Financial Vulnerability and Monetary Policy [*]

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Abstract

We present a microfounded New Keynesian model that features financial vulnerabilities. Financial intermediaries' occasionally binding value at risk constraints give rise to variation in the pricing of risk that generates time varying risk in the conditional mean and volatility of the output gap. The conditional mean and volatility of the output gap are negatively related: during times of easy financial conditions, growth tends to be high, and risk tends to be low. Monetary policy affects output directly via the IS curve, and indirectly via the pricing of risk that relates to the tightness of the value at risk constraint. The optimal monetary policy rule always depends on financial vulnerabilities in addition to the output gap, inflation, and the natural rate. We show that a classic Taylor rule exacerbates deviations of the output gap from its target value of zero relative to an optimal interest rate rule that includes vulnerability. Simulations show that optimal policy significantly increases welfare relative to a classic Taylor rule. The model provides a microfoundation for optimal monetary policy frameworks that include financial stability.

Keywords: optimal monetary policy, macro-finance, financial stability, nonlinear DSGE model, continuous time, closed form
JEL classification: G10, G12, E52

1 Introduction

Many central banks have financial stability alongside inflation and employment mandates. Traditionally, prudential policy (both macro and micro) has been used to ensure financial stability, while monetary policy has focused on macroeconomic fluctuations of inflation, output, and employment. However, it is well recognized that prudential policy is unlikely to eliminate all risks to financial stability. Hence monetary policy frameworks in many countries recognize explicitly that financial stability should feed into the monetary policy mandate. However, to date, the literature has not converged on a modeling framework that captures inflation, output, and financial stability objectives in a tractable and parsimonious yet realistic and empirically relevant manner.

When macro-financial linkages are generated by financial vulnerabilities, they can generate downside risks to growth and thus pose threats to financial stability. Bernanke and Gertler (1989) and Bernanke and Blinder (1992) argue for the credit channel of monetary policy where financial frictions shift credit demand or supply curves, thus generating an amplification mechanism that is transmitted via changes in the pricing of risk. Changes in the pricing of risk are generally caused by deeper frictions linked to leverage in the financial and non-financial sectors and the degree of maturity transformation (see Brunnermeier, Eisenbach, and Sannikov (2013) or Adrian, Covitz, and Liang (2015) for surveys on the role of vulnerabilities for macro-financial linkages). Financial vulnerabilities can cause changes in downside risks to GDP, and the pricing of risk tends to reflect financial vulnerabilities. Recent literature on monetary policy has started to explore the role of financial conditions and vulnerabilities in monetary policy settings (see Adrian and Shin (2010), Borio and Zhu (2012)). Curdia and Woodford (2010) and Gambacorta and Signoretti (2014) consder Taylor rules that are augmented with financial variables and study their quantitative relevance. In this paper, we explicitly study the link of financial conditions to downside risks of GDP, and ask to what extent optimal monetary policy should take such downside risks into account.

We present a parsimonious macroeconomic framework for incorporating financial vulnerabilities in monetary policy. Our starting point is the standard New Keynesian (NK) model of Woodford (2003) and Galí (2008). Households have risk averse utility over differentiated products and supply labor to firms that produce intermediate goods. Intermediate good producers have a constant returns to scale technology with labor as the only input for production, and maximize profits subject to a downward-solping demand curve for differentiated products and Calvo-style price stickiness. Their output is sold to the final good

producers in a monopolistically competitive way. Intermediate good producing firms are fully equity financed. Their profits are distributed as dividends to its shareholders. The assumption of being fully equity financed is without loss of generality, as the Modigliani-Miller theorem holds for these firms. The final good sector is perfectly competitive, uses intermediate goods as its only input and has zero profits.

The point of departure from the standard NK model is the existence of banks. Households cannot directly invest in the equity shares of the intermediate good producing sector, as it is assumed that all financing is intermediated by the banks. There is a continuum of identical banks that issue riskless deposits that pay the risk free rate of return. The deposits, as well as the risky equity of banks, are owned by households. Banks maximize profits by investing in all available risky assets in the economy. The banks' portfolio selection problem is subject to a value at risk (VaR) constraint on their net worth (i.e., on bank equity). Banks do not consume and distribute their profits to their shareholders. Relative to earlier NK models with banks, the VaR constraint is the main difference (see, for example, Curdia and Woodford (2010) and Gertler and Karadi (2011)).

The only source of risk in the economy are shocks to the time preference rate of banks. These shocks can capture differences in beliefs or effective risk aversion between banks and the other agents in the economy – rational or otherwise. Because there is only one source of risk, and because bank equity and bank deposits trade continuously, markets are complete. Markets are complete even from the point of view of the households; despite the households' inability to directly hold the equity shares of intermediate good producers, they can replicate its payoffs by trading other assets with banks. However, because of the banks' preference shocks and VaR constraint, the price of the derivative security that replicates the payoff of intermediate good producers' shares will turn out to be distorted from the point of view of the household.

Using techniques from continuous-time financial economics, we solve for the full stochastic non-linear equilibrium in closed form, and characterize it through the conditional means and conditional volatilities of endogenous variables as a function of state variables. To provide a more transparent explanation of how the model works, and to make it easier build upon and compare our model to existing NK models, we also derive a simple four-equation reduced form approximation to our model. Compared to the standard three-equation NK model, our model has two new features. The first new feature is an equation (the "fourth equation") that describes the evolution of a new state variable, GDP-at-risk. GDP-at-risk is defined as the 5th conditional quantile of the one-year-ahead GDP distribution. It summarizes the tail risks to aggregate economic activity that arise from the amplification

of financial shocks created by the VaR constraint of banks. The second new feature is that the IS curve (derived from the Euler equation) has an endogenously time-varying risk premium. The widely used first-order approximation in discrete time NK models has a risk premium that is identically zero by construction. Preserving time variation in the risk premium using the standard methodology would require a third-order approximation with all of its associated challenges. Instead, we use a novel approximation technique that consists in linearizing *first and second moments* of the continuous-time stochastic processes, which preserves the time variation in the risk premium while remaining just as tractable (because stochastic processes are all linear).

The two new elements we introduce – the GDP-at-risk state variable and the time-varying risk premium – interact to create inefficient fluctuations in the real economy. A binding VaR constraint amplifies the responsiveness of banks' optimal portfolio to economic and financial fluctuations. When the risk premium increases, banks invest more in risky assets not only because the risky assets are themselves more attractive — as they would even in the absence of a VaR constraint — but also because their risk-taking capacity has increased as the VaR constraint is relaxed. In the short-term, realized returns on financial assets are high, volatility is low, and employment and output increase. However, the banks' additional investment in risky assets induced by the relaxation of the VaR constraint is financed by debt. The resulting increase in banks' leverage is thus also magnified by the VaR constraint. With higher leverage, future gains and losses are also magnified. Households hold financial assets and trade with banks in complete markets, so they inherit the increased amplification of higher leverage through wealth effects and the pricing of risk. This increased amplification ultimately translates into larger expected fluctuations in consumption and output in the medium run (three to five years after the initial shock). The higher volatility of output expected in the future implies a higher GDP-at-risk. When GDP-at-risk increases, so does the risk premium, because in equilibrium investors require additional compensation for taking the additional tail risk. In turn, the higher risk premium leads to a higher GDP-at-risk, and so on. The same logic applies to negative shocks, which result in faster and deeper deleveraging, and larger declines in consumption and output, than without the VaR constraint.

Our modeling approach is motivated by the empirical evidence that financial conditions forecast tail risks of macroeconomic outcomes. Estrella and Hardouvelis (1991) and Estrella and Mishkin (1998) show that the term spread, an indicator of the pricing of interest rate risk, forecasts recessions. Gilchrist and Zakrajšek (2012), López-Salido, Stein, and Zakrajšek (2016), and Krishnamurthy and Muir (2016) find that credit spreads forecast

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downside risks to GDP growth. More generally, Adrian, Boyarchenko, and Giannone (2019) document that financial conditions are strong forecasters of downside risks to GDP growth, while other non-financial variables such as lagged GDP growth and inflation, are not. More precisely, they find that the upper quantiles of the one- and two-year-ahead GDP growth distribution are more or less constant, while the lower conditional quantiles – such as GDP-at-risk – respond strongly to financial conditions. They explain this empirical regularity by showing that there is a negative conditional (on financial conditions) correlation between the mean and volatility of the GDP distribution rather than, say, time variation in higher moments. Deteriorating financial conditions give rise to a decline in the conditional mean and an increase in the conditional volatility of GDP, and vice-versa for improving financial conditions. The low-mean high-volatility states thus produce a high GDP-at-risk. In addition, the conditional negative correlation between the mean and the volatility of output imply that the unconditional distribution of GDP is highly skewed to the left as a function of financial conditions.

In this paper, we verify that the same patterns that Adrian, Boyarchenko, and Giannone (2019) found for GDP are also present in the output gap, the welfare-relevant variable in NK models. We also show the new result that inflation, the other important determinant of welfare in NK models, does *not* follow the same patterns. Indeed, financial conditions forecast neither the mean nor the volatility of inflation, providing important restrictions for models that attempt to replicate these empirical patterns.

The four equation reduced form NK model gives rise to the same relationship between the conditional mean and the conditional volatility of output documented empirically by Adrian, Boyarchenko, and Giannone (2019). As a result, the model features strongly time varying downside risks to the output gap that are a function of financial conditions, while upside risks are more or less constant.

The tractability of the solution together with the empirically validated connection between financial conditions and the tails of the output gap distribution allow us to study the risk return tradeoff of monetary policy. Monetary policy impacts not only conditional means through the standard intertemporal substitution channel of the Euler equation, but also conditional volatilities by altering the tightness of the VaR constraint of banks, which in turn affects risk premia for the entire aggregate economy.

The central bank is assumed to minimize a standard loss function with squared deviations of the output gap and inflation from target as arguments. We highlight that this inflation and output gap "dual mandate" loss function does not have financial stability as part of its mandate. We can solve for the optimal policy rules in closed form using

dynamic programming, as our new approximation technique preserves the tractability of the linear-quadratic framework while retaining the time variation dynamics of tail risks.

Our optimal monetary policy rule can be cast in the language of a flexible inflation targeting framework, such as the one in Svensson (1999), Rudebusch and Svensson (1999), Svensson (2002), and Giannoni and Woodford (2012). Relative to the standard New Keynesian model, there are two important differences. First, vulnerability enters the optimal rule, not only because it forecasts the distribution of the output gap, but also because it determines the current efficacy in the transmission of monetary policy through the risk-taking channel. Second, the coefficients in the linear optimal targeting criterion rule that trade off deviations in output and inflation from their desired levels, depend on the parameters that govern vulnerability.

Optimal monetary policy can also be expressed as an augmented Taylor rule. The nominal interest rate not only depends on inflation and output, but also on financial vulnerability. The optimal coefficients on output and inflation are different from the standard ones because they take the parameters that govern vulnerability into account.

Our model captures the intuition that in recent years monetary policy has explicitly taken into account and influenced financial conditions.[1] A deterioration of financial conditions corresponds to an increase in tail risk, as conditional GDP volatility rises, while the conditional growth forecast deteriorates. As a result of such an increase in financial vulnerability, i.e. an increase in the downside risk to GDP growth, monetary policy is relatively easier than under the classic Taylor rule. This results in a concurrent lowering of vulnerability, and hence in less severe left skewness of GDP.

Our model also features a volatility paradox. The dynamics of GDP-at-risk imply that times of easy financial conditions associated with low downside risks in the short term tend to be followed by the buildup of tail risks in the medium term. This intertemporal tradeoff is an important consideration in the setting of optimal policy.

2   Financial Vulnerability

Financial vulnerability refers to the presence of amplification mechanisms that are linked to leverage or maturity transformation. Financial vulnerabiltiy is reflected in the pricing of risk. When financial vulnerability is large, small shocks can have severe aggregate macroeconomic consequences (see Adrian, Covitz, and Liang (2015) or Aikman, Kiley, Lee,

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[1]Dudley (2015, 2017); Yellen (2016).

Palumbo, and Warusawitharana (2015)). We follow Adrian, Boyarchenko, and Giannone (2019) by constructing a measure of financial vulnerability from the National Financial Conditions Index (NFCI) of the Federal Reserve Bank of Chicago. That index aggregates 105 financial market, money market, credit supply, and shadow bank indicators to compute a single index using the filtering methodology of Stock and Watson (1998). Adrian, Boyarchenko, and Giannone (2019) show that the conditional GDP distribution features strong downside risk as a function of financial conditions. We reproduce the main results of Adrian, Boyarchenko, and Giannone (2019) here using a conditionally heteroskedastic model to estimate the conditional first and second moments of the output gap

$$y_t = \gamma_0^y + \gamma_1^y y_{t-1} + \gamma_2^y \pi_{t-1} + \gamma_3^y x_{t-1} + \sigma_t^y \varepsilon_t^y \tag{1}$$

$$\ln\left(\sigma_t^y\right) = \delta_0^y + \delta_1^y x_{t-1} \tag{2}$$

where $\varepsilon_t^y \sim N(0,1)$, x_t denotes the NFCI financial conditions index, and y_t is the output gap. Mean output gap also depends on the lagged quarterly core PCE inflation rate π and on the lagged output gap. In addition to estimating the conditional mean and conditional volatility of the output gap, we also estimate an analogous equation for the inflation rate:

$$\pi_t = \gamma_0^\pi + \gamma_1^\pi y_{t-1} + \gamma_2^\pi \pi_{t-1} + \sigma_t \varepsilon_t^\pi \tag{3}$$

$$\ln\left(\sigma_t^\pi\right) = \delta_0^\pi + \delta_1^\pi \pi_{t-1} \tag{4}$$

The model is estimated via maximum likelihood.

Figure 1: **Estimated Conditional Distribution of One Quarter Ahead output Gap and PCE Inflation**. The figure reports estimates from equations (1), (2), (3), and (4). Panel (a) shows the actual output gap, the conditional mean of output gap, and the 5th and 95th quantile. Panel (b) shows the actual PCE inflation, the conditional mean of inflation, and the 5th and 95th quantile.
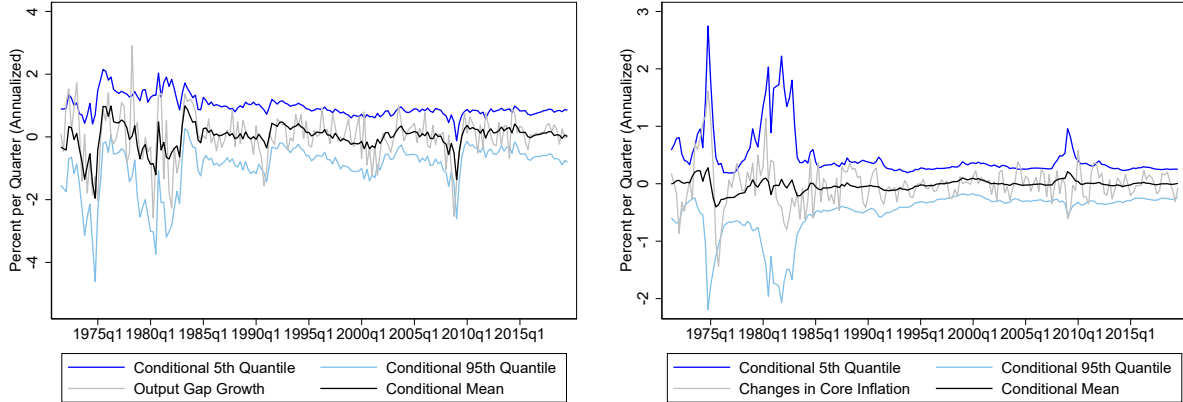

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Table 1: **Output Gap: Conditional Mean and Volatility Estimates**

Output gap	(1) Conditional mean	(2) Conditional volatility
Constant	6.213***	1.785***
	[11.02]	[21.69]
Output gap (lag)	-0.000356	
	[-1.510]	
Inflation (lag)	0.00277	
	[0.0842]	
Financial Conditions (lag)	-1.715***	0.551***
	[-5.096]	[3.765]
Observations	173	173

*** p<0.01, ** p<0.05, * p<0.1

The estimation results are in Figure 1 and Table 1. In Panel (a) of Figure 1, we present the conditional mean of the output gap, the actual realized output gap, and the 5th and 95th quantiles. The distribution is left skewed as deteriorating financial conditions are associated with an increase in conditional volatility, and at the same time a decline in the conditional mean of the output gap (see Table 1). Due the negative correlation of mean and volatility, the unconditional distribution is negatively skewed, even though the conditional distribution is conditionally Gaussian. For inflation, financial conditions are significant neither in the conditional mean nor in the conditional volatility equations. However, the volatility of inflation scales in the level of inflation. Hence, the conditional mean and the conditional volatility are positively correlated. Importantly, financial conditions play no role for inflation dynamics.

