Consultație

29 Jan 2025

ex 1

Studiați convergența simplă și uniformă pt șirul de funcții $(f_n)_{n \in \mathbb{N}^*}$, unde

$f_n : (2, 3) \to \mathbb{R}$

$$f_n(x) = \frac{x^n e^x}{x^n + 1} \quad \forall n \in \mathbb{N}^*$$

ex 2

Determinați mulțimea de convergență a seriei de puteri

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}} \cdot (x+2)^n$$

ex 3

Determinați $\displaystyle\int_0^{64} \frac{x^4}{\sqrt[3]{64 - x}}\, dx$

ex 4

Fie $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \dfrac{(x^2 - y^2)^2}{x^2 + y^2} & ; \ (x, y) \neq (0,0) \\ 0 & ; \ (x, y) = (0,0) \end{cases}$

a) Stud continuitatea lui f

b) Det $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

c) Stud. diferențiabilitatea lui f

ex 5

Fie $f : \mathbb{R} \to \mathbb{R}$, o funcție continuă și neconstantă cu proprietatea $f(x+1) = f(x)$, $\forall x \in \mathbb{R}$

Arătați că funcția $g : (0,1) \to \mathbb{R}$, $g(x) = f\left(\frac{1}{x}\right)$ este continuă dar nu este uniform continuă

ex 6

Arătați că ecuația $5x^2 + 5y^2 + 5z^2 - 2xy - 2xz - 2yz - 9 = 0$ definește într-o vecinătate a punctului $(1,1,1)$ unica funcție implicită $z = z(x,y)$ și determinați

$\frac{\partial z}{\partial x}(1,1)$, $\frac{\partial z}{\partial y}(1,1)$, $dz(1,1)$

ex 7

proiecție pe OY

Det. $\iint_A x \, dx\, dy$, unde

$A = \{ (x,y) \in \mathbb{R}^2 \mid x \geq y^2 - 5,\ x \leq 1 + y,\ x \leq 1 - y \}$

ex 8

Fie funcția $f : [2, \infty) \to (0, \infty)$

$f(x) = \operatorname{arctg}\left(\frac{1}{\sqrt[3]{x}}\right)$

Stud. convergența integralei improprii $\int_2^\infty \left(2^{f(x)} - 1\right) dx$

ex 9

Det mulțimea de convergență a seriei de puteri

$$\sum_{n=1}^{\infty} \frac{5^n \cdot (2+\sqrt{1}) \cdot (2+\sqrt{2}) \cdot \ldots \cdot (2+\sqrt{n})}{(5+\sqrt{1}) \cdot (5+\sqrt{2}) \cdot \ldots \cdot (5+\sqrt{n})} \cdot (x+1)^n$$

ex 1 (2 Feb 2023, ex 1, b))

Studiați convergența simplă și uniformă pt șirul de funcții $(f_n)_{n \in \mathbb{N}^*}$, unde

$f_n : (2, 3) \to \mathbb{R}$

$$f_n(x) = \frac{x^n e^x}{x^n + 1} \quad \forall n \in \mathbb{N}^*$$

Sol:

Convergența simplă

Fie $x \in (2, 3)$

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\cancel{x^n} \cdot e^x}{\cancel{x^n}\left(1 + \frac{1}{x^n}\right)} = e^x$$

$$\Rightarrow f_n \xrightarrow[n \to \infty]{s} f, \quad \text{unde } f : (2, 3) \to \mathbb{R}, \ f(x) = e^x$$

Convergența uniform

$$\sup_{x \in (2,3)} |f_n(x) - f(x)| = \sup_{x \in (2,3)} \left| \frac{e^x \cdot x^n}{x^n + 1} - e^x \right|$$

$$= \sup_{x \in (2,3)} \left| \frac{\cancel{e^x \cdot x^n} - \cancel{e^x \cdot x^n} - e^x}{x^n + 1} \right|$$

$$= \sup_{x \in (2,3)} \frac{e^x}{x^n + 1}$$

Variante: majorări sau minorări

derivate (nu e recomandat pt exponențiale și logaritmice)

$$\sup_{x\in(2,3)} \frac{e^x}{x^n+1} \le \frac{e^3}{2^n+1} \xrightarrow[n\to\infty]{} 0$$

max pos sus

min pos jos

$$\Rightarrow f_n \xrightarrow[n\to\infty]{u} f$$

$\square$

ex 2 (2 Feb 2023 , ex 1, c))

Determinați mulțimea de convergență a seriei de puteri

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1}\cdot\sqrt[4]{n+2}}\cdot(x+2)^n$$

