BASE CHANGE AND LAX KAN EXTENSIONS OF $(\infty,2)$-CATEGORIES.

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ABSTRACT. We do basechange and lax kan extensions

CONTENTS

1. PRELIMINARIES ON FIBRATIONS

1.1. Free fibrations.

Definition 1.1.1. Let $\mathsf{AR}^{(\mathrm{op})\mathrm{lax}}(\mathbb{C}) = \mathsf{FUN}^{(\mathrm{op})\mathrm{lax}}([1],\mathbb{C})$ be the (op)lax arrow $(\infty,2)$-category of $\mathbb{C}$ and denote by $\mathrm{ev}_i : \mathsf{AR}^{(\mathrm{op})\mathrm{lax}}(\mathbb{C}) \to \mathbb{C}$ the functor induced by restriction along the map $\{i\} \to [1]$. We consider a functor

$$\mathfrak{F}_{(i,j)} : \mathsf{CAT}_{(\infty,2)/\mathbb{C}} \to \mathsf{FIB}_{(i,j)}(\mathbb{C}),$$

whose action on objects is as follows:

- Given $p : \mathbb{X} \to \mathbb{C}$ we define $\pi : \mathfrak{F}_{(i,j)}(\mathbb{X}) \to \mathbb{C}$ as the pullback

$$\begin{tikzcd} \mathfrak{F}_{(i,j)}(\mathbb{X}) \arrow[r] \arrow[d] & \mathbb{X} \arrow[d, "p"] \\ \mathsf{AR}^{\epsilon\mathrm{lax}}(\mathbb{C}) \arrow[r, "\mathrm{ev}_i"] & \mathbb{C} \end{tikzcd}$$

  where the map π is induced by ev_{1-i} and $\epsilon = \mathrm{op}$ if $i \neq j$ and $\epsilon = \emptyset$ otherwise.

The universal property of the pullback guarantees that this construction extends to a functor of $(\infty,2)$-categories.

2. BASE CHANGE

Definition 2.0.1. Let $\mathbb{C}$ be an $(\infty,2)$-category . A *fibrational pattern* $\mathfrak{p} := (\mathbb{C},(i,j),E,L)$ is given by:

- A pair (i,j) where $i,j \in \{0,1\}$ which we call the variance.
- A collection of edges E of $\mathbb{C}$ containing all equivalences.
- A collection of 2-simplices $\sigma : [2] \to \mathbb{C}$ containing all commutative triangles.

Given fibrational patterns $\mathfrak{p} = (\mathbb{C},(i,j),E,L)$ and $\mathfrak{q} = (\mathbb{D},(i,j),E',L')$, we say that a functor $f : \mathbb{C} \to \mathbb{D}$ is a morphism of fibrational patterns if $f(E) \subseteq E'$ and $f(L) \subseteq L'$.

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Example 2.0.2. Given an $(\infty,2)$-category $\mathbb{C}$, we denote by $\mathfrak{p}_{\flat}^{(i,j)} := (\mathbb{C},(i,j),\flat,\flat)$ the fibrational pattern with variance (i,j) where the collection of edges is given precisely by the equivalences and the collection L is given by the commuting triangles. Dually, we denote $\mathfrak{p}_{\sharp}^{(i,j)} = (\mathbb{C},(i,j),\sharp,\sharp)$ the fibrational pattern where every edge (resp. every triangle) belongs to E (resp. L). If the variance is clear from the context we will use the abusive notation $\mathfrak{p}_{\flat}$ and $\mathfrak{p}_{\sharp}$.

Definition 2.0.3. Let $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ be the locally full subcategory of $\mathsf{CAT}_{(\infty,2)/\mathbb{C}}$ whose objects are functors $p:\mathbb{X}\to\mathbb{C}$ such that:

Explain this prelim

- There exists i-cartesian lifts of those 1-morphisms in $E_{\mathbb{C}}$.
- There exists j-cartesian lifts of 2-morphisms in $L_{\mathbb{C}}$ which are stable under composition in $\mathbb{X}$.

