How to Escape a Liquidity Trap with Interest Rate Rules

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Abstract

I study how central banks should communicate monetary policy in liquidity trap scenarios in which the zero lower bound on nominal interest rates is binding. Using a standard New Keynesian model, I argue that the key to preventing self-fulfilling deflationary spirals and anchoring expectations is to promise to keep nominal interest rates pegged at zero for a length of time that depends on the state of the economy. I derive necessary and sufficient conditions for this type of state contingent forward guidance to implement the welfare maximizing equilibrium as a globally determinate (i.e., unique) equilibrium. Even though the zero lower bound prevents the Taylor principle from holding, determinacy can be obtained if the central bank sufficiently extends the duration of the zero interest rate peg in response to deflationary or contractionary changes in expectations or outcomes. Fiscal policy is passive, so it plays no role for determinacy. The interest rate rules I consider are easy to communicate, require little institutional change and do not entail any unnecessary social welfare losses.

Keywords: Zero lower bound (ZLB), liquidity trap, New Keynesian model, indeterminacy, monetary policy, Taylor rule, Taylor principle, interest rate rule, forward guidance.

JEL Classification: E43, E52, E58.

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1 Introduction

Short-term nominal interest rates in many advanced economies —including Japan, the US and Europe— have recently spent several years against their zero lower bound (ZLB) or close to it.[1] There is ample awareness by academics and policymakers that visits to the ZLB may be more frequent in the near future, due in part to persistently low natural rates of interest.[2] One tool that central banks have used when constrained by the ZLB is forward guidance, whereby central banks promise to keep short-term nominal interest rates low for an extended period of time.[3] How should this promise be communicated to the public and how should the central bank react —or threaten to react— if outcomes turn out to be different from what it had hoped for? What does it take to "anchor expectations" and prevent self-fulfilling "deflationary traps"?

In this paper, I answer these questions through the lens of a canonical deterministic New Keynesian (NK) model in continuous time with a binding ZLB. The model is identical to that in Werning (2012) and Cochrane (2016). The ZLB binds because the exogenous natural rate of interest is negative for some finite initial period of time, a situation Werning (2012) calls a liquidity trap. Eventually, the natural rate becomes positive and the ZLB ceases to bind, so there is no fundamental reason for inflation to remain below target. However, the economy is susceptible to "deflationary traps" in which the expectation of low inflation can be self-fulfilling, pushing the economy into the ZLB irrespective of the level of the natural rate, a suboptimal outcome for social welfare. Self-fulfilling expectations can also create macroeconomic instability in the form of other kinds of multiple equilibria, including some with chaos (Benhabib, Schmitt-Grohé, and Uribe (2001b, 2002); Schmitt-Grohé and Uribe (2009)).

The main contribution of this paper is to provide necessary and sufficient conditions for a class of monetary policy rules to implement the socially optimal "forward guidance" equilibrium characterized by Werning (2012); Eggertsson and Woodford (2003); Jung, Teranishi, and Watanabe (2005) as a globally determinate (i.e., unique) equilibrium, thereby anchoring expectations and eliminating self-fulfilling deflationary traps. My goal is not to provide an alternative to forward guidance, but rather to show how to communicate it properly. The class of monetary policy rules I consider can be understood in two stages. In the first stage, the central bank promises to keep short-term nominal interest rates pegged at zero for some period of time. I refer to the end of this period —when the promise to keep rates at zero

[1] I use the term ZLB even though the true or "effective" lower bound can certainly be different from zero. All of my results hold as long as there is some lower bound on interest rates.

[2] Kiley and Roberts (2017), Del Negro, Giannone, Giannoni, and Tambalotti (2017), Rogoff (2017).

[3] Forward guidance has been used as a tool to communicate the future path of monetary policy even in periods away from the ZLB, and does not necessarily need to take the form of a promise to keep interest rates low.

ends— as liftoff. The new element I introduce, and the one that is key for determinacy, is that liftoff can be made state-contingent, so that it can depend on past, present and expected values of economic variables. The second stage begins after liftoff. In this stage, the central bank follows a standard Taylor rule that respects the ZLB. Fiscal policy is "passive" or "Ricardian" and hence plays no role in the determinacy of equilibria.

Without a ZLB on interest rates, there is a unique steady state with zero inflation and the Taylor principle —that nominal interest rates react more than one-for-one with inflation— is necessary and sufficient for local determinacy. When the ZLB binds, there can be a second steady state in which inflation is negative. Although the Taylor principle is still necessary and sufficient for local determinacy around the zero-inflation steady state, Benhabib et al. (2001b) show the same is not true for global determinacy by constructing equilibria that start arbitrarily close to the zero-inflation steady state but eventually exit its vicinity and converge to the deflationary steady state.

The study of determinacy that I conduct produces several results that can inform central banks' communication strategy. First, if the central bank announces that liftoff will occur at a fixed future date that is not contingent on the state of the economy, usually referred to as "calendar-based" forward guidance, then depending on its actions after liftoff, the central bank can either implement the optimal equilibrium or attain global determinacy, but not both.[4]

Second, when liftoff does depend on the state of the economy, the optimal equilibrium can be implemented in a globally determinate way without following the Taylor principle after liftoff. In fact, under the appropriate state contingent liftoff policy, the optimal equilibrium is globally determinate even if the central bank pegs interest rates to the optimal path, such that interest rates after liftoff are completely unresponsive to the state of the economy (for example, this can done with a Taylor-type rule that has a time-varying intercept and coefficients of zero on both inflation and the output gap). Together, these first two results imply that the key to eliminating deflationary trap equilibria lies more in how liftoff is communicated and less on what the central bank does after liftoff.

Third, while the necessary and sufficient conditions for global determinacy of the optimal equilibrium that I derive are not as simply stated as the Taylor principle is, a succinct and easy-to-communicate sufficient condition is to promise to sufficiently delay liftoff in response to deflationary and/or contractionary private sector expectations and to then *not* follow the Taylor principle after liftoff. If central bank policy is restricted to be continuous in the state of the economy, then this sufficient principle is also necessary. Continuous rules are an important family not only because they are likely more realistic and easier to communicate,

[4] There is one exception given by the single combination of parameters of the model for which the optimal discretionary policy coincides with the optimal forward guidance equilibrium (which requires commitment).

but also because they are robust to "trembling hand" errors (Loisel (2018)).

Fourth, I show what it takes to "sufficiently delay" liftoff by providing an explicit minimum delay to which the central bank must commit. This minimum delay is a non-linear function of inflation and the output gap. Because any delay in liftoff that is longer than the minimum delay is just as good, communicating the precise shape of this non-linear function to the public is not strictly required. I also show what the "deflationary and/or contractionary private sector expectations" mentioned in the previous paragraph precisely entail. The conditions for determinacy imply that the main gauge of private sector expectations are the time $t = 0$ initial values of inflation and the output gap. To implement the optimal equilibrium with global determinacy, liftoff must depend on these initial levels of economic activity. In other words, monetary policy must be backward-looking (history dependent). This insight is not very surprising, as the optimal equilibrium path is itself history dependent and Eggertsson and Woodford (2003) show that it can be implemented with local determinacy using a price level targeting regime that is history dependent on and off the equilibrium path by design. However, I discover that path dependence is a necessary part of rules that implement the optimal equilibrium with global determinacy. History dependence is not just one possible way to anchor expectations at the ZLB, it is the only way. Unlike price level targeting, the rules I consider can be made memoryless after liftoff if desired —so that all path dependence ends at liftoff— and still implement the optimal equilibrium with global determinacy. Additionally, I find that when the monetary policy rule is continuous in the state of the economy, liftoff must be not only backward-looking but also forward-looking, so it must take into account expectations of future inflation and the output gap.

The class of monetary policy rules I consider closely describe real-world behavior of central banks, so my results can be immediately applied to assess the appropriateness of central banks' communication strategies during the last crisis. For example, in August 2011, the Federal Open Market Committee (FOMC) of the Federal Reserve announced calendar-based forward guidance, specifying near zero interest rates "at least through mid-2013." My results imply that this type of language is not conducive to implementing the optimal equilibrium without indeterminacy. In 2012, the FOMC changed its language and promised to keep the federal funds rate at zero "for [...] at least as long as the unemployment rate remains above 6-1/2 percent [and] inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee's 2 percent longer-run goal," a strategy sometimes referred to as threshold-based forward guidance. My results suggest that the switch from a calendar-based to a state-contingent liftoff announcement is a large step in the right direction. However, the lack of a backward-looking component in the conditions for liftoff mean that threshold-based forward guidance cannot implement the optimal equilibrium without indeterminacy. A practical solution to this problem would have been to add a clause that, for example, liftoff will not occur until average inflation over the last year exceeds 1.5 percent (or some other large enough value).

In this paper, I focus on perfect foresight equilibria only. This equilibrium concept captures the key type of indeterminacy induced by self-fulfilling deflationary expectations at the ZLB that has been highlighted by the recent literature without the unneeded burden of stochasticity.[5] With a deterministic economy and perfect foresight, the only type of indeterminacy that arises in the model is the one associated with the existence of a continuum of time $t = 0$ initial values of inflation and the output gap that are compatible with equilibrium. Many of my results involve not following the Taylor principle after liftoff, which engenders local indeterminacy around the zero inflation steady state. If I relaxed the assumption of perfect foresight and allowed equilibria to be stochastic, then this local indeterminacy would give rise to sunspot equilibria (see Woodford (1984) and Shigoka (1994) for the continuous time version). However, it is possible to modify the interest rate rules I consider to rid the economy of this local indeterminacy by using ideas already present in the literature. For example, the central bank can announce that once the economy is close enough to the zero inflation steady state, it will switch to following the Taylor principle as soon as a sunspot appears. One can also use the idea in Cochrane (2016) that for any given interest rate path, it is possible to implement any equilibrium consistent with that path in a locally determinate way by following a Taylor-type rule that obeys the Taylor principle and has an appropriately chosen time-varying inflation target or intercept. Even though this technique does not work when interest rates are at the ZLB (since the Taylor principle cannot hold when interest rates cannot be lowered in response to lower inflation), it does work after liftoff, when the ZLB is no longer binding. Then, by using Cochrane's technique right after liftoff, the optimal equilibrium can be made locally determinate around the zero inflation steady state. In this case, the state contingent liftoff would eliminate the indeterminacy induced by having a ZLB, and the Taylor rule à la Cochrane would eliminate the more classic indeterminacy associated with passive monetary policy around a steady state that is bounded away from the ZLB. Finally, it is important to mention that even when local indeterminacy is avoided, there may still be other rational expectations equilibria, including sunspot equilibria, that I do not consider.

Related literature. To my knowledge, this paper is the first to present a monetary policy rule that produces global determinacy in a monetary economy in which fiscal policy is passive and the ZLB binds. It is also, again to the best of my knowledge, the first to assess the global determinacy properties of the threshold-based forward guidance pursued by the

[5] Other papers that focus on perfect foresight equilibria in closely related contexts are Cochrane (2016); Carlstrom, Fuerst, and Paustian (2015); Del Negro, Giannoni, and Patterson (2015); McKay, Nakamura, and Steinsson (2016); Farhi and Werning (2017) among many others. See García-Schmidt and Woodford (2019) for a critique of using perfect foresight to study some monetary policy questions.

FOMC.

In the discrete-time version of the model I use, Eggertsson and Woodford (2003) implement the optimal equilibrium as a locally determinate equilibrium by means of an output-gap adjusted price-level targeting rule and as a globally determinate equilibrium by adding a non-Ricardian fiscal policy commitment. My results are different in a few ways and offer some advantages and disadvantages relative to their contribution. First, and perhaps most important, global determinacy for the rules I consider is achieved with Ricardian fiscal policy. However, I only consider a deterministic economy, while they tackle a more general economy with shocks and sunspots. This allows Eggertsson and Woodford (2003) to show that the rule they propose can be implemented without knowledge of the statistical process for the natural rate of interest, a very desirable property for the robustness of the rule. Since I do not consider shocks, it is not possible to evaluate whether the same would hold in a stochastic version of the rules I study. Second, the monetary policy rule I study requires little change in the institutional arrangements of most central banks in advanced economies. Most central banks communicate the level and expected path of a short-term nominal interest rate. Forward guidance, even in its state contingent form, has already been tried before, as mentioned above.[6] A switch to a price-level target such as the one proposed by Eggertsson and Woodford (2003) may entail some initial fixed costs, such as a temporary loss of credibility and central bank resources, although the size and duration of these costs are difficult to assess and could very well be small. Perhaps a more relevant argument is that, rightly or not, central banks do not seem to be seriously considering a switch to price-level targeting.[7]

Third, with price-level targeting, the central bank must implement a specific target, while the interest rate rule I advocate in this paper requires only a "long enough" initial period of zero interest rates. While this makes no difference strictly inside the models, in practice it may reduce the need to estimate private-sector expectations and parameters of the model with high precision. A central bank can judge a certain set of parameters and expectations to be reasonable ex-ante and then enact a policy that works for all such parameters by taking the strongest "strong enough" response across them and without too much fear of overshooting. Fourth, the interest rate rules I consider can be made memoryless after liftoff

[6]Of course, a price-targeting regime can be implemented and explained by means of an interest rate rule and, conversely, the interest rate rule I study can be communicated as a price-targeting rule, or even an inflation targeting rule. However, the essence of the rule in Eggertsson and Woodford (2003) is more naturally explained as a price-level targeting regime, while the rules I examine are more naturally explained as interest rate rules. Since communication is a key aspect of policy rules and the focus of this paper, the distinction seems at least worth mentioning.

[7]Judging from meeting minutes, the Federal Open Market Committee (FOMC) discussed and rejected nominal GDP targeting in 1982, 1992 and 2011 (Federal Open Market Committee of the Federal Reserve (2015a), Federal Open Market Committee of the Federal Reserve (2015b), Federal Open Market Committee of the Federal Reserve (2015c)), but not without noting some of its merits. Bernanke (2015b) and Bernanke (2015a) explain why the FOMC rejected nominal GDP targeting in 2011.

and still guarantee global determinacy of the optimal equilibrium. In price-level targeting, the closing of the price gap that opens while at the ZLB continues after interest rates become positive. This means that the rules I study in this paper are not simply price-level targeting rules in disguise.[8] Finally, the use of continuous versus discrete time is inconsequential for the economics of the model but by using continuous time, I am able to exploit the Poincaré-Bendixson theorem, a powerful tool that classifies all possible dynamics of two-dimensional systems in continuous time.[9]

Indeterminacy is an important issue in all NK models, but Benhabib et al. (2001b) show that self-fulfilling deflationary expectations are particularly difficult to arrest in the presence of a binding ZLB and interest rate feedback rules. While all of the results they obtain also apply to the framework I use, I am able to eliminate the kind of indeterminacy that arises from the self-fulfilling deflationary expectations —and all other perfect foresight equilibria without sunspots, including ones with chaotic trajectories that they study— by considering a broader class of interest rate rules that allow for history dependence and a state-contingent liftoff.

Although Cochrane (2016) considers issues around indeterminacy in the same NK model I use, his focus is neither on how to communicate policy nor on global determinacy.

Schmitt-Grohé and Uribe (2014) propose an interest-rate-based strategy to escape liquidity traps that entails temporarily deviating from a Taylor rule by increasing nominal interest rates in a deterministic and non-state-contingent way until a pre-specified target is reached. This strategy succeeds in setting a floor for inflation without a non-Ricardian fiscal stance but does not lead to globally determinate outcomes.

There are many other proposals on how monetary policy should be conducted in a liquidity trap. Some prominent examples include: Svensson (2004), who advocates an intentional currency depreciation combined with a calibrated crawling peg; McCallum (2011), Sumner (2014) and Romer (2011), who recommend nominal GDP targeting; and Blanchard, Dell'Ariccia, and Mauro (2010) and Ball (2013), who promote increasing the inflation target by contending that the trade-off between higher steady-state inflation and less frequent visits to the ZLB is worth undertaking. In practice, many central banks have advocated and used large-scale asset purchases and negative interest rates. None of the studies mentioned in this paragraph explicitly consider the global determinacy properties of their proposals. For the class of rules I consider, there are no suboptimal trade-offs and there is no need to accommodate new monetary or price aggregates, price-level or inflation targets, "shadow" rates, exchange rates, the central bank's balance sheet, or the quantity or price of other assets.

[8] The version of the rule that becomes memoryless after liftoff more closely resembles the *temporary* price level target advocated by Bernanke (2017), although I suggest a different way to communicate it.

[9] See Appendix C for a precise statement of the Poincaré-Bendixson theorem.

English, López-Salido, and Tetlow (2015); Florez-Jimenez and Parra-Polania (2016); Coenen and Warne (2014) study various aspects of threshold-based forward guidance but not determinacy. Boneva, Harrison, and Waldron (2018) model threshold-based forward guidance as a regime that the central bank enter and exit and argue that to have a unique equilibrium, the exit time must be probabilistic. They also conduct numerical experiments to show that the level of thresholds is important for uniqueness of equilibria.

2 The Canonical New Keynesian Model with a ZLB

I use the framework of Werning (2012), a standard deterministic New Keynesian model in continuous time that, save for the ZLB, is log-linearized around a zero-inflation steady state.[10][11] The economy is described by

$$\dot{x}(t) = \sigma^{-1}\left(i(t) - r(t) - \pi(t)\right), \tag{1}$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t), \tag{2}$$

$$i(t) \geq 0. \tag{3}$$

The variables $x(t)$ and $\pi(t)$ are the output gap and the inflation rate, respectively. The output gap is the log-deviation of actual output from the hypothetical output that would prevail in the flexible price, efficient allocation. Henceforth, for brevity, I refer to the output gap simply as output. The central bank's policy instrument is the path for the nominal short-term (instantaneous) interest rate $i(t)$, which must remain non-negative at all times by the ZLB equation (3). The variable $r(t)$ is the exogenous natural rate of interest, defined as the real interest rate that would prevail in the flexible price, efficient economy with $x(t) = 0$ for all t. The process for $r(t)$ is

$$r(t) = \begin{cases} r_l < 0 & , \quad 0 \leq t < T \\ r_h > 0 & , \quad t \geq T \end{cases} .$$

[10] Although most analysis of determinacy in New Keynesian models is done in log-linearized models, Braun, Körber, and Waki (2016) contend that conclusions would differ in the non-linear model. On the other hand, Christiano and Eichenbaum (2012) show that the additional equilibria that arise from non-linearities in Braun et al. (2016) are not E-learnable. In addition, Christiano and Eichenbaum (2012) show that the linear approximation is accurate except in extreme cases, such as when output deviates by more than 20 percent from steady state. While these issues are important, I do not seek to address them here and simply use the log-linear model (plus the ZLB), a standard practice in the literature. As long as the two steady states of the economy are close to each other, the first-order approximation around one of the steady states should still provide an accurate approximation in a ball that includes both steady states.

[11] By "global determinacy", I do not mean that I consider equilibria in the original non-linear version of the NK model. I use the word global to refer to equilibria that do not necessarily stay in a small neighborhood of a steady state. In linear models, the distinction between local and global determinacy is unnecessary. In the present model, the ZLB introduces a non-linearity that makes global and local determinacy different.

The constants $r_l < 0$, $r_h > 0$ and $T > 0$ are given. I define a *liquidity trap* as the period in which the natural rate is negative, as in Werning (2012). The economy starts in a liquidity trap and exits it with certainty at time T. None of the results in this paper change if the path for $r(t)$ is different as long as $r(t) < 0$ for $t < T$ and $r(t) > 0$ for $t > T$.

Equation (1) is the IS curve, the log-linearized Euler equation of the representative consumer. The constant $\sigma^{-1} > 0$ is the elasticity of intertemporal substitution. Equation (2) is the New Keynesian Phillips curve (NKPC), the log-linear version of firms' first-order conditions when they maximize profits by picking the price of differentiated consumption goods under monopolistic competition while subject to consumers' demand and Calvo pricing. The constant $\rho > 0$ is the representative consumer's discount rate and $\kappa > 0$ is related to the amount of price stickiness in the economy. As $\kappa \to \infty$, the economy converges to a fully flexible price economy while prices become completely rigid when $\kappa \to 0$. Financial markets are complete (there is a complete set of tradable Arrow-Debreu securities).

Definition. *A* ***perfect foresight equilibrium*** *consists of bounded paths for inflation, output and the nominal interest rate* $\{\pi(t), x(t), i(t)\}_{t\geq 0}$ *that, given a path* $\{r(t)\}_{t\geq 0}$ *for the natural rate, satisfy equations (1)-(3).*

Five elements in the definition are worth discussing for the purposes of this paper. First, the requirement that output and inflation remain bounded at all times is equivalent to the asymptotic conditions

$$\lim_{t\to\infty} |x(t)| < \infty, \tag{4}$$

$$\lim_{t\to\infty} |\pi(t)| < \infty. \tag{5}$$

The role that equation (5) plays for determinacy of equilibria has been examined in the literature.[12] In the specific setup of this paper, inflation is unbounded if and only if output is unbounded, making it impossible to differentiate between nominal and real unboundedness of paths. Thus, equation (5) can be omitted from the definition of equilibrium and issues regarding its applicability can be sidestepped.

Second, paths for $\pi(t)$ and $x(t)$ that satisfy equations (1) and (2) must be continuous.[13] With complete markets, if there were any jumps in $x(t)$ or $\pi(t)$, the representative consumer's

[12] Cochrane (2011) argues that there is no obvious economic reason to exclude paths with unbounded inflation from the definition of equilibrium. McCallum (2009) and Atkeson, Chari, and Kehoe (2010) agree and, among others, propose different criteria to eliminate or select equilibria. Woodford (2003), Wren-Lewis (2013) and others defend the approach of using equation (5).

[13] A classical solution to the system of ODEs in equations (1)-(2) would also require differentiability of $x(t)$ and $\pi(t)$ for all t. But if $x(t)$ and $\pi(t)$ were differentiable for all t, the central bank's control problem of Section 3 would have no solution, since any solution necessarily requires a jump in the control $i(t)$.

I instead use "Filippov solutions," also called solutions "in the sense of distribution," a weaker solution concept that allows for non-differentiability in a set of measure zero (Filippov (2013)). Any other weak notion of solution (such as viscosity solutions) would preserve all the results of this paper as long as derivatives are finite everywhere.

Euler equation would be violated in this deterministic economy owing to the existence of arbitrage opportunities. On the other hand, there are no smoothness requirements for $i(t)$ as it is a choice variable for the central bank.

Third, neither the definition of equilibrium nor the dynamics of the economy in equations (1) and (2) make any explicit reference to fiscal policy, although, as stressed by Woodford (1995), Sims (1994), Benhabib, Schmitt-Grohé, and Uribe (2001a), Cochrane (2011) and others, whether determinacy obtains depends on the joint monetary-fiscal regime. The implicit fiscal behavior I assume is that of a "Ricardian" (in the terminology of Woodford (2001)) or "passive" (in the terminology of Leeper (1991)) regime: The fiscal authority always adjusts taxes or spending ex-post to validate any path of the endogenous variables that may arise.

Fourth, the concept of perfect foresight equilibria that I use rules out sunspot equilibria. I briefly return to sunspot equilibria in a later section. Until then, all equilibria should be understood to be non-stochastic.

Fifth, to economize on notation, I do not generally differentiate expected values from realizations, as they are identical when expectations are rational and there is perfect foresight. However, it is useful to keep in mind for the interpretation of many of the results below that $\dot{x}(t)dt = x^e(t+dt) - x(t)$ and $\dot{\pi}(t)dt = \pi^e(t+dt) - \pi(t)$, where $x^e(t+dt)$ and $\pi^e(t+dt)$ are the expected values of $x(t+dt)$ and $\pi(t+dt)$ conditional on time-t information, which makes explicit the role of expectations and the forward-looking nature of equations (1) and (2). Rational expectations requires that $x^e(t) = x(t)$ and $\pi^e(t) = \pi(t)$ for all $t \geq 0$.

3 The Optimal Equilibrium

The social welfare loss function for the economy is

$$V = \frac{1}{2}\int_0^\infty e^{-\rho t}\left(x(t)^2 + \lambda\pi(t)^2\right)dt. \tag{6}$$

The constant $\lambda > 0$ is a preference parameter that dictates the relative importance of deviations of output and inflation from their desired value of zero. This quadratic loss objective function can be obtained as a second-order approximation around zero inflation to the economy's true social welfare function when the flexible price equilibrium is efficient (Woodford (2003)). An *optimal equilibrium* is an equilibrium that minimizes (6).

Werning (2012) solves for the optimal equilibrium $\{\pi^*(t), x^*(t), i^*(t)\}_{t\geq 0}$ when the central bank has perfect commitment and credibility. He finds that it is unique and that the optimal path for the nominal interest rate is

$$i^*(t) = \begin{cases} 0 & , \quad 0 \le t < t^* \\ (1 - \kappa\sigma\lambda)\,\pi^*(t) + r_h & , \quad t \ge t^* \end{cases}, \tag{7}$$

where the optimal liftoff date, t^*, is a constant that can be found as a function of the parameters of the model. Importantly, $t^* > T$.[14] The optimal policy is to commit to zero nominal interest rates for longer than the natural rate $r(t)$ is negative —one of the main elements of forward guidance. In addition, $i^*(t) > 0$ for $t \ge t^*$ so interest rates jump from zero to positive at t^* and never again hit the ZLB. Equation (7) is not a policy *rule*. The optimal path $(1 - \kappa\sigma\lambda)\,\pi^*(t) + r(t)$ is a single fixed path, a function of time only. It describes one particular equilibrium. It is contingent neither on the actual actions of the central bank nor on whether realized inflation, output or private-sector expectations happen to take one value or another (note $\pi(t)$ and $x(t)$ are endogenous variables, while $\pi^*(t)$ and $x^*(t)$ are merely two particular functions of time that are completely pinned down by parameters of the model). As such, it addresses neither the final equilibrium outcome of the economy nor the off-equilibrium behavior of the central bank. Hence, it says nothing about implementability or indeterminacy.

Plugging (7) into (1) and (2) gives the optimal paths for inflation and output for $t > 0$ but not for $t = 0$. Given $\pi^*(0)$ and $x^*(0)$, equations (1), (2) and (7) then determine the entire optimal equilibrium. One way to find $\pi^*(0)$, $x^*(0)$ and t^* is to use the maximum principle, as in Werning (2012).

Figure 1 shows representative optimal paths for three parameter configurations in the π-x plane. The optimal path is most easily understood in three stages, starting from the last one and working backward in time. The beginning and end times for each stage are determined by the discontinuities of $r(t)$ and $i^*(t)$. Because equation (7) is not a policy rule, in the remainder of this section, I describe only the equilibrium path $\{\pi^*(t), x^*(t), i^*(t)\}_{t\ge 0}$ but not the off-equilibrium dynamics of the economy, which would require specifying the central bank's behavior for all $(\pi(t), x(t))$ different from $(\pi^*(t), x^*(t))$.

In the third and last stage, defined by $t \ge t^*$, the economy has positive natural and nominal rates. The path for $(\pi^*(t), x^*(t))$ satisfies

$$x^*(t) = \phi\pi^*(t), \tag{8}$$

where

$$\phi = \frac{1}{2\kappa}\left(\rho + \sqrt{\rho^2 + 4\lambda\kappa^2}\right) > 0. \tag{9}$$

[14] The optimal discretionary equilibrium (i.e., *without* commitment) has $i(t) = 0$ until T and $i(t) = r_h$ thereafter. Inflation and the output gap are at their steady state values of zero immediately after the liquidity trap is over. The optimal equilibrium with and without commitment are identical if and only if $\kappa\sigma\lambda = 1$.

If $\kappa\sigma\lambda \neq 1$, the economy travels along the line defined by equation (8) and converges to the steady state $(\pi_{ss}, x_{ss}) = (0, 0)$ as $t \to \infty$. If $\kappa\sigma\lambda = 1$, $(\pi^*(t), x^*(t))$ reaches (π_{ss}, x_{ss}) exactly at t^*. The blue line in Figure 1 shows the optimal path for an example in which $\kappa\sigma\lambda = 1$. The other two cases keep all parameters unchanged except for λ, the relative weight given to deviations of output and inflation from zero. As can be seen in the figure, the optimal path with a smaller λ tolerates larger deviations of inflation from zero.

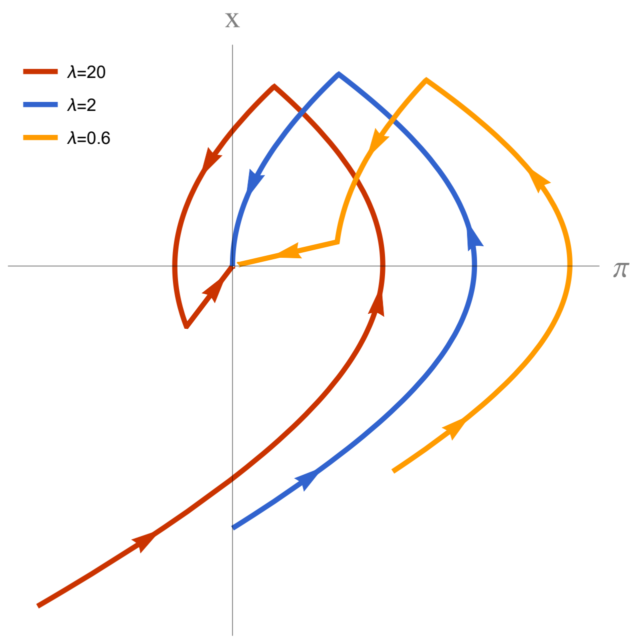


Figure 1: Optimal paths for inflation (horizontal axis) and the output gap (vertical axis) for three values of λ, the weight that the central bank places on inflation relative to output. The rest of the parameters used are taken from Werning (2012) ($T = 2$, $\sigma = 1$, $\kappa = 0.5$, $\rho = 0.01$, $r_h = 0.04$ and $r_l = -0.04$).

The second stage is given by $t \in [T, t^*)$, when the natural rate is positive but the nominal rate is zero. Starting at a given $(\pi^*(t), x^*(t))$ inherited from the first stage, inflation and output move so as to minimize the time it takes to reach the line $x = \phi\pi$. This is accomplished by pegging nominal rates to zero. When $x^*(t)$ and $\pi^*(t)$ hit the line $x = \phi\pi$, the third stage begins.

In the first stage, defined by $t \in [0, T)$, the natural rate is negative and the nominal rate is at the ZLB. The zero nominal rate in the first two stages produces the lowest real interest rate that the central bank can achieve, which reduces the incentive to save and increases the incentive to consume. As a result, inflation and output eventually become positive before T. The initial point $(\pi^*(0), x^*(0))$ is determined by optimality conditions that trade off deviations of inflation and output from zero at each of the three stages. Depending on parameters, the optimal equilibrium can have positive, zero, or negative initial inflation, but it always has a negative initial output, showing that the presence of the ZLB involves recessionary welfare losses even under the best possible monetary policy.

For more details on the optimal equilibrium, see Werning (2012).

4 Indeterminacy

4.1 Interest Rate Pegs

Consider a central bank that follows an interest rate rule in which the time of liftoff is a constant; that is, liftoff occurs at a fixed date that is chosen before $t = 0$ and does not change with the state of the economy. I refer to this type of liftoff as calendar-based liftoff. I show this type of rule can either implement the optimal equilibrium but with indeterminacy or achieve global determinacy at the cost of accepting a suboptimal equilibrium.

An initial natural candidate rule to implement the optimal equilibrium is:

$$i(t) = \begin{cases} 0 & , \quad 0 \le t < t^* \\ (1 - \kappa\sigma\lambda)\, \pi^*(t) + r_h & , \quad t \ge t^* \end{cases} . \tag{10}$$

Although equations (7) and (10) look very similar, they are conceptually different. While equation (7) describes the single optimal path $i^*(t)$, equation (10) is a rule —a policy response function— by which the central bank commits to setting interest rates in all possible states of the world according to $i(t) = i^*(t)$. It therefore provides both the on- and off-equilibrium behavior of the central bank. When combined with the IS and NKPC, they fully specify the dynamic behavior of the economy in all circumstances. For this particular rule, the behavior of the central bank is the same for all states of the world. The central bank announces that nominal interest rates will follow the optimal path described in the last section —which is the same in all states of the world, since the optimal path is just a fixed path and hence not state-contingent— come what may. This is the definition of an interest rate peg (note that a peg does not require interest rates to be constant). If the central bank follows rule (10), then it can clearly implement the optimal equilibrium whenever $(\pi(0), x(0)) = (\pi^*(0), x^*(0))$. However, many other equilibria are also consistent with this rule, leading to an indeterminate outcome.[15]

The intuition for why indeterminacy arises is as follows. Because the IS and NKPC equations are forward-looking, $i(t)$ cannot directly affect contemporaneous inflation and output. However, it can directly affect expectations of future output and inflation (and hence actual future inflation and output since the economy is deterministic). In equation

[15] Cochrane (2016) shows a similar result for the optimal discretionary equilibrium.

