Stable Legendre-Lorentzian Solitons with Bounded External Potential Fields

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Outline

- Non-Linear Schrödinger’s Equation
- Lorentzian Solitons Proposition
- Physical Arguments that Justify their Existence
- Solution and Properties
- Simulations of Propagations
- Stability Analysis
- Stable Regime

Non-Linear Schrödinger's Equation

- Typical definition:

$$i\frac{\partial}{\partial z}U = -\frac{1}{2}\frac{\partial^2}{\partial x^2}U + \kappa|U|^2U \qquad (1)$$

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$$\left(i\frac{\partial}{\partial z} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + |U(x,z)|^2 + V(x)\right)U(x,z) = 0, \tag{2}$$

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  - Computational methods
  - Ansatz approach
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Be careful with possible singular points!

Lorentzian Solitons Proposition

- Soliton is apodised with a Lorentzian function:

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$$\frac{u''(x)}{u(x)} = \frac{\left(b + x^2\right)^2 f''(x) - 4x\left(b + x^2\right) f'(x)}{\left(b + x^2\right)^2 f(x)} - \frac{2\left(b - 3x^2\right)}{\left(b + x^2\right)^2} \tag{6}$$

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Be careful with possible singular points! (again)

Lorentzian Solitons Proposition

- "Leap of faith":

$$\frac{\left(b+x^2\right)^2 f''(x) - 4x\left(b+x^2\right) f'(x)}{\left(b+x^2\right)^2 f(x)} = h(x) = \frac{-c}{\left(b+x^2\right)^2} \tag{7}$$

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- We thus reduce the problem of finding Soliton solutions from equation (2), to simply solving an ODE:

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Looks Legendre-ish!

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$$u(x) = a(b+x^2)^{1/2}\left[c_1 \mathrm{P}_2^m\left(\frac{ix}{\sqrt{b}}\right) + c_2 \mathrm{Q}_2^m\left(\frac{ix}{\sqrt{b}}\right)\right] \tag{9}$$

Evaluated at the imaginary axis!?

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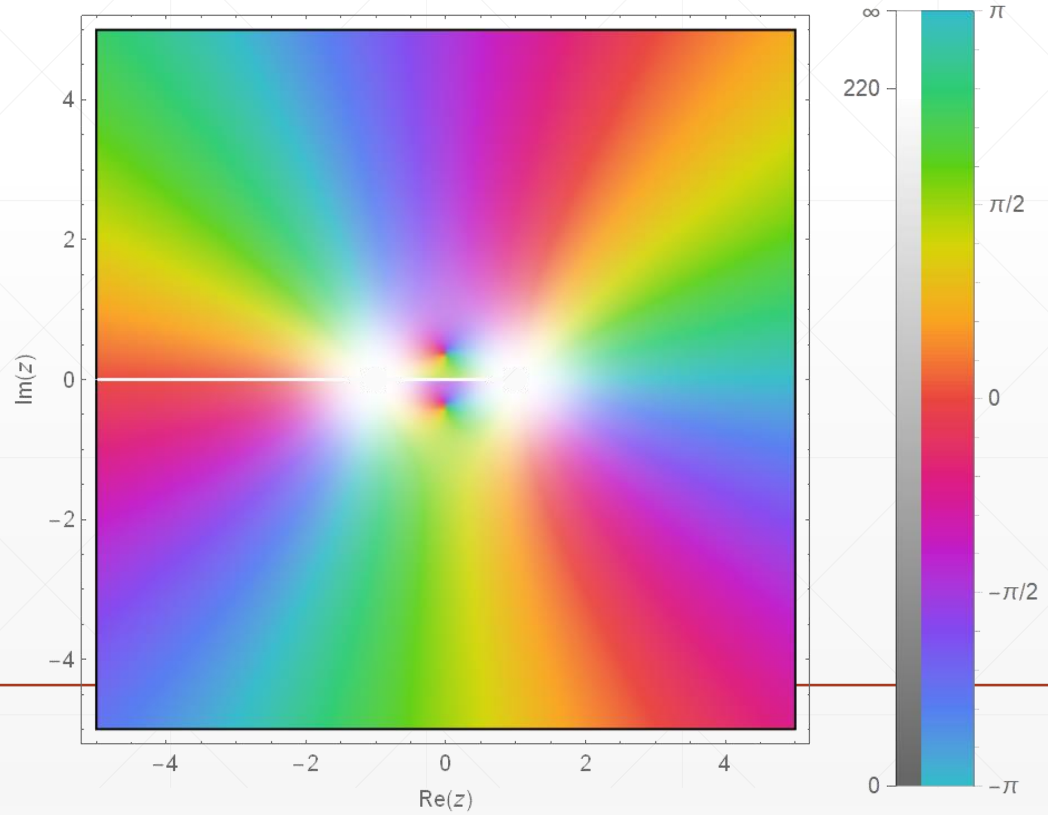
- We use the Ferrers, as they're well defined along the imaginary axis (contrary to Legendre ones).

Legendre's Q(z), {l,m} = {2,5}



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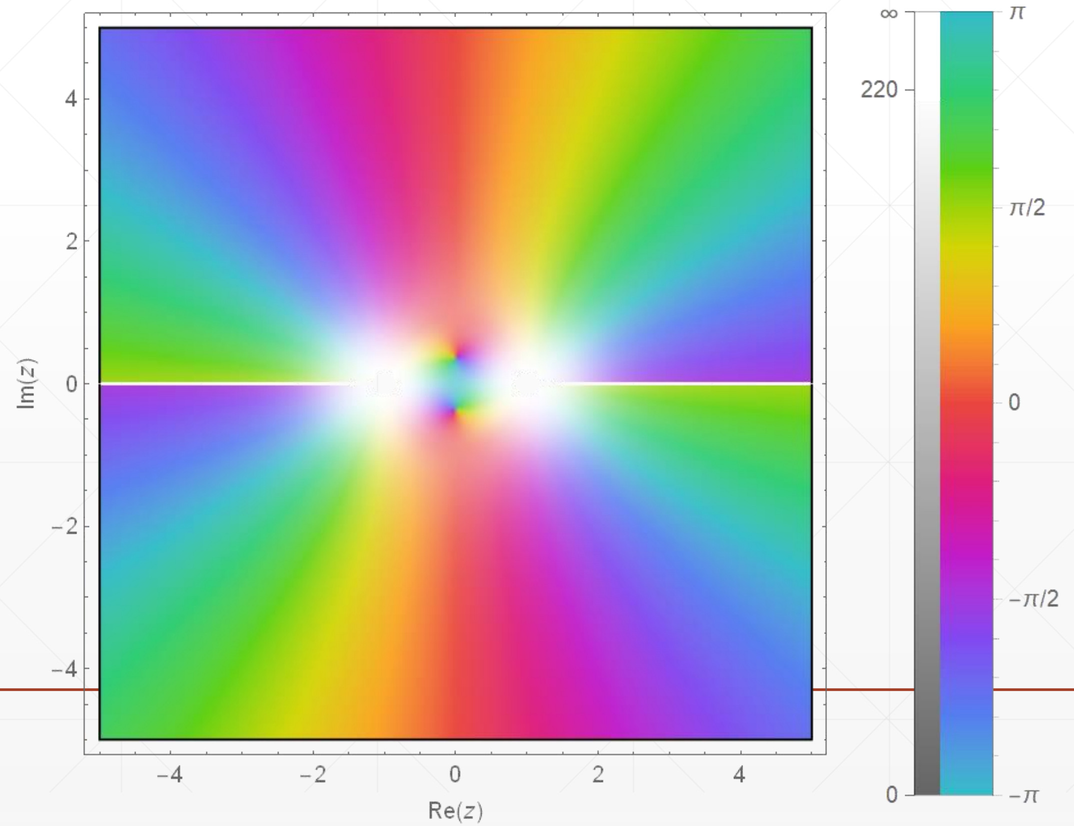

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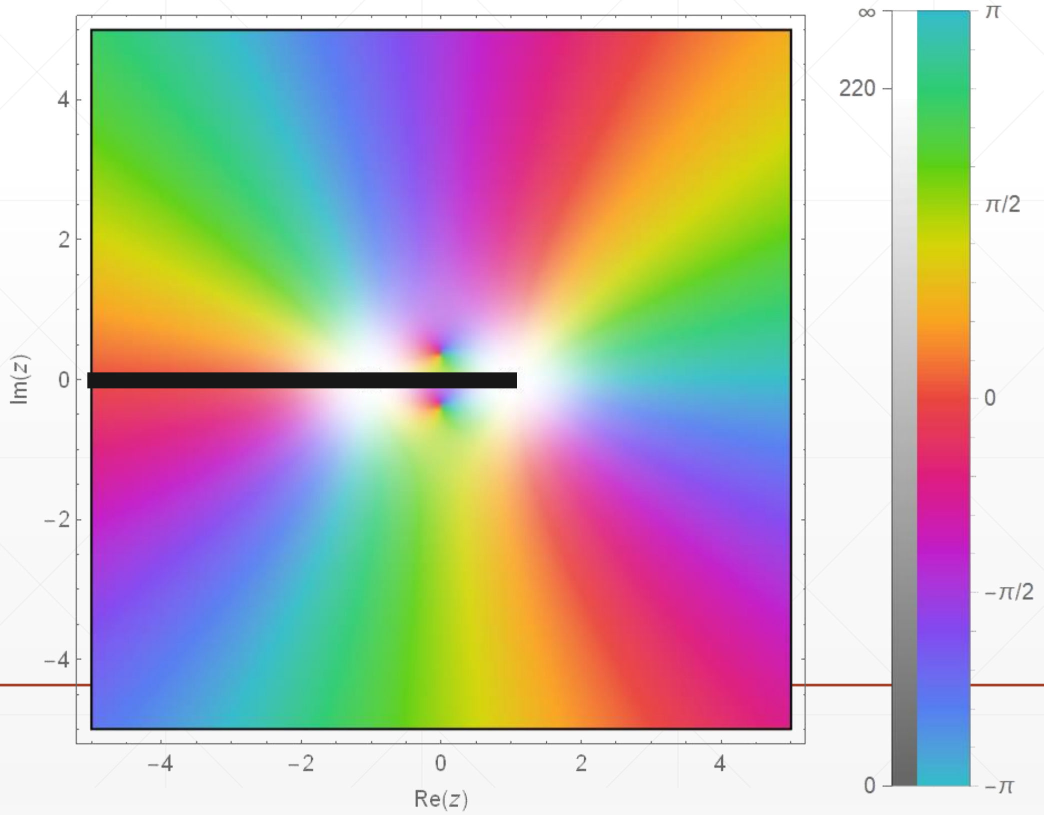

