I = INVESTMENT

Y = OUTPUT

K = CAPITAL

L = LABOR

t = TIME

S = SAVING

SOLOW - SWAN MODEL

- AGGREGATE PROD. FUN.

$$Y = F(K, L, t)$$

- PROPERTIES :

(P1) CONSTANT RET TO SCALE

$$\lambda Y = F(\lambda K, \lambda L, t), \quad \lambda > 0$$
$$= \lambda F(K, L, t)$$

(P2) INADA CONDITIONS

$$\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = +\infty$$

$$\lim_{K \to \infty} F_K = \lim_{L \to \infty} F_L = 0$$

(P3) DERIVATIVES

$$F_K, F_L > 0 \quad ; \quad F_{KK}, F_{LL} < 0$$

$$\Rightarrow \quad F_{KL} > 0$$

- SAVING BEHAVIOR

$$S(t) = s\,Y(t) \quad , \quad s \in (0,1)$$

$$C(t) = (1-s)\,Y(t)$$

- NATIONAL ACCOUNTING IDENTITY

$$Y(t) = C(t) + I(t) \qquad (+\,G(t) + NX(t))$$

- CONTINUOUS TIME : ACCUMMULATION

$$\dot{k}(t) := \frac{dk(t)}{dt}$$

$$\dot{k}(t) = I(t) - \delta k(t) \quad ; \quad \delta > 0$$

- LABOR FORCE

$$\dot{L}(t) = n\,L(t) \quad ; \quad n > 0$$

NO TIME VARIATION IN F

$$F(K, L, t) = F(K, L) \quad (\text{NO } t)$$

- IN EQUILIBRIUM: $S(t) = I(t) \iff$
  $Y(t) = C(t) + I(t)$

"INTENSIVE FORM" $\to$ PER CAPITA / WORKER

$$y(t) := \frac{Y(t)}{L(t)} \qquad k(t) := \frac{K(t)}{L(t)}$$

etc.

- PROD FUN IN INTENSIVE FORM

$$Y = F(K, L)$$

$$\frac{1}{L} Y = \frac{1}{L} F(K, L) = F\left(\frac{K}{L}, \frac{L}{L}\right) = F(k, 1)$$

$$f(k) := F(k, 1)$$

$$\Rightarrow \qquad y = f(k)$$

PROPERTIES OF f:

- $f'(k) > 0$
- $f''(k) < 0$
- $f(0) = 0$

- INTENSIVE FORM FOR

$$\dot{K}(t) = I(t) - \delta K(t)$$

$$\frac{d}{dt}\left(\frac{K(t)}{L(t)}\right) = \frac{1}{L(t)} \frac{dK(t)}{dt} + K(t) \frac{d}{dt}\left(\frac{1}{L(t)}\right)$$

$$= \dots$$

$$= \frac{\dot{K}(t)}{L(t)} + \frac{\dot{L}(t)}{L(t)} \frac{K(t)}{L(t)}$$

$$= \frac{\dot{K}(t)}{L(t)} + n\, k(t)$$

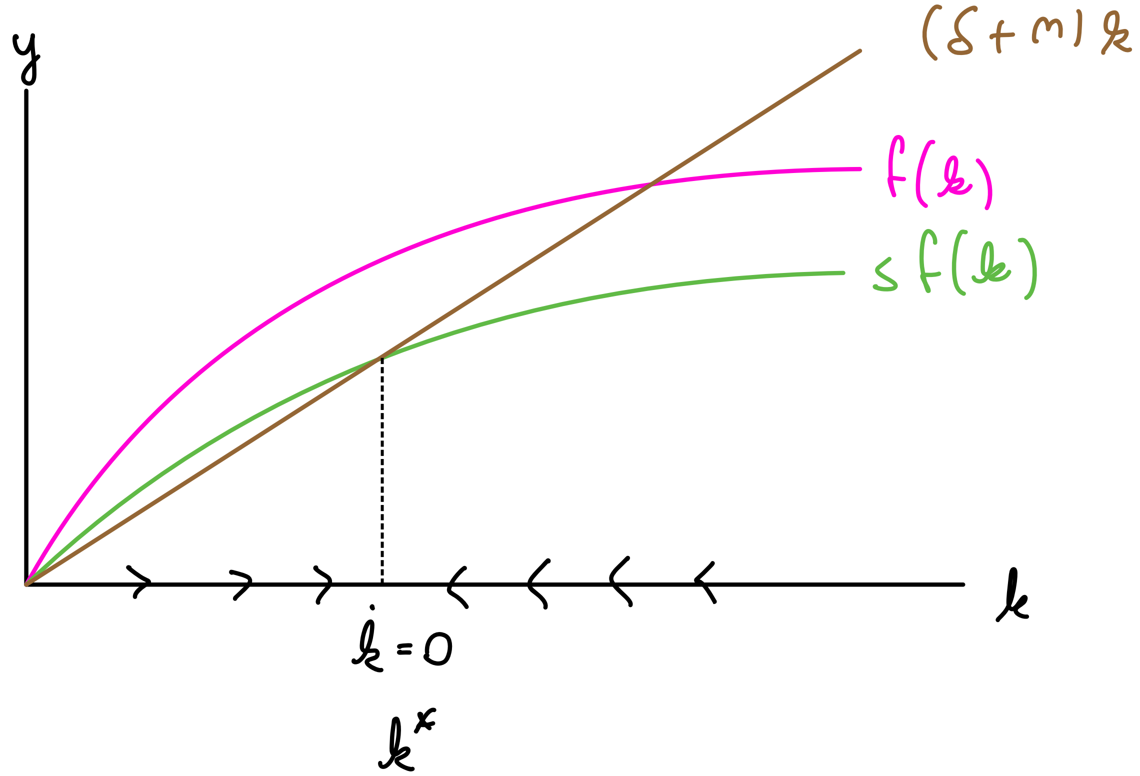
$$\dot{k}(t) = \frac{\dot{K}(t)}{L(t)} + n\, k(t)$$