(1), $i(t)$ can directly control $\dot{x}(t)$. In turn, control over $\dot{x}(t)$ translates, via the NKPC, into control over $\dot{\pi}(t)$. By integrating over time, it follows that $i(t)$ has a direct influence on the levels of inflation and output, $(\pi(t), x(t))$, for all $t > 0$ but not for $t = 0$. Monetary policy influences the economy through an intertemporal channel. Initial inflation and output, $\pi(0)$ and $x(0)$, instead of being control variables as in the last section, are now non-predetermined or "jump" variables that are determined in equilibrium. Any $(\pi(0), x(0))$ that result in continuous bounded paths for inflation and output when the economy follows the IS and NKPC are consistent with an equilibrium in which $i(t)$ is given by equation (10). The only way the central bank can influence $(\pi(0), x(0))$ is indirectly, by "steering" the paths of inflation and output for $t > 0$ into being either bounded or unbounded (continuity, which is the second requirement for paths to be equilibria, cannot be exploited by the central bank, as it arises from complete markets and absence of arbitrage, which are outside the control of the central bank). If monetary policy can make the path that originates at some $(\pi(0), x(0))$ unbounded, then that particular $(\pi(0), x(0))$ is disqualified from being part of an equilibrium path. Instead, if the path remains bounded, then the path starting at $(\pi(0), x(0))$ is an equilibrium. Since the dynamics of $(\pi(t), x(t))$ under rule (10) are saddle-path stable for $t \geq t^*$, there are many initial points $(\pi(0), x(0))$ that the central bank can never steer into being unbounded. The saddle dynamics arise after $t \geq t^*$ because the path for $i(t)$ under rule (10), being a peg, is completely unresponsive to inflation and output. Without any state-contingency, the required asymptotic steering of paths becomes impossible.

To see this, first note that the saddle path is a line through the origin in the π-x plane given by $x = \phi\pi$. Pick a point $(\tilde{\pi}, \tilde{x})$ on the saddle path and consider a candidate equilibrium with $(\pi(t^*), x(t^*)) = (\tilde{\pi}, \tilde{x})$. For $t \geq t^*$, the economy follows the dynamics of the IS equation and the NKPC with $i(t) = (1 - \kappa\sigma\lambda)\, \pi^*(t) + r_h$ by moving along the saddle path toward the steady state (π_{ss}, x_{ss}) (or, if $(\tilde{\pi}, \tilde{x}) = (\pi_{ss}, x_{ss})$, the economy is already in steady state and stays there). Now trace the dynamics of the IS and the NKPC backward in time with $i(t) = 0$, from $t = t^*$ to $t = 0$, starting at $(\pi(t^*), x(t^*)) = (\tilde{\pi}, \tilde{x})$ and ending at some $(\pi(0), x(0))$. The resulting path starting at this $(\pi(0), x(0))$ is bounded, continuous and obeys the IS equation, the NKPC, the ZLB and the central bank's interest rate rule: It is an equilibrium. Because of the linearity of the system (1)-(2), the saddle path and the set of initial conditions $(\pi(0), x(0))$ that put the system on the saddle path at t^* are both lines in the π-x plane. The multiple equilibria under rule (10) are thus indexed by points in a line, which can be taken to be the saddle path or the $\pi(0)$-$x(0)$ line that gets the economy to the saddle path at t^*.

Figure 2 shows these two lines together with paths for inflation and output from equilibria that start at different $\pi(0)$ and $x(0)$. All of these equilibria are obtained using identical parameters and the same interest rate rule, given by (10). They are different equilibria

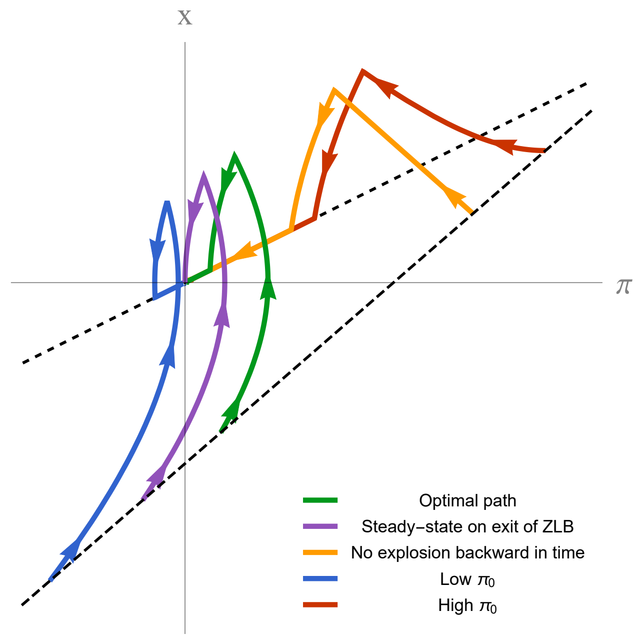


Figure 2: Multiple equilibria when the central bank chooses a rule that sets interest rates equal to the same optimal path in a non-state-contingent way (i.e., the same interest rates for all realizations of inflation, output and their expectations). Interest rate rules for which liftoff is not contingent on the state of the economy never anchor expectations (Proposition 1).

of the same economy, unlike those in Figure 1, which show the optimal equilibrium for different parameter configurations. The green line is the unique optimal equilibrium that starts at $(\pi^*(0), x^*(0))$. The purple line is a suboptimal equilibrium that reaches steady state $(\pi_{ss}, x_{ss}) = (0, 0)$ exactly at liftoff (this is the best possible discretionary equilibrium). The yellow line is one of the "local-to-frictionless" equilibria described in Cochrane (2016) in which inflation and output do not explode backward in time. The remaining paths have two arbitrary values for $\pi(0)$ and illustrate the kinds of behavior that the various equilibria can exhibit.

4.2 Taylor Rules

I now extend these results to Taylor rules. Consider a central bank that commits to keeping interest rates at zero until some fixed liftoff time $\underline{t}$ and follows a Taylor rule that respects the ZLB thereafter. Because the Taylor rule can prescribe zero nominal rates immediately after liftoff, liftoff may or may not coincide with the first time that interest rates become positive after the liquidity trap is over. Liftoff marks the end of the central bank's promise to keep interest rates pegged at zero and the beginning of a promise to follow a Taylor rule, and not necessarily the first time nominal interest rates become positive, since the interest rate rule that the central bank follows after liftoff may very well prescribe zero interest rates for a while longer.

The next proposition establishes that, just as for rules with an interest rate peg considered above, rules that follow a Taylor rule after a calendar-based liftoff cannot implement the optimal equilibrium in a globally determinate way. This impossibility result applies to Taylor rules with *any* coefficients on inflation and output. To further strengthen this intuition, I allow the Taylor rule coefficients to be not just arbitrary constants, but arbitrary functions of the endogenous realizations of inflation and output at time zero, $R_0 = (\pi(0), x(0))$. When, as is the case in this section, the liftoff date is constant and the economy is deterministic, conditioning on R_0 is the same as conditioning on the entire path $\{\pi(t), x(t)\}_{t\in[0,\infty)}$. Although not exactly the same as allowing for a general non-linear rule, being able to have state-dependent Taylor-rule coefficients offers the central bank significant flexibility. For example, it gives the central bank the freedom to pick coefficients as if it knew in advance which of the many potential equilibria will be realized. Despite these faculties, it is still impossible to implement the optimal equilibrium without indeterminacy. Of course, a traditional Taylor rule with constant coefficients is a special case of the one I consider here.

Proposition 1 (Calendar-based forward guidance cannot implement the optimal equilibrium without indeterminacy)**.** *Let* $\underline{t} \geq T$ *be a constant. Let* $\xi_\pi(R_0)$ *and* $\xi_x(R_0)$ *be arbitrary functions of* $R_0 = (\pi(0), x(0))$*. If* $\kappa\sigma\lambda \neq 1$*, the rule*

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < \underline{t} \\ \max\{0, \xi_\pi(R_0)\pi(t) + \xi_x(R_0)x(t) + r_h\} & , \quad \underline{t} \leq t < \infty \end{cases} \tag{11}$$

cannot implement the optimal equilibrium as the unique equilibrium of the economy.

Proof. Appendix B.1 □

A corollary of Proposition 1 is that if the central bank uses calendar-based forward guidance, it can either implement the optimal equilibrium with indeterminacy, or a globally determinate equilibrium that is suboptimal (except for the case $\kappa\sigma\lambda = 1$). Thus, the central bank faces a tradeoff between determinacy and optimality.

The Taylor principle is said to hold if and only if

$$\kappa(\xi_\pi - 1) + \rho\xi_x > 0 \quad \text{and} \quad \xi_x + \sigma\rho > 0.$$

When $\xi_x = 0$, the Taylor principle is equivalent to $\xi_\pi > 1$, one of its most popular forms. The Taylor principle is said to *not* hold if and only if

$$\kappa(\xi_\pi - 1) + \rho\xi_x < 0.$$

When the Taylor principle holds, the economy has explosive dynamics outside the ZLB, while it has a single stable saddle when the Taylor principle does not hold. It is well known that the Taylor principle is necessary and sufficient for local determinacy around the steady state $(x^{ss}, \pi^{ss}) = (0, 0)$.

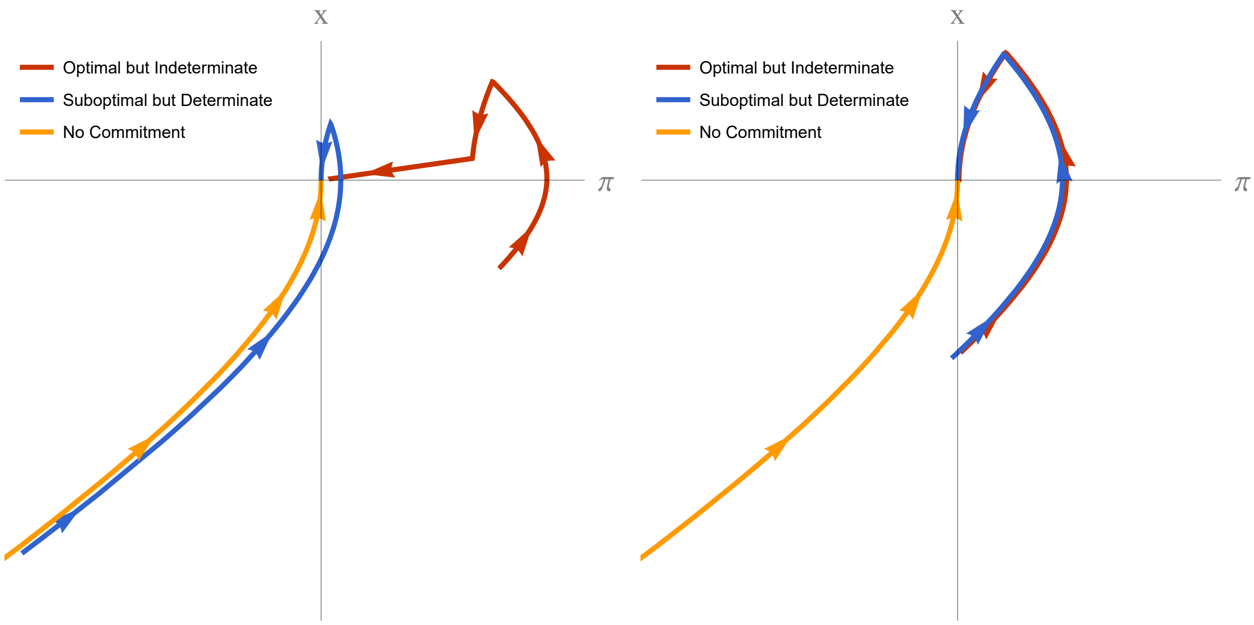
Under rule (11), the Taylor principle is also necessary and sufficient for global determinacy. According to Proposition 1, this means that the Taylor principle always produces a suboptimal equilibrium (unless $\kappa\sigma\lambda = 1$). This globally unique suboptimal equilibrium requires that the economy reaches steady state (x^{ss}, π^{ss}) right after liftoff, at $t = \underline{t}$, since otherwise (4) and (5) would be violated by virtue of the explosive dynamics induced by the Taylor principle. This Taylor principle equilibrium is reminiscent of the no-commitment equilibrium, in which the economy must be at the steady state (x^{ss}, π^{ss}) at $t = T$ (in fact, the two coincide if $\underline{t} = T$).

In sharp contrast to models without a binding ZLB in which the Taylor principle can reap the benefits of commitment, the ZLB can make the Taylor principle produce suboptimality akin to that emerging from *lack* of commitment. Figure 3 displays the optimal equilibrium, the Taylor principle equilibrium with $\underline{t} = t^*$ and the no-commitment equilibrium for two parameter configurations. On the left panel $\kappa\sigma\lambda$ is close to 0, while in the right panel it is close to 1. When $\kappa\sigma\lambda$ is close to zero, inflation and output can deviate substantially from their optimal levels in the Taylor principle equilibrium. As $\kappa\sigma\lambda \to 0$, the Taylor principle equilibrium approaches the no-commitment equilibrium, as the left panel of Figure 3 illustrates. On the other hand, the right panel shows that when $\kappa\sigma\lambda$ is close to 1, the Taylor principle equilibrium implies small welfare losses compared with the optimal path, with the Taylor principle equilibria approaching the optimal one as $\kappa\sigma\lambda \to 1$. Standard calibrations anchored in empirically plausible levels of nominal rigidities tend to give $\kappa\sigma\lambda$ close to 0, which produce large welfare losses for the Taylor principle equilibrium.[16]

5 Rules with a State-Dependent Liftoff Date

In the last section, I showed that calendar-based forward guidance, in which liftoff is pre-announced for a particular date without making it contingent on the state of the economy, cannot implement the optimal equilibrium without global indeterminacy, even when a Taylor rule —with any choice of coefficients— is used after liftoff. In this section, I introduce interest rate rules that have a state-contingent liftoff but follow a non-state-contingent interest

[16]For example, Woodford (2003) calibrates the slope of the NKPC to $\kappa = 0.024$, which combined with the frequently assumed $\sigma = 1$ (log utility) and $\lambda = 1$ (equal weight on the welfare loss function for inflation and output), gives a $\kappa\sigma\lambda$ that is even smaller to the one used in the left panel of Figure 3. On the other hand, if λ is large (such as when the central bank does not care about output), $\kappa\sigma\lambda$ can turn out to be closer to 1. Values of $\kappa\sigma\lambda$ above 1 are also suboptimal, with welfare decreasing as the distance of $\kappa\sigma\lambda$ from 1 increases.



(a) $\kappa\sigma\lambda = 0.05$ (b) $\kappa\sigma\lambda = 0.95$

Figure 3: Comparison between the optimal path, the path without central bank commitment and the path when the central bank satisfies the Taylor principle after t^*. We use $T = 2$, $\sigma = 1$, $\kappa = 0.5$, $\rho = 0.01$, $r_h = 0.04$ and $r_l = -0.04$ for both panels. For the left and right panels, we choose $\lambda = 0.1$ and $\lambda = 1.9$, respectively.

rate peg before and after liftoff. When the dependence of liftoff on the state of the economy is designed correctly, these rules do implement the optimal equilibrium in a globally determinate way. It follows that whether the central bank can implement the optimal equilibrium with global determinacy depends more on the state-contingent form of liftoff and less on the form of state contingency when setting the level of interest rates after liftoff. The main obstacle for global determinacy in the presence of a ZLB is not that interest rates fail to be responsive enough to inflation and output during or after the ZLB binds. Instead, the key challenge is to communicate liftoff in an appropriate way.

Unlike the case of a constant liftoff time, when liftoff is state-contingent, initial inflation and output no longer provide a full description of the state of the economy. With a state-contingent liftoff, expectations of future inflation and output can be important equilibrium determinants of when liftoff occurs, even after $\pi(0)$ and $x(0)$ are realized. I now formalize the the idea of a state-dependent liftoff rule in a mathematically precise way. Let t_1 be the actual endogenous time of liftoff that is realized in the economy. Let $R_{t_1} = (\pi(0), x(0), \pi(t_1), x(t_1))$ be the vector containing realized inflation and output at times 0 and t_1, which fully describes the state of the economy, and let R^e_t be the expectation of R_{t_1} conditional on time-t information.

Before $t = 0$, the central bank announces a liftoff rule and a rule for how it will set interest rates after liftoff. The liftoff rule is a function $f : \mathbb{R}^4 \to [0, \infty)$ that maps expectations, R_t^e, to a promised liftoff date $f(R_t^e)$. The central bank uses the rule f as follows. At $t = 0$, the private sector forms expectations R_0^e of R_{t_1}. These expectations are observed by the central bank. The central bank then computes the number $f(R_0^e)$. If $f(R_0^e) = 0$ the central bank lifts off now (at $t = 0$) and, by definition, $t_1 = 0$. If $f(R_0^e) \neq 0$, the central bank does not lift off and repeats the same procedure in the future. Specifically, for any time s such that liftoff has not yet occurred, the central bank observes time-s private-sector expectations R_s^e of R_{t_1} and computes the number $f(R_s^e)$. If $f(R_s^e) = s$, then the central bank lifts off at time s and consequently $t_1 = s$. If $f(R_t^e) \neq t$ for all t, then the central bank never lifts off and $i(t) = 0$ for all t. In brief, the function f describes when the central bank will lift off for each possible state at each point time – a complete description of the central bank's liftoff plan under any circumstances. The behavior of the private sector and the central bank together determine the actual equilibrium path of the economy, including the endogenous date t_1 when liftoff actually occurs. Along the equilibrium path, by rational expectations and perfect foresight, we must have that $R_{t_1} = R_s^e$ for all s and consequently

$$t_1 = f(R_{t_1}). \tag{12}$$

In addition, $R_{t_1} = R_s^e$ means that the mapping f can be equivalently interpreted as a mapping from states of the economy (instead of expectations) to the liftoff time.

5.1 A Neo-Fisherian Rule

The interest rate rule that I study in this section is

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < f(R_t) \\ (1 - \kappa\sigma\lambda)\,\pi^*(t) + r_h & , \quad t \geq f(R_t) \end{cases} , \tag{13}$$

where the state contingent liftoff is given by

$$f(R_t) = \begin{cases} t^* & , \quad \begin{array}{l} \text{if } (\pi(0), x(0)) = (\pi^*(0), x^*(0)) \text{ or} \\ Ax(0) + B\pi(0) = C \end{array} \\ \\ t^* + 1 & , \quad \begin{array}{l} \text{if } (\pi(0), x(0)) \neq (\pi^*(0), x^*(0)) \\ \text{and } Ax(0) + B\pi(0) \neq C \\ \text{and } Dx(0) + E\pi(0) \neq F \end{array} \\ \\ t^* + 2 & , \quad \text{otherwise.} \end{cases} \tag{14}$$

for constants A, B, C, D, E, F given explicitly in Appendix B.3. Equation (13) says that the central bank commits to zero interest rates until liftoff and to the non-state-contingent interest rate peg $i(t) = i^*(t)$ after liftoff. Equation (14) says that, in contrast, liftoff itself is state-dependent and can take one of three different values: t^*, t^*+1 and t^*+2. This rule shows that a state-dependent liftoff time is a powerful tool to fight indeterminacy at the ZLB. In fact, the only state dependence in this rule is in the time of liftoff; no other state dependence is needed to eliminate indeterminacy. The rule also shows that when a binding ZLB is introduced, following the Taylor principle is no longer necessary for global determinacy of the optimal equilibrium.[17] The rule in equation (13) can be thought of as a Taylor-type rule that does not obey the Taylor principle, since it has coefficients of zero on inflation and output and a time-varying intercept (that is equal to zero before t_1 and $\hat{r}(t) = (1-\kappa\sigma\lambda)\,\pi^*(t) + r_h > 0$ after t_1).

Why does this interest rate rule implement the optimal equilibrium as the unique global equilibrium? When $\pi(0) = \pi^*(0)$ and $x(0) = x^*(0)$, the rule gives $f(R_{t_1}) = t^*$, $i(t) = i^*(t)$ and therefore the optimal equilibrium is implemented. The choice of f also ensures there are no other equilibria. Consider a candidate equilibrium with initial conditions $\pi(0)$ and $x(0)$ different from $(\pi^*(0), x^*(0))$. For $t \geq t_1$, by equation (13), the economy has the same dynamics as when it follows the rule in equation (10) of Section 4. In particular, there is a saddle path given by a straight line that goes through the origin in a π-x plane. Figure 2 is a useful guide.

I consider three cases, defined by the three conditions in equation (14). The first case states that if the economy is either on its optimal path or expected to *not* be on the saddle path at t^*, then liftoff occurs at t^*. The equation $Ax(0) + B\pi(0) \neq C$ in equation (14) describes the set of points $(\pi(0), x(0))$ for which the economy is not expected to be on its saddle path at t^*. The condition $Ax(0)+B\pi(0) \neq C$ means that $(x(0), \pi(0))$ is not on the line where all the paths in Figure 2 begin. If paths do not start on the line $Ax(0) + B\pi(0) = C$, they do not end up on the saddle path at t^*. Because the economy is not on its saddle path at the time of liftoff, either inflation and output instantaneously jump a discrete amount to reach the saddle path, or inflation and output become unbounded. In either case, by definition, an equilibrium cannot form. The second case I consider is given by the second line of equation (14). It corresponds to the non-optimal points $(\pi(0), x(0))$ that take the economy to the saddle path at t^* but not at t^*+1. The condition $Dx(0) + E\pi(0) \neq F$ gives the set of points $(\pi(0), x(0))$ that do not reach the saddle path at t^*+1. For these points, the rule assigns $f(R_{t_1}) = t^*+1$ and equilibria are precluded by the same argument as in the first case: Paths either jump to the saddle path or become unbounded because they are not on the saddle path at t^*+1. The points $(\pi(0), x(0))$ that are not optimal and hit the saddle path at t^* and at t^*+1 define the third case, for which the central bank picks $f(R_{t_1}) = t^*+2$.

[17] In the next section, I show the Taylor principle is also not sufficient for determinacy of the optimal equilibrium.

There is at most one point in this category, since it is given by the intersection of two distinct lines: The line of initial conditions that reach the saddle path at t^* (excluding $(\pi^*(0), x^*(0))$, already analyzed) and the ones that reach it at $t^* + 1$. This point, if it exists, does not reach the saddle path at $t^* + 2$, the time of liftoff, and is therefore not an equilibrium.

The liftoff rule in equation (14) is useful because it showcases the power of a state contingent liftoff in a stark way, especially when compared against the pure peg rule in equation (10). However, it does not readily lend itself to a straightforward intuitive interpretation and comes short in terms real-world applicability. For example, clearly communicating the constants A, B, ..., F and the rationale behind the form of the liftoff rule to the public seems, at least to me, rather quixotic. That the potential liftoff dates $t^* + 1$ and $t^* + 2$ are arbitrary, in the sense that picking any two other numbers that are distinct and larger than t^* will result in the same equilibrium, also appears challenging to explain to the public.

There is one more aspect of the rule is that may complicate its real-world applicability. Along some off-equilibrium paths, the central bank must commit to raising interest rates in many circumstances in which the economy is in the midst of recessions or deflations, an element reminiscent of the proposal in Schmitt-Grohé and Uribe (2014), who exploit the neo-Fisherian effect of NK models that a very persistent increase in interest rates leads to higher inflation not only in the long run, as in the more classical Fisher effect, but also immediately after the interest rates change. While this idea is currently being studied and taken seriously in the literature, there is no consensus on neo-Fisherianism's empirical applicability.A rule like the one in equation (14) seems, at the very least, far from current central bank orthodoxy.

Having used rule (14) to make the theoretical point that a state contingent forward guidance is a powerful tool to fight indeterminacy, I turn next to rules that not only implement the optimal equilibrium in a globally determinate way, but also closely resemble actual central bank behavior and should therefore be of more practical relevance.

5.2 A Practical Rule

For the rest of the paper, I focus on the class of rules given by

$$i(t) = \begin{cases} 0 & , \quad 0 \le t < f(R_{t_1}) \\ \max\left\{0, \xi_\pi(R_{t_1})\pi(t) + \xi_x(R_{t_1})x(t) + r_h\right\} & , \quad f(R_{t_1}) \le t < \infty \end{cases}, \tag{15}$$

where[18] $f : \mathbb{R}^4 \to [T, \infty)$, $\xi_\pi : \mathbb{R}^4 \to \mathbb{R}$ and $\xi_x : \mathbb{R}^4 \to \mathbb{R}$ are functions chosen by the central bank. The rule has a forward guidance period from $t = 0$ to $t = f(R_{t_1})$, followed by a standard Taylor rule period for $t \geq f(R_{t_1})$. The Taylor rule guarantees that interest rates do not become positive for "unconventional" states of the economy, precluding the neo-Fisherian behavior of the rule in equations (13)-(14). In addition to a state-dependent liftoff rule given by $f(R_{t_1})$, I allow for Taylor rule coefficients $\xi_\pi(R_{t_1})$ and $\xi_x(R_{t_1})$ that can depend on R_{t_1}. Of course, traditional Taylor rules with constant coefficients are just a special case of this.

The goal of this section is to develop the prerequisite mathematical notation and economic intuition to understand, in the next section, the necessary and sufficient conditions for global determinacy of the optimal equilibrium under the rule in equation (15). As before, this rule can be understood in three stages.

First stage ($0 \leq t < T$). Figure 11 shows the phase portrait for this stage. Because the natural rate is negative, the unique steady state for the first-stage dynamics, the triangle labeled (π_l, x_l) in the figure, is in the first quadrant and given by $\pi_l = -r_l > 0$ and $x_l = -\rho r_l/\kappa > 0$. Since interest rates are pegged at zero, the Taylor principle does not hold and the dynamics have a stable saddle path, the green line in the figure.

Second stage ($T \leq t < t_1$). The central bank is committed to $i(t) = 0$ between T and $f(R_{t_1})$. Unlike the first stage, the duration of this stage is endogenous. Figure 12 shows the phase portrait, which reveals saddle dynamics since the Taylor principle does not hold, just as in the first stage. Since the natural rate is now positive, the unique steady state (π_{zlb}, x_{zlb}), shown as a black square in the figure, lies in the third quadrant of the π-x plane. In the literature, the steady state (π_{zlb}, x_{zlb}) is variously referred to as the "deflationary steady state," the "liquidity trap steady state," the "expectational trap steady state" or the "unintended steady state." Last, just as in the first stage, there is a single stable saddle path (the red line in figure), which I denote by Υ_{zlb}.

Third stage ($t \geq t_1$). The central bank follows the Taylor rule $i\left(t\right) = \max\left\{0, \xi_\pi(R_{t_1})\pi(t) + \xi_x(R_{t_1})x(t) + r_h\right\}$. I split the π-x plane into two disjoint regions defined by whether the ZLB is binding

$$\Omega_{zlb}(R_{t_1}) = \left\{(x,\pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h \leq 0\right\}, \tag{16}$$

$$\Omega_{ss}(R_{t_1}) = \left\{(x,\pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h > 0\right\}, \tag{17}$$

[18] Throughout the paper, I show results that assume that the liftoff date does not occur before T, as even without commitment a central bank trying to minimize the social loss function in equation (6) would pick a liftoff rule f such that $i(t) = 0$ for $t < T$, when the natural rate is negative. However, none of my results depend on this assumption; extending the results to allow for any liftoff date is straightforward. The key to the relevant proofs is that the liftoff day is bounded below (which it always is, by $t = 0$).

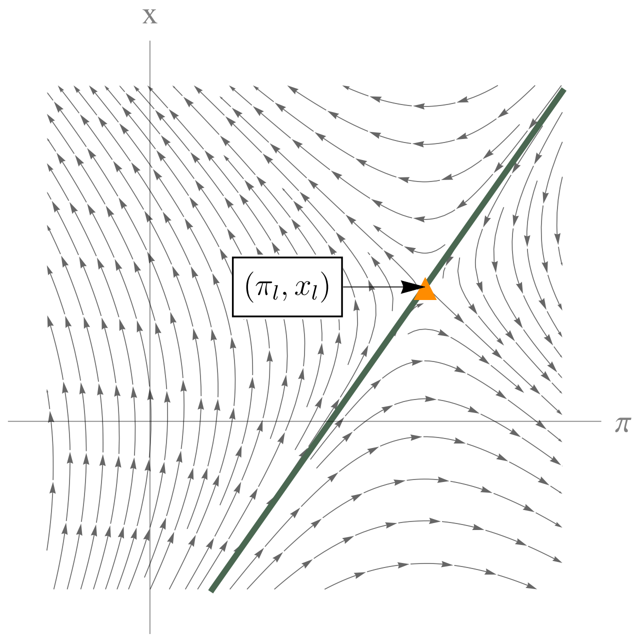


Figure 4: Dynamics of the economy for $t \in [0, T)$, when $i(t) = 0$ for all $\pi(t)$ and $x(t)$ and $r(t) = r_l < 0$. The green line is the saddle path and the orange triangle, labeled (π_l, x_l), is the steady state.

The boundary between the regions $\Omega_{zlb}(R_{t_1})$ and $\Omega_{ss}(R_{t_1})$ is a line

$$\partial\Omega(R_{t_1}) = \left\{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h = 0\right\}. \tag{18}$$

Henceforth, I suppress the dependence of ξ_x, ξ_π, Ω_{zlb}, Ω_{ss} and $\partial\Omega$ on R_{t_1} for ease of notation whenever it does not create confusion.

The dynamics of the economy inside each of the two regions Ω_{zlb} and Ω_{ss} are separately given by a system of linear first-order ordinary differential equations (ODEs) in $x(t)$ and $\pi(t)$, each of which is easy to analyze inside its respective region with standard methods. However, when the two regions are analyzed together, the global dynamics are piecewise linear, with a non-differentiable transition at $\partial\Omega$. The behavior of piecewise linear dynamic systems can exhibit a rich variety of non-linear phenomena such as limit cycles, bifurcations and chaos. I first analyze the properties of each of the two regions separately and then combine them and analyze the resulting global dynamics. Readers familiar with NK models without a ZLB should find the analysis of each of the separate regions familiar. The new results arise when I look at both regions together.

Inside Ω_{ss}, there is a single steady state, $(\pi_{ss}, x_{ss}) = (0, 0)$. The dynamic behavior of the economy depends on the choice of Taylor rule coefficients. The Taylor principle is the key concept needed to differentiate between unstable and saddle dynamics, and between *locally* determinate and indeterminate equilibria.When the Taylor principle holds, if the dynamics

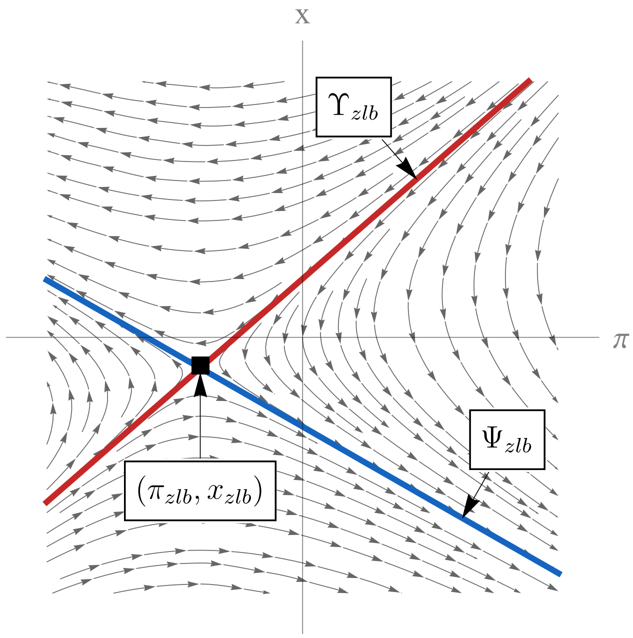


Figure 5: Dynamics of the economy for $t \in [T, t_1)$, when $i(t) = 0$ for all $\pi(t)$ and $x(t)$, and $r(t) = r_h > 0$. The red line, labeled Υ_{zlb}, is the saddle path. If the economy starts on Υ_{zlb}, it converges to the deflationary steady state (π_{zlb}, x_{zlb}). The blue line, labeled Ψ_{zlb}, is the "unstable saddle path." If the economy starts on Ψ_{zlb}, it stays on Ψ_{zlb} and moves away from the deflationary steady state (π_{zlb}, x_{zlb}). Under these dynamics, if the economy is not on Υ_{zlb}, then inflation and output become unbounded.

of the Ω_{ss} region were extended to the entire plane and nominal interest rates were allowed to be negative, or if I considered a small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely inside Ω_{ss}, the dynamics of the system would be unstable. All paths would be unbounded —or exit the small neighborhood— except for the path with $(\pi(t), x(t)) = (\pi_{ss}, x_{ss}) = (0, 0)$ for all t. Figure 13 shows two representative phase portraits, one in which the dynamics "slowly" spiral outward from the steady state and one in which the steady-state is a source.

When the Taylor principle does not hold, if the dynamics of the Ω_{ss} region were extended to the entire plane and interest rates were allowed to be negative, or if I considered a small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely in Ω_{ss}, the system would have saddle dynamics. Figure 14 displays a typical phase diagram. The stable saddle is the green line through the origin, which I denote by Υ_{ss}. Its slope depends on the Taylor rule coefficients ξ_π and ξ_x.[19] Paths are bounded —or stay in the small neighborhood of (π_{ss}, x_{ss})— if and only if they start on the ss saddle path.

Note that because ξ_π and ξ_x can depend on R_{t_1}, whether the Taylor principle holds can depend on R_{t_1}. Within the same economy, there can be a subset of (off-equilibrium) paths for which the Taylor principle holds and a different subset for which it does not hold. Instead,

[19] See equation (A.15) in Appendix A.1 for the explicit formula.

