

I = INVESTMENT

## SOLOW - SWAN MODEL

. AGGREGATE PROD. FUN.

$$Y = F(K, L, t)$$

. PROPERTIES :

(P1) CONSTANT RET TO SCALE

$$\begin{aligned}\lambda Y &= F(\lambda K, \lambda L, t), \quad \lambda > 0 \\ &= \lambda F(K, L, t)\end{aligned}$$

(P2) INADA CONDITIONS

$$\lim_{K \rightarrow 0} F_K = \lim_{L \rightarrow 0} F_L = +\infty$$

$$\lim_{K \rightarrow \infty} F_K = \lim_{L \rightarrow \infty} F_L = 0$$

Y = OUTPUT

K = CAPITAL

L = LABOR

t = TIME

S = SAVING

### (P3) DERIVATIVES

$$F_K, F_L > 0 \quad ; \quad F_{KK}, F_{LL} < 0$$

$$\Rightarrow F_{KL} > 0$$

- SAVING BEHAVIOR

$$S(t) = s Y(t), \quad s \in (0, 1)$$

$$C(t) = (1-s) Y(t)$$

- NATIONAL ACCOUNTING IDENTITY

$$Y(t) = C(t) + I(t) \quad (+ G(t) + NX(t))$$

- CONTINUOUS TIME: ACCUMULATION

$$\dot{k}(t) := \frac{dk(t)}{dt}$$

$$\dot{k}(t) = I(t) - \delta k(t); \quad \delta > 0$$

- LABOR FORCE

$$\dot{L}(t) = m L(t); \quad m > 0$$

## NO TIME VARIATION IN F

$$F(k, L, t) = F(k, L) \quad (\text{no } t)$$

- IN EQUILIBRIUM :  $S(t) = I(t) \Leftrightarrow Y(t) = C(t) + I(t)$

"INTENSIVE FORM"  $\rightarrow$  PER CAPITA / WORKER

$$y(t) := \frac{Y(t)}{L(t)} \quad k(t) := \frac{k(t)}{L(t)}$$

etc.

- PROD FUN IN INTENSIVE FORM

$$Y = F(k, L)$$

$$\frac{1}{L} Y = \frac{1}{L} F(k, L) = F\left(\frac{k}{L}, \frac{L}{L}\right) = F(k, 1)$$

$$f(k) := F(k, 1)$$

$$\Rightarrow y = f(k)$$

PROPERTIES OF  $f$  :

- $f'(k) > 0$
- $f''(k) < 0$
- $f(0) = 0$

• INTENSIVE FORM FOR

$$\dot{k}(t) = I(t) - \delta k(t)$$

$$\frac{d}{dt} \left( \frac{k(t)}{L(t)} \right) = \frac{1}{L(t)} \frac{dk(t)}{dt} + k(t) \frac{d}{dt} \left( \frac{1}{L(t)} \right)$$

= ...

$$= \frac{\dot{k}(t)}{L(t)} + \frac{\dot{L}(t)}{L(t)} \frac{k(t)}{L(t)}$$

$$= \frac{\dot{k}(t)}{L(t)} + m k(t)$$

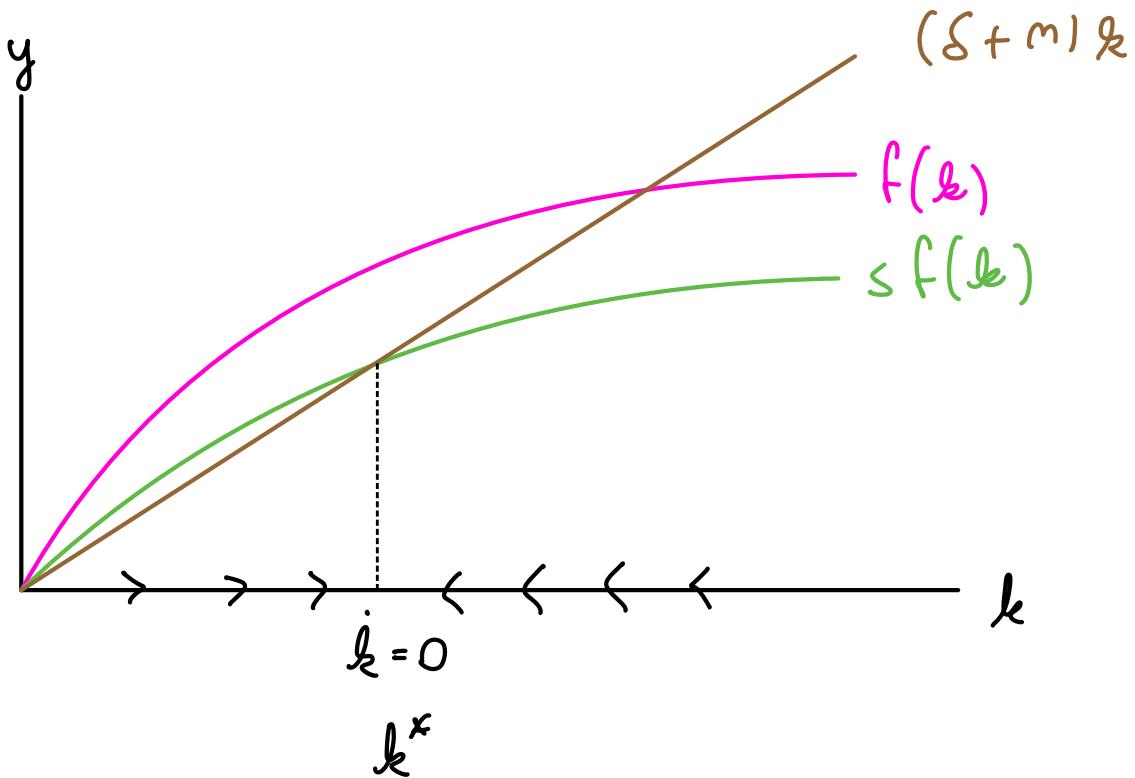
$$\dot{k}(t) = \frac{\dot{k}(t)}{L(t)} + m k(t)$$

