1 Computing the gradient

The loss function is given by

$$\begin{aligned}
L\left(\tilde{\theta}\right) &= E\left[\sum_{k=1}^{n} \min\left(x_k, w_k\right) + \sum_{k=1}^{n}\left(1+\gamma\right)\left(x_k - w_k\right)u_k\left(x\right)\right] \\
& \quad x \text{ drawn from joint cdf } F\left(x;\tilde{\theta}\right) \\
u_k\left(x\right) &= \begin{cases} 0 & , \text{ if } k \notin D \\ e_k\left(I_D - \left(1+\gamma\right)A_D\right)^{-1} 1_D & , \text{ if } k \in D \end{cases} \\
D &= \left\{i : p_i < \bar{p}_i\right\} \\
p_i &= \min\left\{\bar{p}_i, \max\left\{\left(1+\gamma\right)\left(\sum_j p_j a_{ji} + c_i - x_i\right) - \gamma\bar{p}_i, 0\right\}\right\} \\
\underset{|D|\times|D|}{A_D} &= \underbrace{S'}_{|D|\times N}\underbrace{A}_{n\times n}\underbrace{S}_{n\times|D|} \\
S_{ij} &= \begin{cases} 1 & , \text{ if } j \in D \text{ and } i \leq j \\ 0 & , \text{ otherwise} \end{cases} \\
e_k &= \begin{bmatrix} 0 & \ldots & \underbrace{1}_{k^{th}\text{ entry}} & \ldots & 0 \end{bmatrix} \\
I_D &= |D| \times |D| \text{ identity matrix} \\
\tilde{\theta} &= \left(A, b, c, w, \bar{p}, \delta, \gamma\right)
\end{aligned}$$

Let's write it in vector/matrix form

$$L\left(\tilde{\theta}\right) = E\left\{\underbrace{\underbrace{\min\left(\underbrace{x}_{1\times n}, \underbrace{w}_{1\times n}\right)}\underbrace{1_n}_{n\times 1}}_{1\times 1} + \left(1+\gamma\right)\left[\underbrace{\underbrace{\left(x-w\right)}_{1\times n}\underbrace{S}_{n\times|D|}\underbrace{\left(\underbrace{I_D}_{|D|\times|D|} - \left(1+\gamma\right)\underbrace{S^T}_{|D|\times n}\underbrace{A}_{n\times n}\underbrace{S}_{n\times|D|}\right)^{-1}}_{|D|\times|D|}\underbrace{1_D}_{|D|\times 1}}_{1\times 1}\right]\right\}$$

$$
\begin{aligned}
& x \text{ drawn from joint cdf } F\left(x;\tilde{\theta}\right) \text{ and joint pdf } f\left(x;\tilde{\theta}\right) \\
D &= \{i : p_i < \bar{p}_i\} \\
|D| &= \#\text{ elements in } D \\
p &= \min\left\{\bar{p}, \max\left\{(1+\gamma)\left(\underbrace{\underbrace{p}_{1\times n}\underbrace{A}_{n\times n}}_{1\times n} + \underbrace{c-x}_{1\times n}\right) - \gamma\bar{p}, 0\right\}\right\} \\
S_{ij} &= \left\{\begin{array}{ccc} 1 & , & \text{if } j \in D \text{ and } i \leq j \\ 0 & , & \text{otherwise} \end{array}\right. \\
I_D &= |D| \times |D| \text{ identity matrix} \\
1_D &= \left[\begin{array}{ccccc} 1 & ... & \underset{|D|\times 1}{1} & ... & 1 \end{array}\right]^T \\
\tilde{\theta} &= (A, b, c, w, \bar{p}, \delta, \gamma)
\end{aligned}
$$

$$
\begin{aligned}
p_i &< \bar{p}_i \\
p_i &< \bar{p}_i
\end{aligned}
$$

$$
\begin{aligned}
case1 &: \bar{p} \leq \max\left\{(1+\gamma)(pA + c - x) - \gamma\bar{p}, 0\right\} \\
case1a &: (1+\gamma)(pA + c - x) - \gamma\bar{p} > 0 \\
case1b &: (1+\gamma)(pA + c - x) - \gamma\bar{p} \leq 0
\end{aligned}
$$

