Theoretical Framework for Simulating Limit Cycles in Differential Equations

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1 Introduction

This document provides a theoretical framework for simulating and analyzing limit cycles in two well-known nonlinear dynamical systems: the Van der Pol oscillator and the FitzHugh-Nagumo model. Both systems exhibit rich dynamical behaviors, including periodic oscillations represented by limit cycles in their phase spaces.

2 The Van der Pol Oscillator

2.1 Mathematical Model

The Van der Pol oscillator is a second-order nonlinear differential equation originally proposed by Balthasar Van der Pol while studying electrical circuits with nonlinear vacuum tube amplifiers. The equation is given by:

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0, \tag{1}$$

where:

- x is the position variable.
- μ is a scalar parameter indicating the nonlinearity and the strength of the damping.

2.2 Conversion to First-Order System

To simulate this system numerically, we convert it into a system of first-order differential equations by introducing a new variable y:

$$\frac{dx}{dt} = y, \tag{2}$$

$$\frac{dy}{dt} = \mu(1 - x^2)y - x. \tag{3}$$

2.3 Dynamics and Limit Cycles

For $\mu > 0$, the system exhibits a stable limit cycle attractor, meaning that trajectories in the phase space converge to a closed orbit, resulting in sustained oscillations. The nonlinearity parameter μ affects the shape and amplitude of the limit cycle:

- For small μ, the system behaves similarly to a harmonic oscillator.
- As μ increases, the nonlinearity becomes more pronounced, distorting the limit cycle.

3 The FitzHugh-Nagumo Model

3.1 Mathematical Model

The FitzHugh-Nagumo model is a simplification of the Hodgkin-Huxley model of neuron activation and is used to describe excitable systems. The equations are:

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I, \tag{4}$$

$$\frac{dw}{dt} = \gamma(v + a - bw), \tag{5}$$

where:

- v represents the membrane potential.
- w is a recovery variable.
- I is an external current stimulus.
- a, b, and γ are parameters that affect the system dynamics.

In our implementation, we use the following parameter values:

- $a = 0.7$
- $b = 0.8$
- $\gamma = 0.08$

Substituting these values into equations (4) and (5), we get:

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I, \tag{6}$$

$$\frac{dw}{dt} = 0.08(v + 0.7 - 0.8w). \tag{7}$$

3.2 Dynamics and Limit Cycles

The FitzHugh-Nagumo model can exhibit limit cycle behavior depending on the value of the external current I:

- For certain values of I, the system has a stable fixed point.
- As I increases beyond a critical value, the system undergoes a Hopf bifurcation, leading to sustained oscillations represented by a limit cycle.

4 Numerical Simulation

4.1 Numerical Integration Method

We use the Runge-Kutta method of order 5(4), implemented in the `solve_ivp` function from the SciPy library, to numerically integrate the systems. This method is suitable for solving ordinary differential equations (ODEs) and provides a good balance between accuracy and computational efficiency.

4.2 Implementation Details

4.2.1 Time Span and Evaluation Points

The simulations are run over a time interval $t \in [0, 100]$, with 10,000 evenly spaced evaluation points to capture the dynamics accurately.

4.2.2 Initial Conditions

- **Van der Pol Oscillator:** $x(0) = 0.5$, $y(0) = 0.5$.
- **FitzHugh-Nagumo Model:** $v(0) = 0.0$, $w(0) = 0.0$.

4.3 Parameter Variation

We explore how varying the parameters μ and I affects the system dynamics:

- **Van der Pol Oscillator:** We simulate for $\mu = \{0.5, 1.0, 2.0\}$.
- **FitzHugh-Nagumo Model:** We simulate for $I = \{0.5, 0.7, 0.9\}$.

5 Results and Analysis

5.1 Phase Portraits

The phase portraits plot the trajectories in the phase space of each system.

5.1.1 Van der Pol Oscillator

The phase portrait shows how the system evolves over time in the (x, y) plane, eventually converging to a limit cycle.

5.1.2 FitzHugh-Nagumo Model

Similarly, the phase portrait in the (v, w) plane illustrates the oscillatory behavior of the system under different external currents I.

5.2 Time Series Plots

The time series plots display the evolution of the state variables over time, showing the periodic nature of the oscillations.

5.3 Effect of Parameter Variation

5.3.1 Van der Pol Oscillator

Increasing μ leads to:

- A more pronounced nonlinearity.
- A limit cycle with sharper transitions.

5.3.2 FitzHugh-Nagumo Model

Increasing I results in:

- Transition from damped oscillations to sustained oscillations.
- Changes in the amplitude and frequency of the limit cycle.

6 Conclusion

The simulations confirm that both the Van der Pol oscillator and the FitzHugh-Nagumo model exhibit limit cycles under certain conditions. By varying the system parameters, we can observe different dynamical behaviors, providing insights into the nonlinear characteristics of these systems.

7 References

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