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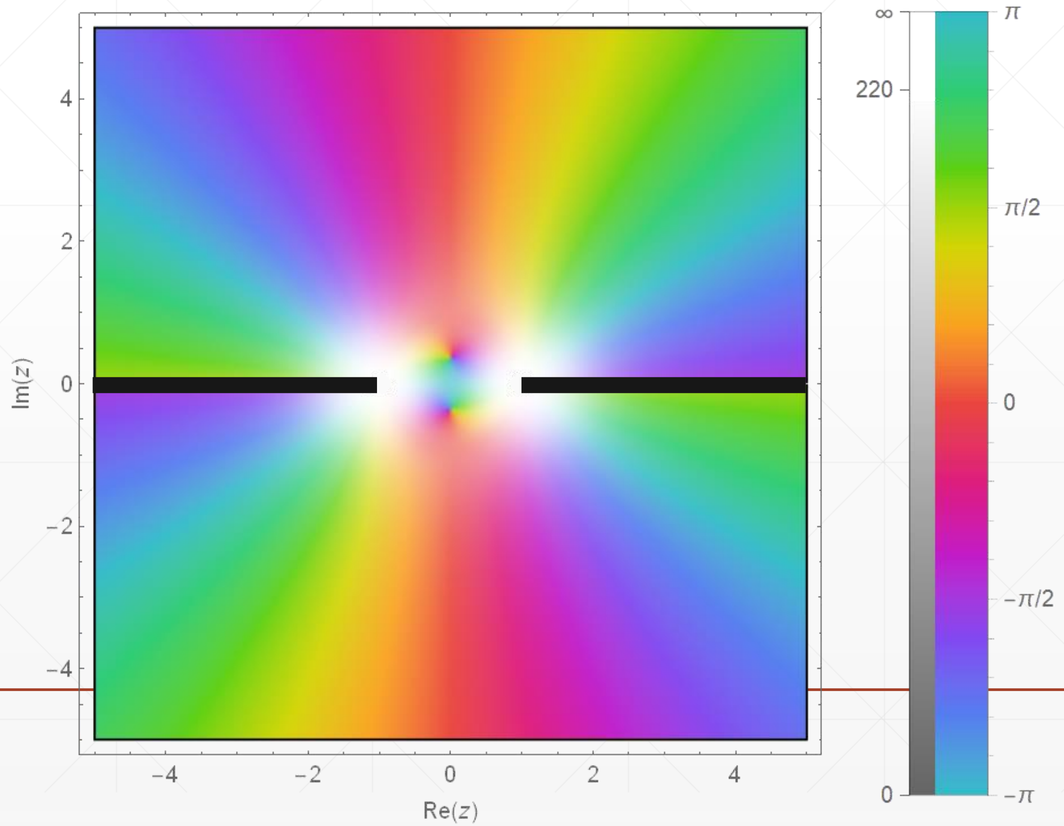

Interlude: Legendre Functions Definitions [8]

- Ferrers' functions definitions in the complex domain (by analytical continuation):

$$\mathrm{P}_l^m(z) = \left(\frac{1+z}{1-z}\right)^{m/2} \mathbf{F}\left(l+1, -l; 1-m; \frac{1-z}{2}\right)$$

$$\mathrm{Q}_l^m(z) = \frac{\pi \csc(m\pi)}{2}\left(\cos(m\pi)\,\mathrm{P}_l^m(z) - \frac{\Gamma(l+m+1)}{\Gamma(l-m+1)}\mathrm{P}_l^{-m}(z)\right)$$

- Olver's hypergeometric function

$$\mathbf{F}(a, b; c; z) = \frac{1}{\Gamma(c)} \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s s!} z^s$$

- Pochhammer symbol

$$(k)_s = \frac{\Gamma(k+s)}{\Gamma(k)}$$

Interlude: Legendre Functions Definitions [8]

- Asymptotic behaviour:

$$\mathrm{P}_l^m(z) \sim (2z)^l \frac{\Gamma(l+1/2)}{\pi^{1/2}\Gamma(l-m+1)} e^{i\pi m/2},$$

$$\mathrm{Im}\{z\} > 0,\ l \neq 1/2, 3/2, 5/2, \ldots,\ m-l \neq 1, 2, 3, \ldots$$

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Physical Arguments that Justify their Existence

$$u(x) = a(b + x^2)^{1/2} \left[c_1 \mathrm{P}_2^m \left(\frac{ix}{\sqrt{b}} \right) + c_2 \mathrm{Q}_2^m \left(\frac{ix}{\sqrt{b}} \right) \right] \tag{9}$$

- Square integrability:

$$m \geq 3, \quad m \in \mathbb{Z} \tag{10}$$

$$c_1 = 0, \quad c_2 \neq 0 \tag{11}$$

- Obtain real function via a scaling factor:

$$\mathrm{T}_l^m(x) = i e^{i\pi(l+m)/2} \, \mathrm{Q}_l^m(ix), \tag{12}$$

Solution and Properties

- Let us define:

$$u_n(x) = -e^{i\pi n/2}\, a(b+x^2)^{1/2}\left[\mathrm{Q}_2^{n+3}\left(\frac{ix}{\sqrt{b}}\right)\right] \tag{13}$$

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- The transverse power:

$$P_n := \int_{-\infty}^{\infty} |u_n(x)|^2 \mathrm{d}x = \frac{16}{7}\pi\, a^2 b^{3/2}\left(n^2+6n+14\right)(1)_n(6)_n, \tag{14}$$

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- Redefine the Legendre-Lorentzian Soliton Solutions:

$$\Pi_n := \frac{P_n}{a^2} \implies \quad u_n(x) = \sqrt{\frac{P_{out}}{\Pi_n}}\left(-e^{i\pi n/2}\,(b+x^2)^{1/2}\left[\mathrm{Q}_2^{n+3}\left(\frac{ix}{\sqrt{b}}\right)\right]\right) \tag{15}$$

$$\implies \int_{-\infty}^{\infty} |u_n(x)|^2 \mathrm{d}x = P_{out}$$

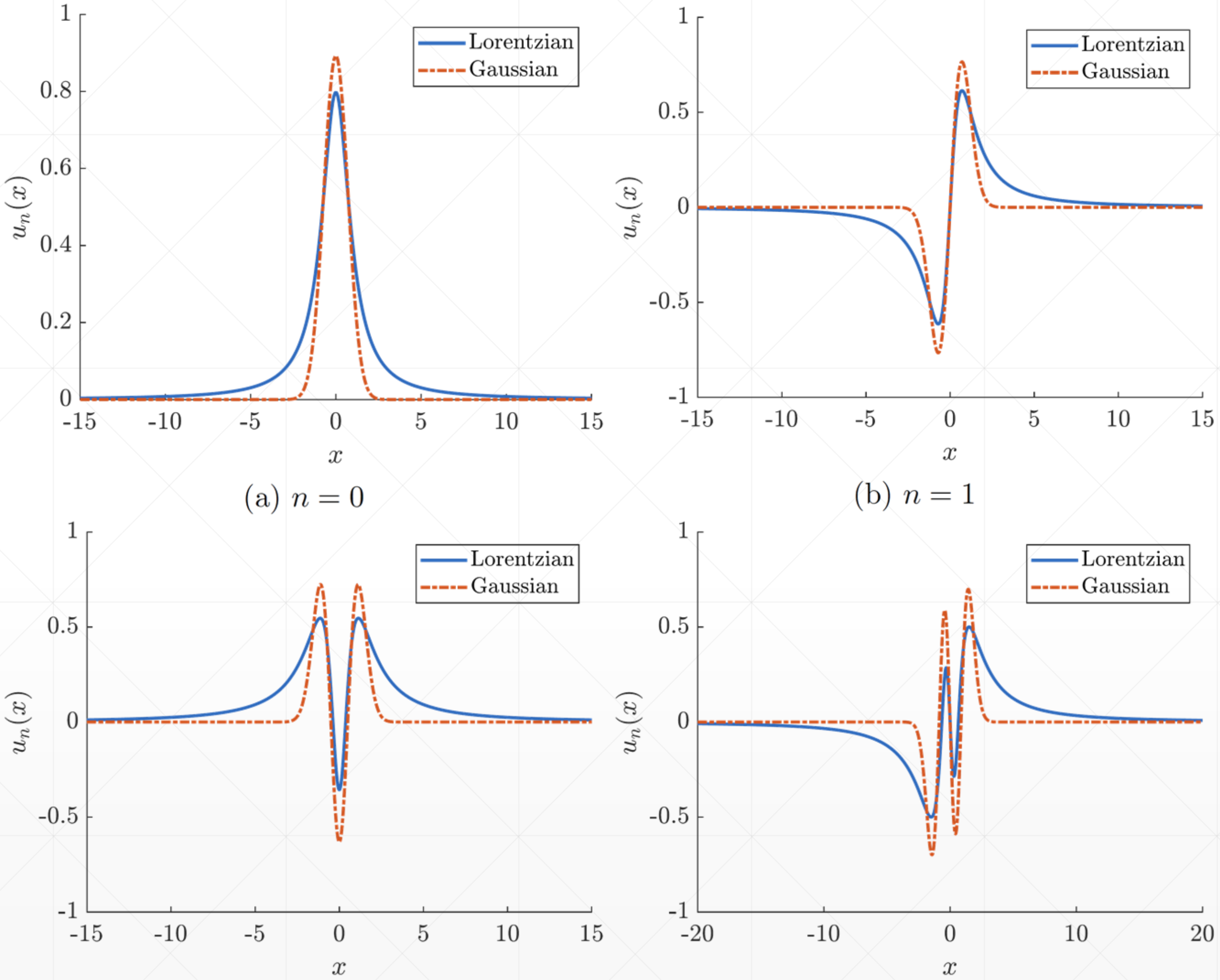
Solution and Properties

- Final expressions:

$$\boxed{U_n(x,z) = \sqrt{\frac{P_{out}}{\Pi_n}}\,(b+x^2)^{1/2}\left[\mathrm{T}_2^{n+3}\left(\frac{x}{\sqrt{b}}\right)\right] e^{i\lambda z}} \tag{16}$$

$$\boxed{V_n(x) = \lambda - \frac{P_{out}}{\Pi_n}\,(b+x^2)\left[\mathrm{T}_2^{n+3}\left(\frac{x}{\sqrt{b}}\right)\right]^2 + \frac{b\left(n^2+6n+2\right)-6x^2}{2\left(b+x^2\right)^2}} \tag{17}$$

