

BASE CHANGE AND LAX KAN EXTENSIONS OF $(\infty, 2)$ -CATEGORIES.

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ABSTRACT. We do basechange and lax kan extensions

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1. PRELIMINARIES

1.1. Elementary definitions and models for $(\infty, 2)$ -categories.

1.2. Fibrations of $(\infty, 2)$ -categories and straightening.

Definition 1.2.1. Let (\mathbb{B}, E) be a marked $(\infty, 2)$ -category and let $\mathbf{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{B})$ denote the wide locally full subcategory of $\mathbf{FIB}_{(i,j)}^{\text{lax}}(\mathbb{B})$ where the morphisms are required to preserve i -cartesian morphisms over E . We denote by $\mathbf{Fun}_{/\mathbb{B}}^{E\text{-cart}}(-, -)$ the mapping ∞ -category functor for $\mathbf{FIB}_{(1,j)}^{E\text{-lax}}(\mathbb{B})$ and similarly $\mathbf{Fun}_{/\mathbb{B}}^{E\text{-coc}}(-, -)$ for $\mathbf{FIB}_{(0,j)}^{E\text{-lax}}(\mathbb{B})$.

1.3. Partially lax limits. In this section, we collect basic results and definitions of the theory of partially lax (co)limits which will be of use later. For a more extensive treatment (and the proofs of the statements below) we refer the reader to (CITE) .

Definition 1.3.1. Let (\mathbb{A}, E) be a marked $(\infty, 2)$ -category and let $F: \mathbb{A} \rightarrow \mathbb{B}$ be a functor of $(\infty, 2)$ -categories. We say that $\text{colim}_{\mathbb{A}}^{\diamond} F \in \mathbb{B}$, where $\diamond \in \{E\text{-lax}, E\text{-oplax}\}$, is the \diamond -colimit of F if we have a natural equivalence of \mathbf{CAT}_{∞} -valued functors

$$\mathbf{Nat}_{\mathbb{A}, \mathbb{B}}^{\diamond} \left(F, \underline{(-)} \right) \simeq \mathbb{B}(\text{colim}_{\mathbb{A}}^{\diamond} F, -)$$

where $\underline{(-)}: \mathbb{B} \rightarrow \mathbf{FUN}(\mathbb{A}, \mathbb{B})^{\diamond}$ is the functor sending each object to a constant diagram. Similarly, we say that $\lim_{\mathbb{A}}^{\diamond} F \in \mathbb{B}$, where $\diamond \in \{E\text{-lax}, E\text{-oplax}\}$, is the \diamond -limit of F if we have a natural equivalence of \mathbf{CAT}_{∞} -valued functors

$$\mathbf{Nat}_{\mathbb{A}, \mathbb{B}}^{\diamond} \left(\underline{(-)}, F \right) \simeq \mathbb{B}(-, \lim_{\mathbb{A}}^{\diamond} F).$$

Proposition 1.3.2. *Let (\mathbb{A}, E) be a marked $(\infty, 2)$ -category and consider a functor $F: \mathbb{A} \rightarrow \mathbf{CAT}_{(\infty, 2)}$ with corresponding (i, j) -fibration $\pi: \mathcal{F}_{i,j} \rightarrow \mathbb{A}^\epsilon$ where $\epsilon \in \{\emptyset, \text{op}, \text{co}, \text{coop}\}$. Write $L_E(\mathcal{F}_{i,j})$ for the $(\infty, 2)$ -category obtained from $\mathcal{F}_{i,j}$ by inverting the j -cartesian 2-morphisms together with those i -cartesian 1-morphisms that lie over E . Then the E -(op)lax colimits of F can be described as*

$$\text{colim}_{\mathbb{A}}^{E\text{-lax}} F \simeq L_E(\mathcal{F}_{0,j}), \quad \text{colim}_{\mathbb{A}}^{E\text{-oplax}} F \simeq L_E(\mathcal{F}_{1,j})$$

for $j = 0, 1$.

