

# BASE CHANGE AND LAX KAN EXTENSIONS OF $(\infty, 2)$ -CATEGORIES.

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ABSTRACT. We do basechange and lax kan extensions

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## 1. PRELIMINARIES

### 1.1. Elementary definitions and models for $(\infty, 2)$ -categories.

### 1.2. Fibrations of $(\infty, 2)$ -categories and straightening.

**Definition 1.2.1.** Let  $(\mathbb{B}, E)$  be a marked  $(\infty, 2)$ -category and let  $\mathbf{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{B})$  denote the wide locally full subcategory of  $\mathbf{FIB}_{(i,j)}^{\text{lax}}(\mathbb{B})$  where the morphisms are required to preserve  $i$ -cartesian morphisms over  $E$ . We denote by  $\mathbf{Fun}_{/\mathbb{B}}^{E\text{-cart}}(-, -)$  the mapping  $\infty$ -category functor for  $\mathbf{FIB}_{(1,j)}^{E\text{-lax}}(\mathbb{B})$  and similarly  $\mathbf{Fun}_{/\mathbb{B}}^{E\text{-coc}}(-, -)$  for  $\mathbf{FIB}_{(0,j)}^{E\text{-lax}}(\mathbb{B})$ .

**1.3. Partially lax limits.** In this section, we collect basic results and definitions of the theory of partially lax (co)limits which will be of use later. For a more extensive treatment (and the proofs of the statements below) we refer the reader to (CITE) .

**Definition 1.3.1.** Let  $(\mathbb{A}, E)$  be a marked  $(\infty, 2)$ -category and let  $F: \mathbb{A} \rightarrow \mathbb{B}$  be a functor of  $(\infty, 2)$ -categories. We say that  $\text{colim}_{\mathbb{A}}^{\diamond} F \in \mathbb{B}$ , where  $\diamond \in \{E\text{-lax}, E\text{-oplax}\}$ , is the  $\diamond$ -colimit of  $F$  if we have a natural equivalence of  $\mathbf{CAT}_{\infty}$ -valued functors

$$\mathbf{Nat}_{\mathbb{A}, \mathbb{B}}^{\diamond} \left( F, \underline{(-)} \right) \simeq \mathbb{B}(\text{colim}_{\mathbb{A}}^{\diamond} F, -)$$

where  $\underline{(-)}: \mathbb{B} \rightarrow \mathbf{FUN}(\mathbb{A}, \mathbb{B})^{\diamond}$  is the functor sending each object to a constant diagram. Similarly, we say that  $\lim_{\mathbb{A}}^{\diamond} F \in \mathbb{B}$ , where  $\diamond \in \{E\text{-lax}, E\text{-oplax}\}$ , is the  $\diamond$ -limit of  $F$  if we have a natural equivalence of  $\mathbf{CAT}_{\infty}$ -valued functors

$$\mathbf{Nat}_{\mathbb{A}, \mathbb{B}}^{\diamond} \left( \underline{(-)}, F \right) \simeq \mathbb{B}(-, \lim_{\mathbb{A}}^{\diamond} F).$$

**Proposition 1.3.2.** *Let  $(\mathbb{A}, E)$  be a marked  $(\infty, 2)$ -category and consider a functor  $F: \mathbb{A} \rightarrow \mathbf{CAT}_{(\infty, 2)}$  with corresponding  $(i, j)$ -fibration  $\pi: \mathcal{F}_{i,j} \rightarrow \mathbb{A}^\epsilon$  where  $\epsilon \in \{\emptyset, \text{op}, \text{co}, \text{coop}\}$ . Write  $L_E(\mathcal{F}_{i,j})$  for the  $(\infty, 2)$ -category obtained from  $\mathcal{F}_{i,j}$  by inverting the  $j$ -cartesian 2-morphisms together with those  $i$ -cartesian 1-morphisms that lie over  $E$ . Then the  $E$ -(op)lax colimits of  $F$  can be described as*

$$\text{colim}_{\mathbb{A}}^{E\text{-lax}} F \simeq L_E(\mathcal{F}_{0,j}), \quad \text{colim}_{\mathbb{A}}^{E\text{-oplax}} F \simeq L_E(\mathcal{F}_{1,j})$$

for  $j = 0, 1$ .

**Definition 1.3.3.** Let  $\mathcal{F}, \mathcal{G} \in \mathbf{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{A})$ . If  $i = 0$ , we denote by  $\mathbf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F}, \mathcal{G})$  the  $(\infty, 2)$ -category characterised by the universal property

$$\mathbf{Map}_{\mathbf{Cat}_{(\infty, 2)}}(\mathbb{X}, \mathbf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F}, \mathcal{G})) \simeq \mathbf{Fun}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F} \times \mathbb{X}, \mathcal{G}),$$

where the right-handside was defined in Definition 1.2.1. We define  $\mathbf{FUN}_{/\mathbb{A}}^{E\text{-cart}}(\mathcal{F}, \mathcal{G})$  analogously whenever  $i = 1$ .