The estimates for the output gap have the unusual property that the shift in the conditional mean and volatility offset each other in such a way that the 95th quantile is close to constant. In contrast, the 5th quantile strongly varies as a function of financial conditions. Importantly, this property only arises when the output gap distribution is estimated as a function of financial conditions—real economic indicators do not contain significant information for the tail of the output gap distribution. This is shown more generally by Adrian, Boyarchenko, and Giannone (2019). This property is visualized in Figure 2, which scatters the conditional mean against the conditional volatility for the output gap and PCE inflation. For the output gap, mean and volatility are strongly negatively correlated, but there is no correlation for PCE inflation.

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Figure 2: **Estimated Conditional Mean and Conditional Volatility of One Quarter Ahead Output Gap and PCE Inflation**. The figure reports estimates from equations (1), (2), (3), and (4). Panel (a) plots the output gap mean against the output gap volatility, panel (b) plots PCE inflation mean against PCE inflation volatility.
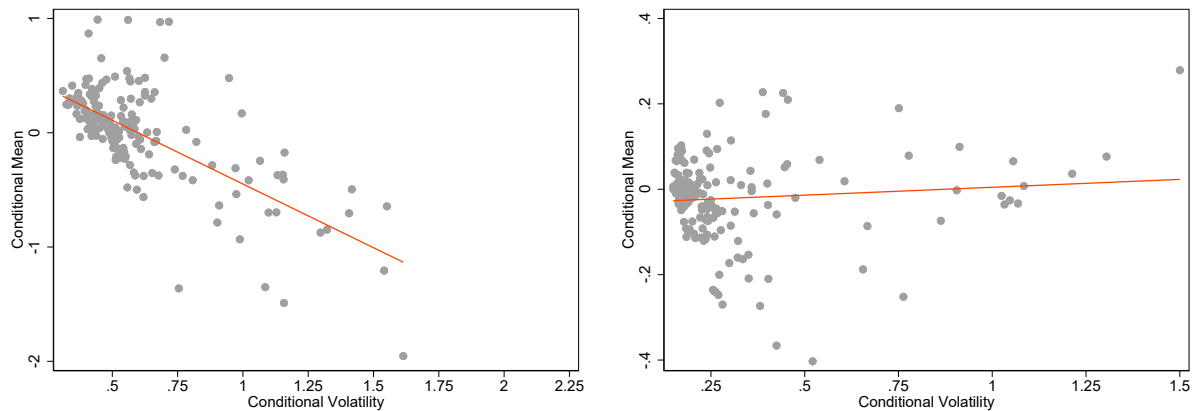
Table 2: **Mean-Volatility tradeoff**

Output Gap	(1) Conditional Mean
Conditional Volatility	-1.115***
	[-9.563]
Constant	0.666***
	[10.95]
Observations	173
R-squared	0.466

*** p<0.01, ** p<0.05, * p<0.1

Inflation	(1) Conditional Mean
Conditional Volatility	0.0369
	[1.007]
Constant	-0.0323***
	[-3.155]
Observations	173
R-squared	0.009

*** p<0.01, ** p<0.05, * p<0.1

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These results suggest that the lower quantiles of the output gap dstributions are related to the link between first and second moments of the GDP gap distribution, conditional on financial conditions. Next, we develop a model that can reproduce the empirical regularities in this section by adding a banking sector with a value-at-risk constraint to a baseline New Keynesian model.

3 Environment

Time is continuous with $t \in [0, \infty)$. There is a continuum of intermediate goods indexed by $i \in [0, 1]$ and a single final good. Output $Y_t(i)$ of each intermediate good of type i can be produced using labor $N_t(i)$ through the constant returns to scale technology

$$Y_t(i) = AN_t(i), \tag{5}$$

where $A > 0$ is the constant aggregate level of TFP. The final good Y_t is produced with the technology

$$Y_t = \left(\int_0^1 Y_t(i)^{1-\frac{1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{6}$$

where $\varepsilon > 1$ is the constant elasticity of substitution across different goods.

There is a continuum of mass one, identical, atomistic, and infinitely lived households who rank consumption streams C_t and labor streams N_t according to

$$\mathbb{E}_0 \int_0^\infty e^{-\beta t} \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\xi}}{1+\xi} \right) dt, \tag{7}$$

where $\beta > 0$ is a time-preference parameter, $\gamma > 0$ is the coefficient of relative risk aversion, and $\xi > 0$ is the inverse of the Frisch elasticity of labor supply. The variable C_t represents a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{8}$$

where $C_t(i)$ is the quantity of differentiated good i consumed by the household. The variable N_t is the labor supplied by the household to all firms and given by

$$N_t \equiv \int_0^1 N_t(i) \, di, \tag{9}$$

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where $N_t(i)$ is the amount of labor supplied to firm i at time t.

The resource constraint of the economy is

$$Y_t(i) = C_t(i), \tag{10}$$

which implies $Y_t = C_t$.

4   The Efficient Allocation (First Best)

The efficient allocation (first best) is obtained by choosing paths of $C_t(i)$, $N_t(i)$ and $Y_t(i)$ to maximize the utility of the representative household in equation (7), subject to the structure of the economy's physical environment described by equations (5), (8), (9), and (10). The solution to this problem is

$$C_t^{FB}(i) = C_t^{FB}(j) = C_t^{FB} \tag{11}$$
$$Y_t^{FB}(i) = Y_t^{FB}(j) = Y_t^{FB} \tag{12}$$
$$N_t^{FB}(i) = N_t^{FB}(j) = N_t^{FB} \tag{13}$$

for all i and j, with aggregate consumption, output, and employment given by

$$C_t^{FB} = Y_t^{FB} = A^{\frac{1-\gamma}{\gamma+\xi}+1} \tag{14}$$
$$N_t^{FB} = A^{\frac{1-\gamma}{\gamma+\xi}} \tag{15}$$

5   The Decentralized Economy

5.1   Overview

There are four types of agents: firms that produce goods, households, banks, and the government.

Good-producing firms are exactly as in the standard New Keynesian (NK) model, except that they are financed by banks instead of households. There are two kinds: intermediate, and final good producers. Final good producers buy intermediate goods from the intermediate good producers and sell the final good to the households. They sell in a perfectly competitive environment and make zero profits, so their ownership and corporate capital structure are irrelevant for equilibrium outcomes. Intermediate good producers

hire labor from the household as the sole input of production in a perfectly competitive and frictionless labor market. Intermediate good producers sell intermediate goods to final good producers in a monopolistically competitive market and set good prices á la Calvo. This structure gives rise to a standard New Keynesian Phillips Curve (NKPC). Intermediate good producers are financed by issuing stocks (equity shares) that pay profits as dividends. The assumption of equity-only financing is without loss of generality since the Modigliani-Miller theorem holds for these firms.

Households are as in the standard NK model with two differences. First, they cannot hold the stocks issued by the intermediate good producers. Second, they can trade a full set of sate-contingent Arrow-Debreu (A-D) securities with the banks without any frictions and, as in the standard NK model, also among themselves.

Banks are intermediation firms that channel households' savings into the financing (that is, the purchasing of the stock) of good producers. Banks are the only agents that have the necessary information, expertise, technology, or relationships to do so.[2] Banks are allowed to trade a full set of A-D securities with the households and among themselves without frictions. Banks finance themselves through the trading of the A-D securities, so their financing can take any state-contigent form, and their capital structure is not a priori restricted. For example, banks could choose to finance themselves with any mix of stocks, short- and long-term debt, and hybrid securities like convertible bonds. Adjusting the capital structure can be done frictionlessly. In particular, the issuance and repurchase of stock are costless.

Banks maximize a risk averse objective function over total distributions to shareholders (dividends plus net stock issuance) by choosing the amount of total distributions and a portfolio of investments in the financial assets available to them, subject to a standard budget constraint and a value-at-risk (VaR) constraint on their wealth[3]. The bank's objective function is subject to preference shocks that are driven by a one-dimensional Brownian motion B_t, which is the only shock in the economy.

The ability of banks and households to trade a full set of A-D securities implies that

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[2]We use the label of "bank" in our model as a short-hand for for any firm that conducts intermediation between households and firms.

Of course, in practice, only a fraction of all firm financing is intermediated. Allowing households to provide a share of financing to firms without any additional frictions is straightforward. However, doing so provides no additional insight in our setup (although it would change the quantitative magnitude of some of our results). See Brunnermeier and Sannikov (2014) and Adrian and Boyarchenko (2012) for examples of households that finance firms along with specialized intermediaries, but with additional frictions that make households less efficient at the task.

[3]Banks' wealth, net worth and equity capital are all the same.

households face complete markets despite the restriction on their holdings of good producers' stocks. Households can trade derivative securities that replicate the payoff of the good producers' stock that they are not allowed to hold directly. However, because of the banks' preference shocks and VaR constraint, the price of such derivative security, and in general of all financial assets, will turn out to be distorted from the point of view of the household.

Government spending is zero. The government provides a proportional subsidy labor income that is financed each period by lump-sum taxes levied on the household. The subsidy is such that it eliminates the distortions arising from the monopolistic market power of intermediate good producers. The central bank sets the short-term (instantaneous) nominal interest rate by paying interest on base money in the cashless limit, as in Woodford (2003).

The social welfare in the decentralized economy can differ from its first-best level due to the presence of three frictions: nominal price rigidities of intermediate good producers, preference shocks to banks and the VaR constraint of banks. The distortions due to the market power of monopolistic firms are corrected by the government subsidy, so they are not a source of inefficiency. Because the first-best levels of output, consumption and employment are constant, all fluctuations in these variables are inefficient. Monetary policy cannot induce the first-best. The first-best would require banks to fully insure households against all risk in all states of the world so that their consumption is constant. But there always exist states of the world in which the risk is too high for banks to insure households and satisfy the VaR constraint at the same time. Monetary policy can shift risk to banks only up to the point allowed by the VaR constraint. If any one of the three frictions that cause inefficiency were removed, then it would be possible for monetary policy to induce the efficient allocation.

5.2   Financial Assets

As mentioned above, the financial assets in the economy are a complete set of A-D securities, and the stocks of the intermediate good producers. Because the only source of uncertainty in this model is a one-dimensional Brownian motion B_t, there is a single A-D security. In addition, a riskless real bond and any single risky asset (that is, exposed to B_t) span all possible payoffs of the economy, including that of the A-D security. We can therefore assume without loss of generality that banks are financed by issuing risky stocks and riskless short-term debt only; we refer to total distributions as dividends from now on. We also correspondingly assume, again without loss of generality, that households

trade the stocks of banks and the riskless bond only. It is then possible to back out the prices, payoffs, and implied portfolio positions in terms of the original A-D security by using no-arbitrage relations and market clearing conditions.

We framed the original discussion of the market structure in terms of "a full set of A-D securities" (rather than the single A-D security that ends up being all that is needed) to emphasize two aspects of our setup. First, our results straightforwardly generalize to a setup with multiple shocks without requiring any additional market-incompleteness friction.[4] The decentralized equilibrium is inefficient despite markets being complete, with one or with many shocks. Second, the capital structure of banks is unrestricted and can be changed costlessly and instantaneously. That banks issue stocks and riskless debt only is a consequence of the single-shock assumption. In a multiple-shock model, the universe of securities available to banks to finance themselves would be larger. In other words, our results do not rely on a specific capital structure for banks.

Without loss of generality, we group all of the stocks of banks into a single banking sector stock, and all intermediate good producer stocks into a single intermediate good producer sector stock. The banking sector stock pays the aggregate dividends of all banks and the producer sector stock pays as dividends the aggregate profits of all producers. Thus, all in all, we characterize the economy in terms of three financial assets: a real riskless bond, the stock of banks and the stock of the intermediate good producers. Households can only hold and trade the riskless bond and the stock of banks, while banks can trade and hold all three assets. Even though banks find one of the stocks redundant (it can be replicated by the other stock and the bond), the restriction on households that they cannot hold the stock of good producers implies that banks must hold both stocks in equilibrium. The bond has a net supply of zero while both of the stocks are in positive net supply normalized to one.

The riskless bond has a price $S_{0,t}$ in real terms that evolves according to $dS_{0,t} = S_{0,t}R_t dt$, where R_t is the equilibrium real short-term interest rate. The two risky assets are indexed by $j \in \{banks, goods\}$. Each stock has real price S_{jt} that satisfies

$$\frac{dS_{jt}}{S_{jt}} = \tilde{\mu}_{jt}dt + \sigma_{jt}dB_t, \tag{16}$$

where $\tilde{\mu}_{jt}$ is the real expected return (including dividends) and σ_{jt} is the volatility.[5] We

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[4]Coletti , Duarte, Feunou, Meh, Zhang (2020).

[5]With some abuse of language, we refer to S_t as the "price" of the stock instead of using the more precise "gain process" terminology.

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define real expected excess returns — the risk premium — by $\mu_{jt} \equiv \tilde{\mu}_{jt} - R_t$, and the market price of risk by $\eta_t \equiv \mu_{jt}/\sigma_{jt}$. We note that η_t is not indexed by j since, in equilibrium, the absence of arbitrage requires that the price of risk is the same for both stocks, that is, $\eta_t = \mu_{banks,t}/\sigma_{banks,t} = \mu_{goods,t}/\sigma_{goods,t}$. The market price of risk is a measure of risk-adjusted expected excess returns and captures the compensation that agents require in equilibrium to hold the risk associated with the Brownian motion B_t. More precisely, the market price of risk is the equilibrium risk premium earned per unit of exposure to B_t.

Last, we define the real state price density (SPD) Q_t as the solution to

$$dQ_t \equiv -Q_t R_t dt - Q_t \eta_t dB_t, \tag{17}$$

$$Q_0 \equiv 1. \tag{18}$$

No arbitrage implies that the price of an asset that pays a stream of cash flows $\{D_s\}_{s \geq t}$ is $\mathbb{E}_t \left[\int_t^\infty (Q_s/Q_t) D_s ds \right]$. The ratio Q_s/Q_t will turn out to be the equilibrium stochastic discount factor (SDF) of the representative household.