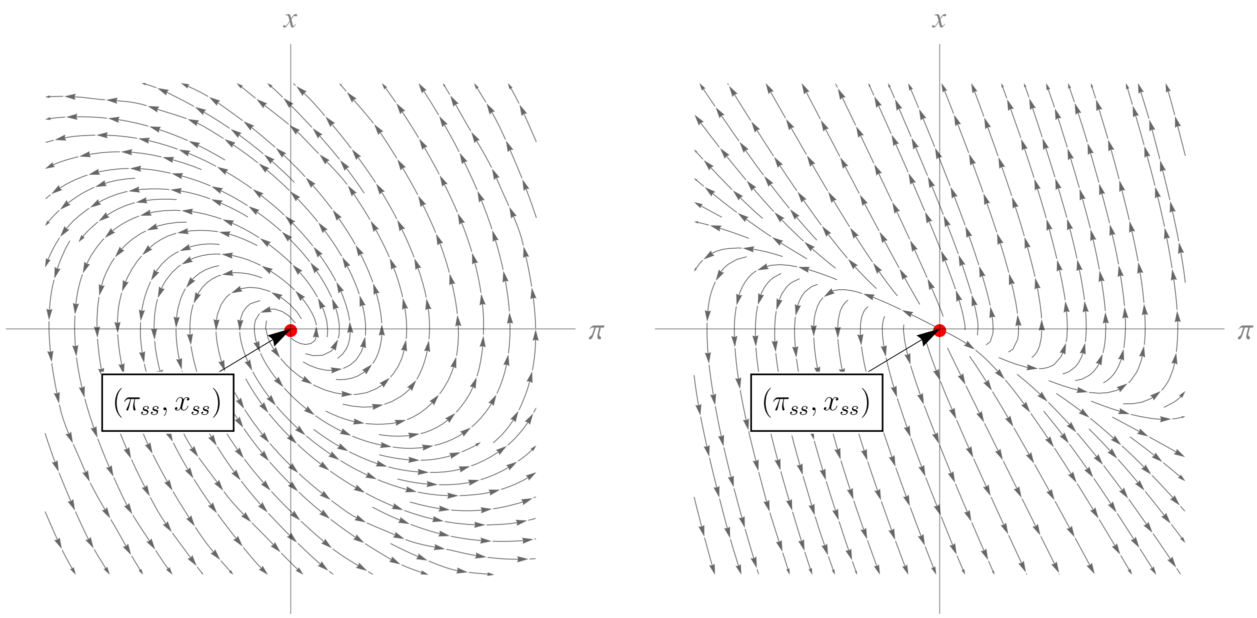


Figure 6: Dynamics of the economy for $t \geq t_1$ when the Taylor principle holds and there is no ZLB. The dynamics of the diagram on the left have imaginary eigenvalues, while the right-hand side have real ones. When there is no ZLB, the Taylor principle is necessary and sufficient for *local* determinacy. Unless the economy starts at $(0, 0)$, inflation and output become unbounded.

when ξ_π and ξ_x are constant, the Taylor principle must hold either for all paths or for no paths. In all cases, the Taylor rule coefficients are fully determined by the time the central bank lifts off and they remain unchanged from then on.

The ZLB region Ω_{zlb} has dynamics identical to those of the second stage analyzed above, with the phase diagram given in Figure 12, with the exception that these dynamics only hold on a half-plane instead of the whole plane.

I now put the dynamics of the regions Ω_{ss} and Ω_{zlb} together and describe some of the global dynamics of the economy. The left panel of Figure 15 shows an example in which the Taylor principle holds, while the right panel shows an example in which the Taylor principle does not hold. Irrespective of whether the Taylor principle holds, the dynamics inside Ω_{zlb} always look like those in Figure 12, as they are not affected by the choice of ξ_x or ξ_π. However, the coefficients ξ_x and ξ_π do have a crucial effect on the ZLB, as they determine the location of the boundary $\partial\Omega$ and, consequently, whether a deflationary steady state can exist in the economy. If the Taylor principle holds, the boundary $\partial\Omega$ is such that the deflationary steady state (π_{zlb}, x_{zlb}) is inside Ω_{zlb}, so (π_{zlb}, x_{zlb}) is indeed a steady state of the global dynamics. On the other hand, if the Taylor principle does not hold, (π_{zlb}, x_{zlb}) is in Ω_{ss}. Under the Ω_{ss} dynamics, the point (π_{zlb}, x_{zlb}) is not a steady state. In this case, the only steady state for the global dynamics is the desired one, (π_{ss}, x_{ss}). The conclusion that following the Taylor

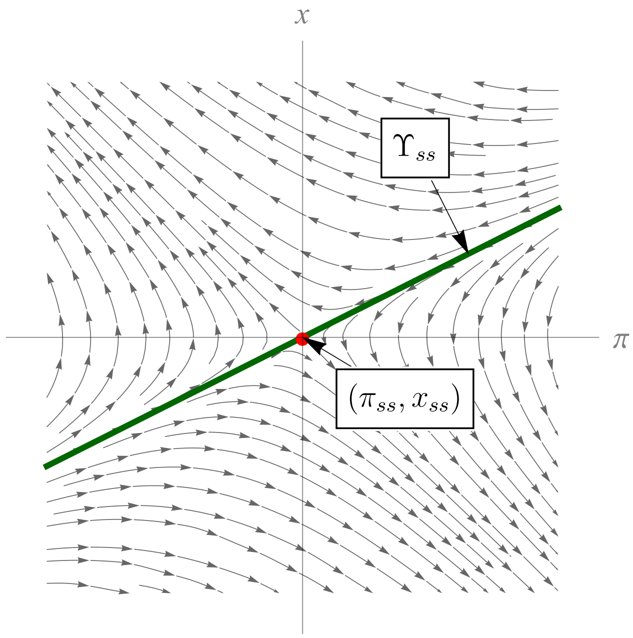


Figure 7: Dynamics of the economy for $t \geq t_1$ when the Taylor principle does not hold and there is no ZLB. Unless the economy starts on its saddle path Υ_{ss}, shown in green, inflation and output become unbounded.

principle outside the ZLB induces the existence of a deflationary steady state at the ZLB is similar to one of the results in Benhabib et al. (2001b). Note that in this case, there are saddle dynamics around (π_{zlb}, x_{zlb}), so the deflationary steady state is locally indeterminate.

6 How to Implement the Optimal Equilibrium without Indeterminacy

In this section, I prove the main result of the paper: A central bank following the rule in equation (15) can implement the optimal equilibrium in a globally determinate way by appropriately choosing the rule for liftoff, given by f, and the Taylor rule coefficients ξ_π and ξ_x.

6.1 Boundedness of Continuous Paths

In this section, I characterize all bounded continuous paths that follow the dynamics given by the IS curve, the NKPC and the ZLB, which constitute the set of paths that are candidate equilibria. It turns out that all candidate equilibria can be classified into four types determined by whether they (a) converge to the zero inflation steady state (π_{ss}, x_{ss}) or the deflationary steady state (π_{zlb}, x_{zlb}) and (b) are inside or outside the ZLB at liftoff (i.e., are

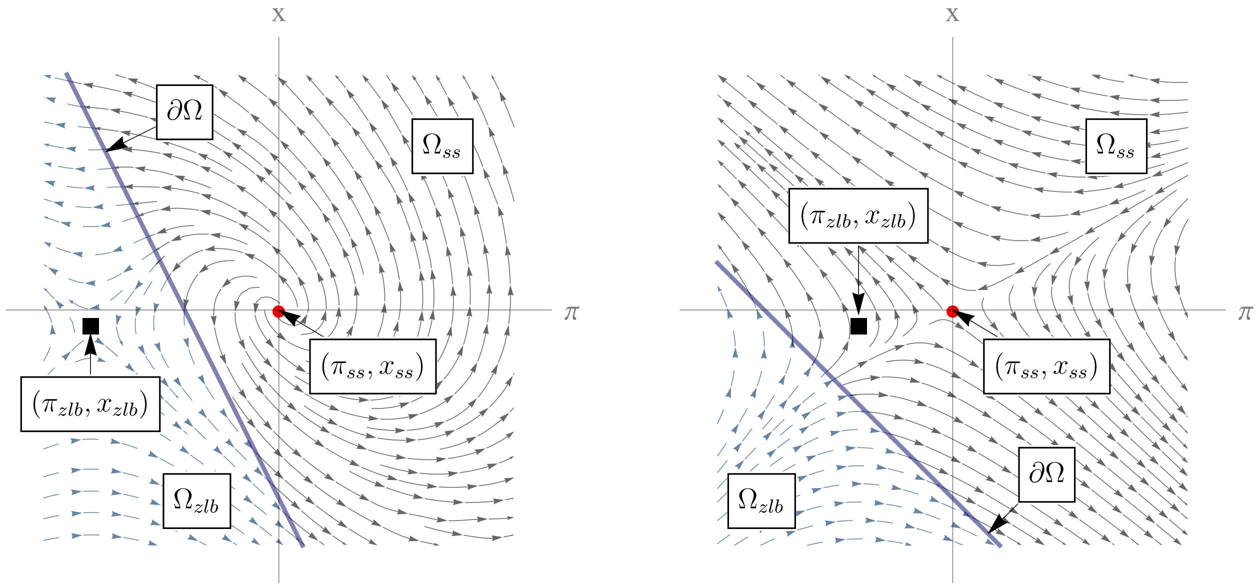


Figure 8: Non-linear dynamics of the economy after liftoff t_1. The central bank follows the Taylor rule $i(t) = \max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\}$. When $\xi_\pi \pi(t)+\xi_x x(t)+r_h > 0$, the economy is in the region Ω_{ss} and follows the solid black flow lines. When $\xi_\pi \pi(t) + \xi_x x(t) + r_h \leq 0$, it is in the region Ω_{zlb} and follows the dashed blue flow lines. The boundary between the two regions is the line $\partial\Omega$. The point (π_{ss}, x_{ss}), shown in red, is always a steady state of the economy. The point (π_{zlb}, x_{zlb}), shown as a black square, is a steady state of the economy if and only if the Taylor principle holds, as in the left panel. When the Taylor principle does not hold, as in the right panel, (π_{zlb}, x_{zlb}) is not in Ω_{zlb} and is therefore not a steady state.

in Ω_{ss} or Ω_{zlb} at t_1).

Proposition 2 (Continuous bounded paths when the Taylor principle does not hold)**.** *For a continuous path, if the Taylor principle does not hold, the path is bounded if and only if there exists $r \geq t_1$ such that $(\pi(r), x(r)) \in \Upsilon_{ss} \cap \overline{\Omega}_{ss}$.*[20]

Proof. Appendix B.4. □

When the Taylor principle does not hold, continuous paths are bounded if and only if they reach the *ss* saddle path outside (or on the boundary of) the ZLB region at t_1 or later. This can happen in one of two ways, depicted in Figure 9. The flow lines in the background show the dynamics after t_1. The green path shows the first case. The economy finds itself outside the ZLB at t_1. If it is on the saddle path Υ_{ss}, then it stays on it and travels toward (π_{ss}, x_{ss}), remaining bounded as in the figure. If the economy is not on the saddle path Υ_{ss}, paths for $(\pi(t), x(t))$ can either become unbounded or eventually enter the region Ω_{zlb}. Because the Taylor principle does not hold, the *zlb* steady state is not inside Ω_{zlb}. Thus, when the economy enters Ω_{zlb}, paths can either become unbounded (when they stay inside

[20] $\overline{\Omega}_{ss} = \Omega_{ss} \cup \partial\Omega$ is the closure of Ω_{ss}.

Ω_{zlb}) or eventually exit the ZLB and return to Ω_{ss}. Proposition 2 states that the process of transitioning from one region to the other has to stop at some point —there are no limit cycles. The proof in the appendix reveals that, in fact, there can be at most one transition; once the economy leaves Ω_{ss} and enters Ω_{zlb}, it remains there.

The red path in Figure 9 shows the second way in which the economy can reach Υ_{ss}. At t_1, the economy is inside the ZLB region Ω_{zlb}. As it follows the dynamics induced by $i(t) = 0$, it eventually hits the boundary $\partial\Omega$ at some time $r > t_1$ exactly where $\partial\Omega$ intersects Υ_{ss}. From there on, the analysis is identical to that of the first case.

Proposition 2 reveals some of the advantages and disadvantages of not following the Taylor principle. On the one hand, continuous bounded paths always exit the ZLB. On the other hand, the saddle dynamics outside the ZLB are conducive to (both local and global) indeterminacy, since being on any point of the saddle Υ_{ss} at t_1, or on any point of the path inside Ω_{zlb} that eventually reaches the saddle, gives continuous bounded paths and hence potential equilibria.

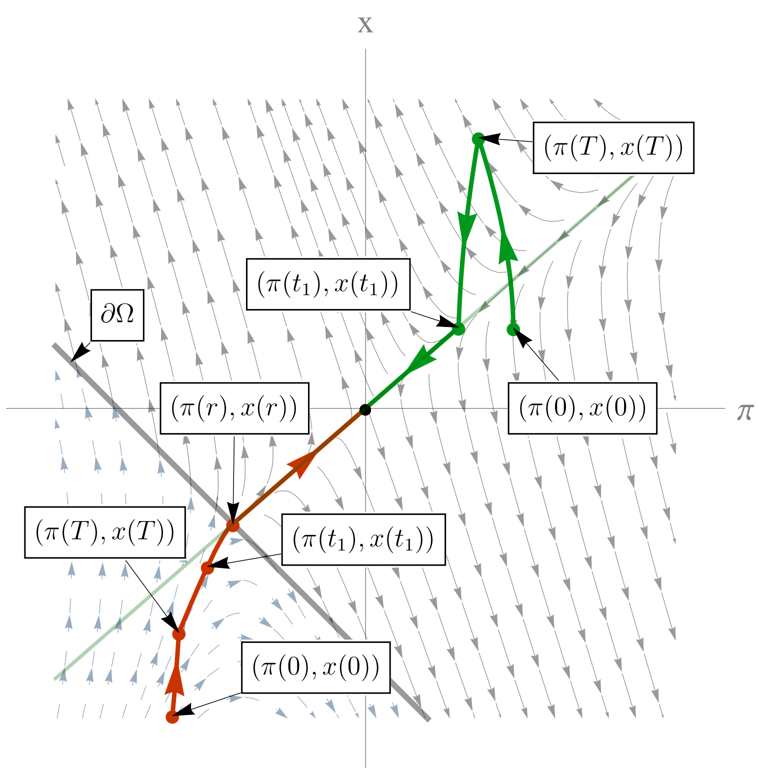


Figure 9: When the Taylor principle does not hold, there are only two kinds of continuous bounded paths, shown in the figure. The flow lines in the background are for the dynamics after liftoff, which occurs at t_1.

In contrast, when the Taylor principle holds, the benefit is to have no saddle path equilibria outside the ZLB, but at the cost of introducing them inside the ZLB, as I show next.

Proposition 3 (Continuous bounded paths when the Taylor principle holds)**.** *For a continuous path, if the Taylor principle holds, the path is bounded if and only if either* $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$ *or there exists* $r \geq t_1$ *such that* $(\pi(r), x(r)) \in \Upsilon_{zlb} \cap \Omega_{zlb}$.

Proof. Appendix B.5. □

When the Taylor principle holds, Proposition 3 states that continuous paths are bounded if and only if they either reach the intended steady state (π_{ss}, x_{ss}) exactly at t_1, or if they eventually reach the *zlb* saddle path Υ_{zlb} inside the ZLB region Ω_{zlb}. If $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, then paths are clearly bounded since (π_{ss}, x_{ss}) is a steady state. If $(\pi(t_1), x(t_1))$ is in Ω_{ss} but is not equal to (π_{ss}, x_{ss}), the only way for continuous paths to remain bounded is to eventually enter Ω_{zlb}, since the dynamics in Ω_{ss} are otherwise explosive. Once the paths are in Ω_{zlb}, boundedness requires that the economy follow the saddle path toward the *zlb* steady state (π_{zlb}, x_{zlb}). The red path in Figure 10 illustrates this case. Because the Taylor principle holds, (π_{zlb}, x_{zlb}) is inside Ω_{zlb} and is therefore a steady state of the system, as discussed first in Section 5.2. The economy converges to (π_{zlb}, x_{zlb}) as $t \to \infty$. This is the type of equilibrium in Benhabib et al. (2001b), also shown in Figure 16. The closer $(\pi(t_1), x(t_1))$ gets to (π_{ss}, x_{ss}), the longer it will take the economy to reach Ω_{zlb}. Although listed separately in Proposition 3 and at first sight different, the case for which $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$ can be seen as the limit of this type of equilibrium when the time to reach Ω_{zlb} goes to infinity.

Proposition 3 reveals another type of potential equilibrium that converges to the unintended steady state (π_{zlb}, x_{zlb}), shown in green in Figure 10. The economy is always at the ZLB. At T, the economy is already on Υ_{zlb}. It stays there for all $t \geq T$, traveling toward (π_{zlb}, x_{zlb}). At t_1, even though liftoff occurs, the Taylor rule prescribes $i(t_1) = 0$, since the economy is in Ω_{zlb} and the economy remains on Υ_{zlb}.

Proposition 3 also proves that there are no continuous bounded paths other than the ones I have described. In particular, there are no closed loops, no limit cycles and no chaotic paths, unlike the setup in Benhabib et al. (2002), in which the Taylor principle gives rise to chaotic trajectories. One direct implication is that whether following the Taylor principle leads to chaotic interest rate rules depends on the specific setup.[21] Without a ZLB, a positive divergence is equivalent to the Taylor principle and to explosive dynamics. Proposition 3 shows that having a ZLB does not break this link. Thus, even though the system includes a binding ZLB, the Taylor principle is still the right concept to assess the tendency of the system to move away from the intended steady state.

[21] For this economy, the Taylor principle is actually the exact condition needed to make the divergence of $(\dot{\pi}(t), \dot{x}(t))$ positive everywhere. By Green's theorem, a positive divergence automatically eliminates bounded orbits, since the line integral around a closed loop is positive. The proof of item (f) in Lemma 2 of Appendix B.5 has more details.

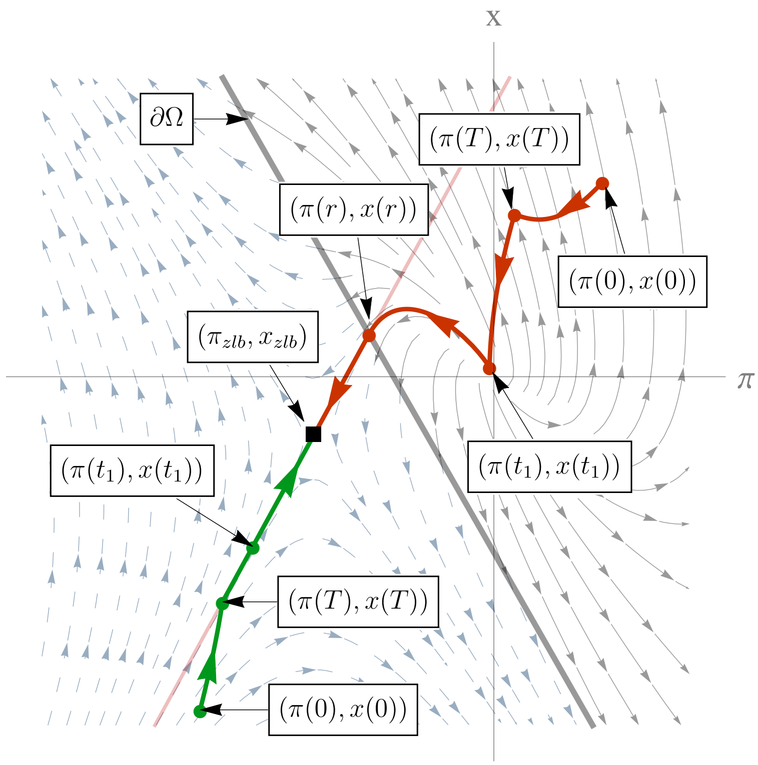


Figure 10: When the Taylor principle holds, there are two types of continuous bounded paths. Two of them are shown in the figure in red and green. The flow lines in the background are for the dynamics after liftoff, which occurs at t_1. The continuous bounded path in which the economy reaches the steady state $(\pi_{ss}, x_{ss}) = (0, 0)$ at t_1 and stays there forever after is a limiting case of the equilibrium shown in red.

6.2 Implementing the Optimal Equilibrium

I now show that, under the rule in equation (15), there are many ways to implement it. Let $R^* = (\pi^*(0), x^*(0), \pi^*(t^*), x^*(t^*))$ be the vector R_{t_1} evaluated at the optimal equilibrium.

Proposition 4 (Implementation of the optimal equilibrium)**.** *If $\kappa\sigma\lambda \neq 1$, the rule in equation (15) implements the optimal equilibrium if and only if*

$$
\begin{aligned}
f(R^*) &= t^*, & (19)\\
\xi_\pi(R^*) + \phi\xi_x(R^*) &= 1 - \kappa\sigma\lambda. & (20)
\end{aligned}
$$

If $\kappa\sigma\lambda = 1$, the rule implements the optimal equilibrium if and only if equation (19) holds (i.e. equation (20) is no longer needed and any values for $\xi_\pi(R^*)$, $\xi_x(R^*)$ implement the optimal equilibrium).

Proof. Appendix B.6. □

Equation (19) states the rather obvious condition that to implement the optimal equilibrium, the central bank must have a liftoff rule that lifts off at optimal time t^* along the optimal path.

Equation (20) shows how the Taylor rule coefficients have to be picked to implement the optimal equilibrium when $\kappa\sigma\lambda \neq 1$, where we recall that ϕ is the slope of the optimal saddle path defined in equation (8) and $1 - \kappa\sigma\lambda$ is the coefficient in front of $\pi^*(t)$ in the optimal interest rate path in equation (7). Thus, equation (20) says that picking any $\xi_\pi(R^*)$ and $\xi_x(R^*)$ that generate the slope for the optimal saddle path is all that is required to implement the optimal equilibrium. Because there are two coefficients to be chosen and a single slope to match, there is one degree of freedom in the choice of coefficients.

When $\kappa\sigma\lambda = 1$, the optimal equilibrium never travels along the saddle path, as the economy reaches the steady state exactly at t^*. In this case, the slope of the saddle path, or even whether a saddle path exists at all, is not important for the implementation of the optimal equilibrium. All that is required is that the economy reach (π_{ss}, x_{ss}) at t^*. Because the optimal equilibrium when $\kappa\sigma\lambda = 1$ is implementable with any Taylor rule coefficients it can, in particular, be implemented while following the Taylor principle. The case $\kappa\sigma\lambda = 1$ is the only case for which this happens.

6.3 Eliminating Non-Optimal Equilibria

What should the central bank do for $R_{t_1} \neq R^*$? What should its off-equilibrium threats be if it wants to eliminate indeterminacy? The first proposition in this section answers these questions by providing necessary and sufficient conditions to rule out non-optimal equilibria, the main contribution of this paper.

Proposition 5 (Eliminating non-optimal equilibria)**.** *The rule in equation (15) implements no suboptimal equilibria (i.e., those with $R_{t_1} \neq R^*$) if and only if the following three statements are true:*

(a) [Strong deflationary expectations]. The Taylor principle does not hold when $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.

(b) [Weak deflationary expectations]. The liftoff rule satisfies

$$f(R_{t_1}) \neq \mathcal{T}(R_{t_1}) \tag{21}$$

for any bounded continuous path that fulfills one of the three conditions below:

i. The Taylor principle holds and there exists $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$;

ii. The Taylor principle does not hold and $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss} \cap \Upsilon_{ss}$;

iii. The Taylor principle holds and $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$.

(c) [Intermediate deflationary expectations]. The liftoff rule satisfies

$$f(R_{t_1}) > \frac{1}{\phi_1} \log \left(\frac{(x(r) - x_{zlb}) - \frac{\phi_1}{\kappa}(\pi(r) - \pi_{zlb})}{(x(t_1) - x_{zlb}) - \frac{\phi_1}{\kappa}(\pi(t_1) - \pi_{zlb})} \right) \tag{22}$$

for any $R_{t_1} \neq R^*$ such that, first, the Taylor principle does not hold and, second, there exists $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$.

Proof. Appendix B.7. □

To understand Proposition 5, consider a central bank facing private-sector expectations about the path of the economy. I define deflationary expectations as *strong* if the private sector forecasts that the economy will never exit the ZLB, as *weak* if the private sector forecasts the economy to be outside the ZLB at liftoff[22] and as *intermediate* in all other cases (which correspond to the private sector forecasting that the economy will be at the ZLB at liftoff but outside the ZLB at some future time). In Proposition 5, items (a), (b) and (c) each address one of these three types of expectations.

When deflationary expectations are strong, items (a) states that *not* following the Taylor principle after liftoff is necessary and sufficient for determinacy. Following the Taylor principle counterproductive because promising to be tough on inflation outside the ZLB prevents the future boom in inflation and output necessary to arrest the self-fulfilling deflationary expectations while at the ZLB. Even though, just as in models without a ZLB, the Taylor principle does anchor expectations outside the ZLB, the private sector's expectations of never being outside the ZLB render this anchoring irrelevant. Delaying liftoff does not help either; it only confirms the private sector's strong deflationary expectations. The only way to avoid the deflationary trap is to eliminate it outright by not following the Taylor principle. Not following the Taylor principle implies $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$, which prevents (π_{zlb}, x_{zlb}) from being a steady state after t_1.

When deflationary expectations are weak, item (b) states that avoiding liftoff at the one particular (state-contingent) liftoff date $\mathcal{T}(R_{t_1})$ is necessary and sufficient for determinacy, irrespective of whether the Taylor principle holds outside the ZLB. The exact functional form of $\mathcal{T}(R_{t_1})$ is in the Appendix. Since the private sector forecasts that the economy will be outside the ZLB at liftoff, expectations are already conducive to avoiding the deflationary

[22]Technically, weak expectations also include the case when the economy is outside the ZLB an instant dt after liftoff to take into account item (b)ii. in Proposition 5.

trap. To end up with multiple equilibria, the central bank must willfully steer the economy into the deflationary trap, which can only be done by lifting off at precisely the wrong time.

Subitem i. corresponds to the deflationary trap equilibria discussed in Benhabib et al. (2001b) in which the economy can initially be arbitrarily close to the desired steady state yet still converge to the deflationary steady state. In the present context, these paths can get arbitrarily close to the zero inflation steady state (π_{ss}, x_{ss}) not at $t = 0$, but at t_1, although they are still conceptually completely analogous. Precluding these paths from being an equilibrium can be done it without having to abandon the Taylor principle if f is chosen so as to satisfy equation (21). This result is different from those in Benhabib et al. (2001b). In Benhabib et al. (2001b), the Taylor principle necessarily implies that these paths always constitute perfect foresight equilibria. The difference in results arises because I consider a broader set of rules that the central bank can follow by allowing a state-dependent liftoff.[23] Subitem ii. describes an economy that converges to the desired steady state (π_{ss}, x_{ss}) after having successfully escaped the ZLB region Ω_{zlb} by time t_1. Even though the economy converges to the intended steady state, it does so by following a suboptimal path. These equilibria can be precluded by either following the Taylor principle or choosing $f(R_{t_1}) \neq \mathcal{T}(R_{t_1})$. Subitem iii. considers equilibria in which the Taylor principle holds and the economy is on (π_{ss}, x_{ss}) exactly at liftoff.

The case described in item (c) corresponds to intermediate deflationary expectations. In this case, the central bank produces global determinacy if and only if it follows one of two strategies. The first strategy is to postpone the date of liftoff far enough into the future and, after that, to not follow the Taylor principle. For each R_{t_1} that satisfies the assumptions in item (c), any low enough value of t_1 gives a continuous bounded path and hence an equilibrium. When the economy is inside Ω_{zlb}, pegging interest rates at zero or following $i(t) = \max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\}$ induces the same outcome. Liftoff while $\xi_\pi \pi(t) + \xi_x x(t) + r_h < 0$ produces the same path as lifting off exactly at time r, when the economy is at the boundary of the ZLB region and also on the stable saddle leading to (π_{ss}, x_{ss}), that is, $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$. For all liftoff times $t_1 \leq r$, the economy ends up following the stable saddle path toward (π_{ss}, x_{ss}), staying bounded and continuous and hence generating a suboptimal equilibrium. To stop this equilibrium from forming, the central bank first computes how long it takes to reach $\partial\Omega \cap \Upsilon_{ss}$ starting at the $(\pi(0), x(0))$ given by the first two components of R_{t_1}. It then picks a liftoff that exceeds that time. Equation (22) expresses exactly this strategy. More intuitively, sufficiently delaying liftoff and not following the Taylor principle induce higher inflation and output, reinforcing the expectations that the

[23] Despite this difference, the requirements of item (a) in Proposition 5 do support the conclusion in Benhabib et al. (2001b) that it is not possible to have determinacy while *always* (i.e., along every path) following the Taylor principle.

economy will exit the ZLB. The central bank must give the economy enough of an escape velocity to permanently exit the ZLB region and to also avoid other suboptimal equilibria that approach the optimal steady state but with too little inflation along the way. The earliest possible time for liftoff that guarantees determinacy is a function of private-sector expectations. The more deflationary the expectations, the longer the minimum time the central bank must wait before it lifts off. Since this lower bound on the time of liftoff depends on expectations, it changes with the state of the economy. Determinacy in this case requires a state-dependent liftoff date. A full description of the state of the economy before liftoff involves not only the levels of inflation and output that are expected to prevail exactly at liftoff but also their paths before then. A purely forward-looking rule that gives liftoff only as a function of economic conditions expected to prevail at the time of liftoff can anchor expectations of inflation and output at liftoff and later, but not before then, resulting in indeterminacy. To get determinacy, the liftoff rule must be a function of the state of the economy during the liquidity trap, before liftoff.

The second strategy to obtain determinacy when deflationary expectations are of intermediate strength is to lift off at any time and to then follow the Taylor principle. This strategy works because following the Taylor principle outside the ZLB negates the intermediate-strength deflationary expectations outright. For times in which the private sector forecasts that the economy will be outside the ZLB, the Taylor principle anchors inflation and the output gap to their steady-state targets of zero, just as in models without the ZLB. As soon as the economy is outside the ZLB, both inflation and the output gap must always equal their steady-state value of zero. This steady state is bounded away from the ZLB, since, in a small neighborhood around it, the nominal rate is close to the natural rate, which by then has turned positive. Therefore, an economy that is initially at the ZLB but expected to be outside of it at some point must jump a discrete amount the instant it exits the ZLB in order to immediately reach steady state. But predictable discrete jumps in inflation and the output gap create arbitrage opportunities, which cannot exist in equilibrium. The economy can therefore never exit the ZLB in the first place, disproving the private sector's initial expectations.

Even though Proposition 5 gives exact ways in which to eliminate each type of potential equilibria, it also offers broader, more conceptual lessons that apply along all paths. It gives general properties that any interest rate rule must have and can have in order to produce a determinate optimal outcome. When I combine Propositions 4 and 5, I obtain the following results.

Proposition 6 (General properties for global determinacy of the optimal equilibrium)**.** *If $\kappa\sigma\lambda \neq 1$, when the optimal equilibrium is the unique equilibrium:*

(a) The liftoff rule cannot be purely forward-looking, i.e., f cannot be constant in its first two arguments (which correspond to $(\pi(0), x(0))$ in equilibrium).

(b) The interest rate rule can be purely backward-looking, i.e., ξ_π, ξ_x and f can all be constant in their last two arguments (which correspond to $(\pi(t_1), x(t_1))$ in equilibrium).

(c) The interest rate rule can be memoryless after liftoff, i.e., the Taylor rule coefficients ξ_π and ξ_x can both be constant (and hence state and path independent).

Proof. Appendix B.8. □

Proposition 7 (Continuous policy rules)**.** *If the interest rate rule is continuous with respect to the state of the economy, i.e., if ξ_π, ξ_x and f are continuous:*

(a) Proposition 5 holds by replacing equation (21) with

$$f(R_{t_1}) > \mathcal{T}(R_{t_1}). \tag{23}$$

If $\kappa\sigma\lambda \neq 1$ and the optimal equilibrium is the unique equilibrium, then, in addition:

(b) The Taylor principle never holds.

(c) The interest rate rule must be forward- and backward-looking, i.e., at least one of the three functions ξ_π, ξ_x and f must be not constant in the first two arguments, and at least one of the three functions must be not constant in the last two arguments.

Proof. Appendix B.9. □

The first two items in Proposition 6 state that an interest rate rule that brings about global determinacy of the optimal equilibrium must be path dependent but need not be forward-looking. Why must the liftoff rule have a backward-looking component? A purely forward-looking rule specifies the length of time for which the central bank promises to keep interest rates at zero for each possible combination of inflation and output expected to prevail at the time of liftoff without referencing how the economy gets there. This strategy anchors expectations at liftoff and onward but not before then. Expectations for $(\pi(0), x(0))$ can freely adjust to make expectations of t_1 and $(\pi(t_1), x(t_1))$ consistent with a rational expectations equilibrium. For example, assume that the rule for liftoff is purely forward-looking and that the private sector expects liftoff to occur when the economy hits the steady state (π_{ss}, x_{ss}). Consider the value of $f(a, b, \pi_{ss}, x_{ss})$, which will be the same for all a and b since the rule is purely forward-looking. Using the continuous pasting conditions, it is always possible to find $(\hat{\pi}_0, \hat{x}_0)$ such that when $(\pi(0), x(0)) = (\hat{\pi}_0, \hat{x}_0)$, the economy follows a continuous path that reaches (π_{ss}, x_{ss}) at $t = f(a, b, \pi_{ss}, x_{ss})$. Indeed, all it takes is to trace the IS and NKPC backward in time starting at (π_{ss}, x_{ss}) for a period of time $f(a, b, \pi_{ss}, x_{ss})$ while setting $i(t) = 0$ throughout. Setting $t_1 = f(a, b, \pi_{ss}, x_{ss})$ produces a rational expectations equilibrium. But when $\kappa\sigma\lambda \neq 1$, this equilibrium is suboptimal, contradicting the assumption that the optimal equilibrium is the unique equilibrium.