Definition 1.3.3. Let $\mathcal{F}, \mathcal{G} \in \mathbf{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{A})$. If $i = 0$, we denote by $\mathbf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F}, \mathcal{G})$ the $(\infty, 2)$ -category characterised by the universal property

$$\mathbf{Map}_{\mathbf{Cat}_{(\infty, 2)}}(\mathbb{X}, \mathbf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F}, \mathcal{G})) \simeq \mathbf{Fun}_{/\mathbb{A}}^{E\text{-coc}}(\mathcal{F} \times \mathbb{X}, \mathcal{G}),$$

where the right-handside was defined in Definition 1.2.1. We define $\mathbf{FUN}_{/\mathbb{A}}^{E\text{-cart}}(\mathcal{F}, \mathcal{G})$ analogously whenever $i = 1$.

Proposition 1.3.4. *Let (\mathbb{A}, E) be a marked $(\infty, 2)$ -category and consider a functor $F: \mathbb{A} \rightarrow \mathbf{CAT}_{(\infty, 2)}$ with corresponding (i, j) -fibration $\pi: \mathcal{F}_{i,j} \rightarrow \mathbb{A}^\epsilon$ where $\epsilon \in \{\neg, \text{op}, \text{co}, \text{coop}\}$. Then the E -(op)lax limits of F in $\mathbf{CAT}_{(\infty, 2)}$ can be described as*

$$\lim_{\mathbb{A}}^{E\text{-lax}} F \simeq \mathbf{FUN}_{/\mathbb{A}}^{E\text{-coc}}(\mathbb{A}, \mathcal{F}_{0,j}), \quad \lim_{\mathbb{A}}^{E\text{-oplax}} F \simeq \mathbf{FUN}_{/\mathbb{A}}^{E\text{-cart}}(\mathbb{A}, \mathcal{F}_{1,j})$$

for $j = 0, 1$.

Proposition 1.3.5. *Let $\Lambda_1^2 = (0 \rightarrow 2 \leftarrow 1)$ denote the walking cospan and consider a functor in $F: \Lambda_1^2 \rightarrow \mathbf{CAT}_\infty$ given by $\mathbf{A} \rightarrow \mathbf{C} \leftarrow \mathbf{B}$. Then we have equivalences of ∞ -categories:*

$$\lim_{\Lambda_1^2}^{E\text{-lax}} F = (\mathbf{C}^{[1]} \times_{\text{ev}_0} \mathbf{B}) \times_{\text{ev}_1} \mathbf{A}, \quad \lim_{\Lambda_1^2}^{E\text{-oplax}} F = (\mathbf{C}^{[1]} \times_{\text{ev}_1} \mathbf{B}) \times_{\text{ev}_0} \mathbf{A},$$

where $E = (0 \rightarrow 2)$.

Proof. We give the proof in the E -lax case since the E -oplax case is formally dual. Let $\mathcal{F} \rightarrow \Lambda_1^2$ be the cocartesian fibration associated to F . It follows from Proposition 1.3.4 that we have a natural equivalence

$$\lim_{\Lambda_1^2}^{E\text{-lax}} F \simeq \mathbf{FUN}_{/\Lambda_1^2}^{E\text{-coc}}(\Lambda_1^2, \mathcal{F}) \simeq \mathbf{NAT}_{\Lambda_1^2, \mathbf{CAT}_\infty}(\mathcal{W}, F)$$

where $\mathcal{W}: \Lambda_1^2 \rightarrow \mathbf{CAT}_\infty$ is given by $\mathcal{W}(0) = \{1\} \rightarrow [1] = \mathcal{W}(2) \leftarrow \{1\} = \mathcal{W}(1)$. Given an $(\infty, 2)$ -category \mathbf{T} , we can construct natural equivalences

$$\mathbf{Map}(\mathbf{T}, \mathbf{NAT}_{\Lambda_1^2, \mathbf{CAT}_\infty}(\mathcal{W}, F)) \simeq \mathbf{Nat}_{\Lambda_1^2, \mathbf{CAT}_\infty}(\mathbf{T} \times \mathcal{W}, F),$$

