Lattice gauge theories simulations

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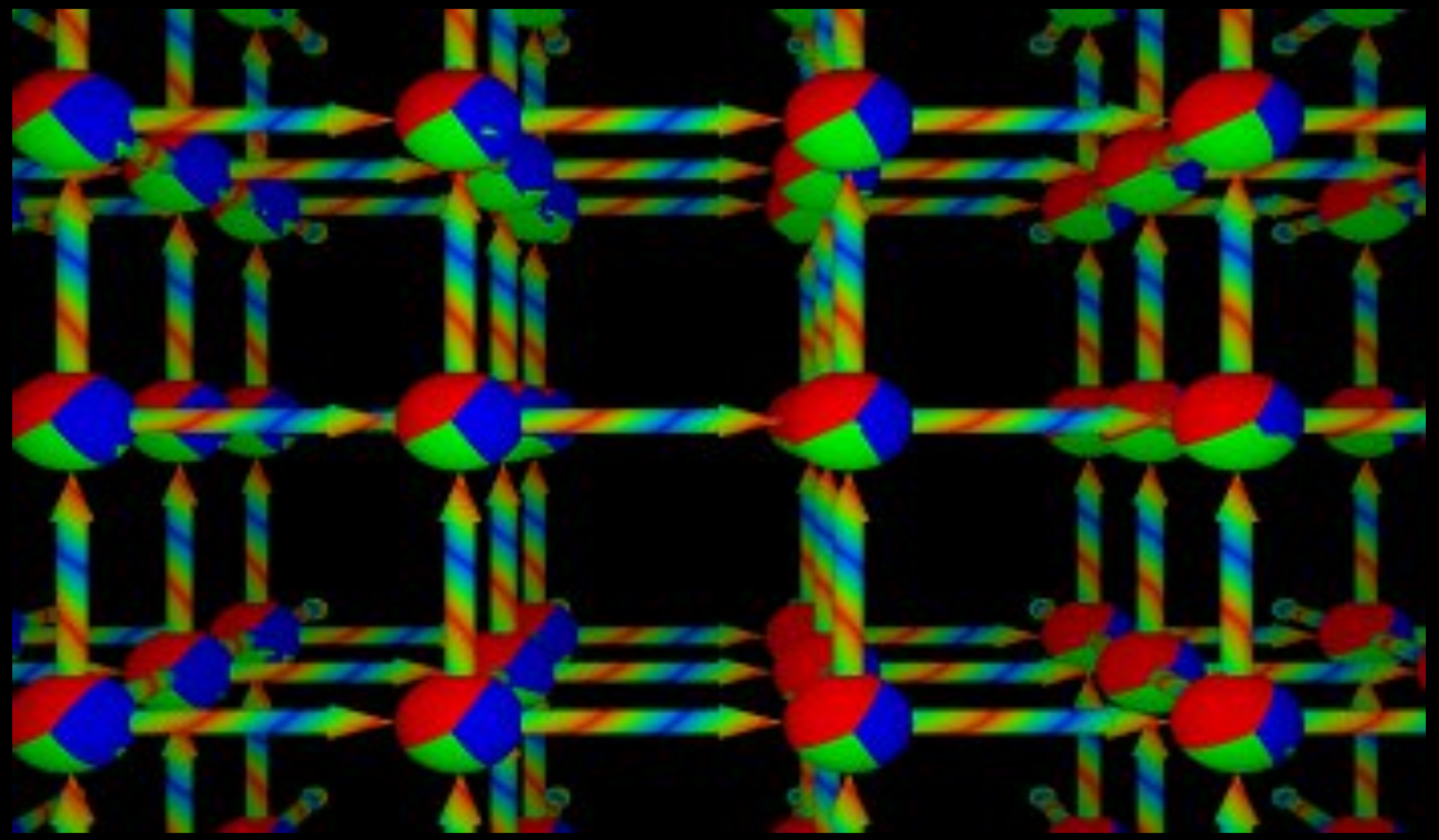
Outline

- Discussion on gauge theories
  - Gauge symmetry in electrodynamics
  - Gauge symmetry in quantum physics
- Wilson's formulation of lattice gauge theories
- Quantum link models of lattice gauge theories
  - U(1) symmetry
  - SU(N) symmetry
- Transformations of quantum operators
- Quantum simulation