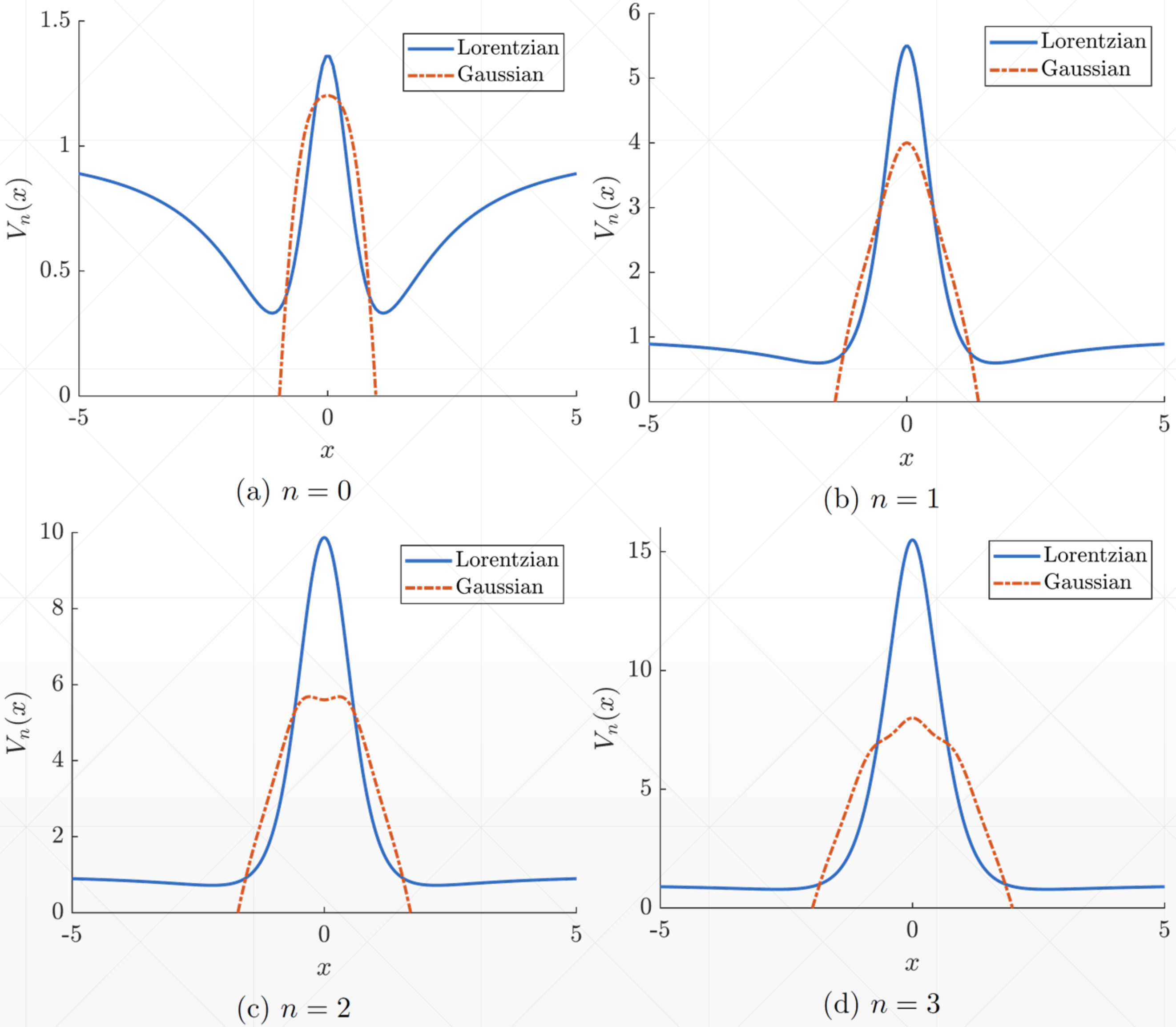
- Some Soliton Wave-function Plots [4]



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$ (d) $n = 3$

$$\lambda = 1, P_{\text{out}} = 1, b = 1.$$

- Some Soliton External Potential Plots [4]



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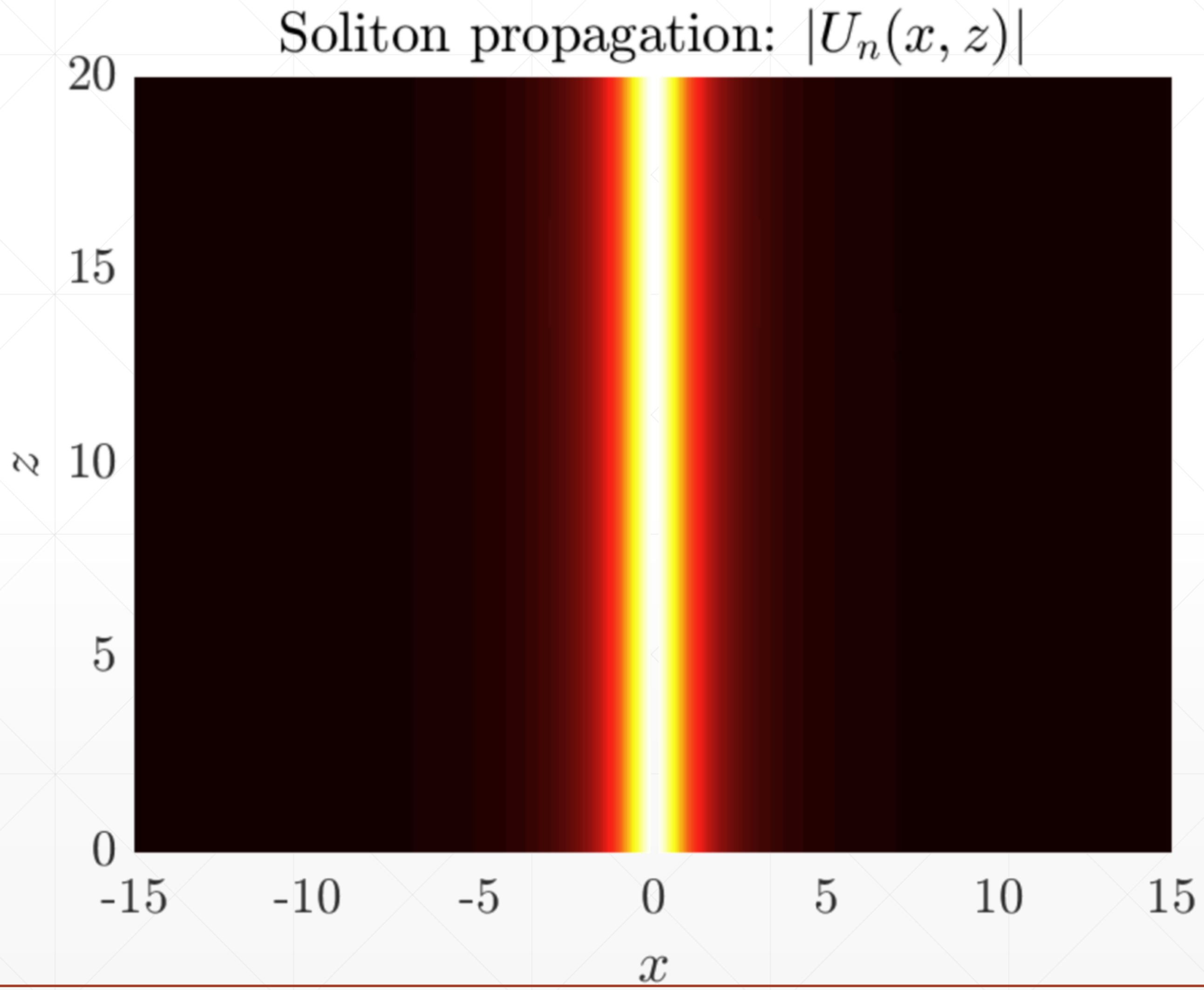
Simulations of Propagations