ACCUMM. EQ IN INTENSIVE FORM:

$$\dot{k}(t) = i(t) - \delta k(t) - n k(t)$$

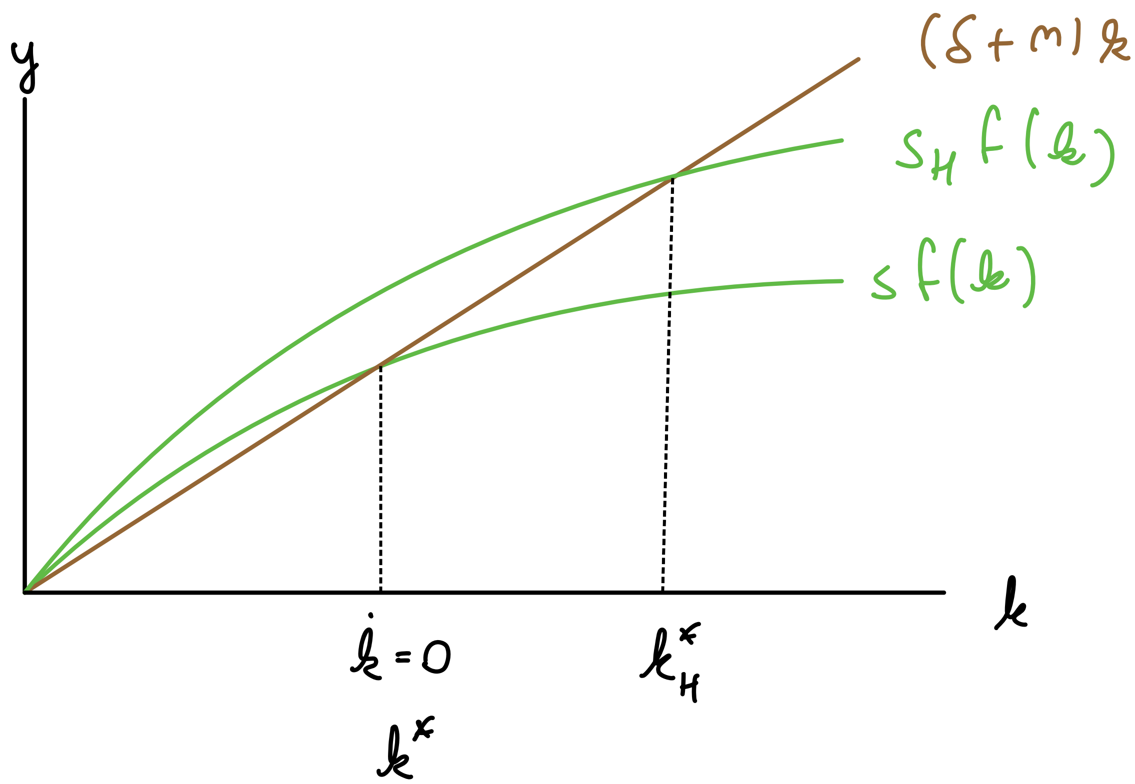
$$\boxed{\dot{k}(t) = s f(k(t)) - (n + \delta) k(t)}$$