__Sol__ :

Notăm $x+2=y$

Seria devine $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1}\cdot\sqrt[4]{n+2}}\cdot y^n$

$$a_n = \frac{(-2)^n}{\sqrt[3]{n+1}\cdot\sqrt[4]{n+2}} \quad \forall\, n\in\mathbb{N}^*$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{\sqrt[3]{n+2}\cdot\sqrt[4]{n+3}}\right|\cdot\left|\frac{\sqrt[3]{n+1}\cdot\sqrt[4]{n+2}}{(-2)^n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{2^{\cancel{n+1}}}{\sqrt[3]{n+2}\cdot\sqrt[4]{n+3}}\cdot\frac{\sqrt[3]{n+1}\cdot\sqrt[4]{n+2}}{\cancel{2^n}}\right)$$

$$= \lim_{n\to\infty} 2\cdot\sqrt[3]{\frac{n+1}{n+2}}\cdot\sqrt[4]{\frac{n+2}{n+3}}\Bigg) = 2\cdot 1\cdot 1 = 2$$

Raza de convergență : $R = \dfrac{1}{\lim\limits_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}}$

$R = \dfrac{1}{2}$

Fie N mulțimea de convergență a seriei de puteri $\sum_{n=1}^{\infty} a_n \cdot y^n$

Avem $(-R, R) \subset N \subset [-R, R]$, i.e.

$(-\frac{1}{2}, \frac{1}{2}) \subset N \subset [-\frac{1}{2}, \frac{1}{2}]$

Studiem dacă $-\frac{1}{2} \in N$ și $\frac{1}{2} \in N$

Dacă $y = \frac{1}{2}$, seria devine $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}} \cdot \left(\frac{1}{2}\right)^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cancel{2^n}}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}} \cdot \frac{1}{\cancel{2^n}}$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}}$ conv. (Crit. lui Leibniz)

Deci $\frac{1}{2} \in N$

Dacă $y = -\frac{1}{2}$ seria devine $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}} \cdot \left(-\frac{1}{2}\right)^n$

$= \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}} \cdot \frac{1}{(-2)^n}$

$= \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}}$ ← ne încurcă $+1$ și $+2$

Fie $x_n = \frac{1}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}}$

$\forall n \in \mathbb{N}^*$

$y_n = \frac{1}{\sqrt[3]{n} \cdot \sqrt[4]{n}}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left(\frac{1}{\sqrt[3]{n+1} \cdot \sqrt[4]{n+2}} \cdot \frac{\sqrt[3]{n} \cdot \sqrt[4]{n}}{1} \right)$$

$$= \lim_{n\to\infty} \left(\sqrt[3]{\frac{n}{n+1}} \cdot \sqrt[4]{\frac{n}{n+2}} \right) = 1 \in (0, \infty)$$

Conform Crit. de comp. cu limită avem că $\sum x_n \sim \sum y_n$

$$\sum y_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} \cdot \sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}} \cdot n^{\frac{1}{4}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}+\frac{1}{4}}} =$$

$$= \sum \frac{1}{n^{\frac{7}{12}}}$$ div (serie armonică gen, $\alpha = \frac{7}{12}$)

Deci $-\frac{1}{2} \notin N$

Prin urmare, $N = \left(-\frac{1}{2}, \frac{1}{2} \right]$

Fie M mulțimea de conv a seriei de puteri din enunț.

$$y \in N \iff -\frac{1}{2} < \underset{x+2}{\underset{\shortparallel}{y}} \le \frac{1}{2} \iff -\frac{1}{2} < x+2 \le \frac{1}{2} \quad \Big| -2$$

$$-\frac{5}{2} < x \le -\frac{3}{2}$$

Asadar $M = \left(-\frac{5}{2}, -\frac{3}{2} \right]$

ex 3 (2 Feb 2023, ex 3, b))

Determinați $\int_0^{64} \frac{x^4}{\sqrt[3]{64-x}}\, dx$

Sol:

Recomandare: înmulțiri, împ

NU adunări scăderi

$$\int_0^{64} \frac{x^4}{\sqrt[3]{64-x}}\, dx = \int_0^{64} \frac{x^4}{\sqrt[3]{64\left(1-\frac{x}{64}\right)}}\, dx$$

$$= \int_0^1 \frac{64^4 \cdot t^4}{\sqrt[3]{64} \cdot \sqrt[3]{1-t}} \cdot 64\, dt =$$