where the later space can be expressed (according to (REF)) as the limit of the following diagram

$$\begin{array}{ccccc} \mathbf{Map}(\mathbf{T}, \mathbf{B}) & & & & \mathbf{Map}(\mathbf{T}, \mathbf{A}) \\ \downarrow & & & & \downarrow \\ \mathbf{Map}(\mathbf{T}, \mathbf{C}) & \xleftarrow{\text{ev}_0^*} & \mathbf{Map}(\mathbf{T} \times [1], \mathbf{C}) & \xrightarrow{\text{ev}_1^*} & \mathbf{Map}(\mathbf{T}, \mathbf{C}). \end{array}$$

It follows from the universal property of the iterated pullback $(\mathbf{C}^{[1]} \times_{\text{ev}_0} \mathbf{B}) \times_{\text{ev}_1} \mathbf{A}$ that the limit of the diagram above is naturally equivalent to the mapping space $\mathbf{Map}(\mathbf{T}, (\mathbf{C}^{[1]} \times_{\text{ev}_0} \mathbf{B}) \times_{\text{ev}_1} \mathbf{A})$. The result now follows by the Yoneda lemma. \square

1.4. Free fibrations and cofinality. $\mathfrak{f}\mathfrak{f}\mathfrak{f}$

Definition 1.4.1. Let $\mathbf{AR}^{(\text{op})\text{lax}}(\mathbb{C}) = \mathbf{FUN}^{(\text{op})\text{lax}}([1], \mathbb{C})$ be the $(\text{op})\text{lax}$ arrow $(\infty, 2)$ -category of \mathbb{C} and denote by $\text{ev}_i : \mathbf{AR}^{(\text{op})\text{lax}}(\mathbb{C}) \rightarrow \mathbb{C}$ the functor induced by restriction along the map $\{i\} \rightarrow [1]$. We consider a functor

$$\mathfrak{F}_{(i,j)} : \mathbf{CAT}_{(\infty,2)/\mathbb{C}} \rightarrow \mathbf{FIB}_{(i,j)}(\mathbb{C}),$$

whose action on objects is as follows:

- Given $p : \mathbb{X} \rightarrow \mathbb{C}$ we define $\pi : \mathfrak{F}_{(i,j)}(\mathbb{X}) \rightarrow \mathbb{C}$ as the pullback

$$\begin{array}{ccc} \mathfrak{F}_{(i,j)}(\mathbb{X}) & \longrightarrow & \mathbb{X} \\ \downarrow & & \downarrow p \\ \mathbf{AR}^{\text{elax}}(\mathbb{C}) & \xrightarrow{\text{ev}_i} & \mathbb{C} \end{array}$$

where the map π is induced by ev_{1-i} and $\epsilon = \text{op}$ if $i \neq j$ and $\epsilon = \emptyset$ otherwise.

The universal property of the pullback guarantees that this construction extends to a functor of $(\infty, 2)$ -categories.

Theorem 1.4.2. Let \mathbb{C} be an $(\infty, 2)$ -category then there exists an adjunction of $(\infty, 2)$ -categories

$$\mathfrak{F}_{(i,j)} : \mathbf{CAT}_{(\infty,2)/\mathbb{C}} \rightleftarrows \mathbf{FIB}_{(i,j)}(\mathbb{C}) : \mathfrak{U}_{(i,j)}$$

where $\mathfrak{U}_{(i,j)}$ denotes the forgetful functor.

Definition 1.4.3. Let $f : \mathbb{X} \rightarrow \mathbb{C}$ be a functor of $(\infty, 2)$ -categories and assume further that (\mathbb{X}, E) is a marked $(\infty, 2)$ -category. Then can we view $\mathfrak{F}_{(i,j)}(\mathbb{X})$ as a marked $(\infty, 2)$ -category by declaring an morphism $e : [1] \rightarrow \mathfrak{F}_{(i,j)}(\mathbb{X})$ marked if:

- (1) The associated diagram $[1] \otimes [1] \rightarrow \mathbb{C}$ factors through $[1] \times [1]$.
- (2) The composite $[1] \rightarrow \mathfrak{F}_{(i,j)}(\mathbb{X}) \rightarrow \mathbb{X}$ belongs to E .

