

I = INVESTMENT

SOLOW - SWAN MODEL

. AGGREGATE PROD. FUN.

$$Y = F(K, L, t)$$

. PROPERTIES :

(P1) CONSTANT RET TO SCALE

$$\begin{aligned}\lambda Y &= F(\lambda K, \lambda L, t), \quad \lambda > 0 \\ &= \lambda F(K, L, t)\end{aligned}$$

(P2) INADA CONDITIONS

$$\lim_{K \rightarrow 0} F_K = \lim_{L \rightarrow 0} F_L = +\infty$$

$$\lim_{K \rightarrow \infty} F_K = \lim_{L \rightarrow \infty} F_L = 0$$

Y = OUTPUT

K = CAPITAL

L = LABOR

t = TIME

S = SAVING

(P3) DERIVATIVES

$$F_K, F_L > 0 \quad ; \quad F_{KK}, F_{LL} < 0$$

$$\Rightarrow F_{KL} > 0$$

- SAVING BEHAVIOR

$$S(t) = s Y(t), \quad s \in (0, 1)$$

$$C(t) = (1-s) Y(t)$$

- NATIONAL ACCOUNTING IDENTITY

$$Y(t) = C(t) + I(t) \quad (+ G(t) + NX(t))$$

- CONTINUOUS TIME: ACCUMULATION

$$\dot{k}(t) := \frac{dk(t)}{dt}$$

$$\dot{k}(t) = I(t) - \delta k(t); \quad \delta > 0$$

- LABOR FORCE

$$\dot{L}(t) = m L(t); \quad m > 0$$

NO TIME VARIATION IN F

$$F(k, L, t) = F(k, L) \quad (\text{no } t)$$

- IN EQUILIBRIUM : $S(t) = I(t) \Leftrightarrow Y(t) = C(t) + I(t)$

"INTENSIVE FORM" \rightarrow PER CAPITA / WORKER

$$y(t) := \frac{Y(t)}{L(t)} \quad k(t) := \frac{k(t)}{L(t)}$$

etc.

- PROD FUN IN INTENSIVE FORM

$$Y = F(k, L)$$

$$\frac{1}{L} Y = \frac{1}{L} F(k, L) = F\left(\frac{k}{L}, \frac{L}{L}\right) = F(k, 1)$$

$$f(k) := F(k, 1)$$

$$\Rightarrow y = f(k)$$

PROPERTIES OF f :

- $f'(k) > 0$
- $f''(k) < 0$
- $f(0) = 0$

• INTENSIVE FORM FOR

$$\dot{k}(t) = I(t) - \delta k(t)$$

$$\frac{d}{dt} \left(\frac{k(t)}{L(t)} \right) = \frac{1}{L(t)} \frac{dk(t)}{dt} + k(t) \frac{d}{dt} \left(\frac{1}{L(t)} \right)$$

= ...

$$= \frac{\dot{k}(t)}{L(t)} + \frac{\dot{L}(t)}{L(t)} \frac{k(t)}{L(t)}$$

$$= \frac{\dot{k}(t)}{L(t)} + m k(t)$$

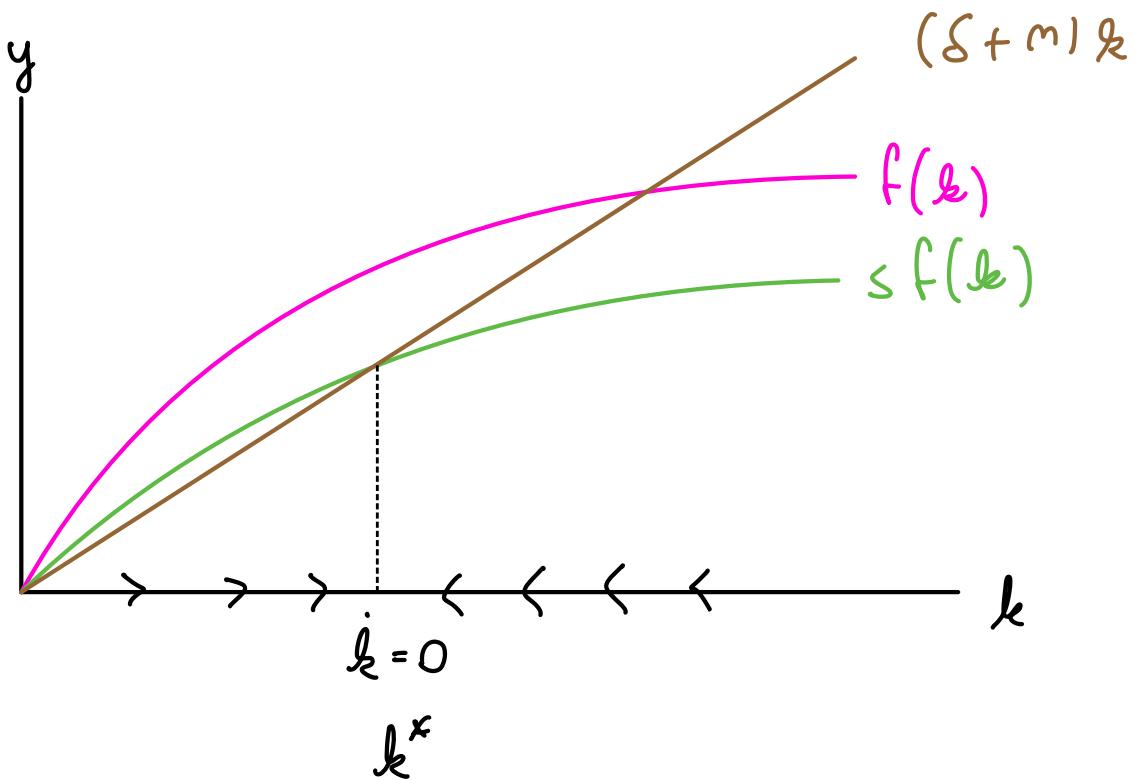
$$\dot{k}(t) = \frac{\dot{k}(t)}{L(t)} + m k(t)$$

ACCUM. EQ IN INTENSIVE FORM:

$$\dot{k}(t) = i(t) - \delta k(t) - n k(t)$$

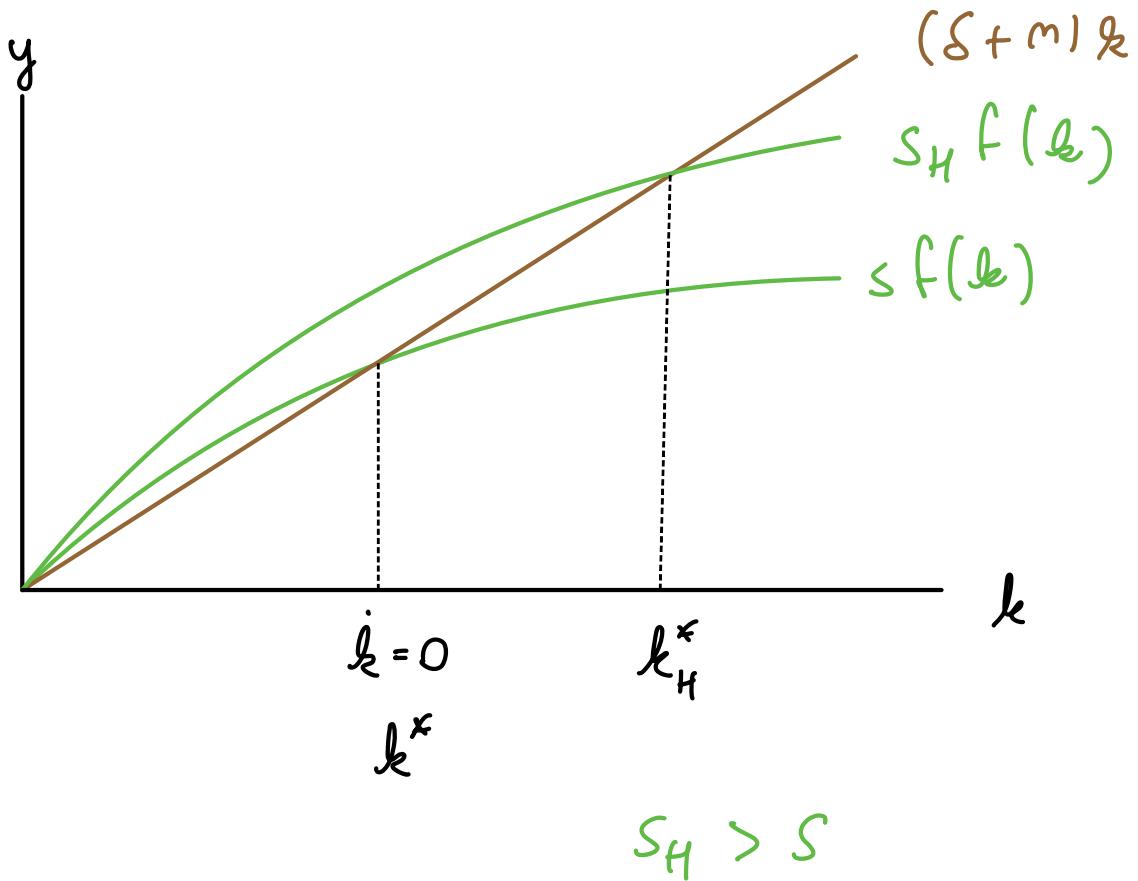
$$\dot{k}(t) = sf(k(t)) - (n + \delta)k(t)$$

$$F(k, L) = k^\alpha L^{1-\alpha} \Rightarrow f(k) = k^\alpha = y$$

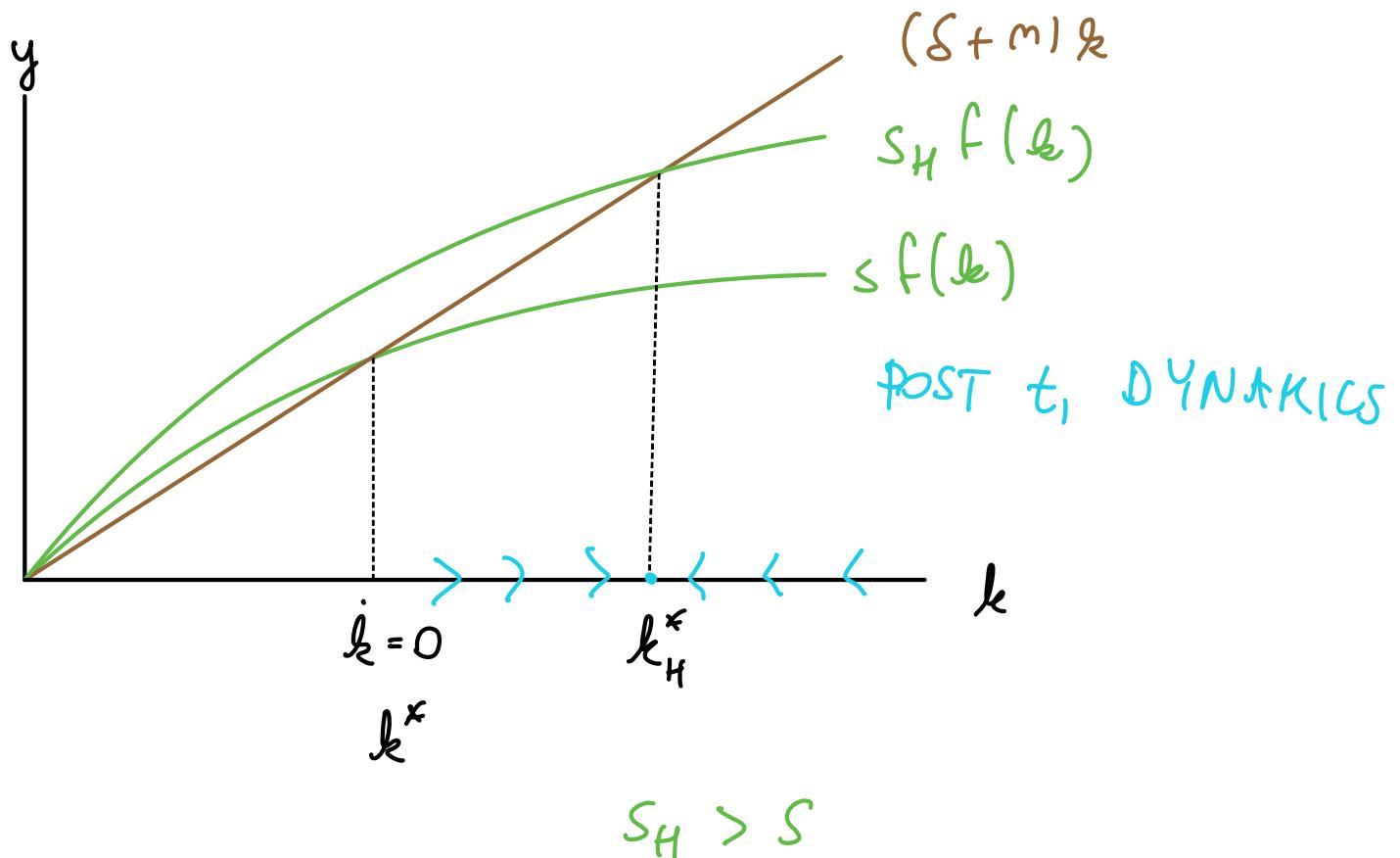


\dot{k} = BALANCED GROWTH PATH

EXPERIMENT : CHANGE s.