In contrast, if the rule has a backward-looking component, finding the path by traveling backward in time from liftoff until $t = 0$ is a fixed-point problem, since the amount of time elapsed before reaching $(\pi(0), x(0))$ depends on $(\pi(0), x(0))$ itself. If the central bank picks an appropriate rule, the fixed-point problem can be guaranteed to have no solution. For the example above, in economic terms, no solution to the fixed-point problem means that, given the central bank's behavior, there exist no rational expectations for $(\pi(0), x(0))$ that get the economy to (π_{ss}, x_{ss}) at time $f\left(\pi(0), x(0), \pi_{ss}, x_{ss}\right)$. This type of logic implies that if $f(R_{t_1})$, instead of being a function of R_{t_1}, were a function of $(\pi(s), x(s), \pi(t_1), x(t_1))$ for some $s \in (0, t_1)$, Propositions 5 and 6 would still apply with only minor modifications. Moreover, if the central bank is willing to accept an indeterminate path between $t = 0$ and s, it can even wait until time s to announce its rule.

It is thus not necessary for the central bank to make its policy contingent exactly on $(\pi(0), x(0))$. It can make it contingent on inflation and output for any $s < T$. This idea may be important for the practical implementation of the rule, as it provides more options to communicate the backward-looking nature of the rule to the public. For example, instead of having to refer to inflation and output that prevail at the single exact moment in time when the economy enters the liquidity trap, the central bank can make liftoff contingent on the average inflation rate before T. Of course, there is no benefit from doing so in the model, but it is easy to imagine situations in which this can be beneficial. For example, if inflation has measurement error, a rule involving averages may be more desirable than one that uses a single inflation observation.

Communication involving past averages of inflation is reminiscent of price-level targeting rules. However, the rule in equation (15) is not, in general, equivalent to price targeting rules like the one studied in Eggertsson and Woodford (2003). Because price-level targeting rules are path dependent for all periods after their inception, an easy way to see that equation (15) is indeed different is to invoke item (c) of Proposition 6.

Item (c) of Proposition 6 shows that monetary policy after liftoff can be made independent of all outcomes and actions before liftoff, yet still provide determinacy. The proof in Appendix B.8 makes clear that making the Taylor rule coefficients constant does not introduce any new constraints or complications. Determinacy of the optimal equilibrium can be obtained with any combination of constant Taylor rule coefficients that implement the optimal equilibrium (characterized by Proposition 4). Lastly, constant Taylor rule coefficients may also be desirable from a communications standpoint, as the complexity of the rule is reduced.

Another way to simplify communication is to have a rule that is continuous with respect to the state of the economy. Empirically, at least during non-crisis times, small changes in economic conditions generally lead to changes in monetary policy of commensurate size. One concrete instance of a discontinuous rule is in equation (14), for which f can jump, for example, from t^* to $t^* + 1$ when $(\pi(0), x(0))$ deviates from $(\pi^*(0), x^*(0))$ by only small amount.

When f, ξ_x and ξ_π are assumed to be continuous, Proposition 7 shows that optimal monetary policy that brings about determinacy becomes more dovish. Items (a) and (b) of Proposition 7 state that continuous rules produce determinacy by promising a long enough period of zero interest rates together with a commitment to never following the Taylor principle once the promise of zero rates expires. The precise meaning of "long enough" is given by equations (22) and (23). Even though these two lower bounds for liftoff are relatively simple functions of current and future expected states of the economy, the freedom to overshoot the lower bound and still get determinacy further reduces the informational requirements of the central bank. It is not necessary to know the precise parameters of the model or the exact functional form of $\mathcal{T}$ to implement a "long enough" period of zero interest rates.

Why does continuity change equation (21) into equation (23)? For paths that fulfill one of the three conditions in subitems i., ii. and iii. of item (b) in Proposition 5, the central bank must avoid lifting off at the critical time $\mathcal{T}$; otherwise, it enables a suboptimal equilibrium. By continuity, having $f < \mathcal{T}$ for one path and $f > \mathcal{T}$ for some other path inevitably implies $f = \mathcal{T}$ for some third path. Therefore, if liftoff occurs before the critical time for any one path, it must occur before the critical time for all paths. But this strategy unravels because there are continuous bounded paths with $\mathcal{T} = T$ so that $f < \mathcal{T}$ implies that $f < \mathcal{T} = T$, contradicting the notion that liftoff never occurs before the natural rate turns positive.[24] One simple example of a bounded continuous path with $\mathcal{T} = T$ is when liftoff occurs at T and $(\pi(T), x(T)) = (\pi_{ss}, x_{ss})$.

A similar argument explains why the Taylor principle can never hold when the policy rule is continuous. If the Taylor principle holds for some paths and does not hold for other paths, continuity implies that at least one path satisfies the intermediate case $\kappa(\xi_\pi - 1) + \rho\xi_x = 0$. For that path, the deflationary steady state (π_{zlb}, x_{zlb}) is exactly on the boundary $\partial\Omega$. A

[24]If f is allowed to occur before T, an analogous argument applies because f is still bounded below by zero.

suboptimal equilibrium arises in which the economy travels inside Ω_{zlb} along Υ_{zlb} and toward (π_{zlb}, x_{zlb}).[25] Increasing the time of liftoff in an attempt to escape the ZLB fails: To exit the ZLB, the economy must go across the boundary $\partial\Omega$ before entering Ω_{ss}, but when the economy reaches $\partial\Omega$, it is in steady state and hence remains there forever after. It follows that to avoid $\kappa(\xi_\pi - 1) + \rho\xi_x = 0$, either all paths obey the Taylor principle, or none of them do. By item (a) of Proposition 5, a unique optimal equilibrium requires some paths to not follow the Taylor principle, irrespective of f, and hence the Taylor principle never holds.

Lastly, item (c) of Proposition 7 states that a continuous rule consistent with global determinacy of the optimal equilibrium is necessarily both forward- and backward-looking. By item (a) of Proposition 6, we already know the rule cannot be purely forward-looking. The reason it can no longer be purely backward-looking is that suboptimal equilibria would emerge in a neighborhood of the optimal equilibrium. Because the optimal equilibrium (with $\kappa\sigma\lambda \neq 1$) must be implemented without following the Taylor principle, it always induces saddle dynamics outside the ZLB. First imagine that Taylor rule coefficients are constant, so that all paths have saddle dynamics in Ω_{ss} and the same slope for the saddle path. With a continuous and purely backward-looking f, points $(\pi(0), x(0))$ around $(\pi^*(0), x^*(0))$ always take the economy close to $(\pi^*(t^*), x^*(t^*))$ at time $f(\pi(0), x(0))$. For a small enough $\varepsilon > 0$, there always is a point $(\pi(t_1), x(t_1)) \in \Upsilon_{ss}$ that is close enough to $(\pi^*(t^*), x^*(t^*))$ and can be reached at time $t^* \pm \varepsilon$. This situation is similar to that in Figure 2, only that instead of a constant liftoff time, liftoff times change a small amount as the economy approaches R^*. The small change is not enough to overcome the indeterminacy. The presence of a saddle path breeds indeterminacy; liftoff slightly earlier or later simply puts the economy on different nearby points on the saddle path. This is one of the reasons why the discontinuous neo-Fisherian rule in equation (14) has discrete jumps (t^*, $t^* + 1$ and $t^* + 2$) for the liftoff times: Jumps can put $(\pi(t_1), x(t_1))$ far from $(\pi^*(t^*), x^*(t^*))$ even when $(\pi(0), x(0))$ is close to $(\pi^*(0), x^*(0))$. When the Taylor rule coefficients are not constant but are instead continuous functions of $(\pi(0), x(0))$, by item (b) in Proposition 7, they still fail to satisfy the Taylor principle. The only change is that now the saddle path associated with starting at $(\pi(0), x(0))$ is slightly different from the one associated with starting at $(\pi^*(0), x^*(0))$. This change is still not enough to eliminate equilibria close to the optimal one. Points near R^* can still be part of an equilibrium with liftoff arbitrarily close to t^* and a saddle path arbitrarily close to the optimal one. Introducing a forward-looking component eliminates this kind of equilibrium not because $(\pi(t_1), x(t_1))$ is suddenly far from $(\pi^*(t^*), x^*(t^*))$, but because f can be chosen so as to always violate the condition in equation (12), a situation analogous to the one previously discussed for item (a) of Proposition 6.

[25] A formal proof is in Lemma 3 of Appendix B.8

7 Conclusion

I have presented necessary and sufficient conditions for global determinacy of the optimal equilibrium in a New Keynesian (NK) economy with a binding ZLB and passive fiscal policy. By using the most basic version of the NK model, I am able to write down the solution of the model in closed form and fully characterize global determinacy. On the other hand, the literature examines a myriad of important ways in which this baseline model can be extended, refined and modified.[26] Nevertheless, this basic version of the model is a natural place to start. The forces highlighted in this paper are likely at play in any model with a NK core. A future necessary step before the rules I propose can be used by a real-world central bank is to investigate their performance in different versions of the NK model —and in other models as well.

[26] A partial list includes: Investment dynamics, trade and capital flows, coordination with fiscal policy, limited commitment, imperfect credibility, heterogeneous agents, financial intermediation, financial stability concerns, informational frictions, learning, and non-rational expectations are some of the realistic components that can change the optimality and determinacy properties of equilibria.

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Internet Appendix for "How to Escape a Liquidity Trap with Interest Rate Rules"

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Abstract

This Internet Appendix provides additional material supporting the main text. Section A derives the solution to the IS and PC system of differential equations, provides additional details on Section 5.2, and characterizes continuous pasting. Section B has the proofs of the propositions in the main text of the paper.

[1]The views expressed in this appendix are those of the author and are not necessarily reflective of views at the Federal Reserve Bank of New York or the Federal Reserve System. Duarte: fernando.duarte@ny.frb.org

A Preliminaries

A.1 Solution to the IS-PC system of equations

Transition times. Let $\tau(t)$ be the largest of t_1 and the last time $(\pi(t), x(t))$ entered the region Ω_{zlb} from Ω_{ss} before or at time t. More formally,[2]

$$\begin{aligned}\tau(t) &= \max\{t_1, \tau_{zlb}(t)\},\\ \tau_{zlb}(t) &= \sup\{s \le t : (\pi(s), x(s)) \in \partial\Omega \text{ and } \exists\varepsilon > 0 \text{ s.t. } (\pi(s+\varepsilon), x(s+\varepsilon)) \in \Omega_{zlb}\}.\end{aligned}$$

Let $\eta(t)$ be the largest of t_1 and the last time $(\pi(t), x(t))$ enters the region Ω_{ss} from Ω_{zlb} before or at time t, i.e.,

$$\begin{aligned}\eta(t) &= \max\{t_1, \eta_{ss}(t)\},\\ \eta_{ss}(t) &= \sup\{s \le t : (\pi(s), x(s)) \in \partial\Omega \text{ and } \exists\varepsilon > 0 \text{ s.t. } (\pi(s+\varepsilon), x(s+\varepsilon)) \in \Omega_{ss}\}.\end{aligned}$$

Note that $\tau(t)$ and $\eta(t)$ are piece-wise constant and thus for all $t \neq \tau(t)$ we have $\dot{\tau}(t) = 0$ and for all $t \neq \eta(t)$ we have $\dot{\eta}(t) = 0$.

Solution to IS and NKPC. Let

$$\begin{aligned}A_{zlb} &= \begin{bmatrix} 0 & -\frac{1}{\sigma} \\ -\kappa & \rho \end{bmatrix},\\ A_{ss} &= \begin{bmatrix} \frac{1}{\sigma}\xi_x & \frac{1}{\sigma}(\xi_\pi - 1) \\ -\kappa & \rho \end{bmatrix}.\end{aligned}$$

The matrix A_{zlb} gives the dynamics of the system of ODEs (1)-(2) when $i(t) = 0$ while the matrix A_{ss} gives the dynamics when $i(t) = r_h + \xi_\pi \pi(t) + \xi_x x(t)$. The matrix A_{zlb} has eigenvalues ϕ_1 and ϕ_2 defined in equations (A.17)-(A.18). The eigenvalues of A_{ss} are

$$\alpha_1 = \frac{1}{2\sigma}\left(\xi_x + \sigma\rho + \sqrt{(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1)}\right), \tag{A.1}$$

$$\alpha_2 = \frac{1}{2\sigma}\left(\xi_x + \sigma\rho - \sqrt{(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1)}\right). \tag{A.2}$$

Because stable dynamics always produce indeterminacy (see Lemma 3 in Appendix B.8 for a proof), I restrict all analysis to cases in which either $\det A_{ss} > 0$ and $\operatorname{trace} A_{ss} > 0$, or $\det A_{ss} < 0$. Below, I use d_{exit} and d_{trap} defined in equations (A.21) and (A.22).

[2] Recall that the supremum of the empty set is $-\infty$. If there is no s such that $(x(s), \pi(s)) \in \partial\Omega$ or $\nexists\varepsilon > 0$ s.t. $(x(s+\varepsilon), \pi(s+\varepsilon)) \in \Omega_{zlb}$, then $\tau_{zlb}(t) = -\infty$ and $\tau(t) = t_1$.

For $t \in [0,T)$ the solution to (1)-(2) under the interest rate rule in equation (15) is

$$\begin{aligned}
x(t) &= -\frac{\phi_2}{(\phi_1-\phi_2)}\left(d_{exit}(0)-\frac{\phi_2}{\kappa}(r_h-r_l)\right)e^{\phi_1 t} \\
&\quad+\frac{\phi_1}{(\phi_1-\phi_2)}\left(d_{trap}(0)-\frac{\phi_1}{\kappa}(r_h-r_l)\right)e^{\phi_2 t}-\frac{\rho}{\kappa}r_l, \qquad \text{(A.3)}\\
\pi(t) &= -\frac{\kappa}{(\phi_1-\phi_2)}\left(d_{exit}(0)-\frac{\phi_2}{\kappa}(r_h-r_l)\right)e^{\phi_1 t} \\
&\quad+\frac{\kappa}{(\phi_1-\phi_2)}\left(d_{trap}(0)-\frac{\phi_1}{\kappa}(r_h-r_l)\right)e^{\phi_2 t}-r_l. \qquad \text{(A.4)}
\end{aligned}$$

For $t \in [T,t_1)$ the solution is

$$x(t) = -\frac{\phi_2 d_{exit}(t)}{(\phi_1-\phi_2)}e^{\phi_1(t-T)}+\frac{\phi_1 d_{trap}(t)}{(\phi_1-\phi_2)}e^{\phi_2(t-T)}-\frac{\rho}{\kappa}r_h \qquad \text{(A.5)}$$

$$\pi(t) = -\frac{\kappa d_{exit}(t)}{(\phi_1-\phi_2)}e^{\phi_1(t-T)}+\frac{\kappa d_{trap}(t)}{(\phi_1-\phi_2)}e^{\phi_2(t-T)}-r_h \qquad \text{(A.6)}$$

For $t \in [t_1,\infty)$, when $(\pi(t),x(t)) \in \Omega_{zlb}$, the solution is

$$x(t) = -\frac{\phi_2 d_{exit}(\tau(t))}{(\phi_1-\phi_2)}e^{\phi_1(t-\tau(t))}+\frac{\phi_1 d_{trap}(\tau(t))}{(\phi_1-\phi_2)}e^{\phi_2(t-\tau(t))}-\frac{\rho}{\kappa}r_h, \qquad \text{(A.7)}$$

$$\pi(t) = -\frac{\kappa d_{exit}(\tau(t))}{(\phi_1-\phi_2)}e^{\phi_1(t-\tau(t))}+\frac{\kappa d_{trap}(\tau(t))}{(\phi_1-\phi_2)}e^{\phi_2(t-\tau(t))}-r_h. \qquad \text{(A.8)}$$

For $t \in [t_1,\infty)$, when $(\pi(t),x(t)) \in \Omega_{ss}$, I distinguish three cases:

Case I: $\xi_\pi \neq 1$ and $4\kappa\sigma(\xi_\pi-1) \neq (\xi_x-\sigma\rho)^2$;

Case II: $\xi_\pi = 1$ and $4\kappa\sigma(\xi_\pi-1) \neq (\xi_x-\sigma\rho)^2$;

Case III: $4\kappa\sigma(\xi_\pi-1) = (\xi_x-\sigma\rho)^2$.

For Case I, the solution is

$$\begin{aligned}
x(t) &= -\frac{(1-\xi_\pi)\pi(\eta(t))+(\sigma\alpha_2-\xi_x)x(\eta(t))}{\sigma(\alpha_1-\alpha_2)}e^{\alpha_1(t-\eta(t))} \\
&\quad+\frac{(1-\xi_\pi)\pi(\eta(t))+(\sigma\alpha_1-\xi_x)x(\eta(t))}{\sigma(\alpha_1-\alpha_2)}e^{\alpha_2(t-\eta(t))}, \qquad \text{(A.9)}\\
\pi(t) &= \frac{(1-\xi_\pi)\pi(\eta(t))+(\sigma\alpha_2-\xi_x)x(\eta(t))}{\sigma(\xi_\pi-1)(\alpha_1-\alpha_2)}(\xi_x-\sigma\alpha_1)e^{\alpha_1(t-\eta(t))} \\
&\quad-\frac{(1-\xi_\pi)\pi(\eta(t))+(\sigma\alpha_1-\xi_x)x(\eta(t))}{\sigma(\xi_\pi-1)(\alpha_1-\alpha_2)}(\xi_x-\sigma\alpha_2)e^{\alpha_2(t-\eta(t))}. \qquad \text{(A.10)}
\end{aligned}$$

For Case II, the solution is

$$x(t) = x(\eta(t))e^{\frac{1}{\sigma}\xi_x(t-\eta(t))}, \tag{A.11}$$

$$\pi(t) = \frac{\pi(\eta(t))\left(\xi_x - \sigma\rho\right) + \kappa\sigma x(\eta(t))}{\xi_x - \sigma\rho} e^{\rho(t-\eta(t))} - \frac{\kappa\sigma x(\eta(t))}{\xi_x - \sigma\rho} e^{\frac{1}{\sigma}\xi_x(t-\eta(t))}. \tag{A.12}$$

For Case III, the solution is

$$\begin{aligned} x(t) &= \left(1 + \frac{1}{2\sigma}\left(\xi_x - \sigma\rho\right)(t-t_1)\right) x(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))} \\ &\quad + \frac{1}{\kappa}\left(\frac{1}{2\sigma}\left(\sigma\rho - \xi_x\right)\right)^2 (t-t_1)\,\pi(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))}, \end{aligned} \tag{A.13}$$

$$\begin{aligned} \pi(t) &= -\kappa\left(t-t_1\right) x(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))} \\ &\quad + \left(1 - \frac{1}{2\sigma}\left(\xi_x - \sigma\rho\right)(t-t_1)\right)\pi(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))}. \end{aligned} \tag{A.14}$$

Saddle path in Ω_{ss}. The saddle path Υ_{ss} is the set of points (π, x) such that

$$\pi = \begin{cases} \frac{(\xi_x - \sigma\alpha_2)}{(1-\xi_\pi)} x & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi \neq 1 \\ \frac{\kappa\sigma}{(\sigma\rho - \xi_x)} x & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi = 1 \\ \frac{\kappa}{\rho} x & , \text{ if } \det A_{ss} = 0 \text{ and } \operatorname{trace} A_{ss} \geq 0 \\ \emptyset & , \text{ otherwise} \end{cases} . \tag{A.15}$$

A.2 Details on Section 5.2

In this section, I expand on the content of Section 5.2 by providing additional details on the derivations and economic intuition.

First stage ($0 \leq t < T$). The dynamics of the economy are exactly as in rule (10), since $i(t) = 0$ and there are no other decisions for the central bank to make (recall liftoff occurs after T by assumption). Each point $(\pi(0), x(0))$ maps to one and only one $(\pi(T), x(T))$ and the mapping is unaffected by expectations or outcomes. Figure 11 shows the phase portrait that can be used to understand this mapping. Because the natural rate is negative, the unique steady state for the first-stage dynamics, labeled (π_l, x_l) in the figure, is in the first quadrant and given by $\pi_l = -r_l > 0$ and $x_l = -\rho r_l/\kappa > 0$. While the dynamics and the location of the steady state do not change with private-sector expectations or central bank actions, the specific $(\pi(0), x(0))$ that is actually realized in equilibrium —and hence which path is realized— does depend on them.

Second stage ($T \leq t < t_1$). The central bank is committed to $i(t) = 0$ between T and $f(R_{t_1})$. The dynamics are identical to those in rule (10), with the only difference being

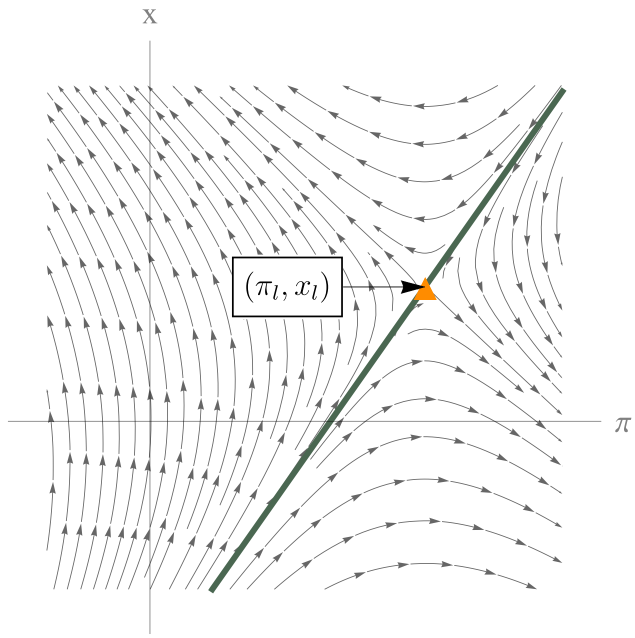

Figure 11: Dynamics of the economy for $t \in [0, T)$, when $i(t) = 0$ for all $\pi(t)$ and $x(t)$ and $r(t) = r_l < 0$. The green line is the saddle path and the orange triangle, labeled (π_l, x_l), is the steady state.

that they are maintained until $f(R_{t_1})$, which depends on R_{t_1}, instead of until the constant liftoff time t^*. In equilibrium, $t_1 = f(R_{t_1})$, so the actual duration of this stage is endogenous and depends not only on the announced liftoff rule but also on private-sector expectations and the realizations of inflation and output in the first stage. Given a known liftoff time t_1, analogous to what happens in the first stage, the dynamics of the economy and the mapping from $(\pi(T), x(T))$ to $(\pi(t_1), x(t_1))$ are always unchanged, while the specific $(\pi(T), x(T))$ and $(\pi(t_1), x(t_1))$ that end up being realized change based on which equilibrium ends up being realized.

Figure 12 shows the phase portrait of $\pi(t)$ and $x(t)$, which reveals saddle dynamics. I denote the stable *zlb saddle path* by

$$\Upsilon_{zlb} = \left\{ (\pi, x) : x = \frac{\phi_1}{\kappa}\pi - \frac{\phi_2}{\kappa} r_h \right\}, \tag{A.16}$$

where

$$\phi_1 = \frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} > 0, \tag{A.17}$$

$$\phi_2 = \frac{1}{2}\rho - \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} < 0, \tag{A.18}$$

are the two eigenvalues of the system. The *unstable zlb saddle path* is given by

$$\Psi_{zlb} = \left\{ (\pi, x) : x = \frac{\phi_2}{\kappa}\pi - \frac{\phi_1}{\kappa} r_h \right\}, \tag{A.19}$$

which is the saddle path that would be stable if the system evolved backward in time.

The *zlb* saddle path Υ_{zlb}, a line with positive slope, intersects the unstable *zlb* saddle path Ψ_{zlb}, a line with negative slope, at the *zlb steady state*

$$(\pi_{zlb}, x_{zlb}) = \left(-r_h, -\frac{\rho r_h}{\kappa} \right). \tag{A.20}$$

The point (π_{zlb}, x_{zlb}) always lies in the third quadrant of the π-x plane. Neither the location of (π_{zlb}, x_{zlb}) nor the slopes of Υ_{zlb} and Ψ_{zlb} depend on policy choices of the central bank; they are fully specified by the parameters κ, ρ, σ and r_h. In the literature, the steady state (π_{zlb}, x_{zlb}) is variously referred to as the "deflationary steady state," the "liquidity trap steady state," the "expectational trap steady state" or the "unintended steady state."

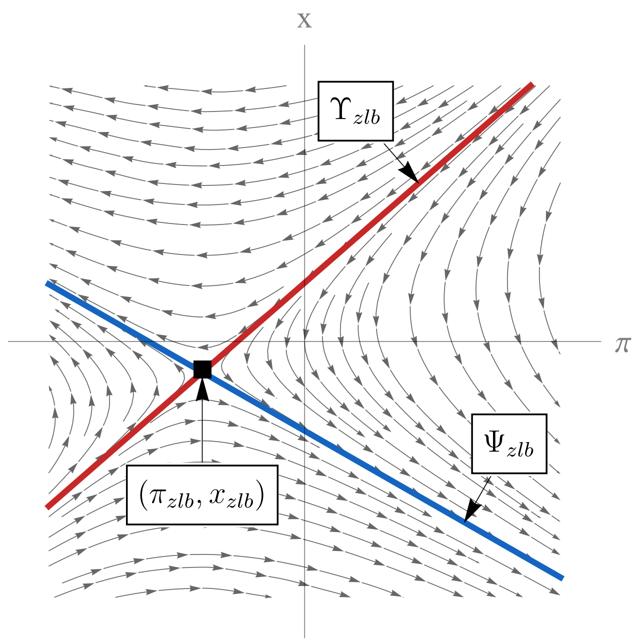

Figure 12: Dynamics of the economy for $t \in [T, t_1)$, when $i(t) = 0$ for all $\pi(t)$ and $x(t)$, and $r(t) = r_h > 0$. The red line, labeled Υ_{zlb}, is the saddle path. If the economy starts on Υ_{zlb}, it converges to the deflationary steady state (π_{zlb}, x_{zlb}). The blue line, labeled Ψ_{zlb}, is the "unstable saddle path." If the economy starts on Ψ_{zlb}, it stays on Ψ_{zlb} and moves away from the deflationary steady state (π_{zlb}, x_{zlb}). Under these dynamics, if the economy is not on Υ_{zlb}, then inflation and output become unbounded.

Two key objects to understanding the behavior of the economy and its determinacy properties are:

$$d_{exit}(t) = x(t) - \frac{\phi_1}{\kappa}\pi(t) + \frac{\phi_2 r_h}{\kappa}, \tag{A.21}$$

$$d_{trap}(t) = x(t) - \frac{\phi_2}{\kappa}\pi(t) + \frac{\phi_1 r_h}{\kappa}. \tag{A.22}$$

The value of d_{exit} is a measure of the distance (with sign) to the stable zlb saddle path Υ_{zlb} defined in equation (A.16). Similarly, d_{trap} gives a measure of distance (with sign) from $(\pi(t), x(t))$ to the unstable saddle path Ψ_{zlb} defined in equation (A.19). A d_{exit} closer to zero means the economy is closer to Υ_{zlb} and hence will behave more similarly to an economy that is on Υ_{zlb}, at least for some initial period of time. A d_{exit} closer to zero also implies that the economy at some point gets closer to the unintended steady state (π_{zlb}, x_{zlb}). In fact, $d_{exit}(t) = 0$ indicates that the economy is exactly on Υ_{zlb} at time t and converging toward (π_{zlb}, x_{zlb}). In contrast, a d_{trap} closer to zero implies that the dynamics of the economy look more like those of the unstable saddle path Ψ_{zlb}, pushing the economy further away from (π_{zlb}, x_{zlb}).

The dynamics of the economy are a tug of war between two competing forces: one driving the economy into the deflationary steady state (π_{zlb}, x_{zlb}) and another pulling the economy away from it. The strength of these two forces is given by d_{exit} and d_{trap}. Indeed, inflation and output can be written as

$$\begin{bmatrix} x(t) - x_{zlb} \\ \pi(t) - \pi_{zlb} \end{bmatrix} = d_{exit}(t)\ v_{exit} + d_{trap}(t)\ v_{trap}, \tag{A.23}$$

where

$$v_{exit} = \begin{bmatrix} -\frac{\phi_2}{\phi_1 - \phi_2} \\ -\frac{\kappa}{\phi_1 - \phi_2} \end{bmatrix} \text{ and } v_{trap} = \begin{bmatrix} \frac{\phi_1}{\phi_1 - \phi_2} \\ \frac{\kappa}{\phi_1 - \phi_2} \end{bmatrix}, \tag{A.24}$$

are the eigenvectors of the system. The eigenvector v_{exit} is associated with the explosive eigenvalue $\phi_1 > 0$ and lies on the unstable saddle path Ψ_{zlb}. The eigenvector v_{trap} is associated with the stabilizing eigenvalue $\phi_2 < 0$ and lies on the stable saddle path Υ_{zlb}. The eigenvalue v_{trap} is the "trap factor" that drives the economy into the expectational trap steady state (π_{zlb}, x_{zlb}), while v_{exit} is the "exit factor" pulling the economy away from it. After a change of coordinates that makes (π_{zlb}, x_{zlb}) the origin and the eigenvectors of the system the coordinate basis vectors, the vector $(\pi(t), x(t))$ has coordinates $(d_{exit}(t), d_{trap}(t))$:

$$\begin{bmatrix} -\frac{\phi_2}{\phi_1 - \phi_2} & \frac{\phi_1}{\phi_1 - \phi_2} \\ -\frac{\kappa}{\phi_1 - \phi_2} & \frac{\kappa}{\phi_1 - \phi_2} \end{bmatrix}^{-1} \begin{bmatrix} x(t) - x_{zlb} \\ \pi(t) - \pi_{zlb} \end{bmatrix} = \begin{bmatrix} d_{exit}(t) \\ d_{trap}(t) \end{bmatrix} \tag{A.25}$$

In other words, projecting $(\pi(t), x(t))$ onto the eigenvalues gives loadings of $(d_{exit}(t), d_{trap}(t))$. This linear two-factor representation of the economy is exact in the sense that there is no residual left once $x(t)$ and $\pi(t)$ are expressed as a linear combination of the factors plus a constant. As already pointed out through various other arguments by Benhabib et al. (2001b), Werning (2012) and others, the current levels of inflation and output are, on their own, not very informative about whether the economy is in a liquidity trap, constrained by the ZLB, at risk of converging to the unintended steady state, or on a desirable policy path. For example, Benhabib et al. (2001b) show that an economy can have inflation and output arbitrarily close to target (in the model I consider here, the target is the intended steady state (π_{ss}, x_{ss})) and yet converge to the unintended steady state (π_{zlb}, x_{zlb}). In contrast, observing a $d_{exit}(t) = 0$ immediately reveals that an economy is headed toward (π_{zlb}, x_{zlb}). In addition, and more important for this paper, the necessary and sufficient conditions for global determinacy in the next section are most simply expressed as functions of $d_{exit}(t)$ and $d_{trap}(t)$, highlighting not just their mathematical convenience but also their economic importance. When a central bank is trying to assess private-sector expectations in order to know what is needed to anchor expectations, the results in this paper suggest that it should focus on the linear combinations of inflation and output given by $d_{exit}(t)$ and $d_{trap}(t)$ rather than on the levels of inflation and output by themselves.

Third stage ($t \geq t_1$). I split the π-x plane into two disjoint regions defined by whether the ZLB is binding

$$\Omega_{zlb}(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h \leq 0\}, \tag{A.26}$$

$$\Omega_{ss}(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h > 0\}, \tag{A.27}$$

where the subscript *zlb* in Ω_{zlb} stands for zero lower bound and the subscript *ss* in Ω_{ss} stands for "intended steady state," as the region $\Omega_{ss}(R_{t_1})$ contains $(\pi_{ss}, x_{ss}) = (0, 0)$, the steady state that the central bank would like the economy to converge to in the long run if the optimal equilibrium is to be achieved. The boundary between the regions $\Omega_{zlb}(R_{t_1})$ and $\Omega_{ss}(R_{t_1})$ is a line

$$\partial\Omega(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h = 0\} \subset \Omega_{zlb}(R_{t_1}). \tag{A.28}$$

Henceforth, I suppress the dependence of ξ_x, ξ_π, Ω_{zlb}, Ω_{ss} and $\partial\Omega$ on R_{t_1} for ease of notation whenever it does not create confusion.