S.V. $\frac{x}{64} = t \Rightarrow x = 64t$

$dx = 64\, dt$

$x = 0 \Rightarrow t = 0$

$x \to 64 \Rightarrow t \to 1$

$$= \frac{64^5}{4} \cdot \int_0^1 t^4 (1-t)^{-\frac{1}{3}}\, dt$$

$$= \frac{64^5}{4} \cdot \int_0^1 t^{5-1} \cdot (1-t)^{\frac{2}{3}-1}\, dt$$

$$= \frac{64^5}{4} \cdot \beta\left(5, \frac{2}{3}\right)$$

$$= \frac{(4^3)^5}{4} \cdot \beta\left(5, \frac{2}{3}\right)$$

$$= 4^{14} \cdot \beta\left(5, \frac{2}{3}\right) = 4^{14} \cdot \frac{24 \cdot 3^5}{14 \cdot 11 \cdot 8 \cdot 5 \cdot 2}$$

$$\beta\left(5, \frac{2}{3}\right) = \frac{\Gamma(5) \cdot \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(5+\frac{2}{3}\right)} = \frac{4! \cdot \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(5+\frac{2}{3}\right)}$$

$$= \frac{24 \cdot \Gamma\left(\frac{2}{3}\right)}{\frac{14 \cdot 11 \cdot 8 \cdot 5 \cdot 2}{3} \cdot \Gamma\left(\frac{2}{3}\right)} = \frac{24 \cdot 3^5}{14 \cdot 11 \cdot 8 \cdot 5 \cdot 2}$$

$$\Gamma(1+x) = x \cdot \Gamma(x)$$

$$\Gamma(5) = 4! = 24$$

$$\Gamma\left(5+\frac{2}{3}\right) = \Gamma\left(1+4+\frac{2}{3}\right) = \left(4+\frac{2}{3}\right)\cdot \Gamma\left(4+\frac{2}{3}\right)$$

$$= \frac{14}{3} \cdot \Gamma\left(1+3+\frac{2}{3}\right)$$

$$= \frac{14}{3} \cdot \left(3+\frac{2}{3}\right) \cdot \Gamma\left(3+\frac{2}{3}\right)$$

$$= \frac{14}{3} \cdot \frac{11}{3} \cdot \Gamma\left(1+2+\frac{2}{3}\right)$$

$$= \frac{14 \cdot 11}{3^2} \cdot \left(2+\frac{2}{3}\right) \cdot \Gamma\left(2+\frac{2}{3}\right)$$

$$= \frac{14 \cdot 11 \cdot 8}{3^3} \cdot \Gamma\left(1+1+\frac{2}{3}\right)$$

$$= \frac{14 \cdot 11 \cdot 8 \cdot}{3} \cdot \left(1+\frac{2}{3}\right) + \Gamma\left(1+\frac{2}{3}\right)$$

$$= \frac{14 \cdot 11 \cdot 8 \cdot 5}{3^4} \cdot \frac{2}{3} \cdot \Gamma\left(\frac{2}{3}\right)$$

$$= \frac{14 \cdot 11 \cdot 8 \cdot 5 \cdot 2}{3^5} \cdot \Gamma\left(\frac{2}{3}\right)$$

$$\Gamma(x) = \int_0^\infty t^{x-1} \cdot e^{-t} \, dt$$

$$B(x,y) = \int_0^1 t^{x-1} \cdot (1-t)^{y-1} \, dt$$

$$\forall \; x, y \in (0, \infty)$$

ex 4

Fie $f : \mathbb{R}^2 \to \mathbb{R}$, $\quad f(x,y) = \begin{cases} \dfrac{(x^2-y^2)^2}{x^2+y^2} & ; \ (x,y) \neq (0,0) \\ 0 & ; \ (x,y) = (0,0) \end{cases}$

a) Stud continuitatea lui f

b) Det $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$

c) Stud. diferențiabilitatea lui f

Sol:

a) f cont pe $\mathbb{R}^2 / (0,0)$ (op. cu func. elementare)

Studiem continuitatea lui f în $(0,0)$

Avem $f(x,y) = \begin{cases} \dfrac{x^4+y^4-2x^2y^2}{x^2+y^2} & ; \ (x,y) \neq (0,0) \\ 0 & ; \ (x,y) = (0,0) \end{cases}$

Fie $x, y \in \mathbb{R}^2 \setminus (0,0)$

$$| f(x,y) - f(0,0) |$$

$$= \left| \frac{x^4+y^4-2x^2y^2}{x^2+y^2} - 0 \right|$$

$$= \frac{|x^4+y^2-2x^2y^2|}{x^2+y^2} \leq \frac{|x^4|}{x^2+y^2} + \frac{|y^4|}{x^2+y^2} + \frac{|-2x^2y^2|}{x^2+y^2} =$$

$$= \frac{x^4}{x^2+y^2} + \frac{y^4}{x^2+y^2} + \frac{2x^2y^2}{x^2+y^2}$$

$$= x^2 \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1} + y^2 \cdot \underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1} + 2x^2 \cdot \underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1} \leq x^2 + y^2 + 2x^2 \xrightarrow[(x,y)\to(0,0)]{} 0$$

