

Phys 512 - PS 3

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P1

We can expand the equation for the paraboloid and make a parameter substitution to get a function that is linear on its parameters:

Expand and change parameters

```
In[145]:= z == a ((x - x0)^2 + (y - y0)^2) + z0;  
Expand[%]  
Collect[Collect[%, -2 a], a]  
Collect[  
  Expand[% /. {x0^2 + y0^2 -> (m0 - z0) / a} /. {x0 -> m1 / (-2 a)} /.  
    {y0 -> m2 / (-2 a)} /. a -> m3], m3]  
Out[146]= z == a x^2 + a y^2 - 2 a x x0 + a x0^2 - 2 a y y0 + a y0^2 + z0  
Out[147]= z == a (x^2 + y^2 + x0^2 + y0^2 - 2 (x x0 + y y0)) + z0  
Out[148]= z == m0 + m1 x + m2 y + m3 (x^2 + y^2)
```

The new parameters (m_0, m_1, m_2, m_3) can be written in terms of the old ones (x_0, y_0, z_0, a) as

```
In[149]:= Solve[x0^2 + y0^2 == (m0 - z0) / a, m0]  
Solve[x0 == m1 / (-2 a), m1]  
Solve[y0 == m2 / (-2 a), m2]  
m3 -> a  
Out[149]= {{m0 -> a x0^2 + a y0^2 + z0}}  
Out[150]= {{m1 -> -2 a x0}}  
Out[151]= {{m2 -> -2 a y0}}  
Out[152]= m3 -> a
```

and the old ones can be recovered by substituting:

$$\text{In[157]:= Solve}\left[\left\{\mathbf{x}_0^2 + \mathbf{y}_0^2 == (\mathbf{m0} - \mathbf{z}_0) / \mathbf{a}, \mathbf{x}_0 == \mathbf{m1} / (-2 \mathbf{a}), \mathbf{y}_0 == \mathbf{m2} / (-2 \mathbf{a})\right\}, \left\{\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0\right\}\right]$$

$$\text{Out[157]= } \left\{\left\{\mathbf{x}_0 \rightarrow -\frac{\mathbf{m1}}{2 \mathbf{a}}, \mathbf{y}_0 \rightarrow -\frac{\mathbf{m2}}{2 \mathbf{a}}, \mathbf{z}_0 \rightarrow -\frac{-4 \mathbf{a} \mathbf{m0} + \mathbf{m1}^2 + \mathbf{m2}^2}{4 \mathbf{a}}\right\}\right\}$$

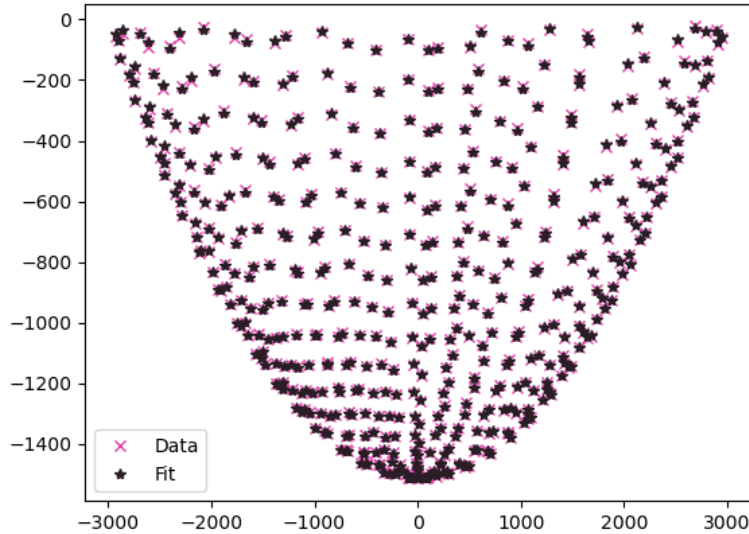
Carrying out the fit, my best-fit new parameters are:

$$\begin{aligned} m0 &= -1512.31182 \\ m1 &= 4.53599028 \times 10^{-4} \\ m2 &= -1.94115589 \times 10^{-2} \\ m3 &= 1.66704455 \times 10^{-4} . \end{aligned}$$

The old parameters are then given by:

$$\begin{aligned} x_0 &= -1.3604886221970875 \\ y_0 &= 58.22147608157978 \\ z_0 &= -1512.8772100367878 \\ a &= 0.00016670445477401347 . \end{aligned}$$

Plotting the fit and the real data, I get:



with an RMS error between fit and data equal to 3.7683386487847277.