The derivatives of inflation and output with respect to time, $\dot{\pi}(t)$ and $\dot{x}(t)$, inherit the properties of $i(t)$ and are therefore not differentiable on $\partial\Omega$ as a function of time (i.e., $\pi(t)$ and $x(t)$ do not have second derivatives on $\partial\Omega$). However, $\dot{\pi}(t)$ and $\dot{x}(t)$ are always continuous with respect to time, ensuring a continuous path for $(\pi(t), x(t))$.[3] By the second line in equation (15), after t_1,

$$
\begin{aligned}
i(t) &= 0 \text{ iff } (\pi(t), x(t)) \in \Omega_{zlb}, &\text{(A.29)}\\
i(t) &= \xi_\pi \pi(t) + \xi_x x(t) + r_h \text{ iff } (\pi(t), x(t)) \in \Omega_{ss}. &\text{(A.30)}
\end{aligned}
$$

When equations (A.29) and (A.30) are used in the IS equation and the NKPC, the dynamics of the economy inside each of the two regions Ω_{zlb} and Ω_{ss} are separately given by a system of linear first-order ordinary differential equations (ODEs) in $x(t)$ and $\pi(t)$, each of which is easy to analyze inside its respective region with standard methods. However, when the two regions are analyzed together, the global dynamics are piecewise linear, with a non-differentiable transition at $\partial\Omega$. The behavior of piecewise linear dynamic systems can, in general, exhibit a rich variety of non-linear phenomena such as limit cycles, bifurcations and chaos. The global properties can also be quite different from those of each individual region. For example, it is possible to construct paths that are globally bounded for systems in which each separate region has explosive dynamics.[4] To tackle the non-linearities of the New Keynesian economy at hand, I first analyze the properties of each of the two regions separately and then combine them and analyze the resulting global dynamics. Readers familiar with New Keynesian models without a ZLB should find the analysis of each of the separate regions familiar. The new results arise when I look at both regions together.

First, consider the behavior of the economy in the region Ω_{ss}. Inside Ω_{ss}, there is always a single steady state, $(\pi_{ss}, x_{ss}) = (0, 0)$. The dynamic behavior of the economy depends on the choice of Taylor rule coefficients ξ_π and ξ_x. I focus on Taylor rule coefficients that, absent the ZLB, give either unstable or saddle dynamics, since the central bank would not pick stable dynamics that have no explosive paths, as they always lead to indeterminacy.[5] The Taylor principle is the key concept needed to differentiate between unstable and saddle dynamics, and between *locally* determinate and indeterminate equilibria. The *Taylor principle* is said to *hold* if and only if

$$\kappa\,(\xi_\pi - 1) + \rho\xi_x > 0 \quad \text{and} \quad \xi_x + \sigma\rho > 0. \tag{A.31}$$

When $\xi_x = 0$, the Taylor principle is equivalent to $\xi_\pi > 1$, one of its most popular forms. When the Taylor principle holds, if the dynamics of the Ω_{ss} region were extended to the entire plane and nominal interest rates were allowed to be negative, or if I considered a

[3]In fact, $(\dot{\pi}(t), \dot{x}(t))$ is Lipschitz continuous in $(\pi(t), x(t))$ and continuous in t, guaranteeing the global existence and uniqueness of the continuous solution for $t \geq t_1$.

[4]For example, see Bernardo, Budd, Champneys, and Kowalczyk (2008).

[5]For stable dynamics, it is immediate that there is indeterminacy for any choice of ξ_π, ξ_x and f. I also exclude the knife-edge case in which the dynamics have a line whose points are all steady states, but the dynamics are otherwise explosive. See Lemma 3 in Appendix B.8 for a proof that, in this case, there also is indeterminacy for any choice of ξ_π, ξ_x and f.

small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely inside Ω_{ss}, the dynamics of the system would be unstable. All paths would be unbounded —or exit the small neighborhood— unless $(\pi(t), x(t)) = (\pi_{ss}, x_{ss}) = (0, 0)$ for all t. Figure 13 shows representative phase portraits of two such economies. In the diagram on the left, the Taylor rule coefficients satisfy $(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1) < 0$ and paths "slowly" spiral outward from the steady state. When the reverse inequality holds, the steady state is instead a source, shown in the diagram on the right. In models without a ZLB, the Taylor principle is a necessary and sufficient condition for local determinacy. When the ZLB is introduced, since there is always a small enough neighborhood of (π_{ss}, x_{ss}) that is contained entirely in Ω_{ss}, the Taylor principle is still a necessary and sufficient condition for local determinacy of the equilibrium with $(\pi(t), x(t)) = (\pi_{ss}, x_{ss})$ for all $t \geq t_1$. As I briefly discussed before and will expand on later, the Taylor principle is, however, neither necessary nor sufficient for *global* determinacy of the optimal equilibrium.

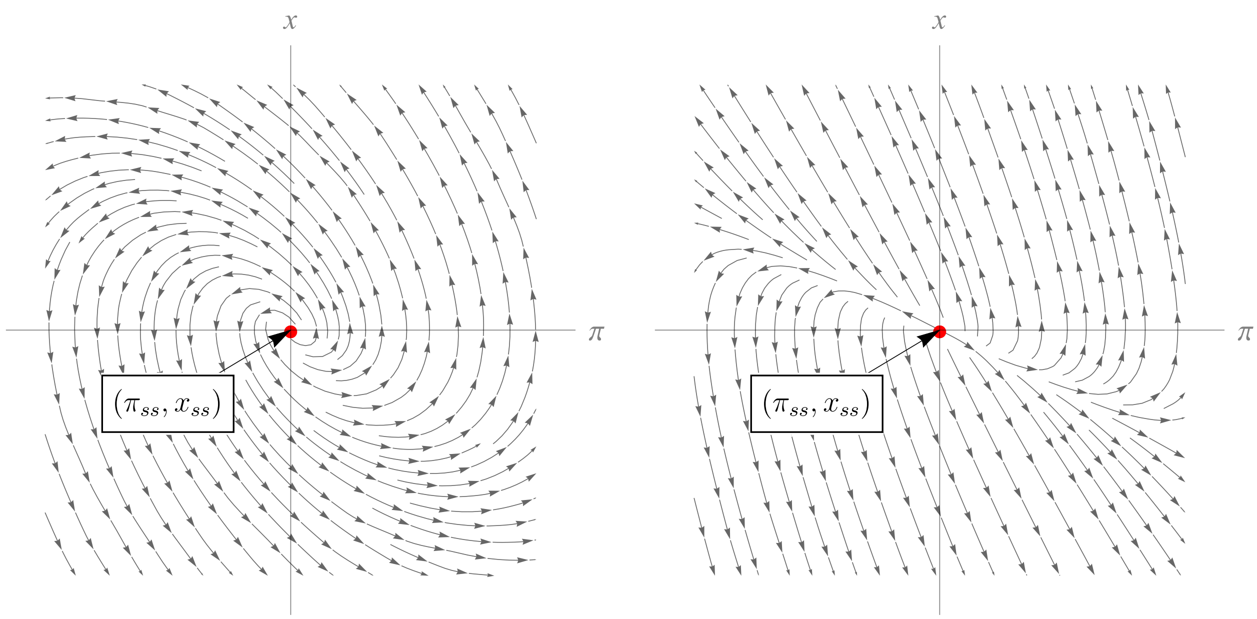

Figure 13: Dynamics of the economy for $t \geq t_1$ when the Taylor principle holds and there is no ZLB. The dynamics of the diagram on the left have imaginary eigenvalues, while the right-hand side have real ones. When there is no ZLB, the Taylor principle is necessary and sufficient for *local* determinacy. Unless the economy starts at $(0, 0)$, inflation and output become unbounded.

The *Taylor principle* is said to *not hold* if and only if

$$\kappa(\xi_\pi - 1) + \rho\xi_x < 0. \tag{A.32}$$

When the Taylor principle does not hold, if the dynamics of the Ω_{ss} region were extended to the entire plane and interest rates were allowed to be negative, or if I considered a small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely in Ω_{ss}, the system would have saddle dynamics. I denote the *ss saddle path* by Υ_{ss}. It is a line through the origin whose slope depends on the Taylor rule coefficients ξ_π and ξ_x.[6] Paths are bounded —or stay in the small neighborhood of (π_{ss}, x_{ss})— if and only if they start on the ss saddle path. Figure 14 displays a typical phase diagram when the Taylor principle does not hold.

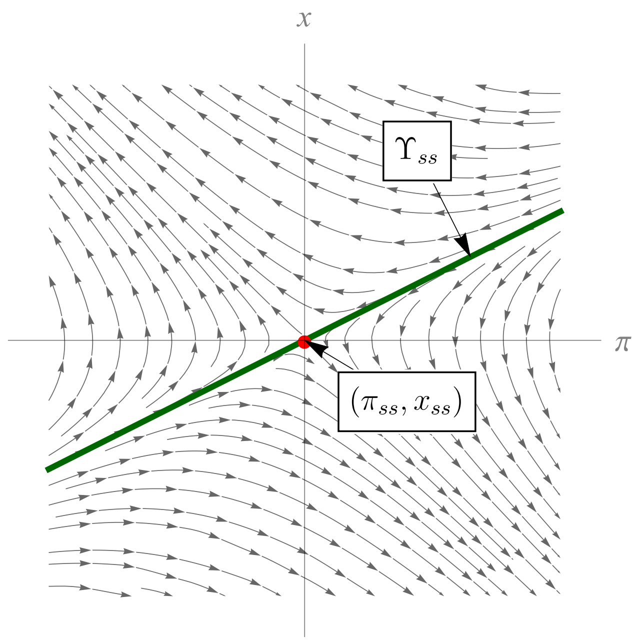

Figure 14: Dynamics of the economy for $t \geq t_1$ when the Taylor principle does not hold and there is no ZLB. Unless the economy starts on its saddle path Υ_{ss}, shown in green, inflation and output become unbounded.

Note that because ξ_π and ξ_x depend on R_{t_1}, whether the Taylor principle holds depends on R_{t_1}. Within the same economy, there can be a subset of (off-equilibrium) paths for which the Taylor principle holds and a different subset for which it does not hold. Instead, when ξ_π and ξ_x are constant, the Taylor principle must hold either for all paths or for no paths. In all cases, the Taylor rule coefficients are fully determined by the time the central bank lifts off and they remain unchanged from then on.

Now consider the behavior in the *zlb* region, Ω_{zlb}. If the dynamics of the Ω_{zlb} region were extended to the entire π-x plane, the dynamics would be identical to those of the second stage analyzed above, with the phase diagram given in Figure 12. Inside Ω_{zlb}, the system always has saddle dynamics with saddles Υ_{zlb} and Ψ_{zlb}.

I now put the dynamics of the regions Ω_{ss} and Ω_{zlb} together and describe some of the global properties of the economy. The left panel of Figure 15 shows an example of the global dynamics in which the Taylor principle holds, while the right panel shows an example in

[6]See equation (A.15) in Appendix A.1 for the explicit formula.

which the Taylor principle does not hold. When the Taylor principle holds, the dynamics inside Ω_{ss} look like those in Figure 13. When the Taylor principle does not hold, they look like those in Figure 14. Of course, whether the Taylor principle holds or not, the dynamics in Ω_{zlb} always look like those in Figure 12, as they are not affected by the choice of ξ_x or ξ_π. However, the coefficients ξ_x and ξ_π do have a crucial effect on the ZLB, as they determine the location of the boundary $\partial\Omega$ and, consequently, whether the undesirable deflationary steady state can exist in the economy. If the Taylor principle holds, the *zlb* steady state (π_{zlb}, x_{zlb}) is in Ω_{zlb}; when the Taylor principle does not hold, it is not. To see this, compute

$$\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h = \xi_\pi (-r_h) + \xi_x \left(-\frac{\rho r_h}{\kappa}\right) + r_h, \tag{A.33}$$

$$= -\frac{r_h}{\kappa} \left(\kappa \left(\xi_\pi - 1\right) + \rho \xi_x\right). \tag{A.34}$$

By definition, the sign of this expression determines whether the *zlb* steady state (π_{zlb}, x_{zlb}) is inside or outside Ω_{zlb}. In turn, the sign of $\kappa(\xi_\pi - 1) + \rho\xi_x$ is determined by whether the Taylor principle holds. When the Taylor principle holds, (π_{zlb}, x_{zlb}) is a steady state of the global dynamics. Because of its saddle dynamics, equilibria are locally indeterminate around (π_{zlb}, x_{zlb}). Together with the intended steady state (π_{ss}, x_{ss}), they are the two global steady states of the economy. On the other hand, if the Taylor principle does not hold, (π_{zlb}, x_{zlb}) is in Ω_{ss}. Under the Ω_{ss} dynamics, the point (π_{zlb}, x_{zlb}) is not a steady state. In this case, the only steady state for the global dynamics is the desired one, (π_{ss}, x_{ss}).

The conclusion that following the Taylor principle outside the ZLB induces the existence of a deflationary steady state at the ZLB is similar to one of the results in Benhabib et al. (2001b). They further show that when the Taylor principle holds, the deflationary steady state engenders an infinite number of suboptimal equilibria. As mentioned before, these equilibria can start arbitrarily close to the intended steady state (π_{ss}, x_{ss}) and still converge to (π_{zlb}, x_{zlb}). The same possibility is present in the setup I consider here. To construct equilibria analogous to those in Benhabib et al. (2001b), I use the dynamics for the three stages described above. For the next steps, refer to Figure 16. First pick two numbers q and r such that $r, q > T$ and $r - q > T$. Let $(\pi_b, x_b) = \partial\Omega \cap \Upsilon_{zlb}$.[7] Assume the Taylor principle holds. Using $(\pi(r), x(r)) = (\pi_b, x_b)$ as the starting point, trace the dynamics of $(\pi(t), x(t))$ backward in time using the interest rate specified by equation (A.30) for a length of time q. As in Benhabib et al. (2001b), these equilibria can get arbitrarily close to the intended steady state: Because the dynamics of $(\pi(t), x(t))$ are unstable when going forward in time,

[7] If $\kappa\xi_\pi + \phi_1\xi_x = 0$, $\partial\Omega \cap \Upsilon_{zlb} = \emptyset$. Albeit not a general strategy to eliminate all non-optimal equilibria, picking ξ_x, ξ_π such that $\kappa\xi_\pi + \phi_1\xi_x = 0$ does preclude this particular class of equilibria from forming for any choice of f. This possibility was not present in Benhabib et al. (2001b), as their model did not have both inflation and output as state variables of the economy.

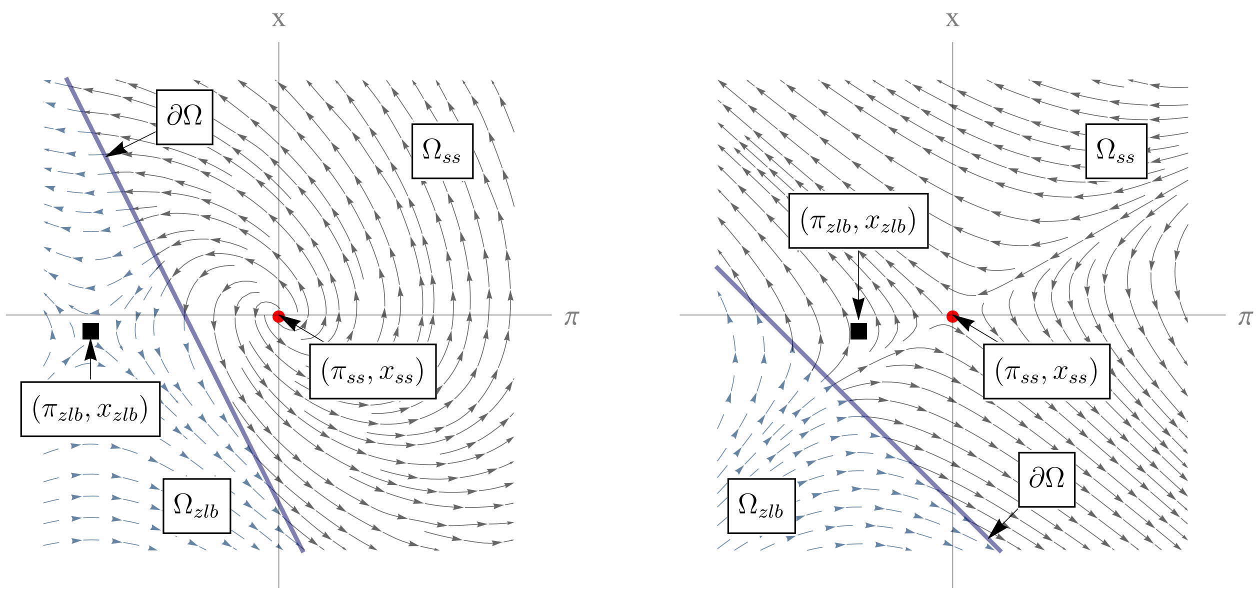

Figure 15: Non-linear dynamics of the economy after liftoff t_1. The central bank follows the Taylor rule $i(t) = \max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\}$. When $\xi_\pi \pi(t) + \xi_x x(t) + r_h > 0$, the economy is in the region Ω_{ss} and follows the solid black flow lines. When $\xi_\pi \pi(t) + \xi_x x(t) + r_h \leq 0$, it is in the region Ω_{zlb} and follows the dashed blue flow lines. The boundary between the two regions is the line $\partial\Omega$. The point (π_{ss}, x_{ss}), shown in red, is always a steady state of the economy. The point (π_{zlb}, x_{zlb}), shown as a black square, is a steady state of the economy if and only if the Taylor principle holds, as in the left panel. When the Taylor principle does not hold, as in the right panel, (π_{zlb}, x_{zlb}) is not in Ω_{zlb} and is therefore not a steady state.

they are stable backward in time and $(\pi(t), x(t))$ converges to (π_{ss}, x_{ss}) as $q \to \infty$.[8] At time $r - q$, trace the dynamics of $(\pi(t), x(t))$ backward in time using $(\pi(r-q), x(r-q))$ as the starting point and $i(t) = 0$ throughout, until $t = 0$, when the path reaches $(\pi(0), x(0))$. Of course, the natural rate is positive after T and negative before T, so the dynamics change from those of the second stage to those of the first. Note that in Figure 16, the gray flow lines in the background reflect the dynamics that prevail for $t \geq t_1$ only. Set $t_1 = r - q > T$. By construction, the path starting at $(\pi(0), x(0))$ reaches (π_b, x_b) at time r when following the interest rate rule in equation (15). Now going forward in time, for $t \geq r$, $(\pi(t), x(t)) \in \Upsilon_{zlb} \subset \Omega_{zlb}$, which means the economy travels on the *zlb* saddle path toward the unintended steady state (π_{zlb}, x_{zlb}). The path constructed is continuous and bounded and has consistent expectations: It is a rational expectations equilibrium. All equilibria in this class can be obtained by picking different q and r.

[8]This result is not immediate, since it may be possible that $(\pi(t), x(t))$ exits Ω_{ss} before getting close to (π_{ss}, x_{ss}) and then follows the Ω_{zlb} dynamics for which (π_{ss}, x_{ss}) is no longer a sink (flowing backward in time). However, I show in Appendix B.5, Lemma 2, item (d) that this never happens. For all q, the path of $(\pi(t), x(t))$ remains entirely in Ω_{ss}.

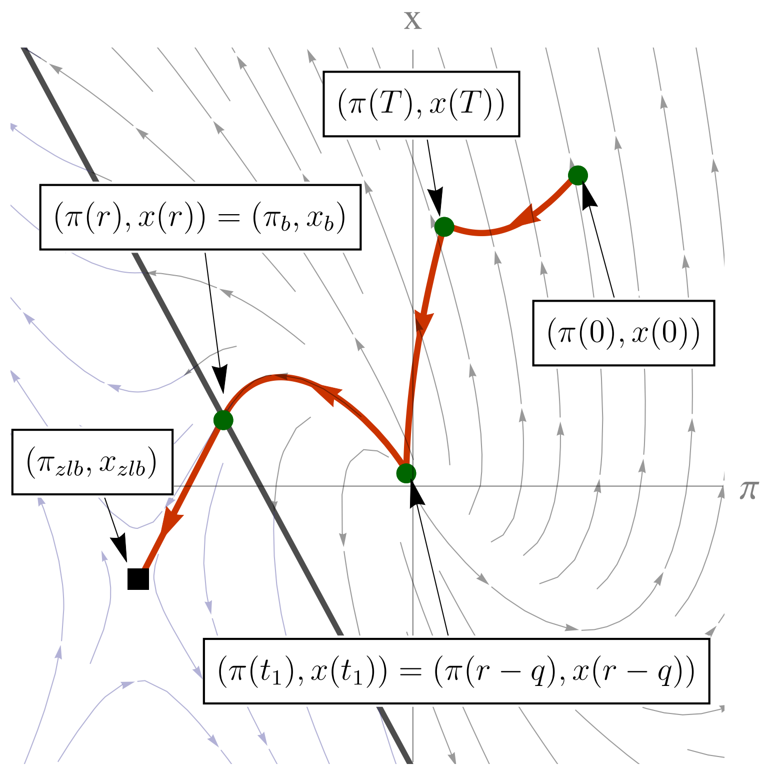

Figure 16: An equilibrium analogous to the one studied by Benhabib et al. (2001b). The flow lines in the background correspond to the dynamics after liftoff, which occurs at t_1. Because the Taylor principle holds, there is a deflationary steady state (π_{zlb}, x_{zlb}), shown as a black square. At time t_1, even though the economy is outside the ZLB and can get arbitrarily close to the "desired" steady state $(\pi_{ss}, x_{ss}) = (0, 0)$, it still converges to the "unintended" steady state (π_{zlb}, x_{zlb}). At time r, the economy enters the ZLB and stays there ($i(t)$=0) forever after.

A.3 Continuous Pasting

By using the explicit solutions in Appendix A.1, the continuous pasting conditions for a path $(\pi(t), x(t))$ with $R_{t_1} = \overline{R}$ and $f(R_{t_1}) = \overline{t_1}$ imply

$$\begin{aligned} x_0 &= \frac{\phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}}}{\phi_1 - \phi_2} x_1 - \frac{1}{\sigma} \frac{e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}}}{\phi_1 - \phi_2} \pi_1 \\ &+ \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 \overline{t_1}} - \phi_2^2 e^{-\phi_1 \overline{t_1}}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \end{aligned} \tag{A.35}$$

$$\begin{aligned} \pi_0 &= -\kappa \frac{e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}}}{\phi_1 - \phi_2} x_1 + \frac{\phi_1 e^{-\phi_1 \overline{t_1}} - \phi_2 e^{-\phi_2 \overline{t_1}}}{\phi_1 - \phi_2} \pi_1 \\ &+ r_h \frac{\phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l. \end{aligned} \tag{A.36}$$

Equations (A.35) and (A.36) are equivalent to continuous pasting at $\overline{t_1}$ if there already is continuous pasting at T. Solving for (π_1, x_1) in equations (A.35) and (A.36) gives, in matrix notation,

$$\left[\begin{array}{c} x_1 \\ \pi_1 \end{array}\right] = \frac{e^{(\phi_1+\phi_2)\overline{t_1}}}{\phi_1 - \phi_2} \left[\begin{array}{cc} \phi_1 e^{-\phi_1 \overline{t_1}} - \phi_2 e^{-\phi_2 \overline{t_1}} & \frac{1}{\sigma}\left(e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}}\right) \\ \kappa \left(e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}}\right) & \phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}} \end{array}\right] \left[\begin{array}{c} x(0) - h\left(\overline{t_1}\right) \\ \pi(0) - m\left(\overline{t_1}\right) \end{array}\right], \tag{A.37}$$

where

$$\begin{aligned} h\left(\overline{t_1}\right) &= r_h \frac{\phi_1^2 e^{-\phi_2 \overline{t_1}} - \phi_2^2 e^{-\phi_1 \overline{t_1}}}{\kappa\left(\phi_1 - \phi_2\right)} + \left(r_h - r_l\right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\kappa\left(\phi_1 - \phi_2\right)} - \frac{r_l \rho}{\kappa}, \\ m\left(\overline{t_1}\right) &= r_h \frac{\phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}}}{\left(\phi_1 - \phi_2\right)} + \left(r_h - r_l\right) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\left(\phi_1 - \phi_2\right)} - r_l. \end{aligned}$$

Solving for $e^{-\phi_1 \overline{t_1}}$ and $e^{-\phi_2 \overline{t_1}}$ in equation (A.37) and then eliminating $\overline{t_1}$ from one of the equations gives that paths that are already continuous at T are also continuous at $\overline{t_1}$ if and only if

$$0 = \mathcal{P}\left(\overline{R}\right), \tag{A.38}$$

$$\overline{t_1} = \mathcal{T}\left(\overline{R}\right), \tag{A.39}$$

where

$$\mathcal{P}(\overline{R}) = \begin{cases} 1\left\{d_{exit}\left(\overline{t_1}\right) \neq 0 \text{ or } d_{trap}\left(\overline{t_1}\right) \neq 0\right\} & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ d_{exit}\left(\overline{t_1}\right) + 1\left\{d_{trap}\left(\overline{t_1}\right) = 0\right\} & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ d_{trap}\left(\overline{t_1}\right) + 1\left\{d_{exit}\left(\overline{t_1}\right) = 0\right\} & , \text{ if } d_{exit}(0) \neq \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ \left(\frac{d_{exit}\left(\overline{t_1}\right)}{d_{exit}(0) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_1} - 1\right)}\right)^{\phi_2} & , \text{ if } d_{exit}(0) \neq \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ - \left(\frac{d_{trap}\left(\overline{t_1}\right)}{d_{trap}(0) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}\right)^{\phi_1} & \quad \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \end{cases}, \tag{A.40}$$

and

$$\mathcal{T}(\overline{R}) = \begin{cases} [T,\infty) & , \quad \text{if } d_{exit}(0) = \frac{\phi_2}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_2}\right) \\ \frac{1}{\phi_2}\log\frac{d_{trap}(\overline{t_1})}{d_{trap}(0)+\frac{\phi_1}{\kappa}(r_h-r_l)\left(e^{-T\phi_2}-1\right)} & , \quad \text{if } d_{exit}(0) = \frac{\phi_2}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_2}\right) \\ \frac{1}{\phi_1}\log\frac{d_{exit}(\overline{t_1})}{d_{exit}(0)+\frac{\phi_2}{\kappa}(r_h-r_l)\left(e^{-T\phi_2}-1\right)} & , \quad \text{otherwise} \end{cases}, \tag{A.41}$$

In the expression for $\mathcal{P}\left(\overline{R}\right)$, I have used the indicator function $1\{E\}$, which is equal to 1 if E is true and zero otherwise.

To derive equations (A.38) and (A.39) and to find the explicit expressions for $\mathcal{P}$ and $\mathcal{T}$ shown in equations (A.40)-(A.41), I consider four cases separately.

The first case corresponds to the economy reaching (π_{zlb}, x_{zlb}) at $\overline{t_1}$; the second and third, to the economy reaching, respectively, Υ_{zlb} and Ψ_{zlb} at $\overline{t_1}$; the fourth case considers all remaining $\overline{R}$.

The first case, shown in green in Figure 17, is defined by (π_0, x_0) such that[9]

$$d_{exit}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_1}{\kappa}[\pi_0 - \pi_{zlb}] = \frac{\phi_2}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_1}\right), \tag{A.42}$$

$$d_{trap}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_2}{\kappa}[\pi_0 - \pi_{zlb}] = \frac{\phi_1}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_2}\right). \tag{A.43}$$

In the figure, the line defined by equation (A.42) is the black dashed line while the line defined by equation (A.43) is the dashed gray line. This first case corresponds to (π_0, x_0) at the intersection of these two lines. The economy reaches the *zlb* steady-state (π_{zlb}, x_{zlb}) at $t = T$. Since between T and $\overline{t_1}$ the point (π_{zlb}, x_{zlb}) is a steady-state, the economy just sits there for all $t \in [T, \overline{t_1})$. The continuous pasting conditions are

$$d_{exit}(\pi(t), x(t)) = d_{trap}(\pi(t), x(t)) = 0, \tag{A.44}$$

$$d_{exit}(\pi_1, x_1) = d_{trap}(\pi_1, x_1) = 0, \tag{A.45}$$

i.e., $(\pi(t), x(t)) = (\pi_1, x_1) = (\pi_{zlb}, x_{zlb})$. In Figure 17, the lines described in equations (A.44) and (A.45) are shown in the solid black and gray lines, and correspond to Υ_{zlb} and Ψ_{zlb}.

[9] With slight abuse of notation, in this section I write $d_{exit}(\pi_0, x_0)$ instead of $d_{exit}(t)$ to emphasize that $d_{exit}(\pi_0, x_0)$ is not a function of time since (π_0, x_0) is a vector of two numbers, as opposed to $(\pi(0), x(0))$, which is a function of time evaluated at $t = 0$. The same notation applies to d_{trap} and to π_T, x_T, π_1, x_1.

Equations (A.44) and (A.45) define the function $\mathcal{P}$ for this case

$$\mathcal{P}(\overline{R}) = 1\left\{d_{exit}\left(\pi_1, x_1\right) \neq 0 \text{ or } d_{trap}\left(\pi_1, x_1\right) \neq 0\right\},$$

We then have $\mathcal{P}\left(\overline{R}\right) = 0$ if and only if $d_{trap}\left(\pi_1, x_1\right)$ and $d_{trap}\left(\pi_1, x_1\right)$ are both zero. Graphically, the set of $\overline{R}$ such that $\mathcal{P}(\overline{R}) = 0$ are the two points in Figure 17 where the lines intersect, that is, where the green path begins and ends. Once (A.44) and (A.45) hold, any $\overline{t_1} \geq T$ is consistent with continuous pasting and thus

$$\mathcal{T}(\overline{R}) = [T, \infty).$$

In the three remaining cases, $\mathcal{T}(\overline{R})$ is single-valued and depends on $\overline{R}$.

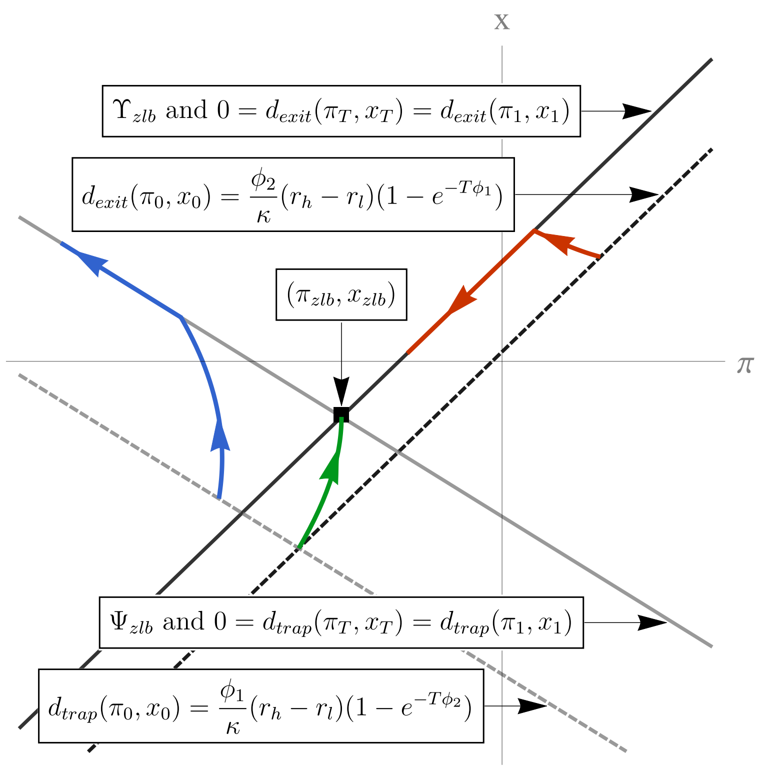

Figure 17: Continuous pasting conditions when (π_1, x_1) is in Υ_{zlb} or Ψ_{zlb}. Continuous pasting requires that paths that are on the *zlb* saddle path (the black solid line) at times T and t_1, such as the red one, be on the $d_{exit}\left(\pi_0, x_0\right) = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right)$ line (the black dashed line) at time $t = 0$. For these paths, expectations of falling into the deflationary equilibrium (π_{zlb}, x_{zlb}) are self-fulfilling. To be continuous, paths like the blue that start on the $d_{trap}\left(\pi_0, x_0\right) = \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right)$ line (the gray dashed line) at $t = 0$ must be on Ψ_{zlb} at times T and t_1. These paths escape the deflationary trap. The green path originates at the intersection of the two dashed lines. Continuous pasting requires that it reaches the deflationary steady-state at T.