$$F(K, L) = K^{\alpha} L^{1-\alpha} \Rightarrow f(k) = k^{\alpha} = y$$



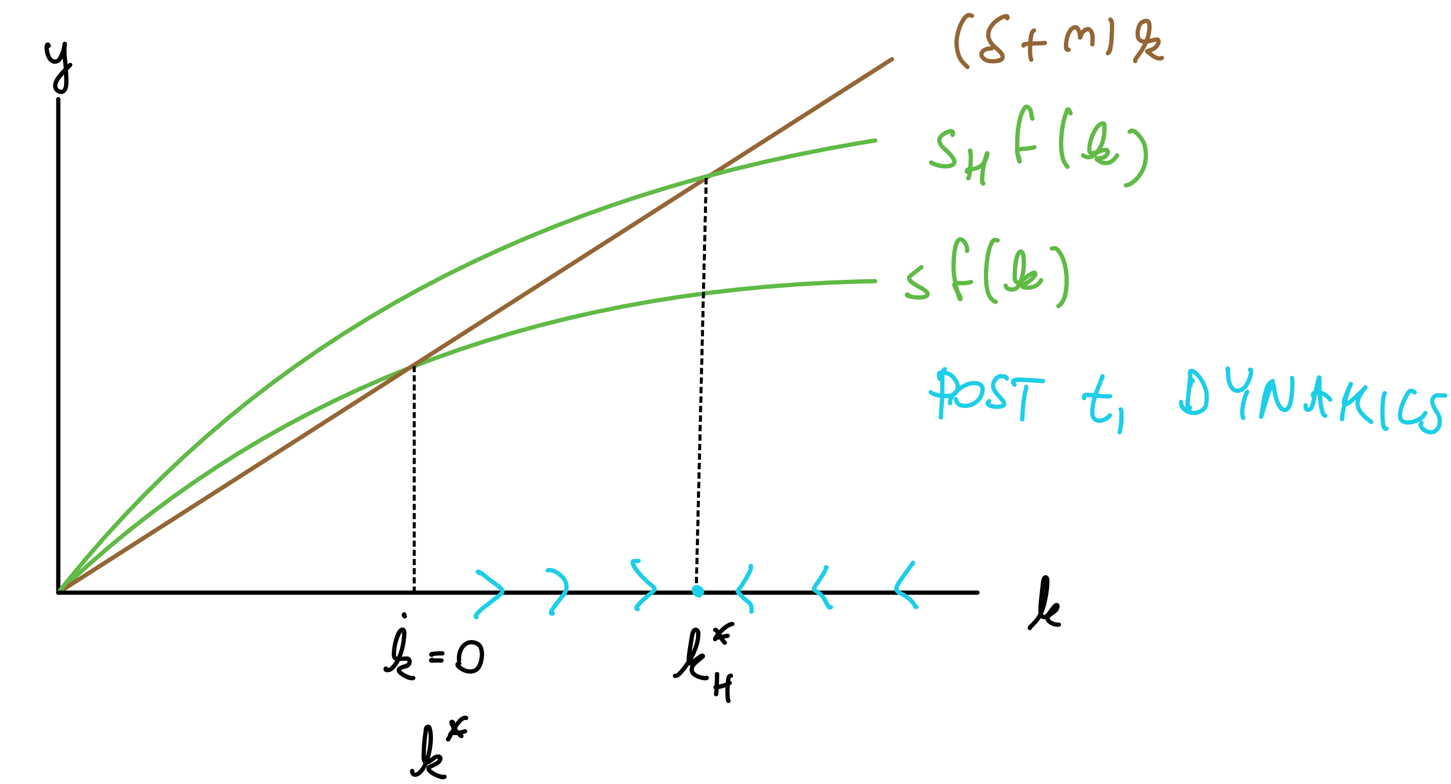
$\dot{k}$ = BALANCED GROWTH PATH

EXPERIMENT: CHANGE s.



$s_H > s$

EXPERIMENT: CHANGE s.



