Maxwell's Equations - Review Notes

BasiCS Physics Program

2022 - 2023

1 Revisao de analise vetorial

1.1 O operador $\vec{\nabla}$

Note 1. Neste texto, ate que se mencione diferentemente, consideraremos o conjunto de coordenadas cartesianas ($\hat{x}$, $\hat{y}$, $\hat{z}$). Para as operacoes vetoriais, utilizaremos $\cdot$ para o produto escalar e $\times$ para o produto vetorial.

Como comumente apresentado nos cursos introdutorios de calculo, uma das primeiras atuacoes do operador $\vec{\nabla}$ e vista por meio do **gradiente**. Que, supondo um escalar A, possui a seguinte forma:

$$\vec{\nabla}A = \left(\frac{\partial A}{\partial x}\hat{x} + \frac{\partial A}{\partial y}\hat{y} + \frac{\partial A}{\partial z}\hat{z}\right) \tag{1}$$

Que e o gradiente de A

Isto pode ser reescrito de um modo mais interessante como:

$$\vec{\nabla}A = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)A \tag{2}$$

O termo entre parentesis e chamado de "del' e assim o denotamos como operador $\vec{\nabla}$:

$$\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \tag{3}$$

1.2 Divergente e Rotacional

Nesta subsecao, considere um vetor $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

Divergente

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \tag{4}$$

Rotacional

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{5}$$

1.3 O Laplaciano - Δ

O laplaciano consiste, basicamente, no operador 'del', porem, no lugar das derivadas primeiras, utilizamos as derivadas segundas. Voces podem ter visto, ao longo de seu percurso ate aqui, diversas notacoes, contudo, as mais usadas sao Δ ou ∇^2. Entretanto, a notacao mais utilizada ao longo dos cursos da CentraleSupelec e Δ e manteremos a mesma aqui neste texto.

1.3.1 Laplaciano de um escalar

Considere ϕ uma quantidade escalar, calculemos, então, o laplaciano deste escalar:

$$\Delta\phi = \vec{\nabla} \cdot \left(\vec{\nabla}\phi\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{6}$$

1.3.2 Laplaciano de um vetor

Considere $\vec{E}$ uma quantidade vetorial, tal que $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$. O laplaciano deste vetor é, simplesmente, o vetor com os laplacianos de cada componente escalar:

$$\Delta\vec{E} = \begin{pmatrix} \Delta E_x & \Delta E_y & \Delta E_z \end{pmatrix} = \Delta E_x\hat{x} + \Delta E_y\hat{y} + \Delta E_z\hat{z} \tag{7}$$

Important relations
Consider the following four vectors: $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{E}$. Then one has:

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \cdot \vec{C}\right)\vec{B} - \left(\vec{A} \cdot \vec{B}\right)\vec{C} \tag{8}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \Delta\vec{E} \tag{9}$$

Note 2. To make the relations easier to read, sometimes, we may use $\vec{grad}\phi = \vec{\nabla}\phi$, $div\vec{E} = \vec{\nabla} \cdot \vec{E}$ and $\vec{rot}\vec{E} = \vec{\nabla} \times \vec{E}$, where ϕ is a scalar quantity.

2 As equacoes de Maxwell

Note 3. A intencao deste texto nao e de atuar como um textbook, mas sim, apenas introduzir relacoes do mundo da eletrodinamica, conhecidas como equacoes de Maxwell. Portanto, caso deseje ir mais alem, eh valida a consulta ao material: livro

2.1 As equacoes de Maxwell microscopicas no vacuo

Uma serie de experimentos conduzidos ao longo do seculo XIX, em especial por Gauss, Faraday e Maxwell, resultaram no seguinte conjunto de equacoes que, por sua vez, descrevem o panorama da teoria eletromagnetica que sera utilizada durante os nossos estudos iniciais na CentraleSupelec. As equacoes sao:

Maxwell's Equations
Considering the electric field as $\vec{E}\,(\vec{r},t)$ and the magnetic field as $\vec{B}\,(\vec{r},t)$, where $\vec{r}$ is the position vector and t indicates the time. Then one has:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \text{ (Gauss' law)} \tag{10}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \text{ (Gauss' law of magnetism)} \tag{11}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ (Faraday's law)} \tag{12}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_o \epsilon_0 \frac{\partial \vec{E}}{\partial t} \text{ (Ampère - Maxwell's equation)} \tag{13}$$

$\mu_0 = 4\pi \times 10^{-7}$H/m indicates the magnetic permeability of vacuum, $\epsilon_0 = \frac{1}{36\pi} \times 10^{-9}$F/m its electric permittivity and $\mu_0 \epsilon_0 c^2 = 1$, where $c = 3 \times 10^8$m/s is the speed of light in the vacuum.

Note 4. As you may know, $\vec{J}$ is the current density vector, then, sometimes, we may indicate the term $\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ at Ampère-Maxwell's equation as a 'displacement current' to rewrite the equation as $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_d = \mu_0 \left(\vec{J} + \vec{J}_d\right)$

2.2 How to deal with these equations?

2.2.1 Obtaining $\vec{B}$ knowing $\vec{E}$

(i) Write $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(ii) Use the relation: $\vec{B}(\vec{r},t) = \vec{B}(\vec{r},t_0) + \int_{t_0}^{t} \frac{\partial \vec{B}}{\partial t'} dt'$

(iii) Finally, verify the Gauss' law of magnetism: $\vec{\nabla} \cdot \vec{B} = 0$

2.2.2 Obtaining $\vec{E}$ knowing $\vec{B}$ and $\vec{J}$

(i) Write $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_o \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

(ii) Use the relation: $\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}, t_0) + \int_{t_0}^{t} \frac{\partial \vec{E}}{\partial t'} dt'$

(iii) Finally, verify the Gauss' law: $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$

Important concepts

Lorentz's force

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \tag{14}$$

Volume current density vector

If the medium contains N families of charged elements, each family characterized at $\vec{r}$ and t by its particular density $n_i \left(\vec{r}, t \right)$ $[m^{-3}]$ and its group speed $\vec{v_i}(\vec{r}, t)$ $[m \cdot s^{-1}]$ considering its particules of charge q_i. Then, we define the volume current density vector $\vec{J}$ as:

$$\vec{J} = \sum_{i=1}^{N} n_i q_i \vec{v_i} \text{ in units of } [A \cdot m^{-2}] \tag{15}$$

Local Ohm's law

$$\vec{J} = \sigma \vec{E} \text{ (inside an ohmic medium)} \tag{16}$$

Where σ stands for the medium's electric condutivity, in units of $\Omega^{-1} \cdot m^{-1}$ or even of $S \cdot m^{-1}$

2.3 Local charge conservation

First, we remind you of a fundamental relation in vector analysis: $div \left(r\vec{o}t\vec{B} \right) = 0$ also written as $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = 0$

After that, we know from section 2.1 the Ampère-Maxwell's equation: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_o \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Using the previous relation, we apply the divergence operator into Ampère-Maxwell:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \left(\vec{\nabla} \cdot \vec{J} \right) + \mu_0 \epsilon_0 \left(\vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} \right) \tag{17}$$

Now, $\vec{\nabla} \cdot \left(\frac{\partial \vec{E}}{\partial t} \right) = \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{E} \right)$

And, using Gauss' law, we have:

$$\vec{\nabla} \cdot \left(\frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t} \tag{18}$$

Finally, using the previous relations, we arrive at the:

Local charge conservation equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{19}$$