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$$Y = C + I \quad (+/ \cancel{X})$$

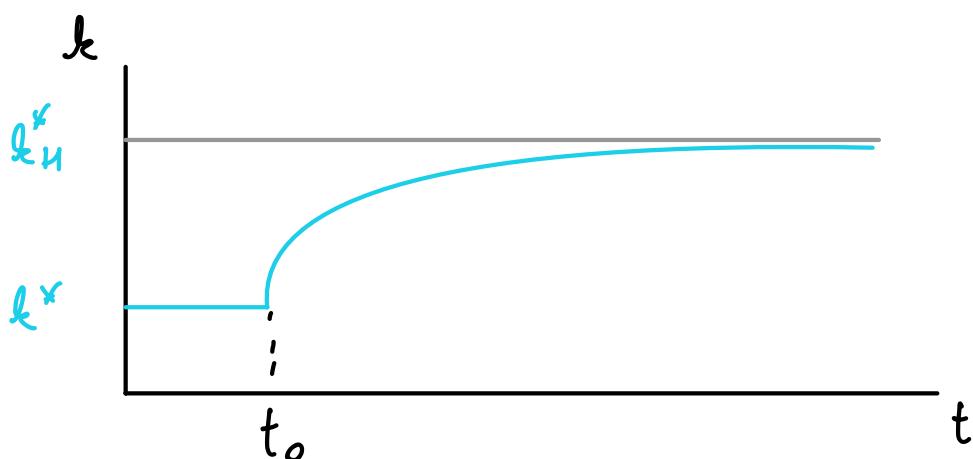
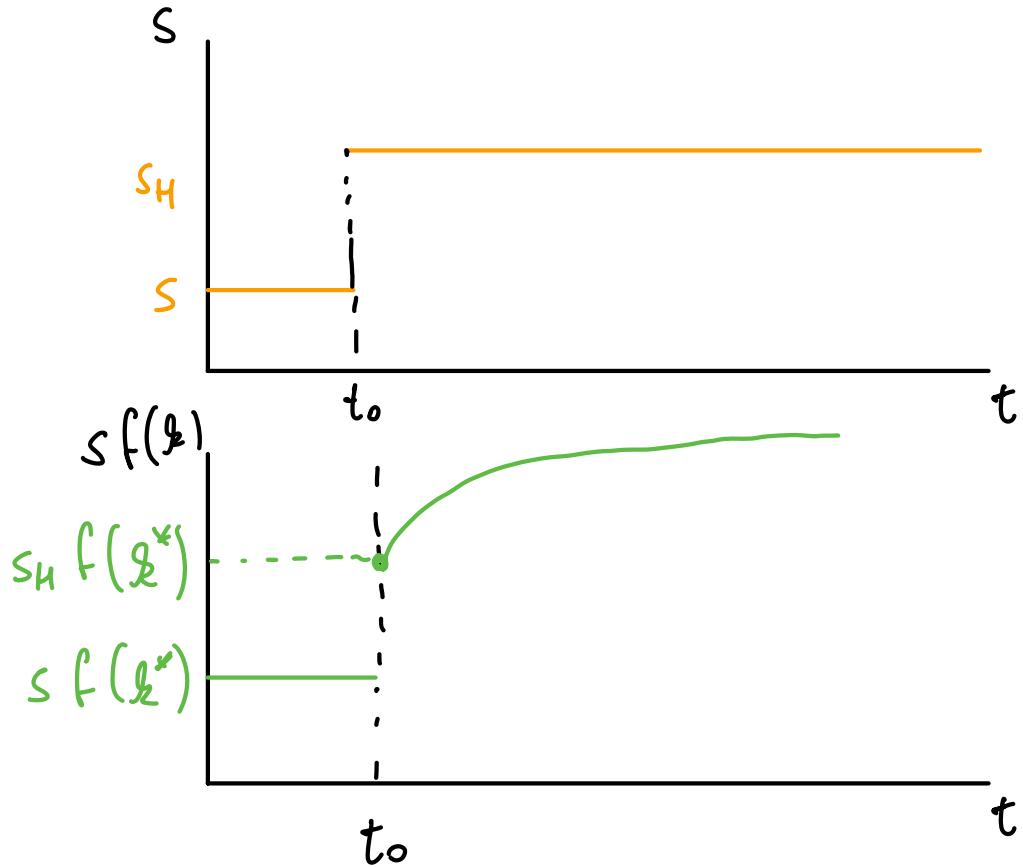
$$G : \mathbb{R}_+ \rightarrow \mathbb{R}$$

$$\dot{k}(t) = G(k(t))$$

$$\dot{k}(t) \approx G(k^*) + G'(k^*)(k - k^*)$$

$$\dot{k}(t) = G'(k^*)(k - k^*) + G''(k^*)(k - k^*)^2$$

$$k(t) = k(0) \exp(G'(k^*)t)$$



$$\dot{k}(t) = s f(k(t)) - (m + \delta) k(t)$$

$$G(k) := s f(k) - (m + \delta) k$$

$$G'(k) = s f'(k) - (m + \delta)$$

$$G'(k^*) = s f'(k^*) - (m + \delta)$$

$$f'(k^*) > 0$$

$$f''(k^*) < 0$$

CONSUMPTION

$$\begin{aligned} c &= (1-s)y = (1-s)f(k) \\ &= f(k) - sf(k) \end{aligned}$$

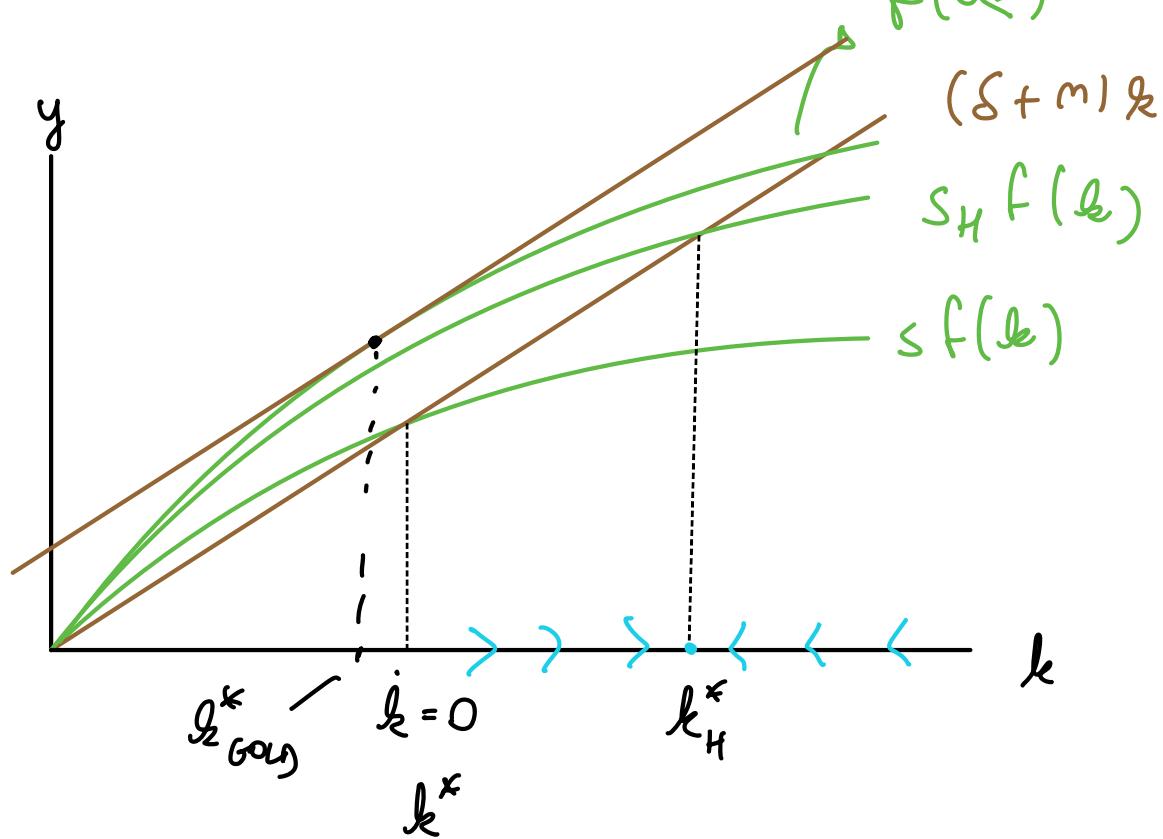
$$\begin{aligned} \text{IN SS: } \dot{k} = 0 &\Leftrightarrow 0 = sf(k^*) - (n + \delta)k^* \\ &\Rightarrow sf(k^*) = (n + \delta)k^* \end{aligned}$$

$$c^* = \underbrace{f(k^*)}_{\substack{\text{INCREASING} \\ \text{IN } k^*}} - \underbrace{(n + \delta)k^*}_{\substack{\text{DECREASING} \\ \text{IN } k^*}}$$

$$\frac{dc^*}{ds} = f'(k^*) \frac{dk^*}{ds} - (n + \delta) \frac{dk^*}{ds}$$

$$= [f'(k^*) - (n + \delta)] \underbrace{\frac{dk^*}{ds}}_{> 0}$$

$f'(k^*)$



S THAT MAXIMIZES C IS CALLED
"GOLDEN RULE LEVEL"

$$k_{\text{GOLD}}^* \\ c_{\text{GOLD}}^*$$

CHARACTERIZED BY $\frac{dc^*}{ds} = 0$

OR

$$f'(k_{\text{GOLD}}^*) = \delta + m$$

TO CHECK MAX AND NOT MIN :

$$\frac{d^2 c^*}{ds^2} = \frac{d}{ds} \left[[f'(k^*) - (\delta + m)] \frac{dk^*}{ds} \right]$$

RAMSEY - CASS - KOOPMAN / NEOCLASSICAL

- CONTINUUM OF HH \rightarrow REPRESENT. HH
- " OF FIRMS \rightarrow " FIRM

HH:

- WORK
- OWN k
- CONSUME
- RENT k TO FIRMS

FIRMS:

- PRODUCE USING LABOR l & k
- PAY w TO LABOR
- PAY r TO k
- ALL MARKETS PERFECTLY COMPETITIVE

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- PRODUCE USING LABOR L & K
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 - ALL MARKETS PERFECTLY COMPETITIVE (ZERO PROFITS AND FREE ENTRY)

$$L(t) = e^{rt} L(0)$$

$$= e^{rt} \quad (\dot{L} = rL)$$

LOWER CASE

FIRMS

NUMERAIRE

$$\pi(K, L) := Y - RK - wL$$

$$\begin{aligned} \max_{\substack{K>0 \\ L \geq 0}} \pi(K, L) &= \max_{\substack{K>0 \\ L \geq 0}} F(K, L) - RK - wL \end{aligned}$$

FOC (NECESSARY AND SUFFICIENT
GIVEN ASSUMPTIONS ON F)

$$\frac{\partial \Pi}{\partial K} = 0 \quad ; \quad \frac{\partial \Pi}{\partial L} = 0$$

$$\frac{\partial F}{\partial K} - R = 0 \quad ; \quad \frac{\partial F}{\partial L} - w = 0$$

$$\frac{\partial F}{\partial K} = R \quad ; \quad \frac{\partial F}{\partial L} = w$$

$$\begin{aligned}\Pi(K, L) &= F\left(\frac{K}{L}, \frac{L}{L}\right)L - RkL - wL \\ &= [f(k) - Rk - w]L \\ &\stackrel{=:}{=} \phi(k)L\end{aligned}$$

$$\underline{\text{FOC}}: \quad \phi'(k) = 0 \quad \Leftrightarrow f'(k) = R$$

$$\text{ZERO PROFITS} \Rightarrow \phi(k) = 0$$

HOUSEHOLD

$u(C(t))$ AT EACH t

$$u' > 0 \quad ; \quad u'' < 0 \quad ; \quad \lim_{c \rightarrow 0} u'(c) = \infty$$

DISCRETE TIME: β^t , $\beta \in (0, 1]$

CONT. TIME: $e^{-\ell t}$, $\ell > 0$ $\lim_{c \rightarrow \infty} u'(c) = 0$

$$\int_0^\infty e^{-\ell t} \mu(c(t)) L(t) dt$$

DISCRETE TIME

$$L(t) = e^{nt} \rightarrow L(t+1) = (1 + g_L) L(t)$$

$$\int_0^\infty e^{-\ell t} \mu(c(t)) L(t) dt \rightarrow \sum_{t=0}^{\infty} p^t \mu(c(t)) L(t)$$

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-(\ell - r)t} \mu(c(t)) dt \\ c(t) > 0 & \\ k(t) > 0 & \text{s.t.} \end{array}$$

BUDGET CONSTRAINT.

- FINANCIAL ASSETS
L BOND (RISKLESS) } ZERO NET SUPPLY

- REAL ASSETS
L CAPITAL (RISKLESS)

$a(t) L(t)$: ASSETS OF HH.

$$\frac{\partial}{\partial t} [a(t)L(t)] = a(t)L(t)r(t) + w(t)L(t)$$

$$- c(t) L(t)$$

$$\dot{a}(t) = a(t)r(t) + \omega(t) - c(t) - ma(t)$$

$$= a(t)(r(t)-m) + \omega(t) - c(t)$$

$$a(t) \leftarrow \begin{array}{l} \text{BONDS} \rightarrow \text{ZERO} \\ \text{CAPITAL} \rightarrow k(t) \end{array}$$

IN EQUILIBRIUM

$$a(t) = k(t)$$

$$r(t) = R(t) - \delta$$

$$\dot{k}(t) = k(t)(R(t) - \delta - m) + \omega(t) - c(t)$$

IF NO BONDS $\rightarrow k(t) \geq 0$

IF BONDS \rightarrow "NO PONZI"

TRANSVERSALITY
CONDITION

$$\lim_{t \rightarrow \infty} a(t) \exp\left(-\int_0^t (r(s)-m) ds\right) \geq 0$$

MH PROBLEM

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-(\ell-m)t} u(C(t)) dt \\ c(t) > 0 & \\ a(t) > 0 & \text{s.t.} \end{array}$$

$$\dot{a}(t) = a(t)(r(t)-m) + w(t) - c(t)$$

$$\lim_{t \rightarrow \infty} a(t) \exp \left(- \int_0^t (r(s)-m) ds \right) \geq 0$$

TO KEEP UTILITY FINITE WE NEED

$$\ell - m > 0$$

MH PROBLEM

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-(\ell-m)t} u(c(t)) dt \\ c(t) > 0 & \\ \alpha(t) > 0 & \text{s.t.} \quad \alpha(0) = \alpha_0 \end{array}$$

$$\dot{\alpha}(t) = \alpha(t)(r(t)-m) + w(t) - c(t)$$

$$\lim_{t \rightarrow \infty} \alpha(t) \exp \left(- \int_0^t (r(s)-m) ds \right) \geq 0$$

TO KEEP UTILITY FINITE WE NEED

$$\ell - m > 0$$

MAXIMUM PRINCIPLE

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-\ell t} f(x(t), u(t)) dt \\ u(\cdot) & \\ & \text{s.t.} \end{array}$$

$$\dot{x}(t) = \mu(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$x(0) = x_0 \quad \text{GIVEN}$$

$$u(\cdot) \in \mathcal{U}$$

NECESSARY CONDITIONS:

1. STATE EQUATION & FEASIBILITY

$$\dot{x}(t) = h(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$x(0) = x_0 \quad ; \quad u(\cdot) \in \mathcal{U}$$

2. Let

$$J(t, x(t), u(t), \lambda(t), \phi(t)) :=$$

$$\begin{aligned} & f(x(t), u(t)) + \lambda(t) h(t, x(t), u(t)) \\ & + \phi(t) G(x(t), u(t)) \end{aligned}$$

$$u^*(t) \in \underset{u}{\operatorname{MAX}} \quad J(t, x^*(t), u, \lambda(t), \phi(t))$$

$$G(x^*(t), u) \geq 0$$

FOC: USUAL CALCULUS CONSTRAINT
OPTIM.

3. DYNAMICS OF $\lambda(t)$

$$\dot{\lambda}(t) = \rho \lambda(t) - \frac{\partial \mathcal{J}_L}{\partial x}(t, x^*(t), u^*(t), \lambda(t), \phi(t))$$

4. COMPLEMENTARY SLACKNESS

$$\phi(t) \geq 0 \quad \phi(t) G(x^*(t), u^*(t)) = 0$$

5. TRANSVERSALITY

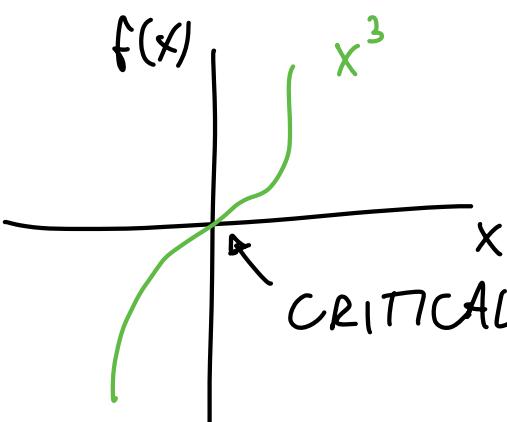
$$\lim_{t \rightarrow \infty} e^{-\rho t} \mathcal{J}_L(t, x^*(t), u^*(t), \lambda(t), \phi(t)) = 0$$

OR

(UNDER SOME ASSUMPTIONS)

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) x(t) = 0$$

COMMENTS



- NECESSARY

CRITICAL POINT : $f'(x) = 0$ AT $x=0$
 BUT $x=0$ NOT
 MAX OR MIN

- λ, ϕ ARE "SHADOW VALUES" OF CONSTRAINTS.
- "PRESENT VALUE \mathcal{H}^{PV} "

$$\mathcal{H} = e^{-\lambda t} f + \lambda^{PV} h + \phi^{PV} \cdot g$$

$$\lambda^{PV} = e^{-\lambda t} \lambda$$

$$\phi^{PV} = e^{-\lambda t} \phi$$

$$\begin{aligned}\dot{\lambda}^{PV} &= e^{-\lambda t} \dot{\lambda} \\ &\quad - e^{-\lambda t} e^{\lambda t} \lambda \\ &= e^{-\lambda t} (\dot{\lambda} - e^{\lambda t} \lambda)\end{aligned}$$

$$\dot{\lambda}^{PV} = - \frac{\partial \mathcal{J}_L}{\partial x}$$

• BACKWARD INDUCTION

IMAGINE HORIZON IS $T < \infty$

At $t = T$

\dot{x} EQUATION DOES NOT
APPLY

\Rightarrow MAX f w/o THINKING
ABOUT FUTURE x

IF AT T, NO CONSTRAINTS ON x

$$\lambda(T) = 0$$

(w/ CONSTRAINTS : $\lambda(T) = G' \dots$)

$\dot{\lambda}$ EQ IS A "BACKWARD EQ"

IN INFINITE HORIZON, WHAT IS

" $\lambda(T)$ " ?

ANSW: TRANSVERSALITY!

NOT

$$\lim_{t \rightarrow \infty} e^{et} \lambda(t) = 0 \quad !!$$

$$\mathcal{H} = \underbrace{\mu(c(t))}_{f} + \lambda(t) \left[\underbrace{\omega(t)(r(t) - m)}_{h} + \omega(t) - c(t) \right]$$

$$\dot{\lambda}(t) = (\ell - m) \lambda(t) - \frac{\partial \mathcal{H}}{\partial a}(t, \dots)$$

$$\dot{\lambda}(t) = (\ell - m) \lambda(t) - (r(t) - m) \lambda(t)$$

(1) $\dot{\lambda}(t) = [\ell - r(t)] \lambda(t)$

(2) $\max_c \mathcal{H} \Rightarrow \frac{\partial \mathcal{H}}{\partial c} = 0 \Rightarrow \mu'(c^*(t)) = \lambda(t)$

From (1) $\frac{\dot{\lambda}(t)}{\lambda(t)} = \ell - r(t)$

$$\frac{\partial}{\partial t} (\log \lambda(t)) = e - r(t)$$

$$\log \lambda(t) = \log \lambda(0) + \int_0^t [e - r(s)] ds$$

$$\int_0^T d \log (\lambda(t)) = \int_0^T [e - r(t)] dt$$

$$\log \lambda(T) - \log \lambda(0) = \int_0^T [e - r(t)] dt$$

USE (2) :

$$\mu'(c^*) = \lambda(0) \exp \left(- \int_0^t [r(s) - e] ds \right)$$

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USE (2) :

$$\lambda(t) = u'(c^*) = \lambda(0) \exp \left(- \int_0^t [r(s) - e] ds \right)$$

(3) TRANSVERSALITY :

$$\lim_{t \rightarrow \infty} e^{-(e-r)} a(t) \lambda(t) = 0$$

(FINITE HORIZON : $a(T) = 0$)

L EITHER NEED $a(t) \geqslant$ LOWER BOUND

OR

L ADMISSIBLE CONTROLS / STATES
ARE FINITE FOR ALL t

COMBINE (2) AND (3). THEN,

NO PONZI HOLDS WITH EQUALITY

COMBINE $\mu'(C(t)) = \lambda(t)$ AND

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -(r(t) - \ell)$$

$$\frac{\partial}{\partial t} \mu'(C(t)) = \mu''(C(t)) \dot{C}(t) = \dot{\lambda}(t)$$

$$\Rightarrow \mu''(C(t)) \dot{C}(t) = -(r(t) - \ell) \lambda(t) \\ = -(r(t) - \ell) \mu'(C(t))$$

$$\frac{\dot{C}(t)}{C(t)} = - \underbrace{\frac{\mu'(C(t))}{C(t) \mu''(C(t))}}_{\text{ELASTICITY OF}} (r(t) - \ell)$$