$s_H > s$

$$Y = C + I \quad (\cancel{+NX})$$

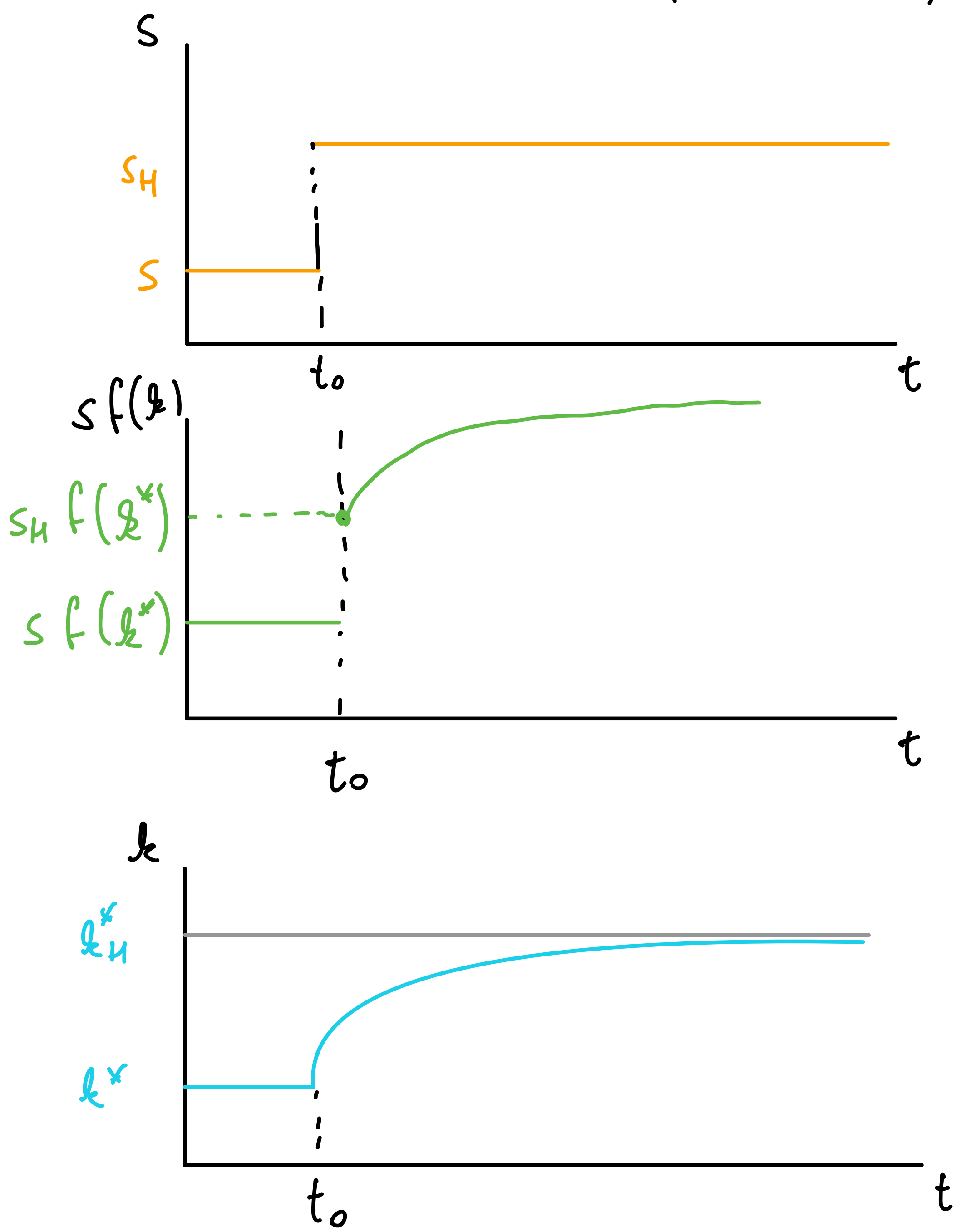
$$G : \mathbb{R}_+ \to \mathbb{R}$$

$$\dot{k}(t) : G(k(t))$$

$$\dot{k}(t) \simeq G(k^*) + G'(k^*)(k - k^*)$$

$$\dot{k}(t) = G'(k^*)(k - k^*) + G''(k^*)(k-k^*)^2$$

$$k(t) = k(0)\exp\left(G'(k^*)\,t\right)$$



$$\boxed{\dot{k}(t) = s f(k(t)) - (n+\delta)\,k(t)}$$

$$G(k) := s f(k) - (n+\delta)\,k$$

$$G'(k) = s f'(k) - (n+\delta)$$

$$G'(k^*) = s f'(k^*) - (n+\delta)$$

$$f'(k^*) > 0$$
$$f''(k^*) < 0$$

CONSUMPTION

$$c = (1-s)y = (1-s)f(k)$$
$$= f(k) - sf(k)$$

IN SS : $\dot{k} = 0 \iff 0 = sf(k^*) - (n+\delta)k^*$

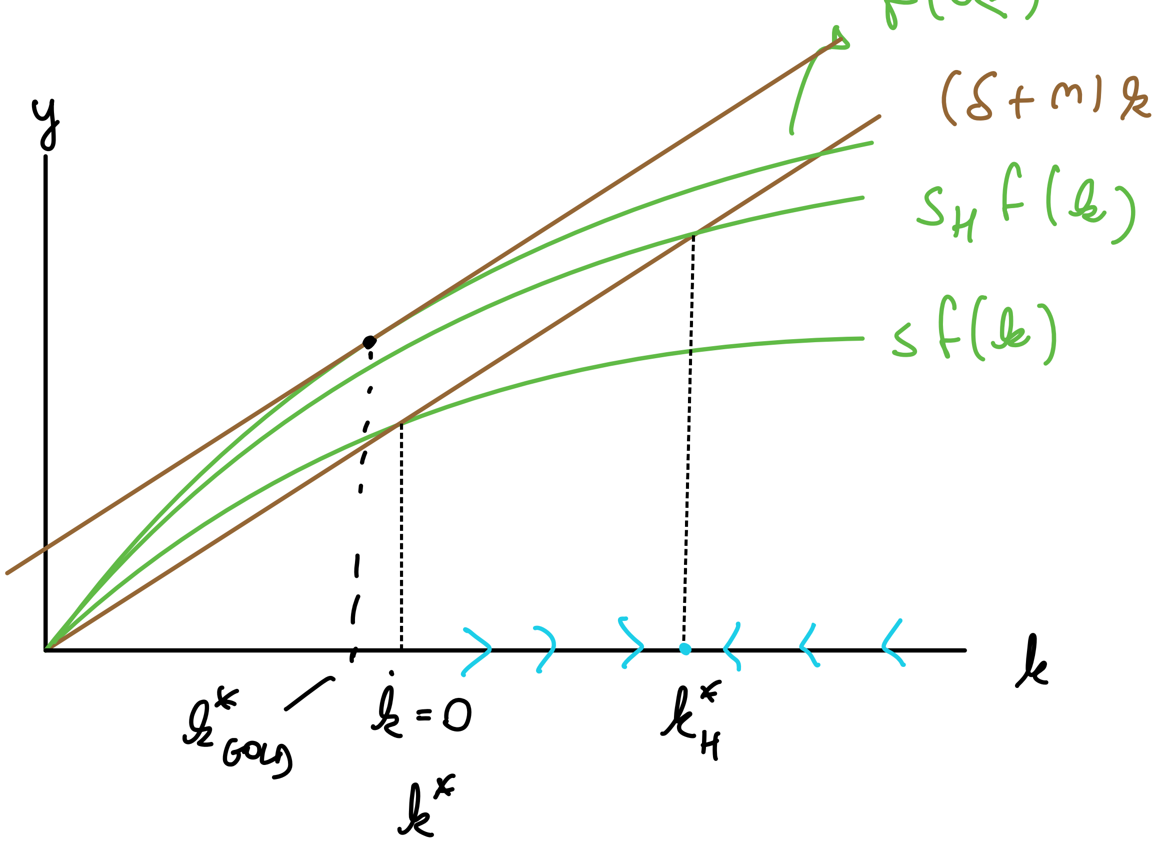
$$\Rightarrow sf(k^*) = (n+\delta)k^*$$

$$c^* = \underbrace{f(k^*)}_{\text{INCREASING IN } k^*} - \underbrace{(n+\delta)k^*}_{\text{DECREASING IN } k^*}$$

$$\frac{dc^*}{ds} = f'(k^*)\frac{dk^*}{ds} - (n+\delta)\frac{dk^*}{ds}$$

$$= \left[f'(k^*) - (n+\delta)\right]\underbrace{\frac{dk^*}{ds}}_{>0}$$

$f(k)$



S THAT MAXIMIZES C IS CALLED "GOLDEN RULE LEVEL"

k^*_{GOLD}

c^*_{GOLD} CHARACTERIZED BY $\frac{dc^*}{ds} = 0$

OR

$$f'(k^*_{GOLD}) = \delta + n$$

TO CHECK MAX AND NOT MIN:

$$\frac{d^2c^*}{ds^2} = \frac{d}{ds}\left[\left[f'(k^*) - (n+\delta)\right]\frac{dk^*}{ds}\right]$$

RAMSEY - CASS - KOOPMAN / NEOCLASSICAL

- CONTINUUM OF HH → REPRESENT. HH
- " OF FIRMS → " FIRM

HH:
- WORK
- OWN K
- CONSUME
- RENT K TO FIRMS

FIRMS:
- PRODUCE USING LABOR & K
- PAY w TO LABOR
- PAY r TO K
- ALL MARKETS PERFECTLY COMPETITIVE

RAMSEY - CASS - KOOPMAN / NEOCLASSICAL

- CONTINUUM OF HH → REPRESENT. HH
