1 Perform calibration using Sum-Product Message Passing algorithm:

- Pass a message from the left clique to the right $\delta_{1\to 2}$: calculate β_2

  Given the following:

ψ	Noun	Verb	Other
Noun	1	2	3
Verb	10	1	3
Other	3	5	2

Considering that next word can be either noun, verb or other, the weight of it consists of the sum of the weights of all possible kinds of previous words which trigger the next one:

$$\sum_{p=noun}^{P} \psi_{p,n=noun} = 1 + 2 + 3 = 6$$
$$\sum_{p=noun}^{P} \psi_{p,n=verb} = 10 + 1 + 3 = 14$$
$$\sum_{p=noun}^{P} \psi_{p,n=other} = 3 + 5 + 2 = 10$$

Therefore, $\delta_{1\to 2}$ = [6, 14, 10]

From the obtained δ, the value of β_2 can be calculated. The previously obtained values will act as weights of the previous word when calculating the values of the next one; they are multiplied by the correspondent original weights shown in the original table:

β_2	Noun	Verb	Other
Noun	6	28	30
Verb	60	14	30
Other	18	70	20

- Pass a message from the left clique to the right $\delta_{2\to 1}$: calculate β_1

  A similar procedure than the followed in the previous point can be followed to calculate β_1, but know, it must be done backwards, from the last third word to the first one:

$$\sum_{n=noun}^{N} \psi_{p=noun,n} = 1 + 10 + 3 = 14$$
$$\sum_{n=noun}^{N} \psi_{p=verb,n} = 2 + 1 + 5 = 8$$
$$\sum_{n=noun}^{N} \psi_{p=other,n} = 3 + 3 + 2 = 8$$

Therefore, $\delta_{2\to 1}$ = [14, 8, 8]

Applying these results, but to the previous word, the following is obtained:

β_1	Noun	Verb	Other
Noun	14	28	42
Verb	80	8	24
Other	24	40	16

- Check if the resulting beliefs β_i are calibrated (if $\mu_{1,2} = \mu_{2,1}$)

  Two beliefs are calibrated if:

$$\sum_{C_i - S_{i,j}} \beta_i(C_i) = \sum_{C_j - S_{i,j}} \beta_j(C_j)$$

  In this case, it can be seen as the sum of all elements composing the rows of β_1 is equal to the sum of all elements composing the columns of β_2.

$$14 + 28 + 42 = 6 + 60 + 18 = 84$$
$$80 + 8 + 24 = 28 + 14 + 70 = 112$$
$$24 + 40 + 16 = 30 + 30 + 20 = 80$$

  As the results are equal, it can be said that the beliefs are calibrated.

2 Calculate the marginal distribution over the third word:

- Marginalize the belief $\beta(\textit{Second}, \textit{Third})$ over the second word

  Employing the results obtained in the previous exercise when calculating β_2, the belief can be marginalized over the second word. To do so, it is necessary to sum over all the possible values the second word can take:

$$\begin{aligned} Noun &\implies 6 + 28 + 30 = 64 \\ Verb &\implies 60 + 14 + 30 = 104 \\ Other &\implies 18 + 70 + 20 = 108 \end{aligned}$$

- Normalize resulting distribution

  The sum of all the values, Z = 84 + 112 + 80 = 276.

  If we divide the previously obtained values by 276, the result is the following:

$$\begin{aligned} Noun &\implies 64/276 = 0.23 \\ Verb &\implies 104/276 = 0.38 \\ Other &\implies 108/276 = 0.39 \end{aligned}$$

3 Make the following queries:

- You know that the first word is noun. What does it tell you about the third word? Find condition distribution $P(Third|First = Noun)$

  Applying the following formula:

$$\tilde{P}(T = N|F = N) = \sum_S \frac{\beta_1(F = N, S) * \beta_2(S, T = N)}{\mu_{1,2}(S)}$$

  We obtain:

  1. Third = noun:

$$\frac{14 * 6}{14 + 28 + 42} + \frac{80 * 28}{80 + 8 + 24} + \frac{24 * 30}{24 + 40 + 16} = 30$$

2. Third = verb:

$$\frac{14*60}{14+28+42}+\frac{80*14}{80+8+24}+\frac{24*30}{24+40+16}=29$$

3. Third = other:

$$\frac{14*18}{14+28+42}+\frac{80*70}{80+8+24}+\frac{24*20}{24+40+16}=59$$

After normalizing, the results are:

1. Third = noun: $30/118 = 0.254$
2. Third = verb: $14/118 = 0.246$
3. Third = other: $70/118 = 0.5$

- Now you also know that the second word is verb. Update your probability that the 3rd word is "other".

  In this case, the values will be, applying β_2 knowledge:

  1. Third = noun: 28
  2. Third = verb: 14
  3. Third = other: 70

  After normalizing, the values are: 0.25, 0.125, 0.625.

  As we know the second word, the knowledge of the first one is implicit in it. Therefore, applying directly the values of the original table will provide the same probabilities.

  1. Third = noun: 2
  2. Third = verb: 1
  3. Third = other: 5

  After normalizing, the values are: 0.25, 0.125, 0.625, same than those calculated before.