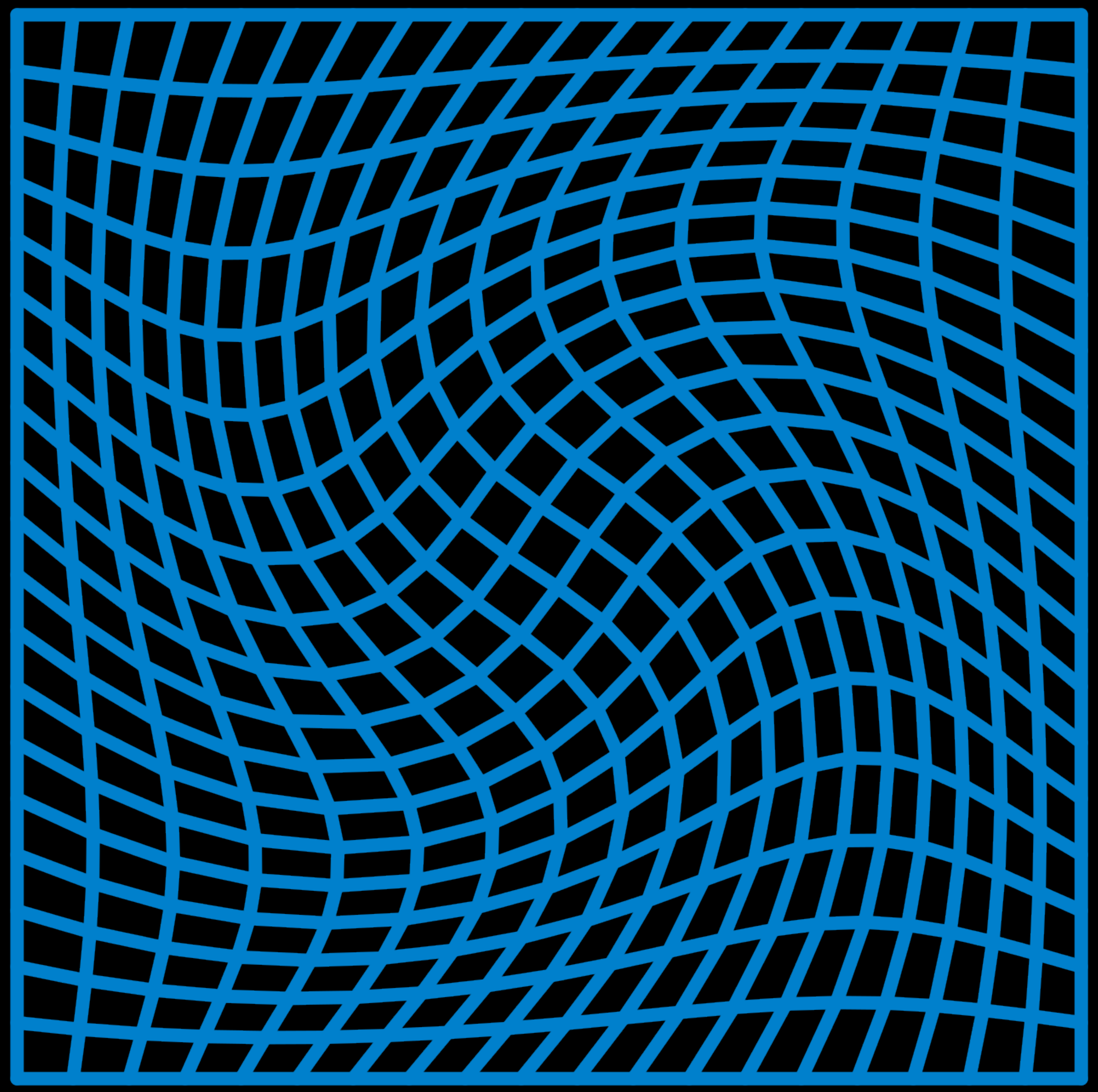
Some starting definitions

- Lattice: ordered array of sites, and links.
- Gauge transformation: A local transformation on the physical objects that leaves dynamics invariant.
- Lattice gauge theory: a physical theory that incorporates gauge invariance (e.g. QED, QCD) on a lattice.
- Quantum Simulations: The processes or algorithms by which a quantum system can reproduce the dynamics of another quantum system.



Why study gauge theories?

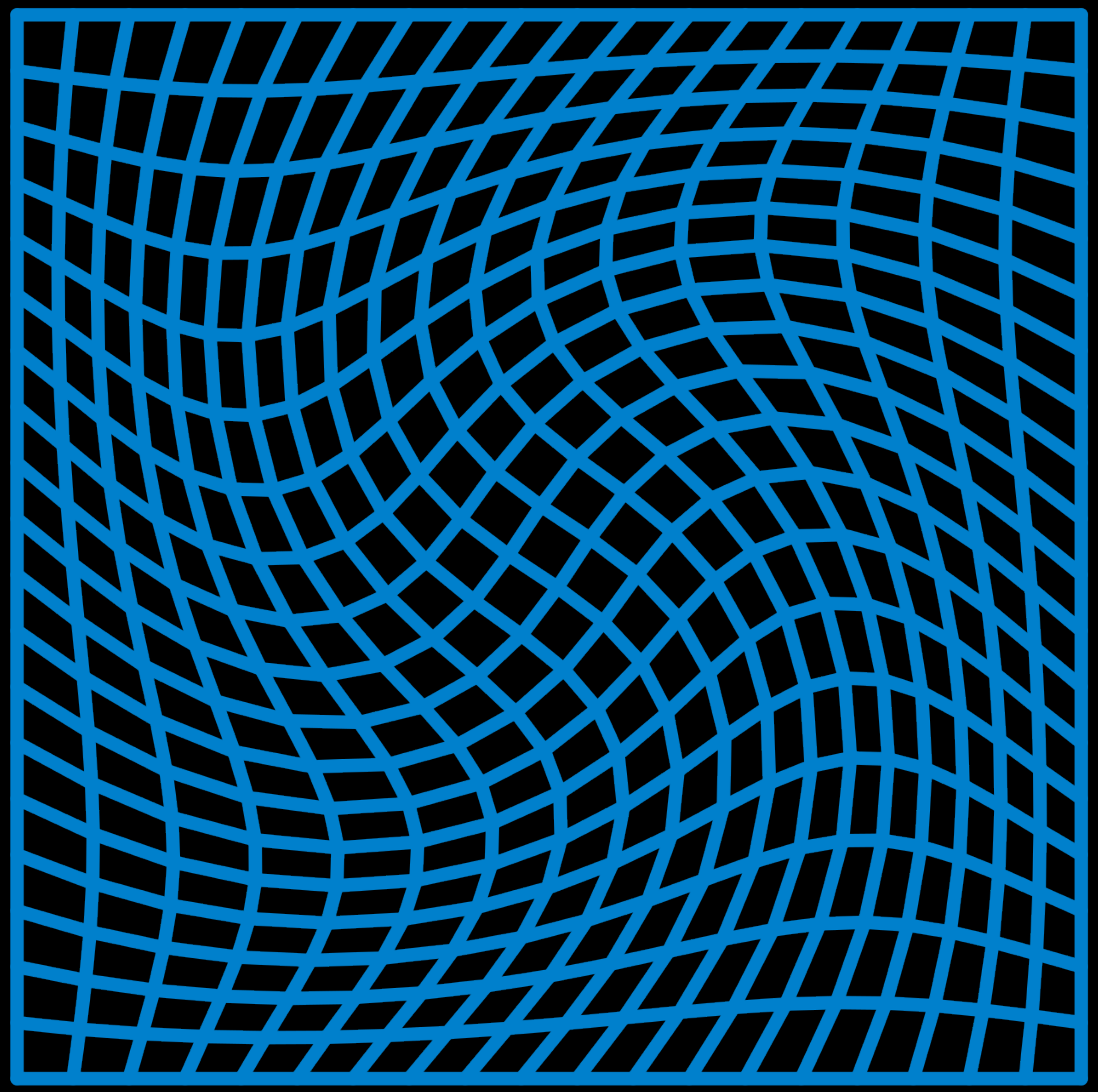
- Dogma: Nature loves gauge symmetries.
- We can find physical descriptions by instead asking: what if the theory can have this certain symmetry? e.g.:
    - Global Poincaré symmetry in rel. QFTs
    - $SU(3) \times SU(2) \times U(1)$ symmetry in the Standard Model



"So, imagine the universe as a big chessboard. I could change every white square on a chessboard to a black square and every black square to a white square and the game would be exactly the same. That's the simple kind of symmetry. Now I can turn it into a gauge symmetry by making it much trickier. I can say, "Let me just change locally, whenever I want, a white square to a black square or a black square to a white square. Not everywhere but place to place." Now the chessboard doesn't look the same at all, so the game can't be the same unless I also have a rule book—a coordinate system for what happens at every point—containing rules for the pieces of the chessboard to follow to keep the game the same"
-Krauss, 2017

(Very) short summary on gauge theories

- A local transformation that leaves physics invariant is a gauge transformation.
- These gauge transformations form a symmetry group.
- Each generator of the symmetry is associated with one bosonic degrees of freedom.