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HH:
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- OWN k
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FIRMS:
- PRODUCE USING LABOR & K
- PAY w TO LABOR
- PAY R TO K
- ALL MARKETS PERFECTLY COMPETITIVE (ZERO PROFITS AND FREE ENTRY)

$$L(t) = e^{nt} L(0)$$
$$= e^{nt} \quad (\dot{L} = nL)$$

LOWER CASE N

FIRMS

NUMERAIRE

$$\pi(K,L) := 1 \cdot Y - RK - wL$$

$$\max_{\substack{K > 0 \\ L \geq 0}} \pi(K,L) = \max_{\substack{K > 0 \\ L \geq 0}} F(K,L) - RK - wL$$

FOC (NECESARY AND SUFFICIENT GIVEN ASSUMPTIONS ON F)

$$\frac{\partial \Pi}{\partial K} = 0 \quad ; \quad \frac{\partial \Pi}{\partial L} = 0$$

$$\frac{\partial F}{\partial K} - R = 0 \quad ; \quad \frac{\partial F}{\partial L} - w = 0$$

$$\frac{\partial F}{\partial K} = R \quad ; \quad \frac{\partial F}{\partial L} = w$$

$$\Pi(K, L) = F\left(\frac{k}{L}, \frac{L}{L}\right)L - RkL - wL$$

$$= \left[f(k) - Rk - w \right] L$$

$$=: \phi(k) L$$

FOC: $\phi'(k) = 0 \iff f'(k) = R$

ZERO PROFITS $\Rightarrow \phi(k) = 0$

HOUSEHOLD

$u(C(t))$ AT EACH t

$u' > 0$; $u'' < 0$; $\lim_{c \to 0} u'(c) = \infty$

DISCRETE TIME: β^t , $\beta \in (0,1]$

CONT. TIME: $e^{-\rho t}$, $\rho > 0$ $\lim_{c \to \infty} u'(c) = 0$

$$\int_0^\infty e^{-\rho t} u(c(t)) L(t)\, dt$$

DISCRETE TIME

$$L(t) = e^{nt} \longrightarrow L(t+1) = (1+g_L) L(t)$$

$$\int_0^\infty e^{-\rho t} u(c(t)) L(t)\, dt \longrightarrow \sum_{t=0}^{\infty} \beta^t u(c(t)) L(t)$$

$$\max_{\substack{c(t) > 0 \\ k(t) > 0}} \int_0^\infty e^{-(\rho - n)t} u(c(t))\, dt$$

s.t.