5.3   Banks

5.3.1   Optimization Problem

There is a continuum of mass one, identical, atomistic, and infinitely lived banks. The representative bank solves a Merton portfolio problem augmented by a VaR constraint and preference shocks, with all variables expressed in real terms:

$$\max_{\{\theta_s, f_s\}_{s \geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-\beta(s-t)} e^{\zeta_s} \log \left(f_s X_s \right) ds \right] \tag{19}$$

$$s.t.$$

$$\frac{dX_t}{X_t} = \left(R_t - f_t + \theta_t \mu_t \right) dt + \theta_t \sigma_t dB_t, \tag{20}$$

$$VaR_{\tau,\alpha} \left(t, \theta_t, f_t \right) \leq a_V X_t, \tag{21}$$

$$d\zeta_t = -\frac{1}{2} g_t^2 dt - g_t dB_t, \tag{22}$$

$$dg_t = -\kappa_g \left(g_t - m_g \right) dt + \sigma_g dB_t. \tag{23}$$

Although at this point we do not know whether in equilibrium $\sigma_t \neq 0$ a.s. (which is required for markets to be complete), we nevertheless proceed as if were true, since it does turn out to be true anyway (which is easy to verify once the model is solved without having assumed $\sigma_t \neq 0$ in the first place).

The variable X_t is the wealth of the bank, e^{ζ_t} is a preference shock driven by the exogenous stochastic processes in equations (22)-(23), f_t is the share of wealth paid out as dividends to shareholders, and θ_t is the share of wealth invested in the two risky stocks. Equation (20) is the budget constraint. Equation (21) is a VaR constraint. In its maximization problem, the bank takes X_t and the paths $\{R_s, \mu_s, \sigma_s, \zeta_s, g_s\}_{s \geq t}$ as given.

Because one of the stocks can always be replicated by the other stock and the bond, the portfolio choice of the bank only determines the allocation of wealth between the bond and a portfolio of the two stocks, but not the allocation of wealth to each of the two stocks individually. Therefore, μ_t and σ_t in the portfolio choice problem of the bank should be interpreted as the drfit and volatility of the protfolio of stocks and not as vectors of drifts and volatilities that contain the drifts and volatilities of each stock. Similarly, θ_t is the share of wealth invested in the risky portfolio (a single number) and not a vector of portfolio weights for each stock. Given θ_t, the amount that the bank invests in each of the two stocks individually is determined by the market clearing condition that banks hold the entire supply of good producers' stock.

The budget constraint in (20) states that wealth changes are equal to changes in the value of the bank's portfolio minus the dividends paid to shareholders. The bank will be levered most of the time with $\theta_t > 1$ (borrowing can be done by short-selling the bond, which can be also interpreted as issuing riskless demand deposits). When $\theta_t > 1$, the balance sheet is[6]

Assets	Liabilities
risky assets $\theta_t X_t$	debt $(\theta_t - 1) X_t$
	equity X_t

We do not distinguish between book and market values, as they are identical in this model. Log-preferences imply that $X_t > 0$, so the representative bank never undergoes bankruptcy, consistent with its ability to lend and borrow at the riskless rate.

In equation (21), $VaR_{\tau,\alpha}(t, \theta_t, f_t)$ is the value-at-risk of the bank's wealth with horizon $\tau > 0$ at level $\alpha \in (0, 1/2]$. The constant $a_V \in (0, 1)$ defines the VaR limit as a share of wealth X_t. Informally, the constraint says that the bank cannot take too much left tail or downside risk. More formally, $VaR_{\tau,\alpha}(t, \theta_t, f_t)$ is defined as the negative of the αth quantile of the distribution of changes in wealth between t and $t + \tau$, conditional on time-t information, assuming the portfolio weight θ_t, dividends f_t, the interest rate R_t, and the risky asset coefficients μ_t and σ_t remain constant at their time-t levels in the

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[6]If the bank is not levered ($\theta_t \leq 1$), debt is zero, assets are given by $\max\{0, \theta_t X_t\} + (1 - \theta_t) X_t$ and liabilities by $\min\{0, \theta_t X_t\} + X_t$.

interval $[t, t + \tau]$.[7] For example, if $\tau = 1$ month, $\alpha = 0.05$ and $VaR_{\tau,\alpha}$ equals \$100 million, wealth will drop by \$100 million or more over the next month with only 5% probability. If $a_V = 0.1$, the VaR constraint is satisfied if and only if \$100 million is less than 10% of the bank's wealth at time t. Lower values of a_V represent a more restrictive VaR constraint that prescribes smaller tail losses.

Since $B_{t+\tau} - B_t$ is normally distributed, the quantile function of the normal distribution gives

$$
\begin{aligned}
VaR_{\tau,\alpha}(t, \theta_t, f_t) &= X_t \left[1 - \exp\left\{ \frac{1}{dt} \mathbb{E}_t \left[d \log X_t \right] \tau + \mathcal{N}^{-1}(\alpha) \sqrt{\frac{1}{dt} Var_t (d \log X_t) \tau} \right\} \right] \\
&= X_t \left[1 - \exp\left\{ (R_t - f_t + \theta_t \mu_t - \frac{1}{2} \theta_t^2 \sigma_t^2) \tau + \mathcal{N}^{-1}(\alpha) |\theta_t \sigma_t| \sqrt{\tau} \right\} \right],
\end{aligned}
$$

where $\mathcal{N}^{-1}(\alpha) < 0$ is the inverse cumulative distribution function of a standard normal random variable.[8]

$VaR_{\tau,\alpha}$ is larger when the conditional distribution of wealth has lower mean or higher variance. The mean is lower when dividends are paid at a higher rate or the portfolio (of bonds and risky assets together) has lower expected return. The variance is higher when the bank's portfolio of risky assets is more volatile, which can occur because the underlying risky assets are themselves more volatile (larger $|\sigma_t|$) or because the bank invests a larger share of wealth in risky assets ($|\theta_t|$ is higher).

An important consideration when we later study monetary policy is how the risk-free rate R_t affects $VaR_{\tau,\alpha}$. When the bank is levered ($\theta_t > 1$), increasing R_t lowers the conditional mean of the wealth distribution ceteris paribus, since the bank has a negative portfolio position in the bond. The "price of leverage" has increased and honoring debt has just become more expensive. This lowering of the conditional mean of the wealth distribution is a budget constraint effect that is present even when there is no VaR constraint.

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[7]This is not only the standard in the literature (see Cuoco, He, and Isaenko (2008) and Leippold, Trojani, and Vanini (2006) for discussions), it is also generally the case in practice. One reason to fix variables at their current levels is to try to avoid the possibility of gaming the constraint through promised yet unenforceable changes in future behavior, or through projections for the price processes of assets that are deceitfully biased. Fixing the relevant variables at their current levels (or at historical averages) is also required whenever the VaR constraint is regulatory in nature. Last, fixing variables at their current levels can be thought of as an approximation of a VaR without fxing variables that is very accurate for small τ and deteriorates as τ increases.

[8]For a generic stochastic process $dx_t = m_t dt + v_t dB_t$, we use the notation $\frac{1}{dt} \mathbb{E}_t [dx_t]$ or $drift(dx_t)$ to mean $\frac{d}{ds} \mathbb{E}_t [x_s]\big|_{s=t} = m_t$. We use $\frac{1}{dt} Var_t (dx_t)$ to mean $\frac{d}{ds} Var_t (x_s)\big|_{s=t} = v_t^2$, $stoch(dx_t)$ to mean v_t, and $Vol_t(dx_t)$ to mean $|v_t|$.

In turn, the lower conditional mean of the wealth distribution implies a higher $VaR_{\tau,\alpha}$. On the other hand, R_t does not affect the conditional variance of the wealth distribution because riskless bonds are not exposed to the Brownian motion B_t. Thus, the direct effect of higher real rates is to tighten the VaR constraint (and to loosen it when real rates are lower). Changes in R_t can also affect $VaR_{\tau,\alpha}$ through general equilibrium effects that we study later on.

The process for g_t in (23) is an Ornstein–Uhlenbeck process — analogous to an $AR(1)$ process in discrete time — with mean-reversion parameter $\kappa_g > 0$, constant volatility $\sigma_g > 0$, and long-run mean m_g. We choose the specific form of the process for ζ_t in (22) so that the preference shock e^{ζ_t} is a change of probability measure (a Radon–Nikodym derivative), which makes the preference shock interpretable as a belief shock. The bank's problem under the "bank measure" defined by e^{ζ_t} is

$$\max_{\{\theta_s,f_s\}_{s\geq t}} \mathbb{E}_t^{bank} \left[\int_t^\infty e^{-\beta(s-t)} \log\left(f_s X_s\right) ds \right] \tag{24}$$

s.t.

$$\frac{dX_t}{X_t} = \left(R_t - f_t + \theta_t\left(\mu_t - \sigma_t g_t\right)\right) dt + \theta_t \sigma_t dB_t^{bank}, \tag{25}$$

$$VaR_{\tau,\alpha}^{bank}\left(t,\theta_t,f_t\right) \leq a_V X_t, \tag{26}$$

$$dg_t = -\left(\kappa_g + \sigma_g\right)\left(g_t - \frac{\kappa_g m_g}{\kappa_g + \sigma_g}\right) dt + \sigma_g dB_t^{bank}, \tag{27}$$

The conditional expectation $\mathbb{E}_t^{bank}[\cdot]$ and the value-at-risk $VaR_{\tau,\alpha}^{bank}\left(t,\theta_t,f_t\right)$ are now computed according to (25) and (27) rather than (20) and (22)-(23) using that, under the bank measure, $dB_t^{bank} = dB_t + g_t dt$ are increments of a standard Brownian motion.

5.3.2 Solution

The solution to the bank problem (24)-(27) is[9]

$$\theta_t = \frac{1}{\gamma_t}\theta_{M,t}, \tag{28}$$

$$f_t = u\left(\gamma_t, \eta_t^{bank}\right) f_M, \tag{29}$$

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[9]The derivation of the solution to the bank's problem can be found in Appendix C.

where $\eta_t^{bank} \equiv \eta_t - g_t$ is the bank's "effective" price of risk under the bank measure, γ_t is defined by $\gamma_t \equiv \max\{1, \hat{\gamma}_t\}$ with $\hat{\gamma}_t$ such that

$$VaR_{\tau,\alpha}^{bank}\left(t, \frac{1}{\hat{\gamma}_t}\theta_{M,t}, u\left(\hat{\gamma}_t, \eta_t^{bank}\right) f_M\right) = a_V X_t, \tag{30}$$

and the function u is defined by

$$u(\gamma, \eta) \equiv 1 + \frac{\sqrt{\tau}\,|\eta|}{\mathcal{N}^{-1}(\alpha)}\left(1 - \frac{1}{\gamma}\right). \tag{31}$$

To understand this solution, we first look at the solution to the bank problem if the VaR constraint were removed:

$$\begin{aligned} \theta_{M,t} &= \frac{\eta_t^{bank}}{\sigma_t} = \frac{\eta_t - g_t}{\sigma_t}, \\ f_M &= \beta. \end{aligned}$$

This solution is identical to the familiar solution of a Merton portfolio problem. The optimal share of wealth invested in risky assets, $\theta_{M,t}$, is higher when the price of risk η_t^{bank} is higher, and when volatility σ_t is lower. Given η_t and σ_t, the value of g_t shifts the bank's demand for the risky asset by shifting the bank's perceived price of risk; under the bank measure, the bank has no preference shocks but believes expected returns are $\mu_t - \sigma_t g_t$ rather than μ_t. The optimal portfolio is the mean-variance efficient portfolio. Hedging demand is zero due to log-utility.[10] Also because of log-utility, the income and substitution effects for dividends cancel each other, and dividends are paid at a constant rate equal to the time-preference rate $f_M = \beta$.

We now return to the discussion of the full bank problem that includes the VaR constraint. According to (86), the optimal share of wealth invested in risky assets is the same as in the Merton case, $\theta_{M,t}$, but divided by $\gamma_t \geq 1$. We refer to γ_t as the bank's effective risk aversion, since $\theta_{M,t}/\gamma_t$ mirrors the static demand $\theta_{M,t}/\gamma$ (the demand for the mean-variance efficient portfolio) of the optimal portfolio in a standard Merton problem without a VaR constraint but with CRRA utility with coefficient γ. As in the uncon-

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[10]Hedging demand, sometimes also called "dynamic" or "intertemporal" demand, is the demand for risky assets arising from changes in the investment opportunity set. With log-utility, the wealth effects exactly cancel out the re-allocation effects, giving a hedging demand of zero irrespective of the investment opportunity set.

The remaining demand for risky assets is called the "static" or "myopic" demand. It is the demand for the mean-variance efficient portfolio.

strained Merton case, log-utility continues to produce no hedging demand, which means that the optimal portfolio is still the mean-variance efficient portfolio. Therefore, the VaR constraint distorts the size, but not the composition, of the bank's portfolio.

Equation (30) determines the level of effective risk aversion $\hat{\gamma}_t$ that would make the VaR constraint (26) hold with equality under optimal bank behavior. The actual level of effective risk aversion is $\gamma_t = \max\{1, \hat{\gamma}_t\}$, which takes into account that $\hat{\gamma}_t < 1$ means that the VaR constraint is not binding. The degree of effective risk aversion of the bank can be interpreted as a measure of the tightness of the VaR constraint. Indeed, γ_t is a one-to-one function of the Lagrange multiplier λ_t associated to the VaR constraint:

$$1 + \beta\tau\lambda_t = \frac{1}{u\left(\gamma_t, \eta_t^{bank}\right)} \tag{32}$$

We can therefore interpret the function u in equation (31) as providing a conversion between the Lagrange multiplier of the VaR constraint and effective risk aversion.

For a given η_t^{bank}, when the multiplier λ_t is larger, the constraint is tighter and effective risk aversion is higher. The multiplier is zero and the constraint not binding if and only if $\gamma_t = 1$. When the bank is unconstrained, its effective risk aversion of $\gamma_t = 1$ is equal to its structural risk aversion of 1 given by the log-utility. Since $\gamma_t = \max\{1, \hat{\gamma}_t\} \geq 1$, the bank never invests a larger share of wealth in risky assets than when the VaR constrained is removed from the problem. In this sense, the VaR is thus effective at curbing risk-taking.

For a given Lagrange multiplier λ_t, (32) shows that the bank's effective risk aversion γ_t is decreasing in the market price of risk η_t^{bank}. Keeping the tightness of the VaR constraint constant, higher risk-adjusted excess returns are associated with a lower effective risk aversion. One implication is that a binding VaR constraint amplifies the responsiveness of the bank's optimal portfolio to fluctuations in the price of risk. An increase in η_t induces changes in θ_t for two reasons. First, $\theta_{M,t}$ increses. Second, γ_t decreases. Equation (86) shows that these two changes reinforce each other and create a larger increase in θ_t than if γ_t had remained fixed. The intuition is that when risk-adjusted expected returns increase, the bank invests more in risky assets not only because the risky assets are themselves more attractive — as it would even in the absence of a VaR constraint —but also because its risk capacity has increased as the VaR constraint is relaxed.