$$
\begin{aligned}
case2 &: \bar{p} > \max\left\{(1+\gamma)(pA + c - x) - \gamma\bar{p}, 0\right\} \\
case2a &: (1+\gamma)(pA + c - x) - \gamma\bar{p} > 0 \\
case2b &: (1+\gamma)(pA + c - x) - \gamma\bar{p} \leq 0
\end{aligned}
$$

$$
\begin{aligned}
& A \text{ is } n \times n \\
& b, c, w, \bar{p}, \delta, x \text{ are } 1 \times n \\
& \gamma \text{ is } 1 \times 1 \\
& F\left(x;\tilde{\theta}\right) \text{ is } 1 \times 1 \\
& vec\left(\tilde{\theta}\right) \text{ is } \left(n^2 + 5n + 1\right) \times 1
\end{aligned}
$$

We solve

$$
\begin{aligned}
& \max_{\theta} L\left(x;\tilde{\theta}\right) \\
& s.t. \\
& ...
\end{aligned}
$$

where θ is a subset of $\tilde{\theta}$.

To compute numerically, we want to compute

$$\nabla_{\theta} L\left(x;\tilde{\theta}\right)$$

Let's make a change of variables so that we can draw the random variables from the uniform distribution U and then transform them

$$x = F^{-1}\left(u;\tilde{\theta}\right)$$

$$\begin{aligned} L\left(\tilde{\theta}\right) &= \int_{[0,1]^n} \left\{\min\left(x,w\right)1_n + \left(1+\gamma\right)\left[\left(x-w\right)S\left(I_D-\left(1+\gamma\right)S^TAS\right)^{-1}1_D\right]\right\} f\left(x;\tilde{\theta}\right)dx \\ &= \int_{[0,1]^n} \left\{\min\left(F^{-1}\left(u;\tilde{\theta}\right),w\right)1_n + \left(1+\gamma\right)\left[\left(F^{-1}\left(u;\tilde{\theta}\right)-w\right)S\left(I_D-\left(1+\gamma\right)S^TAS\right)^{-1}1_D\right]\right\} c\left(u;\right. \end{aligned}$$

where $c\left(u;\tilde{\theta}\right)=1$ for iid uniform. Now let's consider the beta distribution with parameters α and β such that $P\left(c_iX_i > w_i\right)=\delta_i$. Then we have

$$\begin{aligned} F\left(x;\tilde{\theta}\right) &= \frac{1}{B\left(\alpha,\beta\right)}x^{\alpha-1}\left(1-x\right)^{\beta-1} \\ &= \frac{1}{B\left(1,\beta\right)}\left(1-x\right)^{\beta-1} \\ &= 1-\left(1-x\right)^{\beta} \end{aligned}$$

Therefore

$$\begin{aligned} \delta_i &= P\left(c_iX_i > w_i\right) \\ &= 1-P\left(c_iX_i \leq w_i\right) \\ &= 1-F_i\left(\frac{w_i}{c_i};\tilde{\theta}\right) \\ &= 1-\left(1-\left(1-\frac{w_i}{c_i}\right)^{\beta_i}\right) \\ &= \left(1-\frac{w_i}{c_i}\right)^{\beta_i} \end{aligned}$$

We then deduce that

$$\beta_i = \frac{\log\delta_i}{\log\left(1-\frac{w_i}{c_i}\right)}$$

The marginal cdf are

$$F\left(x;\tilde{\theta}\right) = 1-\left(1-x\right)^{\frac{\log\delta_i}{\log\left(1-\frac{w_i}{c_i}\right)}}$$

The inverse marginal cdf is

$$F^{-1}\left(u_i;\tilde{\theta}\right) = 1-\left(1-u_i\right)^{\frac{\log\left(1-\frac{w_i}{c_i}\right)}{\log\delta_i}}$$