The covariance matrix $N = (A^T N^{-1} A)^{-1}$ gives the correlated noise $\langle n_i n_j \rangle$. Since we have $a = m3$, the uncertainty in a is given by $n_3 = \sqrt{\langle n_3 n_3 \rangle}$, where $\langle n_3 n_3 \rangle$ is just the last input of the matrix N . In this way, I get the uncertainty $\delta a = 1.712133743432042 \times 10^{-8}$.

The focal length in the paraboloid is given by

$$f = \frac{1}{4a}$$

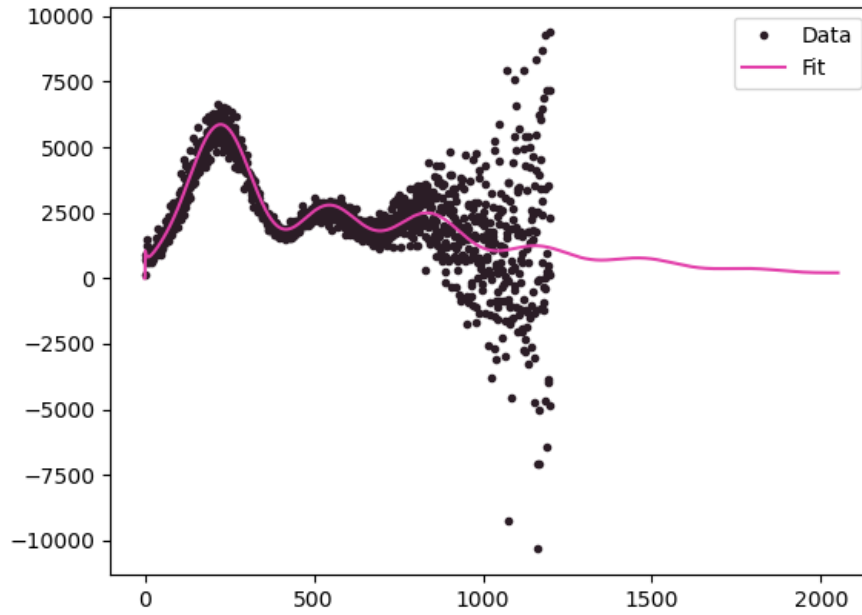
and its uncertainty (by error propagation) is

$$\delta f = \left| \frac{\delta a}{4a^2} \right|.$$

Substituting values, we get $f = 1.49966 \pm 0.00015$ m.

P2

With the given parameters, we get a $\chi^2 = 1588.2376465828454$.



P3

Using Newton's method. With optical depth fixed at $\tau = 0.05$, the best-fit values for the other parameters are:

$$\begin{aligned} H_0 &= 69.3273685 \pm 2.40207892 \\ \omega_b h^2 &= (2.24913957 \pm 0.0539639860) \times 10^{-2} \\ \omega_c h^2 &= 0.113912236 \pm 0.00522680121 \\ A_s &= (2.04250577 \pm 0.0389837370) \times 10^{-9} \\ n_s &= 0.969769787 \pm 0.0135890077 \end{aligned}$$

and the new minimized χ^2 is equal to 1227.935636135719.

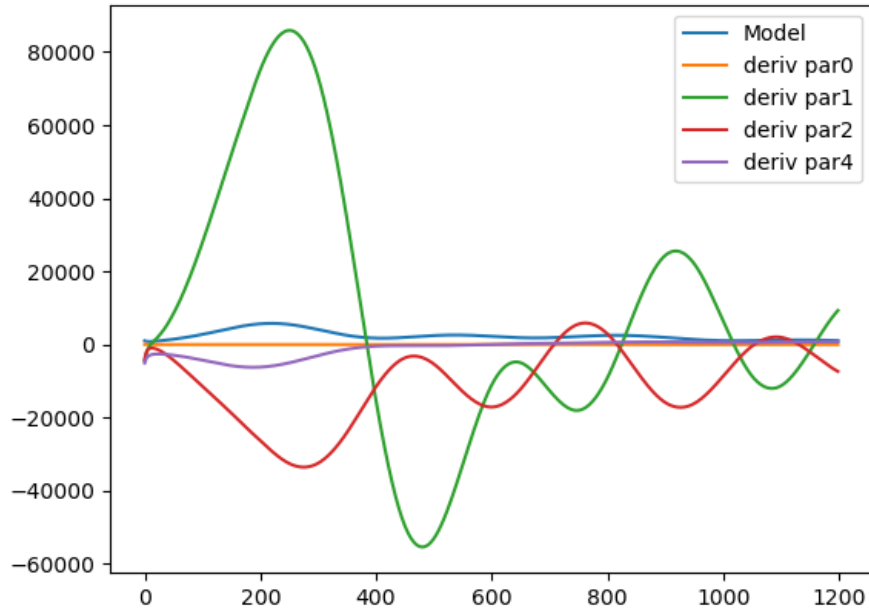


Figure 1: CMB and the derivatives for each of the parameters above, keeping tau constant. The derivative w.r.t. parameter 3 (A_s) was omitted due to its scaling.