**Proposition 1.3.4.** *Let  $(\mathbb{A}, E)$  be a marked  $(\infty, 2)$ -category and consider a functor  $F: \mathbb{A} \rightarrow \mathbf{CAT}_{(\infty, 2)}$  with corresponding  $(i, j)$ -fibration  $\pi: \mathcal{F}_{i,j} \rightarrow \mathbb{A}^\epsilon$  where  $\epsilon \in \{\neg, \text{op}, \text{co}, \text{coop}\}$ . Then the  $E$ -(op)lax limits of  $F$  in  $\mathbf{CAT}_{(\infty, 2)}$  can be described as*

$$\lim_{\mathbb{A}}^{E\text{-lax}} F \simeq \mathbf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathbb{A}, \mathcal{F}_{0,j}), \quad \lim_{\mathbb{A}}^{E\text{-oplax}} F \simeq \mathbf{FUN}_{/\mathbb{A}}^{E\text{-cart}}(\mathbb{A}, \mathcal{F}_{1,j})$$

for  $j = 0, 1$ .

#### 1.4. Cofinality. $\mathfrak{ff}$

**Definition 1.4.1.** Let  $\mathbf{AR}^{(\text{op})\text{lax}}(\mathbb{C}) = \mathbf{FUN}^{(\text{op})\text{lax}}([1], \mathbb{C})$  be the (op)lax arrow  $(\infty, 2)$ -category of  $\mathbb{C}$  and denote by  $\text{ev}_i: \mathbf{AR}^{(\text{op})\text{lax}}(\mathbb{C}) \rightarrow \mathbb{C}$  the functor induced by restriction along the map  $\{i\} \rightarrow [1]$ . We consider a functor

$$\mathfrak{F}_{(i,j)}: \mathbf{CAT}_{(\infty, 2)/\mathbb{C}} \rightarrow \mathbf{FIB}_{(i,j)}(\mathbb{C}),$$

whose action on objects is as follows:

- Given  $p: \mathbb{X} \rightarrow \mathbb{C}$  we define  $\pi: \mathfrak{F}_{(i,j)}(\mathbb{X}) \rightarrow \mathbb{C}$  as the pullback

$$\begin{array}{ccc} \mathfrak{F}_{(i,j)}(\mathbb{X}) & \longrightarrow & \mathbb{X} \\ \downarrow & & \downarrow p \\ \mathbf{AR}^{E\text{lax}}(\mathbb{C}) & \xrightarrow{\text{ev}_i} & \mathbb{C} \end{array}$$

where the map  $\pi$  is induced by  $\text{ev}_{1-i}$  and  $\epsilon = \text{op}$  if  $i \neq j$  and  $\epsilon = \emptyset$  otherwise.

The universal property of the pullback guarantees that this construction extends to a functor of  $(\infty, 2)$ -categories.

## 2. BASE CHANGE

**Definition 2.0.1.** Let  $\mathbb{C}$  be an  $(\infty, 2)$ -category. A *fibrational pattern*  $\mathfrak{p} := (\mathbb{C}, (i, j), E, L)$  is given by:

- A pair  $(i, j)$  where  $i, j \in \{0, 1\}$  which we call the variance.
- A collection of edges  $E$  of  $\mathbb{C}$  containing all equivalences.
- A collection of 2-simplices  $\sigma: [2] \rightarrow \mathbb{C}$  containing all commutative triangles.

Given fibrational patterns  $\mathfrak{p} = (\mathbb{C}, (i, j), E, L)$  and  $\mathfrak{q} = (\mathbb{D}, (i, j), E', L')$ , we say that a functor  $f: \mathbb{C} \rightarrow \mathbb{D}$  is a morphism of fibrational patterns if  $f(E) \subseteq E'$  and  $f(L) \subseteq L'$ .

**Example 2.0.2.** Given an  $(\infty, 2)$ -category  $\mathbb{C}$ , we denote by  $\mathfrak{p}_b^{(i,j)} := (\mathbb{C}, (i, j), b, b)$  the fibrational pattern with variance  $(i, j)$  where the collection of edges is given precisely by the equivalences and the collection  $L$  is given by the commuting triangles. Dually, we denote  $\mathfrak{p}_\#^{(i,j)} = (\mathbb{C}, (i, j), \#, \#)$  the fibrational pattern where every edge (resp. every triangle) belongs to  $E$  (resp.  $L$ ). If the variance is clear from the context we will use the abusive notation  $\mathfrak{p}_b$  and  $\mathfrak{p}_\#$ .