$n = 0$



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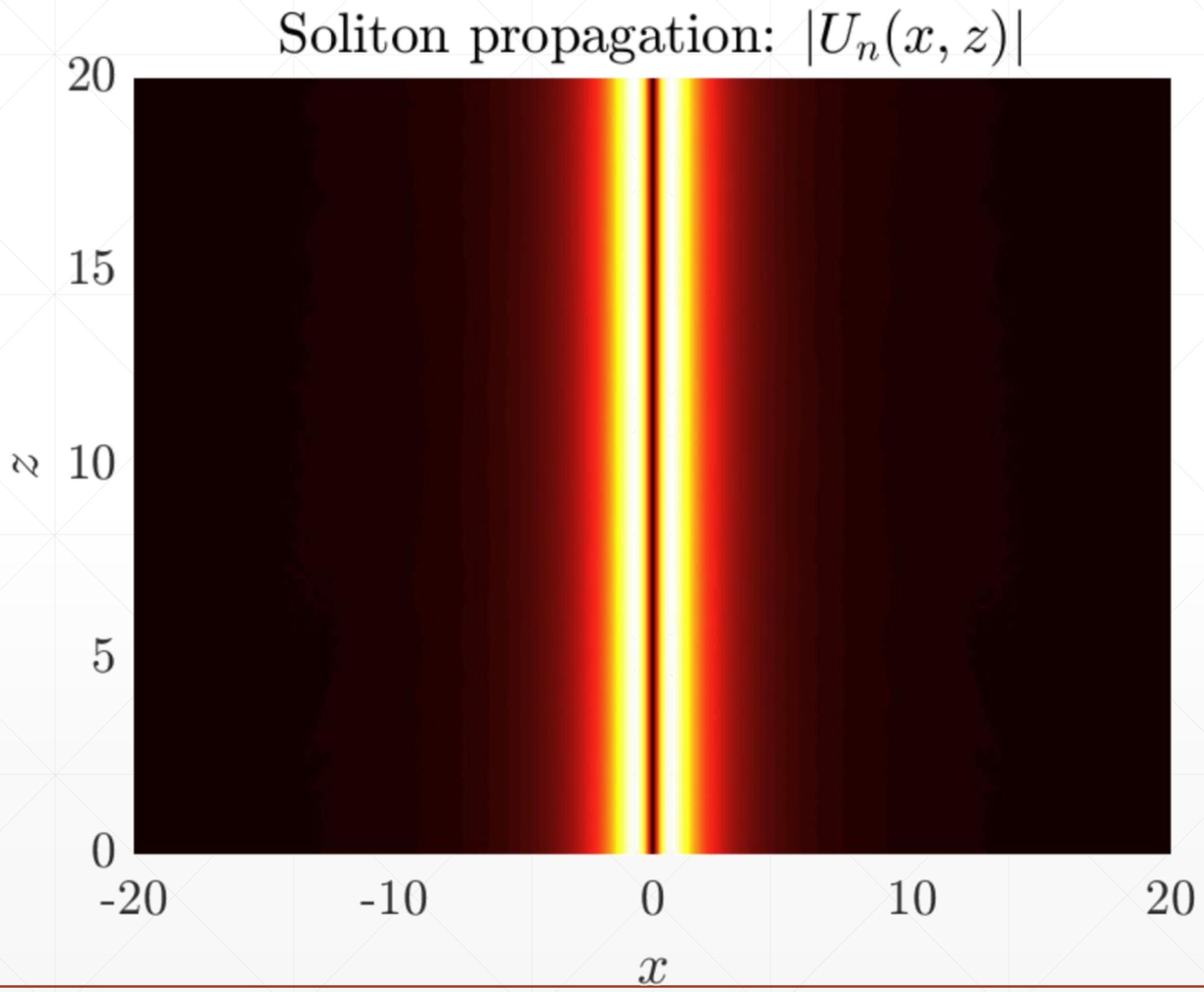
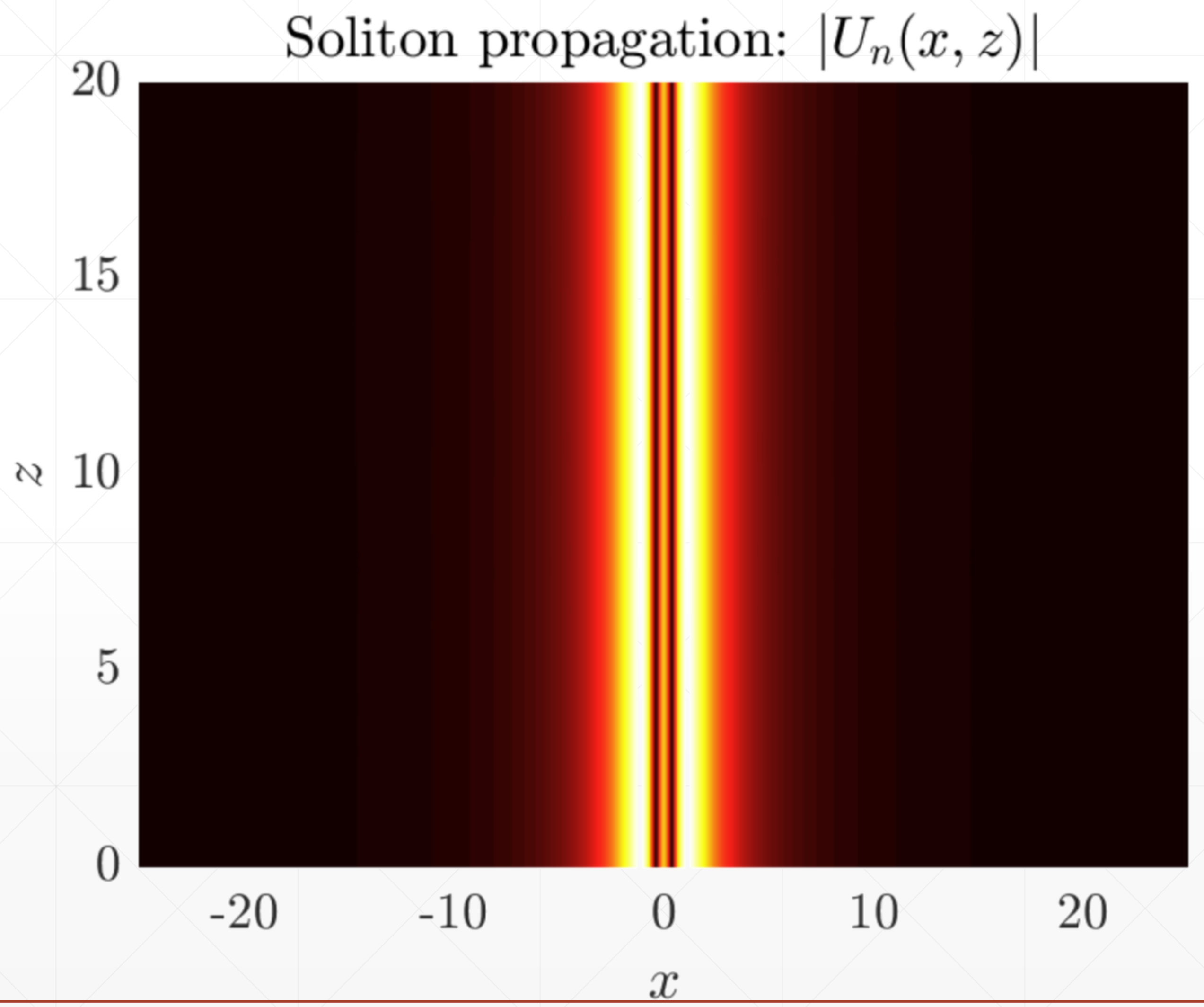


Simulations of Propagations

$n = 2$



Stability Analysis

- Norm definition

$$||f(x,z)||_x = \sqrt{\int_{-\infty}^{\infty} |f(x,z)|^2 \mathrm{d}x}$$

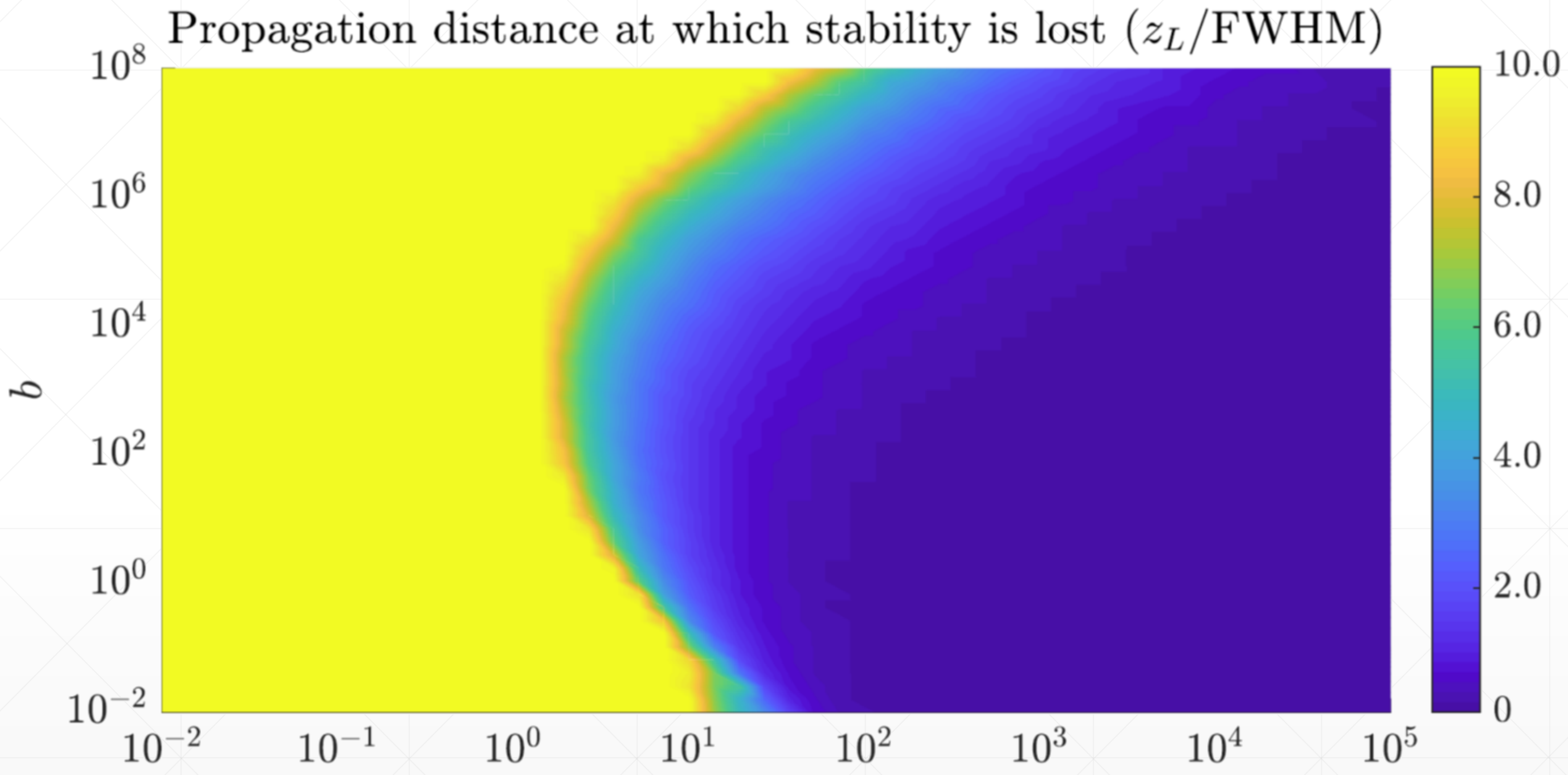
- Propagation error definition

$$\delta(z) = \frac{||U(x,z) - U(x,0)e^{i\lambda z}||_x}{||U(x,0)||_x} \tag{18}$$

