Multiconvex Optimization in Julia

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Documentation for Multiconvex.jl

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Multiconvex functions

Definition (Restriction)

For $f : \mathbf{R}^n \to \mathbf{R}$ and $\omega \subseteq \{1, \dots, n\}$, define the **restriction** $f_\omega(\cdot, \bar{x}) : \mathbf{R}^{|\omega|} \to \mathbf{R}$ of f to ω to be the function obtained by fixing the coefficients in ω^C to their values in $\bar{x} \in \mathbf{R}^n$: $x \mapsto f_\omega(x; \bar{x})$.

Definition (Multiconvex function)

A function $f : \mathbf{R}^n \to \mathbf{R}$ is k-**convex** if there exists a partition $\Omega = \{\omega_1, \dots, \omega_k\}$ of $\{1, \dots, n\}$ so that f_{ω_j} is convex for every $j = 1, \dots, k$.

Multiconvex functions generalize **biconvex** and **multilinear** functions.

- A 1-convex function is convex; a 2-convex function is biconvex; a 3-convex function is triconvex; etc.
- A multilinear function is multiconvex.

Multiconvex problems

Consider a (nonconvex) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x_{\beta_i}) \leq 0, \quad i = 1, \ldots, m \end{array} \qquad (\mathcal{P})$$

with variable $x \in \mathbf{R}^n$.

Definition (Multiconvex problem)

An optimization problem is k-**convex** if there exists a partition $\Omega = \{\omega_1, \ldots, \omega_k\}$ of $\{1, \ldots, n\}$ with the following properties:

- f_0 is k-convex with partition Ω;
- f_i is convex for every $i = 1, \ldots, m$;
- for every constraint $i = 1, \ldots, m$, there is an element j of the partition with $\beta_i \subseteq \omega_j$.

MultiConvex.jl

`MultiConvex.jl` extends `Convex.jl` to detect and (heuristically) solve multiconvex optimization problems using **disciplined** multiconvex programming:

- simple: less than 300 lines of code
- heuristic solution method: alternating minimization

Definition (Disciplined multiconvex problem)

A multiconvex optimization problem is a **disciplined multiconvex problem** if

- f_0 is k-**convex** with partition $\Omega = \{\omega_1, \dots, \omega_k\}$
- f_0 restricted to ω_j is a disciplined convex function for every $j = 1, \dots, k$
- f_i is a disciplined convex function for $i = 1, \dots, m$

MultiConvex.jl in action

```
using MultiConvex

# initialize nonconvex problem
n, k = 10, 1
A = rand(n, k) * rand(k, n)
x = Variable(n, k)
y = Variable(k, n)
problem = minimize(sum_squares(A - x*y), x>=0, y>=0)

# perform alternating minimization on the problem
altmin!(problem)
```

Conflict graphs

Definition

The **conflict graph** $G = (V, E)$ of a multiconvex expression e is a graph on the variables in the expression:

$$V = \textbf{variablesin}(e), \qquad E \subseteq V \times V$$

with the property that for any independent set of variables ω in the graph, the restriction f_ω of f to ω is convex.

Every multiconvex expression has a (unique) conflict graph.

Conflict graphs: recursion

- *Constant.* A constant c is multiconvex with conflict graph $(\emptyset, \emptyset)$
- *Variable.* A variable v is multiconvex with conflict graph $(v, \emptyset)$
- *Expressions.* The conflict graph of a composite expression is the union of the conflict graphs of its arguments, together with (possibly) a few more edges.
  - multiplication $(*, (x, y))$ adds complete bipartite graph on **variablesin**(x) and **variablesin**(y)
- *Constraints.* A constraint is multiconvex iff it is convex.
- *Problems.* Problems check their convexity by constructing a certifying partition Ω of the conflict graph of the objective that respects the constraints (if one exists).

Alternating minimization

Now that we've found a partition Ω, we can use alternating minimization:

```
for iter=1:AMiters
    for ω in Ω
        # free the variables in ω to optimize over just those variables
        for v in ω
            free!(v)
        end
        solve!(problem, warmstart=true)
        # now that we've found their values, fix them again
        for v in ω
            fix!(v)
        end
    end
end
```

(or ADMM, or ...)

More information (and code!)

- `Convex.jl`: `http://www.github.com/JuliaOpt/Convex.jl`
- `MultiConvex.jl`: `http://www.github.com/madeleineudell/MultiConvex.jl`
- Convex.jl paper: `http://arxiv.org/abs/1410.4821`