BUDGET CONSTRAINT.

- FINANCIAL ASSETS
  - BOND (RISKLESS) } ZERO NET SUPPLY
- REAL ASSETS
  - CAPITAL (RISKLESS)

$a(t) L(t)$: ASSETS OF HH.

$$\frac{\partial}{\partial t}[a(t) L(t)] = a(t) L(t) r(t) + w(t) L(t)$$

$- \ c(t)L(t)$

$$\dot{a}(t) = a(t)r(t) + w(t) - c(t) - m\,a(t)$$
$$= a(t)\left(r(t) - m\right) + w(t) - c(t)$$

$a(t)$ $\begin{cases} \text{BONDS} \rightarrow \text{ZERO} \\ \text{CAPITAL} \rightarrow k(t) \end{cases}$

$a(t) = k(t)$

$r(t) = R(t) - \delta$

IN EQUILIBRIUM

$$\dot{k}(t) = k(t)\left(R(t) - \delta - m\right) + w(t) - c(t)$$

IF NO BONDS $\rightarrow$ $k(t) \geq 0$

IF BONDS $\rightarrow$ "NO PONZI" TRANSVERSALITY CONDITION

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t \left(r(s) - m\right) ds\right) \geq 0$$

HH PROBLEM

$$\max_{\substack{c(t)>0 \\ a(t)>0}} \int_0^\infty e^{-(\rho-n)t} u(C(t))\, dt$$

s.t.

$$\dot{a}(t) = a(t)\left(r(t)-n\right) + w(t) - c(t)$$

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)\, ds\right) \geq 0$$

TO KEEP UTILITY FINITE WE NEED

$$\rho - n > 0$$

HH PROBLEM

$$\max_{\substack{c(t)>0\\ a(t)>0}} \int_0^\infty e^{-(\rho-n)t} u(C(t))\,dt$$

$$\text{s.t.} \quad a(0) = a_0$$

$$\dot{a}(t) = a(t)(r(t)-n) + w(t) - c(t)$$

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)\,ds\right) \geq 0$$

TO KEEP UTILITY FINITE WE NEED

$$\rho - n > 0$$

MAXIMUM PRINCIPLE

$$\max_{u(\cdot)} \int_0^\infty e^{-\rho t} f(x(t), u(t))\,dt$$

s.t.

$$\dot{x}(t) = \mu(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$X(0) = X_0 \quad \text{GIVEN}$$

$$u(\cdot) \in \mathcal{U}$$

NECESSARY CONDITIONS:

1. STATE EQUATION & FEASABILITY

$$\dot{x}(t) = h(t, x(t), u(t))$$

$$G(x(t), u(t)) \geq 0$$

$$X(0) = x_0 \quad ; \quad u(\cdot) \in \mathcal{U}$$

2. Let

$$\mathcal{H}(t, x(t), u(t), \lambda(t), \phi(t)) :=$$

$$f(x(t), u(t)) + \lambda(t)\, h(t, x(t), u(t)) + \phi(t)\, G(x(t), u(t))$$

$$u^*(t) \in \max_{u} \mathcal{H}(t, x^*(t), u, \lambda(t), \phi(t))$$

$$G(x^*(t), u) \geq 0$$

FOC: USUAL CALCULUS CONSTRAINED OPTIM.

3. DYNAMICS OF $\lambda(t)$

$$\dot{\lambda}(t) = \rho \lambda(t) - \frac{\partial \mathcal{H}}{\partial x}(t, x^*(t), u^*(t), \lambda(t), \phi(t))$$

4. COMPLEMENTARY SLACKNESS

$$\phi(t) \geq 0 \qquad \phi(t)\, G(x^*(t), u^*(t)) = 0$$

5. TRANSVERSALITY

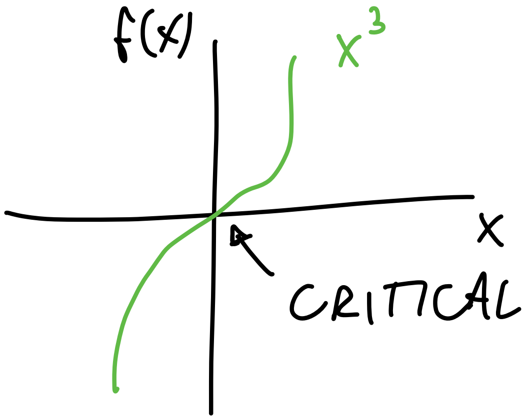
$$\lim_{t \to \infty} e^{-\rho t} \mathcal{H}(t, x^*(t), u^*(t), \lambda(t), \phi(t)) = 0$$

OR

(UNDER SOME ASSUMPTIONS)

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t)\, x(t) = 0$$

COMMENTS



CRITICAL POINT : $f'(x) = 0$ AT $x=0$

- NECESSARY

BUT $x=0$ NOT MAX OR MIN

- λ, ϕ ARE "SHADOW VALUES" OF CONSTRAINTS.