A classical gauge symmetry

- Recall Maxwell's equations in classical Electrodynamics:

$$\nabla \cdot E(t, \mathbf{x}) = \rho(t, \mathbf{x})$$
$$\nabla \cdot B(t, \mathbf{x}) = 0$$
$$\nabla \times E(t, \mathbf{x}) + \partial_t B(t, \mathbf{x}) = 0$$
$$\nabla \times B(t, \mathbf{x}) - \partial_t E(t, \mathbf{x}) = j(t, \mathbf{x})$$

- We can introduce a vector potential, such that we conveniently solve 2 of the equations "automatically"

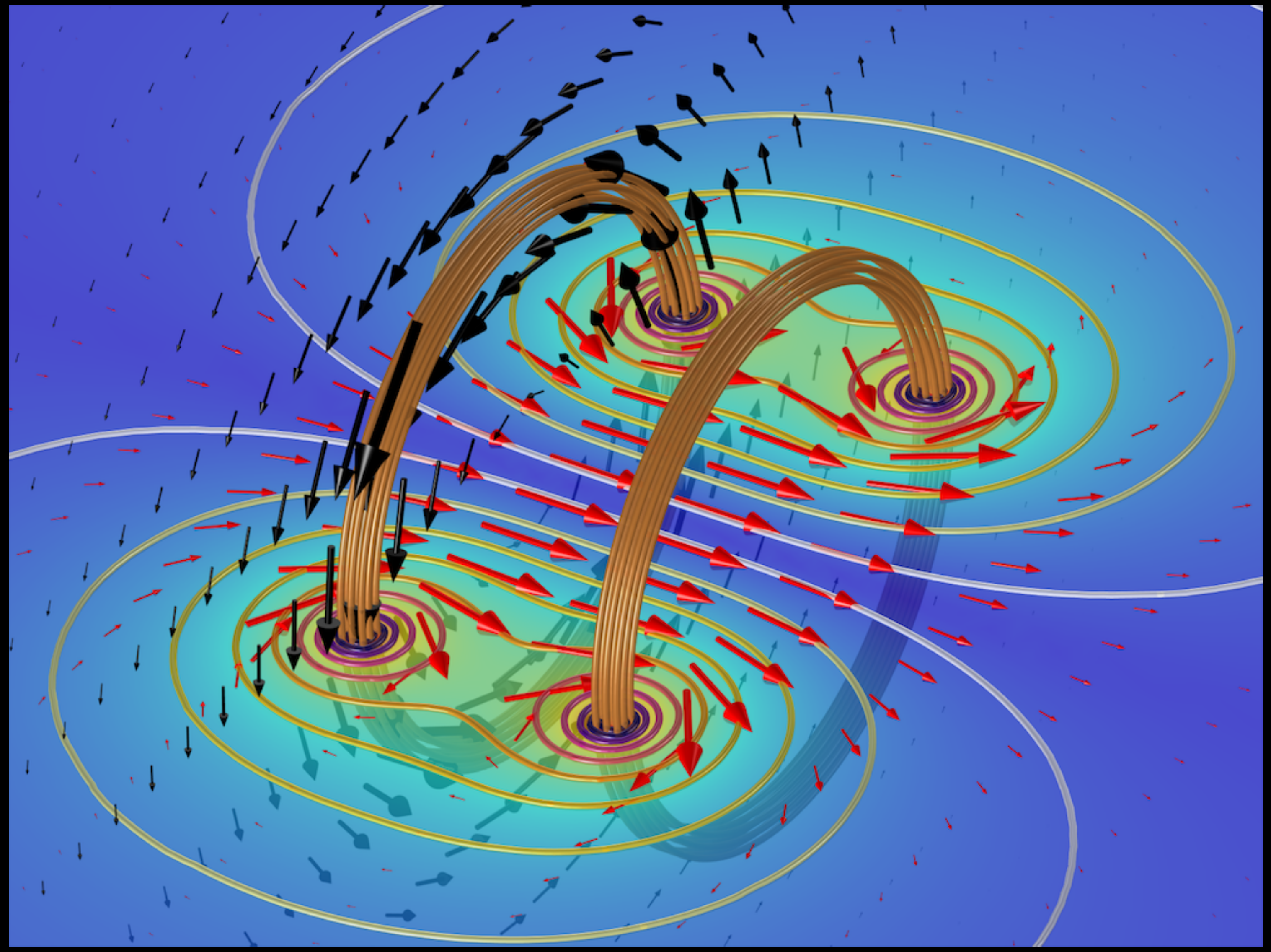
$$E = -\nabla \Phi - \partial_t \mathbf{A}$$
$$B = \nabla \times \mathbf{A}$$

- However, we end up paying the price of adding redundancy to the theory.

A classical gauge symmetry

- There is a gauge transformation:

$$\mathbf{A}(t,\mathbf{x}) \longmapsto \mathbf{A}(t,\mathbf{x}) - e\,\nabla\alpha(t,\mathbf{x})$$
$$\Phi(t,\mathbf{x}) \longmapsto \Phi(t,\mathbf{x}) + e\,\partial_t\alpha(t,\mathbf{x})$$
$$A_\mu(x) \longmapsto A'_\mu(x) = A_\mu(x) + e\,\partial_\mu\alpha(x)$$



- The redundancy is removed in a gauge fixing process to obtain a physical solution.
- Although at this moment it is a classical example, we will see that the photon field in QED transforms precisely like this.

Dirac free electron

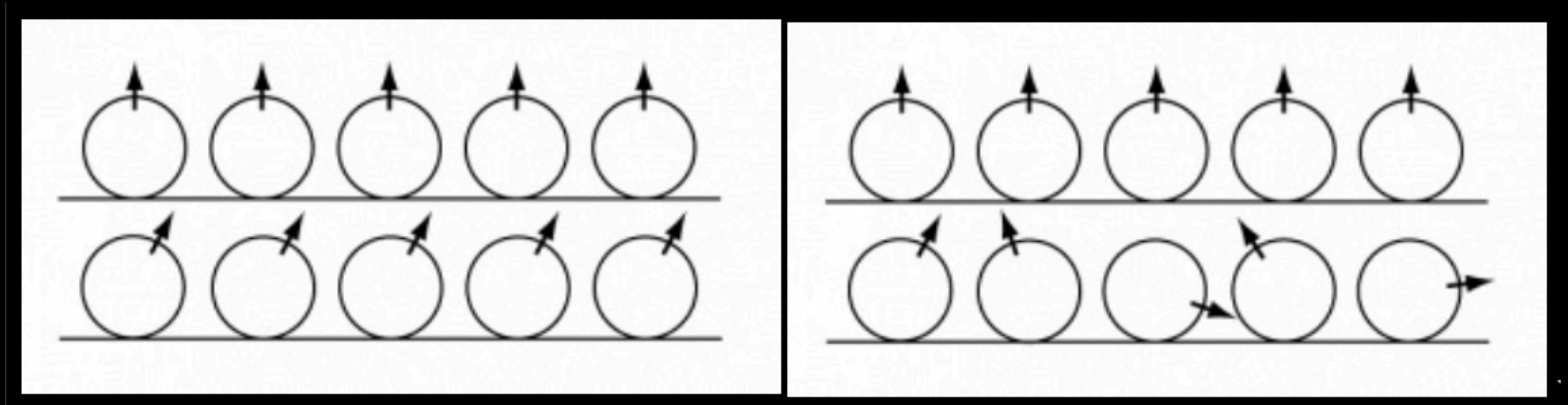
- Consider now the Lagrangian of the free electron

$$\mathcal{L} = \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x)$$

- Notice that the system posses $U(1)$ global symmetry

$$\psi(x) \mapsto \psi'(x) = \exp(ie\alpha)\psi(x)$$

But what if we force it to have $U(1)$ local symmetry?