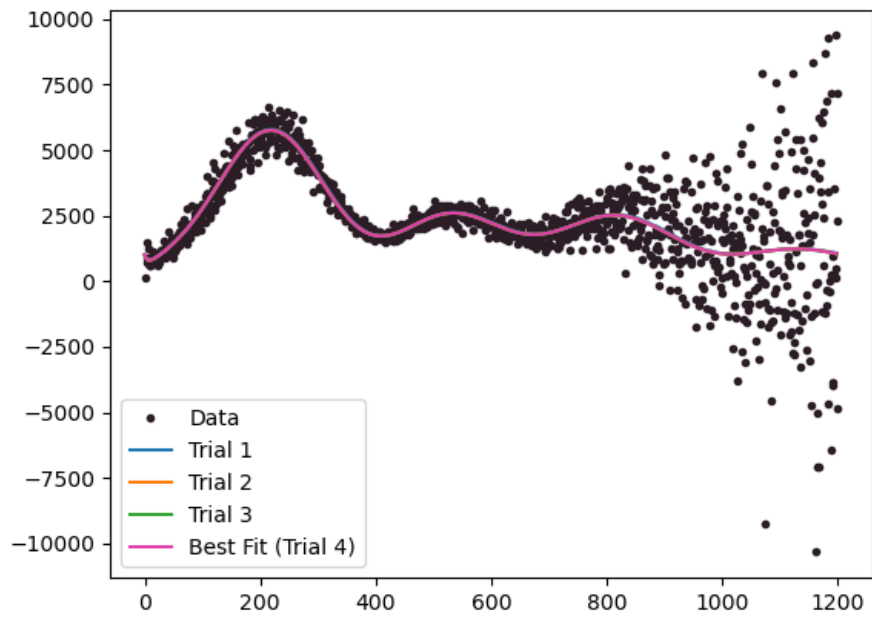


Figure 2: CMB data and 4 fits.

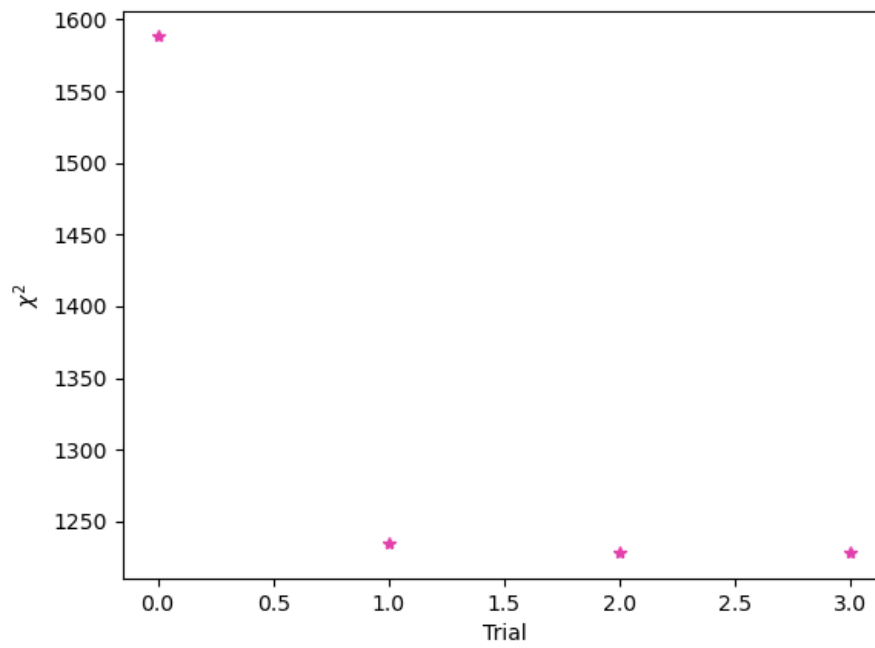


Figure 3: χ^2 for each of the 4 trials.