The morphisms in $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ are precisely those which preserve the i-cartesian (resp. j-cartesian) 1-morphisms (resp. 2-morphisms) above. Given $\mathbb{X}\to\mathbb{C}$ and $\mathbb{Y}\to\mathbb{C}$ in $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ we denote by $\mathsf{MAP}_{/(\mathbb{C},\mathfrak{p})}(\mathbb{X},\mathbb{Y})$ the mapping ∞-category in $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$.

Remark 2.0.4. Observe that $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p}_{\flat}^{(i,j)})}$ is simply given by the slice $(\infty,2)$-category $\mathsf{CAT}_{(\infty,2)/\mathbb{C}}$ and that if $\mathfrak{p}=(\mathbb{C},(i,j),E,\sharp)$ then $\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$ is given by $\mathsf{FIB}_{(i,j)}^{E\text{-lax}}(\mathbb{C})$.

Remark 2.0.5. Let $f:\mathbb{C}\to\mathbb{D}$ be functor inducing a map of fibrational patterns $\mathfrak{p}\to\mathfrak{q}$. Then pullback along f induces a functor of $(\infty,2)$-categories

$$f^*:\mathsf{CAT}_{(\infty,2)/(\mathbb{D},\mathfrak{q})}\to\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$$

which we call the base change functor along f.

Definition 2.0.6. Let $\mathfrak{p}=(\mathbb{C},(i,j),E,L)$ be a fibrational pattern and consider a functor $f:\mathbb{C}\to\mathbb{D}$. We define the lax basechange functor as the composite

$$f_{\mathfrak{p}}^{\mathrm{lax}}:\mathsf{CAT}_{(\infty,2)/\mathbb{D}}\to\mathsf{FIB}_{(i,j)}(\mathbb{D})\to\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}$$

where the first map was given in Definition 1.1.1 and the second map is base change along the map $(\mathbb{C},(i,j),E,L)\to(\mathbb{D},(i,j),\sharp,\sharp)$ induced by f.

Definition 2.0.7. Let $\mathfrak{p}=(\mathbb{C},(i,j),E,L)$ be a fibrational pattern and consider a functor $f:\mathbb{C}\to\mathbb{D}$. Then there exists a functor

$$Rf_*:\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}\to\mathsf{FUN}((\mathsf{CAT}_{(\infty,2)/\mathbb{D}})^{\mathrm{op}},\mathsf{CAT}_\infty),\quad(\mathbb{X}\to\mathbb{C})\mapsto\mathsf{MAP}_{/(\mathbb{C},\mathfrak{p})}(f_{\mathfrak{p}}^{\mathrm{lax}}(-),\mathbb{X}).$$

Proposition 2.0.8. *The functor*

$$Rf_*:\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}\to\mathsf{FUN}((\mathsf{CAT}_{(\infty,2)/\mathbb{D}})^{\mathrm{op}},\mathsf{CAT}_\infty)$$

factors through the composite $\mathsf{FIB}_{(i,j)}(\mathbb{D})\to\mathsf{CAT}_{(\infty,2)/\mathbb{D}}\to\mathsf{FUN}((\mathsf{CAT}_{(\infty,2)/\mathbb{D}})^{\mathrm{op}},\mathsf{CAT}_\infty)$ *where the second functor is the Yoneda embedding.*

Proof. ss □

Definition 2.0.9. We will call the functor $f_*:\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}\to\mathsf{FIB}_{(i,j)}(\mathbb{D})$ the *fibrational* pushforward functor.

Theorem 2.0.10. *Let* $\mathfrak{p}=(\mathbb{C},(i,j),E,L)$ *be a fibrational pattern and let* $f:\mathbb{C}\to\mathbb{D}$ *be a functor. Then there exists an adjunction of* $(\infty,2)$*-categories:*

$$f^*:\mathsf{FIB}_{(i,j)}(\mathbb{D})\rightleftarrows\mathsf{CAT}_{(\infty,2)/(\mathbb{C},\mathfrak{p})}:f_*$$

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