Global Gauge Transformation　Local Gauge Transformation

Dirac free electron

- A $U(1)$ local (gauge) transformation is

$$\psi(x) \mapsto \psi'(x) = \exp(ie\alpha(x))\psi(x)$$

$$\not{\partial}\psi(x) \mapsto e^{ie\alpha(x)}\not{\partial}\psi(x) + (\not{\partial}e^{ie\alpha(x)})\psi(x)$$

- Define a covariant derivative and absorb extra terms into f_μ:

$$D_\mu = \partial_\mu + f_\mu(x)$$

- The question is now to define f_μ in a convenient way such that

$$\not{D}\psi \mapsto e^{ie\alpha}\not{D}\psi$$

How does the gauge field transforms?

- We observe the following:

$$D'_\mu \psi' = e^{ie\alpha(x)} D_\mu \psi(x)$$

$$(\partial_\mu + f'_\mu(x))(e^{ie\alpha(x)}\psi(x)) = e^{ie\alpha(x)}(\partial_\mu + f_\mu(x))\psi(x)$$

$$(\partial_\mu e^{ie\alpha(x)})\psi(x) + f'_\mu(x)e^{ie\alpha(x)}\psi(x) = f_\mu(x)e^{ie\alpha(x)}\psi(x)$$

$$f'_\mu(x)\psi(x) = (f_\mu(x) - ie\partial_\mu\alpha(x))\psi(x)$$

- Meaning that we can absorb the extra terms by making f_μ transform as

$$f_\mu(x) \mapsto f'_\mu(x) = f_\mu(x) - ie\,\partial_\mu\alpha(x)$$

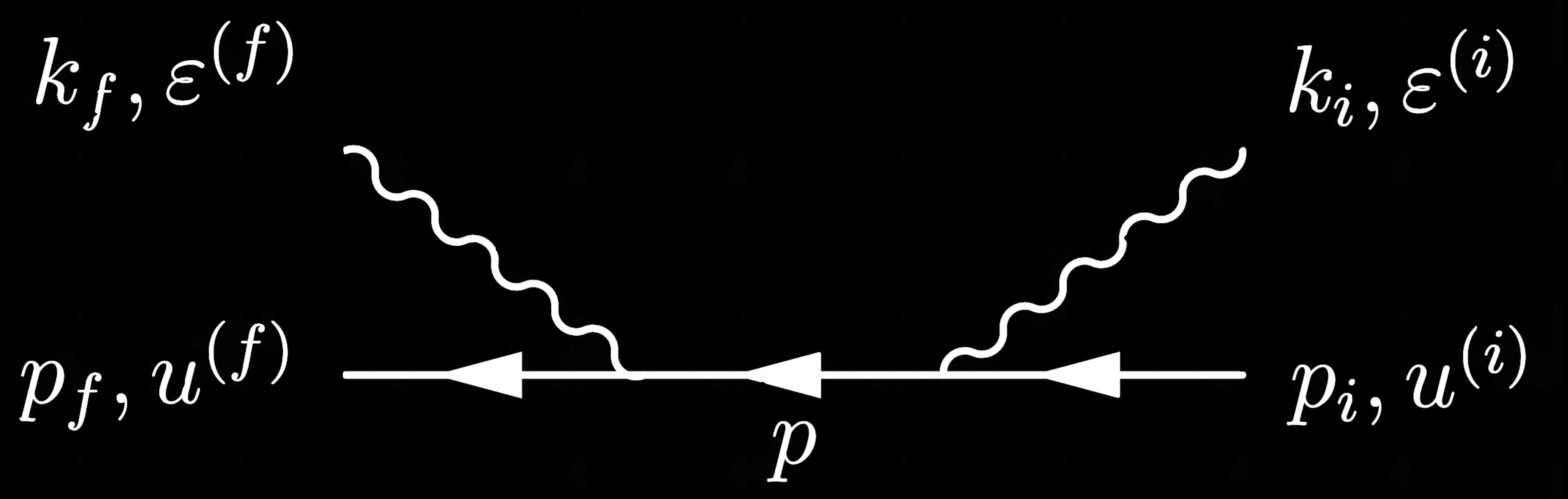
- Rescaling the function, we find that our gauge field transforms like the classical electromagnetic field:

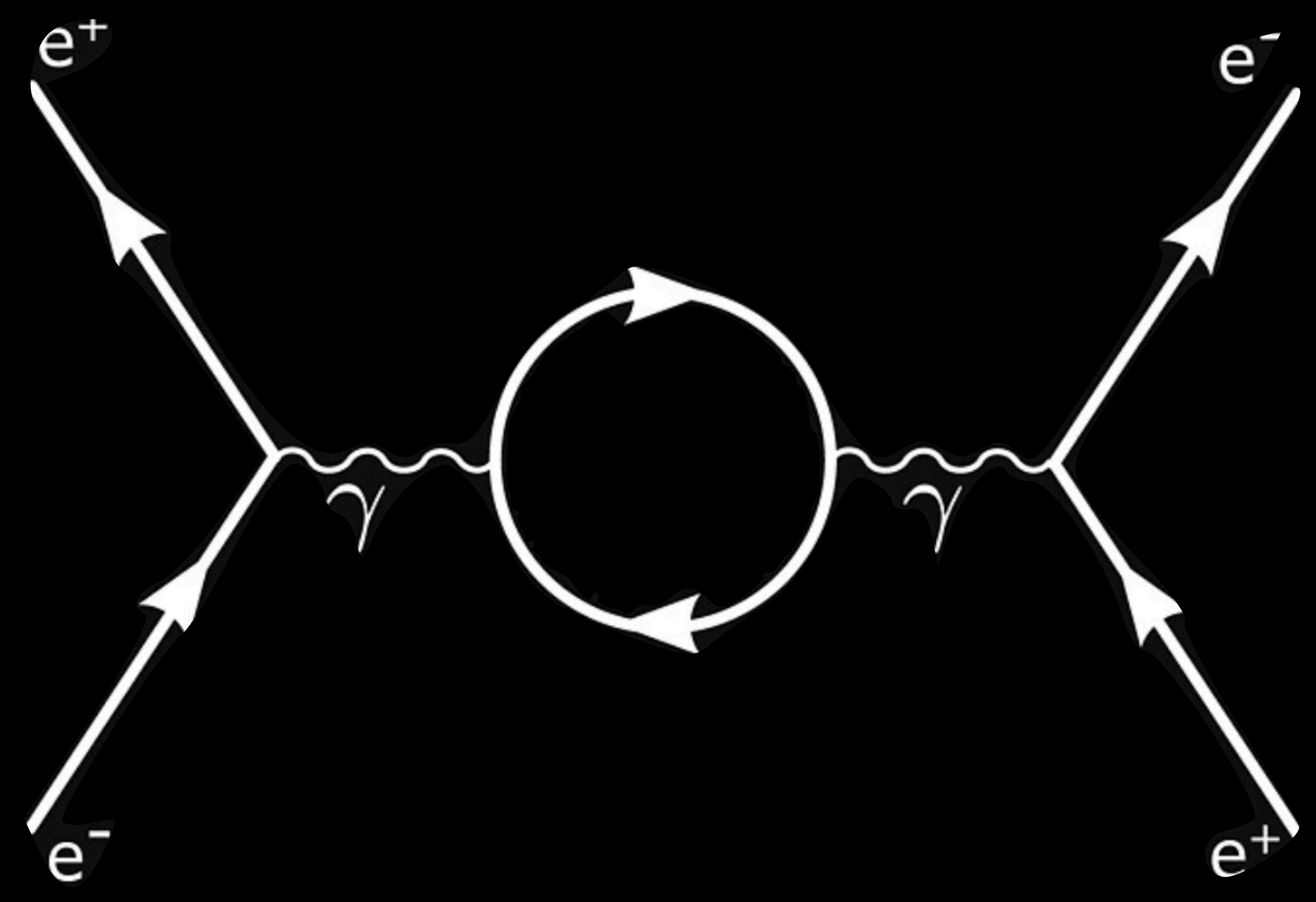
$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + e\,\partial_\mu\alpha(x)$$

QED

- From imposing $U(1)$ gauge symmetry, we recovered the Lagrangian term for an electron in an electromagnetic field. Adding to it the term of the field itself, we can write QED's Lagrangian:

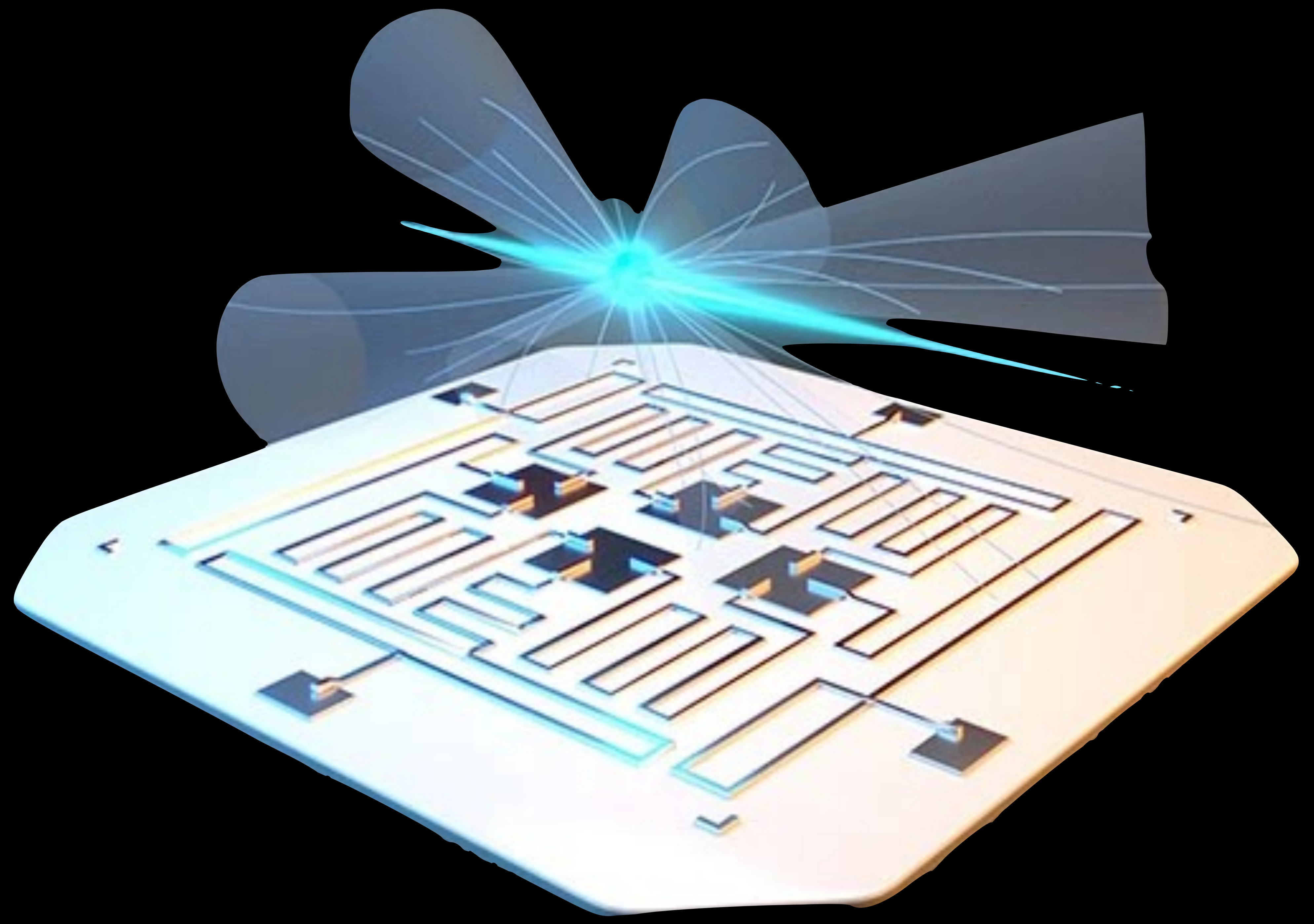
$$\mathcal{L} = \overline{\psi}(i\not{D} - m) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$





Why study lattice gauge theories?

- More ameanable to simulations.
- Lattice parameters provide the theory with a cutoff and regulatization procedure.
- Quantum simulation is arguably the most natural way of simulating a quantum theory.