(Explicatie : $x^2 \leq x^2 + y^2 \quad | : (x^2+y^2)$)

$$\Leftrightarrow \frac{x^2}{x^2+y^2} \leq 1$$

$\Rightarrow f$ cont în $(0,0)$

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y) = \left(\frac{(x^2-y^2)^2}{x^2+y^2} \right)'_x$$

$$= \frac{2(x^2-y^2)\cdot 2x(x^2+y^2) - (x^2-y^2)^2 \cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2(x^2-y^2)\cdot(-2y)\cdot(x^2+y^2) - (x^2 \cdot y^2)\cdot 2y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0) + t \cdot e_1) - f(0,0)}{t} \qquad e_1 = (1,0)$$

$$= \lim_{t\to 0} \frac{f((0,0) + t(1,0)) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f((0,0) + (t,0)) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{(t^2-0^2)^2}{t^2+0^2} - 0}{t} = \lim_{t\to 0} \frac{\frac{t^4}{t^2}}{t} = \lim_{t\to 0} t = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot e_2) - f(0,0)}{t} \qquad e_2 = (0,1)$$

$$= \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{(0^2-t^2)^2}{0^2+t^2} - 0}{t} = \lim_{t\to 0} \frac{t^2}{t} = 0$$

c) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$\mathbb{R}^2 \setminus \{(0,0)\}$ deschisă $\quad \Rightarrow$

$\Rightarrow f$ dif $\mathbb{R}^2 \setminus \{(0,0)\}$ (Crit de diferențiabilitate)

Studiem dif. lui f în $(0,0)$

Dacă f ar fi diferențiabilă în $(0,0)$,

atunci $\underbrace{df(0,0)}_{T} : \mathbb{R}^2 \to \mathbb{R}$

$$\underbrace{df(0,0)}_{T}(u,v) = \left[\left(\underset{\substack{\| \\ 0}}{\frac{\partial f}{\partial x}(0,0)} \quad \underset{\substack{\| \\ 0}}{\frac{\partial f}{\partial y}(0,0)} \right) \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right]^t$$

$$= 0 \cdot u + 0 \cdot v$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \overbrace{df(0,0)}^{T}((x,y)-(0,0))}{\| (x,y) - (0,0) \|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{(x^2-y^2)^2}{x^2+y^2} - 0 - 0}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4+y^4-2x^2y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\left| \frac{x^4+y^4-2x^2y^2}{(x^2+y^2)\sqrt{x^2+y^2}} - 0 \right| = \frac{|x^4+y^4-2x^2y^2|}{(x^2+y^2)\sqrt{x^2+y^2}} \leq$$

$$\leq \frac{|x^4|+|y^4|+|-2x^2y^2|}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{(x^2+y^2)}{(x^2+y^2)\sqrt{x^2+y^2}} = \quad x^2$$

Deci $\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{(x^2+y^2)\sqrt{x^2+y^2}} = 0$

Prin urmare f este dif în $(0,0)$

$\square$

Model examen (b) 2

ex 5

Fie $f : \mathbb{R} \to \mathbb{R}$, o funcție continuă și neconstantă cu proprietatea $f(x+1) = f(x)$, $\forall x \in \mathbb{R}$

Arătați că funcția $g : (0,1) \to \mathbb{R}$, $g(x) = f\left(\frac{1}{x}\right)$ este continuă dar nu este uniform continuă

Sol:

$$x_n \longrightarrow a \Rightarrow g(x_n) \longrightarrow g(a)$$

Fie $a \in (0,1)$

Fie $(x_n)_n \subset (0,1)$ a.î. $\lim\limits_{n \to \infty} x_n = a$

Avem $g(x_n) = f\left(\frac{1}{x_n}\right)$ $\forall n \in \mathbb{N}$

Deoarece f este continuă, avem că $\lim\limits_{n \to \infty} f\left(\frac{1}{x_n}\right) = f\left(\frac{1}{a}\right)$

Așadar $\lim\limits_{n \to \infty} g(x_n) = \lim\limits_{n \to \infty} f\left(\frac{1}{x_n}\right) = f\left(\frac{1}{a}\right) = g(a)$

Prin urmare g este cont în a

Deoarece am ales în mod arbitrar pe a, rezultă că g este continuă (pe $(0,1)$)

Arătam în continuare că g nu este uniform continuă.

g nu e uniform cont $\Leftrightarrow$ $\exists (x_n)_n \subset (0,1)$, $\exists (y_n)_n \subset (0,1)$ a.î. $\lim\limits_{n\to\infty} (x_n - y_n) = 0$ şi $\lim\limits_{n\to\infty} g(x_n) - g(y_n) \neq 0$

f neconstantă $\Rightarrow \exists \underset{\alpha \neq \beta}{\alpha, \beta} \in \mathbb{R}$ a.î. $f(\alpha) \neq f(\beta)$