ACCUM. EQ IN INTENSIVE FORM:

$$\dot{k}(t) = i(t) - \delta k(t) - n k(t)$$

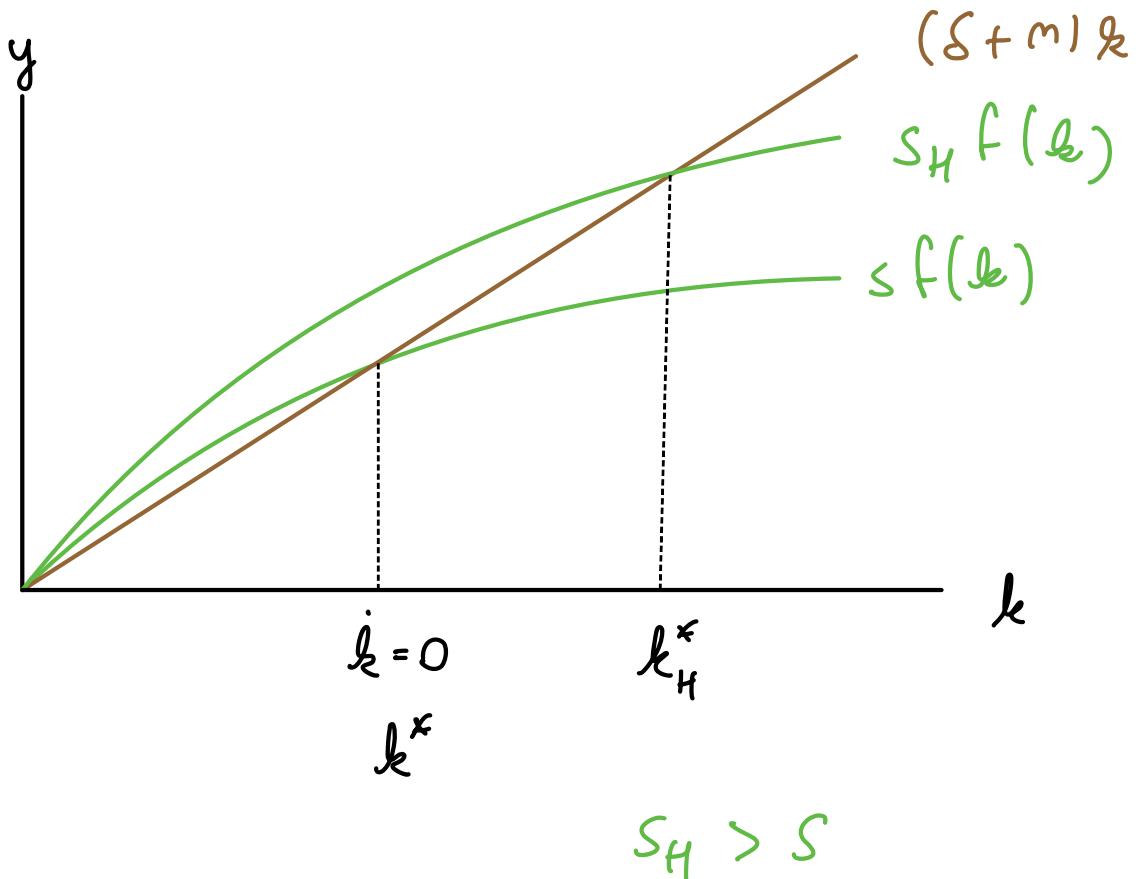
$$\dot{k}(t) = sf(k(t)) - (n + \delta)k(t)$$