Recipe for discretizing a theory
on the board

Free fermions in a lattice

- A lattice with staggered fermions can be implemented:

$$\mathcal{H}_{\text{free}} = -t \sum_{\langle xy \rangle} s_{xy} \left(\psi_x^\dagger \psi_y + \psi_y^\dagger \psi_x \right) + m \sum_x s_x \psi_x^\dagger \psi_y$$

$$s_x = (-1)^{x_1+\cdots+x_d} \quad \text{for particles/anti-particles checkered pattern}$$

$$s_{xy} = (-1)^{x_1+\cdots+x_{k-1}} \quad \text{for links in } k^{\text{th}} \text{ direction}$$

- Where s_x realizes the role of γ^0 , and s_{xy} realises the role of $\gamma^0\gamma^k$ for the staggered sites.

How to incorporate the electromagnetic field?

Wilson's formulation

- The gauge field on the lattice can be encoded as

$$\hat{A}_{xy} = \int_{x_k}^{y_k} \mathrm{d}x \hat{A}_k(x)$$

- So we can define the transport and electric field operators

$$U_{xy} = \exp\{ieA_{xy}\}$$

$$E_{xy} = -i\ \partial/\partial aeA_{xy}$$

- Which are defined this way to ensure the following commutation relations:

$$[E_{xy}, U_{x'y'}] = \delta_{xx'}\delta_{yy'}U_{xy},$$
$$\left[E_{xy}, U^{\dagger}_{x'y'}\right] = -\delta_{xx'}\delta_{yy'}U^{\dagger}_{xy}.$$

Wilson's formulation

- The Magnetic field enters by considering Stokes law on the lattice's smallest finite loop

$$\iint_S \mathrm{d}\mathbf{S} \cdot (\nabla \times \mathbf{A}) = \oint_{\partial S} \mathrm{d}\mathbf{r} \cdot \mathbf{A}$$

- This loop integral can be realized discretely by stacking the transport operator

$$U_\square = U_{wx} U_{xy} U_{zy}^\dagger U_{wz}^\dagger$$
$$U_\square^\dagger = U_{wz} U_{zy} U_{xy}^\dagger U_{wx}^\dagger$$

- Such that the magnetic field energy density takes the form

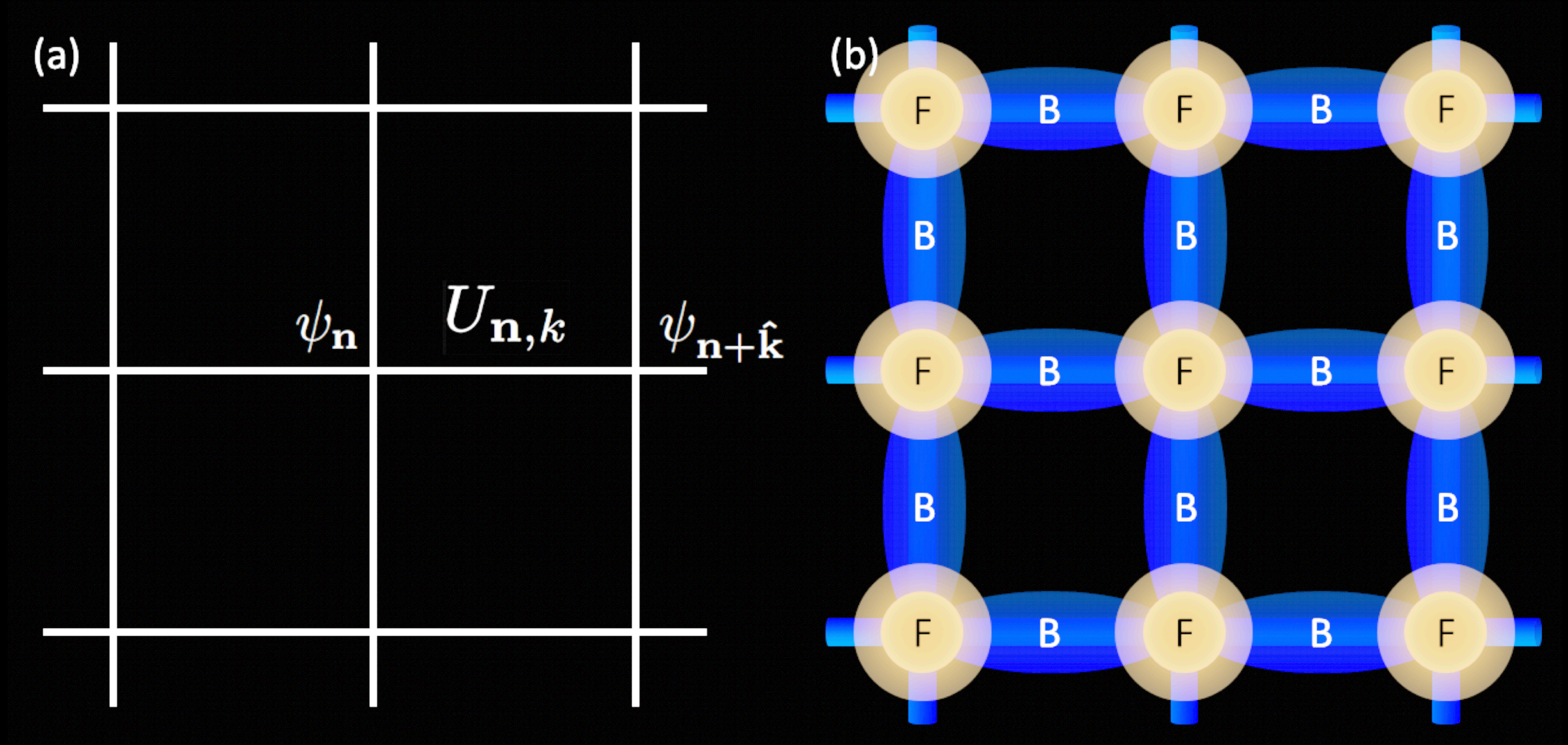
$$-\left(U_\square + U_\square^\dagger\right) = -2\cos\left(e \oint_\square \mathrm{d}\mathbf{r} \cdot \mathbf{A}\right) = -2 + \left(e \oint_\square \mathrm{d}\mathbf{r} \cdot \mathbf{A}\right)^2 = -2 + \left(e \left\langle B \right\rangle_\square\right)^2$$

Wilson's formulation

- The final Wilson QED Hamiltonian is thus written as

$$\mathcal{H}_{QED} = -t \sum_{\langle xy \rangle} s_{xy} \left(\psi_x^\dagger U_{xy} \psi_y + \psi_y^\dagger U_{xy}^\dagger \psi_x \right) + m \sum_x s_x \psi_x^\dagger \psi_x +$$

$$+ \frac{e^2}{2} \sum_{\langle xy \rangle} E_{xy}^2 - \frac{1}{4e^2} \sum_{\square} \left(U_\square + U_\square^\dagger \right)$$



- Notice it is gauge invariant, where the gauge transformation is

$$U'_{xy} = \Omega_x U_{xy} \Omega_y^\dagger, \quad \text{for} \quad \Omega_x = e^{iea(x)} \in U(1)$$

Wilson’s formulation

- Furthermore, we can write down the generator of the gauge symmetry

$$G_x = \psi_x^\dagger \psi_x + \sum_k \left(E_{x,x+\hat{k}} - E_{x-\hat{k},x} \right), \quad [\mathcal{H}, G_x] = 0.$$

- This generator commutes with the Hamiltonian, so it is a conservation law. Notice this conservation law is precisely Gauss’ law for the electric field.