Taking derivatives, we have

$$\begin{aligned}
\frac{\partial F^{-1}\left(u_i;\tilde{\theta}\right)}{\partial c_i} &= -\left(1-u_i\right)^{\frac{\log\left(1-\frac{w_i}{c_i}\right)}{\log\delta_i}}\frac{w_i\log\left(1-u_i\right)}{c_i^2\log\delta_i}\left(1-\frac{w_i}{c_i}\right)^{-1} \\
\frac{\partial F^{-1}\left(u_i;\tilde{\theta}\right)}{\partial c_j} &= 0
\end{aligned}$$

Now for the whole loss

$$\begin{aligned}
&\frac{\partial}{\partial c_i}E\left[\sum_{k=1}^{n}\min\left(c_kF_k^{-1}\left(u_k;c_k\right),w_k\right)+\sum_{k=1}^{n}\left(1+\gamma\right)\left(c_kF_k^{-1}\left(u_k;c_k\right)-w_k\right)u_k\left(x\right)\right] \\
&E\left[\sum_{k=1}^{n}\frac{\partial}{\partial c_i}\left\{\min\left(c_kF_k^{-1}\left(u_k;c_k\right),w_k\right)\right\}+\sum_{k=1}^{n}\left(1+\gamma\right)\frac{\partial}{\partial c_i}\left\{\left(c_kF_k^{-1}\left(u_k;c_k\right)-w_k\right)u_k\left(x\right)\right\}\right] \\
&E\left[\sum_{k=1}^{n}\frac{\partial}{\partial c_i}\left\{\min\left(c_kF_k^{-1}\left(u_k;c_k\right),w_k\right)\right\}+\sum_{k=1}^{n}\left(1+\gamma\right)\left[\frac{\partial}{\partial c_i}\left\{\left(c_kF_k^{-1}\left(u_k;c_k\right)-w_k\right)\right\}u_k\left(x\right)+\left(c_kF_k^{-1}\left(u_k;c_k\right.\right.\right.\right. \\
&E\left[\sum_{k=1}^{n}\frac{\partial\left[c_kF_k^{-1}\left(u_k;c_k\right)\right]}{\partial c_i}1\left\{c_kF_k^{-1}\left(u_k;c_k\right)<w_k\right\}+\left(1+\gamma\right)\sum_{k=1}^{n}\frac{\partial\left[c_kF_k^{-1}\left(u_k;c_k\right)\right]}{\partial c_i}u_k\left(x\right)\right] \\
&E\left[\sum_{k=1}^{n}\left\{1\left\{c_kx_k<w_k\right\}+\left(1+\gamma\right)u_k\left(x\right)\right\}\left(c_k\frac{\partial F_k^{-1}\left(u_k;c_k\right)}{\partial c_i}+F_k^{-1}\left(u_k;c_k\right)\frac{\partial c_k}{\partial c_i}\right)\right] \\
&E\left[\sum_{k=1}^{n}\left\{1\left\{c_kx_k<w_k\right\}+\left(1+\gamma\right)u_k\left(x\right)\right\}\left(c_k\frac{\partial F_k^{-1}\left(u_k;c_k\right)}{\partial c_k}+x_k\right)\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial c_i}\left\{\left(c_kF_k^{-1}\left(u_k;c_k\right)-w_k\right)\right\} &= \frac{\partial}{\partial c_i}c_kF_k^{-1}\left(u_k;c_k\right) \\
&= c_k\frac{\partial F_k^{-1}\left(u_k;c_k\right)}{\partial c_i}+x_k\frac{\partial c_k}{\partial c_i} \\
&= \begin{matrix} -\left(1-u_i\right)^{\frac{\log\left(1-\frac{w_i}{c_i}\right)}{\log\delta_i}}\frac{w_i\log(1-u_i)}{c_i\log\delta_i}\left(1-\frac{w_i}{c_i}\right)^{-1}+F_i^{-1}\left(u_i;c_i\right) & , & i=k \\ 0 & , & \text{otherwise} \end{matrix} \\
\frac{\partial}{\partial c_i}\left\{\min\left(c_kF_k^{-1}\left(u_k;c_k\right),w_k\right)\right\} &= \begin{matrix} F_k^{-1}\left(u_k;c_k\right)+c_k\frac{\partial}{\partial c_i}F_k^{-1}\left(u_k;c_k\right) & , & c_kF_k^{-1}\left(u_k;c_k\right)<w_k \\ 0 & , & \text{otherwise} \end{matrix} \\
&= \begin{matrix} x_k+c_k\frac{\partial}{\partial c_i}F_k^{-1}\left(u_k;c_k\right) & , & c_kx_k<w_k \\ 0 & , & \text{otherwise} \end{matrix} \\
\frac{\partial u_k\left(c\circ x\right)}{\partial c_j} &= 0
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial\left[\left(c_k x_k-w_k\right) u_k(c \circ x)\right]}{\partial c_j} &= u_k(c \circ x) \frac{\partial\left(c_k x_k-w_k\right)}{\partial c_j}+\left(c_k x_k-w_k\right) \frac{\partial u_k(c \circ x)}{\partial c_j} \\
\frac{\partial\left(c_k x_k-w_k\right)}{\partial c_j} &= x_k \frac{\partial c_k}{\partial c_j}=\left\{\begin{array}{ccc}
x_k & , & j=k \\
0 & , & \text{otherwise}
\end{array}\right. \\
\frac{\partial u_k(c \circ x)}{\partial c_j} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left(\nabla_c L\right)_i &= \left\{x_i+c_i \frac{\partial F_i^{-1}\left(u_i ; c_i\right)}{\partial c_i}\right\} 1\left\{c_i x_i<w_i\right\}+\left\{x_i+c_i \frac{\partial F_k^{-1}\left(u_i ; c_i\right)}{\partial c_i}\right\} \\
&= \left(1\left\{c_i x_i<w_i\right\}+1\right)\left\{x_i+c_i \frac{\partial F_k^{-1}\left(u_i ; c_i\right)}{\partial c_i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\left[e_k\left(I_D-(1+\gamma) A_D\right)^{-1} 1_D\right]}{\partial A_D} &= (1+\gamma)\left(I_D-(1+\gamma) A_D\right)^{-T} e_k^T 1_D^T\left(I_D-(1+\gamma) A_D\right)^{-T} \\
&= (1+\gamma)\left[e_k\left(I_D-(1+\gamma) A_D\right)^{-1}\right]^T\left[\left(I_D-(1+\gamma) A_D\right)^{-1} 1_D\right]^T \\
&= (1+\gamma)\left(I_D-(1+\gamma) A_D\right)^{-T} e_k^T u_D^T
\end{aligned}
$$