$$F(k, L) = k^\alpha L^{1-\alpha} \Rightarrow f(k) = k^\alpha = y$$

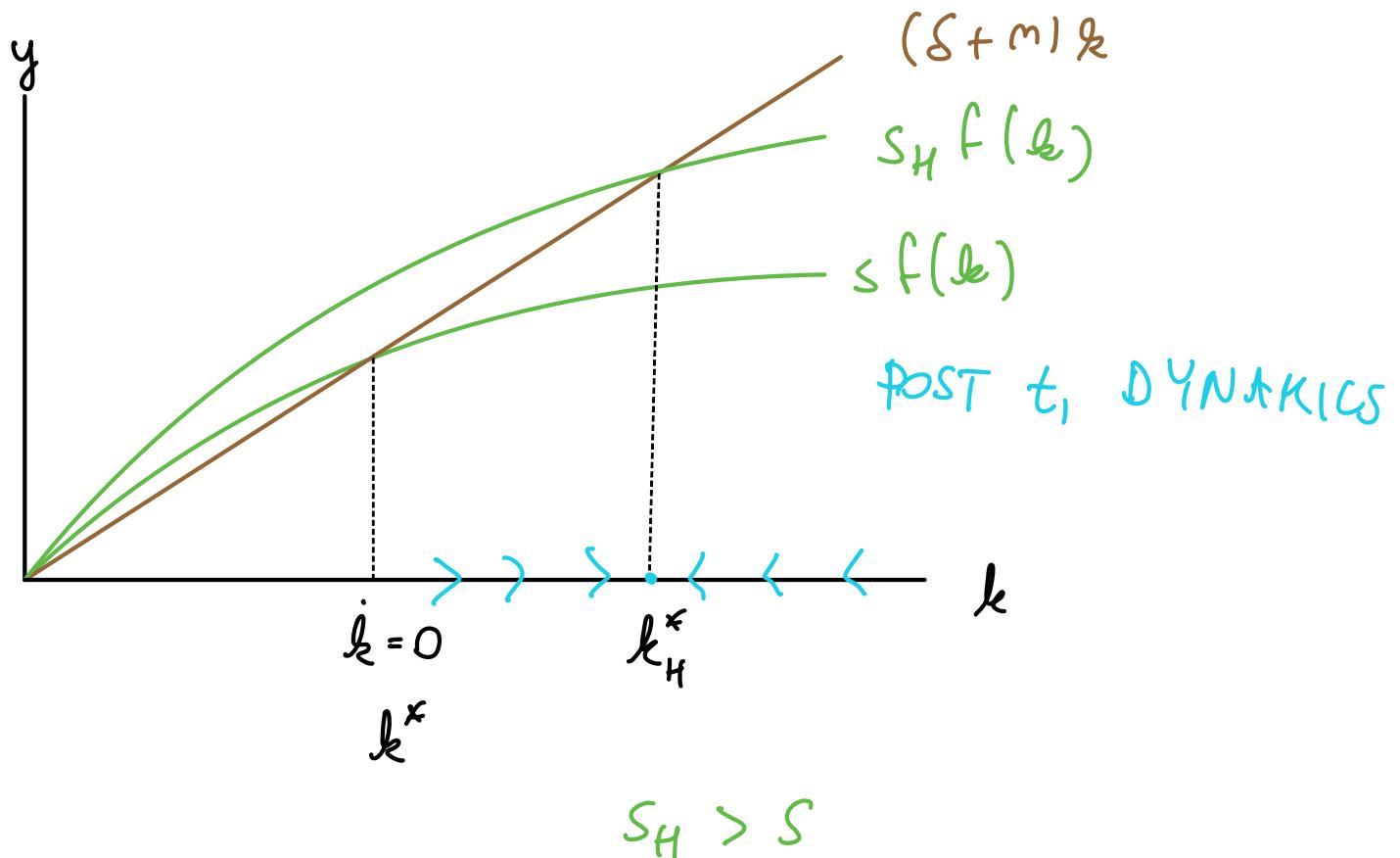


$\dot{k}$  = BALANCED GROWTH PATH

# EXPERIMENT : CHANGE s.



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$$Y = C + I \quad (+/ \cancel{X})$$