Alegem $x_n = \frac{1}{n+\alpha}$ $\forall n \in \mathbb{N}^*$, $n \geq |[\alpha]| + 2$

$y_n = \frac{1}{n+\beta}$ $\forall n \in \mathbb{N}^*$, $n \geq |[\beta]| + 2$

Fie $n \in \mathbb{N}^*$, $n \geq \max\{|[\alpha]| + 2, |[\beta]| + 2\}$

Avem $x_n \in (0,1)$ şi $y_n \in (0,1)$

Avem $\lim\limits_{n\to\infty} (x_n - y_n) = \lim\limits_{n\to\infty} \left(\frac{1}{n+\alpha} - \frac{1}{n+\beta}\right) = 0$

$g(x_n) = g\left(\frac{1}{n+\alpha}\right) = f(n+\alpha) = f(1+n-1+\alpha) \underset{f(x+1)=f(x)}{=} f(n-1+\alpha)$

$= \dots = f(\alpha)$

$\forall n \geq \max\{|[\alpha]| + 2, |[\beta]| + 2\}$

$g(y_n) = g\left(\frac{1}{n+\beta}\right) = f(n+\beta) = f(1+n-1+\beta) \underset{f(x+1)=f(x)}{=} f(n-1+\beta)$

$= \dots = f(\beta)$

$\forall n \geq \max\{|[\alpha]| + 2, |[\beta]| + 2\}$

$\lim\limits_{n\to\infty} (g(x_n) - g(y_n)) = \lim\limits_{n\to\infty} (f(\alpha) - f(\beta)) \neq 0$

Prin urmare g nu este uniform cont $\square$

ex 6

Arătați că ecuația $5x^2 + 5y^2 + 5z^2 - 2xy - 2xz - 2yz - 9 = 0$ definește într-o vecinătate a punctului $(1,1,1)$ unica funcție implicită $z = z(x,y)$ $\wedge$ determinați

$\frac{\partial z}{\partial x}(1,1)$, $\frac{\partial z}{\partial y}(1,1)$, $dz(1,1)$

Sol:

Fie $D = \mathbb{R}^3$ și $F : D \to \mathbb{R}$.

$F(x,y,z) = 5x^2 + 5y^2 + 5z^2 - 2xy - 2xz - 2yz - 9$

D deschisă

1) $F(1,1,1) = 5+5+5-2-2-2-9 = 0$

2) $\frac{\partial F}{\partial x}(x,y,z) = 10x - 2y - 2z$

$\frac{\partial F}{\partial y}(x,y,z) = 10y - 2x - 2z \qquad \forall (x,y,z) \in \mathbb{R}^2$

$\frac{\partial F}{\partial z}(x,y,z) = 10z - 2x - 2y$

$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ cont pe $\mathbb{R}^3$

$\mathbb{R}^3$ deschisă

$\Rightarrow F$ de clasă C^1 pe $\mathbb{R}^3$

3) $\frac{\partial F}{\partial x}(1,1,1) = 10 - 2 - 2 = 6 \neq 0$

Conform T.F.I. $\exists U = \mathring{U} \in \mathcal{V}_{(1,1)}$, $\exists V = \mathring{V} \in \mathcal{V}_1$

$\exists ! z : U \to V$ (z unica funcție implicită) a.î.

a) $z(1,1) = 1$

b) $F(x, y, z(x,y)) = 0 \quad \forall (x,y) \in U$

c) z este de clasă C^1 și

$$\frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x, y, z(x,y))}{\frac{\partial F}{\partial z}(x, y, z(x,y))} \quad \forall (x,y) \in U$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x, y, z(x,y))}{\frac{\partial F}{\partial z}(x, y, z(x,y))} \quad \forall (x,y) \in U$$

$$\frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x, y, z(x,y))}{\frac{\partial F}{\partial z}(x, y, z(x,y))}$$

$$= -\frac{10x - 2y - 2z(x,y)}{10z(x,y) - 2x - 2y} \quad \forall (x,y) \in U$$

$$\Rightarrow \frac{\partial z}{\partial x}(1,1) = -\frac{10 \cdot 1 - 2 \cdot 1 - 2z(1,1)}{10 \cdot z(1,1) - 2 \cdot 1 - 2 \cdot 1} \underset{z(1,1)=1}{=} -\frac{10 - 2 - 2}{10 - 2 - 2} = -1$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x, y, z(x,y))}{\frac{\partial F}{\partial z}(x, y, z(x,y))}$$