We denote the resulting marked $(\infty, 2)$ -category by $(\mathfrak{F}_{(i,j)}(\mathbb{X}), E_{\square})$.

Definition 1.4.4. Let $f : \mathbb{X} \rightarrow \mathbb{C}$ be a functor of $(\infty, 2)$ -categories and let $\mathfrak{F}_{(i,j)}(\mathbb{X}) \rightarrow \mathbb{C}$. Given $c \in \mathbb{C}$ we fix the notation $\mathbb{X}_{\mathcal{F}_c}$

Definition 1.4.5. Let $f : (\mathbb{C}, E) \rightarrow (\mathbb{D}, E')$ be a functor of marked $(\infty, 2)$ -categories. We say that:

- The functor f is \diamond -cofinal where $\diamond \in \{\text{lax}, \text{oplax}\}$ if given a diagram $L : \mathbb{D} \rightarrow \mathbb{A}$ then the induced map

$$\text{colim}_{\mathbb{D}}^{E-\diamond} L \xrightarrow{\sim} \text{colim}_{\mathbb{C}}^{E'-\diamond} L \circ f$$

is an equivalence whenever either those colimits is defined.

- The functor f is \diamond -initial where $\diamond \in \{\text{lax}, \text{oplax}\}$ if given a diagram $L : \mathbb{D} \rightarrow \mathbb{A}$ then the induced map

$$\lim_{\mathbb{C}}^{E'-\diamond} L \circ f \xrightarrow{\sim} \lim_{\mathbb{D}}^{E-\diamond} L$$

is an equivalence whenever either those limits is defined.

2. BASE CHANGE

Definition 2.0.1. Let \mathbb{C} be an $(\infty, 2)$ -category. A *fibrational pattern* $\mathfrak{p} := (\mathbb{C}, (i, j), E, L)$ is given by:

- A pair (i, j) where $i, j \in \{0, 1\}$ which we call the variance.
- A collection of edges E of \mathbb{C} containing all equivalences.
- A collection of 2-simplices $\sigma : [2] \rightarrow \mathbb{C}$ containing all commutative triangles.

Given fibrational patterns $\mathbf{p} = (\mathbb{C}, (i, j), E, L)$ and $\mathbf{q} = (\mathbb{D}, (i, j), E', L')$, we say that a functor $f : \mathbb{C} \rightarrow \mathbb{D}$ is a morphism of fibrational patterns if $f(E) \subseteq E'$ and $f(L) \subseteq L'$.

Example 2.0.2. Given an $(\infty, 2)$ -category \mathbb{C} , we denote by $\mathbf{p}_b^{(i, j)} := (\mathbb{C}, (i, j), b, b)$ the fibrational pattern with variance (i, j) where the collection of edges is given precisely by the equivalences and the collection L is given by the commuting triangles. Dually, we denote $\mathbf{p}_\#^{(i, j)} = (\mathbb{C}, (i, j), \#, \#)$ the fibrational pattern where every edge (resp. every triangle) belongs to E (resp. L). If the variance is clear from the context we will use the abusive notation \mathbf{p}_b and $\mathbf{p}_\#$.

Definition 2.0.3. Let $\mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})}$ be the locally full subcategory of $\mathbf{CAT}_{(\infty, 2)/\mathbb{C}}$ whose objects are functors $p : \mathbb{X} \rightarrow \mathbb{C}$ such that:

- ▶ There exists i -cartesian lifts of those 1-morphisms in $E_{\mathbb{C}}$.
- ▶ There exists j -cartesian lifts of 2-morphisms in $L_{\mathbb{C}}$ which are stable under composition in \mathbb{X} .

The morphisms in $\mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})}$ are precisely those which preserve the i -cartesian (resp. j -cartesian) 1-morphisms (resp. 2-morphisms) above. Given $\mathbb{X} \rightarrow \mathbb{C}$ and $\mathbb{Y} \rightarrow \mathbb{C}$ in $\mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})}$ we denote by $\mathbf{MAP}_{/(\mathbb{C}, \mathbf{p})}(\mathbb{X}, \mathbb{Y})$ the mapping ∞ -category in $\mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})}$.