- Threshold distance for lost of stability

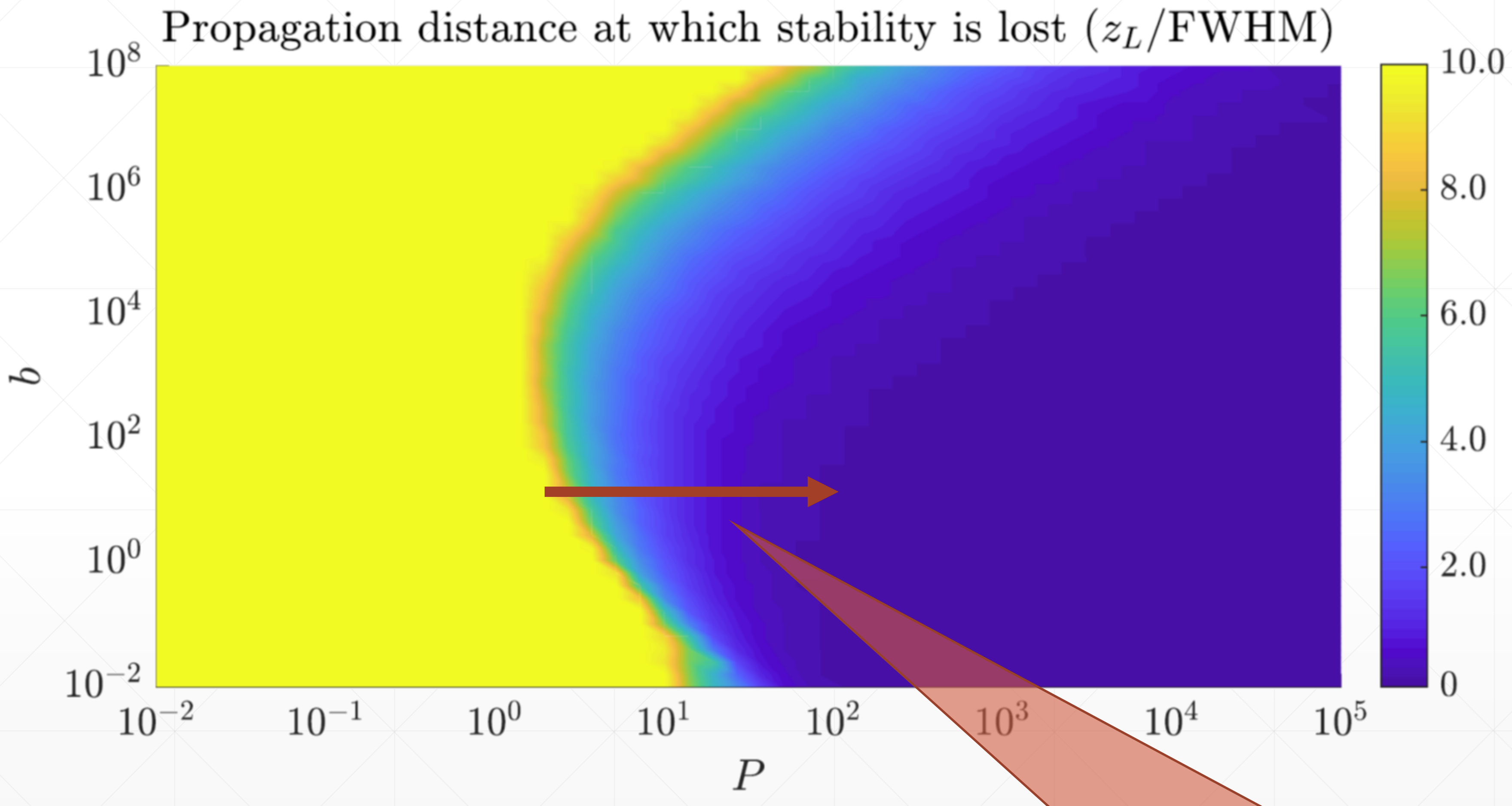
$$\delta(z_L) = 0.05$$

Stable Regimes



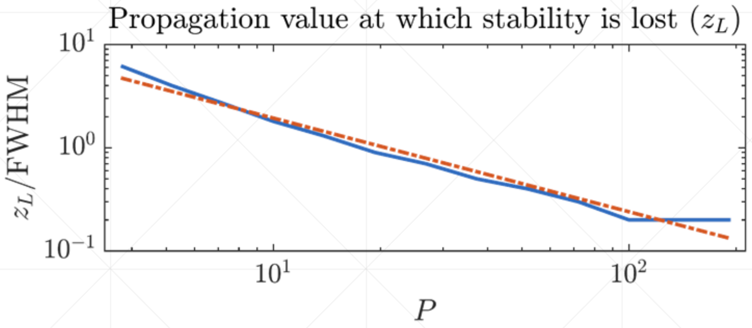
(a) $n = 0$

Stable Regimes: Increasing Power

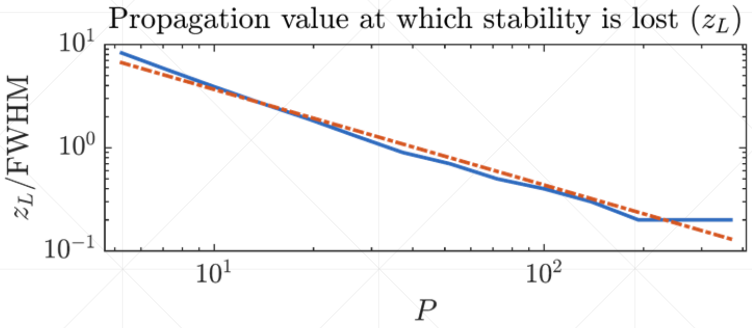


(a) $n = 0$

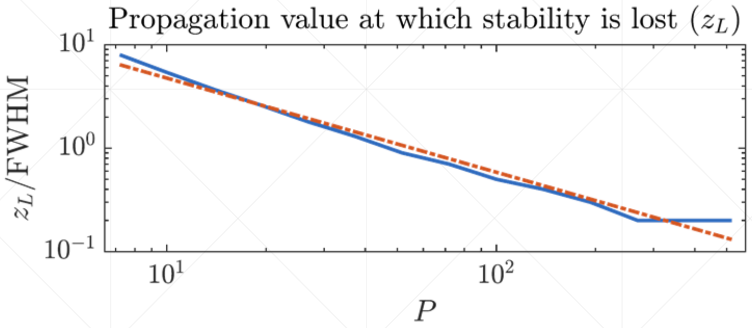
And similarly for next order LL solitons



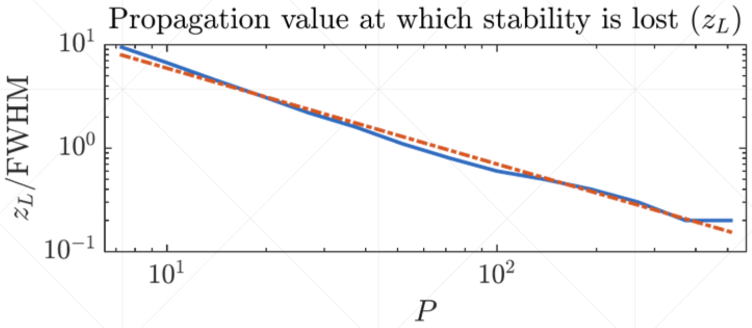
(a) $n = 0$, $z_l \sim \text{FWHM}\,(P^{-0.90724})$.



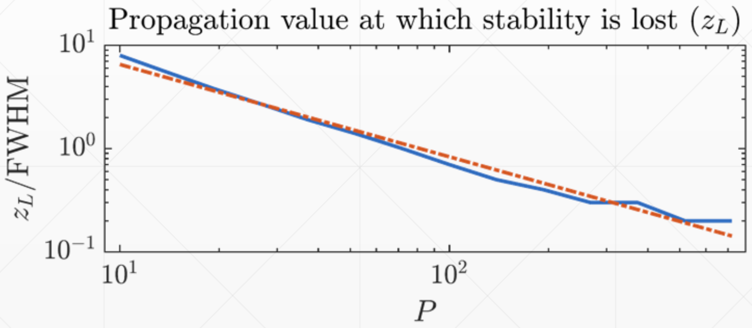
(b) $n = 1$, $z_l \sim \text{FWHM}\,(P^{-0.92512})$.



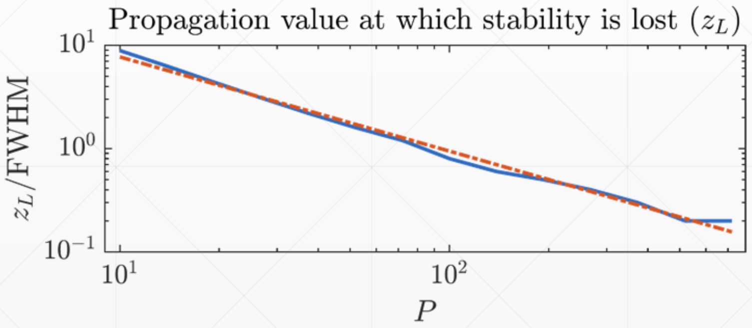
(c) $n = 2$, $z_l \sim \text{FWHM}\,(P^{-0.91018})$.



(d) $n = 3$, $z_l \sim \text{FWHM}\,(P^{-0.92704})$.