- "PRESENT VALUE $\mathcal{H}^{PV}$"

$$\mathcal{H} = e^{-\rho t} f + \lambda^{PV} h + \phi^{PV} \cdot G$$

$$\lambda^{PV} = e^{-\rho t} \lambda$$

$$\phi^{PV} = e^{-\rho t} \phi$$

$$\dot{\lambda}^{PV} = e^{-\rho t} \dot{\lambda} - \rho e^{\rho t} \lambda = e^{-\rho t}(\dot{\lambda} - \rho \lambda)$$

$$\dot{\lambda}^{PV} = - \frac{\partial \mathcal{H}}{\partial x}$$

- BACKWARD INDUCTION

IMAGINE HORIZON IS $T < \infty$

AT $t = T$

$\dot{x}$ EQUATION DOES NOT APPLY

$\Rightarrow$ MAX f w/o THINKING ABOUT FUTURE x

IF AT T, NO CONSTRAINTS ON x

$\lambda(T) = 0$

(w/ CONSTRAINTS: $\lambda(T) = G' \ldots$)

$\dot{\lambda}$ EQ IS A "BACKWARD EQ"

IN INFINITE HORIZON, WHAT IS

" $\lambda(T)$ " ?

ANSW: TRANSVERSALITY!

NOT $\lim_{t\to\infty} e^{-\rho t}\lambda(t) = 0$!!

$$\mathcal{H} = \underbrace{u(c(t))}_{f} + \lambda(t)\underbrace{\left[a(t)(r(t)-m) + w(t) - c(t)\right]}_{h}$$

$$\dot{\lambda}(t) = (\rho - m)\lambda(t) - \frac{\partial \mathcal{H}}{\partial a}(t, \dots)$$

$$\dot{\lambda}(t) = (\rho - m)\lambda(t) - (r(t) - m)\lambda(t)$$

(1) $\dot{\lambda}(t) = [\rho - r(t)]\lambda(t)$

(2) $\max_c \mathcal{H} \Rightarrow \frac{\partial \mathcal{H}}{\partial c} = 0 \Rightarrow u'(c^*(t)) = \lambda(t)$

FROM (1) $\frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - r(t)$

$$\frac{\partial}{\partial t}\left(\log \lambda(t)\right) = \rho - r(t)$$

$$\log \lambda(t) = \log \lambda(0) + \int_0^t [\rho - r(s)]ds$$

$$\int_0^T d \log(\lambda(t)) = \int_0^T [\rho - r(t)]dt$$

$$\log \lambda(T) - \log \lambda(0) = \int_0^T [\rho - r(t)]dt$$

USE (2) :

$$u'(c^*) = \lambda(0) \exp\left(-\int_0^t [r(s) - \rho]ds\right)$$

$$\frac{\partial}{\partial t}\left(\log \lambda(t)\right) = \rho - r(t)$$

$$\log \lambda(t) = \log \lambda(0) + \int_0^t [\rho - r(s)]ds$$

$$\int_0^T d \log(\lambda(t)) = \int_0^T [\rho - r(t)]dt$$

$$\log \lambda(T) - \log \lambda(0) = \int_0^T [\rho - r(t)]dt$$

USE (2) :

$$\lambda(t) = u'(c^*) = \lambda(0) \exp\left(-\int_0^t [r(s) - \rho] ds\right)$$

(3) TRANSVERSALITY :

$$\lim_{t \to \infty} e^{-(\rho - m)} a(t) \lambda(t) = 0$$

(FINITE HORIZON : $a(T) = 0$)

$\llcorner$ EITHER NEED $a(t) \geq$ LOWER BOUND

OR

$\llcorner$ ADMISSIBLE CONTROLS / STATES ARE FINITE FOR ALL t

COMBINE (2) AND (3). THEN,

NO PONZI HOLDS WITH EQUALITY

COMBINE $u'(c(t)) = \lambda(t)$ AND

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -(r(t) - \rho)$$

$$\frac{\partial}{\partial t} u'(c(t)) = u''(c(t))\,\dot{c}(t) = \dot{\lambda}(t)$$

$$\Rightarrow u''(c(t))\,\dot{c}(t) = -(r(t) - \rho)\,\lambda(t)$$

$$= -(r(t) - \rho)\,u'(c(t))$$

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'(c(t))}{\underbrace{c(t)\,u''(c(t))}}\,(r(t) - \rho)$$

