Heterogeneity and incomplete markets: a dual approach

Fernando Duarte

MIT

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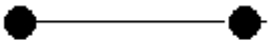
Introduction

- Broad goal: understand heterogeneity + incomplete markets in macroeconomics.
- Prominent example: Krusell and Smith (1998), growth model with aggregate shocks and idiosyncratic labor shocks.
- When is heterogeneity important? When do we obtain approximate aggregation?
- Computational challenges of high dimensional state-spaces.
- I present a different way to look at these class of problems and begin to explore its usefulness.

A simple example: prices

- Uncertainty tree:



- One asset with price (B_1, B_2) has span:

$$M_C = \{c \in \mathbb{R}^2 \mid c_1 = -\theta_1 B_1, c_2 = \theta_1 B_2 \text{ for some } \theta_1 \in \mathbb{R}\}$$

- No assets:

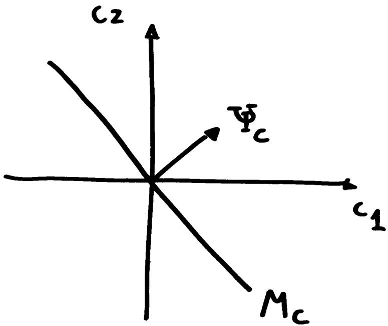
$$M_I = (0, 0).$$

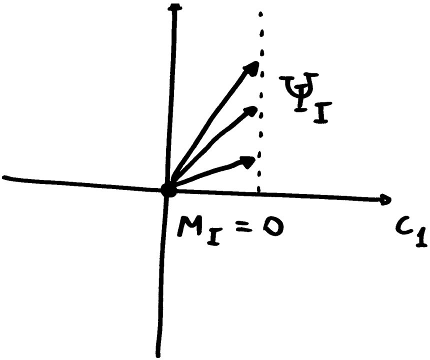
- State or Arrow-Debreu prices:

$$\Psi = \{\psi \in \mathbb{R}^2_{++} \mid \psi_1 c_1 + \psi_2 c_2 = \psi \cdot c = 0 \text{ for all } c \in M \text{ and } \psi_1 = 1\}$$

- $\Psi_C = \left(1, \frac{B_1}{B_2}\right)$ and $\Psi_I = (1, \text{ any } \psi_2 > 0)$

A simple example: prices





A simple example: consumers

- One consumer with endowment process $e \in \mathbb{R}^2_+$, $e \neq 0$.
- Utility maximization program is:

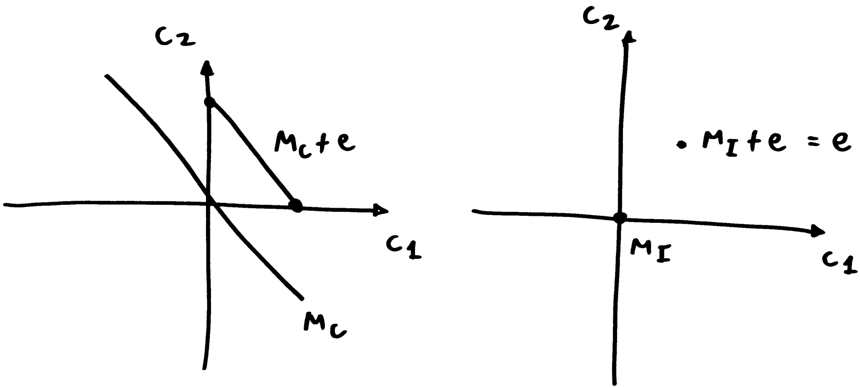
$$\begin{aligned} U^* &= \max_c U(c) = \max_{c_1, c_2} \log(c_1) + \log(c_2) \\ & \quad s.t \\ c &\in M + e = \left\{ c \in \mathbb{R}^2_+ \mid c - e \in M \right\} \end{aligned}$$

- For complete markets, the dynamic budget constraint is

$$M_C + e = \left\{ \begin{array}{c} c_1 = -\theta_1 B_1 + e_1 \\ c_2 = \theta_1 B_2 + e_2 \\ c_1, c_2 \geq 0 \end{array} \right\} = \left\{ \begin{array}{c} \frac{(c_1 - e_1)}{B_1} + \frac{(c_2 - e_2)}{B_2} = 0 \\ c_1, c_2 \geq 0 \end{array} \right\}$$

- For incomplete markets: $c_1 = e_1$ and $c_2 = e_2$.