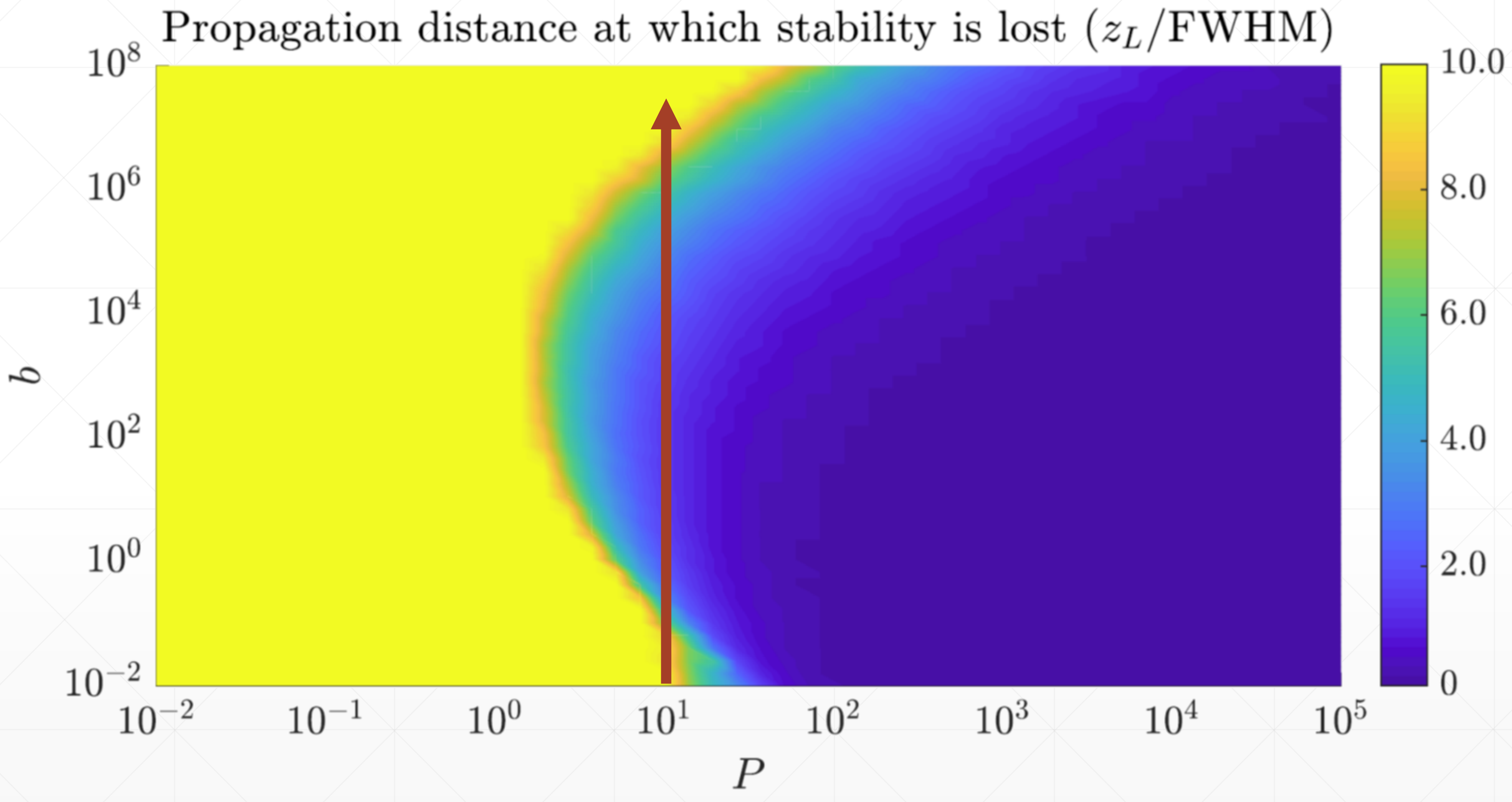


(e) $n = 4$, $z_l \sim \text{FWHM}\,(P^{-0.89451})$.



(f) $n = 5$, $z_l \sim \text{FWHM}\,(P^{-0.91236})$.

Stable Regimes: Increasing Soliton width



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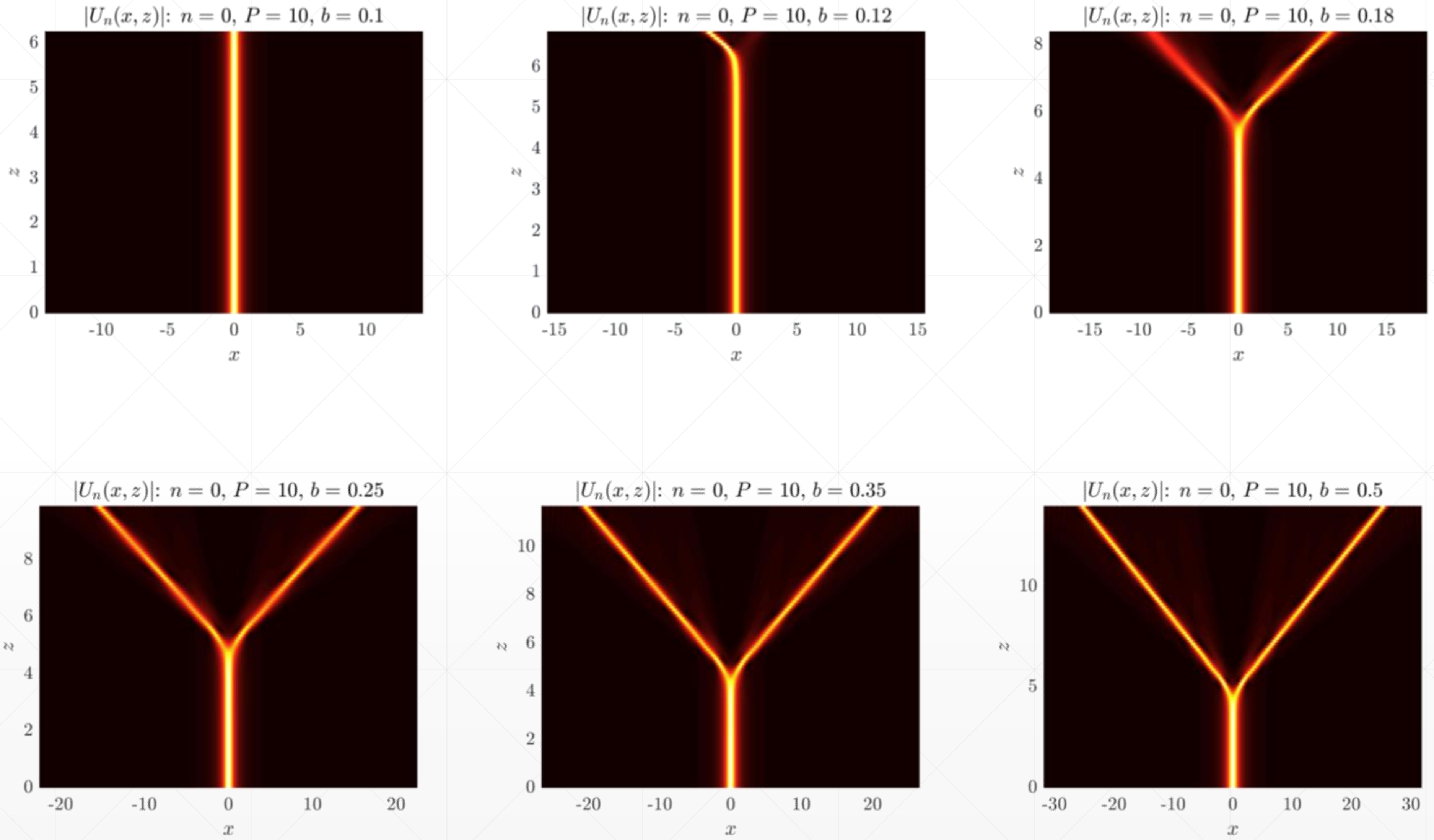
Stable Regimes: Increasing Soliton width



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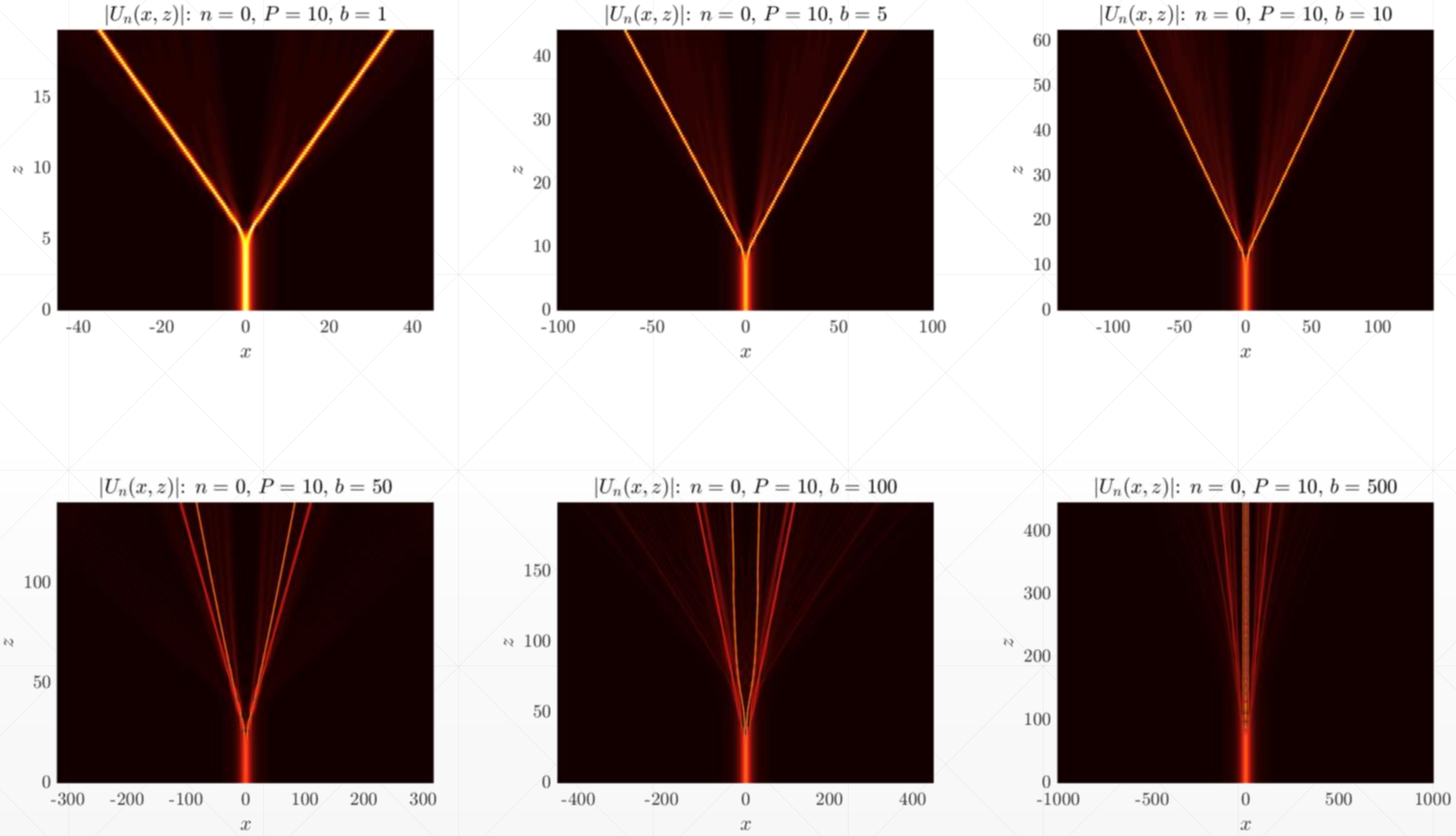