2 Old

We would like to compute the gradient of expected losses

$$
\nabla_A\left\{E\left[\sum_{k=1}^n \min \left(x_k, w_k\right)+\sum_{k=1}^n\left(x_k-w_k\right) u_k(x)\right]\right\}
$$

where ∇_A means the gradient with respect to the entries of the matrix A. Since x_k and w_k do not depend on A, and expectations are linear,

$$
\nabla_A\left\{E\left[\sum_{k=1}^n \min \left(x_k, w_k\right)+\sum_{k=1}^n\left(x_k-w_k\right) u_k(x)\right]\right\}=E \sum_{k=1}^n\left(x_k-w_k\right) \nabla_A u_k(x)
$$

To compute $\nabla_A u_k(x)$, we use that

$$
u_k(x)=\left\{\begin{array}{ccc}
0 & , & \text{if } k \notin D(A) \\
e_k\left(I_D-(1+\gamma) A_D\right)^{-1} 1_D & , & \text{if } k \in D(A)
\end{array}\right.
$$

where D is the set of defaulting nodes, I_D is the $|D| \times|D|$ identity matrix, e_k is a row vector with a 1 in column k and zeros otherwise, and A_D is the matrix A restricted to D. We can write

$$
A_D=S^{\prime} A S
$$

for a matrix S with dimension $n \times |D|$

$$S_{ij} = \left\{ \begin{array}{ccc} 1 & , & \text{if } j \in D \text{ and } i \leq j \\ 0 & , & \text{otherwise} \end{array} \right.$$

so that A_D has dimension $|D| \times |D|$. For example, if $n = 3$ and $D = \{1, 3\}$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} A_D &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \end{aligned}$$