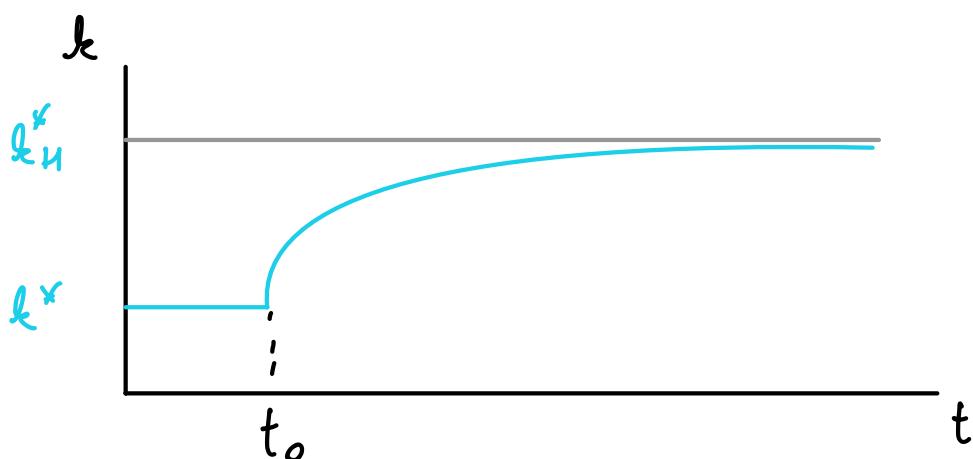
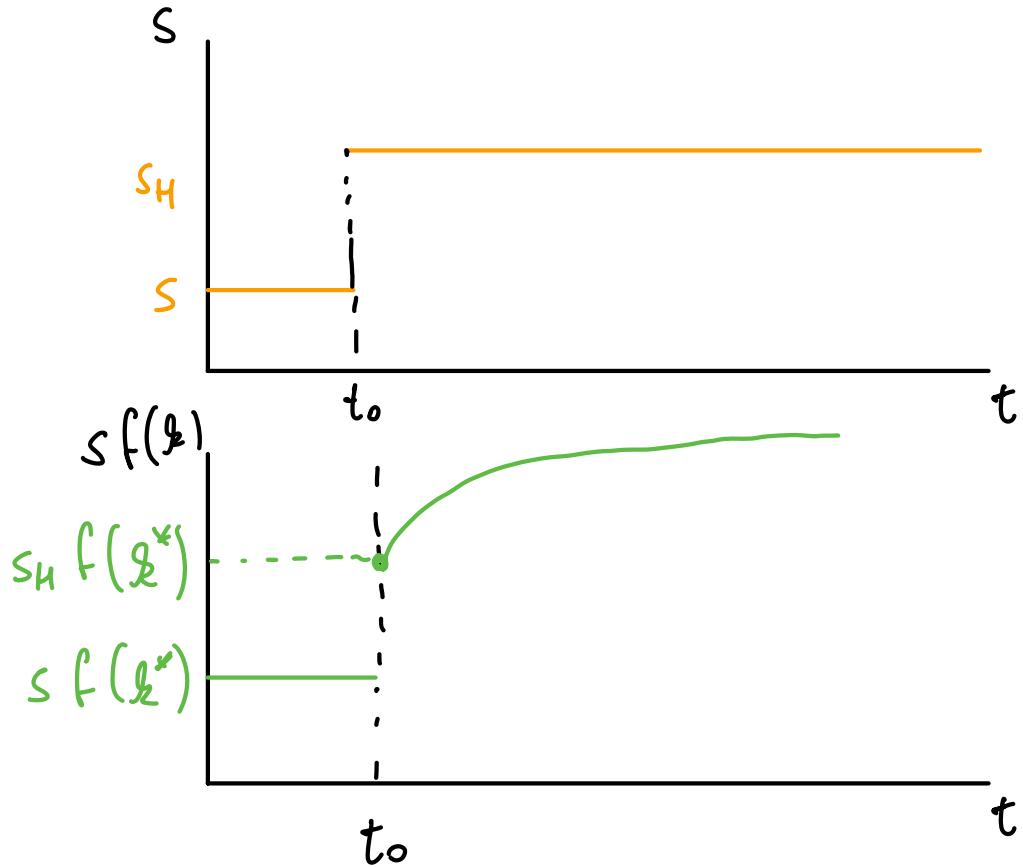
$$G : \mathbb{R}_+ \rightarrow \mathbb{R}$$

$$\dot{k}(t) = G(k(t))$$

$$\dot{k}(t) \approx G(k^*) + G'(k^*)(k - k^*)$$

$$\dot{k}(t) = G'(k^*)(k - k^*) + G''(k^*)(k - k^*)^2$$

$$k(t) = k(0) \exp(G'(k^*)t)$$



$$\dot{k}(t) = s f(k(t)) - (m + \delta) k(t)$$

$$G(k) := s f(k) - (m + \delta) k$$

$$G'(k) = s f'(k) - (m + \delta)$$

$$G'(k^*) = s f'(k^*) - (m + \delta)$$

$$f'(k^*) > 0$$

$$f''(k^*) < 0$$

## CONSUMPTION

$$\begin{aligned} c &= (1-s)y = (1-s)f(k) \\ &= f(k) - sf(k) \end{aligned}$$

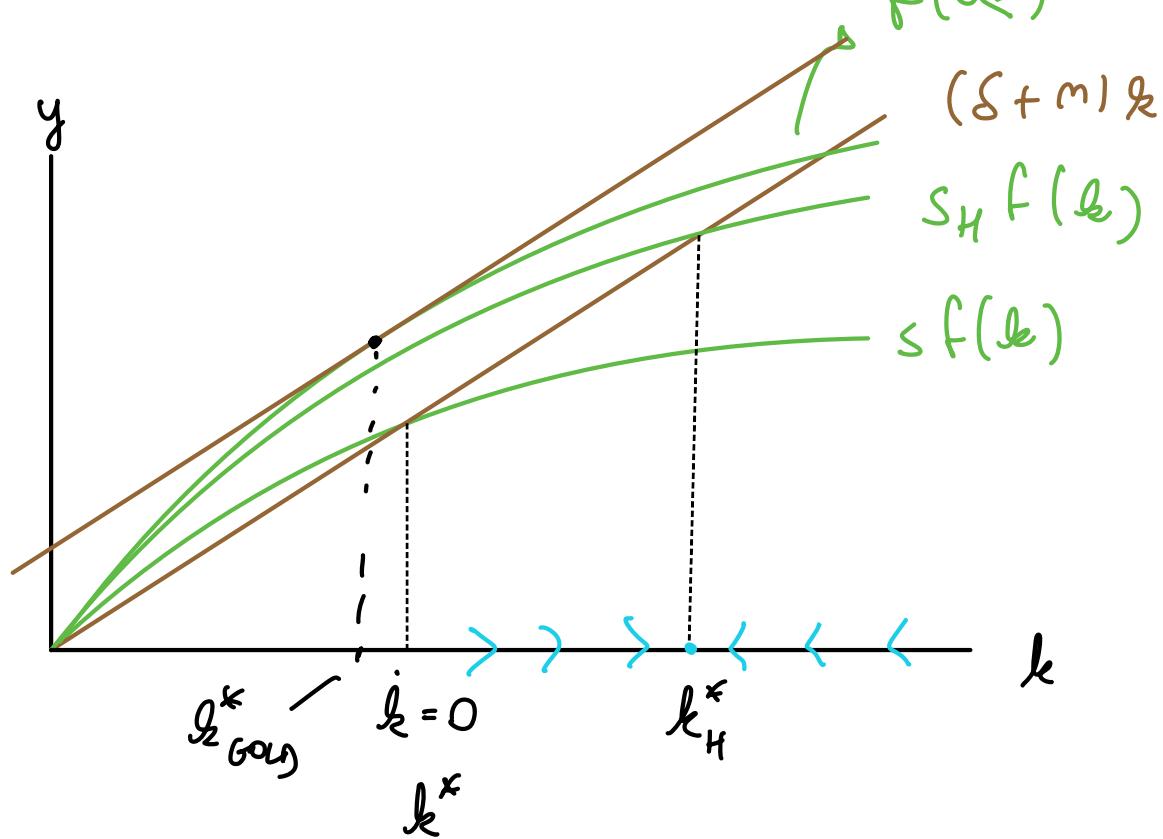
$$\begin{aligned} \text{IN SS: } \dot{k} = 0 &\Leftrightarrow 0 = sf(k^*) - (n + \delta)k^* \\ &\Rightarrow sf(k^*) = (n + \delta)k^* \end{aligned}$$