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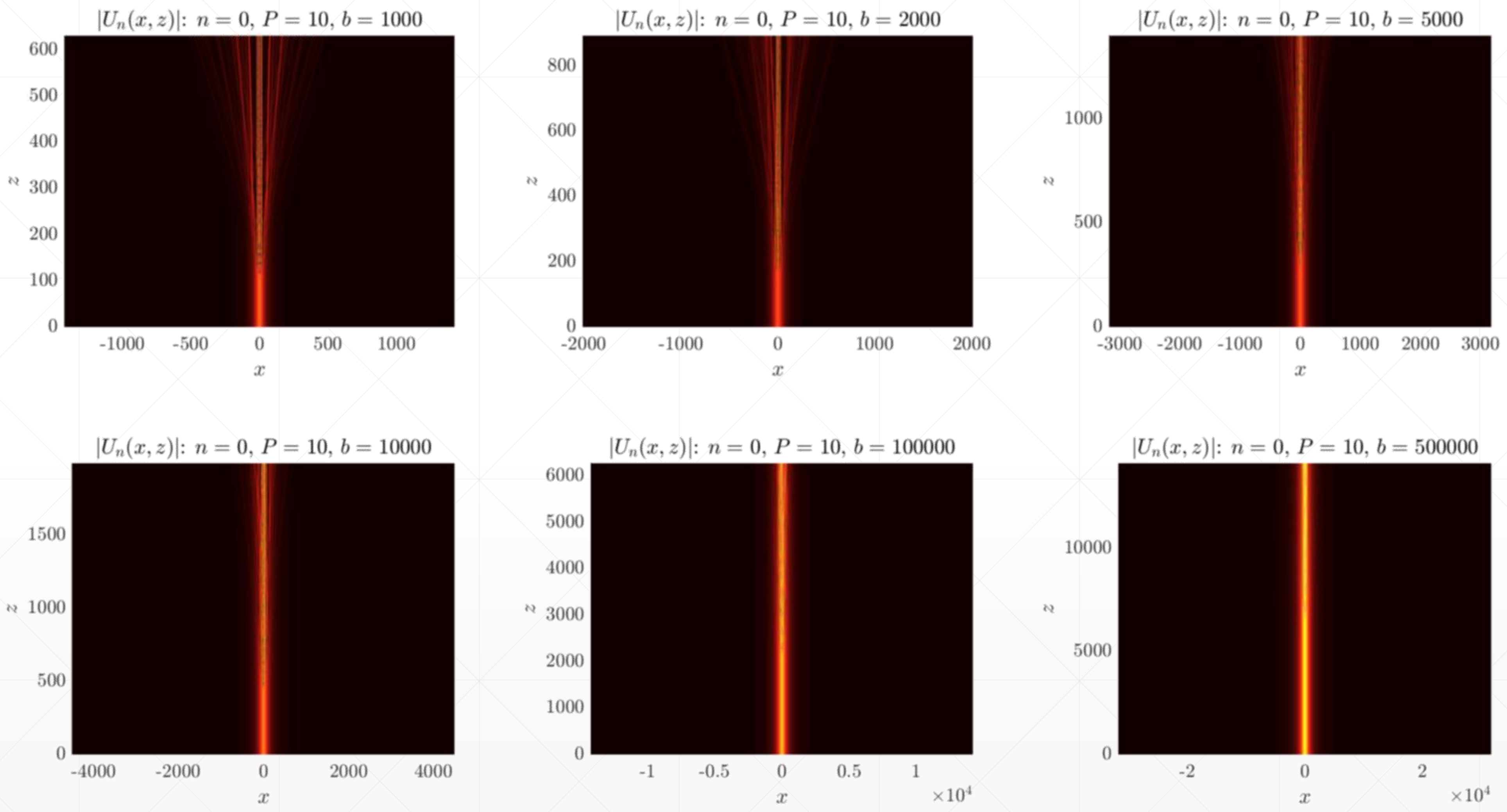
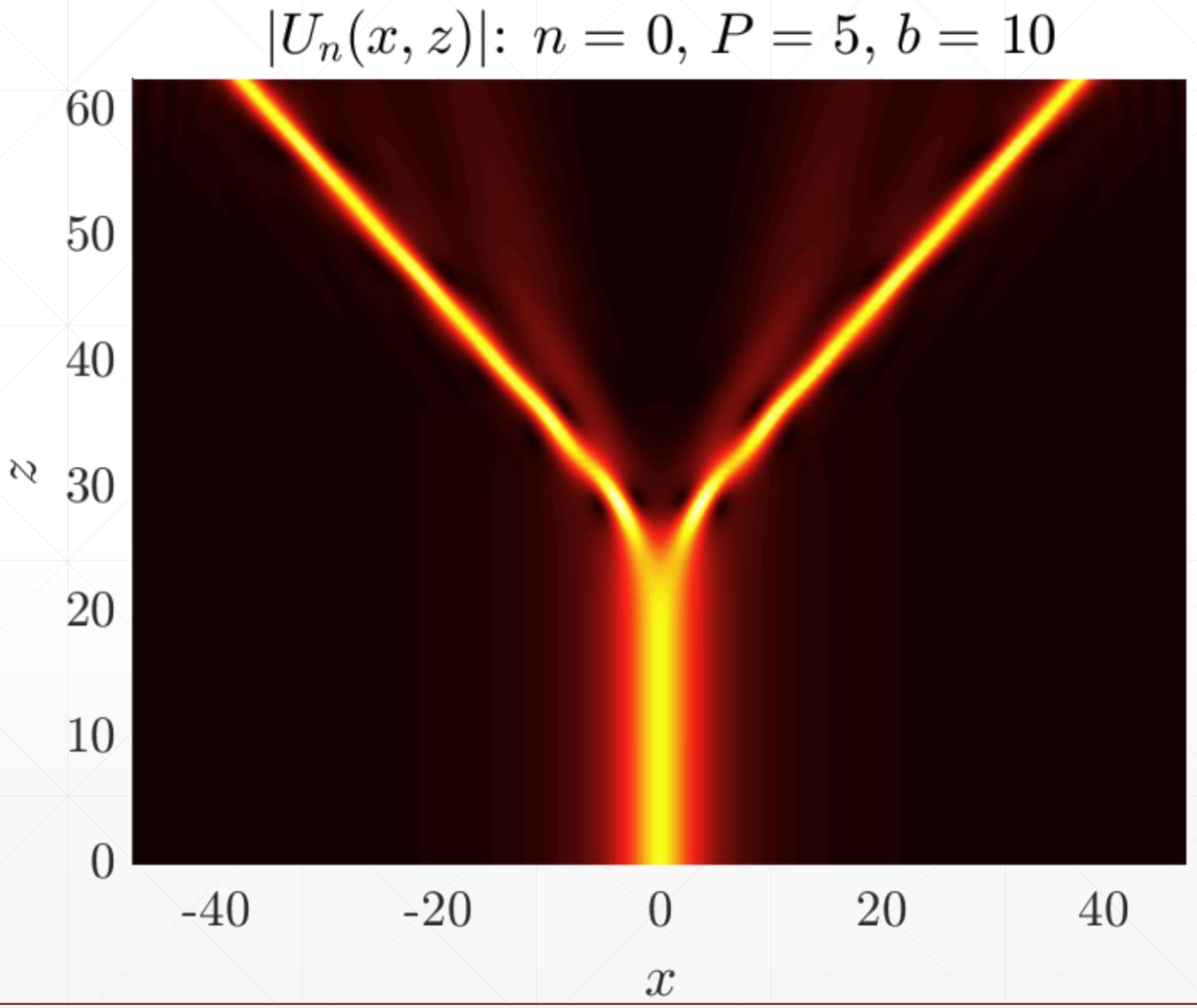
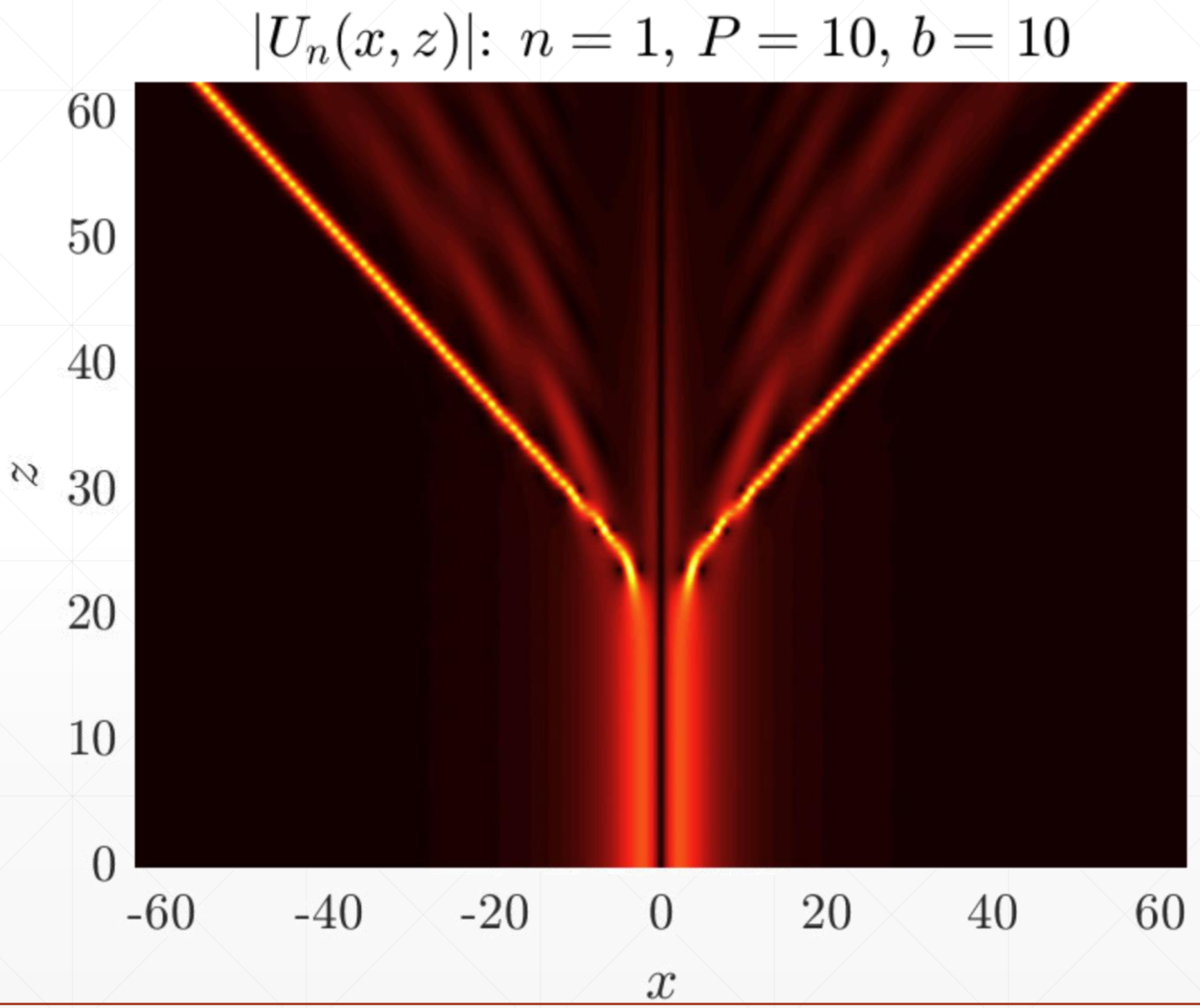