We also note that D depends on A

$$\begin{aligned} D(A) &= \{i : p_i(A) < \bar{p}_i\} \\ &= \left\{ i : \min \left\{ \bar{p}_i, \max \left\{ \sum_j p_j(A) a_{ji} + c_i - x_i, 0 \right\} \right\} < \bar{p}_i \right\} \end{aligned}$$

However, $p_i(A)$ is continuous in A, so small changes in A can only affect D if

$$p_i(A) = \bar{p}_i$$

We now compute

$$\nabla_A u_k(x) = \left\{ \begin{array}{ccc} 0 & , & \text{if } k \notin D(A) \\ \nabla_A e_k \left(I_D - S^T A S\right)^{-1} 1_D & , & \text{if } k \in D(A) \end{array} \right.$$

Use the formulas

$$\begin{aligned} \frac{\partial a^T X^{-1} b}{\partial X} &= -X^{-T} a b^T X^{-T} \\ \partial(XY) &= (\partial X) Y + X \partial Y \\ \partial\left(S^T A S\right) &= \partial\left(S^T A\right) S + \left(S^T A\right) \partial S \\ &= \left[\partial\left(S^T\right) A + S^T \partial A\right] S + \left(S^T A\right) \partial S \\ &= S^T (\partial A) S \end{aligned}$$

to get

$$\begin{aligned} \frac{\partial \left[e_k \left(I_D - (1+\gamma) A_D\right)^{-1} 1_D\right]}{\partial A_D} &= (1+\gamma)\left(I_D - (1+\gamma) A_D\right)^{-T} e_k^T 1_D^T \left(I_D - (1+\gamma) A_D\right)^{-T} \\ &= (1+\gamma)\left[e_k \left(I_D - (1+\gamma) A_D\right)^{-1}\right]^T \left[\left(I_D - (1+\gamma) A_D\right)^{-1} 1_D\right]^T \\ &= (1+\gamma)\left(I_D - (1+\gamma) A_D\right)^{-T} e_k^T u_D^T \end{aligned}$$

and

$$\frac{\partial\left[e_k\left(I_D-A_D\right)^{-1}1_D\right]}{\partial A_{D^c}}=0$$

where D^c is the complement of D.

$$\nabla_A u_k\left(x\right)=S\left[\nabla_{A_D}u_k\left(x\right)\right]S^T$$

Finally,

$$E\sum_{k=1}^{n}\left(x_k-w_k\right)S\left[\nabla_{A_D}u_k\left(x\right)\right]S^T$$

Now we compute the gradient with respect to c, assuming that x_k is a random variable between 0 and 1 and thus total losses are

$$E\left[\sum_{k=1}^{n}\min\left(c_kx_k,w_k\right)+\sum_{k=1}^{n}\left(c_kx_k-w_k\right)u_k\left(c\circ x\right)\right]$$

where $c\circ x$ denotes element-wise multiplication.

We compute

$$\nabla_c\left\{E\left[\sum_{k=1}^{n}\min\left(c_kx_k,w_k\right)+\sum_{k=1}^{n}\left(c_kx_k-w_k\right)u_k\left(c\circ x\right)\right]\right\}$$

