BASE CHANGE AND LAX KAN EXTENSIONS OF $(\infty,2)$-CATEGORIES.

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Abstract. We do basechange and lax kan extensions

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1. Preliminaries

1.1. Elementary definitions and models for $(\infty,2)-$categories.

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Definition 1.2.1. Let $(\mathbb{B},E)$ be a marked $(\infty,2)$-category and let $\mathsf{FIB}^{E\text{-lax}}_{(i,j)}(\mathbb{B})$ denote the wide locally full subcategory of $\mathsf{FIB}^{\text{lax}}_{(i,j)}(\mathbb{B})$ where the morphisms are required to preserve i-cartesian morphisms over E. We denote by $\mathsf{Fun}^{E\text{-cart}}_{/\mathbb{B}}(-,-)$ the mapping ∞-category functor for $\mathsf{FIB}^{E\text{-lax}}_{(1,j)}(\mathbb{B})$ and similarly $\mathsf{Fun}^{E\text{-coc}}_{/\mathbb{B}}(-,-)$ for $\mathsf{FIB}^{E\text{-lax}}_{(0,j)}(\mathbb{B})$.

1.3. Partially lax limits. In this section, we collect basic results and definitions of the theory of partially lax (co)limits which will be of use later. For a more extensive treatment (and the proofs of the statements below) we refer the reader to (CITE) .

Definition 1.3.1. Let $(\mathbb{A},E)$ be a marked $(\infty,2)$-category and let $F\colon \mathbb{A}\to\mathbb{B}$ be a functor of $(\infty,2)$-categories. We say that $\operatorname{colim}^{\diamond}_{\mathbb{A}} F\in\mathbb{B}$, where $\diamond\in\{E\text{-lax},E\text{-oplax}\}$, is the $\diamond$-*colimit* of F if we have a natural equivalence of CAT_∞-valued functors

$$\mathsf{Nat}^{\diamond}_{\mathbb{A},\mathbb{B}}\left(F,\underline{(-)}\right)\simeq\mathbb{B}(\operatorname{colim}^{\diamond}_{\mathbb{A}}F,-)$$

where $\underline{(-)}\colon\mathbb{B}\to\mathsf{FUN}(\mathbb{A},\mathbb{B})^{\diamond}$ is the functor sending each object to a constant diagram. Similarly, we say that $\lim^{\diamond}_{\mathbb{A}}F\in\mathbb{B}$, where $\diamond\in\{E\text{-lax},E\text{-oplax}\}$, is the $\diamond$-*limit* of F if we have a natural equivalence of CAT_∞-valued functors

$$\mathsf{Nat}^{\diamond}_{\mathbb{A},\mathbb{B}}\left(\underline{(-)},F\right)\simeq\mathbb{B}(-,\lim^{\diamond}_{\mathbb{A}}F).$$

Date: May 9, 2024.

Proposition 1.3.2. *Let $(\mathbb{A}, E)$ be a marked $(\infty,2)$-category and consider a functor $F\colon \mathbb{A} \to \mathsf{CAT}_{(\infty,2)}$ with corresponding (i,j)-fibration $\pi\colon \mathcal{F}_{i,j} \to \mathbb{A}^{\epsilon}$ where $\epsilon \in \{\emptyset, \mathrm{op}, \mathrm{co}, \mathrm{coop}\}$. Write $L_E(\mathcal{F}_{i,j})$ for the $(\infty,2)$-category obtained from $\mathcal{F}_{i,j}$ by inverting the j-cartesian 2-morphisms together with those i-cartesian 1-morphisms that lie over E. Then the E-(op)lax colimits of F can be described as*

$$\operatorname{colim}_{\mathbb{A}}^{E\text{-lax}} F \simeq L_E(\mathcal{F}_{0,j}), \quad \operatorname{colim}_{\mathbb{A}}^{E\text{-oplax}} F \simeq L_E(\mathcal{F}_{1,j})$$

for $j = 0, 1$.

Definition 1.3.3. Let $\mathcal{F}, \mathcal{G} \in \mathsf{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{A})$. If $i = 0$, we denote by $\mathsf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F}, \mathcal{G})$ the $(\infty,2)$-category characterised by the universal property

$$\mathsf{Map}_{\mathsf{Cat}_{(\infty,2)}}(\mathbb{X}, \mathsf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F}, \mathcal{G})) \simeq \mathsf{Fun}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F} \times \mathbb{X}, \mathcal{G}),$$

where the right-handside was defined in Definition 1.2.1. We define $\mathsf{FUN}_{/\mathbb{A}}^{E\text{-cart}}(\mathcal{F}, \mathcal{G})$ analogously whenever $i = 1$.