$$= -\frac{10y - 2x - 2z(x,y)}{10z(x,y) - 2x - 2y} \quad \forall (x,y) \in U$$

$$\Rightarrow \frac{\partial z}{\partial y}(1,1) = - \frac{10\cdot 1 - 2\cdot 1 - 2z(1,1)}{10\cdot z(1,1) - 2\cdot 1 - 2\cdot 1} \underset{z(1,1)=1}{=} - \frac{10-2-2}{10-2-2} = -1$$

z de clasă C^1 (pe U) $\Rightarrow$ z dif (pe U) $\Rightarrow$

$\Rightarrow$ z dif în $(1,1)$

$dz(1,1) : \mathbb{R}^2 \to \mathbb{R}$

$$dz(1,1)(u,v) = \left[\left(\underset{-1}{\underset{\shortparallel}{\frac{\partial z}{\partial x}(1,1)}} \quad \underset{-1}{\underset{\shortparallel}{\frac{\partial z}{\partial y}(1,1)}} \right) \begin{pmatrix} u \\ v \end{pmatrix} \right]$$

$= -u - v$, i.e. $dz(1,1) = -dx - dy$

$\square$

5 Feb 2024

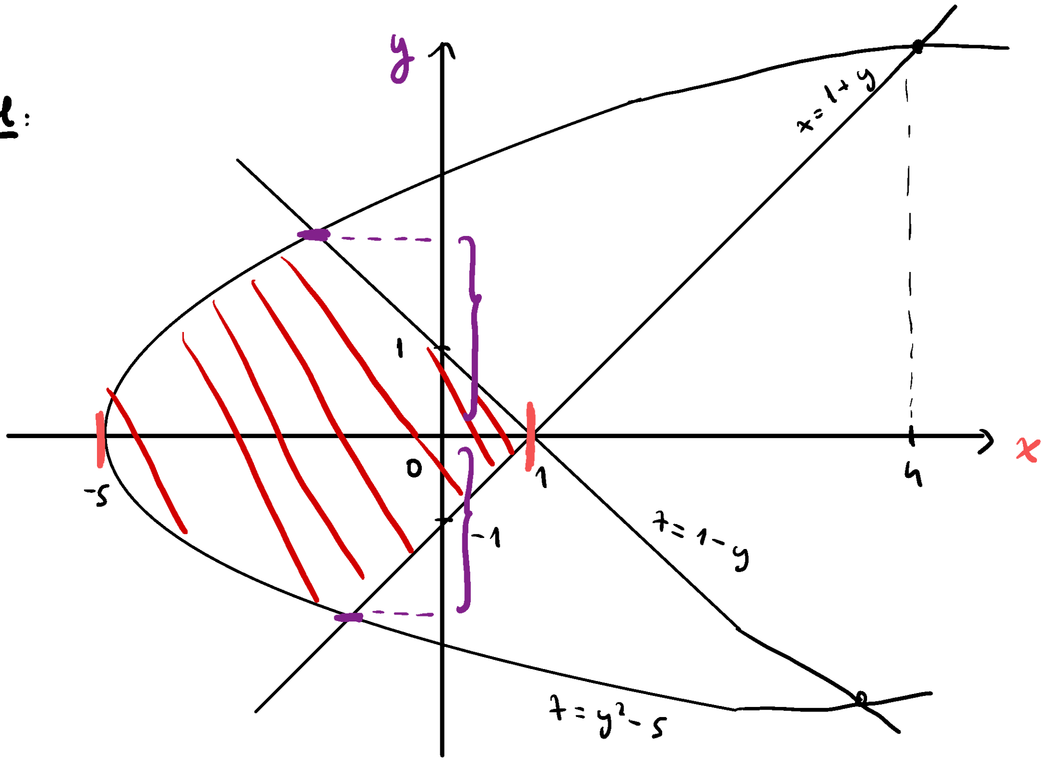
ex 7

Det. $\iint_A x \, dx \, dy$, unde

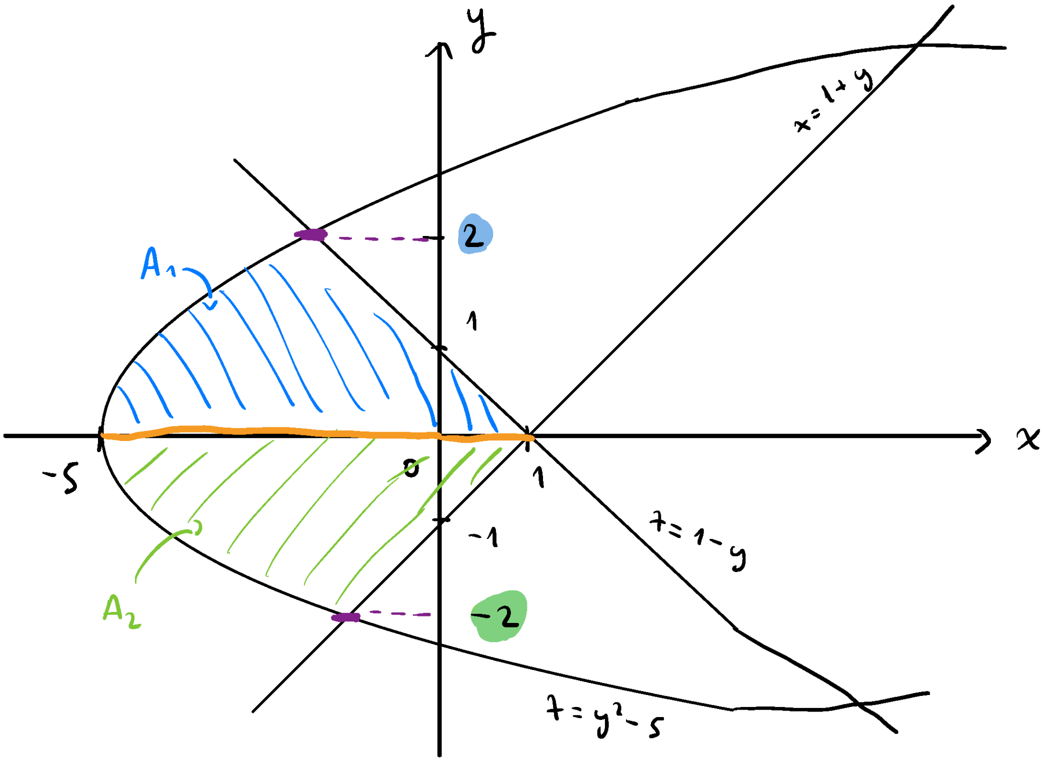
proiecție pe Oy

$A = \{ (x,y) \in \mathbb{R}^2 \mid x \geq y^2 - 5, \ x \leq 1 + y, \ x \leq 1 - y \}$

Sol:



Proiecție pe Oy



Det. punctele de intersecție dintre $x = y^2 - 5$ și $x = 1 - y$

$$\begin{cases} x = y^2 - 5 \\ x = 1 - y \end{cases} \Rightarrow \quad y^2 - 5 = 1 - y$$

$$y^2 + y - 6 = 0$$

$$\Delta = 1 + 24 = 25$$