$$c^* = \underbrace{f(k^*)}_{\substack{\text{INCREASING} \\ \text{IN } k^*}} - \underbrace{(n + \delta)k^*}_{\substack{\text{DECREASING} \\ \text{IN } k^*}}$$

$$\frac{dc^*}{ds} = f'(k^*) \frac{dk^*}{ds} - (n + \delta) \frac{dk^*}{ds}$$

$$= [f'(k^*) - (n + \delta)] \underbrace{\frac{dk^*}{ds}}_{> 0}$$

$f'(k^*)$



S THAT MAXIMIZES C IS CALLED  
"GOLDEN RULE LEVEL"

$$k_{\text{GOLD}}^* \\ c_{\text{GOLD}}^*$$

CHARACTERIZED BY  $\frac{dc^*}{ds} = 0$

OR

$$f'(k_{\text{GOLD}}^*) = \delta + m$$

TO CHECK MAX AND NOT MIN :

$$\frac{d^2c^*}{ds^2} = \frac{d}{ds} \left[ [f'(k^*) - (\delta + m)] \frac{dk^*}{ds} \right]$$

# RAMSEY - CASS - KOOPMAN / NEOCLASSICAL

- CONTINUUM OF HH  $\rightarrow$  REPRESENT. HH
- " OF FIRMS  $\rightarrow$  " FIRM

HH:

- WORK
- OWN  $k$
- CONSUME
- RENT  $k$  TO FIRMS

FIRMS:

- PRODUCE USING LABOR  $l$  &  $k$
- PAY  $w$  TO LABOR
- PAY  $r$  TO  $k$
- ALL MARKETS PERFECTLY COMPETITIVE

# RAMSEY - CASS - KOOPMAN / NEOCLASSICAL

- CONTINUUM OF HH  $\rightarrow$  REPRESENT. HH
- " OF FIRMS  $\rightarrow$  " FIRM

- HH:
- WORK
  - OWN K
  - CONSUME
  - RENT K TO FIRMS

- FIRMS:
- PRODUCE USING LABOR L & K
  - PAY w TO LABOR
  - PAY R TO K
  - ALL MARKETS PERFECTLY COMPETITIVE (ZERO PROFITS AND FREE ENTRY)

$$L(t) = e^{rt} L(0)$$

$$= e^{rt} \quad (\dot{L} = rL)$$

LOWER CASE

FIRMS

NUMERAIRE

$$\pi(K, L) := Y - RK - wL$$

$$\begin{aligned} \max_{\substack{K>0 \\ L \geq 0}} \pi(K, L) &= \max_{\substack{K>0 \\ L \geq 0}} F(K, L) - RK - wL \end{aligned}$$

FOC (NECESSARY AND SUFFICIENT  
GIVEN ASSUMPTIONS ON F)

$$\frac{\partial \Pi}{\partial K} = 0 \quad ; \quad \frac{\partial \Pi}{\partial L} = 0$$

$$\frac{\partial F}{\partial K} - R = 0 \quad ; \quad \frac{\partial F}{\partial L} - w = 0$$

$$\frac{\partial F}{\partial K} = R \quad ; \quad \frac{\partial F}{\partial L} = w$$

$$\begin{aligned}\Pi(K, L) &= F\left(\frac{K}{L}, \frac{L}{L}\right)L - RkL - wL \\ &= [f(k) - Rk - w]L \\ &\stackrel{=:}{=} \phi(k)L\end{aligned}$$

$$\underline{\text{FOC}}: \quad \phi'(k) = 0 \quad \Leftrightarrow f'(k) = R$$

$$\text{ZERO PROFITS} \Rightarrow \phi(k) = 0$$

## HOUSEHOLD

$u(C(t))$  AT EACH  $t$

$$u' > 0 \quad ; \quad u'' < 0 \quad ; \quad \lim_{c \rightarrow 0} u'(c) = \infty$$