ELASTICITY OF

INTERTEMPORAL

SUBSTITUTION

$$:= [\Theta(t)]^{-1}$$

ELASTICITY B/W t, s :

$$\Theta(t, s) := \frac{\partial (\log [u'(C(t)) / u'(C(s))])}{\partial (\log [C(t) / C(s)])}$$

$$\lim_{s \rightarrow t} \Theta(t, s) = \Theta(t)$$

USE FIRM OPTIMALITY:

$$f'(k) = R(t)$$

$$r(t) = f'(k) - \delta$$

ZERO PROFITS:

$$w(t) = f(k) - f'(k)k$$

USE MARKET CLEARING:

$$a(t) = k(t)$$

(SINCE BONDS IN ZERO NET SUPPLY)

BUDGET CONSTRAINT:

$$\dot{a}(t) = \alpha(t)(r(t) - \gamma) + \omega(t) - c(t)$$

$$\begin{aligned}\dot{k}(t) &= k(t)(f'(k) - \delta - \gamma) + \\ &+ f(k) - f'(k)k(t) - c(t)\end{aligned}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + \gamma)k(t)$$

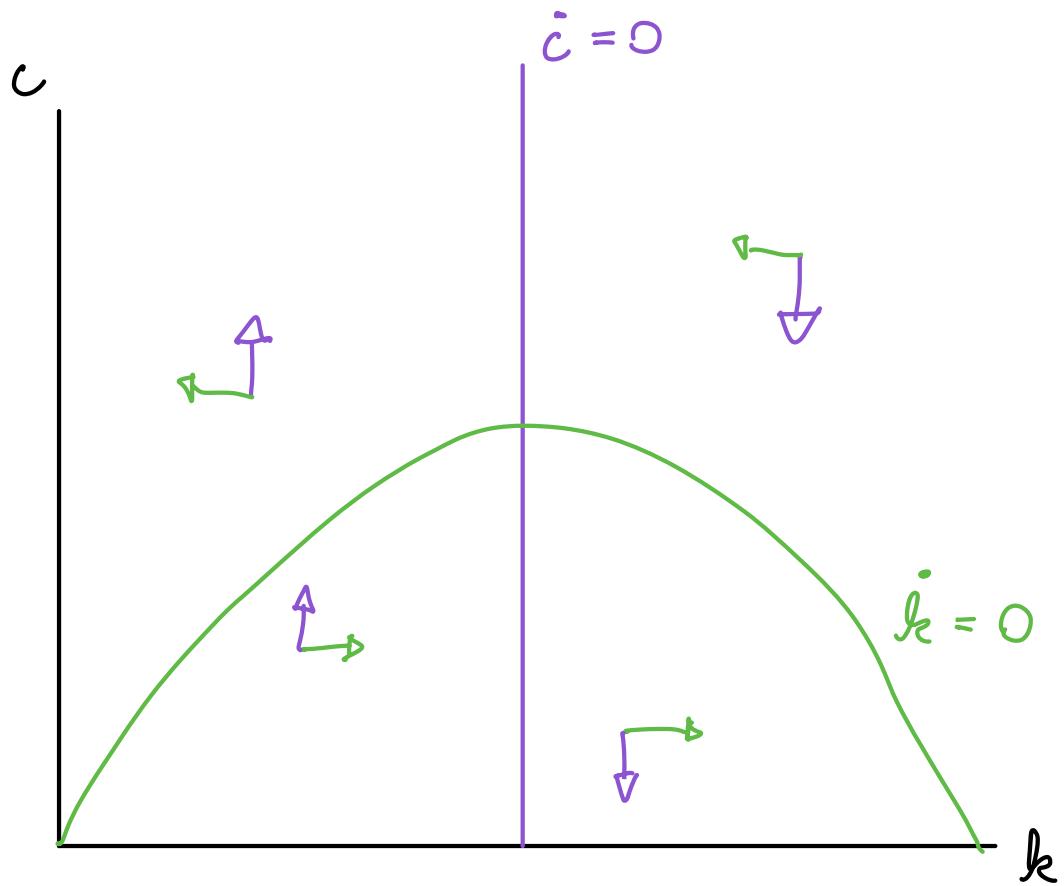
$$k(0) = k_0 \quad + \quad \text{TRANSVERSALITY}$$

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'(c(t))}{c(t) u''(c(t))} (r(t) - \rho)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\Theta(t)} (f'(k(t)) - \delta - \rho)$$

$$\dot{c}=0 \Rightarrow f'(k) = \delta + \rho$$

PHASE DIAGRAM



$$\dot{k} = 0 \Rightarrow 0 = f(k) - c - (\delta + m)k$$

$$\Rightarrow c = f(k) - (\delta + m)k$$

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etc.

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= ...

$$= \frac{\dot{k}(t)}{L(t)} + \frac{\dot{L}(t)}{L(t)} \frac{k(t)}{L(t)}$$

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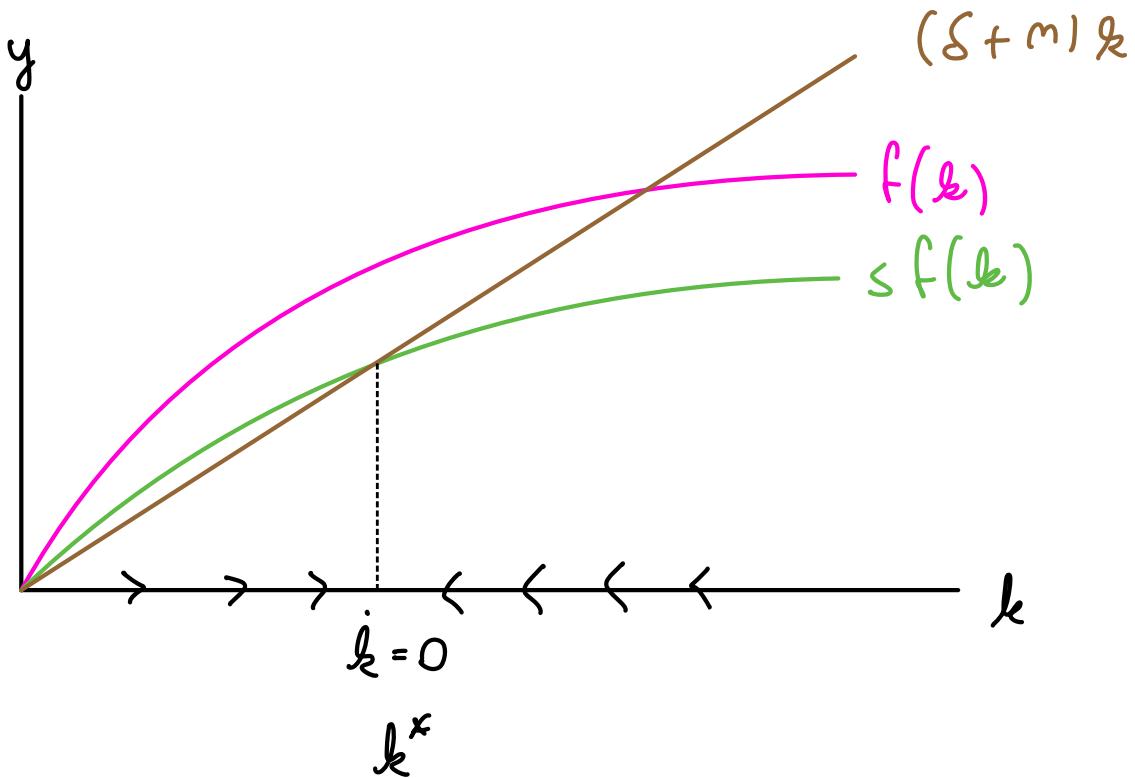
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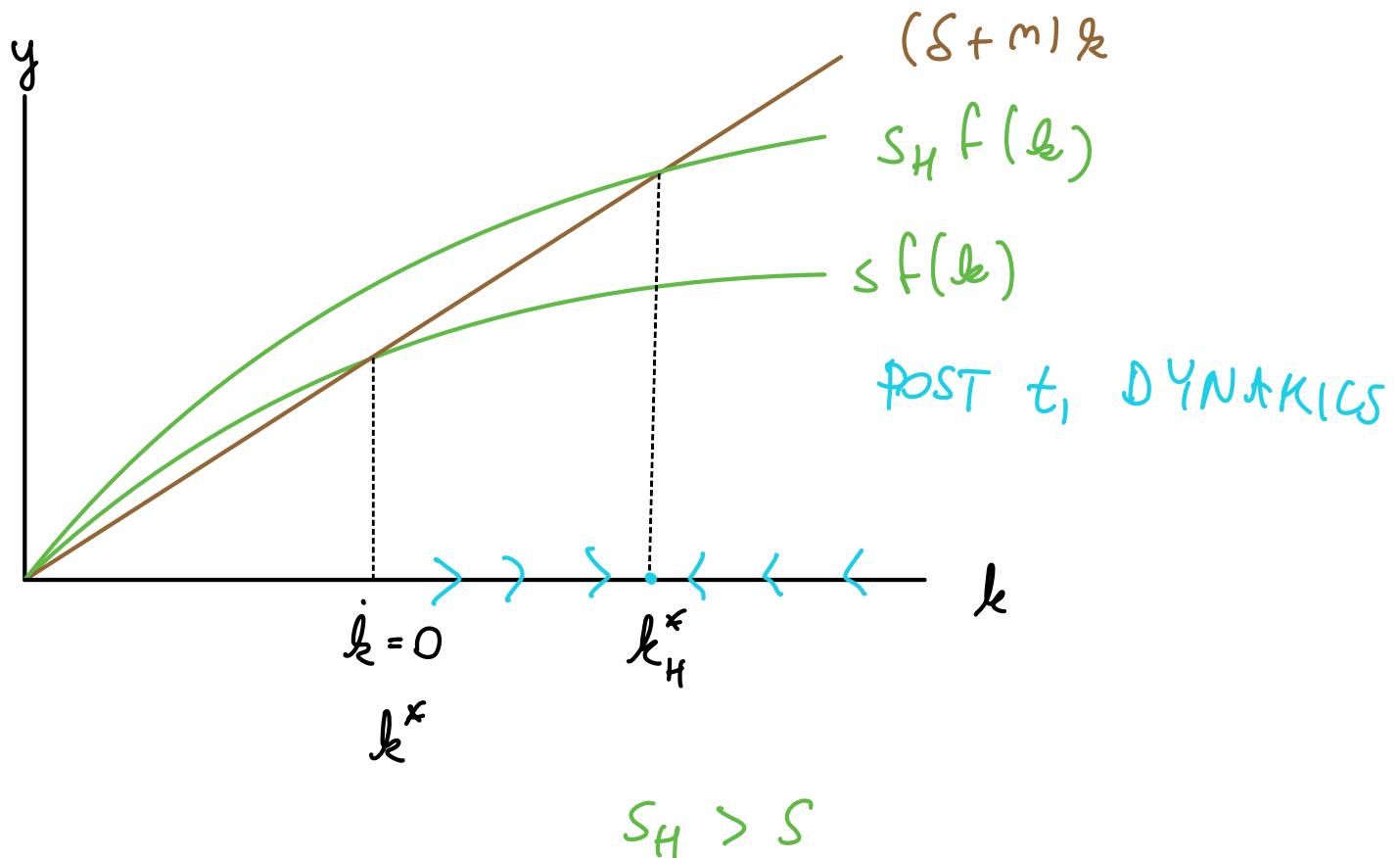
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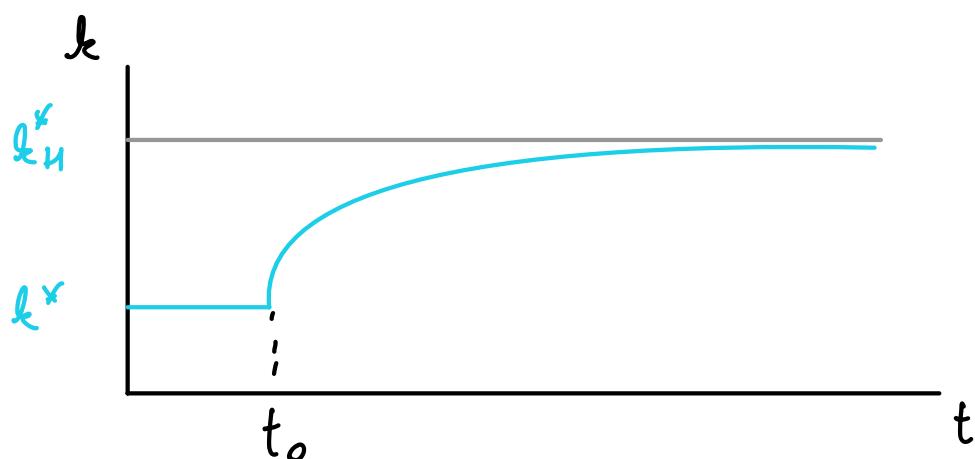
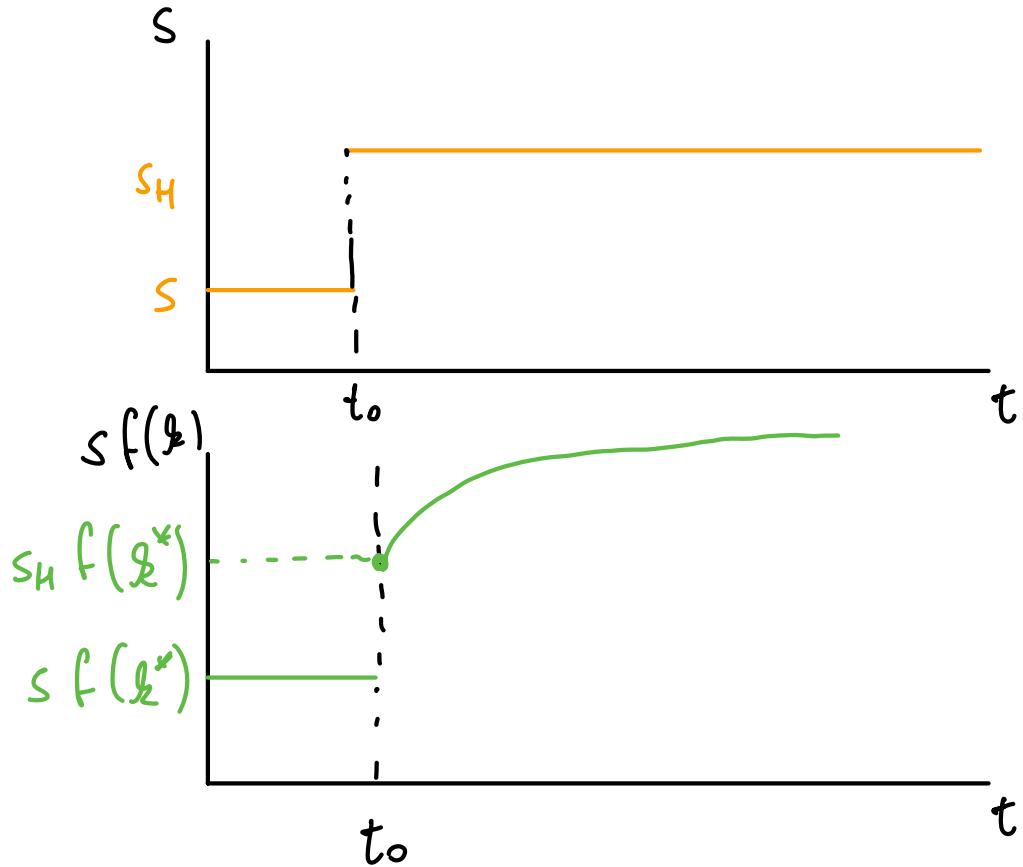
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CONSUMPTION