$$y_{1,2} = \frac{-1 \pm 5}{2} \left\langle \begin{array}{l} 2 \\ -3 \end{array} \right.$$

$y_1 = 2 \Rightarrow x_1 = 4 - 5 = -1$

$y_2 = -3 \Rightarrow x_1 = 9 - 5 = 4$

Det. punctele de intersecție dintre $x = y^2 - 5$ și $x = 1 + y$

$$\begin{cases} x = y^2 - 5 \\ x = 1 + y \end{cases} \Rightarrow \quad y^2 - 5 = 1 + y$$

$$y^2 - y - 6 = 0$$

$$\Delta = 1 + 24 = 25$$

$$y_{1,2} = \frac{1 \pm 5}{2} \quad \begin{matrix} 3 \\ -2 \end{matrix}$$

$y_1 = 3 \Rightarrow x_1 = 9 - 5 = 4$

$y_2 = -2 \Rightarrow x_2 = 4 - 5 = 1$

$A = A_1 \cup A_2$, unde

$A_1 = \{ (x, y) \mid y \in [0, 2],\ y^2 - 5 \leq x \leq 1 - y \}$

$A_2 = \{ (x, y) \mid y \in [-2, 0],\ y^2 - 5 \leq x \leq 1 + y \}$

Fie $\varphi_1, \psi_1 : [0, 2]$, $\varphi_1(y) = y^2 - 5$, $\psi_1(y) = 1 - y$

φ_1, ψ_1 continue

$A_1 \in J(\mathbb{R}^2)$ și A_1 compactă

Fie $\varphi_2, \psi_2 : [-2, 0] \to \mathbb{R}$, $\varphi_2(y) = y^2 - 5$, $\psi_2 = 1 + y$

φ_2, ψ_2 cont

$A_2 \in J(\mathbb{R}^2)$ și A_2 compacta

Deci $A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = x$

f cont

$A_1 \cap A_2 = [-5, 1] \times [0] \Rightarrow \mu(A_1 \cap A_2) = \mu([-5,1] \times [0])$

