regresión.

$$Y|X=x \sim \mathcal{N}(x,2)$$

$$\varphi(x) = \mathbb{E}[Y|X=x] = \frac{2+x}{2} = 1 + x/2.$$

$$\rightarrow \mathbb{E}[Y|X] = \varphi(X) = 1 + X/2$$

Sean X, Y dos v.a. con función de densidad conjunta

$$f_{X,Y}(x,y) = \frac{e^{-x/2y}}{4y} \mathbf{1}\{0 < x, 1 < y < 3\}$$

Hallar la función de densidad de X|Y=y

$X|Y=y.$

obj: $f_{X,Y}(x,y) = f_Y(y) \underbrace{f_{X|Y=y}(x)}$

$$\boxed{\lambda e^{-\lambda x} \mathbb{1}\{x>0\}}$$

$$\frac{e^{-x/2y}}{4y\}2 \cdot 2y} \mathbb{1}\{0<x\} \mathbb{1}\{1<y<3\}$$

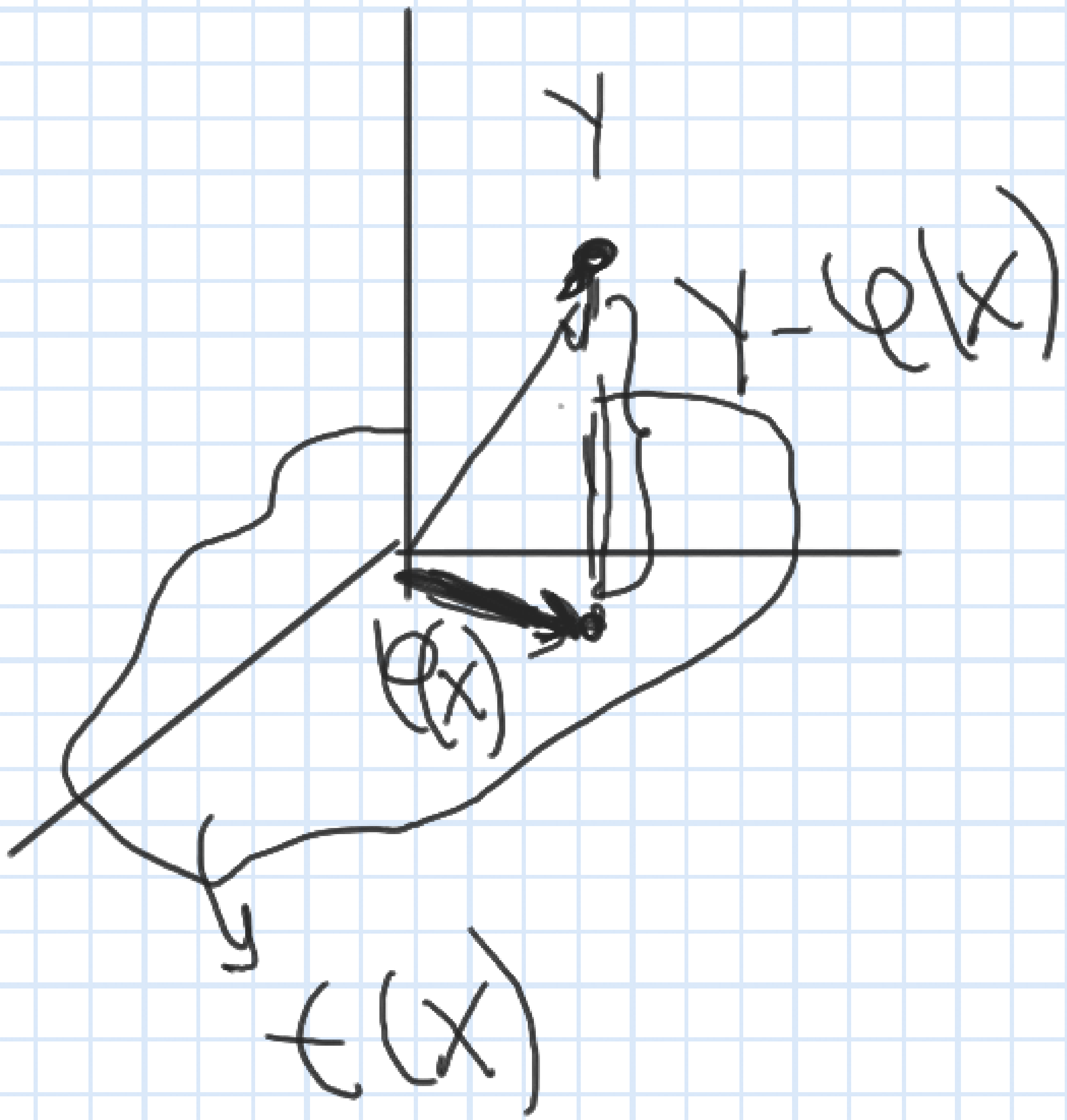
$$\underbrace{\frac{e^{-x/2y}}{2y} \mathbb{1}\{x>0\}}_{X|Y=y\,(1/2y)} \underbrace{\frac{1}{2} \mathbb{1}\{1<y<3\}}_{Y \sim \mathcal{U}(1,3)}$$

func de regresión

$$X|Y=y \sim \mathcal{E}\left(\frac{1}{2y}\right)$$

$$\varphi(y) = \mathbb{E}[X|Y=y] = \frac{1}{1/2y} = 2y \rightarrow \mathbb{E}[X|Y] = \varphi(Y) = 2Y$$