DISCRETE TIME:  $\beta^t$ ,  $\beta \in (0, 1]$

CONT. TIME:  $e^{-\ell t}$ ,  $\ell > 0$   $\lim_{c \rightarrow \infty} u'(c) = 0$

$$\int_0^\infty e^{-\ell t} \mu(c(t)) L(t) dt$$


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## DISCRETE TIME

$$L(t) = e^{nt} \rightarrow L(t+1) = (1 + g_L) L(t)$$

$$\int_0^\infty e^{-\ell t} \mu(c(t)) L(t) dt \rightarrow \sum_{t=0}^{\infty} p^t \mu(c(t)) L(t)$$


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$$\begin{aligned} & \text{MAX} && \int_0^\infty e^{-(\ell - r)t} \mu(c(t)) dt \\ & c(t) > 0 && \\ & k(t) > 0 && \text{s.t.} \end{aligned}$$

BUDGET CONSTRAINT.

- FINANCIAL ASSETS  
L BOND (RISKLESS) } ZERO NET SUPPLY

- REAL ASSETS  
L CAPITAL (RISKLESS)

$a(t) L(t)$  : ASSETS OF HH.

$$\frac{\partial}{\partial t} [a(t)L(t)] = a(t)L(t)r(t) + w(t)L(t)$$

$$- c(t) L(t)$$

$$\dot{a}(t) = a(t)r(t) + \omega(t) - c(t) - ma(t)$$

$$= a(t)(r(t)-m) + \omega(t) - c(t)$$

$$a(t) \leftarrow \begin{array}{l} \text{BONDS} \rightarrow \text{ZERO} \\ \text{CAPITAL} \rightarrow k(t) \end{array}$$

IN EQUILIBRIUM

$$a(t) = k(t)$$

$$r(t) = R(t) - \delta$$

$$\dot{k}(t) = k(t)(R(t) - \delta - m) + \omega(t) - c(t)$$

IF NO BONDS  $\rightarrow k(t) \geq 0$

IF BONDS  $\rightarrow$  "NO PONZI"

TRANSVERSALITY  
CONDITION

$$\lim_{t \rightarrow \infty} a(t) \exp\left(-\int_0^t (r(s)-m) ds\right) \geq 0$$

## MH PROBLEM

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-(\ell-m)t} u(C(t)) dt \\ c(t) > 0 & \\ a(t) > 0 & \text{s.t.} \end{array}$$

$$\dot{a}(t) = a(t)(r(t)-m) + w(t) - c(t)$$

$$\lim_{t \rightarrow \infty} a(t) \exp \left( - \int_0^t (r(s)-m) ds \right) \geq 0$$

TO KEEP UTILITY FINITE WE NEED

$$\ell - m > 0$$

## MH PROBLEM

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-(\ell-m)t} u(c(t)) dt \\ c(t) > 0 & \\ \alpha(t) > 0 & \text{s.t.} \quad \alpha(0) = \alpha_0 \end{array}$$

$$\dot{\alpha}(t) = \alpha(t)(r(t) - m) + w(t) - c(t)$$

$$\lim_{t \rightarrow \infty} \alpha(t) \exp \left( - \int_0^t (r(s) - m) ds \right) \geq 0$$

TO KEEP UTILITY FINITE WE NEED

$$\ell - m > 0$$

## MAXIMUM PRINCIPLE

$$\begin{array}{ll} \text{MAX} & \int_0^\infty e^{-\ell t} f(x(t), u(t)) dt \\ u(\cdot) & \\ & \text{s.t.} \end{array}$$

$$\dot{x}(t) = \mu(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$x(0) = x_0 \quad \text{GIVEN}$$

$$u(\cdot) \in \mathcal{U}$$

NECESSARY      CONDITIONS:

1. STATE EQUATION & FEASIBILITY

$$\dot{x}(t) = h(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$x(0) = x_0 \quad ; \quad u(\cdot) \in \mathcal{U}$$

2. Let

$$J(t, x(t), u(t), \lambda(t), \phi(t)) :=$$

$$\begin{aligned} & f(x(t), u(t)) + \lambda(t) h(t, x(t), u(t)) \\ & + \phi(t) G(x(t), u(t)) \end{aligned}$$

$$u^*(t) \in \underset{u}{\operatorname{MAX}} \quad J(t, x^*(t), u, \lambda(t), \phi(t))$$

$$G(x^*(t), u) \geq 0$$

FOC: USUAL CALCULUS CONSTRAINT  
OPTIM.