Equation (29) gives the optimal share of wealth that the bank pays out as dividends. It is equal to the Merton solution $f_M = \beta$ multiplied by $u\left(\gamma_t, \eta_t^{bank}\right)$, defined in (31). Since $u\left(\gamma_t, \eta_t^{bank}\right) \leq 1$, dividends are paid at a lower rate than in the Merton case, consistent with the bank having become more conservative regarding risk. However, as was the

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case for the optimal portfolio θ_t, dividends are more responsive to fundamentals vis-a-vis the case without the VaR constraint. Without the VaR constraint, the dividend rate $f_M = \beta$ is constant, so all fluctuations in f_t stem from the presence of the VaR constraint. Everything else equal, dividends are higher when effective risk aversion γ_t is lower and the VaR constraint less binding (lower λ_t). Dividends are also higher when the market price of risk η_t^{bank} is lower since, everything else equal, investing in the risky asset is less attractive, making dividend payouts more attractive by comparison.

5.3.3 The Bank's State Price Density

The bank's SPD implied by its optimization problem is

$$Q_t^{bank} = \frac{1}{\beta} \frac{e^{-\beta t} e^{\zeta_t}}{\lambda_{bc} X_t} + 2\tau \lambda_t \frac{e^{-\beta t} e^{\zeta_t}}{\lambda_{bc} X_t}$$

The first term captures the marginal value of issuing dividends. The second term gives the marginal value of relaxing the VaR constraint. The constant $\lambda_{bc} > 0$ is such that $Q_0^{bank} = 1$ and can be interpreted as the time-0 Lagrange multiplier on the bank's static budget constraint. That the SPD of the bank moves one-for-one with e^{ζ_t} and is proportional to $1/X_t$ follow directly from the preferences of the bank in equation (24).

5.3.4 Some Thoughts on the Bank's Preference Shocks and VaR Constraint

By writing $\mu_t - \sigma_t g_t = \sigma_t(\eta_t - g_t)$ in the drift of the wealth process in (25) and in the corresponding expression inside $VaR_{\tau,\alpha}^{bank}(t, \theta_t, f_t)$ in (25), we see that problem (24)-(27) has the same structure as problem (19)-(23) under the physical measure with four differences: (i) it has no preference shocks, (ii) the price of risk is $\eta_t^{bank} = \eta_t - g_t$, (iii) in (27), the mean-reversion parameter is higher and the long-run mean is lower and (iv) the probability measure is different. We can then interpret the bank's preference shock not just as a generic belief shock, but as one that makes the bank behave as if it did not have any preference shocks but believed that the market price of risk is η_t^{bank}, and that g_t is a less persistent process with a lower mean.

There are other ways to interpret the preference shocks. For example, they can represent shocks to time-preference rates, risk aversion, habits, optimism, information sets, biases or mistakes; they can be fully rational or behavioral. If the preference shock were removed from banks and its inverse included as preference shocks to households, all equilibrium outcomes would be identical, opening up the possibility for additional interpretations,

such as consumer "demand" shocks. Some of the many alternatives just mentioned lend themselves to a structural interpretation, while others may be the reduced form of a deeper structure. We do not take a stance regarding which interpretation is correct, but point out that since we are interested in monetary policy, non-structural interpretations are adequate only to the extent that the underlying microfoundation of the preference shock remains unaffected by changes in monetary policy.

The same logic applies to interpretations of the VaR constraint. It can be understood as a literal VaR constraint, or as the reduced-form of other risk-management constraints, other regulations, stress-tests, agency problems, and so on. Again, any interpretation can be applied to our results as long as whatever mechanism gives rise to the VaR constraint remains unchanged by monetary policy.

Adrian and Shin (2013) offer empirical evidence and a theoretical model that jointly microfounds a VaR constraint and shocks that, like the bank preference shocks, shift its tightness. In their model, the VaR constraint is the optimal incentive-compatible contract between an itermediary and its creditor when faced with a risk-shifting moral hazard problem. The shocks in their model are to the state of the business cycle. Their results apply to a large family of shock distributions determined by extreme-value theory. The shape of the optimal contract is not affected by monetary policy, as would also be aguably the case for private contracts in actuality.[11] Therefore, for our purpose of studying monetary policy, were this our preferred interpretation, there would be no need to embed the microfoundations of the VaR constraint into the bank problem. Our preference shocks can also be directly mapped to the business cycle shocks in Adrian and Shin (2013) by interpreting them as shocks that affect the business cycle through broad financial conditions (in the case of shocks to banks) or through broad economic conditions (if the shocks were interpreted as demand shocks from households). In our empirical results, we focus on financial conditions, which motivates our choice of having the shocks appear on the bank side of the economy rather than in households, and to interpret them as shocks to financial conditions. The results in Adrian and Shin (2013) therefore show that, at the very least, there exist coherent and empirically relevant microfoundations for our bank setting.

The reduction in risk induced by the VaR constraint is compatible with the risk-shifting moral hazard interpretation of Adrian and Shin (2013). In their model, the optimal contract

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[11]We note that monetary policy can and indeed does change the tightness of the constraint — the numerical value of VaR itself — in both their model and ours. What remains invariant under changes in monetary policy is that the optimal contract places a limit on VaR, that is, has the functional form of a VaR constraint.

reduces the amount of risk taken by the intermediary to levels closer to what the creditor prefers. In our model, the household performs the role of the creditor, delegating the financing of good producers to the bank. The household can have a CRRA coefficient larger than the bank's ($\gamma \geq 1$). If this indeed the case, the bank would, absent the VaR constraint, take too much risk from the point of view of the household, just as in the risk-shifting problem. The VaR constraint acts as the contract that induces the bank to have effective risk aversion of γ_t rather than the log-utility risk aversion of 1 when choosing its portfolio, reducing the risks it takes from $\theta_{M,t}$ to $\theta_{M,t}/\gamma_t$. The closer γ_t is to the household's γ, the closer the alignment between the household's and the bank's preferred size of investment in the mean-variance efficient portfolio.

5.4 Good-producing firms

Since the structure of good-producing firms is exactly as in the standard New Keynesian (NK), we relegate details to Appendix A. Their monopolistically competitive market structure and nominal price rigidities a la Calvo give rise to a non-linear Phillips curve that can be linearized to

$$d\pi_t = (\beta\pi_t - \kappa y_t)\, dt \tag{33}$$

where π_t is inflation, y_t is log-output, and $\kappa > 0$ is a reduced-form parameter that depends on the structural parameters of the model and captures, among other things, the degree of price stickiness.

Intermediate good producers are financed by issuing stocks that are bought by banks. The real profits for the producer of intermediate good i are

$$D_{t,goods}(i) = \frac{P_t(i) Y_t(i)}{P_t} - MC_t Y_t(i)$$

Aggregating across firms gives the aggregate profits for the sector, which are paid out as dividends to shareholders

$$
\begin{aligned}
D_{t,goods} &= \int_0^1 D_{t,goods}(i)\, di \\
&= \frac{1}{P_t} \int_0^1 P_t(i) Y_t(i)\, di - MC_t \int_0^1 Y_t(i)\, di \\
&= \frac{1}{P_t}(P_t Y_t) - MC_t (v_t Y_t) \\
&= (1 - v_t MC_t) Y_t \tag{34}
\end{aligned}
$$

where

$$v_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di$$

is a measure of the aggregate loss of output due to price distortions.

5.5 Households

There is a continuum of mass one, identical, atomistic, and infinitely lived households. The representative household solves

$$\max_{\{C_s, N_s, \omega_s\}_{s \geq t}} \mathbb{E}_t \int_t^\infty e^{-\beta(s-t)} \left(\frac{C_s^{1-\gamma}}{1-\gamma} - \frac{N_s^{1+\xi}}{1+\xi} \right) ds, \tag{35}$$

s.t.

$$\frac{dF_t}{F_t} = \left(R_t + \omega_t \mu_{banks,t} - \frac{1}{F_t} \left(C_t - (1-s_t) \frac{W_t}{P_t} N_t + T_t \right) \right) dt + \omega_t \sigma_{banks,t} dB_t, \tag{36}$$

$$\lim_{s \to \infty} \mathbb{E}_t [Q_s F_s] = 0. \tag{37}$$

The household maximizes utility by choosing the path of final good consumption C_t, supply of labor N_t, and share of wealth (portfolio weight) on the risky asset ω_t, subject to the budget constraint (36), and a solvency constraint (transversality condition). The variable F_t is the household's real financial wealth, R_t is the riskless real rate, $\mu_{banks,t}$ and $\sigma_{banks,t}$ are the is the expected returns and volatility of the bank stock, s_t is a labor tax (or subsidy, if negative), T_t are lump-sum taxes, and Q_t is the real state-price density (SPD). In its maximization, the household takes $\left\{ P_t, R_t, W_t, \mu_{banks,t}, \sigma_{banks,t}, s_t, T_t, Q_t \right\}_{s \geq t}$ and F_t as given.

The dynamic flow budget constraint in equation (36) states that changes in the household's financial wealth must be equal to the payoff on financial assets (which can be negative), minus nominal consumption expenditures, plus after-tax nominal labor income, plus lump-sum taxes. The transversality condition in equation (37) is a no-Ponzi condition for the household.

The FOC for consumption and labor are

$$Q_t = e^{-\beta t} C_t^{-\gamma}, \tag{38}$$

$$C_t^\gamma N_t^\xi = (1-s_t) \frac{W_t}{P_t}. \tag{39}$$

With complete markets, the flow budget constraint (35) and the transversality condition

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(37) are equivalent to the static budget constraint

$$F_t = \mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \left(C_s - (1 - s_s) \frac{W_s}{P_s} N_s + T_s \right) ds \qquad (40)$$

Given the optimal choice of consumption and labor, the portfolio weight ω_t is implicitly determined by equation (36).

6 Asset Pricing

6.1 Stock Prices and Portfolio Positions

The prices $S_{banks,t}$ and $S_{goods,t}$ of the two stocks in the economy are equal to the expected present discounted value of their dividends:

$$
\begin{aligned}
S_{t,goods} &= E_t \left[\int_t^\infty \frac{Q_s}{Q_t} D_{s,goods} ds \right] \\
S_{t,banks} &= E_t \left[\int_t^\infty \frac{Q_s}{Q_t} (f_s X_s) ds \right]
\end{aligned}
$$

Since θ_t finances the cash flow $f_t X_t$ and $S_{banks,t}$ pays dividends $f_t X_t$, the absence of arbitrage implies that

$$S_{banks,t} = X_t$$

In addition, equation (34), goods market clearing, and the optimality conditions of the household and firms gives

$$D_{t,goods} = C_s - (1 - s_s) \frac{W_s}{P_s} N_s + T_s$$

which implies that the dividends of the stock of good producers are equal to the representative household's cash flows, and hence

$$S_{t,goods} = F_t$$

Market clearing of stocks and bonds, together with the above relations, allow us to split the share of wealth that the bank invests in its porfolio of risky assets, θ_t, into the portions

invested in the stocks of good producers and banks,

$$\theta_{goods,t} = \frac{F_t}{X_t}$$

$$\theta_{banks,t} = \theta_t - \frac{F_t}{X_t}$$

where $\theta_t = \theta_{goods,t} + \theta_{banks,t}$. The bank holds all of the good producer's stock, investing and amount $\theta_{goods,t} X_t = S_{t,goods} = F_t$ on it. We note that $\theta_{goods,t}$ can be written without direct reference to θ_t, which reflects that, in equilibrium, banks must be the ones holding all the good producer's stock, irrespective of optimality or its VaR constraint. The bank also invests $\theta_t X_t - F_t$ on the bank's stock. The remaining wealth, $X_t - (\theta_{goods,t} X_t) - (\theta_t X_t - F_t) = (1 - \theta_t) X_t$, is invested in riskless bonds.

Similarly, the household's portfolio position can be written as

$$\omega_t = 1 + (1 - \theta_t) \frac{X_t}{F_t}$$

The household invests $\omega_t F_t$ in the bank stock, and the rest of its wealth in the riskless bond. When the bank is levered with $\theta_t > 1$, the household must be long (have a positive position on) the riskless bond, since it is in zero net supply. A household with constant levels of consumption and employment —as, for example, in the first best allocation— must have no risk and $\omega_t = 0$. In this case, $\theta_{banks,t} = 1$ and $\theta_{goods,t} = F_t/X_t$.

6.2 The Market Price of Risk

We now find the price of risk η_t, as a function of the exogenous variables ζ_t and g_t. The key relation we use is that, because households and banks trade with each other in a complete market, their state price densities (equivalently, their stochastic discount factors) must agree

$$Q_t = Q_t^{bank}$$

Equating the stochastic parts of $-dQ_t/Q_t$ and $-dQ_t^{bank}/Q_t^{banks}$ gives

$$\eta_t = \frac{\eta_t - g_t}{\gamma_t} + g_t - stoch\left(d\log\left(\frac{1}{\beta} + 2\tau\lambda_t\right)\right) \tag{41}$$

The left-hand side of the equation gives η_t, the household's required compensation for taking on one unit of B_t-risk. The right-hand side gives the bank's required compensation

for risk. The first term, $(\eta_t - g_t)/\gamma_t$, gives the bank's required risk compensation for changes in its wealth. Instead of η_t, the bank uses its "perceived" $\eta_t^{bank} = \eta_t - g_t$. Similarly, it uses its effective risk aversion γ_t, rather than its log-utility risk aversion of 1. The second term is the volatility comes from the direct effect of the preference shock e^{ζ_t} in the bank's preferences in equation (19). Becuase e^{ζ_t} appears directly in the preferences of the bank, its volatility of g_t shifts the volatility of bank's SPD one-for-one. The last term gives the risk compensation required from changes in the tightness of the VaR constraint.

Equation (41) shows that prefence shocks and the VaR constraint are both necessary for the bank's behavior to have any influence on the household's pricing of risk. If the VaR constraint is not binding, $\gamma_t = 1$ and $\lambda_t = 0$, and the right-hand side becomes η_t. In other words, if the VaR were never binding, or absent altogether, the bank's pricing of risk would be identical to the household's at all times, and the bank's behavior would impose no restrictions on the equilibrium price of risk. If, instead, there were no preference shocks, $g_t = 0$. Because there is no risk, the VaR constraint would either not bind, in which case $\gamma_t = 1$ and $\lambda_t = 0$, or it would bind in a non-time varying way, giving a constant $\gamma_t = \bar{\gamma} > 1$ and a constant $\lambda_t > 0$. A constant λ_t makes the last term on the left-hand side of (41) equal to zero. Therefore, in the absence of shocks, equation (41) would read $\eta_t = \eta_t/\bar{\gamma}$, which implies that the VaR would never bind in equilibrium and, once again, make the bank's behavior irrelevant for the household's pricing of risk.