The second case is defined by the economy reaching some point in the *zlb* saddle path

Υ_{zlb} at $\overline{t_1}$, except for (π_{zlb}, x_{zlb}), which was already analyzed. This case, shown in red in Figure 17, occurs when

$$\begin{aligned} d_{exit}\left(\pi_0, x_0\right) &= \left[x_0 - x_{zlb}\right] - \frac{\phi_1}{\kappa}\left[\pi_0 - \pi_{zlb}\right] = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right), && \text{(A.46)} \\ d_{trap}\left(\pi_0, x_0\right) &= \left[x_0 - x_{zlb}\right] - \frac{\phi_2}{\kappa}\left[\pi_0 - \pi_{zlb}\right] \neq \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right). && \text{(A.47)} \end{aligned}$$

Continuous pasting at T requires

$$\begin{aligned} d_{exit}\left(\pi(t), x(t)\right) &= \left[x(t) - x_{zlb}\right] - \frac{\phi_1}{\kappa}\left[\pi(t) - \pi_{zlb}\right] = 0, && \text{(A.48)} \\ T &= \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_T, x(t)\right)}{d_{trap}\left(\pi_0, x_0\right) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}\right). && \text{(A.49)} \end{aligned}$$

Continuous pasting at $\overline{t_1}$ requires

$$\begin{aligned} d_{exit}\left(\pi_1, x_1\right) &= \left[x_1 - x_{zlb}\right] - \frac{\phi_1}{\kappa}\left[\pi_1 - \pi_{zlb}\right] = 0, && \text{(A.50)} \\ \overline{t_1} &= T + \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_1, x_1\right)}{d_{trap}\left(\pi(t), x(t)\right)}\right). && \text{(A.51)} \end{aligned}$$

Equations (A.46), (A.47), (A.48) and (A.50) describe the continuous pasting constraints on (π_0, x_0) without any reference to $\overline{t_1}$. Combinations (π_0, x_0) that satisfy these equations can be part of a continuous path for some $\overline{t_1}$. Equations (A.49) and (A.51) then show which particular point is reachable with a specific $\overline{t_1}$. Any continuous path in this case must start in the black dashed line of Figure 17 and be on the solid black line at times T and $\overline{t_1}$. For a specific $\overline{t_1}$, or for a specific point in one of the two lines, only one path is continuous.

Combining equations (A.46)-(A.51) gives

$$\begin{aligned} \mathcal{P}(\overline{R}) &= d_{exit}\left(\pi_1, x_1\right) + 1\left\{d_{trap}\left(\pi_1, x_1\right) = 0\right\}, \\ \mathcal{T}(\overline{R}) &= \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_1, x_1\right)}{d_{trap}\left(\pi_0, x_0\right) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}\right). \end{aligned}$$

The indicator $1\left\{d_{trap}\left(\pi_1, x_1\right) = 0\right\}$ in the equation for $\mathcal{P}$ is there to guarantee that equation (A.46) holds. In Figure 17, the points $\overline{R}$ such that $\mathcal{P}(\overline{R}) = 0$ are given by the dashed and solid black lines, with the exception of the points where the black lines intersect the gray lines.

The third case is similar to the second and is represented by the blue line in Figure 17. Instead of reaching Υ_{zlb} at $\overline{t_1}$, the economy reaches the unstable *zlb* saddle path Ψ_{zlb} at $\overline{t_1}$, with the exception of (π_{zlb}, x_{zlb}), which was already studied. This case is defined by

$$d_{exit}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_1}{\kappa}[\pi_0 - \pi_{zlb}] \neq \frac{\phi_2}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_1}\right), \tag{A.52}$$

$$d_{trap}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_2}{\kappa}[\pi_0 - \pi_{zlb}] = \frac{\phi_1}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_2}\right). \tag{A.53}$$

Continuous pasting at T occurs if and only if

$$d_{trap}(\pi(t), x(t)) = [x(t) - x_{zlb}] - \frac{\phi_2}{\kappa}[\pi(t) - \pi_{zlb}] = 0, \tag{A.54}$$

$$T = \frac{1}{\phi_1}\log\left(\frac{d_{exit}(\pi_T, x(t))}{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)}\right), \tag{A.55}$$

while continuous pasting at t_1 occurs if and only if

$$d_{trap}(\pi_1, x_1) = [x_1 - x_{zlb}] - \frac{\phi_2}{\kappa}[\pi_1 - \pi_{zlb}] = 0, \tag{A.56}$$

$$\overline{t_1} = T + \frac{1}{\phi_1}\log\left(\frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi(t), x(t))}\right). \tag{A.57}$$

It follows that

$$\mathcal{P}(\overline{R}) = d_{trap}(\pi_1, x_1) + 1\{d_{exit}(\pi_1, x_1) = 0\},$$

$$\mathcal{T}(\overline{R}) = \frac{1}{\phi_1}\log\left(\frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)}\right).$$

The fourth and last case corresponds to all remaining choices for $\overline{R}$ that can be part of a continuous path. The continuous pasting conditions are

$$\left(\frac{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa}(r_h - r_l)\left(e^{-T\phi_2} - 1\right)}{d_{trap}(\pi(t), x(t))}\right)^{\phi_1} = \left(\frac{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{d_{exit}(\pi(t), x(t))}\right)^{\phi_2}, \tag{A.58}$$

$$T = \frac{1}{\phi_2}\log\frac{d_{trap}(\pi(t), x(t))}{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)}, \tag{A.59}$$

for T and

$$\left(\frac{d_{trap}\left(\pi_1, x_1\right)}{d_{trap}\left(\pi_T, x(t)\right)}\right)^{\phi_1} = \left(\frac{d_{exit}\left(\pi_1, x_1\right)}{d_{exit}\left(\pi(t), x(t)\right)}\right)^{\phi_2}, \tag{A.60}$$

$$t_1 = T + \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_1, x_1\right)}{d_{trap}\left(\pi(t), x(t)\right)}\right), \tag{A.61}$$

for $\overline{t_1}$.

Assuming continuous pasting at T, equations (A.58)-(A.60) reveal the set of points (π_0, x_0, π_1, x_1) that can be reached through continuous paths for some $\overline{t_1}$, which give

$$\mathcal{P}(\overline{R}) = \left(\frac{d_{exit}\left(\pi_1, x_1\right)}{d_{exit}\left(\pi_0, x_0\right) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_1} - 1\right)}\right)^{\phi_2} - \left(\frac{d_{trap}\left(\pi_1, x_1\right)}{d_{trap}\left(\pi_0, x_0\right) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}\right)^{\phi_1} + 1\left\{d_{exit}\left(\pi_1, x_1\right) = 0 \text{ or } d_{trap}\left(\pi_1, x_1\right) = 0\right\}.$$

Finally, equations (A.59) and (A.61) give

$$\mathcal{T}\left(\overline{R}\right) = \frac{1}{\phi_1}\log\frac{d_{exit}\left(\overline{t_1}\right)}{d_{exit}\left(0\right) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}.$$

B Proofs of Propositions

B.1 Proof of Proposition 1

Proof. Assume that rule (11) implements the optimal equilibrium. I show that it also implements a different second equilibrium. Let $R_0^* = (\pi^*(0), x^*(0))$. Because the rule (11) implements the optimal equilibrium and $\max\left\{0, \xi_\pi(R_0^*)\pi^*(t^*) + \xi_x(R_0^*)x^*(t^*) + r_h\right\} > 0$, we have $\underline{t} = t^*$.

Consider the path that starts at $(\pi(0), x(0))$ and reaches $(\pi_{ss}, x_{ss}) = (0, 0)$ at $t = t^*$ when following (1), (2) and (11). This path always exists, since we can find it by positioning the economy on $(0, 0)$ at $t = t^*$ and running time backward until $t = 0$ using $i(t) = 0$ throughout. Since the point $(0, 0)$ is a steady state after t^* for any choice of $\xi_\pi(R_0)$ and $\xi_x(R_0)$, $(\pi(t), x(t))$ remains bounded. If $\kappa\sigma\lambda \neq 1$, $\left(x\left(\underline{t}\right), \pi\left(\underline{t}\right)\right) = (\pi(t^*), x(t^*)) = (0, 0) \neq (\pi^*(t^*), x^*(t^*))$ (see Werning (2012) for a proof that $(0, 0) \neq (\pi^*(t^*), x^*(t^*))$ when $\kappa\sigma\lambda \neq 1$). Hence, when $\kappa\sigma\lambda \neq 1$, the path that starts at the $(\pi(0), x(0))$ that reaches $(0, 0)$ at $t = t^*$ when $i(t) = 0$ between $t = 0$ and $t = t^*$ constitutes an equilibrium different from the optimal one for any choice of functions $\xi_\pi(R_0)$ and $\xi_x(R_0)$. When $\kappa\sigma\lambda = 1$, the optimal equilibrium happens to have $(x^*(t), \pi^*(t)) = (0,0)$ for all $t \geq t^*$ and the optimal equilibrium is indeed implementable as the unique equilibrium (Appendix B.2 shows how). $\square$

B.2 Case $\kappa\sigma\lambda = 1$ in Proposition 1

The next example shows that when $\kappa\sigma\lambda = 1$, it is indeed possible to implement the optimal equilibrium uniquely with a constant liftoff time. Let $\underline{t}= t^*$ and $\kappa\sigma\lambda = 1$. Pick

$$(\xi_\pi(R_0), \xi_x(R_0)) = \begin{cases} (0,0) & , \text{ if } R_0 \text{ is such that } (\pi(\underline{t}), x(\underline{t})) = (0,0) \text{ or } \rho\pi(\underline{t}) \neq \kappa x(\underline{t}) \\ \left(1, -\frac{\rho}{\kappa}\right) & , \text{ otherwise} \end{cases} \tag{B.1}$$

Note that when the time of liftoff is constant, it is equivalent to write ξ_π and ξ_x as a function of R_0 or as a function of $(\pi(s), x(s))$ for any $s > 0$. Hence, equation (B.1) can be written as

$$(\xi_\pi, \xi_x) = \begin{cases} (0,0) & , \text{ if } (x(\underline{t}), \pi(\underline{t})) = (0,0) \text{ or } \rho\pi(\underline{t}) \neq \kappa x(\underline{t}) \\ \left(1, -\frac{\rho}{\kappa}\right) & , \text{ otherwise} \end{cases}. \tag{B.2}$$

I now show that the rule

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < \underline{t} \\ \max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r(t)\} & , \quad \underline{t} \leq t < \infty \end{cases} \tag{B.3}$$

implements the optimal equilibrium as the unique equilibrium of the economy.

When $(\pi(\underline{t}), x(\underline{t})) = (0,0)$, the rule implements the optimal path. Werning (2012) shows that when $\kappa\sigma\lambda = 1$, $(\pi^*(t^*), x^*(t^*)) = (0,0)$. Since $\underline{t}= t^*$ and $(\pi(\underline{t}), x(\underline{t})) = (\pi^*(t^*), x^*(t^*)) = (0,0)$, $(\pi(t), x(t)) = (\pi^*(t), x^*(t))$ for $t < t^*$. By equation (B.2), $\xi_x = \xi_\pi = 0$ and thus $i(t) = i^*(t) = r_h > 0$ for $t \geq t_1$. As $(0,0)$ is a steady state, $(\pi(t), x(t)) = (0,0)$ for all $t \geq t_1$, which shows that $(\pi(t), x(t)) = (\pi^*(t), x^*(t))$ for $t \geq t_1$.

No other equilibrium exists since, for all $R_0 \neq (\pi^*(0), x^*(0))$, continuous paths are unbounded. If $\rho\pi(\underline{t}) \neq \kappa x(\underline{t})$, equation (B.2) gives $(\xi_x, \xi_\pi) = (0,0)$ and, by equation (A.15), the saddle path is $\rho\pi = \kappa x$. It follows that $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$. In addition, $i(t) = r_h > 0$ for $t \geq\underline{t}$ and thus $(\pi(t), x(t)) \in \Omega_{ss}$ for all $t \geq\underline{t}$ since Ω_{zlb} is empty. The global saddle path dynamics and $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$ imply that $(\pi(t), x(t))$ explodes as $t \to \infty$.

If $(\pi(\underline{t}), x(\underline{t})) \neq (0,0)$ and $\rho\pi(\underline{t}) = \kappa x(\underline{t})$, $(\xi_x, \xi_\pi) = \left(1, -\frac{\rho}{\kappa}\right)$ implies that the Taylor principle does not hold, since $\kappa(\xi_\pi - 1) + \rho\xi_x = -\kappa < 0$. In addition, $(\pi(\underline{t}), x(\underline{t})) \in \Omega_{ss}$ since $\xi_\pi \pi(\underline{t}) + \xi_x x(\underline{t}) + r(\underline{t}) = r_h > 0$ and $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$ by equation (A.15). Because the dynamics are saddle path stable and $(\pi(\underline{t}), x(\underline{t}))$ is not on the saddle path, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb} in finite time. By item (c) in Lemma 1 of Appendix B.4, if $(\pi(t), x(t))$ enters Ω_{zlb} it also explodes.

B.3 Constants in the Neo-Fisherian Rule of Section 5.1

To be on the saddle path at time t_1, $\pi(t_1) = \phi x\,(t_1)$. Using the continuous pasting conditions in equation (A.37) to express $\pi(t_1) = \phi x\,(t_1)$ in terms of $x(0)$ and $\pi(0)$ gives

$$p(t_1)x(0) + q(t_1)\pi\,(0) = v(t_1),$$

where

$$\begin{aligned}
p(t_1) &= \kappa\sigma\left(\left(\phi_1 - \kappa\phi\right)e^{-\phi_1 t_1} - \left(\phi_2 - \kappa\phi\right)e^{-\phi_2 t_1}\right), \\
q(t_1) &= \kappa\left(\left(\sigma\phi\phi_2 + 1\right)e^{-\phi_1 t_1} - \left(\sigma\phi\phi_1 + 1\right)e^{-\phi_2 t_1}\right), \\
v(t_1) &= -\left(\left(\kappa + \sigma\phi_1\left(\rho - \kappa\phi\right)\right)r_l + \frac{\left(r_h - r_l\right)\phi_1}{\phi_1 - \phi_2}\left(\kappa + \sigma\phi_1^2 - \kappa\sigma\phi\left(\phi_1 - \phi_2\right)\right)e^{-T\phi_2}\right)e^{-\phi_1 t_1} \\
&\quad + \left(\left(\kappa + \sigma\phi_2\left(\rho - \kappa\phi\right)\right)r_l - \frac{\left(r_h - r_l\right)\phi_2}{\phi_1 - \phi_2}\left(\kappa + \sigma\phi_2^2 + \kappa\sigma\phi\left(\phi_1 - \phi_2\right)\right)e^{-T\phi_1}\right)e^{-\phi_2 t_1} \\
&\quad + \frac{\left(\sigma\rho^2 + 4\kappa\right)\left(\rho - \kappa\phi\right)r_h}{\left(\phi_1 - \phi_2\right)}e^{-\phi_1 t_1}e^{-\phi_2 t_1}.
\end{aligned}$$

Evaluating these functions at $t_1 = t^*$ and $t_1 = t^* + 1$ determines the constants A, B, C and D, E, F, respectively.

B.4 Proof of Proposition 2

I first prove a lemma and then proceed to the proof of Proposition 2.

Lemma 1. *When the Taylor principle does not hold, the following are true:*

(a) $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$.

(b) If $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q \geq t_1$*,* $(\pi(t), x(t))$ *either explodes as* $t \to \infty$ *or exits* Ω_{zlb} *in finite time.*

(c) If $(\pi(s), x(s)) \in \Omega_{ss}$ *for some* $s \geq t_1$ *and* $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q > s$*, then* $(\pi(t), x(t))$ *explodes as* $t \to \infty$.

(d) If $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q \geq t_1$ *and there exist no* $r \geq q$ *such that* $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$*, then* $(\pi(t), x(t))$ *explodes as* $t \to \infty$.

(e) If $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q \geq t_1$ *and there exist* $r > q$ *such that* $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$*, then there exist no* $t \in [q, r)$ *such that* $(\pi(t), x(t)) \in \Upsilon_{ss}$.

Proof of Lemma 1. (a) Plugging the steady-state from equation (A.20) into the Taylor rule gives

$$\begin{aligned}
\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h &= \xi_\pi \left(-r_h\right) + \xi_x \left(-\frac{\rho}{\kappa} r_h\right) + r_h, \\
&= -\frac{r_h}{\kappa}\left(\kappa\left(\xi_\pi - 1\right) + \rho \xi_x\right), \\
&= -\frac{r_h \sigma}{\kappa} \det A_{ss}, \\
&> 0.
\end{aligned}$$

where $\det A_{ss} < 0$ because the Taylor principle does not hold.

(b) The saddle path stable dynamics in Ω_{zlb} and $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$ immediately imply that paths starting in Ω_{zlb} either explode or exit Ω_{zlb} in finite time.

(c) Let $\hat{n}$ be a unit vector normal to $\partial\Omega$ pointing towards Ω_{ss}. Because $(\pi(t), x(t))$ transitions from Ω_{ss} to Ω_{zlb} and its path is continuous, there exist $\omega \in (s, q]$ such that

$$\xi_\pi \pi(\omega) + \xi_x x(\omega) + r_h = 0, \tag{B.4}$$

$$\hat{n} \cdot \left(\dot{\pi}(\omega), \dot{x}(\omega)\right) \leq 0. \tag{B.5}$$

Equation (B.4) says that $(\pi(\omega), x(\omega)) \in \partial\Omega$. The non-positive dot product in equation (B.5) says that $(\pi(\omega), x(\omega))$ is not moving towards Ω_{ss} (and moving towards Ω_{zlb} when the dot product is negative). Writing out the dot product gives

$$\begin{bmatrix} \frac{\xi_x}{\sqrt{\xi_\pi^2+\xi_x^2}} \\ \frac{\xi_\pi}{\sqrt{\xi_\pi^2+\xi_x^2}} \end{bmatrix}^T \begin{bmatrix} -\frac{1}{\sigma}\left(\pi(\omega) + r_h\right) \\ \rho\pi(\omega) - \kappa x(\omega) \end{bmatrix} = -\frac{\pi(\omega)\left(\xi_x - \sigma\rho\xi_\pi\right) + \xi_x r_h + \kappa\sigma\xi_\pi x(\omega)}{\sigma\sqrt{\xi_\pi^2 + \xi_x^2}} \leq 0,$$

or, simplifying,

$$\pi(\omega)\left(\xi_x - \sigma\rho\xi_\pi\right) + \xi_x r_h + \kappa\sigma\xi_\pi x(\omega) \geq 0. \tag{B.6}$$

The Taylor principle (TP) not holding, equation (B.4), equation (B.6) and $\phi_1, r_h > 0$ imply that

$$\begin{aligned}
&- r_h \underbrace{\left(\kappa\left(\xi_\pi - 1\right) + \rho\xi_x\right)}_{<0 \text{ as TP does not hold}} - \kappa \underbrace{\left(r_h + \xi_\pi \pi(\omega) + \xi_x x(\omega)\right)}_{=0 \text{ by eq. (B.4)}} \\
&\qquad + \phi_1 \underbrace{\left(\pi(\omega)\left(\xi_x - \sigma\rho\xi_\pi\right) + \xi_x r_h + \kappa\sigma\xi_\pi x(\omega)\right)}_{\geq 0 \text{ by eq. (B.6)}} > 0.
\end{aligned} \tag{B.7}$$

Equation (B.7) is a sufficient condition for $\big(\pi_(t), x_(t)\big)$ to be in Ω_{zlb} for all $t \geq \omega$. To see this, use the dynamics in equations (A.7) and (A.8) to write

$$\xi_\pi \pi(t) + \xi_x x(t) + r_h = W(t - \omega),$$

where

$$\begin{aligned} W(t) &= Ae^{\phi_1(t-\omega)} + Be^{\phi_2(t-\omega)} + C, \\ A &= -\left(\phi_1 \pi(\omega) - \kappa x(\omega) - \phi_2 r_h\right) \frac{\xi_x - \sigma\phi_1\xi_\pi}{\sigma\phi_1\left(\phi_1 - \phi_2\right)}, \\ B &= \left(\phi_2 \pi(\omega) - \kappa x(\omega) - \phi_1 r_h\right) \frac{\xi_x - \sigma\phi_2\xi_\pi}{\sigma\phi_2\left(\phi_1 - \phi_2\right)}, \\ C &= \frac{r_h}{\kappa}\left(\kappa\left(1 - \xi_\pi\right) - \rho\xi_x\right). \end{aligned}$$

By definition, $(\pi(t), x(t)) \in \Omega_{zlb}$ iff $W(t) \leq 0$. Therefore, if $W(t)$ has no zeros for $t > \omega$, $(\pi(t), x(t))$ remains in Ω_{zlb} forever. Since $\phi_2 < 0 < \phi_1$ and $W(\omega) = 0$ by (B.4) and $W'(\omega) \leq 0$ by (B.5), a sufficient condition for $W(u)$ to have no zeros for $u > \omega$ is that $A < 0$. After some manipulations, it can be seen that (B.7) is equivalent to $A < 0$.

By (b), since $(\pi(t), x(t))$ never transitions to Ω_{ss} after ω, it follows that $(\pi(t), x(t))$ explodes as $t \to \infty$.

(d) By (b), if $(\pi(t), x(t))$ does not exit Ω_{zlb}, it explodes. If $(\pi(t), x(t))$ exits Ω_{zlb} at some time η and $(\pi(\eta), x(\eta))$ is not on the ss saddle path, due to the saddle path dynamics inside Ω_{ss}, $(\pi(t), x(t))$ either explodes or returns to Ω_{zlb} in finite time. If $(\pi(t), x(t))$ returns to Ω_{zlb}, by item (c), it explodes.

(e) By equation (A.15), Υ_{ss} is a line through the origin, which can be written as $A\pi - x = 0$ with $A \neq 0$. If $\xi_\pi \neq 1$ then $A = \left(1 - \xi_\pi\right) / \left(\xi_x - \sigma\alpha_2\right)$ and if $\xi_\pi = 1$ then $A = \left(\sigma\rho - \xi_x\right) / \kappa\sigma$. Let

$$F(t) = A\pi(t) - x(t). \tag{B.8}$$

The path of $(\pi(t), x(t))$ intersects Υ_{ss} at some time $\bar{t}$ iff $F\left(\bar{t}\right) = 0$.

Plugging $\xi_\pi \pi(t) + \xi_x x(t) + r_h = 0$ in the IS and NKPC after t_1, it can be seen that $(\dot{\pi}(t), \dot{x}(t))$ is continuous on $\partial\Omega$ for all $t \geq t_1$. It follows that the right and left derivatives of $F(t)$ are equal at $t = r$. By equations (A.9)-(A.14) and (A.15), $(\pi(t), x(t))$ remains on Υ_{ss} after intersecting it at $t = r$. Hence, the right derivative of $F(t)$ at $t = r$ is zero.

Now I find the left derivative. If $(\pi(t), x(t))$ exits Ω_{zlb} without intersecting $\partial\Omega \cap \Upsilon_{ss}$, by uniqueness inside Ω_{ss}, it won't intersect $\Upsilon_{ss} \cap \Omega_{ss}$. If $(\pi(t), x(t))$ re-enters Ω_{zlb}

after being in $\Omega_{ss} \setminus \Upsilon_{ss}$, by item (c), it explodes. Hence, between times q and r, $(\pi(t), x(t)) \in \Omega_{zlb}$ and its dynamics are given by equations (A.7) and (A.8).

Using these dynamics in equation (B.8) gives

$$F(t) = Pe^{\phi_1(t-\tau(t))} + Qe^{\phi_2(t-\tau(t))} + R, \tag{B.9}$$

where

$$\begin{aligned} P &= \frac{\phi_2 - A\kappa}{\kappa(\phi_1 - \phi_2)} (\phi_2 r_h - \phi_1 \pi(\tau(t)) + \kappa x(\tau(t))), \\ Q &= -\frac{\phi_1 - A\kappa}{\kappa(\phi_1 - \phi_2)} (\phi_1 r_h - \phi_2 \pi(\tau(t)) + \kappa x(\tau(t))), \\ R &= \frac{r_h(\rho - A\kappa)}{\kappa}. \end{aligned}$$

Note that P and Q cannot both be zero. Indeed, $P = Q = 0$ implies $R = 0$, since $F(r) = 0$. But $R = 0$ implies $A = \rho/\kappa$, which in turn implies $(x(\tau(t)), \pi(\tau(t))) = (\pi_{zlb}, x_{zlb}) = (x(q), \pi(q)) \in \Omega_{zlb}$, contradicting item (a). Thus, P and Q cannot both be zero.

Using equation (B.9) to compute the left derivative of $F(t)$ at $t = r$ and setting it equal to the value of the right derivative, which is zero as shown above, gives

$$F'(r) = 0 = \phi_1 Pe^{\phi_1(r-\tau(r))} + \phi_2 Qe^{\phi_2(r-\tau(r))}. \tag{B.10}$$

In other words, the path for $(\pi(t), x(t))$ must be tangent to the ss saddle path Υ_{ss} at $t = r$. Since $\phi_2 < 0 < \phi_1$, equation (B.10) implies that P and Q have the same sign (and the sign is not zero since P and Q cannot both be zero). In turn, P and Q having the same (non-zero) sign implies that

$$F''(t) = \phi_1^2 Pe^{\phi_1(t-\tau(t))} + \phi_2^2 Qe^{\phi_2(t-\tau(t))} \tag{B.11}$$

has the same (non-zero) sign for all $t \in [q, r]$, so $F'(t)$ is strictly monotonic. A continuous and strictly monotonic $F'(t)$ in $t \in [q, r]$, together with $F(r) = F'(r) = 0$, imply that the only solution to $F(t) = 0$ for $t \in [q, r]$ is r. $\square$

Proof of Proposition 2. Consider the following condition:

$$\begin{gathered} (\pi(t_1), x(t_1)) \in \Omega_{ss} \cap \Upsilon_{ss}, \\ \text{or} \\ (\pi(t_1), x(t_1)) \in \Omega_{zlb} \text{ and } (\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss} \text{ for some } r \in [t_1, \infty). \end{gathered} \tag{B.12}$$

I first show that condition (B.12) implies paths are not explosive. If $(\pi(t_1), x(t_1)) \in \Omega_{ss} \cap \Upsilon_{ss}$, then $(\pi(t), x(t)) \in \Upsilon_{ss}$ for all $t \geq t_1$. If $(x(t_1), \pi(t_1)) \in \Omega_{zlb}$ and $(x(r), \pi(r)) \in \partial\Omega \cap \Upsilon_{ss}$ for some $r \in [t_1, \infty)$, $(\pi(t), x(t)) \in \Upsilon_{ss}$ for all $t \geq r$. In either case, the path converges to $(0, 0)$ and therefore does not explode.

To prove the converse, I prove the contrapositive. There are two cases to consider.

Case 1: If $(\pi(t_1), x(t_1)) \notin \Omega_{ss}$ and there exist no $r \in [t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$, then $(\pi(t), x(t))$ explodes by item (e) of Lemma 1.

Case 2: If $(\pi(t_1), x(t_1)) \notin \Upsilon_{ss}$ and $(\pi(t_1), x(t_1)) \notin \Omega_{zlb}$, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb}. If it enters Ω_{zlb}, it explodes by item (c) of Lemma 1.

Note that cases 1 and 2 above also cover the case in which $(\pi(t_1), x(t_1)) \notin \Upsilon_{ss}$ and there exists no $r \in [t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{ss}$. Indeed, if $(\pi(t_1), x(t_1)) \notin \Omega_{ss}$, case 1 applies. And if $(\pi(t_1), x(t_1)) \notin \Omega_{zlb}$, case 2 applies. □

B.5 Proof of Proposition 3

I first prove a lemma and then proceed to the proof of Proposition 3.

Lemma 2. *When the Taylor principle holds, the following are true:*

(a) $(\pi_{zlb}, x_{zlb}) \in \Omega_{zlb}$.

(b) If $(\pi(m), x(m)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ *with* $m \geq T$, *then* $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ *for all* $t \geq m$.

(c) There exist $(\pi(0), x(0))$ *such that* $(\pi(T), x(T)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.

(d) If $(\pi(s), x(s)) \in \partial\Omega$ *for some* $s \geq t_1$, *there is no* $p > 0$ *such that* $(\pi(t), x(t)) \in \Omega_{ss}$ *for* $t \in (s, s+p)$ *and* $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$.

(e) If $(\pi(q), x(q)) \in \Omega_{ss}$ *for* $q \geq t_1$ *with* $(\pi(q), x(q)) \neq (\pi_{ss}, x_{ss})$ *and there is no* $p > 0$ *such that* $[(\pi(t), x(t)) \in \Omega_{ss}$ *for* $t \in (q, q+p)$ *and* $(\pi(q+p), x(q+p)) \in \Upsilon_{zlb} \cap \partial\Omega]$, *then* $(\pi(t), x(t))$ *explodes as* $t \to \infty$.

(f) There is no chaos (in the sense of R. Devaney[10]).

Proof of Lemma 2. (a) Plugging the steady-state (A.20) into the Taylor rule gives

$$\begin{aligned}
\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h &= \xi_\pi (-r_h) + \xi_x \left(-\frac{\rho}{\kappa} r_h\right) + r_h, \\
&= -\frac{1}{\kappa} r_h \left(\kappa (\xi_\pi - 1) + \rho \xi_x\right), \\
&= -\frac{r_h \sigma}{\kappa} \det A_{ss}, \\
&< 0.
\end{aligned}$$

[10] See Banks, Brooks, Cairns, Davis, and Stacey (1992) for a definition.

where $\det A_{ss} > 0$ because the Taylor principle holds.

(b) Let (x, π) be a point in the line segment with endpoints $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}), i.e. (x, π) is in the portion of the *zlb* saddle path between $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}). Then

$$(x, \pi) = a(\pi(m), x(m)) + (1-a)(\pi_{zlb}, x_{zlb}),$$

for some $a \in [0, 1]$. It follows that

$$\begin{aligned} \xi_\pi \pi + \xi_x x + r_h &= \xi_\pi (a\pi(m) + (1-a)\pi_{zlb}) + \xi_x (ax(m) + (1-a) x_{zlb}) + r_h, \\ &= a(\xi_\pi \pi(m) + \xi_x x(m) + r_h) + (1-a)(\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h), \\ &< 0, \end{aligned} \tag{B.13}$$

where the last line uses that $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}) are both in Ω_{zlb}. The line segment with endpoints $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}) is thus entirely in Ω_{zlb}. For $t \in [T, t_1)$, the dynamics of $(\pi(t), x(t))$ are given by (A.5)-(A.6) and thus $(\pi(t), x(t))$ travels along the *zlb* saddle path. For $t \geq t_1$, equation (B.13) implies that $\max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\} = 0$ so that $(\pi(t), x(t))$ follows the same dynamics given by (A.7)-(A.8), which means $(\pi(t), x(t))$ stays on the *zlb* saddle path and travels on it towards (π_{zlb}, x_{zlb}).

(c) Because $(\pi(T), x(T))$ is in Ω_{zlb} and in Υ_{zlb}, it satisfies

$$\begin{aligned} 0 &\geq \xi_\pi \pi(t) + \xi_x x(t) + r_h, \\ x(t) &= \frac{\phi_1}{\kappa}\pi(t) - \frac{\phi_2}{\kappa} r_h, \end{aligned}$$

which is equivalent to

$$x(t) = \frac{\phi_1}{\kappa}\pi(t) - \frac{\phi_2}{\kappa} r_h, \tag{B.14}$$

$$(\kappa \xi_\pi + \xi_x \phi_1)\pi(t) \leq r_h (\phi_2 \xi_x - \kappa). \tag{B.15}$$

If $\kappa\xi_\pi + \xi_x\phi_1 \neq 0$, it is easy to find $(\pi(t), x(t))$ that satisfies (B.14) and (B.15). If $\kappa\xi_\pi + \xi_x\phi_1 = 0$, equation (B.15) holds because the Taylor principle holds. Any pair $(\pi(t), x(t))$ that satisfies equation (B.14) will be in Ω_{zlb} and in Υ_{zlb}. To find the corresponding $(\pi(0), x(0))$, use the dynamics of $(\pi(t), x(t))$ for $t \in [0, T)$ given by (A.3)-(A.4).

(d) By direct computation, the set of points $(x, \pi) \in \Upsilon_{zlb} \cap \partial\Omega$ are

$$(x, \pi) = \begin{cases} \left(-r_h \frac{\phi_1 + \phi_2 \xi_\pi}{\kappa \xi_\pi + \phi_1 \xi_x}, -r_h \frac{\kappa - \phi_2 \xi_x}{\kappa \xi_\pi + \phi_1 \xi_x}\right) & \text{if} \quad \kappa \xi_\pi + \phi_1 \xi_x \neq 0 \\ \emptyset & \text{if} \quad \kappa \xi_\pi + \phi_1 \xi_x = 0 \end{cases}. \tag{B.16}$$

If $\kappa \xi_\pi + \phi_1 \xi_x = 0$, there is clearly no $p > 0$ such that $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$. If $\kappa \xi_\pi + \phi_1 \xi_x \neq 0$, I analyze three cases according to the three different dynamics that $(\pi(t), x(t))$ can follow in Ω_{ss} given in Section A.1.