Remark 2.0.4. Observe that $\mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p}_b^{(i, j)})}$ is simply given by the slice $(\infty, 2)$ -category $\mathbf{CAT}_{(\infty, 2)/\mathbb{C}}$ and that if $\mathbf{p} = (\mathbb{C}, (i, j), E, \#)$ then $\mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})}$ is given by $\mathbf{FIB}_{(i, j)}^{E-\text{lax}}(\mathbb{C})$.

Remark 2.0.5. Let $f : \mathbb{C} \rightarrow \mathbb{D}$ be functor inducing a map of fibrational patterns $\mathbf{p} \rightarrow \mathbf{q}$. Then pullback along f induces a functor of $(\infty, 2)$ -categories

$$f^* : \mathbf{CAT}_{(\infty, 2)/(\mathbb{D}, \mathbf{q})} \rightarrow \mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})}$$

which we call the base change functor along f .

Definition 2.0.6. Let $\mathbf{p} = (\mathbb{C}, (i, j), E, L)$ be a fibrational pattern and consider a functor $f : \mathbb{C} \rightarrow \mathbb{D}$. We define the lax basechange functor as the composite

$$f_{\mathbf{p}}^{\text{lax}} : \mathbf{CAT}_{(\infty, 2)/\mathbb{D}} \rightarrow \mathbf{FIB}_{(i, j)}(\mathbb{D}) \rightarrow \mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})}$$

where the first map was given in Definition 1.4.1 and the second map is base change along the map $(\mathbb{C}, (i, j), E, L) \rightarrow (\mathbb{D}, (i, j), \#, \#)$ induced by f .

Definition 2.0.7. Let $\mathbf{p} = (\mathbb{C}, (i, j), E, L)$ be a fibrational pattern and consider a functor $f : \mathbb{C} \rightarrow \mathbb{D}$. Then there exists a functor

$$Rf_* : \mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})} \rightarrow \mathbf{FUN}((\mathbf{CAT}_{(\infty, 2)/\mathbb{D}})^{\text{op}}, \mathbf{CAT}_{\infty}), \quad (\mathbb{X} \rightarrow \mathbb{C}) \mapsto \mathbf{MAP}_{/(\mathbb{C}, \mathbf{p})}(f_{\mathbf{p}}^{\text{lax}}(-), \mathbb{X}).$$

Proposition 2.0.8. *The functor*

$$Rf_* : \mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})} \rightarrow \mathbf{FUN}((\mathbf{CAT}_{(\infty, 2)/\mathbb{D}})^{\text{op}}, \mathbf{CAT}_{\infty})$$

factors through the composite $\mathbf{FIB}_{(i, j)}(\mathbb{D}) \rightarrow \mathbf{CAT}_{(\infty, 2)/\mathbb{D}} \rightarrow \mathbf{FUN}((\mathbf{CAT}_{(\infty, 2)/\mathbb{D}})^{\text{op}}, \mathbf{CAT}_{\infty})$ where the second functor is the Yoneda embedding.

Proof. ss □

Definition 2.0.9. We will call the functor $f_* : \mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})} \rightarrow \mathbf{FIB}_{(i, j)}(\mathbb{D})$ the *fibrational pushforward* functor.

Theorem 2.0.10. *Let $\mathbf{p} = (\mathbb{C}, (i, j), E, L)$ be a fibrational pattern and let $f : \mathbb{C} \rightarrow \mathbb{D}$ be a functor. Then there exists an adjunction of $(\infty, 2)$ -categories:*

$$f^* : \mathbf{FIB}_{(i, j)}(\mathbb{D}) \rightleftarrows \mathbf{CAT}_{(\infty, 2)/(\mathbb{C}, \mathbf{p})} : f_*$$

3. A FIBREWISE CRITERION FOR KAN EXTENSIONS

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