Proposition 1.3.4. *Let $(\mathbb{A}, E)$ be a marked $(\infty,2)$-category and consider a functor $F\colon \mathbb{A} \to \mathsf{CAT}_{(\infty,2)}$ with corresponding (i,j)-fibration $\pi\colon \mathcal{F}_{i,j} \to \mathbb{A}^{\epsilon}$ where $\epsilon \in \{-, \mathrm{op}, \mathrm{co}, \mathrm{coop}\}$. Then the E-(op)lax limits of F in $\mathsf{CAT}_{(\infty,2)}$ can be described as*

$$\lim_{\mathbb{A}}^{E\text{-lax}} F \simeq \mathsf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathbb{A}, \mathcal{F}_{0,j}), \quad \lim_{\mathbb{A}}^{E\text{-oplax}} F \simeq \mathsf{FUN}_{/\mathbb{A}}^{E\text{-cart}}(\mathbb{A}, \mathcal{F}_{1,j})$$

for $j = 0, 1$.

Proposition 1.3.5. *Let $\Lambda_1^2 = (0 \to 2 \leftarrow 1)$ denote the walking cospan and consider a functor in $F : \Lambda_1^2 \to \mathsf{CAT}_\infty$ given by $\mathsf{A} \to \mathsf{C} \leftarrow \mathsf{B}$. Then we have equivalences of ∞-categories:*

$$\lim^{E\text{-lax}} F = (\mathsf{C}^{[1]} \times_{\mathrm{ev}_0} \mathsf{B}) \times_{\mathrm{ev}_1} \mathsf{A}, \qquad \lim^{E\text{-oplax}} F = (\mathsf{C}^{[1]} \times_{\mathrm{ev}_1} \mathsf{B}) \times_{\mathrm{ev}_0} \mathsf{A},$$

where $E = (0 \to 2)$.

Proof. We give the proof in the E-lax case since the E-oplax case is formally dual. Let $\mathcal{F} \to \Lambda_1^2$ be the cocartesian fibration associated to F. It follows from Proposition 1.3.4 that we have a natural equivalence

$$\lim_{/\Lambda_1^2}^{E\text{-lax}} F \simeq \mathsf{FUN}_{/\Lambda_1^2}^{E\text{-coc}}(\Lambda_1^2, \mathcal{F}) \simeq \mathsf{NAT}_{\Lambda_1^2, \mathsf{CAT}_\infty}(\mathcal{W}, F)$$

where $\mathcal{W} : \Lambda_1^2 \to \mathsf{CAT}_\infty$ is given by $\mathcal{W}(0) = \{1\} \to [1] = \mathcal{W}(2) \leftarrow \{1\} = \mathcal{W}(1)$. Given an $(\infty,2)$-category T, we can construct natural equivalences

$$\mathsf{Map}(\mathsf{T}, \mathsf{NAT}_{\Lambda_1^2, \mathsf{CAT}_\infty}(\mathcal{W}, F)) \simeq \mathsf{Nat}_{\Lambda_1^2, \mathsf{CAT}_\infty}(\mathsf{T} \times \mathcal{W}, F),$$

where the later space can be expressed (according to (REF)) as the limit of the following diagram

$$\begin{tikzcd}
\mathsf{Map}(\mathsf{T},\mathsf{B}) \arrow[d] & & \mathsf{Map}(\mathsf{T},\mathsf{A}) \arrow[d] \\
\mathsf{Map}(\mathsf{T},\mathsf{C}) & \mathsf{Map}(\mathsf{T}\times[1], C) \arrow[l, "\mathrm{ev}_0^*"'] \arrow[r, "\mathrm{ev}_1^*"'] & \mathsf{Map}(\mathsf{T},\mathsf{C}).
\end{tikzcd}$$

It follows from the universal property of the iterated pullback $(\mathsf{C}^{[1]} \times_{\mathrm{ev}_0} \mathsf{B}) \times_{\mathrm{ev}_1} \mathsf{A}$ that the limit of the diagram above is naturally equivalent to the mapping space $\mathsf{Map}(\mathsf{T}, (\mathsf{C}^{[1]} \times_{\mathrm{ev}_0} \mathsf{B}) \times_{\mathrm{ev}_1} \mathsf{A})$. The result now follows by the Yoneda lemma. □

1.4. **Free fibrations and cofinality.** fff

Definition 1.4.1. Let $\mathsf{AR}^{(\mathrm{op})\mathrm{lax}}(\mathbb{C}) = \mathsf{FUN}^{(\mathrm{op})\mathrm{lax}}([1],\mathbb{C})$ be the (op)lax arrow $(\infty,2)$-category of $\mathbb{C}$ and denote by $\mathrm{ev}_i : \mathsf{AR}^{(\mathrm{op})\mathrm{lax}}(\mathbb{C}) \to \mathbb{C}$ the functor induced by restriction along the map $\{i\} \to [1]$. We consider a functor

$$\mathfrak{F}_{(i,j)} : \mathsf{CAT}_{(\infty,2)/\mathbb{C}} \to \mathsf{FIB}_{(i,j)}(\mathbb{C}),$$

whose action on objects is as follows:

- Given $p : \mathbb{X} \to \mathbb{C}$ we define $\pi : \mathfrak{F}_{(i,j)}(\mathbb{X}) \to \mathbb{C}$ as the pullback

$$\begin{tikzcd} \mathfrak{F}_{(i,j)}(\mathbb{X}) \arrow[r] \arrow[d] & \mathbb{X} \arrow[d, "p"] \\ \mathsf{AR}^{\epsilon\mathrm{lax}}(\mathbb{C}) \arrow[r, "\mathrm{ev}_i"] & \mathbb{C} \end{tikzcd}$$

  where the map π is induced by ev_{1-i} and $\epsilon = \mathrm{op}$ if $i \neq j$ and $\epsilon = \emptyset$ otherwise.

The universal property of the pullback guarantees that this construction extends to a functor of $(\infty,2)$-categories.

Theorem 1.4.2. *Let $\mathbb{C}$ be an $(\infty,2)$-category then there exists an adjunction of $(\infty,2)$-categories*

$$\mathfrak{F}_{(i,j)} : \mathsf{CAT}_{(\infty,2)/\mathbb{C}} \rightleftarrows \mathsf{FIB}_{(i,j)}(\mathbb{C}) : \mathfrak{U}_{(i,j)}$$

where $\mathfrak{U}_{(i,j)}$ denotes the forgetful functor.

Definition 1.4.3. Let $f : \mathbb{X} \to \mathbb{C}$ be a functor of $(\infty,2)$-categories and assume further that $(\mathbb{X}, E)$ is a marked $(\infty,2)$-category. Then can we view $\mathfrak{F}_{(i,j)}(\mathbb{X})$ as a marked $(\infty,2)$-category by declaring an morphism $e : [1] \to \mathfrak{F}_{(i,j)}(\mathbb{X})$ marked if:

(1) The associated diagram $[1] \otimes [1] \to \mathbb{C}$ factors through $[1] \times [1]$.
(2) The composite $[1] \to \mathfrak{F}_{(i,j)}(\mathbb{X}) \to \mathbb{X}$ belongs to E.

We denote the resulting marked $(\infty,2)$-category by $(\mathfrak{F}_{(i,j)}(\mathbb{X}), E_\square)$.

Definition 1.4.4. Let $f : (\mathbb{C}, E) \to (\mathbb{D}, E')$ be a functor of marked $(\infty,2)$-category. We say that:

- The functor f is $\diamond$-cofinal where $\diamond \in \{\mathrm{lax}, \mathrm{oplax}\}$ if given a diagram $L : \mathbb{D} \to \mathbb{A}$ then the induced map

$$\mathrm{colim}^{\mathrm{E}-\diamond}_{\mathbb{D}} L \xrightarrow{\simeq} \mathrm{colim}^{\mathrm{E}'-\diamond}_{\mathbb{C}} L \circ f$$

  is an equivalence whenever either those colimits is defined.
- The functor f is $\diamond$-initial where $\diamond \in \{\mathrm{lax}, \mathrm{oplax}\}$ if given a diagram $L : \mathbb{D} \to \mathbb{A}$ then the induced map

$$\lim^{\mathrm{E}'-\diamond}_{\mathbb{C}} L \circ f \xrightarrow{\simeq} \lim^{\mathrm{E}-\diamond}_{\mathbb{D}} L$$

  is an equivalence whenever either those colimits is defined.

2. Base change

Definition 2.0.1. Let $\mathbb{C}$ be an $(\infty,2)$-category . A *fibrational pattern* $\mathfrak{p} := (\mathbb{C}, (i,j), E, L)$ is given by:

- A pair (i,j) where $i,j \in \{0,1\}$ which we call the variance.
- A collection of edges E of $\mathbb{C}$ containing all equivalences.
- A collection of 2-simplices $\sigma : [2] \to \mathbb{C}$ containing all commutative triangles.

Given fibrational patterns $\mathfrak{p} = (\mathbb{C}, (i,j), E, L)$ and $\mathfrak{q} = (\mathbb{D}, (i,j), E', L')$, we say that a functor $f : \mathbb{C} \to \mathbb{D}$ is a morphism of fibrational patterns if $f(E) \subseteq E'$ and $f(L) \subseteq L'$.