Case I. Let $t = s + p$. Then $\eta(t) = s$ and $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$ gives

$$\begin{aligned} x(s+p) = -r_h \frac{\phi_1 + \phi_2 \xi_\pi}{\kappa \xi_\pi + \phi_1 \xi_x} = &-\frac{(1 - \xi_\pi)\, \pi(s) + (\sigma\alpha_2 - \xi_x)\, x(s)}{\sigma (\alpha_1 - \alpha_2)} e^{\alpha_1 p} \\ &+ \frac{(1 - \xi_\pi)\, \pi(s) + (\sigma\alpha_1 - \xi_x)\, x(s)}{\sigma (\alpha_1 - \alpha_2)} e^{\alpha_2 p}, \end{aligned}$$

$$\begin{aligned} \pi(s+p) = -r_h \frac{\kappa - \phi_2 \xi_x}{\kappa \xi_\pi + \phi_1 \xi_x} = &\frac{(1 - \xi_\pi)\, \pi(s) + (\sigma\alpha_2 - \xi_x)\, x(s)}{\sigma (\xi_\pi - 1)(\alpha_1 - \alpha_2)} (\xi_x - \sigma\alpha_1)\, e^{\alpha_1 p} \\ &- \frac{(1 - \xi_\pi)\, \pi(s) + (\sigma\alpha_1 - \xi_x)\, x(s)}{\sigma (\xi_\pi - 1)(\alpha_1 - \alpha_2)} (\xi_x - \sigma\alpha_2)\, e^{\alpha_2 p}. \end{aligned}$$

Solving for $(\pi(s), x(s))$ as a function of p gives

$$\begin{aligned} x(s)(p) = &-\frac{r_h}{\sigma} \frac{(-\kappa + \kappa\xi_\pi + \phi_1 \xi_x + \phi_2 \xi_x - \sigma\alpha_2\phi_1 - \sigma\alpha_2\phi_2\xi_\pi)}{(\kappa \xi_\pi + \phi_1 \xi_x)(\alpha_1 - \alpha_2)} e^{-p\alpha_1} \\ &- \frac{r_h}{\sigma} \frac{(\kappa - \kappa\xi_\pi - \phi_1 \xi_x - \phi_2 \xi_x + \sigma\alpha_1\phi_1 + \sigma\alpha_1\phi_2\xi_\pi)}{(\kappa \xi_\pi + \phi_1 \xi_x)(\alpha_1 - \alpha_2)} e^{-p\alpha_2}, \end{aligned}$$

$$\begin{aligned} \pi(s)(p) = &\frac{r_h}{\sigma} \frac{(\xi_x - \sigma\alpha_1)(-\kappa + \kappa\xi_\pi + \phi_1 \xi_x + \phi_2 \xi_x - \sigma\alpha_2\phi_1 - \sigma\alpha_2\phi_2\xi_\pi)}{(\xi_\pi - 1)(\kappa \xi_\pi + \phi_1 \xi_x)(\alpha_1 - \alpha_2)} e^{-p\alpha_1} \\ &+ \frac{r_h}{\sigma} \frac{(-\xi_x + \sigma\alpha_2)(-\kappa + \kappa\xi_\pi + \phi_1 \xi_x + \phi_2 \xi_x - \sigma\alpha_1\phi_1 - \sigma\alpha_1\phi_2\xi_\pi)}{(\xi_\pi - 1)(\kappa \xi_\pi + \phi_1 \xi_x)(\alpha_1 - \alpha_2)} e^{-p\alpha_2}. \end{aligned}$$

Let

$$F(p) = -\frac{r_h\left(-\xi_x + \sigma\alpha_1\xi_\pi\right)\left(-\kappa + \kappa\xi_\pi + \phi_1\xi_x + \phi_2\xi_x - \sigma\alpha_2\phi_1 - \sigma\alpha_2\phi_2\xi_\pi\right)}{\sigma\left(\xi_\pi - 1\right)\left(\kappa\xi_\pi + \phi_1\xi_x\right)\left(\alpha_1 - \alpha_2\right)}e^{-p\alpha_1}$$
$$- \frac{r_h\left(\xi_x - \sigma\alpha_2\xi_\pi\right)\left(-\kappa + \kappa\xi_\pi + \phi_1\xi_x + \phi_2\xi_x - \sigma\alpha_1\phi_1 - \sigma\alpha_1\phi_2\xi_\pi\right)}{\sigma\left(\xi_\pi - 1\right)\left(\kappa\xi_\pi + \phi_1\xi_x\right)\left(\alpha_1 - \alpha_2\right)}e^{-p\alpha_2}$$
$$+ \frac{r_h\sigma\left(\xi_\pi - 1\right)\left(\alpha_1 - \alpha_2\right)\left(\kappa\xi_\pi + \phi_1\xi_x\right)}{\sigma\left(\xi_\pi - 1\right)\left(\kappa\xi_\pi + \phi_1\xi_x\right)\left(\alpha_1 - \alpha_2\right)}.$$

Then,

$$F(p) = \xi_\pi\pi(s)\,(p) + \xi_x x(s)\,(p) + r_h,$$

and since $(\pi(s), x(s)) \in \partial\Omega$, it follows that $\xi_\pi\pi(s)(0) + \xi_x x(s)(0) + r_h = 0 = F(0)$. I show there is no $p > 0$ that satisfies $F(p) = 0$.

First, note that $\pi(s)$ and $x(s)$ are always real, even when α_1 and α_2 are complex. By direct computation, I find that

$$\begin{aligned} F(0) &= 0. && \text{(B.17)} \\ F'(p) &= 0 \text{ has at most one solution for } p > 0, && \text{(B.18)} \\ F'(0) &= \frac{r_h}{\sigma\phi_1}\left(\kappa\left(\xi_\pi - 1\right) + \rho\xi_x\right) > 0, && \text{(B.19)} \\ \lim_{p\to\infty} F(p) &= r_h > 0. && \text{(B.20)} \end{aligned}$$

$F'(0) > 0$ because the Taylor principle holds. Together, equations (B.17)-(B.20) and continuity of $F\,(p)$ show that there is no solution to $F(p) = 0$ for $p > 0$.

Case II. Let $t = s + p$. Then $\eta(t) = s$ and

$$\begin{aligned} -r_h\frac{\rho}{\kappa + \phi_1\xi_x} &= x(s)e^{\frac{1}{\sigma}\xi_x p}, && \text{(B.21)} \\ -r_h\frac{\kappa - \phi_2\xi_x}{\kappa + \phi_1\xi_x} &= \frac{\pi(s)\left(\xi_x - \sigma\rho\right) + \kappa\sigma x(s)}{\xi_x - \sigma\rho}e^{\rho p} - \frac{\kappa\sigma x(s)}{\xi_x - \sigma\rho}e^{\frac{1}{\sigma}\xi_x p}. && \text{(B.22)} \end{aligned}$$

Using equation (B.16) and that $\xi_\pi\pi(s) + \xi_x x(s) + r_h = 0$, equation (B.22) becomes

$$\begin{aligned} -r_h\frac{\kappa - \phi_2\xi_x}{\kappa + \phi_1\xi_x} &= \left(\frac{\left(\xi_x - \sigma\rho\right)\xi_x - \kappa\sigma}{\left(\xi_x - \sigma\rho\right)\left(\kappa + \phi_1\xi_x\right)}r_h\rho e^{-\frac{1}{\sigma}\xi_x p} - r_h\right)e^{\rho p} \\ &\quad + \frac{r_h\kappa\sigma\rho}{\left(\xi_x - \sigma\rho\right)\left(\kappa + \phi_1\xi_x\right)}. && \text{(B.23)} \end{aligned}$$

Solving for $x(s)$ in equation (B.21) and plugging it into equation (B.23) gives

$$x(s) = \xi_x\left(\kappa - \phi_2\xi_x + \sigma\rho\phi_2\right), \qquad \text{(B.24)}$$

and

$$\begin{aligned}\xi_x\left(\kappa-\phi_2\xi_x+\sigma\rho\phi_2\right) &= e^{p\rho}\left(\xi_x-\sigma\rho\right)\left(\kappa+\phi_1\xi_x\right) \\ &\quad +e^{\frac{p}{\sigma}\left(\sigma\rho-\xi_x\right)}\rho\left(-\xi_x^2+\kappa\sigma+\sigma\rho\xi_x\right). \qquad \text{(B.25)}\end{aligned}$$

I now show that there is no $p>0$ such that equations (B.24)-(B.25) hold. If $\xi_x=\sigma\phi_1$, then $-\xi_x^2+\kappa\sigma+\sigma\rho\xi_x=0$ and equations (B.24)-(B.25) become

$$\begin{aligned}x(s) &= -\rho r_h\frac{e^{-p\phi_1}}{\sigma\phi_1^2+\kappa}, \\ 1 &= e^{p\rho}.\end{aligned}$$

The last equation has no solution for $p>0$. If $\xi_x\neq\sigma\phi_1$, and recalling that $\xi_x\neq\sigma\phi_2$ so that $\Upsilon_{zlb}\cap\partial\Omega$ is non-empty, then $-\xi_x^2+\kappa\sigma+\sigma\rho\xi_x\neq 0$, and equations (B.24)-(B.25) become

$$\begin{aligned}x(s) &= -\frac{r_h\rho}{\kappa+\phi_1\xi_x}e^{-\frac{1}{\sigma}\xi_x p}, \\ 0 &= \frac{\phi_2\xi_x}{\xi_x-\sigma\phi_1}\left(e^{p\rho}-1\right)+\rho e^{p\rho}\left(e^{-\frac{\xi_x}{\sigma}p}-1\right).\end{aligned}$$

Let

$$F(p)=\frac{\phi_2\xi_x}{\xi_x-\sigma\phi_1}\left(e^{p\rho}-1\right)+\rho e^{p\rho}\left(e^{-\frac{\xi_x}{\sigma}p}-1\right).$$

Compute

$$\begin{aligned}F'(p) &= \frac{\rho\xi_x}{\sigma}\left(e^{-\frac{p}{\sigma}\xi_x}-\frac{\sigma\phi_2}{\left(\xi_x-\sigma\phi_1\right)}e^{-p\rho}\right), \\ F''(p) &= -\frac{\rho\xi_x}{\sigma^2}\left(\xi_x e^{-\frac{p}{\sigma}\xi_x}-\frac{\sigma^2\rho\phi_2}{\left(\xi_x-\sigma\phi_1\right)}e^{-p\rho}\right),\end{aligned}$$

and

$$\begin{aligned}F(0) &= 0, && \text{(B.26)} \\ F'(0) &= \frac{\rho\xi_x\left(\xi_x-\sigma\rho\right)}{\sigma\left(\xi_x-\sigma\phi_1\right)}, && \text{(B.27)} \\ F'(p) &= 0\Rightarrow e^{\left(\rho-\frac{\xi_x}{\sigma}\right)p}=\frac{\sigma\phi_2}{\left(\xi_x-\sigma\phi_1\right)}, && \text{(B.28)} \\ \lim_{p\to\infty}F(p) &= \phi_1\frac{\xi_x-\sigma\rho}{\xi_x-\sigma\phi_1}, && \text{(B.29)} \\ \lim_{p\to\infty}F'(p) &= 0. && \text{(B.30)}\end{aligned}$$

If $\xi_x - \sigma\phi_1 > 0$, $F(p)$ is monotonic, which combined with $F(0) = 0$ gives no solutions to $F(p) = 0$ for $p > 0$. If $\xi_x - \sigma\phi_1 < 0$ and $\xi_x - \sigma\rho < 0$, then the unique local maximum occurs for some $p > 0$ and F is positive at that maximum. Using (B.29) and (B.30) then shows that there is no solution to $F(p) = 0$ for $p > 0$. If $\xi_x - \sigma\phi_1 < 0$ and $\xi_x - \sigma\rho > 0$, an analogous argument applies but instead of a unique maximum, there is a unique minimum.

Case III. Let $t = s + p$. Then $\eta(t) = s$ and

$$x(p) = \left(\left(1 + \frac{1}{2\sigma}\left(\xi_x - \sigma\rho\right)p\right)x(s) + \frac{1}{\kappa}\left(\frac{1}{2\sigma}\left(\sigma\rho - \xi_x\right)\right)^2 p\pi(s)\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}, \tag{B.31}$$

$$\pi(p) = \left(-\kappa p x(s) + \left(1 - \frac{1}{2\sigma}\left(\xi_x - \sigma\rho\right)p\right)\pi(s)\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}. \tag{B.32}$$

Using equation (B.16) and that $\xi_\pi \pi(s) + \xi_x x(s) + r_h = 0$, equations (B.31)-(B.32) become

$$\begin{aligned} -r_h\frac{\phi_1 + \phi_2\xi_\pi}{\kappa\xi_\pi + \phi_1\xi_x} &= \left(1 + \frac{1}{2\sigma}\left(\xi_x - \sigma\rho\right)p\right)x(s)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p} \\ &\quad + \frac{1}{\kappa}\left(\frac{1}{2\sigma}\left(\sigma\rho - \xi_x\right)\right)^2 p\left(-\frac{\left(\xi_x x(s) + r_h\right)}{\xi_\pi}\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}, \end{aligned} \tag{B.33}$$

$$\begin{aligned} -r_h\frac{\kappa - \phi_2\xi_x}{\kappa\xi_\pi + \phi_1\xi_x} &= -\kappa p x(s)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p} \\ &\quad + \left(1 - \frac{1}{2\sigma}\left(\xi_x - \sigma\rho\right)p\right)\left(-\frac{\left(\xi_x x(s) + r_h\right)}{\xi_\pi}\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}. \end{aligned} \tag{B.34}$$

Combining equations (B.33)-(B.34), I solve for $x(s)$ as a function of p

$$x(s) = \frac{A_0 + A_1 p}{B_0 + B_1 p}, \tag{B.35}$$

where

$$\begin{aligned} A_0 &= -4\sigma r_h\left(\phi_2\xi_x^2 + \sigma^2\rho^2\phi_2 + 4\kappa\sigma\rho - 2\sigma\rho\phi_2\xi_x\right), \\ A_1 &= 2r_h\left(\xi_x^2 - \sigma^2\rho^2\right)\left(2\kappa - \phi_2\xi_x + \sigma\rho\phi_2\right), \\ B_0 &= 4\kappa\sigma\left(\xi_x^2 + \sigma^2\rho^2 + 4\kappa\sigma - 2\sigma\rho\xi_x + 4\sigma\phi_1\xi_x\right), \\ B_1 &= \left(\xi_x + \sigma\rho\right)\left(2\kappa - \phi_2\xi_x + \sigma\rho\phi_2\right)\left(\sigma^2\rho^2 + 4\kappa\sigma - \xi_x^2\right). \end{aligned}$$

Plugging equation (B.35) into equation (B.33), I get

$$F(p) = 0,$$

where

$$F(p) = e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p} - 1 + \frac{(\xi_x + \sigma\rho)(2\kappa - \phi_2\xi_x + \sigma\rho\phi_2)}{4\kappa\sigma}\left(\frac{\xi_x - \sigma(\phi_1 - \phi_2)}{\xi_x + \sigma(\phi_1 - \phi_2)}\right)p.$$

Since

$$\begin{aligned} F(0) &= 0, \\ F'(0) &= -\frac{\phi_2}{4\kappa\sigma}(\xi_x + \sigma\rho)^2 > 0, \\ F''(p) &= \frac{1}{4}\left(\rho + \frac{1}{\sigma}\xi_x\right)^2 e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p} > 0, \end{aligned}$$

the equation $F(p) = 0$ has no solution for $p > 0$.

(e) The assumptions required for Theorem 3 in Appendix C, the Poincaré-Bendixson Theorem, hold. Indeed, because

$$\dot{x}(t) = \sigma^{-1}\left(\max\{0, \xi_x x(t) + \xi_\pi \pi(t) + r(t)\} - r(t) - \pi(t)\right), \tag{B.36}$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t), \tag{B.37}$$

are, as functions of $\pi(t)$ and $x(t)$, continuous and differentiable almost everywhere, they are Lipschitz. The rest of the conditions are easy to check.

I show that the ω-limit set[11] of $(\pi(q), x(q))$ contains no steady-states and is not a periodic orbit. By Theorem 3, $(\pi(t), x(t))$ then explodes.

Because (π_{ss}, x_{ss}) is a locally unstable steady-state (by the Taylor principle) and $(\pi(q), x(q)) \neq (\pi_{ss}, x_{ss})$, the ω-limit set of $(\pi(q), x(q))$ does not contain (π_{ss}, x_{ss}), as $(\pi(t), x(t))$ is bounded away from (π_{ss}, x_{ss}) for all $t \geq q$. Because (π_{zlb}, x_{zlb}) is locally a saddle-path steady-state, the only paths converging to (π_{zlb}, x_{zlb}) as $t \to \infty$ must eventually be in $\Upsilon_{zlb} \cap \Omega_{zlb}$. By hypothesis, $(\pi(\tau(t)), x(\tau(t))) \notin \Upsilon_{zlb} \cap \partial\Omega$, where recall $\tau(t)$ is the time of first entry into Ω_{zlb} after t. By item (d), if $(\pi(t), x(t))$ enters Ω_{zlb} a second time after $\tau(t)$ (of course, by first visiting Ω_{ss}) it is not through $\Upsilon_{zlb} \cap \partial\Omega$. It follows that the ω-limit set of $(\pi(q), x(q))$ does not contain (π_{zlb}, x_{zlb}), as the orbit of $(\pi(q), x(q))$ never intersects $\Upsilon_{zlb} \cap \Omega_{zlb}$.

[11] See Appendix C for definitions of ω-limit sets and other concepts needed to state the Poincaré-Bendixson Theorem.

I now show that there are no closed orbits. The divergence of $(\dot{\pi}(t), \dot{x}(t))$ computed in the distribution sense is

$$\operatorname{div}(\dot{\pi}(t), \dot{x}(t)) = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{\pi}}{\partial \pi} = \begin{cases} \rho & , \quad \text{if } (\pi(t), x(t)) \in \Omega_{zlb} \setminus \partial\Omega \\ \frac{\xi_x}{2\sigma} + \rho & , \quad \text{if } (\pi(t), x(t)) \in \partial\Omega \\ \frac{1}{\sigma}\xi_x + \rho & , \quad \text{if } (\pi(t), x(t)) \in \Omega_{ss} \end{cases},$$

where $\Omega_{zlb} \setminus \partial\Omega$ denotes the interior of Ω_{zlb}.

The Taylor principle and $\rho > 0$ imply that $\operatorname{div}(\pi(t), x(t)) > 0$ for all $(\pi(t), x(t))$. By Theorem 2, there are no closed orbits[12].

(f) The result that there is no chaos is a direct consequence of Theorem 3, which tightly restricts the behavior of bounded solutions to two cases, none of which is chaotic. For continuous systems, strange attractors and other chaotic behavior can only emerge when the dimension of the phase space is three or more. Note that the concept of chaos I consider here is different from chaos in the sense of Li and Yorke (1975) used in Benhabib et al. (2002), which is more appropriate for a discrete time setting.

□

Proof of Proposition 3. Consider the following condition:

$$\begin{aligned} (\pi(t_1), x(t_1)) &= (\pi_{ss}, x_{ss}), \\ &\text{or} \\ (\pi(t_1), x(t_1)) &\in \Omega_{zlb} \cap \Upsilon_{zlb}, \\ &\text{or} \\ (\pi(t_1), x(t_1)) &\in \overline{\Omega}_{ss} \text{ and } (\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb} \text{ for some } r \in (t_1, \infty). \end{aligned} \tag{B.38}$$

I first prove that if condition (B.38) holds, then $(\pi(t), x(t))$ is bounded. I consider three cases.

Case 1: If $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, then $(\pi(t), x(t))$ is bounded because (π_{ss}, x_{ss}) is a steady-state.

[12] The version of the Poincaré-Bendixson theorem I have used is stronger than needed since our vector field is continuous (but non-differentiable) in $\partial\Omega$ while the theorem allows for discontinuities across the boundary between regions.

In addition, I have used one particular generalized derivative, the "derivative in the distribution sense." However, since the vector field under consideration is continuous, any generalized derivative (such as viscosity solutions) would still give a finite value for $(\dot{\pi}(t), \dot{x}(t))$. When the value of $(\dot{\pi}(t), \dot{x}_t)$ is finite along $\partial\Omega$, because $\partial\Omega$ has measure zero, its value does not contribute to the line integral along a closed loop. By Green's theorem, it then does not matter which concept of generalized derivative I use for this particular purpose.

Case 2: If $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$, then item (b) of Lemma 2 shows, by picking $m = t_1$, that $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq t_1$. The dynamics in equations (A.7)-(A.8) then show $(\pi(t), x(t)) \to (\pi_{zlb}, x_{zlb})$.

Case 3: If $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$ and $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$ for some $r \in (t_1, \infty)$, item (b) of Lemma 2 shows, by picking $m = r$, that $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq r$. The dynamics in equations (A.7)-(A.8) then show $(\pi(t), x(t)) \to (\pi_{zlb}, x_{zlb})$.

To prove the converse, I prove the contrapositive. Assume $(\pi(t_1), x(t_1)) \neq (\pi_{ss}, x_{ss})$ and $(\pi(t_1), x(t_1)) \notin \Omega_{zlb} \cap \Upsilon_{zlb}$. I consider two cases.

Case 1: $(\pi(t_1), x(t_1)) \notin \overline{\Omega}_{ss}$. Because of the saddle path dynamics in Ω_{zlb}, if $(\pi(t_1), x(t_1)) \notin \Upsilon_{zlb}$, then $(\pi(t), x(t))$ either explodes or enters Ω_{ss} in finite time. If it enters Ω_{ss} by intersecting $\partial\Omega$ at some time $r > t_1$, item (d) of Lemma 2 shows that there is no $p > 0$ such that $(\pi(t), x(t)) \in \Omega_{ss}$ for $t \in (r, r+p)$ and $(\pi(r+p), x(r+p)) \in \Upsilon_{zlb} \cap \partial\Omega$. Then item (e) of Lemma 2 shows that $(\pi(t), x(t))$ explodes.

Case 2: There is no $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$. If $(\pi(t_1), x(t_1)) \notin \overline{\Omega}_{ss}$, case 1 shows $(\pi(t), x(t))$ explodes. If $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$, given that $(\pi(t_1), x(t_1)) \neq (\pi_{ss}, x_{ss})$, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb}. By assumption, if it enters Ω_{zlb}, it does not intersect Υ_{zlb}. This means $(\pi(t), x(t))$ is eventually in the interior of Ω_{zlb} but not in Υ_{zlb}. The same logic applied in case 1 shows that $(\pi(t), x(t))$ explodes. $\square$

B.6 Proof of Proposition 4

Assume the rule implements the optimal equilibrium, i.e. $\{x(t), \pi_t, i_t\} = \{x^*(t), \pi^*(t), i^*(t)\}$ when the central bank follows the rule in equation (15). Werning (2012) shows that $i^*(t) = (1 - \kappa\sigma\lambda)\, \pi^*(t) + r(t) > 0$ for $t \geq t^*$. It follows that $f(R^*) \leq t^*$. In addition, $f(R^*) \geq s$ for all $s \leq t^*$ such that $(1 - \kappa\sigma\lambda)\, \pi^*(t) + r(t) > 0$, since otherwise the rule (15) would prescribe $i_t > 0$ while $i^*(t) = 0$. Pick $s = t^*$ to get $f(R^*) \geq t^*$ since $(1 - \kappa\sigma\lambda)\, \pi^*_{t^*} + r_{t^*} > 0$. Because $f(R^*) \leq t^*$ and $f(R^*) \geq t^*$, it follows that $f(R^*) = t^*$, and equation (19) holds.

To prove (20), I use $f(R^*) = t^*$ to get that for all $t \geq t^*$

$$
\begin{aligned}
\max\{0, \xi_\pi(R^*)\pi^*(t) + \xi_x(R^*)x^*(t) + r(t)\} &= \xi_\pi(R^*)\pi^*(t) + \xi_x(R^*)x^*(t) + r_h, \\
&= (1 - \kappa\sigma\lambda)\, \pi^*(t) + r_h, \qquad \text{(B.39)}
\end{aligned}
$$

since otherwise $i_t = i^*(t)$ would not hold. If $\kappa\sigma\lambda \neq 1$, use $x^*(t) = \phi\pi^*(t)$ in equation (B.39) and then equation (20) follows immediately, as $\pi^*(t) \neq 0$ for $t \in [t^*, \infty)$. If $\kappa\sigma\lambda = 1$, any $\xi_\pi(R^*), \xi_x(R^*)$ implement the optimal equilibrium as $(0,0)$ is a steady-state for all $\xi_\pi(R^*), \xi_x(R^*)$.

Now assume that equations (19)-(20) hold. I show rule (15) implements the optimal equilibrium. When $(\pi_0, x_0) = (x_0^*, \pi_0^*)$, clearly $i_t = i^*(t) = 0$ and $(\pi(t), x(t)) = (x^*(t), \pi^*(t))$ for $t < t^*$. Because $(\pi_t, x(t))$ and $(x^*(t), \pi^*(t))$ are continuous as a function of time and their paths coincide in $[t^* - \varepsilon, t^*)$ for any $\varepsilon > 0$, $(x(t^*), \pi(t^*)) = (x^*(t^*), \pi^*(t^*))$. As $x^*(t^*) = \phi\pi^*(t^*)$ for $t = t^*$,

$$x(t^*) = \phi\pi(t^*). \tag{B.40}$$

If $\kappa\sigma\lambda = 1$, $(x(t^*), \pi(t^*)) = (x^*(t^*), \pi^*(t^*)) = (0,0)$, because $(0,0)$ is a steady-state, $(\pi_t, x(t)) = (x^*(t), \pi^*(t)) = (0,0)$ for all $t \geq t^*$ and any $\xi_\pi(R^*), \xi_x(R^*)$. In addition, if $(\pi_t, x(t)) = (0,0)$,

$$i_t = \max\{0, \xi_\pi(R^*)\pi(t) + \xi_x(R^*)x(t) + r(t)\} = r_h = i^*(t)$$

for all $t \geq t^*$.

When $\kappa\sigma\lambda \neq 1$, using equations (A.9)-(A.14), it can be checked by direct computation that $(\pi(t), x(t)) = (x^*(t), \pi^*(t))$ for all $t \geq t^*$ where $(x^*(t), \pi^*(t))$ is given by

$$\begin{aligned} x^*(t) &= x_1^* \exp\left(-\frac{\kappa\lambda}{\phi}(t - t_1)\right), & \text{(B.41)} \\ \pi^*(t) &= \pi_1^* \exp\left(-\frac{\kappa\lambda}{\phi}(t - t_1)\right). & \text{(B.42)} \end{aligned}$$

The following relations may be helpful for the computations: If $\xi_\pi(R^*) < \kappa\sigma\lambda + \sigma\phi\rho + 1$, then

$$\begin{aligned} \alpha_1 &= \rho + \frac{1 - \xi_\pi(R^*)}{\sigma\phi}, & \text{(B.43)} \\ \alpha_2 &= -\frac{\kappa\lambda}{\phi}. & \text{(B.44)} \end{aligned}$$

If $\xi_\pi(R^*) > \kappa\sigma\lambda + \sigma\phi\rho + 1$, then

$$\begin{aligned} \alpha_1 &= -\frac{\kappa\lambda}{\phi}, & \text{(B.45)} \\ \alpha_2 &= \rho + \frac{1 - \xi_\pi(R^*)}{\sigma\phi}. & \text{(B.46)} \end{aligned}$$

If $\xi_\pi(R^*) = \kappa\sigma\lambda + \sigma\phi\rho + 1$

$$\alpha_1 = \alpha_2 = -\frac{\kappa\lambda}{\phi}. \tag{B.47}$$

B.7 Proof of Proposition 5

I first assume the rule implements no equilibrium with $R \neq R^*$ and prove items (a)-(c) hold.

Item (a): By Proposition 3, if the Taylor principle holds, there exist continuous bounded paths with $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$. Since for these paths $(\pi(t), x(t)) \to (\pi_{zlb}, x_{zlb})$, they constitute non-optimal equilibria. By item (b) of Lemma 2, $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq t_1$, irrespective of the choice of t_1. It follows that the only way to preclude these type of equilibria is to have the Taylor principle not hold for $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.

Item (b): If there are continuous paths that satisfy the hypotheses of items i., ii. or iii., they are bounded by Propositions 2 and 3 and constitute non-optimal equilibria. Thus, all paths that satsify the hypothesis in items i., ii. and iii. must be discontinuous, which implies equation (21) holds.

Item (c): If $\partial\Omega \cap \Upsilon_{ss} = \emptyset$, then the item is vacuously true. If $\partial\Omega \cap \Upsilon_{ss}$ is non-empty then

$$(\pi(r), x(r)) = \begin{cases} \left(r_h \frac{(\xi_\pi - 1)}{\xi_x - \sigma\alpha_2\xi_\pi}, -r_h \frac{(\xi_x - \sigma\alpha_2)}{\xi_x - \sigma\alpha_2\xi_\pi}\right) & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi = 1 \\ \left(-r_h \frac{(\xi_x - \sigma\rho)}{\xi_x^2 - \kappa\sigma - \sigma\rho\xi_x}, \kappa\sigma \frac{r_h}{\xi_x^2 - \sigma\rho\xi_x - \kappa\sigma}\right) & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi \neq 1 \end{cases} \tag{B.48}$$

Assume that the Taylor principle does not hold for $(x(t_1), \pi_{t_1}) \in \Omega_{zlb}$ and that there exist some $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega_{zlb} \cap \Upsilon_{ss}$. I show that if equation (21) does not hold, then there exist a non-optimal equilibrium. By assumption, $t_1 \in [T, r)$. Let P be the set of points in the continuous path between $(\pi(T), x(T))$ and $(\pi(r), x(r))$, which can be obtained by running the system dynamics backward in time while respecting continuous pasting. By the dynamics in equations (A.7)-(A.8) and equation (B.48), the time q at which $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ reaches $(\pi(r), x(r))$ while following a continuous path is

$$q = \frac{1}{\phi_1} \log\left(\frac{d_{exit}(r)}{d_{exit}(t_1)}\right) \tag{B.49}$$

Note that q is not necessarily equal to r since the hypotheses of item (c) do not require that paths are continuous. Because equation (22) does not hold, $T \leq t_1 \leq q$ and thus $(\pi(t_1), x(t_1)) \in P$. By Theorem 2, the continuous path going through $(\pi(t_1), x(t_1))$ and $(\pi(r), x(r))$ is bounded for $t \geq t_1$. Using the continuous pasting conditions in Section A.3, the path can be continuously extended from $(\pi(t_1), x(t_1))$ to $(\pi(0), x(0))$ to get a continuous bounded path for all $t \geq 0$. This equilibrium is non-optimal since no optimal path has $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \setminus \partial\Omega$.

Conversely, I now assume items (a)-(c) hold and prove that the rule implements no equilibria with $R_{t_1} \neq R^*$. By Proposition 3 and item (a), there are no equilibria with $(x(t_1), \pi(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$. By Propositions 2 and 3, items i.-iii. and the continuous pasting conditions in Section A.3, there are no equilibria when: The Taylor principle holds for $(\pi(t_1), x(t_1)) \in \Omega_{ss}$ and there exist some $r \in (t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{zlb}$, the Taylor principle holds for $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, or the Taylor principle does not hold for $(x(t_1), \pi(t_1)) \in \overline{\Omega}_{ss} \cap \Upsilon_{ss}$. Item (e) of Proposition 2 and item (c) imply that there is no continuous path from $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ to $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$ that remains bounded after r, and thus there are no equilibria when the Taylor principle does not hold for $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ and there exist some $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega_{zlb} \cap \Upsilon_{ss}$. By Propositions 2 and 3, all other cases lead to paths that are discontinuous or unbounded.

B.8 Proof of Proposition 6

Item (a). Assume $f(R_{t_1})$ is constant in its first two arguments. I show there always exist a non-optimal equilibrium. Denote the value of $f(\cdot, \cdot, 0, 0)$ by $\hat{t}$ (because f is constant in its first two arguments, $f(a, b, 0, 0) = \hat{t}$ for all a, b). Define $(\hat{\pi}_0, \hat{x}_0)$ by

$$\hat{x}_0 = \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 \hat{t}} - \phi_2^2 e^{-\phi_1 \hat{t}}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa}\right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \tag{B.50}$$

$$\hat{\pi}_0 = r_h \frac{\phi_1 e^{-\phi_2 \hat{t}} - \phi_2 e^{-\phi_1 \hat{t}}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l. \tag{B.51}$$

The continuous path starting at $(\hat{\pi}_0, \hat{x}_0)$ reaches $(0, 0)$ at time $\hat{t}$ by equations (A.35) and (A.36). Since $(0, 0)$ is a steady-state, $(\pi(t), x(t)) = (0, 0)$ for all $t \geq \hat{t}$. The path for $(\pi(t), x(t))$ is continuous, bounded and follows the IS, the NKPC and the interest rate rule: It is an equilibrium. If $\kappa\sigma\lambda \neq 1$, the equilibrium is not optimal.

Item (b). Consider a rule with

$$\begin{aligned}
\xi_\pi(R_{t_1}) &= 1 - \kappa\sigma\lambda \\
\xi_x(R_{t_1}) &= 0 \\
f(R^*) &= t^* \\
f(R_{t_1}) &= \tau(\pi(0), x(0))
\end{aligned}$$

for some function $\tau : \mathbb{R}^2 \to \mathbb{R}$. I use Proposition 5 to show that there exists a choice of τ compatible with the optimal equilibrium being a unique equilibrium.

The choice of ξ_x, ξ_π implies

$$\alpha_1 = \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\kappa^2\lambda}\right) = \kappa\phi > 0 \tag{B.52}$$

$$\alpha_2 = \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\kappa^2\lambda}\right) < 0 \tag{B.53}$$

$$\alpha_1\alpha_2 = -\kappa^2\lambda < 0 \tag{B.54}$$

$$\alpha_1 + \alpha_2 = \rho \tag{B.55}$$

The Taylor principle never holds, as $\kappa(\xi_\pi - 1) + \rho\xi_x = -\kappa^2\sigma\lambda < 0$.