If I let all parameters float, I get as best-fit values:

$$\begin{aligned}
H_0 &= 68.8015331 \pm 3.25778331 \\
\omega_b h^2 &= (2.23770286 \pm 0.0765761984) \times 10^{-2} \\
\omega_c h^2 &= 0.114822064 \pm 0.00658297176 \\
\tau &= 0.0194766363 \pm 0.150860338 \\
A_s &= (1.92249969 \pm 0.598622072) \times 10^{-9} \\
n_s &= 0.965829151 \pm 0.0228317247
\end{aligned}$$

and χ^2 equal to 1235.5552207546873. These errors and χ^2 are worse than the case above, where τ was constant.

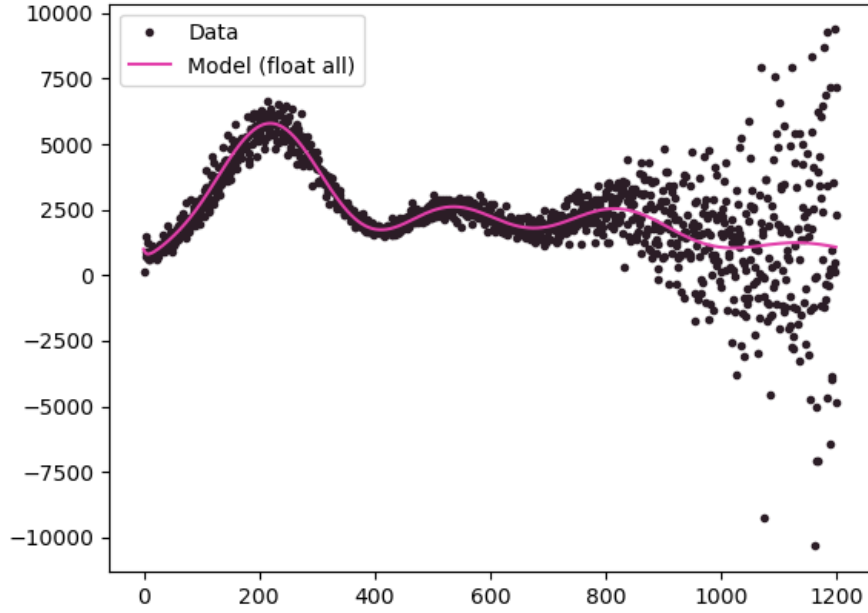


Figure 4: CMB data and fit with all parameters floating.

P4

Using a Markov-chain Monte Carlo with 5000 steps¹, I get the new parameters:

$$\begin{aligned} H_0 &= 70.3112543 \pm 2.62403787 \\ \omega_b h^2 &= (2.26686607 \pm 0.0588006489) \times 10^{-2} \\ \omega_c h^2 &= 0.112163971 \pm 0.00519791688 \\ \tau &= 0.0810869807 \pm 0.0473308810 \\ A_s &= (2.17328497 \pm 0.196466768) \times 10^{-9} \\ n_s &= 0.977929447 \pm 0.0154416237 \end{aligned}$$

and χ^2 equal to 1233.099189381654. Since we could only accept steps that would give positive τ , the actual number of accepted steps was equal to 1784, giving an acceptance rate of 0.3568.

When comparing to the parameters given above for Newton's method floating τ , there is a clear improvement in parameters errors.