Now returning to the general case with the VaR constraint and preference shocks, the last term on the left-hand side of (41) can be written explicitly as a function of g_t and R_t

$$stoch\left(d\log\left(\frac{1}{\beta} + 2\tau\lambda_t\right)\right) = \frac{\Phi_g(g_t, R_t)}{\Phi(g_t, R_t)}$$

where Φ_g denotes the partial derivative with respect to g_t and

$$\Phi(g_t, R_t) = \begin{cases} K_2 & , \text{ if } \gamma_t = 1 \\ \frac{1}{2\beta^2\lambda_0^2}\left(e^{K_1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}\sigma_g}g} + 2\beta\lambda_0 + e^{\frac{1}{2}\left(K_1 - \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}\sigma_g}g\right)}\sqrt{e^{K_1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}\sigma_g}g} + 4\beta\lambda_0}\right) & , \text{ if } 1 < \gamma_t < \infty \\ \frac{1}{\lambda_0}\left(\frac{2}{R_t - \frac{1}{\tau}\log(1 - a_V)} - \frac{1}{\beta}\right) & , \text{ if } \gamma_t = \infty \end{cases}$$

where K_1 and K_2 are integration constants that make Φ a continuous function of g_t. The function Φ has different behavior in the three different regions that are determined by γ_t. When $\gamma_t = 1$, the VaR constraint does not bind, Φ is constant, and $\Phi_g/\Phi = 0$, similar to the case discussed above. On the other extreme, the VaR can be so tigh that the bank

only holds riskless bonds, corresponding to the case $\gamma_t = \infty$. Because the bank's portfolio consists only of bonds, the behavior of Φ is fully determined by the interest rate R_t. The bank's risk compensation required by changes in the tightness of the VaR constraint, Φ_g/Φ, depends on R_t and on how R_t responds to g_t. In the intermediate case in which the VaR is binding but the bank holds some risky assets, $1 < \gamma_t < \infty$. In this region, $\Phi(g_t, R_t)$ is only a function of g_t. However, R_t does have an indirect influence on Φ in this region, as it determines K_1. If $\underline{g}$ a point in the boundary between the cases $1 < \gamma_t < \infty$ and $\gamma_t = \infty$ then the value of K_1 is found by making sure that $\Phi(g_t, R_t)$ is equal to

$$\frac{1}{\lambda_0} \left(\frac{2}{R\left(\underline{g}\right) - \frac{1}{\tau}\log\left(1 - a_V\right)} - \frac{1}{\beta} \right)$$

as $\gamma_t \to \infty$. We note that writing R_t solely as a function of g_t is without loss of generality within the class of Markov equilibria, since g_t fully determines ζ_t and, together, g_t and ζ_t are the only exogenous variables.

7 GDP-at-Risk

To connect the empirical results in Section 2 to the model, just as we defined VaR for the bank's wealth, we can define "GDP-at-risk" by

$$GaR_t \equiv Y_t \left[1 - \exp\left\{ \frac{1}{dt}\mathbb{E}_t\left[d\log Y_t\right]\tau + \mathcal{N}^{-1}\left(\alpha\right)\sqrt{\frac{1}{dt}Var_t\left(d\log Y_t\right)\tau} \right\} \right]$$

Or, in logs,

$$
\begin{aligned}
V_t &\equiv -\log\left(1 - \frac{VaR_{\tau,\alpha}\left(Y_t\right)}{Y_t}\right) \\
&= -\frac{1}{dt}\mathbb{E}_t\left[d\log Y_t\right]\tau - \mathcal{N}^{-1}\left(\alpha\right)\sqrt{\frac{1}{dt}Var_t\left(d\log Y_t\right)\tau}
\end{aligned}
$$

Differentiating the household's Euler equation in (38) gives

$$d\log Y_t = \frac{1}{\gamma}\left(R_t - \beta + \frac{1}{2}\eta_t^2\right)dt + \frac{1}{\gamma}\eta_t dB_t \tag{42}$$

so that

$$V_t = \frac{1}{\gamma}\left(R_t - \beta + \frac{1}{2}\eta_t^2\right)\tau + \frac{\mathcal{N}^{-1}\left(\alpha\right)\sqrt{\tau}}{\gamma}\eta_t$$

Solving for R_t gives

$$R_t = -\frac{\gamma}{\tau}V_t - \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}}|\eta_t| + \beta - \frac{1}{2}\eta_t^2 \tag{43}$$

Plugging this equation in the bank's VaR constraint, equation (21), with equality, and using the relation between the bank's and household's SDF in equation (41) gives η_t as a function of g_t and V_t

$$\eta_t = -\frac{\gamma}{\tau g_t}V_t + S(g_t) \tag{44}$$

where S is a function of g_t only, given by

$$\begin{aligned}
S(g_t) &= -\frac{1}{\tau g_t}\log(1 - a_V) - \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}} + \frac{1}{2}g_t \\
&\quad -\frac{\sigma_g}{\sqrt{\tau}}\left(\frac{\tau\beta}{\mathcal{N}^{-1}(\alpha)} + \mathcal{N}^{-1}(\alpha)\right)\frac{|\Phi'(g_t)|}{\Phi(g_t)}\frac{1}{g_t} - \frac{\sigma_g^2}{2g_t}\left(\frac{\Phi'(g_t)}{\Phi(g_t)}\right)^2
\end{aligned}$$

Equation (44) is not a new result, it is just a way to express some of the results in the previous section in terms of V_t rather than R_t.

Linearizing equation (44) around $\hat{\eta}$ and plugging into (42) gives

$$d\log Y_t = \frac{1}{\gamma}\left(R_t - \beta + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right)dt + \xi(V_t - s_t)dZ_t$$

where $\hat{\eta}$ and ξ are linearization constants and s_t is a random variable that depends linearly on g_t. We therefore have

$$\begin{aligned}
\mathbb{E}_t[dy_t] &= \frac{1}{\gamma}\left(R_t - \beta + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right) \\
\mathbb{V}_t[dy_t] &= \xi(V_t - s_t)
\end{aligned}$$

Solving for R_t and V_t in (42) to get

$$R_t = \beta - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right)s_t + \frac{1}{2}\eta^2 \tag{45}$$

Plug in (45) into (7) to get

$$dy_t = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi(V_t - s_t)dZ_t \tag{46}$$

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Use

$$\mathbb{E}_t[dy_t] = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)$$

$$\mathbb{V}_t[dy_t] = \xi(V_t - s_t)$$

and then eliminating V_t to get

$$\mathbb{E}_t[dy_t] = -\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi}\mathbb{V}_t[dy_t] - \frac{1}{\tau}s_t \tag{47}$$

We have thus obtained the mean-volatility line of Figure 2. Equation (47) also makes clear that the shocks s_t are shifts to vulnerability that shift the mean-volatility line up and down, while all other changes in the economy involve moving along the mean-volaitlity line. Empirically, slope is negative and intercept is positive on average

$$-\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi} < 0$$

$$-\frac{\bar{s}}{\tau} > 0$$

This implies $\bar{s} < 0$ and

$$\xi > 0 \text{ and } 1 + \alpha\sqrt{\tau} > 0$$

$$\text{or}$$

$$\xi < 0 \text{ and } 1 + \alpha\sqrt{\tau} < 0$$

To match empirical estimates, we set

$$\alpha = -1.645$$

$$\sqrt{\tau} = 1$$

To match the actual slope and intercept

$$-\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi} = -1.15$$

$$\bar{s} = -0.67\tau$$

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which gives

$$\xi = 0.36$$
$$\bar{s} = -0.67$$

We identify $s_t - \bar{s}$ with the residuals of the regression of $\mathbb{V}_t[dy_t]$ on $\mathbb{V}_t[dy_t]$. The standard deviation and $AR(1)$ coefficient of these residuals then identify σ_s and κ, respectively. Since

$$Std\left(-\frac{1}{\tau}(s_t - \bar{s})\right) = 0.62$$
$$AR(1) = 0.12$$

we get, converting to annualized values

$$\kappa = -\log(0.12) = 2.12$$
$$\sigma_s = 0.31$$

8 Monetary Policy

8.1 Optimal Monetary Policy

The central bank is minimizing a quadratic loss function over the output gap and inflation

$$L(y_t, \pi_t) = \min_{i_t} \mathbb{E}_t \int_t^\infty e^{-t\beta} \left(y_t^2 + \pi_t^2\right) dt. \tag{48}$$

subject to the dynamics of the economy. Minimizing the quadratic loss function is a standard approach in the NK literature, as Rotemberg and Woodford (1997), Rotemberg and Woodford (1999) and Woodford (2003) have shown that aggregate welfare can be approximated by such a loss function.

We focus on the case with fully fixed prices first. The interest rate R_t can be eliminated

from the optimization problem, so that the central bank's problem can be written as

$$L\left(y_t, s_t\right) = \min_{\{V_s\}_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-s\beta} y_s^2 ds \tag{49}$$

$$s.t.$$

$$V_t = \frac{-\gamma^{-1}\left(R_t - \beta\right) + \alpha\xi s_t\sqrt{\tau} + \hat{\eta}\xi\left(s_t + \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\tau}{1 + \alpha\xi\sqrt{\tau} + \hat{\eta}\xi\tau} \tag{50}$$

$$dy_t = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{51}$$

$$ds_t = -\kappa\left(s_t - \bar{s}\right) + \sigma_s dZ_t \tag{52}$$

The central bank thus effectively picks V_t, which is connected to R_t in a one-to-one fashion by

$$R_t = \beta - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right)s_t + \frac{1}{2}\eta^2$$

The Hamilton-Jacobi-Bellman (HJB) equation for the central banker's optimization is

$$
\begin{aligned}
0 = & \min_V \left\{ y^2 - \beta L - \frac{\partial L}{\partial y}\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s\right) + \frac{1}{2}\frac{\partial^2 L}{\partial y^2}\xi^2\left(V - s\right)^2 \right\} \\
& -\kappa\left(s - \bar{s}\right)\frac{\partial L}{\partial s} + \frac{1}{2}\frac{\partial^2 L}{\partial s^2}\sigma_s^2
\end{aligned}
$$

Intuitively, the HJB takes into account the current value of welfare, as well as the change in welfare associated with changes in the state variables y and s.

The first order condition is

$$0 = -\frac{\partial L}{\partial y}\frac{\alpha\sqrt{\tau}\xi + 1}{\tau} + \frac{\partial^2 L}{\partial y^2}\xi^2\left(V - s\right) \tag{53}$$

$$V = \frac{\partial L}{\partial y}\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi^2}\left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} + s \tag{54}$$

Hence at the optimum, vulnerability is proportional to s, and depends on the first and second derivative of welfare with respect to output. It is also noteworthy that $\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi^2}$, which defines the slope of output volatility with respect to expected output, appears in the FOC.

We look for a quadratic solution of the form

$$L\left(y, x\right) = c_0 + c_1 y + c_2 y^2 + c_3 s + c_4 s^2 + c_5 ys$$

where $c.$ are constants.

Plugging into the HJB, and using

$$\frac{\partial L}{\partial y} = c_1 + 2c_2 y + c_5 s$$

$$\frac{\partial^2 L}{\partial y^2} = 2c_2$$

$$\frac{\partial L}{\partial s} = c_3 + 2c_4 s + c_5 y$$

$$\frac{\partial^2 L}{\partial s^2} = 2c_4$$

we get the following system of equations on the coefficients $c_0, ..., c_5$

$$[y^2] \quad : \quad 0 = \left(-\beta - \frac{1}{\tau^2 \xi^2} \left(\alpha \sqrt{\tau} \xi + 1 \right)^2 \right) c_2 + 1$$

$$[ys] \quad : \quad 0 = \left(-\frac{2}{\tau} \right) c_2 + \left(-\frac{1}{\tau^2 \xi^2} \left(2\alpha \sqrt{\tau} \xi + \alpha^2 \tau \xi^2 + \beta \tau^2 \xi^2 + 1 \right) \right) c_5$$

$$[y] \quad : \quad 0 = -\frac{1}{\tau^2 \xi^2} c_1 \left(2\alpha \sqrt{\tau} \xi + \alpha^2 \tau \xi^2 + \beta \tau^2 \xi^2 + 1 \right)$$

$$[s^2] \quad : \quad 0 = -\frac{1}{4\tau^2 \xi^2 c_2} \left(c_5^2 \left(2\alpha \sqrt{\tau} \xi + \alpha^2 \tau \xi^2 + 1 \right) + 4\tau \xi^2 c_2 c_5 + 4\beta \tau^2 \xi^2 c_2 c_4 \right)$$

$$[s] \quad : \quad 0 = -\frac{1}{2\tau^2 \xi^2 c_2} \left(c_1 c_5 \left(2\alpha \sqrt{\tau} \xi + \alpha^2 \tau \xi^2 + 1 \right) + 2\tau \xi^2 c_1 c_2 + 2\beta \tau^2 \xi^2 c_2 c_3 \right)$$

$$[const] \quad : \quad 0 = -\frac{1}{4\tau^2 \xi^2 c_2} \left(c_1^2 \left(2\alpha \sqrt{\tau} \xi + \alpha^2 \tau \xi^2 + 1 \right) + 4\beta \tau^2 \xi^2 c_0 c_2 - 4\tau^2 \xi^2 \sigma_s^2 c_2 c_4 \right)$$

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with solution

$$c_0 = \frac{\tau^2 \xi^4 \sigma_s^2 \left((\alpha\sqrt{\tau}\xi + 1)^2 + 2\beta\tau^2\xi^2 \right)}{\beta^2 \left((\alpha\sqrt{\tau}\xi + 1)^2 + \beta\tau^2\xi^2 \right)^3} > 0$$

$$c_1 = 0$$

$$c_2 = \frac{\tau^2\xi^2}{\tau^2\xi^2\beta + (\alpha\sqrt{\tau}\xi + 1)^2} > 0$$

$$c_3 = 0$$

$$c_4 = \frac{\xi^4\tau^2 \left((\alpha\sqrt{\tau}\xi + 1)^2 + 2\beta\tau^2\xi^2 \right)}{\beta \left((\alpha\sqrt{\tau}\xi + 1)^2 + \beta\tau^2\xi^2 \right)^3} > 0$$

$$c_5 = -\frac{2\tau^3\xi^4}{\left((\alpha\sqrt{\tau}\xi + 1)^2 + \beta\tau^2\xi^2 \right)^2} < 0$$

To pick the optimal initial conditions, we minimize L with respect to y_0 taking s_0 as given

$$L(y_0, s_0) = c_0 + c_1 y_0 + c_2 y_0^2 + c_3 s_0 + c_4 s_0^2 + c_5 y_0 s_0$$

$$FOC : \frac{\partial L}{\partial y_0} = 0$$

$$SOC : \frac{\partial^2 L}{\partial y_0^2} > 0$$

The FOC and SOC can be solved to get

$$y_0^* = -\left(\frac{c_1}{2c_2} + \frac{c_5}{2c_2} s_0 \right)$$

$$= \frac{\tau\xi^2 s_0}{(\alpha\sqrt{\tau}\xi + 1)^2 + \beta\tau^2\xi^2}$$

$$c_2 > 0$$

The optimal policy in terms of V_t is given by plugging in the optimal solution into the FOC in equation (53):

$$V = \frac{(\alpha\sqrt{\tau}\xi + 1)}{\tau\xi^2} y + \left(1 - \frac{(\alpha\sqrt{\tau}\xi + 1)}{(\alpha\sqrt{\tau}\xi + 1)^2 + \beta\tau^2\xi^2} \right) s \tag{55}$$

This can be viewed as a "flexible inflation targeting rule" (see Svensson (1999), Svensson (2002) and Rudebusch and Svensson (1999)) or, more generally, as a linear optimal targeting criterion (Giannoni and Woodford (2012)). Even though vulnerability and its shocks, V_t and s_t, are not target variables, i.e., they do not appear in the loss function equation (48), they still enter the inflation targeting rule, the first-order condition given by equation (55). There are no independent target values for V_t and s_t that the central bank hopes to achieve. The reason V_t and s_t enter the targeting rule is that they forecast the conditional mean and variance of y_t even after controlling for the information already contained in the mean of y_t itself (more generally, in the means of y_t and π_t when a Phillips Curve is included). This is consistent with the empirical results in Table 1 and with the findings in Adrian, Boyarchenko, and Giannone (2019), who show that financial conditions are excellent predictors of the tail of the GDP distribution in a way that non-financial variables are not. Alternatively, equation (55) can be interpreted as a traditional flexible inflation targeting rule in which the targets for inflation and/or output are time-varying and depend on V_t and s_t. It also important to note that even if a central bank decided not to condition its actions on V_t and s_t, the tradeoff between inflation and output –reflected in the coefficients of the rule in equation (55)– now depends on γ and ξ, the parameters that dictate the strength of the mean-variance tradeoff of output.