**Definition 2.0.3.** Let  $\text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})}$  be the locally full subcategory of  $\text{CAT}_{(\infty, 2)/\mathbb{C}}$  whose objects are functors  $p : \mathbb{X} \rightarrow \mathbb{C}$  such that:

- ▶ There exists  $i$ -cartesian lifts of those 1-morphisms in  $E_{\mathbb{C}}$ .
- ▶ There exists  $j$ -cartesian lifts of 2-morphisms in  $L_{\mathbb{C}}$  which are stable under composition in  $\mathbb{X}$ .

[Explain this prelim](#)

The morphisms in  $\text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})}$  are precisely those which preserve the  $i$ -cartesian (resp.  $j$ -cartesian) 1-morphisms (resp. 2-morphisms) above. Given  $\mathbb{X} \rightarrow \mathbb{C}$  and  $\mathbb{Y} \rightarrow \mathbb{C}$  in  $\text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})}$  we denote by  $\text{MAP}_{/(\mathbb{C}, \mathfrak{p})}(\mathbb{X}, \mathbb{Y})$  the mapping  $\infty$ -category in  $\text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})}$ .

**Remark 2.0.4.** Observe that  $\text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p}_b^{(i,j)})}$  is simply given by the slice  $(\infty, 2)$ -category  $\text{CAT}_{(\infty, 2)/\mathbb{C}}$  and that if  $\mathfrak{p} = (\mathbb{C}, (i, j), E, \#)$  then  $\text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})}$  is given by  $\text{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{C})$ .

**Remark 2.0.5.** Let  $f : \mathbb{C} \rightarrow \mathbb{D}$  be functor inducing a map of fibrational patterns  $\mathfrak{p} \rightarrow \mathfrak{q}$ . Then pullback along  $f$  induces a functor of  $(\infty, 2)$ -categories

$$f^* : \text{CAT}_{(\infty, 2)/(\mathbb{D}, \mathfrak{q})} \rightarrow \text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})}$$

which we call the base change functor along  $f$ .

**Definition 2.0.6.** Let  $\mathfrak{p} = (\mathbb{C}, (i, j), E, L)$  be a fibrational pattern and consider a functor  $f : \mathbb{C} \rightarrow \mathbb{D}$ . We define the lax basechange functor as the composite

$$f_{\mathfrak{p}}^{\text{lax}} : \text{CAT}_{(\infty, 2)/\mathbb{D}} \rightarrow \text{FIB}_{(i,j)}(\mathbb{D}) \rightarrow \text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})}$$

where the first map was given in Definition 1.4.1 and the second map is base change along the map  $(\mathbb{C}, (i, j), E, L) \rightarrow (\mathbb{D}, (i, j), \#, \#)$  induced by  $f$ .

**Definition 2.0.7.** Let  $\mathfrak{p} = (\mathbb{C}, (i, j), E, L)$  be a fibrational pattern and consider a functor  $f : \mathbb{C} \rightarrow \mathbb{D}$ . Then there exists a functor

$$Rf_* : \text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})} \rightarrow \text{FUN}((\text{CAT}_{(\infty, 2)/\mathbb{D}})^{\text{op}}, \text{CAT}_{\infty}), \quad (\mathbb{X} \rightarrow \mathbb{C}) \mapsto \text{MAP}_{/(\mathbb{C}, \mathfrak{p})}(f_{\mathfrak{p}}^{\text{lax}}(-), \mathbb{X}).$$

**Proposition 2.0.8.** *The functor*

$$Rf_* : \text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})} \rightarrow \text{FUN}((\text{CAT}_{(\infty, 2)/\mathbb{D}})^{\text{op}}, \text{CAT}_{\infty})$$

*factors through the composite  $\text{FIB}_{(i,j)}(\mathbb{D}) \rightarrow \text{CAT}_{(\infty, 2)/\mathbb{D}} \rightarrow \text{FUN}((\text{CAT}_{(\infty, 2)/\mathbb{D}})^{\text{op}}, \text{CAT}_{\infty})$  where the second functor is the Yoneda embedding.*

*Proof.* ss □

**Definition 2.0.9.** We will call the functor  $f_* : \text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})} \rightarrow \text{FIB}_{(i,j)}(\mathbb{D})$  the *fibrational pushforward* functor.

**Theorem 2.0.10.** *Let  $\mathfrak{p} = (\mathbb{C}, (i, j), E, L)$  be a fibrational pattern and let  $f : \mathbb{C} \rightarrow \mathbb{D}$  be a functor. Then there exists an adjunction of  $(\infty, 2)$ -categories:*

$$f^* : \text{FIB}_{(i,j)}(\mathbb{D}) \rightleftarrows \text{CAT}_{(\infty, 2)/(\mathbb{C}, \mathfrak{p})} : f_*$$

### 3. A FIBREWISE CRITERION FOR KAN EXTENSIONS