### 3. DYNAMICS OF $\lambda(t)$

$$\dot{\lambda}(t) = \rho \lambda(t) - \frac{\partial \mathcal{J}_L}{\partial x}(t, x^*(t), u^*(t), \lambda(t), \phi(t))$$

### 4. COMPLEMENTARY SLACKNESS

$$\phi(t) \geq 0 \quad \phi(t) G(x^*(t), u^*(t)) = 0$$

### 5. TRANSVERSALITY

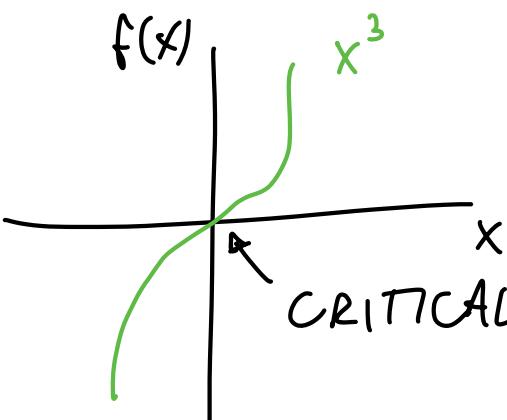
$$\lim_{t \rightarrow \infty} e^{-\rho t} \mathcal{J}_L(t, x^*(t), u^*(t), \lambda(t), \phi(t)) = 0$$

OR

(UNDER SOME ASSUMPTIONS)

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) x(t) = 0$$

## COMMENTS



- NECESSARY

CRITICAL POINT :  $f'(x) = 0$  AT  $x=0$   
BUT  $x=0$  NOT  
MAX OR MIN

- $\lambda, \phi$  ARE "SHADOW VALUES" OF CONSTRAINTS.
- "PRESENT VALUE  $\mathcal{H}^{PV}$ "

$$\mathcal{H} = e^{-\lambda t} f + \lambda^{PV} h + \phi^{PV} \cdot g$$

$$\lambda^{PV} = e^{-\lambda t} \lambda$$

$$\phi^{PV} = e^{-\lambda t} \phi$$

$$\begin{aligned}\dot{\lambda}^{PV} &= e^{-\lambda t} \dot{\lambda} \\ &\quad - e^{-\lambda t} e^{\lambda t} \lambda \\ &= e^{-\lambda t} (\dot{\lambda} - e^{\lambda t} \lambda)\end{aligned}$$

$$\dot{\lambda}^{PV} = - \frac{\partial \mathcal{J}_L}{\partial x}$$

## • BACKWARD INDUCTION

IMAGINE HORIZON IS  $T < \infty$

At  $t = T$

$\dot{x}$  EQUATION DOES NOT  
APPLY

$\Rightarrow$  MAX f w/o THINKING  
ABOUT FUTURE x

IF AT T, NO CONSTRAINTS ON x

$$\lambda(T) = 0$$

( w/ CONSTRAINTS :  $\lambda(T) = G' \dots$  )

$\dot{\lambda}$  EQ IS A "BACKWARD EQ"

IN INFINITE HORIZON, WHAT IS

" $\lambda(T)$ " ?

ANSW: TRANSVERSALITY!

NOT

$$\lim_{t \rightarrow \infty} e^{et} \lambda(t) = 0 \quad !!$$

$$\mathcal{H} = \underbrace{\mu(c(t))}_{f} + \lambda(t) \left[ \underbrace{\alpha(t)(r(t) - m)}_{h} + \omega(t) - c(t) \right]$$

$$\dot{\lambda}(t) = (\ell - m) \lambda(t) - \frac{\partial \mathcal{H}}{\partial a}(t, \dots)$$

$$\dot{\lambda}(t) = (\ell - m) \lambda(t) - (r(t) - m) \lambda(t)$$

(1)  $\dot{\lambda}(t) = [\ell - r(t)] \lambda(t)$

(2)  $\max_c \mathcal{H} \Rightarrow \frac{\partial \mathcal{H}}{\partial c} = 0 \Rightarrow \mu'(c^*(t)) = \lambda(t)$

FROM (1)  $\frac{\dot{\lambda}(t)}{\lambda(t)} = \ell - r(t)$

$$\frac{\partial}{\partial t} (\log \lambda(t)) = e - r(t)$$

$$\log \lambda(t) = \log \lambda(0) + \int_0^t [e - r(s)] ds$$

$$\int_0^T d \log (\lambda(t)) = \int_0^T [e - r(t)] dt$$

$$\log \lambda(T) - \log \lambda(0) = \int_0^T [e - r(t)] dt$$

USE (2) :

$$\mu'(c^*) = \lambda(0) \exp \left( - \int_0^t [r(s) - e] ds \right)$$

$$\frac{\partial}{\partial t} (\log \lambda(t)) = e - r(t)$$

$$\log \lambda(t) = \log \lambda(0) + \int_0^t [e - r(s)] ds$$

$$\int_0^T d \log(\lambda(t)) = \int_0^T [e - r(t)] dt$$

$$\log \lambda(T) - \log \lambda(0) = \int_0^T [e - r(t)] dt$$

USE (2) :

$$\lambda(t) = u'(c^*) = \lambda(0) \exp \left( - \int_0^t [r(s) - e] ds \right)$$

(3) TRANSVERSALITY :

$$\lim_{t \rightarrow \infty} e^{-(e-r)} a(t) \lambda(t) = 0$$

(FINITE HORIZON :  $a(T) = 0$ )

L EITHER NEED  $a(t) \geqslant$  LOWER BOUND

OR

L ADMISSIBLE CONTROLS / STATES  
ARE FINITE FOR ALL  $t$

COMBINE (2) AND (3). THEN,

NO PONZI HOLDS WITH EQUALITY

COMBINE  $\mu'(C(t)) = \lambda(t)$  AND

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -(r(t) - \ell)$$

$$\frac{\partial}{\partial t} \mu'(C(t)) = \mu''(C(t)) \dot{C}(t) = \dot{\lambda}(t)$$

$$\Rightarrow \mu''(C(t)) \dot{C}(t) = -(r(t) - \ell) \lambda(t) \\ = -(r(t) - \ell) \mu'(C(t))$$

$$\frac{\dot{C}(t)}{C(t)} = - \underbrace{\frac{\mu'(C(t))}{C(t) \mu''(C(t))}}_{\text{ELASTICITY OF}} (r(t) - \ell)$$