Using the optimal solution in the process for the output gap in equation (51), we then find that

$$
\begin{aligned}
dy_t &= -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi\left(V_t - s_t\right)dZ_t \\
&= -\left(\frac{(\alpha\sqrt{\tau}\xi + 1)^2}{\tau^2\xi^2}y_t + \frac{\beta\tau\xi^2}{(\alpha\sqrt{\tau}\xi + 1)^2 + \beta\tau^2\xi^2}s_t\right)dt \\
&\quad + \left(\frac{(\alpha\sqrt{\tau}\xi + 1)}{\tau\xi}y_t - \frac{\xi\left(\alpha\sqrt{\tau}\xi + 1\right)}{(\alpha\sqrt{\tau}\xi + 1)^2 + \beta\tau^2\xi^2}s_t\right)dZ_t
\end{aligned}
$$

Recalling that

$$
\mathbb{E}_t\left[dy_t\right] = -\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi}\mathbb{V}_t\left[dy_t\right] - \frac{1}{\tau}s_t \tag{56}
$$

And defining the slope as

$$
M \equiv -\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi}
$$

we get

$$
V = -\frac{M}{\xi}y + \left(1 + \frac{M}{\tau\xi\left(M^2 + \beta\right)}\right)s
$$

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and
$$dy_t = -\left(M^2 \times y_t + \frac{\beta/\tau}{M^2 + \beta} \times s_t\right) dt - \left(M \times y_t - \frac{M/\tau}{M^2 + \beta} \times s_t\right) dZ_t \qquad (57)$$

The last equation makes clear that the magnitude of the tradeoff between stabilizing the mean and variance of the output gap is given by the slope M of the mean-volatility line in Figure 2.

We can also express monetary policy as an interest rate rule. Using the FOC for V, the optimal interest rate is

$$
\begin{aligned}
R_t &= \beta - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right) V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right) s_t + \frac{1}{2}\eta^2 \\
&= \beta + \frac{1}{2}\eta^2 - \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \frac{1}{\tau} + \eta\right) V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right) s_t \\
&= \gamma M\left(\eta + \frac{\alpha}{\sqrt{\tau}}\right)\frac{M^2 + \beta}{\beta + M^2 + M/\tau\xi} y_t - \gamma\xi\left(\left(\eta + \frac{\alpha}{\sqrt{\tau}}\right)\frac{M/\tau\xi}{\beta + M^2 + M/\tau\xi} + \frac{1}{\tau}\right) V_t \\
&\quad + \left(\frac{1}{2}\eta^2 + \beta\right)
\end{aligned}
$$

The optimal interest rule can thus be viewed as an augmented Taylor rule. In addition to the output gap y and the equilibrium rate of interest r (and inflation π_t in the more general case), the level of vulnerability V enters the optimal rule. As before, the coefficients on y (and π in the more general case) depend on the parameters that define vulnerability ξ and γ and thus monetary policy is different from the typical NK model without vulnerabilities not only because vulnerability enters the augmented Taylor rule directly, but also because the presence of vulnerabilities alter the optimal response of interest rates to changes in output and inflation.

8.2 Alternative Monetary Policy Rules

In general, the central bank might follow other monetary policy rules. We consider alternative linear rules that do not explicitly condition on vulnerability or its shocks:

$$i_t = \psi_0 + \psi_y y_t \qquad (58)$$

We show that even after picking the coefficients ψ_0, ψ_y in an optimal way, the rule in equation (58) implies quantitatively large welfare losses compared to the optimal monetary policy found in the last section. To find the coefficients ψ_0, ψ_y that minimize welfare losses,

we solve

$$\min_{\left(\psi_0, \psi_y\right)} L\left(y_0, s_0\right) \tag{59}$$

$$s.t.$$

$$dy_t = \frac{1}{\gamma}\left(i_t - r + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right) dt + \xi\left(V_t - s_t\right) dZ_t \tag{60}$$

$$i_t = \psi_0 + \psi_y y_t \tag{61}$$

$$V_t = -\mathbb{E}_t\left[dy_t\right]\tau - \alpha\mathbb{V}_t\left[dy_t\right]\sqrt{\tau} \tag{62}$$

$$ds_t = -\kappa\left(s_t - \overline{s}\right) + \sigma_s dZ_t \tag{63}$$

Figure 3 shows the steady-state distribution of the output gap y_t using the optimal policy rule that explicitly takes vulnerability into account (using equation (57)), and the Taylor-type rule that does not condition on vulnerability V_t, given by equation (58) with coefficients found by solving (59)-(63). Intuitively, shocks to vulnerability s contain information about the conditional distribution of the output gap that the policy maker should take into account in setting optimal policy. For a given level of the output gap, a higher vulnerability –a larger VaR of output– calls for higher interest rates. Higher interest rates induce the private sector to save more and consume less, thus shifting the conditional future distribution of y_t upwards by shifting its conditional mean upwards. Given the link between the expected mean and the expected volatility of output induced by the presence of vulnerability, a higher conditional mean induces a lower volatility of y_t. Together, higher mean and lower volatility mean lower vulnerability – lower VaR for output. For the suboptimal Taylor rule that ignores vulnerability, interest rates remain unchanged when, for a given level of y_t, V_t changes. Compared to the optimal rule, when V_t increases but i_t remains unchanged, the conditional mean of output is lower and its conditional volatility is higher. Over time, more frequent visit to states of lower mean and higher volatility create an unconditional distribution that is more negatively skewed. When instead V_t decreases, the optimal rule and the suboptimal Taylor rule produce similar right tails for the unconditional distribution of output. The reason is that lower V_t induces both higher mean and lower volatility of output. Therefore, even though the changes in mean and volatility of y_t are different for the two different rules, the actual differences in outcomes for y_t are small because the lower volatility minimizes all fluctuations.

Figure 3: **Probability Density Functions of the Output under the Optimal Policy Rule and a Standard Taylor Rule**. The figure shows the PDFs using the optimal policy rule and the standard Taylor rule. The standard Taylor rule coefficients are calculated for the economy assuming that the policy maker is ignoring the presence of financial vulnerability.
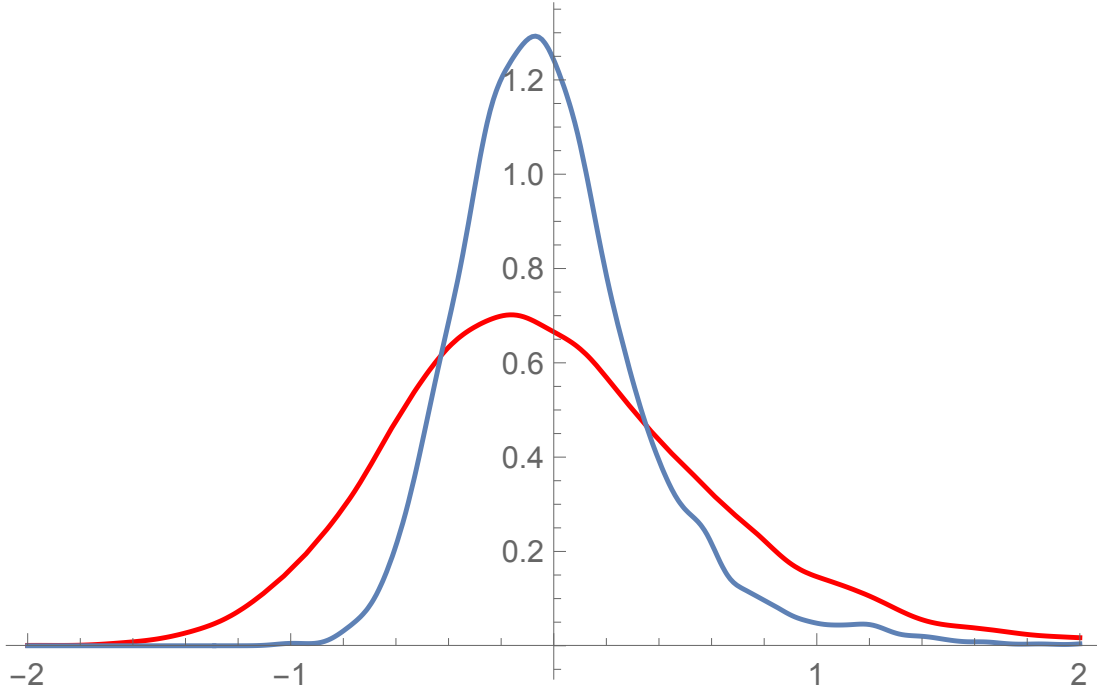
9 Conclusion

The degree to which financial stability considerations should be incorporated in the conduct of monetary policy has long been debated, see Adrian and Liang (2016) for an overview. In this paper, we extend the basic, two equation New Keynesian model to incorporate a notion of financial vulnerability. Shocks to risk premia impact aggregate demand via the Euler equation. The shocks to risk premia are assumed to impact the volatility of output, which is motivated from the empirical observation by Adrian, Boyarchenko, and Giannone (2019) that financial conditions forecast both the mean and the volatility of output. Importantly, our framework reproduces the stylized fact that the conditional mean and the conditional volatility of output are strongly negatively correlated, giving rise to a sharply negatively skewed unconditional output distribution. Vulnerability thus captures movements in the conditional GDP distribution that correspond to the downside risk of growth.

We further assume that the central bank minimizes the expected discounted sum of squared output gaps and squared inflation, which is standard in the literature. This is therefore a central bank that is subject to a dual mandate, without an independent financial

stability objective. Despite that narrow objective function, the optimal flexible inflation targeting rule conditions on the level of vulnerability. Intuitively, all variables that provide information about the conditional distribution of GDP should be taken into account in setting optimal monetary policy. This translates into an augmented Taylor rule, where financial vulnerability—as measured by output gap tail risk as a function of financial variables—is an input into the Taylor rule. Furthermore, the magnitude of the Taylor rule coefficients on output gap and inflation depend on the parameters that determine vulnerability.

The striking result from our setup is that the central bank should **always** condition monetary policy on financial vulnerability. Relative to earlier literature that has made similar arguments (e.g. Curdia and Woodford (2010), Cúrdia and Woodford (2016) and Gambacorta and Signoretti (2014), our modeling approach is deeply rooted in empirical observations which capture macoreconomic shocks of the 2008 crisis very well. Through the negative correlation between conditional mean and conditional variance, our setup captures nonlinearity in macro dynamics in a tractable linear-quadratic setting. The implications of our results for the conduct of monetary policy are in line with the arguments or Adrian and Shin (2010) and Borio and Zhu (2012).

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A   Appendix: Details for Good Producing Firms

A.1   Final Good Producers

There is continuum of measure one of final good producers. Firms in the final good sector produce a homogeneous final good, Y_t, using intermediate goods $Y_t(i)$ of different varieties indexed by $i \in [0, 1]$. The production function for each final good producer is

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{64}$$

where $\varepsilon > 1$ is the constant elasticity of substitution for differentiated goods (which is equal to the elasticity of substitution across goods for consumers).

The representative final good producer chooses inputs $Y_t(i)$ to maximize real profits

$$Y_t - \frac{1}{P_t} \int_0^1 P_t(i) Y_t(i) \, di,$$

where the first term is real revenue and the second term represents real costs. Because final good producers are competitive, they take $P_t(i)$ and P_t as given. Because of constant returns and competition, the size of any one final good firm is indeterminate. However, their input demand is determined by the following cost minimization problem

$$\min_{Y_t(i)} \int_0^1 P_t(i) Y_t(i) \, di,$$
$$s.t$$
$$Y_t \leq \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

The cost minimization yields the demand for intermediate good i

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t. \tag{65}$$

A.2 Intermediate Good Producers

There is continuum of mass one of monopolistically competitive atomistic firms indexed by $i \in [0, 1]$. Each firm faces a demand curve given by equation (65). Firms use labor $N_t(i)$ to produce output according to the technology

$$Y_t(i) = AN_t(i). \tag{66}$$

Labor is hired in a competitive market with perfect mobility.

Firms set prices according to Calvo staggered pricing. Receiving the signal that allows a firm to change its price is independent of the last time the firm received the signal and across firms. The probability density of receiving a signal after an amount of time h has elapsed since the last signal is $\delta e^{-\delta h}$, where $\delta > 0$ is the Calvo parameter. Firms that are able to adjust the price choose the price optimally. Firms that cannot change their price adjust output to meet demand at the pre-established price. Both types of firms choose inputs to minimize costs, given output demand.

We characterize first the input choice problem conditional on output. We then characterize the optimal price adjustment and output decisions.

Input Demand and Marginal Cost   Firm i chooses $N_t(i)$ to minimize total cost, given by

$$\frac{W_t}{P_t} N_t(i)$$

subject to

$$AN_t(i) - Y_t(i) \geq 0 \tag{67}$$

where W_t/P_t is the real wage and s_t is the government labor tax. Let MC_t denote the Lagrange multiplier with respect to the constraint. Note that MC_t is the firm's real marginal cost (the derivative of total cost with respect to $Y_t(i)$). The FOC with respect to $N_t(i)$ is

$$MC_t = \frac{1}{A}\frac{W_t}{P_t} \tag{68}$$

Since the firm takes W_t/P_t as given, real marginal cost is constant across firms. Equation (67) with equality gives labor demand

$$N_t(i) = \frac{Y_t(i)}{A} \tag{69}$$

Optimal Price Setting Intermediate good producers maximize real expected profits discounted using the household's stochastic discount factor, subject to their production technology (66), the demand curve (65) and the Calvo constraint on price adjustment. A firm that is allowed to change its price at time t picks $P_t(i)$ to maximize

$$\mathbb{E}_t \int_t^\infty SDF_{s,t}\delta e^{-\delta(s-t)}\left(\frac{P_t(i)}{P_s}Y_{s|t}(i) - MC_s Y_{s|t}(i)\right)ds$$

subject to

$$Y_{s|t}(i) = \left(\frac{P_t(i)}{P_s}\right)^{-\varepsilon}Y_s,$$

where $SDF_{s,t}$ is the stochastic discount factor used by the household at time t to discount time-s payoffs and $Y_{s|t}(i)$ is the demand of good i at time s conditional on firm i having changed prices for the last time at time t. In the optimal price setting decision, the firm takes the paths $\left\{SDF_{s,t}, P_s, Y_{s|t}(i), Y_s, MC_s\right\}_{s \geq t}$ as given. The maximization yields an optimal price

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1}\mathbb{E}_t \int_t^\infty \Upsilon_{s,t}MC_s ds \tag{70}$$

where $\varepsilon/(\varepsilon - 1)$ is the gross markup and

$$\Upsilon_{s,t} \equiv \frac{Q_s e^{-\delta s}P_s^\varepsilon Y_s}{\mathbb{E}_t \int_t^\infty Q_s e^{-\delta s}P_s^{\varepsilon - 1}Y_s ds}.$$

Equation (70) show that the optimal price is a weighted average of real marginal costs times the markup. Because the optimal price $P_t^*(i)$ depends only on aggregate variables, all firms that are allowed to change the price pick the same optimal price, so we drop the index i from $P_t^*(i)$ and simply write P_t^*. Firms who are not allowed to reset their prices at time t solve the same problem but instead of picking $P_t(i)$ they keep it constant at its pre-existing level.