Using

$$\begin{aligned}
\frac{\partial\min\left(c_kx_k,w_k\right)}{\partial c_j} &= \left\{\begin{array}{ccc} x_k & , & c_k<\frac{w_k}{x_k}\text{ and } j=k \\ 0 & , & otherwise\end{array}\right. \\
\nabla_c\left\{\min\left(c_kx_k,w_k\right)\right\} &= \left(\begin{array}{ccccc} 0 & \cdots & 1_{\left\{c_k<\frac{w_k}{x_k}\right\}}x_k & \cdots & 0\end{array}\right) \\
\sum_{k=1}^{n}\nabla_c\left\{\min\left(c_kx_k,w_k\right)\right\} &= \left(\begin{array}{ccccc} 1_{\left\{c_1<\frac{w_1}{x_1}\right\}}x_1 & \cdots & 1_{\left\{c_k<\frac{w_k}{x_k}\right\}}x_k & \cdots & 1_{\left\{c_n<\frac{w_n}{x_n}\right\}}x_n\end{array}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\left[\left(c_kx_k-w_k\right)u_k\left(c\circ x\right)\right]}{\partial c_j} &= u_k\left(c\circ x\right)\frac{\partial\left(c_kx_k-w_k\right)}{\partial c_j}+\left(c_kx_k-w_k\right)\frac{\partial u_k\left(c\circ x\right)}{\partial c_j} \\
\frac{\partial\left(c_kx_k-w_k\right)}{\partial c_j} &= x_k\frac{\partial c_k}{\partial c_j}=\left\{\begin{array}{ccc} x_k & , & j=k \\ 0 & , & otherwise\end{array}\right. \\
\frac{\partial u_k\left(c\circ x\right)}{\partial c_j} &= 0
\end{aligned}$$

we get

$$
\begin{aligned}
& \nabla_c \left\{ E\left[\sum_{k=1}^{n} \min\left(c_k x_k, w_k\right) + \sum_{k=1}^{n} \left(c_k x_k - w_k\right) u_k\left(c \circ x\right) \right] \right\} \\
= \quad & E\left[\sum_{k=1}^{n} \nabla_c \min\left(c_k x_k, w_k\right) + \sum_{k=1}^{n} \nabla_c \left[\left(c_k x_k - w_k\right) u_k\left(c \circ x\right)\right] \right] \\
= \quad & E\left[\begin{array}{l} \left(\begin{array}{ccccc} 1_{\left\{c_1 < \frac{w_1}{x_1}\right\}} x_1 & \cdots & 1_{\left\{c_k < \frac{w_k}{x_k}\right\}} x_k & \cdots & 1_{\left\{c_n < \frac{w_n}{x_n}\right\}} x_n \end{array} \right) \\ + \left(\begin{array}{ccccc} x_1 u_1\left(c \circ x\right) & \cdots & x_k u_k\left(c \circ x\right) & \cdots & x_n u_n\left(c \circ x\right) \end{array} \right) \end{array} \right] \\
= \quad & E\left[\left(\begin{array}{ccccc} \left[1_{\left\{c_1 < \frac{w_1}{x_1}\right\}} + u_1\left(c \circ x\right)\right] x_1 & \cdots & \left[1_{\left\{c_k < \frac{w_k}{x_k}\right\}} + u_k\left(c \circ x\right)\right] x_k & \cdots & \left[1_{\left\{c_n < \frac{w_n}{x_n}\right\}} + u_n\left(c \circ x\right)\right] x_n \end{array} \right) \right.
\end{aligned}
$$

We pick the parameter β of the beta distribution with pdf

$$
f\left(x\right) = \frac{1}{B\left(\alpha, \beta\right)} x^{\alpha - 1} x^{\beta - 1}
$$

and cdf

$$
F\left(x\right) = \frac{1}{B\left(\alpha, \beta\right)} \int_0^x x^{\alpha - 1} x^{\beta - 1} dx
$$

so that

$$
\begin{aligned}
\delta &= P\left(cx > w\right) \\
&= 1 - F\left(\frac{w}{c}, \alpha, \beta\right) \\
&= 1 - \frac{1}{B\left(\alpha, \beta\right)} \int_0^{\frac{w}{c}} x^{\alpha - 1} x^{\beta - 1} dx
\end{aligned}
$$

Solving for β gives a solution

$$
\beta = \beta\left(c, w, \delta, \alpha\right)
$$

The CDF and PDF are then

$$
\begin{aligned}
F\left(x; c\right) &= \frac{1}{B\left(\alpha, \beta\left(c, w, \delta, \alpha\right)\right)} \int_0^x x^{\alpha - 1} x^{\beta\left(c, w, \delta, \alpha\right) - 1} dx \\
f\left(x; c\right) &= \frac{1}{B\left(\alpha, \beta\left(c, w, \delta, \alpha\right)\right)} x^{\alpha - 1} x^{\beta\left(c, w, \delta, \alpha\right) - 1}
\end{aligned}
$$