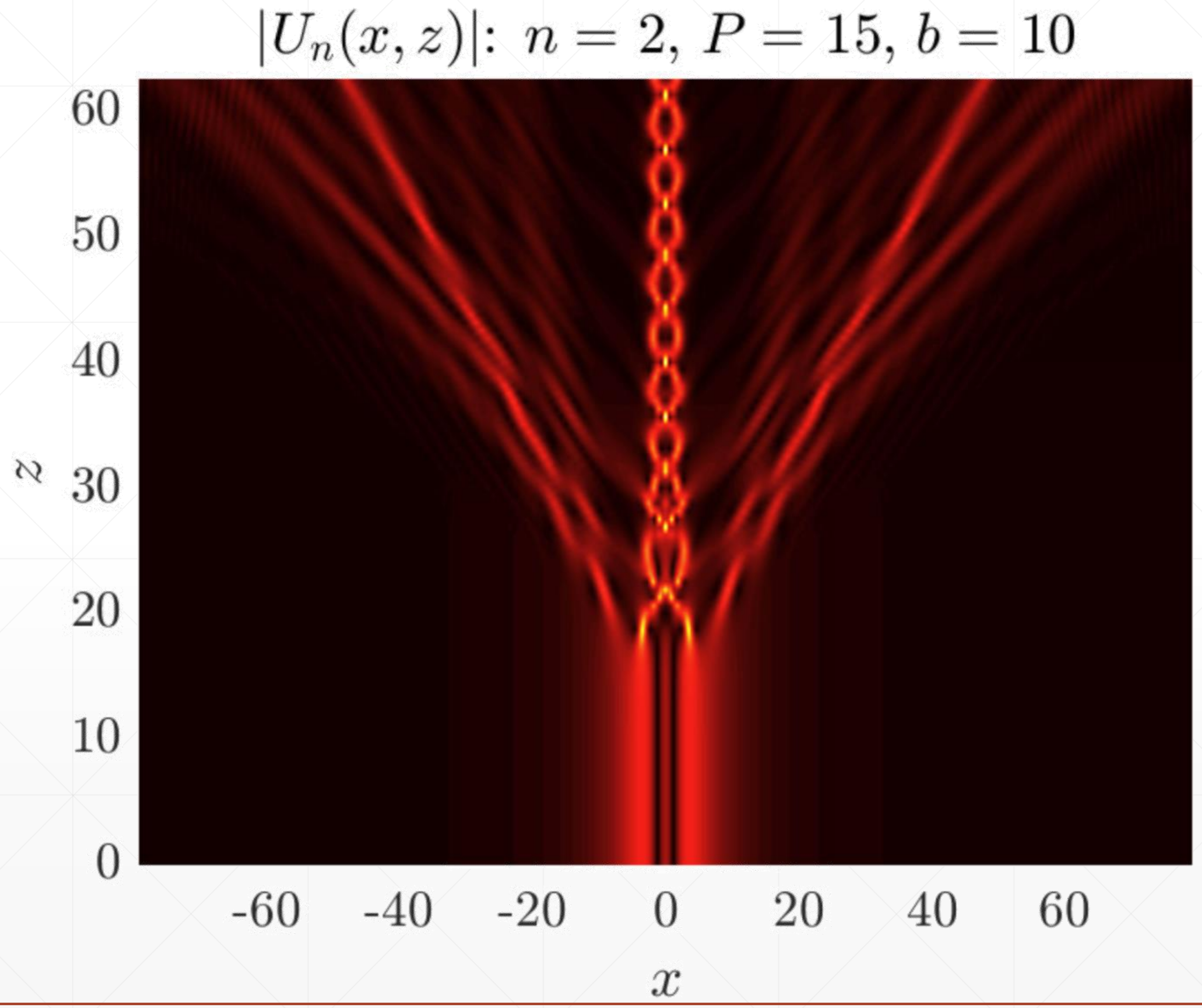
Unstable Regime



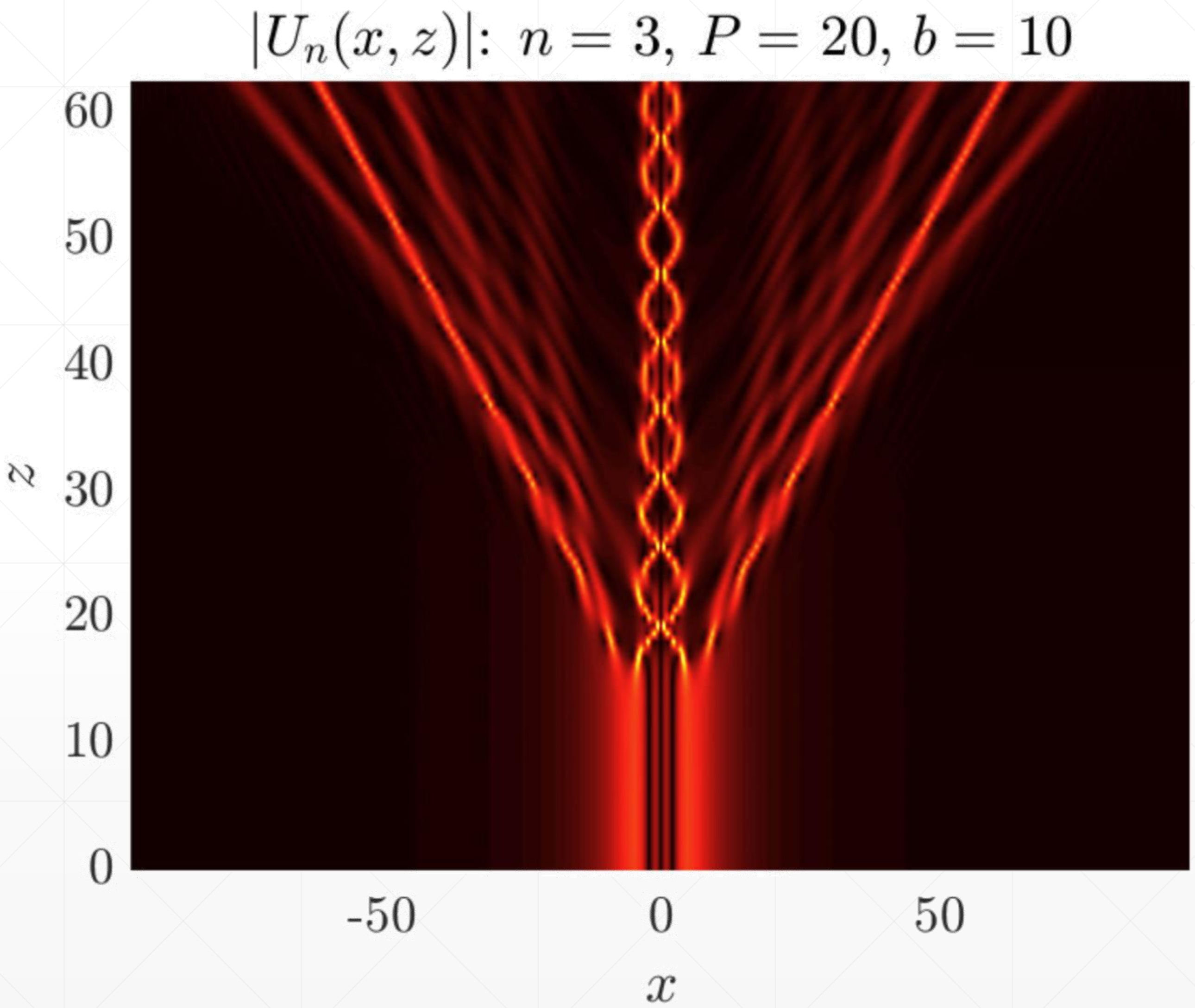
Unstable Regime: Evanescent modes



Unstable Regime: Breathing modes



Unstable Regime: Breathing modes



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