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[2Y] = 2\underbrace{\mathbb{E}[Y]}_{2} = 4$$



Y
Y - φ(x)
φ(x)
t(x)

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \mathbb{E}[\bar{X}] = \frac{1}{n}\mathbb{E}\left[\sum X_i\right] = \frac{1}{n}\sum \underbrace{\mathbb{E}[X_i]}_{\mu}$$

$$= \frac{1}{n} n\mu = \mu$$

$$\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \approx \mathcal{N}(0,1) \qquad \mathbb{E}\left[(\bar{X}-\mu)^2\right]$$

$$= \mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{\mathrm{Var}(X_i)}_{\sigma^2}$$

indep.

$$= \frac{n\sigma^2}{n^2}$$

$$= \frac{\sigma^2}{n}$$

$$W \sim \mathcal{N}(\mu, \sigma^2) \rightarrow \frac{W-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

Se desea estimar la media de una variable con distribución N(5,9) a partir del promedio de n realizaciones. Hallar el ECM para distintos valores de n.

$$X \sim N(\mu, 9) \qquad \underline{X} = (X_1, \dots, X_n) \text{ m.a. } X_i \overset{iid}{\sim} N(\mu, 9)$$

$$\hat{\mu} = \delta(\underline{X}) = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

porque $\mathbb{E}[\overline{X}] = \mu$

$$ECM(\mu) = \mathbb{E}\left[(\overline{X}-\mu)^2\right] = Var(\overline{X}) = \frac{9}{n} \xrightarrow{n\to\infty} 0$$

$$\mathbb{E}[\overline{X}] = \mu$$

En espacio de probab.

P.i $\langle X, Y\rangle = \mathbb{E}[X \cdot Y]$.

$\rightarrow \quad d^2(X,Y) = \mathbb{E}[(X-Y)(X-Y)]$.

Mejor predictor lineal (Recta de regresión)

$$\hat{X} = \frac{\operatorname{Cov}(X,Y)}{\operatorname{var}(Y)}\left(Y - \mathbb{E}[Y]\right) + \mathbb{E}[X]$$

Sea X~U(0,1) e Y=X^2. Hallar la mejor aproximación lineal de Y basada en X. Comparar con la mejor estimación de Y basada en X.

$X \sim U(0,1)$ $\qquad$ $Y = X^2$

(es lo mejor estimación de Y basado en X)

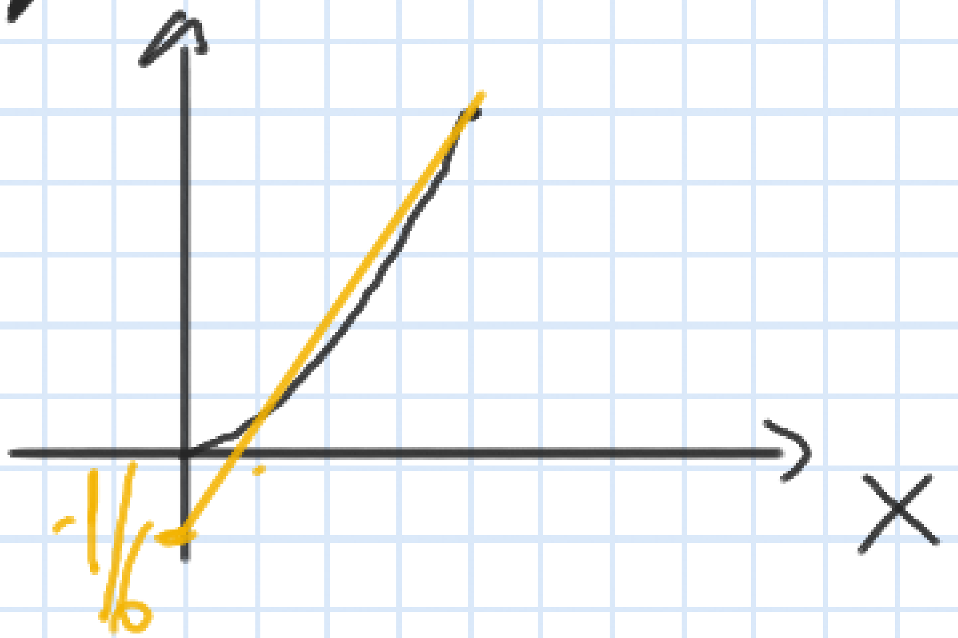
$$\hat{Y} = \frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}\left(X - \mathbb{E}[X]\right) + \mathbb{E}[Y]$$

$E[X] = 0{,}5 \qquad \operatorname{var}(X) = 1/12$

$$E[Y] = \mathbb{E}[X^2] = \int_0^1 x^2 \cdot 1 \, dx$$

$$= \operatorname{var}(X) + \mathbb{E}[X]^2 = \frac{1}{12} + \frac{1}{4} = 1/3$$

$$\operatorname{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$



$$\operatorname{cov}(X, Y) = \operatorname{cov}(X, X^2) = \mathbb{E}[X \cdot X^2] - \mathbb{E}[X]\,\mathbb{E}[X^2]$$

$$= \int_0^1 x^3 \cdot 1 \, dx - 1/2 \cdot 1/3$$

$$= \frac{1}{4} - 1/6 = \frac{3}{12} - \frac{2}{12} = 1/12.$$

$$\Rightarrow \hat{Y} = \frac{\cancel{1/12}}{\cancel{1/12}} \left(X - 1/2 \right) + 1/3 = X - 1/6$$