To compute

$$
\nabla_c \left\{ E\left[\sum_{k=1}^{n} \min\left(c_k x_k, w_k\right) + \sum_{k=1}^{n} \left(c_k x_k - w_k\right) u_k\left(c \circ x\right) \right] \right\}
$$

we then note that

$$
\begin{aligned}
& \nabla_c \left\{ E \left[\sum_{k=1}^{n} \min \left(c_k x_k, w_k \right) + \sum_{k=1}^{n} \left(c_k x_k - w_k \right) u_k \left(c \circ x \right) \right] \right\} \\
= \; & \nabla_c \int_0^1 ... \int_0^1 \left[\sum_{k=1}^{n} \min \left(c_k x_k, w_k \right) + \sum_{k=1}^{n} \left(c_k x_k - w_k \right) u_k \left(c \circ x \right) \right] f \left(x; c \right) dx_1 ... dx_n \\
= \; & \int_0^1 ... \int_0^1 \nabla_c \left\{ \left[\sum_{k=1}^{n} \min \left(c_k x_k, w_k \right) + \sum_{k=1}^{n} \left(c_k x_k - w_k \right) u_k \left(c \circ x \right) \right] f \left(x; c \right) \right\} dx_1 ... dx_n \\
= \; & \int_0^1 ... \int_0^1 \nabla_c \left[\sum_{k=1}^{n} \min \left(c_k x_k, w_k \right) + \sum_{k=1}^{n} \left(c_k x_k - w_k \right) u_k \left(c \circ x \right) \right] f \left(x; c \right) dx_1 ... dx_n \\
& + \int_0^1 ... \int_0^1 \left[\sum_{k=1}^{n} \min \left(c_k x_k, w_k \right) + \sum_{k=1}^{n} \left(c_k x_k - w_k \right) u_k \left(c \circ x \right) \right] \nabla_c f \left(x; c \right) dx_1 ... dx_n \\
= \; & \int_0^1 ... \int_0^1 \nabla_c \left[\sum_{k=1}^{n} \min \left(c_k x_k, w_k \right) + \sum_{k=1}^{n} \left(c_k x_k - w_k \right) u_k \left(c \circ x \right) \right] f \left(x; c \right) dx_1 ... dx_n \\
& + \int_0^1 ... \int_0^1 L \left(x \right) \frac{\nabla_c f \left(x; c \right)}{f \left(x; c \right)} f \left(x; c \right) dx_1 ... dx_n \\
= \; & E \left[\nabla_c \left[\sum_{k=1}^{n} \min \left(c_k x_k, w_k \right) + \sum_{k=1}^{n} \left(c_k x_k - w_k \right) u_k \left(c \circ x \right) \right] \right] + E \left[L \left(x \right) \frac{\nabla_c f \left(x; c \right)}{f \left(x; c \right)} \right]
\end{aligned}
$$

For independent random variables x

$$
f \left(x; c \right) = \prod_{k=1}^{n} f \left(x_k; c_k \right)
$$

so that

$$
\frac{\nabla_c f \left(x; c \right)}{f \left(x; c \right)} = \left(\frac{1}{f \left(x_1; c_1 \right)} \frac{df \left(x_1; c_1 \right)}{dc_1}, ..., \frac{1}{f \left(x_k; c_k \right)} \frac{df \left(x_k; c_k \right)}{dc_k}, ..., \frac{1}{f \left(x_n; c_n \right)} \frac{df \left(x_n; c_n \right)}{dc_n} \right)
$$