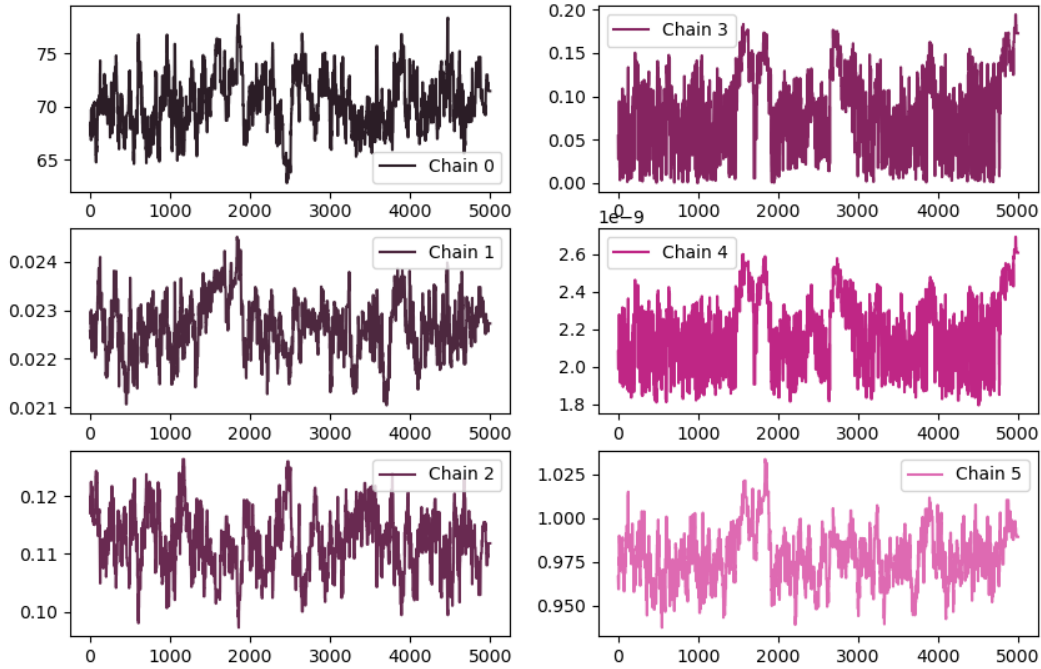


Figure 5: Chains for each parameter.

¹I am doing only 5000 steps and only one chain because I am using a > 10 years old MacBook, and it took around 18h to run this.

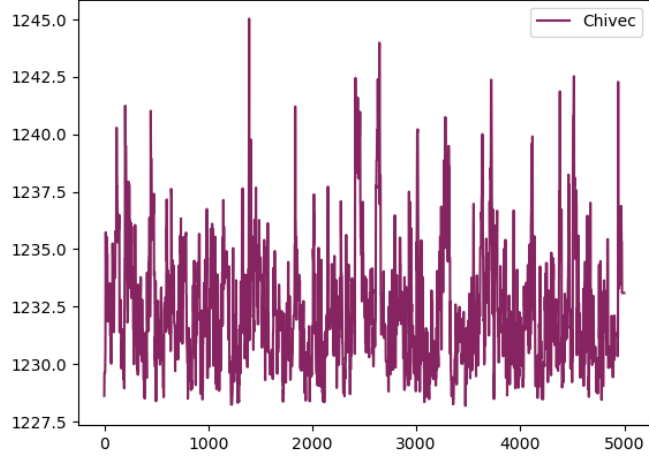


Figure 6: χ^2 evolution during chain.

From the FFTs of the chains (Fig. 7, it seems that the convergence was not very good for this number of steps. I am not sure why I am getting these mirrored spectra.

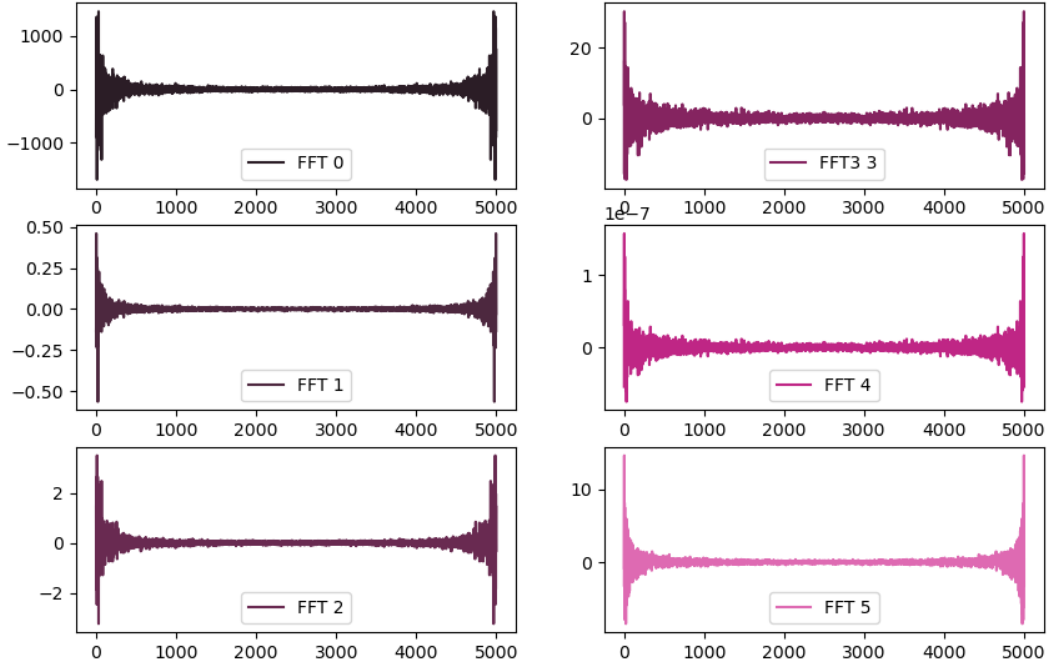


Figure 7: FFTs of chains.