- For completeness, a general gauge transformation can be written down with the generator as

$$V = \prod_x \exp(ie\alpha_x G_x),$$

$$V\psi_x V^\dagger = \Omega_x \psi_x, \quad V\psi_x^\dagger V^\dagger = \psi_x^\dagger \Omega_x^\dagger, \quad VU_{xy}V^\dagger = \Omega_x U_{xy} \Omega_y^\dagger,$$

$$\Omega_x = \exp(ie\alpha_x).$$

The Hilbert space in each link and site from Wilson’s formulation is unbounded.

Is there a generalization that allows us to use operators with a bounded Hilbert space?

QLM in (N+1) dimensions with U(1)

- The continuous $U(1)$ operators are recast into spin operators in the following way:

$$U_{xy} \mapsto S^1_{ij} + iS^2_{ij} = S^+_{ij},$$
$$U^\dagger_{xy} \mapsto S^1_{ij} - iS^2_{ij} = S^-_{ij},$$
$$E_{xy} \mapsto S^3_{ij}.$$

- The spin operators recover the commutation relations from $\left[E_{xy}, U_{x'y'}\right] = \delta_{xx}\delta_{yy}U_{xy}$.

- However, we do not recover $\left[U_{xy}, U^\dagger_{x'y'}\right] = 0$. Instead:

$$\left[S^+_{ij}, S^-_{i'j'}\right] = 2\delta_{ii'}\delta_{jj'}S^3_{ij}$$

QLM in (N+1) dimensions with U(1)

- The fermions on the lattice sites are fermionic creation and annihilation operators (only a change in notation):

$$\psi_x \mapsto c_i$$

$$\psi_y^\dagger \mapsto c_j^\dagger$$

- Thus, the Hamiltonian terms dependent on the fermions are rewritten as

$$H_t = -t \sum_{\langle ij \rangle} \left[c_i^\dagger S_{ij}^+ c_j + \text{h.c.} \right],$$

$$H_m = m \sum_i (-1)^i c_i^\dagger c_i.$$

QLM in (N+1) dimensions with U(1)

- The Hamiltonian terms coming from the fields are rewritten as

$$H_E = \kappa \sum_{\langle ij \rangle} (S^3_{ij})^2,$$

$$H_\square = -J \sum_\square \left[S^+_1 S^+_2 S^-_3 S^-_4 + \text{h.c.} \right].$$

- The final lattice QED Hamiltonian for a quantum link model is thus

$$H_{LQED} = H_t + H_m + H_E + H_\square$$

QLM in (N+1) dimensions with U(1)

- The symmetry on this system has the generator

$$G_i = c_i^\dagger c_i + \frac{1}{2}\sum_{\langle ij\rangle}\left(S_{ij}^3 + \mathrm{h.c.}\right), \quad [H, G_i] = 0.$$

- Where we observe that G_i separates the gauge invariant states in the Hilbert space:

$$\mathcal{H}_G = \{|\psi\rangle \mid G_i\,|\psi\rangle = 0 \quad \forall\, i\}$$

- For $|\psi\rangle$ being the many-body wavefunction of all sites and links in the lattice. Only states that comply with the gauge symmetry are physical.

Insight for the (1+1) U(1) lattice gauge theory on the board

QLM in (1+1) dimensions with SU(N)

- The complete Hamiltonian for the SU(N) quantum link model can be written as

$$H_{SU(N)} = -t \sum_j \left[\left(\psi_j^{\alpha\dagger} c_{R;j}^{\alpha} \right) \left(c_{L;j+1}^{\beta\dagger} \psi_{j+1}^{\beta} \right) + \text{h.c.} \right] + m \sum_j (-1)^j \ \psi_j^{\alpha\dagger} \psi_j^{\alpha}$$
$$+ \frac{g^2}{2} \sum_j \left[L_j^{\alpha} L_j^{\alpha} + R_j^{\alpha} R_j^{\alpha} \right] + \epsilon \sum_j \left(\prod_{\kappa=1}^{N} c_{R;j}^{\kappa\dagger} c_{L;j+1}^{\kappa} + \text{h.c.} \right)$$

- Each of the $N^2 - 1$ symmetry generators implies a conservation law, and thus restrict the Hilbert space of physical quantum states:

$$G_j^{\gamma} \left| \psi \right\rangle = 0, \quad \forall\, j, \ \forall \gamma = \{1, 2, \ldots, N^2 - 1\}.$$

QLM in (1+1) dimensions with SU(N)

- The $N^2 - 1$ generators of the symmetry are

$$G_j^\gamma = \psi_j^{\alpha\dagger} \lambda^\gamma_{\alpha\beta} \psi_j^\beta + L_j^\gamma + R_j^\gamma$$