Example 2.0.2. Given an $(\infty,2)$-category $\mathbb{C}$, we denote by $\mathfrak{p}_{\flat}^{(i,j)} := (\mathbb{C},(i,j),\flat,\flat)$ the fibrational pattern with variance (i,j) where the collection of edges is given precisely by the equivalences and the collection L is given by the commuting triangles. Dually, we denote $\mathfrak{p}_{\sharp}^{(i,j)} = (\mathbb{C},(i,j),\sharp,\sharp)$ the fibrational pattern where every edge (resp. every triangle) belongs to E (resp. L). If the variance is clear from the context we will use the abusive notation $\mathfrak{p}_{\flat}$ and $\mathfrak{p}_{\sharp}$.

Definition 2.0.3. Let $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ be the locally full subcategory of $\mathsf{CAT}_{(\infty,2)/\mathbb{C}}$ whose objects are functors $p : \mathbb{X} \to \mathbb{C}$ such that:

Explain this prelim

- There exists i-cartesian lifts of those 1-morphisms in $E_{\mathbb{C}}$.
- There exists j-cartesian lifts of 2-morphisms in $L_{\mathbb{C}}$ which are stable under composition in $\mathbb{X}$.

The morphisms in $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ are precisely those which preserve the i-cartesian (resp. j-cartesian) 1-morphisms (resp. 2-morphisms) above. Given $\mathbb{X} \to \mathbb{C}$ and $\mathbb{Y} \to \mathbb{C}$ in $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ we denote by $\mathsf{MAP}_{/(\mathbb{C},\mathfrak{p})}(\mathbb{X},\mathbb{Y})$ the mapping ∞-category in $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$.

Remark 2.0.4. Observe that $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p}_{\flat}^{(i,j)})}$ is simply given by the slice $(\infty,2)$-category $\mathsf{CAT}_{(\infty,2)/\mathbb{C}}$ and that if $\mathfrak{p} = (\mathbb{C},(i,j),E,\sharp)$ then $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ is given by $\mathsf{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{C})$.

Remark 2.0.5. Let $f : \mathbb{C} \to \mathbb{D}$ be functor inducing a map of fibrational patterns $\mathfrak{p} \to \mathfrak{q}$. Then pullback along f induces a functor of $(\infty,2)$-categories

$$f^* : \mathsf{CAT}_{(\infty,2)/(\mathbb{D},\mathfrak{q})} \to \mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$$

which we call the base change functor along f.

Definition 2.0.6. Let $\mathfrak{p} = (\mathbb{C},(i,j),E,L)$ be a fibrational pattern and consider a functor $f : \mathbb{C} \to \mathbb{D}$. We define the lax basechange functor as the composite

$$f_{\mathfrak{p}}^{\mathrm{lax}} : \mathsf{CAT}_{(\infty,2)/\mathbb{D}} \to \mathsf{FIB}_{(i,j)}(\mathbb{D}) \to \mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$$

where the first map was given in Definition 1.4.1 and the second map is base change along the map $(\mathbb{C},(i,j),E,L) \to (\mathbb{D},(i,j),\sharp,\sharp)$ induced by f.

Definition 2.0.7. Let $\mathfrak{p} = (\mathbb{C},(i,j),E,L)$ be a fibrational pattern and consider a functor $f : \mathbb{C} \to \mathbb{D}$. Then there exists a functor

$$Rf_* : \mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})} \to \mathsf{FUN}((\mathsf{CAT}_{(\infty,2)/\mathbb{D}})^{\mathrm{op}},\mathsf{CAT}_{\infty}), \quad (\mathbb{X} \to \mathbb{C}) \mapsto \mathsf{MAP}_{/(\mathbb{C},\mathfrak{p})}(f_{\mathfrak{p}}^{\mathrm{lax}}(-),\mathbb{X}).$$

Proposition 2.0.8. *The functor*

$$Rf_* : \mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})} \to \mathsf{FUN}((\mathsf{CAT}_{(\infty,2)/\mathbb{D}})^{\mathrm{op}},\mathsf{CAT}_{\infty})$$

factors through the composite $\mathsf{FIB}_{(i,j)}(\mathbb{D}) \to \mathsf{CAT}_{(\infty,2)/\mathbb{D}} \to \mathsf{FUN}((\mathsf{CAT}_{(\infty,2)/\mathbb{D}})^{\mathrm{op}},\mathsf{CAT}_{\infty})$ *where the second functor is the Yoneda embedding.*

Proof. ss □

Definition 2.0.9. We will call the functor $f_* : \mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})} \to \mathsf{FIB}_{(i,j)}(\mathbb{D})$ the *fibrational* pushforward functor.

Theorem 2.0.10. *Let $\mathfrak{p} = (\mathbb{C},(i,j),E,L)$ be a fibrational pattern and let $f : \mathbb{C} \to \mathbb{D}$ be a functor. Then there exists an adjunction of $(\infty,2)$-categories:*

$$f^* : \mathsf{FIB}_{(i,j)}(\mathbb{D}) \rightleftarrows \mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})} : f_*$$

3. A fibrewise criterion for Kan extensions

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