Item (a) of Proposition 5 is true because the Taylor principle does not hold. Subitems (b)i. and (b)iii. of Proposition 5 do not apply, since the Taylor principle does not hold.

I now analyze subitem (b)ii. and item (c) of Proposition 5, and show they can be satisfied with an appropriate choice of τ.

First, consider item (b)ii.. Because $(\pi(t_1), x(t_1)) \in \Upsilon_{ss}$,

$$\pi(t_1) = \frac{1}{\phi}x(t_1) \tag{B.56}$$

and because $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$

$$(1 - \kappa\sigma\lambda)\,\pi(t_1) + r_h \geq 0.$$

Consider the four cases of the continuous pasting condition in equation A.38 given by equation (A.40).

Case 1: If $d_{exit}(t_1) = 0$ and $d_{trap}(t_1) = 0$, $x(t_1) = x_{zlb} = -\frac{1}{\kappa}r_h\rho$ and $\pi(t_1) = \pi_{zlb} = -r_h$, which contradicts equation (B.56) and hence there is no equilibrium for this case.

Case 2: If $d_{exit}(t_1) = 0$ and $d_{trap}(t_1) \neq 0$,

$$x(t_1) = \frac{\phi_1}{\kappa}\pi(t_1) - \frac{r_h\phi_2}{\kappa}. \tag{B.57}$$

If $\sigma\lambda\kappa = 1$, there is no $(\pi(t_1), x(t_1))$ that satisfies equations (B.57) and (B.56) simultaneously. If $\sigma\lambda\kappa \neq 1$, equations (B.57) and (B.56) imply

$$\begin{aligned} x(t_1) &= r_h\frac{\phi - \sigma\lambda\phi_2}{(\kappa\sigma\lambda - 1)}, \\ \pi(t_1) &= \frac{r_h}{\phi}\frac{\phi - \sigma\lambda\phi_2}{(\kappa\sigma\lambda - 1)} \end{aligned}$$

But then, since $\phi_2 < 0$,

$$\xi_x x(t_1) + \xi_\pi\pi(t_1) + r_h = (1 - \kappa\sigma\lambda)\,\pi(t_1) + r_h = \frac{\lambda\sigma\phi_2 r_h}{\phi} < 0$$

contradicts that $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$ and thus there is no equilibrium for this case.

Case 3: If $d_{exit}(t_1) \neq 0$ and $d_{trap}(t_1) = 0$,

$$x(t_1) = \frac{\phi_2}{\kappa}\pi(t_1) - \frac{r_h\phi_1}{\kappa}. \tag{B.58}$$

If $\sigma\lambda\kappa = 1$, there is no $(\pi(t_1), x(t_1))$ that satisfies equations (B.57) and (B.56) simultaneously.

If $\sigma\lambda\kappa \neq 1$, equations (B.57) and (B.56) imply

$$x(t_1) = r_h \frac{\phi - \sigma\lambda\phi_1}{(\kappa\sigma\lambda - 1)}, \tag{B.59}$$

$$\pi(t_1) = \frac{r_h}{\kappa\sigma\lambda - 1} \frac{(\phi - \sigma\lambda\phi_1)}{\phi}. \tag{B.60}$$

The pasting condition in equations (A.39) and (A.41), and equation (B.60), give

$$\begin{aligned}
\mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1} \log \frac{\pi(0) + \left(r_l + (r_h - r_l)\frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2}\right)}{(\pi(t_1) + r_h)} && \text{(B.61)}\\
&= -\frac{1}{\phi_1} \log \frac{\pi(0) + \left(r_l + (r_h - r_l)\frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2}\right)}{\left(\frac{r_h}{\kappa\sigma\lambda - 1}\frac{(\phi - \sigma\lambda\phi_1)}{\phi} + r_h\right)} && \text{(B.62)}\\
&= \mathcal{T}(\pi(0), x(0)) && \text{(B.63)}
\end{aligned}$$

Setting

$$\tau(\pi(0), x(0)) \neq \mathcal{T}(\pi(0), x(0))$$

precludes any equilibrium for this case.

Case 4: If $d_{exit}(t_1) \neq 0$ and $d_{trap}(t_1) \neq 0$, using equation (B.56), the continuous pasting condition in equations (A.38) and (A.40) is

$$\begin{aligned}
&-\frac{1}{\phi_1} \log \frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\pi(t_1) + \frac{\phi_2 r_h}{\kappa}}\\
&\qquad = -\frac{1}{\phi_2} \log \frac{x(0) - \frac{\phi_2}{\kappa}\pi(0) + \frac{\phi_1 r_h}{\kappa} + \frac{\phi_1}{\kappa}(r_h - r_l)\left(e^{-T\phi_2} - 1\right)}{\left(\phi - \frac{\phi_2}{\kappa}\right) + \frac{\phi_1 r_h}{\kappa}} \qquad \text{(B.64)}
\end{aligned}$$

Let

$$\begin{aligned}
H(\pi(t_1), \pi(0), x(0)) = &-\frac{1}{\phi_1} \log \frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\pi(t_1) + \frac{\phi_2 r_h}{\kappa}}\\
&+ \frac{1}{\phi_2} \log \frac{x(0) - \frac{\phi_2}{\kappa}\pi(0) + \frac{\phi_1 r_h}{\kappa} + \frac{\phi_1}{\kappa}(r_h - r_l)\left(e^{-T\phi_2} - 1\right)}{\left(\phi - \frac{\phi_2}{\kappa}\right)\pi(t_1) + \frac{\phi_1 r_h}{\kappa}}
\end{aligned}$$

Then $G(\pi(t_1), \pi(0), x(0)) = 0$ iff equation (B.64) holds. Since

$$\frac{\partial H(\pi(t_1), \pi(0), x(0))}{\partial \pi(t_1)} = 0 \iff r_h = (\kappa\sigma\lambda - 1)\,\pi(t_1)$$

the implicit function theorem implies that we can write $H(\pi(t_1), \pi(0), x(0)) = 0$ as

$$\pi(t_1) = G(\pi(0), x(0))$$

for some function G, except when

$$\kappa\sigma\lambda - 1 \neq 0 \text{ and } \pi(t_1) = \frac{r_h}{(\kappa\sigma\lambda - 1)} \tag{B.65}$$

If equation (B.65) does not hold, then the continuous pasting conditions are given by

$$\begin{aligned}
\pi(t_1) &= G(\pi(0), x(0)) \\
\mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1}\log\frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\pi(t_1) + \frac{\phi_2 r_h}{\kappa}} \\
&= -\frac{1}{\phi_1}\log\frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right) g(\pi_0, x_0) + \frac{\phi_2 r_h}{\kappa}} \\
&= \mathcal{T}(\pi(0), x(0))
\end{aligned}$$

If equation (B.65) holds, $H(\pi(t_1), \pi(0), x(0)) = H\left(\frac{r_h}{\kappa\sigma\lambda - 1}, \pi(0), x(0)\right)$. If there is no $(\pi(0), x(0))$ so that $H\left(\frac{r_h}{(\kappa\sigma\lambda-1)}, \pi(0), x(0)\right) = 0$, then there are no equilibria since no path is continuous. If there exists $(\pi(0), x(0))$ such that $H\left(\frac{r_h}{(\kappa\sigma\lambda-1)}, \pi(0), x(0)\right) = 0$, continuous pasting gives

$$\begin{aligned}
\mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1}\log\frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\frac{r_h}{(\kappa\sigma\lambda-1)} + \frac{\phi_2 r_h}{\kappa}} \\
&= \mathcal{T}(\pi(0), x(0))
\end{aligned}$$

In either case (when equation (B.65) holds and when it does not hold), setting

$$\tau(\pi(0), x(0)) \neq \mathcal{T}(\pi(0), x(0))$$

precludes any equilibrium for case 4. This concludes the analysis of item (b)ii. of Proposition 5.

Now consider item (c) of Proposition 5. If $\kappa\sigma\lambda = 1$, $\Upsilon_{ss} \cap \partial\Omega_{zlb} = \emptyset$ and thus there are no equilibria. If $\kappa\sigma\lambda \neq 1$, $(\pi(r), x(r)) \in \Upsilon_{ss} \cap \partial\Omega_{zlb}$ implies

$$x(r) = \frac{1}{2\kappa}\frac{r_h}{\kappa\sigma\lambda - 1}\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right) \tag{B.66}$$

$$\pi(r) = \frac{r_h}{\kappa\sigma\lambda - 1} \tag{B.67}$$

Using (B.66)-(B.67) and that $(\pi(t), x(t)) \in \Omega_{zlb}$ for $t \in [t_1, r]$, the continuous pasting equations (A.35) and (A.36) imply that

$$
\begin{aligned}
x(0) = \frac{\phi_1 e^{-\phi_2 r} - \phi_2 e^{-\phi_1 r}}{\phi_1 - \phi_2} & \left(\frac{1}{2\kappa} \frac{r_h}{\kappa\sigma\lambda - 1} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2} \right) \right) \\
& - \frac{1}{\sigma} \frac{e^{-\phi_1 r} - e^{-\phi_2 r}}{\phi_1 - \phi_2} \frac{r_h}{\kappa\sigma\lambda - 1} + \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 r} - \phi_2^2 e^{-\phi_1 r}}{\phi_1 - \phi_2} \\
& + \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa},
\end{aligned}
$$

$$
\begin{aligned}
\pi(0) = -\kappa \frac{e^{-\phi_1 r} - e^{-\phi_2 r}}{\phi_1 - \phi_2} & \left(\frac{1}{2\kappa} \frac{r_h}{\kappa\sigma\lambda - 1} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2} \right) \right) \\
& + \frac{\phi_1 e^{-\phi_1 r} - \phi_2 e^{-\phi_2 r}}{\phi_1 - \phi_2} \frac{r_h}{\kappa\sigma\lambda - 1} + r_h \frac{\phi_1 e^{-\phi_2 r} - \phi_2 e^{-\phi_1 r}}{\phi_1 - \phi_2} \\
& + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l.
\end{aligned}
$$

Solving for r gives

$$
r = \nu(\pi(0), x(0))
$$

where

$$
\begin{aligned}
A &= \frac{1}{\kappa} \rho r_l + \frac{1}{\kappa} (r_h - r_l) \frac{\phi_1^2 e^{-T\phi_2} - \phi_2^2 e^{-T\phi_1}}{\phi_1 - \phi_2} \\
B &= r_l + (r_h - r_l) \frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2}
\end{aligned}
$$

are two constants and

$$
\nu(\pi(0), x(0)) = -\frac{1}{\phi_2} \log \left(\kappa \frac{\sigma\phi\phi_2 + 1}{r_h (\rho - \kappa\phi)} (x(0) + A) - \frac{\phi_2 + \kappa\phi + \sigma\rho\phi\phi_2}{r_h (\rho - \kappa\phi)} (\pi(0) + B) \right)
$$

is a function of $x(0)$ and $\pi(0)$ only (not of $x(t_1)$, $\pi(t_1)$ or t_1). Setting

$$
\tau(\pi(0), x(0)) > \nu(\pi(0), x(0))
$$

precludes all equilibria for the case in which (c) of Proposition 5 applies.
Item (c). The rule in the last item has constant Taylor rule coefficients.

B.9 Proof of Proposition 7

I start with a Lemma.

Lemma 3. *If* $\det A_{ss}(r) = 0$ *for some* $R = (\pi_0, x_0)$*, then there exist a non-optimal equilibrium.*

Proof of Lemma 3. When $\det A_{ss}(r) = 0$, $(\pi_{zlb}, x_{zlb}) \in \partial\Omega$. A continuous path with $(\pi(t), x(t)) \in \Upsilon_{zlb} \cap \Omega_{zlb}$ is bounded for any choice of f, since $(\pi(t), x(t)) \in \Omega_{zlb}$ for all t and it converges to (π_{zlb}, x_{zlb}), which is a steady-state of the economy. □

Now I prove Proposition 7. Item (a). By Proposition 5, if items (a)-(c) hold but with equation (21) replaced with (23), then there is no equilibrium with $R \neq R^*$, since (23) implies (21).

Conversely, assume there is no equilibrium with $R \neq R^*$. I show equation (23) holds. To do so, I first show that the Intermediate Value Theorem is applicable and then use it to show equation (23) holds. Let

$$\Theta = \left\{ R \in \mathbb{R}^4 : \mathcal{P}(r) = 0 \text{ and } R \neq R^* \right\}$$

Because f, ξ_x and ξ_π are continuous, their restriction to Θ are also continuous. In addition, Θ is path-connected because the solution to the ODE (1)-(2) is continuous with respect to time, the mapping from $(\pi(0), x(0))$ to $(\pi(t_1), x(t_1))$ is a continuous bijection for a fixed t_1, $f(R_{t_1}) = t_1$ is continuous in R_{t_1}, and the exclusion of R^* from Θ does not destroy path-connectedness because it is a zero-dimensional set while the dimension of Θ is 3. Because f, ξ_x and ξ_π are continuous in Θ and Θ is path-connected, we can apply the Intermediate Value Theorem.

Assume, for the sake of contradiction, that there exists $R_{low} \in \Theta$ with $f(R_{low}) < \mathcal{T}(R_{low})$. The inequality $f(R_{low}) < \mathcal{T}(R_{low})$ implies $f(r) < \mathcal{T}(r)$ for all $R \in \Theta$ since otherwise, by the Intermediate Value Theorem, there would be some $R_0 \in \Theta$ with $f(R_0) = \mathcal{T}(R_0)$, contradicting that there is no equilibrium with $R \neq R^*$. Consider the point $R_T = (\hat{\pi}_0, \hat{x}_0, \hat{\pi}_1, \hat{x}_1)$ defined by

$$\begin{aligned}
\hat{x}_0 &= \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 T} - \phi_2^2 e^{-\phi_1 T}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa}\right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \\
\hat{\pi}_0 &= r_h \frac{\phi_1 e^{-\phi_2 T} - \phi_2 e^{-\phi_1 T}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l, \\
\hat{x}_1 &= 0, \\
\hat{\pi}_1 &= 0.
\end{aligned}$$

By the continuous pasting conditions in equations (A.35)-(A.36), $R_T \in \Theta$ and $f(R_T) = T = \mathcal{T}(R_T)$, contradicting that $f(r) < \mathcal{T}(r)$ for all $R \in \Theta$.

Item (b). Since $\xi_\pi(R_{t_1})$ and $\xi_x(R_{t_1})$ are continuous, then either the Taylor principle holds for all R_{t_1}, or the Taylor principle does not hold for all R_{t_1}. To see this, assume for the sake of contradiction that there exists R_{TP} that satisfies the Taylor principle and $R_{no\text{-}TP}$ that does not. Then

$$\begin{aligned}
\det A_{ss}(R_{TP}) &= \kappa\left(\xi_\pi(R_{TP})-1\right)+\rho\xi_x(R_{TP})>0,\\
\det A_{ss}(R_{no\text{-}TP}) &= \kappa\left(\xi_\pi(R_{no\text{-}TP})-1\right)+\rho\xi_x(R_{no\text{-}TP})<0.
\end{aligned}$$

By the Intermediate Value Theorem, there exist an R_0 such that $\det A_{ss}(R_0)=\kappa\left(\xi_\pi(R_0)-1\right)+\rho\xi_x(R_0)=0$. By Lemma 3, there exist a non-optimal equilibrium.

By Proposition 4, when $\kappa\sigma\lambda\neq 1$, the Taylor principle does not hold for R^*. Because the Taylor principle does not hold for one R, then it does not hold for all R.

Item (c). By item (a) of Proposition 6, the rule cannot be purely forward-looking.

I show that if the rule is purely backward-looking, that is, if f, ξ_x and ξ_π are all constant in their last two arguments, then there exists an equilibrium with $R\neq R^*$. By item (b) of Proposition 7 just proved above, the Taylor principle never holds. I look for an equilibrium with

$$\pi(t_1)=c(R_{t_1})x(t_1) \tag{B.68}$$

with the function $c(R_{t_1})>0$ defined by equation (A.15). Because ξ_x and ξ_π are continuous in R_{t_1} and constant in $x(t_1)$, $\pi(t_1)$, so is c. To see that c is continuous when $\xi_\pi=1$, compute

$$\begin{aligned}
\lim_{\xi_\pi\to 1} c(R_{t_1}) &= \lim_{\xi_\pi\to 1}\frac{(\xi_x-\sigma\alpha_2)}{(1-\xi_\pi)}\\
&= \lim_{\xi_\pi\to 1} -\frac{1}{2\left(\xi_\pi-1\right)}\left(\xi_x-\sigma\rho+\sqrt{\xi_x^2+\sigma^2\rho^2+4\kappa\sigma-4\kappa\sigma\xi_\pi-2\sigma\rho\xi_x}\right)\\
&= \lim_{\xi_\pi\to 1}\kappa\sigma\left(\xi_x^2+\sigma^2\rho^2+4\kappa\sigma-4\kappa\sigma\xi_\pi-2\sigma\rho\xi_x\right)^{-\frac{1}{2}}\\
&= \frac{\kappa\sigma}{|\xi_x-\sigma\rho|}\\
&= \frac{\kappa\sigma}{\sigma\rho-\xi_x}
\end{aligned}$$

The third line follows by L'Hospital's Rule; in a small enough neighborhood of $\xi_\pi=1$, the Taylor principle not holding implies $\xi_x<0$ and thus both numerator and denominator in the second line go to zero as $\xi_\pi\to 1$. The last line follows because $\xi_x<0$ when $\xi_\pi=1$, again because the Taylor principle does not hold. When $\xi_\pi\neq 1$, c is continuous by equation (A.15).

Let

$$
\begin{aligned}
M\left(\pi(0), x(0)\right) &= \frac{\phi_1 e^{-\phi_2 f} - \phi_2 e^{-\phi_1 f}}{\phi_1 - \phi_2} c - \frac{1}{\sigma} \frac{e^{-\phi_1 f} - e^{-\phi_2 f}}{\phi_1 - \phi_2} \\
N\left(\pi(0), x(0)\right) &= -\kappa \frac{e^{-\phi_1 f} - e^{-\phi_2 f}}{\phi_1 - \phi_2} c + \frac{\phi_1 e^{-\phi_1 f} - \phi_2 e^{-\phi_2 f}}{\phi_1 - \phi_2} \\
P\left(\pi(0), x(0)\right) &= \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 f} - \phi_2^2 e^{-\phi_1 f}}{\phi_1 - \phi_2} \\
Q\left(\pi(0), x(0)\right) &= r_h \frac{\phi_1 e^{-\phi_2 f} - \phi_2 e^{-\phi_1 f}}{\phi_1 - \phi_2} \\
A &= \left(\frac{r_h - r_l}{\kappa}\right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa} \\
B &= \left(r_h - r_l\right) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l
\end{aligned}
$$

The functions M, N, P and Q are continuous and depend only on $x(0), \pi(0)$ (and not on $x(t_1), \pi(t_1)$) because f and c are continuous and constant in $\pi(t_1)$, $x(t_1)$. The continuous pasting conditions in equations (A.35)-(A.36) give

$$
x(0) = M\pi(t_1) + P + A \tag{B.69}
$$

$$
\pi(0) = N\pi(t_1) + Q + B \tag{B.70}
$$

If $M \neq 0$ and $N \neq 0$, the last two equations give

$$
\pi(t_1) = \frac{x(0) - P - A}{M} \tag{B.71}
$$

$$
\pi(0) = \frac{N}{M}\left(x(0) - P - A\right) + Q + B \tag{B.72}
$$

Fix $x(0)$ to $\hat{x}_0 = x^*(0) + \varepsilon$ with $\varepsilon > 0$. The right hand-side of equation (B.72) is a function of $\pi(0)$ only. It is bounded above and below, as $f \in [T, \infty)$ and

$$
\begin{aligned}
& \lim_{f\to\infty} \frac{N}{M}\left(\hat{x}_0 - P - A\right) + Q + B, \\
= \; & \frac{\kappa\sigma\left(\rho - 2\phi_1\right)\left(A - \hat{x}_0\right) + B\left(2\kappa + \sigma\rho\phi_1\right)}{\phi_1 - \phi_2} \lim_{f\to\infty} \frac{c}{\left(c\sigma\phi_1 + 1\right)} \\
& - \frac{\left(2\kappa + \sigma\rho\phi_2\right)\left(A - \hat{x}_0\right) + B\left(\rho - 2\phi_1\right)}{\left(\phi_1 - \phi_2\right)} \lim_{f\to\infty} \frac{1}{\left(c\sigma\phi_1 + 1\right)},
\end{aligned}
$$

is finite since $c > 0$. The left-hand side of equation (B.72), on the other hand, tends to $\pm\infty$ as $\pi(0) \to \pm\infty$. This means, since N, M, Q and B are continuous in $\pi(0)$, that there is at least one $\pi(0)$, say $\hat{\pi}_0$, that satisfies equation (B.72). Plugging $(\hat{\pi}_0, \hat{x}_0)$ into equations (B.68) and (B.71) give values for $(\pi(t_1), x(t_1))$, say $(\hat{\pi}_1, \hat{x}_1)$. By construction, the path defined by $(\hat{\pi}_0, \hat{x}_0)$ is continuous. Picking ε small enough guarantees that $(\hat{\pi}_1, \hat{x}_1) \in \Omega_{ss}$, since $(\pi^*(t^*), x^*(t^*)) \in \Omega_{ss}$ is bounded away from $\partial\Omega$. Equation (B.68) implies $(\hat{\pi}_1, \hat{x}_1) \in \Upsilon_{ss}$. Proposition 2 then shows the path defined by $(\hat{\pi}_0, \hat{x}_0)$ is bounded and hence an equilibrium. Because $\varepsilon \neq 0$, the equilibrium is not the optimal equilibrium.

If $M = 0$ and $N \neq 0$, the continuous pasting conditions (A.35)-(A.36) give

$$x(0) = P + A \tag{B.73}$$

$$\pi(t_1) = \frac{\pi(0) - (Q + B)}{N} \tag{B.74}$$

But $M = 0$ implies

$$e^{-f\phi_2} = \frac{c\sigma\phi_2 + 1}{c\sigma\phi_1 + 1} e^{-f\phi_1}$$

and thus

$$\begin{aligned}
\lim_{f\to\infty} P + A &= \lim_{f\to\infty} \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 f} - \phi_2^2 e^{-\phi_1 f}}{\phi_1 - \phi_2} + A \\
&= \lim_{f\to\infty} \frac{r_h(\rho - c\kappa)}{\kappa + c\kappa\sigma\phi_1} e^{-f\phi_1} + A \\
&= A
\end{aligned}$$

is finite. An argument analogous to the one used for the case in which $M \neq 0$ and $N \neq 0$ shows the existence of a non-optimal equilibrium. The case $M \neq 0$ and $N = 0$ can be treated the same way and $M = N = 0$ cannot happen.

C Non-Linear dynamics – Poincaré-Bendixson Theorem

Assume $f : \mathbb{R}^2 \to \mathbb{R}^2$. Consider the two-dimensional system

$$\dot{x}(t) = f(x(t)). \tag{C.1}$$

Let $\phi_t(p)$ be a solution to (C.1) for $t \geq 0$ with initial condition $x_0 = p$. We assume that for each p, there is a unique solution $\phi(t, p)$. This is the case, for example, if f is Lipschitz.

The *positive semi-orbit* of f through p is defined as

$$\gamma^+(p) = \left\{x \in \mathbb{R}^2 : x = \phi_t(p) \text{ for some } t \in [0, \infty)\right\}.$$

Similarly, the *negative semi-orbit* though p is

$$\gamma^{-}(p) = \left\{x \in \mathbb{R}^2 : x = \phi_t(p) \text{ for some } t \in (-\infty, 0]\right\}.$$

The *orbit* of f through p is the union

$$\gamma(p) = \gamma^{+}(p) \cup \gamma^{-}(p).$$

A *periodic solution* is one for which $\phi_{t+T}(p) = \phi_t(p)$ for some $T > 0$ and all $t \in R$. A *periodic orbit* is the orbit $\gamma(p)$ of periodic solution $\phi_t(p)$.

The ω*-limit set* of p, denoted by $\omega(p)$, is the set

$$\omega(p) = \left\{x \in \mathbb{R}^2 : \exists \{t_k\}_{k=0}^{\infty},\ t_k \in R \text{ with } t_k \to \infty \text{ such that } \phi_{t_k}(p) \to x \text{ as } k \to \infty\right\}.$$

Consider the following four assumptions:

(a) Ω is an open domain in $\mathbb{R}^2$, divided into a finite number of open sub-domains Ω_i such that $\bigcup \overline{\Omega}_i = \overline{\Omega}$.

(b) If $\overline{\Omega}_i$ and $\overline{\Omega}_j$ are not disjoint and $i \neq j$, then $\overline{\Omega}_i \cap \overline{\Omega}_j = \Gamma_{ij}$, where Γ_{ij} (joint boundaries) are piecewise smooth.

(c) f is Lipschitz in all sub-domains Ω_i and possibly discontinuous along Γ_{ij}.

(d) The vector field f defines a direction at each point in Ω. In particular, at every point of Γ_{ij} the vector field $f(x)$ specifies into which Ω_i the flow is directed.

Theorem 1 (Extension of the Poincaré-Bendixson theorem)**.** *Consider the planar autonomous system (C.1). Let the conditions 1-4 be satisfied and let f be bounded in Ω. Suppose that K is a compact region in Ω, containing no fixed points of (C.1). If the solution of (C.1) is in K for all $t \geq t_0$, then (C.1) has a closed orbit in K.*

Theorem 2 (Extension of the Bendixson criterion)**.** *Consider the planar autonomous system (C.1). Let the conditions 1-4 be satisfied and let f be bounded in the simply connected region Ω and C^1 in each Ω_i. If $\operatorname{div} f$ (the divergence of f calculated in the distribution sense) is of the same sign and is not identically zero in Ω, then (C.1) has no closed orbit in Ω.*

Remark The requirement that f is bounded is too strong; it suffices that

$$\iint\limits_{D} \operatorname{div} f \text{ and } \int\limits_{C} f \cdot n \, ds$$

are well-defined (in the distribution sense) for all smooth closed curves C, where D is the region enclosed by C and n is a unit vector normal to C.

A proof of both theorems can be found in Melin (2005). Compared to the classical Poincaré-Bendixson theorem, Melin (2005) allows for some discontinuities in f.

We have cited the theorems exactly as they appear in Melin (2005). However, in this context, it is perhaps more familiar for economists to refer to points for which $f = 0$ as steady-states instead of fixed points and to periodic orbits instead of closed orbits.

We now prove an immediate consequence of this "extended" Poincaré-Bendixson theorem.

Theorem 3. *Assume Theorem 1 holds. If a solution φ_t is bounded for all $t \geq 0$, then either*

(a) $\omega(\varphi)$ contains a steady-state

or

(b) $\omega(\varphi)$ is a periodic orbit

Proof. First, note that because φ is bounded, $\omega(\varphi)$ is non-empty. Indeed, consider a sequence $x_i = \varphi_{t_i}(x)$ for some x. The sequence $\{x_i\}$ is bounded and infinite, so there exist a convergent subsequence. If such convergent subsequence converges to p, then $p \in \omega(\varphi)$ and thus $\omega(\varphi)$ is non-empty.

If $\omega(\varphi)$ contains a steady-state, item (a) obtains. If $\omega(\varphi)$ contains no steady-states (no fixed points), then Theorem 1 implies that $\omega(\varphi)$ is a periodic orbit, corresponding to item (b) (note that because φ is bounded we can always find a compact set K that contains it). □

D BSGU Equilibria

The conclusion that following the Taylor principle outside the ZLB induces the existence of a deflationary steady state at the ZLB is similar to one of the results in Benhabib et al. (2001b). They further show that when the Taylor principle holds, the deflationary steady state engenders an infinite number of suboptimal equilibria. As mentioned before, these equilibria can start arbitrarily close to the intended steady state (π_{ss}, x_{ss}) and still converge to (π_{zlb}, x_{zlb}). The same possibility is present in the setup I consider here. To construct equilibria analogous to those in Benhabib et al. (2001b), I use the dynamics for the three stages described above. For the next steps, refer to Figure 16. First pick two numbers q and r such that $r, q > T$ and $r - q > T$. Let $(\pi_b, x_b) = \partial\Omega \cap \Upsilon_{zlb}$.[13] Assume the Taylor principle

[13] If $\kappa\xi_\pi + \phi_1\xi_x = 0$, $\partial\Omega \cap \Upsilon_{zlb} = \emptyset$. Albeit not a general strategy to eliminate all non-optimal equilibria, picking ξ_x, ξ_π such that $\kappa\xi_\pi + \phi_1\xi_x = 0$ does preclude this particular class of equilibria from forming for any choice of f. This possibility was not present in Benhabib et al. (2001b), as their model did not have both inflation and output as state variables of the economy.

holds. Using $(\pi(r), x(r)) = (\pi_b, x_b)$ as the starting point, trace the dynamics of $(\pi(t), x(t))$ backward in time using the interest rate specified by equation (A.30) for a length of time q. As in Benhabib et al. (2001b), these equilibria can get arbitrarily close to the intended steady state: Because the dynamics of $(\pi(t), x(t))$ are unstable when going forward in time, they are stable backward in time and $(\pi(t), x(t))$ converges to (π_{ss}, x_{ss}) as $q \to \infty$.[14] At time $r - q$, trace the dynamics of $(\pi(t), x(t))$ backward in time using $(\pi(r - q), x(r - q))$ as the starting point and $i(t) = 0$ throughout, until $t = 0$, when the path reaches $(\pi(0), x(0))$. Of course, the natural rate is positive after T and negative before T, so the dynamics change from those of the second stage to those of the first. Note that in Figure 16, the gray flow lines in the background reflect the dynamics that prevail for $t \geq t_1$ only. Set $t_1 = r - q > T$. By construction, the path starting at $(\pi(0), x(0))$ reaches (π_b, x_b) at time r when following the interest rate rule in equation (15). Now going forward in time, for $t \geq r$, $(\pi(t), x(t)) \in \Upsilon_{zlb} \subset \Omega_{zlb}$, which means the economy travels on the zlb saddle path toward the unintended steady state (π_{zlb}, x_{zlb}). The path constructed is continuous and bounded and has consistent expectations: It is a rational expectations equilibrium. All equilibria in this class can be obtained by picking different q and r.

[14] This result is not immediate, since it may be possible that $(\pi(t), x(t))$ exits Ω_{ss} before getting close to (π_{ss}, x_{ss}) and then follows the Ω_{zlb} dynamics for which (π_{ss}, x_{ss}) is no longer a sink (flowing backward in time). However, I show in Appendix B.5, Lemma 2, item (d) that this never happens. For all q, the path of $(\pi(t), x(t))$ remains entirely in Ω_{ss}.

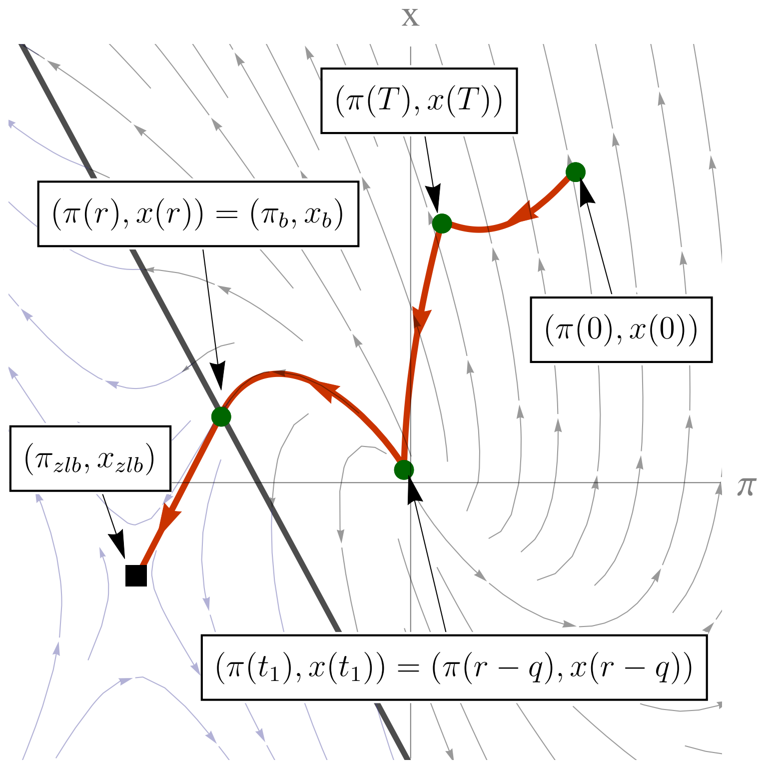

Figure 18: An equilibrium analogous to the one studied by Benhabib et al. (2001b). The flow lines in the background correspond to the dynamics after liftoff, which occurs at t_1. Because the Taylor principle holds, there is a deflationary steady state (π_{zlb}, x_{zlb}), shown as a black square. At time t_1, even though the economy is outside the ZLB and can get arbitrarily close to the "desired" steady state $(\pi_{ss}, x_{ss}) = (0, 0)$, it still converges to the "unintended" steady state (π_{zlb}, x_{zlb}). At time r, the economy enters the ZLB and stays there ($i(t)$=0) forever after.