Maxwell's Equations - Review Notes

BasiCS Physics Program

2022 - 2023

1 Review of vector analysis

1.1 The operator $\vec{\nabla}$

Note 1. In this text, until otherwise mentioned, we will consider the set of Cartesian coordinates ($\hat{x}$, $\hat{y}$, $\hat{z}$). For the vector operations, we will use $\cdot$ for the scalar product and $\times$ for the vector product.

As commonly presented in introductory calculus courses, one of the first actions of the operator $\vec{\nabla}$ is seen by means of the **gradient**. Which, assuming a scalar A, has the following form:

$$\vec{\nabla}A = \left(\frac{\partial A}{\partial x}\hat{x} + \frac{\partial A}{\partial y}\hat{y} + \frac{\partial A}{\partial z}\hat{z}\right) \tag{1}$$

Which is the gradient of A

This can be rewritten in a more interesting way as:

$$\vec{\nabla}A = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) A \tag{2}$$

The term in parentheses is called 'del' and we denote it as the $na\vec{b}la$ operator:

$$\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \tag{3}$$

1.2 Divergence and Curl

In this subsection, consider a vector $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

Divergence

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \tag{4}$$

Curl

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{5}$$

1.3 The Laplacian - Δ

The Laplacian consists basically of the operator 'del', but instead of the first derivatives, we use the second derivatives. You may have seen, along the way, several notations, however, the most used are Δ or ∇^2. However, the notation most often used throughout the CentraleSupélec courses is Δ and we will keep it the same here in this text.

1.3.1 Laplacian of a scalar

Consider ϕ a scalar quantity, evaluate, then, the Laplacian of this scalar:

$$\Delta\phi = \vec{\nabla} \cdot \left(\vec{\nabla}\phi\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{6}$$

1.3.2 Laplacian of a vector

Consider $\vec{E}$ a vector quantity, such that $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$. The Laplacian of this vector is, simply, the vector composed of the laplacian of each scalar component:

$$\Delta\vec{E} = \begin{pmatrix} \Delta E_x & \Delta E_y & \Delta E_z \end{pmatrix} = \Delta E_x\hat{x} + \Delta E_y\hat{y} + \Delta E_z\hat{z} \tag{7}$$

Important relations
Consider the following four vectors: $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{E}$. Then one has:

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \cdot \vec{C}\right)\vec{B} - \left(\vec{A} \cdot \vec{B}\right)\vec{C} \tag{8}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \Delta\vec{E} \tag{9}$$

Note 2. To make the relations easier to read, sometimes, we may use $\vec{grad}\phi = \vec{\nabla}\phi$, $div\vec{E} = \vec{\nabla} \cdot \vec{E}$ and $\vec{rot}\vec{E} = \vec{\nabla} \times \vec{E}$, where ϕ is a scalar quantity.

2 Maxwell's Equations

Note 3. The intention of this text is not to act as a textbook, but only to introduce relations from the world of electrodynamics, known as Maxwell's equations. Therefore, if you want to go further, it is valid to consult the material: Griffiths reference

2.1 Microscopic Maxwell's equations in the vacuum

A series of experiments conducted during the nineteenth century, especially by Gauss, Faraday and Maxwell, resulted in the following set of equations which, in turn, describe the background of the electromagnetic theory that will be used during our initial studies at CentraleSupélec. The equations are:

Maxwell's Equations
Considering the electric field as $\vec{E}\left(\vec{r},t\right)$ and the magnetic field as $\vec{B}\left(\vec{r},t\right)$, where $\vec{r}$ is the position vector and t indicates the time. Then one has:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \text{ (Gauss' law)} \tag{10}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \text{ (Gauss' law of magnetism)} \tag{11}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ (Faraday's law)} \tag{12}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_o\epsilon_0\frac{\partial\vec{E}}{\partial t} \text{ (Ampère - Maxwell's equation)} \tag{13}$$

$\mu_0 = 4\pi\times10^{-7}$H/m indicates the magnetic permeability of vacuum, $\epsilon_0 = \frac{1}{36\pi}\times10^{-9}$F/m its electric permittivity and $\mu_0\epsilon_0c^2 = 1$, where $c = 3\times10^8$m/s is the speed of light in the vacuum.