$$\begin{aligned} c &= (1-s)y = (1-s)f(k) \\ &= f(k) - sf(k) \end{aligned}$$

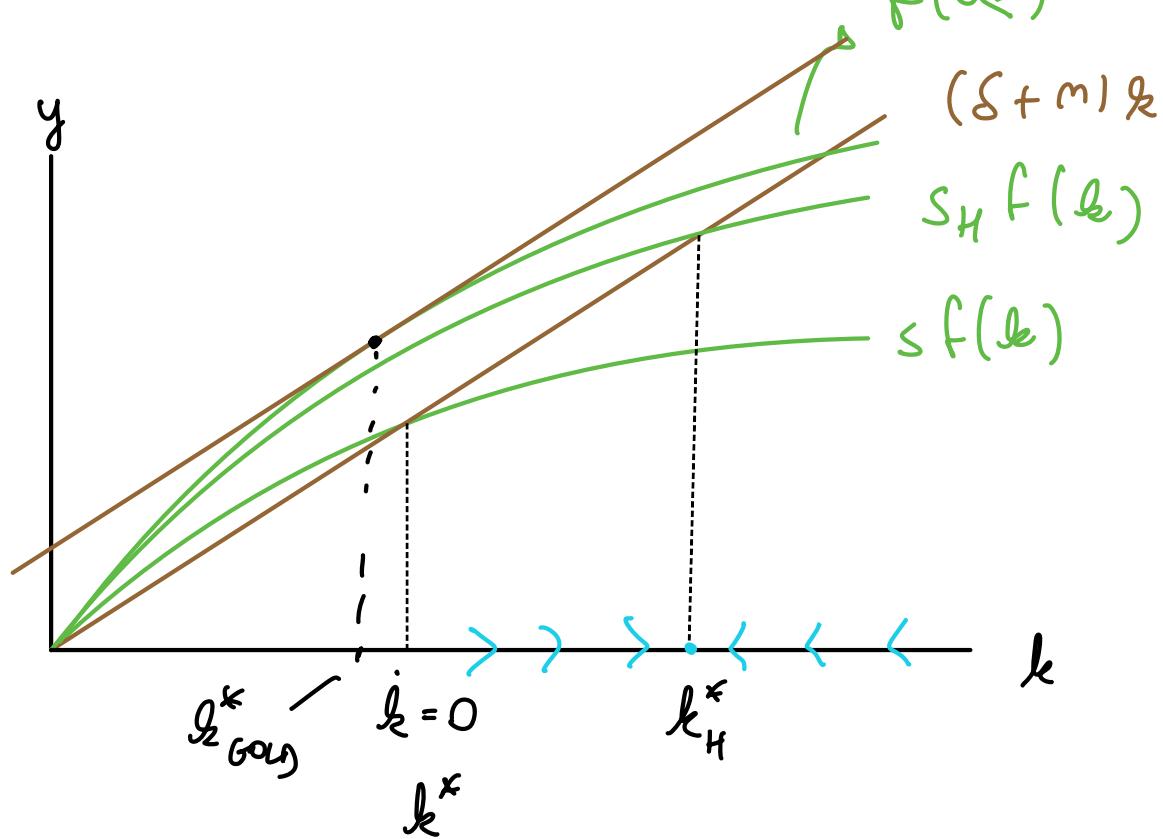
$$\begin{aligned} \text{IN SS: } \dot{k} = 0 &\Leftrightarrow 0 = sf(k^*) - (n + \delta)k^* \\ &\Rightarrow sf(k^*) = (n + \delta)k^* \end{aligned}$$

$$c^* = \underbrace{f(k^*)}_{\substack{\text{INCREASING} \\ \text{IN } k^*}} - \underbrace{(n + \delta)k^*}_{\substack{\text{DECREASING} \\ \text{IN } k^*}}$$

$$\frac{dc^*}{ds} = f'(k^*) \frac{dk^*}{ds} - (n + \delta) \frac{dk^*}{ds}$$

$$= [f'(k^*) - (n + \delta)] \underbrace{\frac{dk^*}{ds}}_{> 0}$$

$f'(k^*)$



S THAT MAXIMIZES C IS CALLED
"GOLDEN RULE LEVEL"

$$k_{\text{GOLD}}^* \\ c_{\text{GOLD}}^*$$

CHARACTERIZED BY $\frac{dc^*}{ds} = 0$

OR

$$f'(k_{\text{GOLD}}^*) = \delta + m$$

TO CHECK MAX AND NOT MIN :

$$\frac{d^2 c^*}{ds^2} = \frac{d}{ds} \left[[f'(k^*) - (\delta + m)] \frac{dk^*}{ds} \right]$$

RAMSEY - CASS - KOOPMAN / NEOCLASSICAL

- CONTINUUM OF HH \rightarrow REPRESENT. HH
- " OF FIRMS \rightarrow " FIRM

- HH:
- WORK
 - OWN K
 - CONSUME
 - RENT K TO FIRMS

- FIRMS:
- PRODUCE USING LABOR L & K
 - PAY w TO LABOR
 - PAY R TO K
 - ALL MARKETS PERFECTLY COMPETITIVE (ZERO PROFITS AND FREE ENTRY)

$$L(t) = e^{rt} L(0)$$

$$= e^{rt} \quad (\dot{L} = rL)$$

LOWER CASE

FIRMS

NUMERAIRE

$$\pi(K, L) := Y - RK - wL$$

$$\begin{aligned} \max_{\substack{K>0 \\ L \geq 0}} \pi(K, L) &= \max_{\substack{K>0 \\ L \geq 0}} F(K, L) - RK - wL \end{aligned}$$

FOC (NECESSARY AND SUFFICIENT
GIVEN ASSUMPTIONS ON F)

$$\frac{\partial \Pi}{\partial K} = 0 \quad ; \quad \frac{\partial \Pi}{\partial L} = 0$$

$$\frac{\partial F}{\partial K} - R = 0 \quad ; \quad \frac{\partial F}{\partial L} - w = 0$$

$$\frac{\partial F}{\partial K} = R \quad ; \quad \frac{\partial F}{\partial L} = w$$

$$\begin{aligned}\Pi(K, L) &= F\left(\frac{K}{L}, \frac{L}{L}\right)L - RkL - wL \\ &= [f(k) - Rk - w]L \\ &\stackrel{=:}{=} \phi(k)L\end{aligned}$$

$$\underline{\text{FOC}}: \quad \phi'(k) = 0 \quad \Leftrightarrow f'(k) = R$$

$$\text{ZERO PROFITS} \Rightarrow \phi(k) = 0$$

HOUSEHOLD

$u(C(t))$ AT EACH t

$$u' > 0 \quad ; \quad u'' < 0 \quad ; \quad \lim_{c \rightarrow 0} u'(c) = \infty$$

DISCRETE TIME: β^t , $\beta \in (0, 1]$

CONT. TIME: $e^{-\ell t}$, $\ell > 0$ $\lim_{c \rightarrow \infty} u'(c) = 0$

$$\int_0^\infty e^{-\ell t} \mu(c(t)) L(t) dt$$

DISCRETE TIME

$$L(t) = e^{nt} \rightarrow L(t+1) = (1 + g_L) L(t)$$

$$\int_0^\infty e^{-\ell t} \mu(c(t)) L(t) dt \rightarrow \sum_{t=0}^{\infty} p^t \mu(c(t)) L(t)$$

$$\begin{aligned} & \text{MAX} && \int_0^\infty e^{-(\ell - r)t} \mu(c(t)) dt \\ & c(t) > 0 && \\ & k(t) > 0 && \text{s.t.} \end{aligned}$$

BUDGET CONSTRAINT.