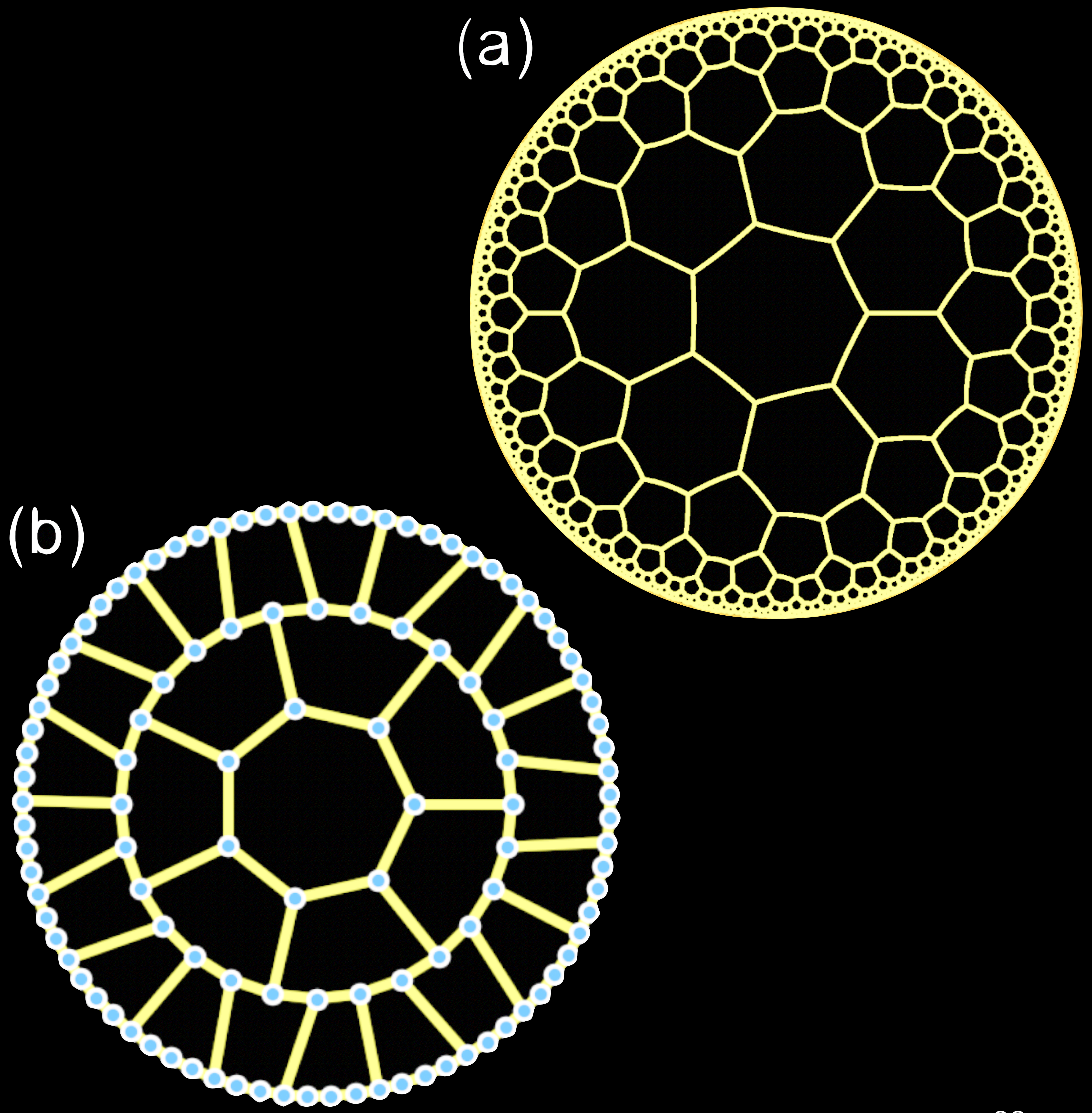
- $\lambda^\gamma_{\alpha\beta}$ are the elements of the $SU(N)$ Gell-Mann matrices. L_j^γ and R_j^γ the non-Abelian flux operators:

$$L_j^\gamma = c_{L;j}^{\alpha\dagger} \lambda^\gamma_{\alpha\beta} c_{L;j}^\beta$$
$$R_j^\gamma = c_{R;j}^{\alpha\dagger} \lambda^\gamma_{\alpha\beta} c_{R;j}^\beta$$

- These flux operators are analogous to the electric field in the $U(1)$ symmetry case.

Quantum Simulations

- Overcome limitations of classical algorithms by simulating directly in a quantum system.
- We have transformations to rewrite Hamiltonians in terms of different operators.
- We have algorithms to simulate the time evolution of a theory by splitting the Hamiltonian.
- Allows us to simulate general theories and study them.



Holstein-Primakoff transformation

- To reproduce expectation values of bosonic wavefunctions with up to $N = 2s + 1$ spin-s terms.

- The vacuum is made equivalent to the $|s, m = +s\rangle$ spin state, then the Fock space is approximated via

$$|s, s-n\rangle \mapsto \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n \, |0\rangle_B$$

- The equivalence between creation & annihilation operators and spin operators becomes

$$S^+ = S^1 + iS^2 = \sqrt{2s}\,\sqrt{1 - \frac{a^\dagger a}{2s}}\, a,$$
$$S^- = S^1 - iS^2 = \sqrt{2s}\, a^\dagger \sqrt{1 - \frac{a^\dagger a}{2s}},$$
$$S^3 = (s - a^\dagger a).$$

Holstein-Primakoff transformation

- Furthermore, we recover the commutation relations

$$
\begin{aligned}
\left[S^+, S^-\right] &= 2S^3,\\
\left[a, a^\dagger\right] &= 1,\\
[a, a] = \left[S^+, S^+\right] &= 0.
\end{aligned}
$$

Procedure
on the board

Jordan-Wigner transformation

- To transform spin operators into fermionic creation & annihilation operators and viceversa.
- We need to ensure

$$\{a_i^\dagger, a_j\} = \delta_{ij}$$

$$\{a_i, a_j\} = 0,$$

$$\{a_i^\dagger, a_j^\dagger\} = 0.$$

- This can be achieved by writting the operators in term of spin-$1/2$ operators

$$a_j^\dagger = \exp\left[+i\pi \sum_{k=1}^{j-1} S_k^+ S_k^-\right] S_j^+,$$

$$a_j = \exp\left[-i\pi \sum_{k=1}^{j-1} S_k^+ S_k^-\right] S_j^-,$$

$$a_j^\dagger a_j = S_j^+ S_j^-.$$

Jordan-Wigner transformation

- Inversely, we can write the spin operators in terms of fermionic operators

$$S_j^+ = \exp\left[-i\pi \sum_{k=1}^{j-1} a_k^\dagger a_k\right] a_j^\dagger,$$

$$S_j^- = \exp\left[+i\pi \sum_{k=1}^{j-1} a_k^\dagger a_k\right] a_j,$$

$$S_j^3 = 2a_j^\dagger a_j - 1.$$

- Notice that the tranformation is not local (for the interested reader: it is also a ’t Hooft operator).

Rewriting a (1+1) QLM with spin operators on the board

Product formula expansion

- Inspire definition from Zassenhaus' formula

$$e^{t(X+Y)} = e^{tX}\, e^{tY}\, e^{-t^2[X,Y]/2}\, e^{t^3(2[Y,[X,Y]]+[X,[X,Y]])/6} \ldots$$

- Decompose Hamiltonian into a sum of L independent (non-commuting) terms

$$H = \sum_{j=1}^{L} \alpha_j H_j$$

- Approximate the time evolution in r steps, where each term of the Hamiltonian is done sequentially

$$\exp(-itH) \approx \left[\prod_{j=1}^{L} \exp\left(-\frac{it}{r}\alpha_j H_j\right)\right]^r$$

- In limit $r \to \infty$ this becomes an equality (Suzuki-Trotter expansion).

Conclusions

- Gauge theories are useful for modelling the fundamental processes in nature.
- There is a way that allows us to discretize continuous physical models to a lattice: Wilson’s formulation.
- With quantum link models, we can model lattice gauge theories using sites and links with bounded Hilbert spaces.
- There is a transformation that can approximate bosonic creation & annihilation operators with spin operators.
- There is a transformation that maps fermionic creation & annihilation operators to spin operators, and viceversa.
- With quantum simulations, we could be able to study general lattice gauge theories.

Thank you