Note 4. As you may know, $\vec{J}$ is the current density vector, then, sometimes, we may indicate the term $\vec{J}_d = \epsilon_0\frac{\partial\vec{E}}{\partial t}$ at Ampère-Maxwell's equation as a 'displacement current' to rewrite the equation as $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\vec{J}_d = \mu_0\left(\vec{J} + \vec{J}_d\right)$

2.2 How to deal with these equations?

2.2.1 Obtaining $\vec{B}$ knowing $\vec{E}$

(i) Write $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

(ii) Use the relation: $\vec{B}(\vec{r},t) = \vec{B}(\vec{r},t_0) + \int_{t_0}^{t}\frac{\partial\vec{B}}{\partial t'}dt'$

(iii) Finally, verify the Gauss' law of magnetism: $\vec{\nabla}\cdot\vec{B} = 0$

2.2.2 Obtaining $\vec{E}$ knowing $\vec{B}$ and $\vec{J}$

(i) Write $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_o \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

(ii) Use the relation: $\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}, t_0) + \int_{t_0}^{t} \frac{\partial \vec{E}}{\partial t'} dt'$

(iii) Finally, verify the Gauss' law: $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$

Important concepts

Lorentz's force

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \tag{14}$$

Volume current density vector

If the medium contains N families of charged elements, each family characterized at $\vec{r}$ and t by its particular density $n_i(\vec{r}, t)$ $[m^{-3}]$ and its group speed $\vec{v_i}(\vec{r}, t)$ $[m \cdot s^{-1}]$ considering its particules of charge q_i. Then, we define the volume current density vector $\vec{J}$ as:

$$\vec{J} = \sum_{i=1}^{N} n_i q_i \vec{v_i} \text{ in units of } [A \cdot m^{-2}] \tag{15}$$

Local Ohm's law

$$\vec{J} = \sigma \vec{E} \text{ (inside an ohmic medium)} \tag{16}$$

Where σ stands for the medium's electric condutivity, in units of $\Omega^{-1} \cdot m^{-1}$ or even of $S \cdot m^{-1}$

2.3 Local charge conservation

First, we remind you of a fundamental relation in vector analysis: $div \left(\vec{rot} \vec{B} \right) = 0$ also written as $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = 0$

After that, we know from section 2.1 the Ampère-Maxwell's equation: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_o \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Using the previous relation, we apply the divergence operator into Ampère-Maxwell:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \left(\vec{\nabla} \cdot \vec{J} \right) + \mu_0 \epsilon_0 \left(\vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} \right) \tag{17}$$

Now, $\vec{\nabla} \cdot \left(\frac{\partial \vec{E}}{\partial t} \right) = \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{E} \right)$

And, using Gauss' law, we have:

$$\vec{\nabla} \cdot \left(\frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t} \tag{18}$$

Finally, using the previous relations, we arrive at the:

Local charge conservation equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{19}$$

3 Poynting's Theorem

Note 5. Consider that we are still working in the free space - or vacuum

As you may have seen during the previous sections or even during your studies at you home university, it is possible to associate an energy density (per unit volume) to the electric ($\vec{E}$) and magnetic ($\vec{B}$) fields, defined as (consider E and B as the absolute value of the electric $\vec{E}$ and $\vec{B}$ fields, respectively):

Electric energy density

$$\mu_E = \frac{\epsilon_0}{2} E^2 \tag{20}$$

Magnetic energy density

$$\mu_B = \frac{1}{2\mu_0} B^2 \tag{21}$$

Then, from (20) and (21), we can easily infer that the total energy that can be stored in the electromagnetic field, per unit volume, is given by:

Electromagnetic energy density

$$w = \mu_E + \mu_B = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \tag{22}$$