- FINANCIAL ASSETS
L BOND (RISKLESS) } ZERO NET SUPPLY

- REAL ASSETS
L CAPITAL (RISKLESS)

$a(t) L(t)$: ASSETS OF HH.

$$\frac{\partial}{\partial t} [a(t)L(t)] = a(t)L(t)r(t) + w(t)L(t)$$

$$- c(t) L(t)$$

$$\dot{a}(t) = a(t)r(t) + \omega(t) - c(t) - ma(t)$$

$$= a(t)(r(t)-m) + \omega(t) - c(t)$$

$$a(t) \leftarrow \begin{array}{l} \text{BONDS} \rightarrow \text{ZERO} \\ \text{CAPITAL} \rightarrow k(t) \end{array}$$

IN EQUILIBRIUM

$$a(t) = k(t)$$

$$r(t) = R(t) - \delta$$

$$\dot{k}(t) = k(t)(R(t) - \delta - m) + \omega(t) - c(t)$$

IF NO BONDS $\rightarrow k(t) \geq 0$

IF BONDS \rightarrow "NO PONZI"

TRANSVERSALITY
CONDITION

$$\lim_{t \rightarrow \infty} a(t) \exp\left(-\int_0^t (r(s)-m) ds\right) \geq 0$$

MH PROBLEM

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-(\ell-m)t} u(c(t)) dt \\ c(t) > 0 & \text{s.t.} \quad a(0) = a_0 \\ \text{NOT NEEDED } [\cancel{a(t) > 0}] & \end{array}$$

$$\dot{a}(t) = a(t)(r(t)-m) + w(t) - c(t)$$

$$\lim_{t \rightarrow \infty} a(t) \exp \left(- \int_0^t (r(s)-m) ds \right) \geq 0$$

TO KEEP UTILITY FINITE WE NEED

$$\ell - m > 0$$

MAXIMUM PRINCIPLE

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-\ell t} f(x(t), u(t)) dt \\ u(\cdot) & \text{s.t.} \end{array}$$

$$\dot{x}(t) = \mu(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$x(0) = x_0 \quad \text{GIVEN}$$

$$u(\cdot) \in \mathcal{U}$$

NECESSARY CONDITIONS:

1. STATE EQUATION & FEASIBILITY

$$\dot{x}(t) = h(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$x(0) = x_0 \quad ; \quad u(\cdot) \in \mathcal{U}$$

2. Let

$$J(t, x(t), u(t), \lambda(t), \phi(t)) :=$$

$$\begin{aligned} & f(x(t), u(t)) + \lambda(t) h(t, x(t), u(t)) \\ & + \phi(t) G(x(t), u(t)) \end{aligned}$$

$$u^*(t) \in \underset{u}{\operatorname{MAX}} \quad J(t, x^*(t), u, \lambda(t), \phi(t))$$

$$G(x^*(t), u) \geq 0$$

FOC: USUAL CALCULUS CONSTRAINT
OPTIM.

3. DYNAMICS OF $\lambda(t)$

$$\dot{\lambda}(t) = \rho \lambda(t) - \frac{\partial \mathcal{J}_L}{\partial x}(t, x^*(t), u^*(t), \lambda(t), \phi(t))$$

4. COMPLEMENTARY SLACKNESS

$$\phi(t) \geq 0 \quad \phi(t) G(x^*(t), u^*(t)) = 0$$

5. TRANSVERSALITY

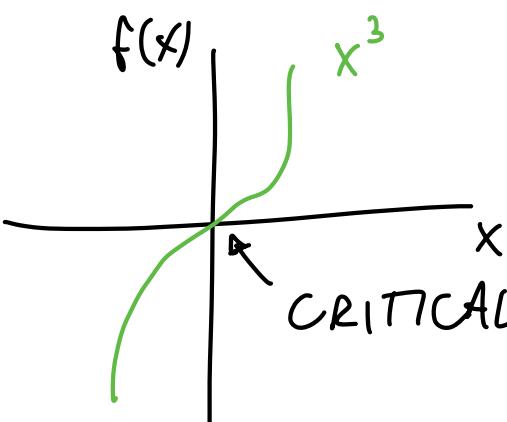
$$\lim_{t \rightarrow \infty} e^{-\rho t} \mathcal{J}_L(t, x^*(t), u^*(t), \lambda(t), \phi(t)) = 0$$

OR

(UNDER SOME ASSUMPTIONS)

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) x(t) = 0$$

COMMENTS



- NECESSARY

CRITICAL POINT : $f'(x) = 0$ AT $x=0$
 BUT $x=0$ NOT
 MAX OR MIN

- λ, ϕ ARE "SHADOW VALUES" OF CONSTRAINTS.
- "PRESENT VALUE \mathcal{H}^{PV} "

$$\mathcal{H} = e^{-\lambda t} f + \lambda^{PV} h + \phi^{PV} \cdot g$$

$$\lambda^{PV} = e^{-\lambda t} \lambda$$

$$\phi^{PV} = e^{-\lambda t} \phi$$

$$\begin{aligned}\dot{\lambda}^{PV} &= e^{-\lambda t} \dot{\lambda} \\ &\quad - e^{-\lambda t} e^{\lambda t} \lambda \\ &= e^{-\lambda t} (\dot{\lambda} - e^{\lambda t} \lambda)\end{aligned}$$

$$\dot{\lambda}^{PV} = - \frac{\partial \mathcal{J}_L}{\partial x}$$

• BACKWARD INDUCTION

IMAGINE HORIZON IS $T < \infty$

At $t = T$

\dot{x} EQUATION DOES NOT
APPLY

\Rightarrow MAX f w/o THINKING
ABOUT FUTURE x

IF AT T, NO CONSTRAINTS ON x

$$\lambda(T) = 0$$

(w/ CONSTRAINTS : $\lambda(T) = G' \dots$)

$\dot{\lambda}$ EQ IS A "BACKWARD EQ"

IN INFINITE HORIZON, WHAT IS

" $\lambda(T)$ " ?

ANSW: TRANSVERSALITY!

NOT

$$\lim_{t \rightarrow \infty} e^{et} \lambda(t) = 0 \quad !!$$

$$\mathcal{H} = \underbrace{\mu(c(t))}_{f} + \lambda(t) \left[\underbrace{\omega(t)(r(t) - m)}_{h} + \omega(t) - c(t) \right]$$

$$\dot{\lambda}(t) = (\ell - m) \lambda(t) - \frac{\partial \mathcal{H}}{\partial a}(t, \dots)$$

$$\dot{\lambda}(t) = (\ell - m) \lambda(t) - (r(t) - m) \lambda(t)$$

(1) $\dot{\lambda}(t) = [\ell - r(t)] \lambda(t)$

(2) $\max_c \mathcal{H} \Rightarrow \frac{\partial \mathcal{H}}{\partial c} = 0 \Rightarrow \mu'(c^*(t)) = \lambda(t)$

FROM (1) $\frac{\dot{\lambda}(t)}{\lambda(t)} = \ell - r(t)$

$$\frac{\partial}{\partial t} (\log \lambda(t)) = e - r(t)$$

$$\log \lambda(t) = \log \lambda(0) + \int_0^t [e - r(s)] ds$$

$$\int_0^T d \log(\lambda(t)) = \int_0^T [e - r(t)] dt$$

$$\log \lambda(T) - \log \lambda(0) = \int_0^T [e - r(t)] dt$$

USE (2) :

$$\lambda(t) = u'(c^*) = \lambda(0) \exp \left(- \int_0^t [r(s) - e] ds \right)$$

(3) TRANSVERSALITY :

$$\lim_{t \rightarrow \infty} e^{-(e-m)} a(t) \lambda(t) = 0$$

(FINITE HORIZON : $a(T) = 0$)

L EITHER NEED $a(t) \geqslant$ LOWER BOUND

OR

L ADMISSIBLE CONTROLS / STATES
ARE FINITE FOR ALL t

COMBINE (2) AND (3). THEN,

NO PONZI HOLDS WITH EQUALITY

COMBINE $\mu'(C(t)) = \lambda(t)$ AND

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -(r(t) - \ell)$$

$$\frac{\partial}{\partial t} \mu'(C(t)) = \mu''(C(t)) \dot{C}(t) = \dot{\lambda}(t)$$

$$\Rightarrow \mu''(C(t)) \dot{C}(t) = -(r(t) - \ell) \lambda(t) \\ = -(r(t) - \ell) \mu'(C(t))$$

$$\frac{\dot{C}(t)}{C(t)} = - \underbrace{\frac{\mu'(C(t))}{C(t) \mu''(C(t))}}_{\text{ELASTICITY OF}} (r(t) - \ell)$$

ELASTICITY OF

INTERTEMPORAL

SUBSTITUTION

$$:= [\Theta(t)]^{-1}$$

ELASTICITY B/W t, s :

$$\Theta(t, s) := \frac{\partial (\log [u'(C(t)) / u'(C(s))])}{\partial (\log [C(t) / C(s)])}$$

$$\lim_{s \rightarrow t} \Theta(t, s) = \Theta(t)$$

USE FIRM OPTIMALITY:

$$f'(k) = R(t)$$

$$r(t) = f'(k) - \delta$$

ZERO PROFITS:

$$w(t) = f(k) - f'(k)k$$

USE MARKET CLEARING:

$$a(t) = k(t)$$

(SINCE BONDS IN ZERO NET SUPPLY)

BUDGET CONSTRAINT:

$$\dot{a}(t) = \alpha(t)(r(t) - \gamma) + \omega(t) - c(t)$$

$$\begin{aligned}\dot{k}(t) &= k(t)(f'(k) - \delta - \gamma) + \\ &+ f(k) - f'(k)k(t) - c(t)\end{aligned}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + \gamma)k(t)$$