Assume $\alpha = 1$. Then

$$
\begin{aligned}
\delta &= 1 - \frac{1}{B \left(1, \beta \right)} \int_0^{\frac{w}{c}} x^{\beta - 1} dx \\
\delta &= \left(1 - \frac{w}{c} \right)^{\beta} \\
\beta &= \frac{\log \delta}{\log \left(1 - \frac{w}{c} \right)}
\end{aligned}
$$

and

$$\begin{aligned}
f_k\left(x;1,\beta\right) &= \beta\left(1-x\right)^{\beta-1}\\
f_k\left(x;1,\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}\right) &= \frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}\left(1-x\right)^{\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}-1}\\
\frac{1}{f\left(x_k;c_k\right)}\frac{df\left(x_k;c_k\right)}{dc_k} &= \frac{1}{\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}\left(1-x\right)^{\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}-1}}\left(\left\{\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}\right\}\frac{d}{dc}\left\{\left(1-x\right)^{\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}-1}\right\}+\left\{(1-x\right.\right.\\
&= \frac{1}{\left(1-x\right)^{\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}}}\frac{d}{dc}\left\{\left(1-x\right)^{\frac{\log\delta}{\log\left(1-\frac{w}{c}\right)}}\right\}-\frac{1}{\log\left(1-\frac{w}{c}\right)}\frac{1}{\left(c-w\right)}\frac{w}{c}\\
&= -\frac{w\left(\log\left(1-\frac{w}{c}\right)+\log\left(1-x\right)\log\left(\delta\right)\right)}{c\left(c-w\right)\log\left(1-\frac{w}{c}\right)^2}
\end{aligned}$$

If we use a gaussian copula with correlation matrix R and marginals

$$f\left(x_k;c_k\right)=\frac{1}{B\left(\alpha_k,\beta\left(c_k,w_k,\delta_k,\alpha_k\right)\right)}x^{\alpha_k-1}x^{\beta\left(c_k,w_k,\delta_k,\alpha_k\right)-1}$$

the density of the copula is

$$c_R^{Gauss}\left(u\right)=\frac{1}{\sqrt{\det R}}\exp\left(-\frac{1}{2}\begin{pmatrix}\Phi^{-1}\left(u_1\right)\\ ...\\ \Phi^{-1}\left(u_n\right)\end{pmatrix}^T\left(R^{-1}-I\right)\begin{pmatrix}\Phi^{-1}\left(u_1\right)\\ ...\\ \Phi^{-1}\left(u_n\right)\end{pmatrix}\right)$$

Thus,

$$f\left(x;c\right)=c_R^{Gauss}\left(F\left(x_1;c_1\right),...,F\left(x_n;c_n\right)\right)f\left(x_1;c_1\right)...f\left(x_n;c_n\right)$$

We can now compute the i^{th} component of $\frac{\nabla_c f(x;c)}{f(x;c)}$:

$$c_{R,k}^{Gauss}\left(F\left(x_1;c_1\right),...,F\left(x_n;c_n\right)\right)f\left(x_k;c_k\right)\frac{df\left(x_k;c_k\right)}{dc_k}+c_R^{Gauss}\left(F\left(x_1;c_1\right),...,F\left(x_n;c_n\right)\right)\frac{1}{f\left(x_k;c_k\right)}\frac{df\left(x_k;c_k\right)}{dc_k}$$

Try

$$\begin{aligned}
&\nabla_c\left\{E\left[\sum_{k=1}^n\min\left(c_kF^{-1}\left(u_k;c_k\right),w_k\right)+\sum_{k=1}^n\left(c_kF^{-1}\left(u_k;c_k\right)-w_k\right)u_k\left(c\circ F^{-1}\left(x\right)\right)\right]\right\}\\
=\;&\nabla_c\left\{E\left[\sum_{k=1}^n\min\left(c_kx_k,w_k\right)+\sum_{k=1}^n\left(c_kx_k-w_k\right)u_k\left(c\circ x\right)\right]\right\}\\
&+\left\{E\left[\sum_{k=1}^n\min\left(c_k\nabla_cF^{-1}\left(u_k;c_k\right),w_k\right)+\sum_{k=1}^n\left(c_k\nabla_cF^{-1}\left(u_k;c_k\right)-w_k\right)u_k\left(c\circ F^{-1}\left(x\right)\right)\right]\right\}
\end{aligned}$$