ELASTICITY OF

INTERTEMPORAL

SUBSTITUTION

$$:= [\Theta(t)]^{-1}$$

ELASTICITY B/W  $t, s$ :

$$\Theta(t, s) := \frac{\partial (\log [u'(c(t)) / u'(c(s))])}{\partial (\log [c(t) / c(s)])}$$

$$\lim_{s \rightarrow t} \Theta(t, s) = \Theta(t)$$

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USE FIRM OPTIMALITY:

$$f'(k) = R(t)$$

$$r(t) = f'(k) - \delta$$

ZERO PROFITS:

$$w(t) = f(k) - f'(k)k$$

USE MARKET CLEARING:

$$a(t) = k(t)$$

(SINCE BONDS IN ZERO NET SUPPLY)

# BUDGET CONSTRAINT:

$$\dot{a}(t) = \alpha(t)(r(t) - \gamma) + \omega(t) - c(t)$$

$$\begin{aligned}\dot{k}(t) &= k(t) \left( f'(k) - \delta - \gamma \right) + \\ &+ f(k) - f'(k)k(t) - c(t)\end{aligned}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + \gamma)k(t)$$

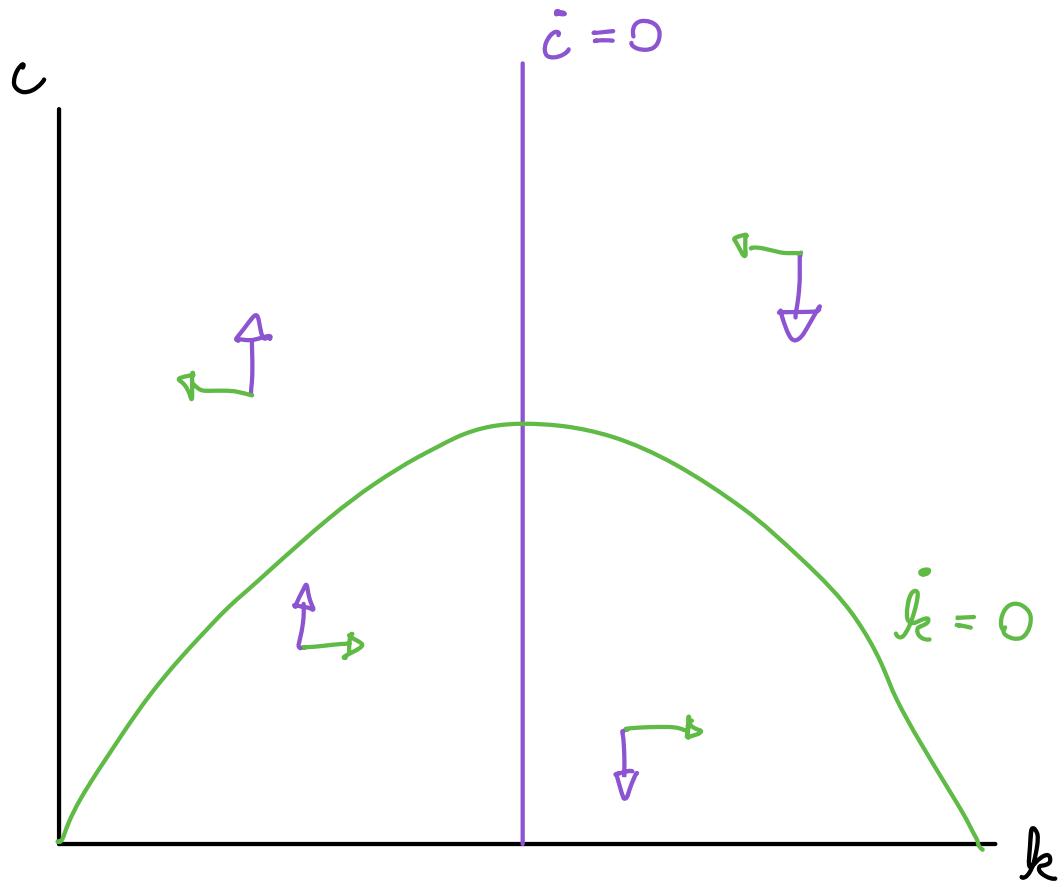
$$k(0) = k_0 \quad + \quad \text{TRANSVERSALITY}$$

$$\frac{\dot{c}(t)}{c(t)} = - \frac{u'(c(t))}{c(t) u''(c(t))} (r(t) - \rho)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\Theta(t)} (f'(k(t)) - \delta - \rho)$$

$$\dot{c}=0 \Rightarrow f'(k) = \delta + \rho$$

## PHASE    DIAGRAM



$$\dot{k} = 0 \Rightarrow 0 = f(k) - c - (\delta + m)k$$

$$\Rightarrow c = f(k) - (\delta + m)k$$