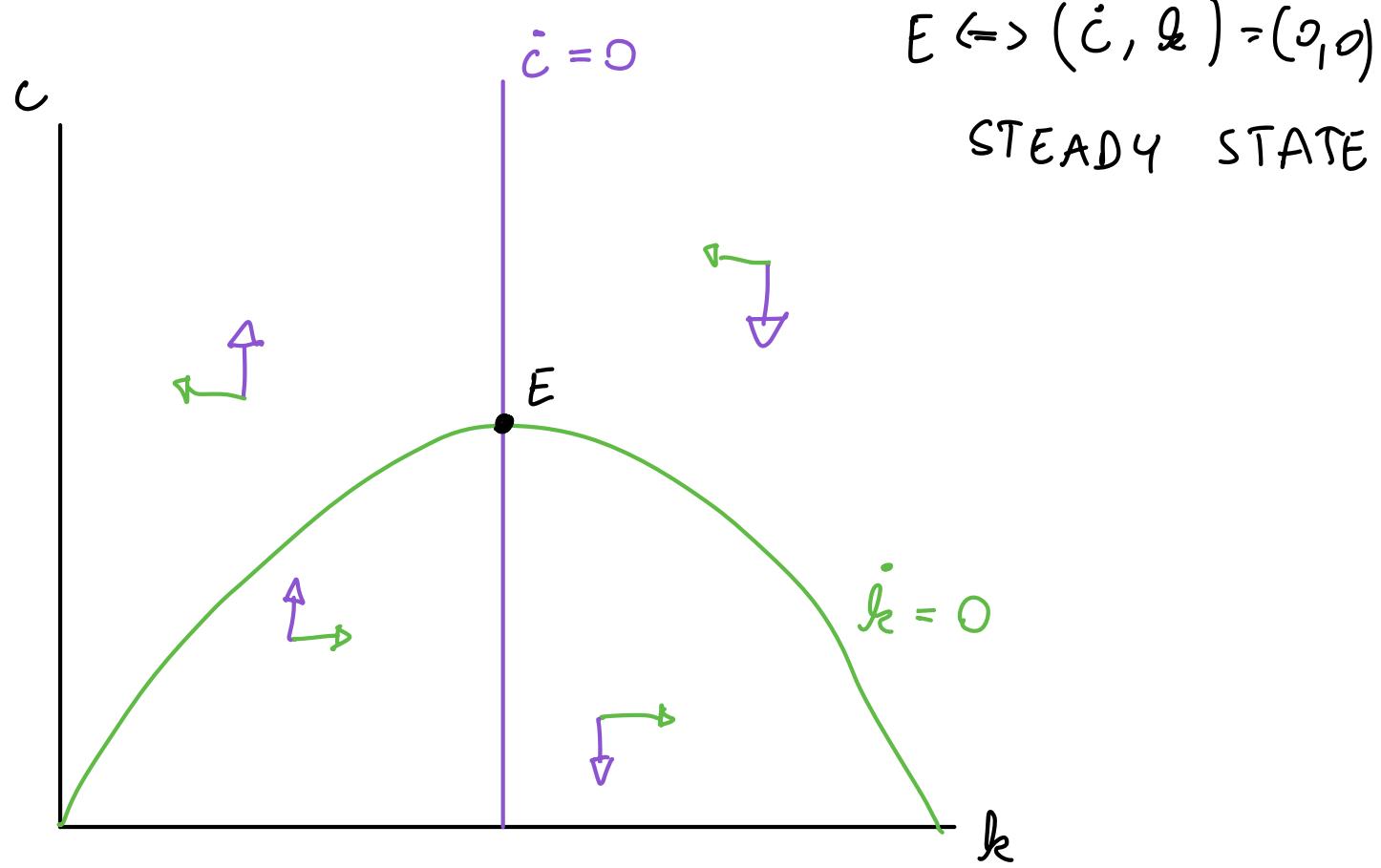
$$k(0) = k_0 \quad + \quad \text{TRANSVERSALITY}$$

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'(c(t))}{c(t) u''(c(t))} (r(t) - \rho)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\Theta(t)} (f'(k(t)) - \delta - \rho)$$

$$\dot{c}=0 \Rightarrow f'(k) = \delta + \rho$$

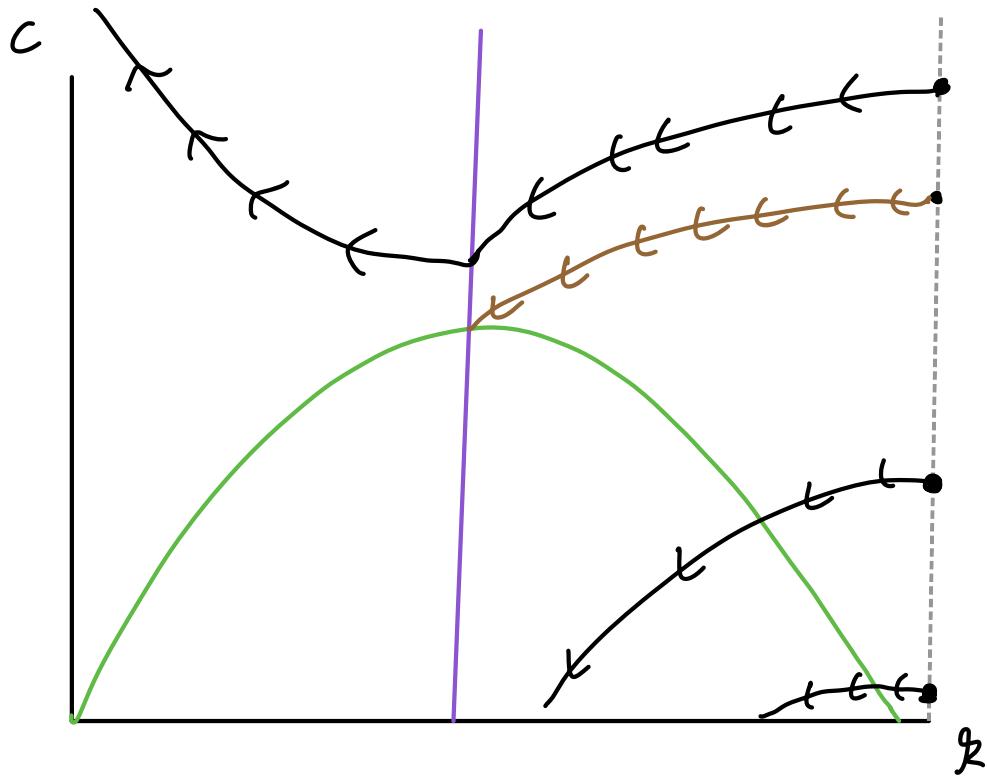
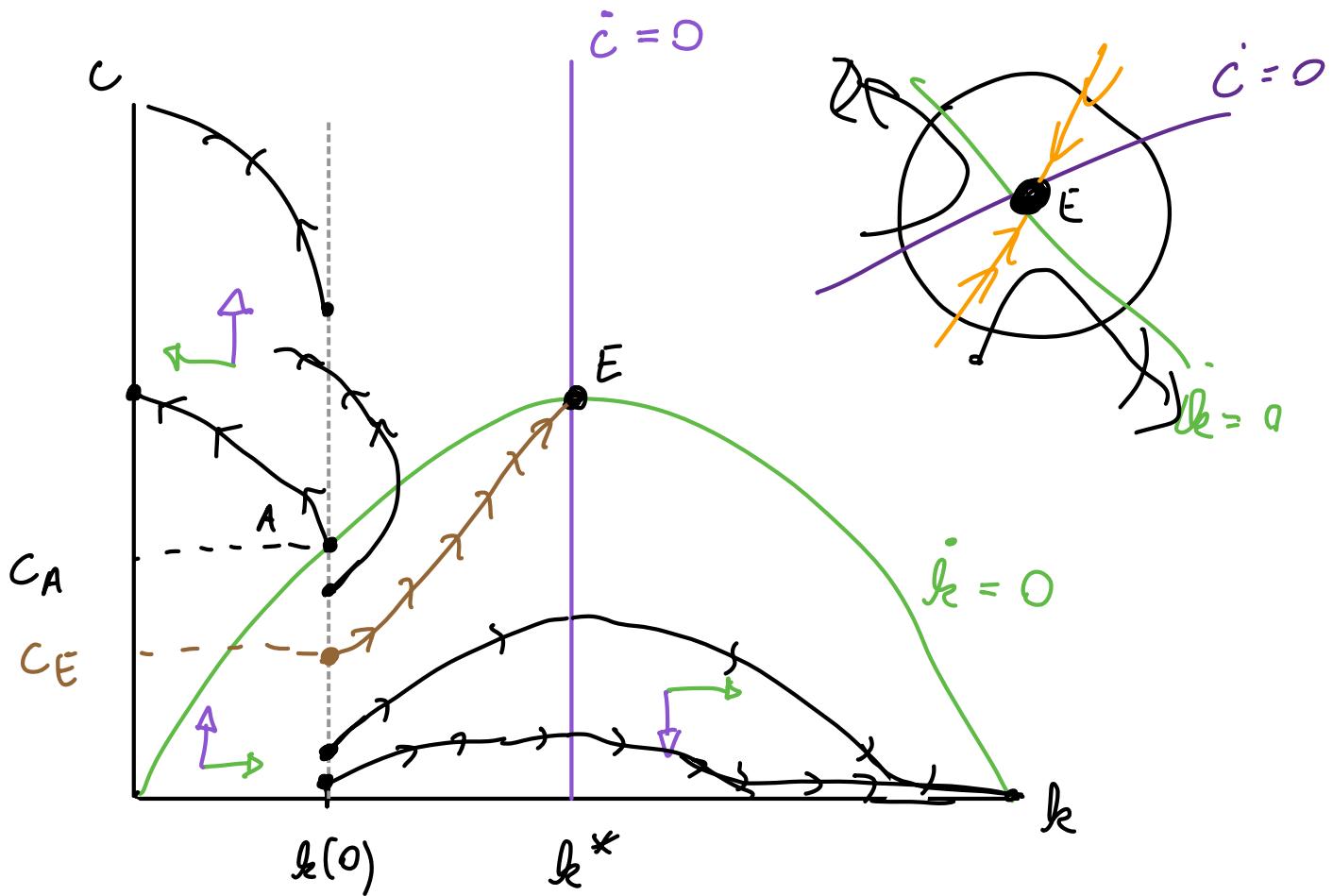
PHASE DIAGRAM



$E \Leftrightarrow (\dot{c}, \dot{k}) = (0, 0)$
STEADY STATE

$$\dot{k} = 0 \Rightarrow 0 = f(k) - c - (\delta + m)k$$

$$\Rightarrow c = f(k) - (\delta + m)k$$



TECH PROGRESS

$$F(k, AL) = K^\alpha (AL)^{1-\alpha}$$

INSTEAD OF $k = \frac{K}{L}$ "PER CAPITA"

DO $k = \frac{K}{AL}$ "EFFECTIVE UNITS

$$y = \frac{Y}{AL}$$
 "OF LABOR"

2 MAIN CONCLUSIONS:

- IN BALANCED GROWTH PATH
(STADY STATE IN UNITS OF
EFFECTIVE LABOR)

$\frac{Y}{L}$, $\frac{k}{L}$ GROW OVER TIME

LONG-RUN GROWTH DUE TO

TECHNOLOGY (A)

• IN SOLOW VS NEOCLASSICAL

INTRODUCING TECH GROWTH $\frac{\dot{A}}{A} = g$

SLOW BGP \neq NEOCLASSICAL BGP

BECUSE k^* STEADY STATE

DEPENDS ON PREFERENCES IN
NEOCLASSICAL MODEL

(NOT IN SOLOW)

$$U = \int_0^\infty e^{-(\ell-m)} \underbrace{u(C(t))}_{\text{NOT IMPORTANT FOR}} dt$$

NOT IMPORTANT FOR
BGP IN NEOCLASSIC.
w/o TECH PROGRESS

PROPERTIES:

- (PSET 2) BGP REQUIRES

$$F(K, L, A) = \tilde{F}(K, AL)$$

"HARROD" NEUTRAL

OR LABOR-AUG.