By equation (**??**) and the Calvo price setting assumption, the aggregate price P_t is given by

$$
\begin{aligned}
P_t^{1-\varepsilon} &= \int_0^1 P_t(i)^{1-\varepsilon}\, di \\
&= \int_{-\infty}^t \delta e^{-\delta(t-s)} P_s^*(i)^{1-\varepsilon}\, ds \tag{71}
\end{aligned}
$$

Intuitively, the last expression says that a mass $\delta e^{-\delta(t-s)}$ of firms changed their price to $P_s^*(i)$ at time $t-s$. Differentiating both sides of (71) with respect to time and defining inflation π_t by $\pi_t \equiv (1/dt)(dP_t/P_t)$ we get

$$
\pi_t = \frac{\delta}{1-\varepsilon}\left(\left(\frac{P_t^*}{P_t}\right)^{1-\varepsilon} - 1\right). \tag{72}
$$

The real profits for the producer of intermediate good i are

$$
D_{t,goods}(i) = \frac{P_t(i)\,Y_t(i)}{P_t} - MC_t Y_t(i). \tag{73}
$$

Phillips Curve   Differentiating (72) with respect to time gives the non-linear NKPC

$$
d\pi_t = \delta\left(\frac{x_{2,t}}{x_{1,t}}\right)^{1-\varepsilon}\left(\frac{\delta}{x_{2,t}}\frac{MC_t}{MC} - \frac{\delta}{x_{1,t}} - \pi_t\right)dt \tag{74}
$$

where $x_{1,t}$ and $x_{2,t}$ have dynamics given by

$$
\begin{aligned}
\frac{dx_{1,t}}{x_{1,t}} &= \left(1 + \frac{1}{x_{1,t}}\right)\delta dt + (1-\varepsilon)\pi_t dt + Q_t Y_t d\left(\frac{1}{Q_t Y_t}\right) \\
\frac{dx_{2,t}}{x_{2,t}} &= \left(1 + \frac{1}{x_{2,t}}\frac{MC_t}{MC}\right)\delta dt - \varepsilon\pi_t dt + Q_t Y_t d\left(\frac{1}{Q_t Y_t}\right)
\end{aligned}
$$

and satisfy $P_t^*/P_t = x_{2,t}/x_{1,t}$.

B   Appendix: Definition of Equilibrium

An equilibrium is a collection of paths $\{N_t, W_t, P_t, C_t, Q_t, F_t, R_t, \omega_t, \mu_t, \sigma_t, Y_t, Y_t(i), P_t(i), P_t, f_t, \theta_t, X_t, \gamma_t, S_t$ such that, for all realizations of the exogenous processes ζ_t and g_t, the following conditions hold:

1. Households optimize

$$\text{(labor supply)} \quad : \quad N_t = \left((1-s_t)\frac{W_t}{P_t}C_t^{-\gamma}\right)^{\frac{1}{\xi}}$$

$$\text{(demand for final goods)} \quad : \quad C_t = \left(e^{\beta t}Q_t\right)^{-\frac{1}{\gamma}}$$

$$\text{(demand for risky asset)} \quad : \quad \frac{dF_t}{F_t} = \left(R_t + \omega_t\mu_{banks,t} - \frac{1}{F_t}\left(C_t - (1-s_t)\frac{W_t}{P_t}N_t + T_t\right)\right)dt + \omega_t\sigma_{banks}$$

$$\text{(demand for riskless bond)} \quad : \quad (1-\omega_t)F_t$$

$$\text{(transversality condition)} \quad : \quad \lim_{s\to\infty}\mathbb{E}_t\left[Q_sF_s\right] = 0$$

2. Good producers optimize

   (a) Final good producers

$$\text{(demand for intermediate good } i) \quad : \quad Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}Y_t$$

$$\text{(supply of final goods)} \quad : \quad Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

   (b) Intermediate goods producers

$$\text{(labor demand)} \quad : \quad N_t(i) = \frac{Y_t(i)}{A}$$

$$\text{(supply of intermediate goods)} \quad : \quad Y_t(i) = AN_t(i)$$

$$\text{(price setting)} \quad : \quad d\pi_t = \delta\left(\frac{x_{2,t}}{x_{1,t}}\right)^{1-\varepsilon}\left(\frac{\delta\varepsilon}{\varepsilon-1}\frac{1}{A}\frac{W_t}{P_t}\frac{1}{x_{2,t}} - \frac{\delta}{x_{1,t}} - \pi_t\right)dt$$

$$\frac{dx_{1,t}}{x_{1,t}} = \left(1 + \frac{1}{x_{1,t}}\right)\delta dt + (1-\varepsilon)\pi_t dt + Q_tY_t d\left(\frac{1}{Q_tY_t}\right)$$

$$\frac{dx_{2,t}}{x_{2,t}} = \left(1 + \frac{\varepsilon}{\varepsilon-1}\frac{1}{A}\frac{W_t}{P_t}\frac{1}{x_{2,t}}\right)\delta dt - \varepsilon\pi_t dt + Q_tY_t d\left(\frac{1}{Q_tY_t}\right)$$

3. Banks optimize

$$\text{(dividends)} \quad : \quad f_t = u\left(\gamma_t, \eta_t - g_t\right)\beta$$

$$\text{(portfolio)} \quad : \quad \theta_t = \frac{1}{\gamma_t}\frac{\eta_t - g_t}{\sigma_t}$$

$$\text{(wealth)} \quad : \quad \frac{dX_t}{X_t} = (R_t - f_t + \theta_t\mu_t)dt + \theta_t\sigma_t dB_t$$

$$\text{(VaR constraint)} \quad : \quad \gamma_t = \max\{1,\hat{\gamma}_t\} \text{ with } \hat{\gamma}_t \text{ such that: } VaR_{\tau,\alpha}^{bank}\left(t, \frac{\eta_t - g_t}{\hat{\gamma}_t\sigma_t}, u\left(\hat{\gamma}_t, \eta_t - g_t\right)\beta\right) = a_V X$$

4. Markets clear

$$\text{(intermediate goods)} \quad : \quad \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}Y_t = A_tN_t(i)$$

$$\text{(final goods)} \quad : \quad C_t = Y_t$$

$$\text{(labor)} \quad : \quad \int_0^1 N_t(i)di = N_t$$

$$\text{(risky asset)} \quad : \quad \frac{F_t\omega_{banks,t}}{S_{banks,t}} + \frac{X_t\theta_{banks,t}}{S_{banks,t}} = 1$$

$$\text{(riskless bond)} \quad : \quad \frac{X_t\theta_{goods,t}}{S_{goods,t}} = 1$$

5. The central bank sets nominal interest rates $i_t \equiv R_t + \pi_t$.

C   Solution to the bank problem

The bank problem under the bank measure is

$$V\left(X_t\right) \;=\; \max_{\{\theta_s, f_s\}_{s \geq t}} \mathbb{E}_t^{bank} \left[\int_t^\infty e^{-\beta(s-t)} \log\left(f_s X_s\right) ds\right] \tag{75}$$

$$s.t.$$

$$\frac{dX_t}{X_t} \;=\; \left(R_t - f_t + \theta_t \left(\mu_t - \sigma_t g_t\right)\right) dt + \theta_t \sigma_t dB_t^{bank}, \tag{76}$$

$$VaR_{\tau,\alpha}^{bank}\left(t, \theta_t, f_t\right) \;\leq\; a_V X_t, \tag{77}$$

$$dg_t \;=\; -\left(\kappa_g + \sigma_g\right) g_t dt + \sigma_g dB_t^{bank}, \tag{78}$$

with X_t and g_t given. Note that $E_s^{bank}\left[\cdot\right]$ and $VaR_{\tau,\alpha}^{bank}$ denote, respectively, conditional expectations and the value-at-risk under the bank measure.

Let

$$U\left(t, \theta, f, g\right) \equiv R_t - f + \theta\left(\mu_t - \sigma_t g\right) - \frac{1}{2}\left(\theta \sigma_t\right)^2$$

be the drift of $d \log X_t$. Then, the dynamic budget constraint of the bank in equation (76) has a strong solution given by

$$X_t = X_0 \exp\left\{\int_0^t U\left(s, \theta_s, f_s, g_s\right) ds + \int_0^t \theta_s \sigma_s dB_s^{bank}\right\}$$

with X_0 given. Projected wealth loss between t and $t+\tau$ when keeping $\left(R_t, \mu_t, \sigma_t, g_t, f_t, \theta_t\right)$ constant at their time-t values during the interval $t \in [t, t+\tau]$ is

$$X_t - X_{t+\tau} = X_t \left[1 - \exp\left\{U\left(t, \theta_t, f_t, g_t\right) \tau + \theta_t \sigma_t \left(B_{t+\tau}^{bank} - B_t^{bank}\right)\right\}\right]$$

Value-at-risk at level α and horizon τ is defined as the α-percentile of the projected wealth loss, $X_t - X_{t+\tau}$, conditional on time-t information, and is given by

$$VaR_{\tau,\alpha}^{bank}\left(t, \theta_t, f_t, X_t, g_t\right) = X_t \left[1 - \exp\left\{U\left(t, \theta_t, f_t, g_t\right) \tau + \mathcal{N}^{-1}\left(\alpha\right) \left|\theta_t \sigma_t\right| \sqrt{\tau}\right\}\right]$$

where $\mathcal{N}^{-1}$ is the inverse cumulative distribution function of a standard normal distribution. Define

$$g_V\left(t, \theta, f, g\right) \equiv -U\left(t, \theta, f, g\right) \tau - \mathcal{N}^{-1}\left(\alpha\right) \left|\theta \sigma_t\right| \sqrt{\tau}$$

Then

$$VaR_{\tau,\alpha}^{bank}\left(t, \theta, f, x, g\right) \leq x a_V$$

if, and only if,

$$g_V\left(t, \theta, f, g\right) \leq \log \frac{1}{1 - a_V}.$$

Since $\alpha \in (0, 1/2]$, we have that $\mathcal{N}^{-1}(\alpha) \leq 0$ and that $g_V(t, \theta, f, g)$ is convex in (θ, f).

Log-utility allows us to transform the bank's optimization problem into a non-stochastic problem. Indeed, the objective function can be written as

$$
\begin{aligned}
\int_0^\infty e^{-\beta t} \log\left(f_t X_t\right) dt \;=\; & \int_0^\infty e^{-\beta t} \log\left(X_t\right) dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt \\[2mm]
=\; & \int_0^\infty e^{-\beta t} \log\left(X_0\right) dt + \int_0^\infty e^{-\beta t}\left\{\int_0^t U\left(s, \theta_s, f_s, g_s\right) ds + \int_0^t \theta_s \sigma_s dB_s^{bank}\right\} dt + \int_0 \\[2mm]
=\; & \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty \int_0^t e^{-\beta t} U\left(s, \theta_s, f_s, g_s\right) ds dt + \int_0 \\[2mm]
=\; & \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty \int_s^\infty e^{-\beta t} U\left(s, \theta_s, f_s, g_s\right) dt ds + \int_0 \\[2mm]
=\; & \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty U\left(s, \theta_s, f_s, g_s\right) \left[\int_s^\infty e^{-\beta t} dt\right] ds +
\end{aligned}
$$

where the change in the order of integration follows from Fubini's theorem. We assume all the usual regularity conditions. In particular, we assume that

$$
\int_0^\infty \left|\sigma_t^{-1}\mu_t\right|^2 dt < \infty \tag{80}
$$

Under the regularity condition in equation (80), the stochastic part of the bank's objective function in (79) is a martingale and not just a local martingale, so

$$
E_0^{bank} \int_0^\infty \int_0^t e^{-\beta t} \theta_s \sigma_s dB_s^{bank} dt = 0
$$

Therefore, taking expectations in (79) gives

$$
\begin{aligned}
E_0^{bank} \int_0^\infty e^{-\beta t} \log\left(f_t X_t\right) dt \;=\; & \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + E_0^{bank} \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + E_0^{bank} \int_0^\infty U\left(s, \theta_s, f_s, g_s\right) \\[2mm]
=\; & \frac{\log\left(X_0\right)}{\beta} + E_0^{bank} \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \frac{1}{\beta} E_0^{bank} \int_0^\infty U\left(s, \theta_s, f_s, g_s\right) e^{-\beta s} ds \\[2mm]
=\; & \frac{\log\left(X_0\right)}{\beta} + E_0^{bank} \int_0^\infty e^{-\beta t}\left(\log\left(f_t\right) + \frac{1}{\beta} U\left(t, \theta_t, f_t, g_t\right)\right) dt
\end{aligned}
$$

Thus, to maximize

$$
E_0^{bank} \int_0^\infty e^{-\beta t} \log\left(f_t X_t\right) dt
$$

over the constrained set, it suffices to maximize

$$
h\left(t, \theta_t, f_t, g_t\right) \equiv \log\left(f_t\right) + \frac{1}{\beta} U\left(t, \theta_t, f_t, g_t\right)
$$

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pathwise over the constrained set. For a fixed path, at time t the bank then solves

$$\max_{\theta_t, f_t} h\left(t, \theta_t, f_t, g_t\right)$$
$$s.t.$$
$$g_V\left(t, \theta_t, f_t, g_t\right) \leq \log \frac{1}{1 - a_V} \tag{81}$$

The function $h\left(t, \theta_t, f_t, g_t\right)$ is concave in $\left(\theta_t, f_t\right)$ and maximized over $\left(\theta_t, f_t\right)$ by

$$f_t = f_{M,t}$$
$$\theta_t = \theta_{M,t}$$

when the VaR constraint is not binding, where we derive $f_{M,t}, \theta_{M,t}$ using the FOC

$$
\begin{aligned}
\left[f_t\right] \;:\; & 0 = \frac{\partial}{\partial f_t} h\left(t, \theta_t, f_{M,t}, g_t\right) \\
\;:\; & 0 = \frac{1}{f_{M,t}} - \frac{1}{\beta} \\
\;:\; & f_{M,t} = \beta \\
\left[\theta_t\right] \;:\; & 0 = \nabla_\theta h\left(t, \theta_{M,t}, f_t, g_t\right) \\
\;:\; & 0 = \frac{1}{\beta}\left(\mu_t - \sigma_t g_t - \sigma_t^2 \theta_{M,t}\right) \\
\;:\; & \theta_{M,t} = \left(\sigma_t\right)^{-1}\left(\sigma_t^{-1}\mu_t - g_t\right)
\end{aligned}
$$

Using the definition of the market price of risk η_t, we can also write

$$\theta_{M,t} = \sigma_t^{-1}\left(\eta_t - g_t\right)$$

If $\left(\theta_{M,t}, f_{M,t}\right)$ satisfy

$$g_V\left(t, \theta_{M,t}, f_{M,t}, g_t\right) \leq \log \frac{1}{1 - a_V}$$

then $\left(\theta_t^*, f_t^*\right) = \left(\theta_{M,t}, f_{M,t}\right)$ is the solution to the problem with the VaR constraint. Otherwise, because the constraint set is compact and convex, and the objective is continuous, there will be a unique solution $\left(\theta_t^*, f_t^*\right) \neq \left(\theta_{M,t}, f_{M,t}\right)$. Moreover, in this case, $\left(\theta_t^*, f_t^*\right) \neq \left(\theta_{M,t}, f_{M,t}\right)$ must be such that the VaR constraint holds with equality.