A simple example: consumers



A simple example: static budget constraint

- For a given $\psi \in \mathbb{R}^2$, the static budget constraint is

$$B_{\psi,e} = \left\{ c \in \mathbb{R}^2_+ \mid c_1 + \psi_2 c_2 = e_1 + \psi_2 e_2 \right\}$$

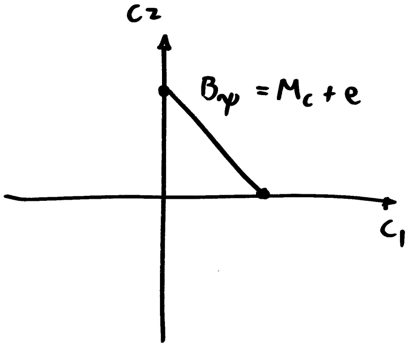
- For both complete and incomplete markets, we have

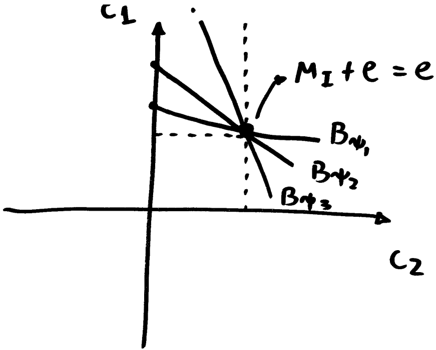
$$\bigcap_{\psi \in \Psi} B_{\psi,e} = M + e$$

- For complete markets, Ψ_C has a single element. For incomplete markets

$$\bigcap_{\psi \in \Psi} B_{\psi,e} = \bigcap_{\psi_2 > 0} \left\{ c \in \mathbb{R}^2_+ \mid c_1 - e_1 = -\psi_2 \left(c_2 - e_2 \right) \right\}$$

A simple example: static and dynamic budget constraints





A simple example: the dual

- We can now solve one of two problems:

$$U^* = \max_c U(c) \quad \text{s.t.} \quad c \in M + e \qquad \text{or} \qquad U^*_\Psi = \max_c U(c) \quad \text{s.t.} \quad c \in \cap_{\psi \in \Psi} B_{\psi,e}$$

- I will show we have a third way, the "dual"

$$\inf_{\psi \in \Psi} U^*_\psi$$

where

$$U^*_\psi = \max_c U(c)$$

$$\text{s.t.} \quad c \in B_{\psi,e}$$

- Thus, we solve in two steps: first, solve the easy static problem U^*_ψ for given state prices. Second, minimize over all allowable ones.

A simple example: solving the dual

- For incomplete markets

$$U^*_\psi = \max_{c_1, c_2} \log(c_1) + \log(c_2)$$

$$\text{s.t.}$$
$$c_1 + \psi_2 c_2 = e_1 + \psi_2 e_2$$

- The Lagrangean is $\mathcal{L} = \log(c_1) + \log(c_2) + \lambda\left(e_1 + \psi_2 e_2 - c_1 - \psi_2 c_2\right)$ and the FOC give

$$0 = \nabla \mathcal{L}\left(c^*\right) = \nabla U\left(c^*\right) - \lambda \psi = \left(\frac{1}{c_1^*}, \frac{1}{c_2^*}\right) - \lambda\left(1, \psi_2\right)$$

A simple example: solving the dual

- Then the dual is

$$\inf_{\psi \in \Psi} U_{\psi}^{*} = \inf_{\psi_2 > 0} \log\left(\frac{e_1 + \psi_2 e_2}{2}\right) + \log\left(\frac{e_1 + \psi_2 e_2}{2}\frac{1}{\psi_2}\right)$$

  which indeed gives $c = e$.

The dual: proof

- First problem: what is the right underlying state space S?
- A consumption plan is a real number on each node: $c \in \mathbb{R}^S$, and S is "big".
- For the proof, it is enough that $\mathbb{R}^S$ has a dot product, i.e. you can "sum" over all nodes (still, unclear how to do this sometimes).

The dual: proof

- Since $M + e \subseteq B_{\psi,e}$ for any $\psi \in \Psi$,

$$M + e \subseteq \bigcap_{\psi \in \Psi} B_{\psi,e} \subseteq B_{\tilde{\psi},e}$$

for any $\tilde{\psi} \in \Psi$, so

$$U^* \leq U^*_{\Psi} \leq \inf_{\psi \in \Psi} U^*_{\psi}.$$

- If I show $\inf_{\psi \in \Psi} U^*_{\psi} \leq U^*$, then we're done.

The dual: proof

- If c^* is an interior solution to U^*

$$\nabla U(c^*) \cdot (c - c^*) = 0 \text{ for all } c \in M + e$$

or

$$\nabla U(c^*) \cdot c = 0 \text{ for all } c \in M$$

- This shows $\nabla U(c^*) = \psi^* \in \Psi$, so c^* is the solution to $U^*_{\psi^*}$.
- Finally

$$\inf_{\psi \in \Psi} U^*_\psi \leq U^*_{\psi^*} = U^*.$$

Krusell and Smith: consumers

- Consumer's maximize:

$$\begin{aligned} U_i^* &= \max_{c,k} E_0 \sum_{t=0}^{\infty} \beta^t \log c_t \\ & \quad s.t \end{aligned}$$

$$\begin{aligned} c_t + k_{t+1} &= [r_t + 1 - \delta]\, k_t + w_t \varepsilon_t^i \\ c, k &\geq 0 \\ & \quad k_0^i \text{ given.} \end{aligned}$$

KS: prices and market clearing

- Prices are given by firm maximization:

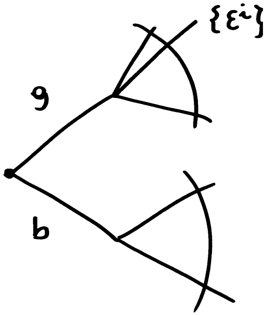
$$\begin{aligned} Y_t &= z_t K_t^{\alpha} L_t^{1-\alpha} \\ r &= Y_K \\ w &= Y_L \end{aligned}$$

- Market clearing:

$$\begin{aligned} \int k_t^i di &= K_t \\ \int \varepsilon_t^i di &= L_t \end{aligned}$$

Uncertainty

- Aggregate shocks: $z_t \in \{z_g, z_b\}$ follows a Markov structure.
- Idiosyncratic shocks: $\varepsilon_t^i \in \{0, l\}$ such that $L_t = L_g$ if $z_t = z_g$ and $L_t = L_b$ if $z_t = z_b$.
- Conditional on z_t, the shocks ε_{it} are uncorrelated.
- Uncertainty tree:



Approximate aggregation

- Krusell and Smith assume agents perceive a simple law of motion for the aggregate state:

$$\log K_{t+1} = a + b \log K_t$$

- Benefits:
  - State variables go from $(k_{it}, \varepsilon_{it}; \Gamma_t, z_t)$ to $(k_{it}, \varepsilon_{it}; K_t, z_t)$, where Γ_t is the entire distribution of capital (and is infinite-dimensional).
  - Excellent goodness of fit for most cases: approximate aggregation.

The dual in KS: static problem

- The static part is

$$
\begin{aligned}
U_\psi^* &= \max_c E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t) \\
& \quad s.t \\
\psi \cdot c &= k_0 + \psi \cdot e
\end{aligned}
$$

  where $e_t = w_t \varepsilon_t$.

- FOC:

$$
\begin{aligned}
0 &= \nabla U\left(c^*\right) - \lambda \psi \\
c_t^* &= \left(k_0 + \psi \cdot e\right)\left(1-\beta\right) \frac{\beta^t p_t}{\psi_t}
\end{aligned}
$$

The dual in KS: minimization

- The second step is to solve: $U^* = \inf_{\psi \in \Psi} U^*_\psi$
- What is Ψ? Its elements have to price capital, the only asset:

$$\begin{aligned} \psi_0 &= 1 \\ \psi_t(s^t) &= \int R_t\left(s^{t+1} \mid s^t\right) \psi_t(s^{t+1} \mid s^t) dP \end{aligned}$$

- This is simply the Euler equation $1 = E_t$ [stoch disc * returns]
- Now solve

$$\begin{aligned} U^* &= \inf_{\psi} E_0 \sum_{t=0}^{\infty} \beta^t \log\left((1-\beta)(k_0 + \psi \cdot e) \frac{\beta^t p_t}{\psi_t} \right) \\ & \text{s.t.} \\ \psi_t &= \int_{s^{t+1}|s^t} R_{t+1} \psi_{t+1} dP \end{aligned}$$

The dual in KS: minimization

- FOC give dynamics of ψ state by state:

$$\frac{e_t}{(1-\beta)(k_0+\psi\cdot e)}-\frac{\beta^t p_t}{\psi_t}=\lambda_t-\lambda_{t-1}p_{t-1}(z_t)R_t$$

  where λ_t is the Lagrange multiplier associated to the constraint for ψ_t.
- The state is given by $\left(k_t^i,\varepsilon_t^i;\Gamma_t,z_t\right)$, where Γ_t is the distribution of capital.
- Aggregation: how do decisions change if Γ_t changes, keeping $\left(k_t^i,\varepsilon_t^i;K_t,z_t\right)$ constant?

The dual in KS: aggregation

- For two states s^t and $\tilde{s}^t$ that share s^{t-1} and have

$$\left(k_t^i, \varepsilon_t^i; K_t, z_t\right) = \left(\tilde{k}_t^i, \tilde{\varepsilon}_t^i; \tilde{K}_t, \tilde{z}_t\right)$$

  the FOC gives:

$$c_t\left(s^t\right) + \lambda_t\left(s^t\right) = c_t\left(\tilde{s}^t\right) + \lambda_t\left(\tilde{s}^t\right).$$

- The Lagrange multiplier λ_t captures the effect of changing the distribution of wealth Γ_t, keeping aggregate capital constant.
- If $\lambda_t = 0$, there would be perfect aggregation.

The dual in KS: aggregation

- If now aggregate productivity is also different $z_t = z_g$ but $\tilde{z}_t = z_b$

$$c_t\left(s^t\right) + \lambda_t\left(s^t\right) = c_t\left(\tilde{s}^t\right) + \lambda_t\left(\tilde{s}^t\right) + a\left[c_t\left(\tilde{s}^t\right) + \lambda_t\left(\tilde{s}^t\right)\right] + bK_t^{\alpha}\varepsilon_t^i$$

  for two constants a, b.
- These expressions give the value of good vs. bad aggregate state, employment vs. unemployment.

Conclusion

- A dual approach to problems with heterogeneity and incomplete markets could be useful to understand aggregation.