- (NOT IN PSET) BGP REQUIRES

$\theta(t) = \text{CONSTANT AS } t \rightarrow \infty$



EIS

WITH $\theta(t) \rightarrow \bar{\theta}$

k^* SATISFIES

$$f'(k^*) = \ell + \delta + \bar{\theta} \cdot g$$

$$\dot{A} = Ag$$

NEW TERM

WITH θ

PREF.
↓

PLAN FOR MON OCT 6

- ISSUES

(1) SPEED OF CONVERGENCE

- EMPIRICALLY 2% OF GAP $k - k^*$ CLOSES PER YEAR
- NEED LOW δ OR HIGH α TO MATCH

(2) CROSS-SECTIONAL DISPERSION

- EMPIRICALLY : "FLAT" OVER LAST 50 YRS
- MODEL : XS STD DEV 1 TO 0.3 IN 50 YRS
- OTHER ATTEMPTS

$$\frac{1+\gamma}{1-\gamma} = 8 ; \alpha = \frac{1}{3}$$

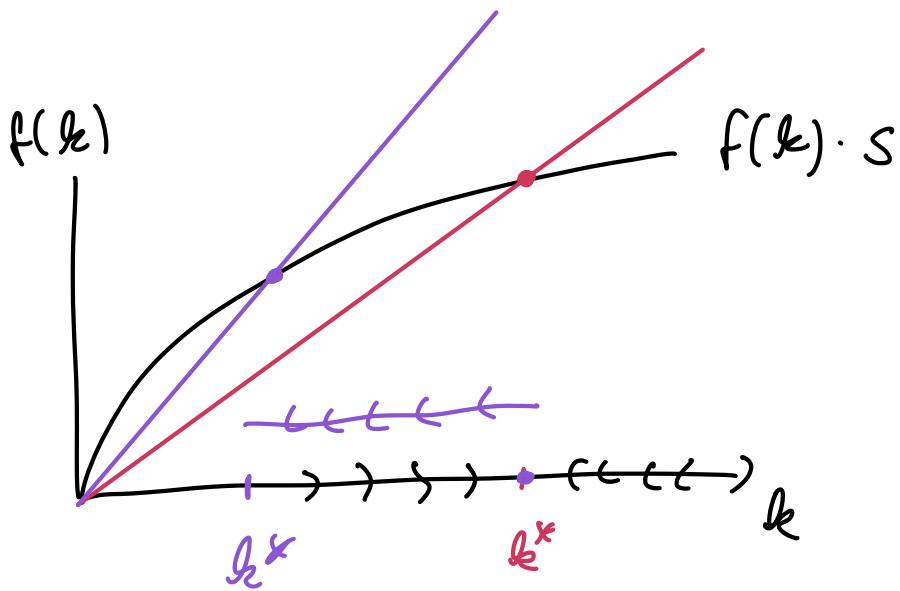
$\Rightarrow \frac{Y}{Y_1} \approx 3$

L PRICE OF INV
 L TAXES
 L MULTIPLE FACTORS

} WEDGE IN
 } $r(t)$, CAN
 } CONFFOUND
 } WITH
 } OTHER
 } EFFECTS

NOT BIG ENOUGH

$r(t)(1+\gamma) \rightarrow \text{OBSERVED}$



NEGATIVE GROWTH OF Y/L :

- $k > k^*$
-

$$\dot{k} = sf(k) - (\delta + m + g)k$$

BGP : $\dot{k} = 0$

$$\frac{f(k)}{k} = \frac{\delta + m + g}{s}$$

$$k^{\alpha-1} = \frac{s + n + g}{s}$$

$$k = \left(\frac{s + n + g}{s} \right)^{\frac{1}{\alpha-1}}$$

$$\frac{K}{LA} = \left(\frac{s + n + g}{s} \right)^{\frac{1}{\alpha-1}}$$

$$\frac{Y}{L} \frac{K}{Y_A} = \frac{Y}{L} \frac{k}{k^\alpha (AL)^{1-\alpha}} \cdot \frac{1}{A}$$

$$= \frac{Y}{L} \left(\frac{k}{AL} \right)^{1-\alpha} \frac{1}{A}$$

$$= \frac{Y}{L} k^{1-\alpha} \frac{1}{A}$$

$$\Rightarrow \frac{Y}{L} = \left(\frac{s + n + g}{s} \right)^{\frac{1}{\alpha-1}} k^{\alpha-1} A$$

$$\frac{d}{dt} \log \left(\frac{Y}{L} \right) = \frac{d}{dt} \log A = g$$

"NEW" MODELS

- HUMAN CAPITAL
 - EXTERNALITIES
 - ↳ "LEARNING BY DOING"
 - ↳ CAN HAVE LONG RUN GROWTH w/o A
 - RESEARCH & DEVELOPMENT
 - INSTITUTIONS
 - ↳ LONG LASTING.
 - "SEMI-ENDOGENOUS" GROWTH
 - ↳ MORE PEOPLE = MORE IDEAS = MORE GROWTH
- EXTERNALITIES:
FIRM: PROD FUN

$$F(K, k_i) = k_i^\alpha K^{1-\alpha}$$

↑ ↑

AGGREGATE CAPITAL

CAPITAL FOR FIRM i

AGGREGATE PROD FUN:

$$F(K) = K^\alpha K^{1-\alpha} = K$$

INFO REVELATION

TIME : $t \in \{0, 1, 2, \dots, T\}$

STATES : SET Ω FINITE
 $\omega \in \Omega$

INFO OVER TIME :

PARTITIONS OF Ω

↳ NON-OVERLAPPING SUBSETS
OF SET, WITH UNION
EQUAL TO SET

EX . TOSSING A COIN 3 TIMES

$$\omega = HHT$$

$$\omega = TTH$$

$$\Omega = \{HHH, HHT, \dots, TTH, TTT\}$$

$$|\Omega| = \# ELEMENTS = 2^3 = 8$$

. INFO REVEALED IS FIRST TOSS = H

$$\mathcal{F}_1^0 = \{A_H, A_T\}$$

$$A_H = \{HHH, HHT, HTT, HTT\}$$

$$A_T = \{TTT, TTH, THT, THH\}$$

"TRUE / REALIZED" STATE $\omega \in \Omega$

- IMAGINE TRUE STATE IS HHH

$$\tilde{A}_H = \{HHH, HHT, HT\};$$

$$A_{HTT} = \{HTT\};$$

$$A_T = \{TTT, TTH, THT, THH\}.$$

. SECOND COIN TOSS IS REVEALED

$$\mathcal{F}_2^0 = \{A_{HH}, A_{HT}, A_{TH}, A_{TT}\}$$

$$A_{HH} = \{HHH, HHT\}; A_{HT} = \{HTH, HTT\}$$

...

. KNOW EVERYTHING AT $t = T = 3$

$$\mathcal{F}_3^0 = \{\{\omega\} \mid \omega \in \Omega\}$$

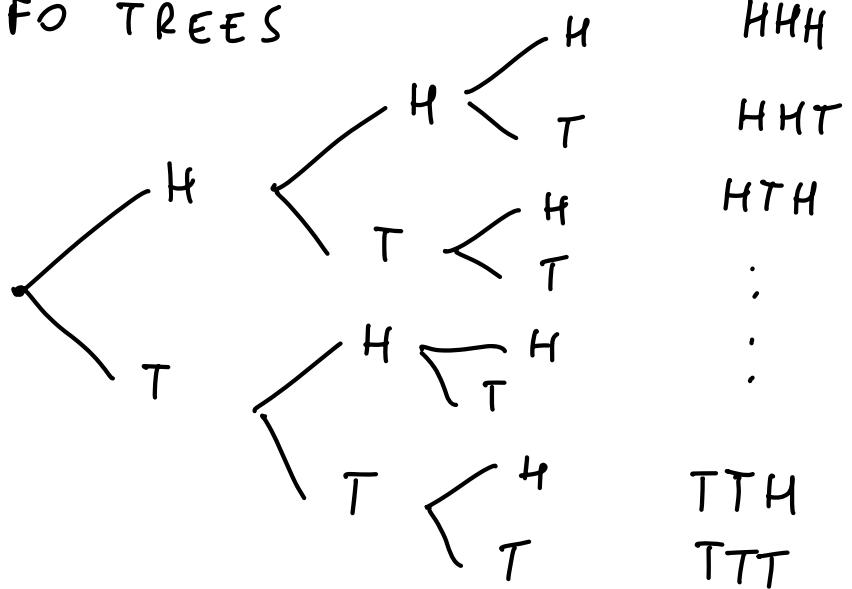
$$= \{\{HHH\}, \{HHT\}, \dots, \{TTT\}\}$$

BACK TO GENERAL CASE

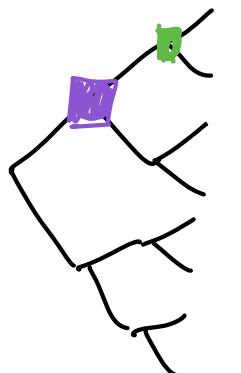
ASSUME $\mathcal{F}_0^0 = \Omega$

$$\mathcal{F}_T^0 = \{\{\omega\} \mid \omega \in \Omega\}$$

• INFO TREES



NODES OF TREE \leftrightarrow PARTITIONS



$\leftrightarrow A_H$ OF \mathcal{F}_1^0

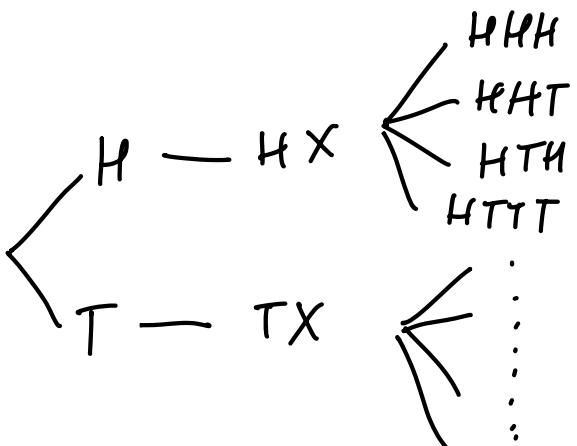
$\leftrightarrow A_{HH}$ OF \mathcal{F}_2^0

$$\mathcal{F}_1^0 = \{ A_H, A_T \}$$

$$A_H = \{ HHH, HHT, HTH, HTT \}$$

$$A_T = \{ TTT, TTH, THT, THH \}$$

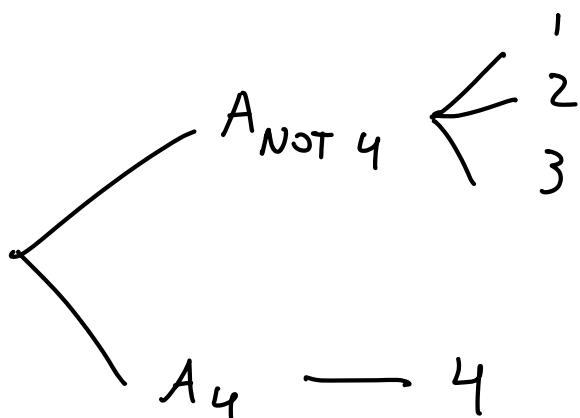
EX NO INFO RECEIVED AT $t=2$



$$\mathcal{F}_1^o = \mathcal{F}_2^o$$

EX $\mathcal{S} = \{1, 2, 3, 4\}$

I KNOW AT $t=1$ THAT 4 DID
OR DID NOT HAPPEN



$$A_{NOT 4} = \{\{1, 2, 3\}\}$$

$$A_4 = \{\{4\}\}$$

- PARTITIONS DON'T WORK FOR UNCOUNTABLE Σ
- GENERAL : FILTRATIONS / σ -ALGEBRAS

DEF A σ -ALGEBRA (σ -FIELD) OVER A NON-EMPTY Σ IS A FAMILY \mathcal{A} OF SUBSETS OF Σ S.T.:

1. $\emptyset \in \mathcal{A}$

2. $A \in \mathcal{A}$ THEN $A^c \in \mathcal{A}$

3. IF $(A_m, m \geq 1)$ IS A COUNTABLE FAMILY WITH $A_m \in \mathcal{A}$, THEN

$$\bigcup_{m=1}^{\infty} A_m \in \mathcal{A}$$

DEF A σ -ALGEBRA GENERATED

BY SOME COLLECTION A OF EVENTS IS THE INTERSECTION OF ALL σ -ALGEBRAS THAT CONTAIN A

DEF LET γ BE AN INDEXED SET A FILTRATION IS A COLLECTION OF σ -ALGEBRAS \mathcal{F}_t WITH $t \in \gamma$. S.T.

1. $\mathcal{F}_0 = \{\emptyset, \Omega\}$

2. FOR ALL $s > t$, $\mathcal{F}_t \subseteq \mathcal{F}_s$

3. $\mathcal{F}_T = \mathcal{F}$

WHERE \mathcal{F} IS A σ -ALGEBRA

ON Ω (FOR FINITE Ω ,

INDEXED MEANS:

$(A_t, t \in \gamma)$.

$\mathcal{F} = 2^\Omega$)