If $\left|\theta \sigma_t\right| = 0$, then (81) gives

$$f_t = R_t - \frac{1}{\tau}\log\left(1 - a_V\right)$$

If $\left|\theta \sigma_t\right| \neq 0$, $g_V\left(t, \theta_t, f_t, g_t\right)$ is differentiable in θ_t and thus we can solve (81) using the

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Karush–Kuhn–Tucker conditions. Set up the Lagrangian

$$\mathcal{L} = h\left(t, \theta_t, f_t, g_t\right) - \lambda \left(g_V \left(t, \theta_t, f_t, g_t\right) - \log \frac{1}{1 - a_V} \right)$$

Direct computation shows that $\nabla g_V \left(t, \theta_t, f_t, g_t\right) \neq 0.$[12] Thus, $\lambda \neq 0$ and the FOC is

$$\nabla h\left(t, \theta_t, f_t, g_t\right) = \lambda \nabla g_V \left(t, \theta_t, f_t, g_t\right) \tag{82}$$

We compute

$$
\begin{aligned}
\nabla_{\theta_t} h &= \frac{1}{\beta} \left(\mu_t - \sigma_t g_t - \sigma_t^2 \theta_t \right) \\
\nabla_{\theta_t} g_V &= - \left(\mu_t - \sigma_t g_t - \sigma_t^2 \theta_t \right) \tau - \mathcal{N}^{-1} \left(\alpha \right) \frac{\sigma_t^2 \theta_t}{|\theta_t \sigma_t|} \sqrt{\tau} \\
\nabla_{f_t} h &= \frac{1}{f_t} - \frac{1}{\beta} \\
\nabla_{f_t} g_V &= \tau
\end{aligned}
$$

so that the FOC become

$$
\begin{aligned}
\nabla_{\theta_t} h\left(t, \theta_t, f_t, g_t\right) &= \lambda \nabla_{\theta_t} g_V \left(t, \theta_t, f_t, g_t\right) \\
\frac{1}{\beta} \left(\mu_t - \sigma_t g_t - \sigma_t^2 \theta_t \right) &= \lambda \left(- \left(\mu_t - \sigma_t g_t - \sigma_t^2 \theta_t \right) \tau - \mathcal{N}^{-1} \left(\alpha \right) \frac{\sigma_t^2 \theta_t}{|\theta_t \sigma_t|} \sqrt{\tau} \right) \\
\left(1 + \beta \tau \lambda \right) \left(\mu_t - \sigma_t g_t \right) &= \left(1 + \left(\tau - \frac{\sqrt{\tau} \mathcal{N}^{-1} \left(\alpha \right)}{|\theta_t \sigma_t|} \right) \beta \lambda \right) \sigma_t^2 \theta_t \tag{83}
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{f_t} h\left(t, \theta_t, f_t, g_t\right) &= \lambda \nabla_{f_t} g_V \left(t, \theta_t, f_t, g_t\right) \\
\frac{1}{f_t} - \frac{1}{\beta} &= \lambda \tau \\
f_t &= \frac{\beta}{\beta \lambda \tau + 1}
\end{aligned}
$$

Since $\theta_{M,t} = \sigma_t^{-2} \left(\mu_t - \sigma_t g_t \right)$, equation (83) shows that θ_t is parallel to $\theta_{M,t}$. Then, to solve the maximization problem in (81), all we need is to find λ_1, λ_2 that solve

$$
\begin{aligned}
\max_{\lambda_1, \lambda_2} &\ h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) \\
&s.t. \\
g_V \left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) &\leq \log \frac{1}{1 - a_V}
\end{aligned}
$$

---

[12]The gradient is taken with respect to θ_t and f_t. We also use the notation ∇_x to denote $\partial/\partial x$.

Again, it can be checked that the constraint holds with equality. The Lagrangian is

$$\mathcal{L} = h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) - \psi \left(g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) - \log \frac{1}{1 - a_V}\right)$$

The FOC are

$$\frac{\partial}{\partial \lambda_1} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) = \psi \frac{\partial}{\partial \lambda_1} g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right)$$

$$\frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) = \psi \frac{\partial}{\partial \lambda_2} g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right)$$

Computing the derivatives gives

$$\frac{\partial}{\partial \lambda_1} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) = \psi \frac{\partial}{\partial \lambda_1} g_V\left(t, \theta, f, g\right)$$

$$\frac{1}{\beta}\left(\theta_{M,t}\left(\mu_t - \sigma_t g_t\right) - \lambda_1 \left(\theta_{M,t}\sigma_t\right)^2\right) = \psi \left(-\left(\theta_{M,t}\left(\mu_t - \sigma_t g_t\right) - \lambda_1 \left(\theta_{M,t}\sigma_t\right)^2\right)\tau - \mathcal{N}^{-1}\left(\alpha\right)|\theta_{M,t}\sigma_t|\sqrt{\tau}\right)$$

and

$$\frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, g_t\right) = \psi \frac{\partial}{\partial \lambda_2} g_V\left(t, \theta, f, g\right)$$

$$\frac{1}{\lambda_2} - \frac{f_{M,t}}{\beta} = \psi f_{M,t}\tau$$

Eliminating ψ and using

$$\theta_{M,t}\sigma_t = \sigma_t^{-1}\left(\mu_t - \sigma_t g_t\right)$$

$$|\theta_{M,t}\sigma_t| = \left|\sigma_t^{-1}\left(\mu_t - \sigma_t g_t\right)\right|$$

$$\theta_{M,t}\left(\mu_t - \sigma_t g_t\right) = \sigma_t^{-2}\left(\mu_t - \sigma_t g_t\right)^2 = \left(\theta_{M,t}\sigma_t\right)^2$$

we get

$$\lambda_2 = u\left(t, \lambda_1\right) f_{M,t}$$

where the function u is defined by

$$u\left(t, z\right) \equiv 1 + \frac{\sqrt{\tau}|\theta_{M,t}\sigma_t|}{\mathcal{N}^{-1}\left(\alpha\right)}\left(1 - z\right) \tag{84}$$

and λ_1 makes the VaR constraint hold with equality. For the VaR constraint to hold with equality, $\lambda_1 > 0$ must satisfy

$$g_V\left(t, \lambda_1 \theta_{M,t}, u\left(t, \lambda_1\right) f_{M,t}, g_t\right) = -\log\left(1 - a_V\right) \tag{85}$$

If there are no positive roots, it means we are in the case

$$
\begin{aligned}
\theta_t &= \mathbf{0} \\
f_t &= R_t - \frac{1}{\tau} \log (1 - a_V)
\end{aligned}
$$

considered before.

Equation (85) has the form

$$
0 = c + b_1 \lambda_1 + b_2 |\lambda_1| + a \lambda_1^2
$$

with

$$
\begin{aligned}
a &= \frac{1}{2}\tau \left(\eta_t - g_t \right)^2 \\
b_1 &= - \left(\frac{\sqrt{\tau} |\eta_t - g_t|}{\mathcal{N}^{-1}(\alpha)} \tau \beta + \tau \left(\eta_t - g_t \right)^2 \right) \\
b_2 &= -\sqrt{\tau} |\eta_t - g_t| \mathcal{N}^{-1}(\alpha) \\
c &= \log (1 - a_V) + \tau \left(\beta - R_t \right) + \tau \beta \frac{\sqrt{\tau} |\eta_t - g_t|}{\mathcal{N}^{-1}(\alpha)}
\end{aligned}
$$

If $\eta_t \neq g_t$, solutions to (85) have the form

$$
\lambda_1 = -\frac{b_1 \pm b_2}{2a} \pm \frac{\sqrt{\left(b_1 + b_2 \right)^2 - 4ac}}{2a}
$$

Since $a, b_2 > 0$, the only solutions that can be positive are

$$
\lambda_1 = -\frac{b_1 + b_2}{2a} \pm \frac{\sqrt{\left(b_1 + b_2 \right)^2 - 4ac}}{2a}
$$

However, when $\lambda_1 > 0$, the smallest of these two solutions is not optimal since $\nabla_{\theta_t} h \left(t, \lambda_1 \theta_{M,t}, u \left(t, \lambda_1 \right) f_{M,t}, g_t \right.$
0. Therefore, the only solution that can be optimal and positive is

$$
\varphi_t \equiv \lambda_1 = -\frac{b_1 + b_2}{2a} + \frac{\sqrt{\left(b_1 + b_2 \right)^2 - 4ac}}{2a}
$$

which we label φ_t to distinguish it from the other potential solutions. To summarize, we have the following cases:

- If $\eta_t = g_t$ or $\varphi_t \leq 0$, then $\theta_t = \mathbf{0}$ and $f_t = R_t - \frac{1}{\tau} \log (1 - a_V)$.

- If $\eta \neq g$, we have two cases

    - If $\varphi_t \in (0, 1]$, the VaR constraint holds with equality and we have $\theta_t = \varphi_t \theta_{M,t}$ and $f_t = u \left(t, \varphi_t \right) f_{M,t}$

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– If $\varphi_t > 1$, the VaR constraint does not bind and $\theta_t = \theta_{M,t}$ and $f_t = f_{M,t}$.

Putting everything together, the optimal portfolio when is then characterized by

$$\theta_t = \min\{1, \max\{0, \varphi_t\}\}\,\theta_{M,t} \tag{86}$$

$$f_t = u(t, \min\{1, \varphi_t\})\,f_{M,t}1_{\{\varphi_t>0\}} + \left(R_t - \frac{1}{\tau}\log(1 - a_V)\right)1_{\{\varphi_t\leq 0\}}$$

φ_t is the largest root of:
$$g_V(t, \varphi_t\theta_{M,t}, u(t, \varphi_t)\,f_{M,t}, g_t) = -\log(1 - a_V) \tag{87}$$

We now study when $\varphi_t > 0$ and $\varphi_t > 1$. If any one of the four conditions

1. $0 < c$ and $b_1 + b_2 < 0$ and $(b_1 + b_2)^2 - 4ac > 0$

2. $b_1 > 0$ and $c < 0$

3. $b_1 + b_2 < 0$ and $c < 0$

4. $b_1 < 0$ and $-b_2 < b_1$ and $c < 0$

is satisfied, then $\varphi_t > 0$. If any of the two conditions

1. $a + b_1 + b_2 + c < 0$ and $((b_1 + b_2 > 0$ and $b_1 < 0)$ or $(b_1 + 2b_2 > 0$ and $b_1 + b_2 < 0)$ or $(c > 0$ and $b_1 + 2b_2 < 0)$ or $b_1 > 0$ or $(a + b_1 + b_2 > 0$ and $b_1 + 2b_2 \leq 0))$

2. $b_1 + 2b_2 < 0$ and $c < 0$ and $a + b_1 + b_2 \leq 0$

is satisfied, then $\varphi_t > 1$.
Let $\hat{\gamma} = 1/\varphi_t$. We define the bank's effective risk aversion by

$$\gamma_t \equiv \frac{1}{\min\left\{1, \max\left\{0, \frac{1}{\hat{\gamma}}\right\}\right\}} \in [1, \infty)$$

$$= \begin{cases} \infty & , & \text{if} & \hat{\gamma} < 0 \\ 1 & , & \text{if} & 0 \leq \hat{\gamma} < 1 \\ \hat{\gamma} & , & \text{if} & \hat{\gamma} \geq 1 \end{cases}$$

Finally, we note that λ is the VaR Lagrange multiplier for the deterministic path-by-path problem, so the Lagrange multiplier for the original problem under the bank's probability measure is

$$\lambda_{VaR,t}^{bank} = \lambda e^{-\beta t}$$

$$= \frac{1}{\tau}\left(\frac{1}{f_t} - \frac{1}{\beta}\right)e^{-\beta t}$$

and under the physical measure is

$$\begin{aligned} \lambda_{VaR,t} &= \lambda e^{-\beta t} e^{\zeta_t} \\ &= \frac{1}{\tau} \left(\frac{1}{f_t} - \frac{1}{\beta} \right) e^{-\beta t} e^{\zeta_t} \end{aligned} \tag{88}$$

Note that since

$$f_t \leq \beta$$

we have

$$\lambda_{VaR,t} \geq 0$$

C.1 State price density of the bank under complete markets

The solution to the bank problem derived above did not require complete markets. We now derive the bank's state price density (SPD) assuming that markets are complete. The bank problem under the physical measure is

$$\begin{aligned} V(X_0) &= \max_{\{\theta_t, f_t\}_{t \geq 0}} E_0 \left[\int_0^\infty e^{-\beta t} e^{\zeta_t} \log \left(f_t X_t \right) dt \right] \\ &\quad s.t. \\ \frac{dX_t}{X_t} &= \left(R_t - f_t + \theta_t \mu_t \right) dt + \theta_t \sigma_t dB_t \tag{89} \\ g_V \left(t, \theta_t, f_t \right) &= - \left(R_t - f_t + \theta_t \mu_t - \frac{1}{2} \left(\theta_t \sigma_t \right)^2 \right) \tau - \mathcal{N}^{-1} \left(\alpha \right) |\theta_t \sigma_t| \sqrt{\tau} \leq \log \frac{1}{1 - a_V} \end{aligned}$$

Complete markets implies that the dynamic budget constraint (89) is equivalent to the static one[13]

$$X_0 = E_0 \left[\int_0^\infty Q_t f_t X_t dt \right] \tag{90}$$

where the banks take Q_t as given. The Lagrangian is :

$$\mathcal{L} = E_0 \left[\int_0^\infty e^{-\beta t} e^{\zeta_t} \log \left(f_t X_t \right) dt \right] + \lambda_{bc} \left(X_0 - E_0 \left[\int_0^\infty Q_t f_t X_t dt \right] \right) - \int_0^\infty \lambda_{VaR,t} \left(g_V \left(t, \theta_t, f_t \right) - \log \frac{1}{1 - a_V} \right)$$

where $\lambda_{bc} > 0$ is a number but $\lambda_{VaR,t} > 0$ is a function of time since we have one VaR constraint for each t. The FOC for an interior solution are

$$[f_t] \quad : \quad 0 = \frac{e^{-\beta t} e^{\zeta_t}}{f_t X_t} - \lambda_{bc} Q_t + \lambda_{VaR,t} \frac{\tau}{X_t} \tag{91}$$

$$[\theta_t] \quad : \quad 0 = \frac{e^{-\beta t} e^{\zeta_t}}{\beta} \nabla_\theta U \left(t, \theta_t, f_t \right) + \lambda_{VaR,t} \nabla_\theta g_V \left(t, \theta_t, f_t \right)$$

---

[13]See Huang and Pages (1992).

Re-arranging (91) gives

$$
\begin{aligned}
Q_t &= \frac{e^{-\beta t}e^{\zeta_t}}{\lambda_{bc}}\frac{1}{f_t X_t} + \frac{\lambda_{VaR,t}\tau}{\lambda_{bc}}\frac{1}{X_t} \\
&= \frac{1}{\lambda_{bc}X_t}\left(e^{-\beta t}e^{\zeta_t}\frac{1}{f_t} + \lambda_{VaR,t}\tau\right)
\end{aligned}
$$

Using $\lambda_{VaR,t}$ from equation (88)

$$
\lambda_{VaR,t} = \frac{e^{-\beta t}e^{\zeta_{b,t}}}{\tau}\left(\frac{1}{f_t} - \frac{1}{\beta}\right)
$$

gives

$$
Q_t = \frac{e^{-\beta t}e^{\zeta_t}}{\lambda_{bc}X_t}\left(\frac{2}{f_t} - \frac{1}{\beta}\right)
$$

The multiplier λ_{bc} can be found from noting that we must have $Q_0 = 1$, which gives

$$
\lambda_{bc} = \frac{e^{\zeta_0}}{X_0}\left(\frac{2}{f_0} - \frac{1}{\beta}\right)
$$