ELASTICITY OF

INTERTEMPORAL

SUBSTITUTION

$:= [\theta(t)]^{-1}$

ELASTICITY B/W t, s:

$$\theta(t,s) := \frac{\partial\left(\log[u'(c(t))/u'(c(s))]\right)}{\partial\left(\log[c(t)/c(s)]\right)}$$

$$\lim_{s \to t} \theta(t,s) = \theta(t)$$

USE FIRM OPTIMALITY:

$$f'(k) = R(t)$$

$$r(t) = f'(k) - \delta$$

ZERO PROFITS:

$$w(t) = f(k) - f'(k)k$$

USE MARKET CLEARING:

$$a(t) = k(t)$$

(SINCE BONDS IN ZERO NET SUPPLY)

BUDGET CONSTRAINT:

$$\dot{a}(t) = a(t)\left(r(t) - n\right) + w(t) - c(t)$$

$$\dot{k}(t) = k(t)\left(f'(k) - \delta - n \right) +$$

$$+ f(k) - f'(k)\,k(t) - c(t)$$

$$\boxed{\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)\,k(t)}$$

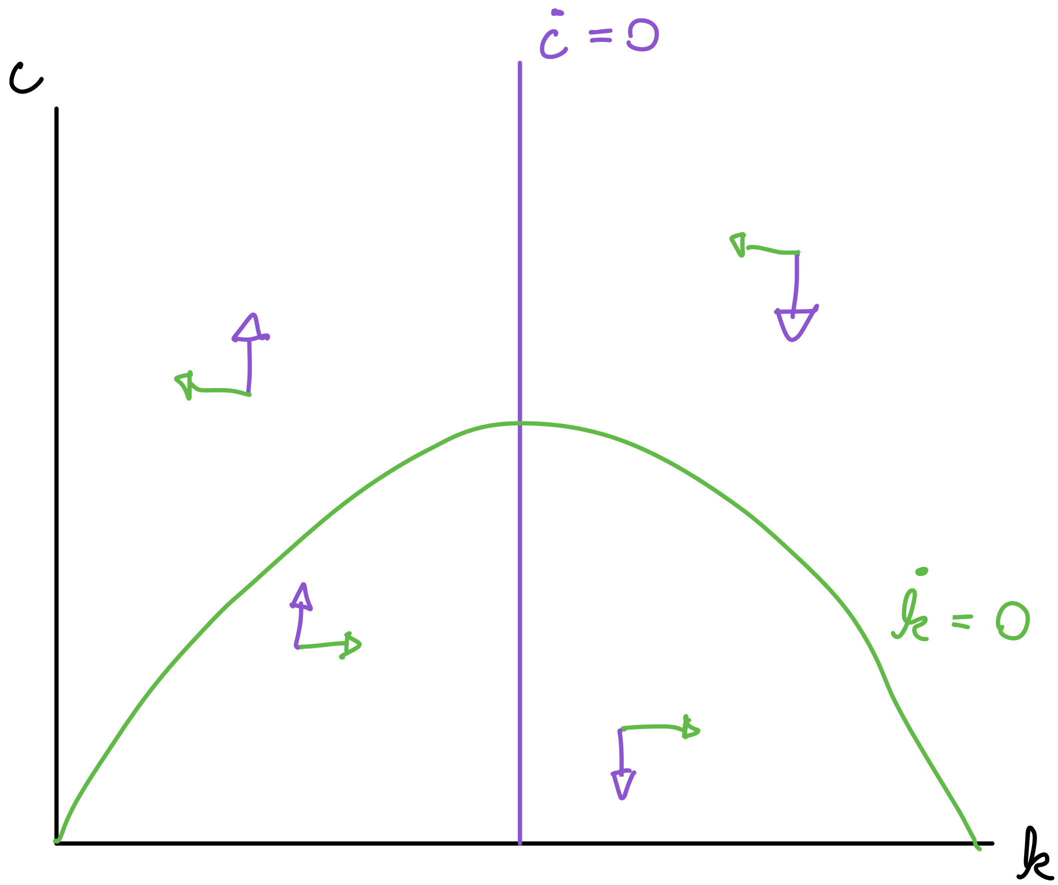
$$\boxed{k(0) = k_0} \quad + \quad \boxed{\text{TRANSVERSALITY}}$$

$$\frac{\dot{c}(t)}{c(t)} = - \frac{u'(c(t))}{c(t)\,u''(c(t))} \left(r(t) - \rho \right)$$

$$\boxed{\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(t)} \left(f'(k(t)) - \delta - \rho \right)}$$

$$\dot{c} = 0 \Rightarrow f'(k) = \delta + \rho$$

PHASE DIAGRAM



$$\dot{k} = 0 \Rightarrow 0 = f(k) - c - (\delta + m)k$$

$$\Rightarrow c = f(k) - (\delta + m)k$$