$= \mu([-5,1]) \cdot \mu(\{0\}) = (1+5) \cdot 0 = 0$

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$$

$$\iint_{A_1} f(x,y)\,dx\,dy$$

$$= \int_0^2 \left(\int_{y^2-5}^{1-y} x\,dx \right) dy$$

$$= \int_0^2 \frac{x^2}{2} \Big|_{x=y^2-5}^{x=1-y} dy$$

$$= \int_0^2 \frac{1}{2} \cdot \left[(1-y)^2 - (y^2-5)^2 \right] dy$$

$$= \int_0^2 \frac{1}{2} \cdot (1 + y^2 - 2y - y^4 + 10y^2 - 25)\,dy$$

$$= \frac{1}{2} \int_0^2 (-y^4 + 11y^2 - 2y - 24)\,dy$$

$$= \frac{1}{2} \left(-\frac{y^5}{5}\Big|_{y=0}^{y=2} + 11 \frac{y^3}{3}\Big|_{y=0}^{y=2} - 2\frac{y^2}{2}\Big|_{y=0}^{y=2} - 24y\Big|_{y=0}^{y=2} \right)$$

$$= \frac{1}{2} \left(-\frac{32}{5} + \frac{88}{3} - 4 - 48 \right)$$

$$= \frac{-16}{5} + \frac{44}{3} - 26$$

$$= \frac{\underbrace{-48 + 220 - 390}}{15} = \frac{-438 + 220}{15} = -\frac{218}{15}$$

$$\iint_{A_2} f(x,y)\,dx\,dy$$

$$= \int_{-2}^{0} \left(\int_{y^2-5}^{1+y} x\,dx \right) dy$$

$$= \int_{-2}^{0} \left(\frac{x^2}{2} \Big|_{x=y^2-5}^{x=1+y} \right) dy$$

$$= \int_{-2}^{0} \frac{1}{2} \left(1 + y^2 + 2y - y^4 + 10y^2 - 25 \right) dy$$

$$= \frac{1}{2} \int_{-2}^{0} \left(-y^4 + 11y^2 + 2y - 24 \right) dy$$

$$= \frac{1}{2} \left(-\frac{y^5}{5} \Big|_{y=-2}^{y=0} + 11\frac{y^3}{3} \Big|_{y=-2}^{y=0} + \frac{2y^2}{2} \Big|_{y=-2}^{y=0} - 24y \Big|_{y=-2}^{y=0} \right)$$

$$= \frac{1}{2} \left(\frac{32}{5} - \frac{88}{3} - 4 - 48 \right)$$

Deci $\iint_A f(x,y)\,dx\,dy = -\frac{218}{15} - \frac{218}{15} = -\frac{536}{15}$

$\square$

ex 8

Fie funcția $f: [2,\infty) \to (0,\infty)$

$f(x) = \operatorname{arctg}\left(\frac{1}{\sqrt[3]{x}}\right)$

Stud. convergența integralei improprii $\int_2^{\infty} (2^{f(x)} - 1)\,dx$

Sol:

$$\lim_{y \to 0} \frac{2^y - 1}{y} = \ln 2$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \operatorname{arctg}\left(\frac{1}{\sqrt[3]{x}}\right) = 0$$

$\Rightarrow$

$$\Rightarrow \lim_{x\to\infty} \frac{2^{f(x)}-1}{f(x)} = \ln 2 \in (0,\infty)$$

Conform Crit de comp. cu limită avem că

$$\int_2^\infty \left(2^{f(x)}-1\right) dx \sim \int_2^\infty f(x)\, dx = \int_2^\infty \operatorname{arctg}\left(\frac{1}{\sqrt[3]{x}}\right) dx$$

$$\left.\begin{aligned} &\lim_{y\to 0} \frac{\operatorname{arctg} y}{y} = 1 \\ &\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \operatorname{arctg}\left(\frac{1}{\sqrt[3]{x}}\right) = 0 \end{aligned}\right| \Rightarrow$$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{\frac{1}{\sqrt[3]{x}}} = 1 \in (0,\infty)$$

Conform Crit de comp. cu limită avem că

$$\int_2^\infty f(x)\, dx \sim \int_2^\infty \frac{1}{\sqrt[3]{x}}\, dx$$

Fie $g : [2,\infty) \to (0,\infty)$, $g(x) = \frac{1}{\sqrt[3]{x}}$

Cum g (strict) descresc. avem, conform criteriul integral al lui Cauchy, că

$$\int_2^\infty g(x)\, dx \sim \sum_{n=2}^{\infty} g(n) = \sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{3}}} \text{ div}$$

(serie arm. gen, $\alpha = \frac{1}{3}$)

Deci $\int_2^\infty \left(2^{f(x)}-1\right) dx$ este div $\square$

ex 9

Det mulțimea de convergență a seriei de puteri

$$\sum_{n=1}^{\infty} \frac{5^n \cdot (2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot \ldots \cdot(5+\sqrt{n})} \cdot (x+1)^n$$

Sol:

Notam $x+1=y$

Seria devine $\displaystyle\sum_{n=1}^{\infty} \frac{5^n \cdot (2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot \ldots \cdot(5+\sqrt{n})} \cdot y^n$

$$a_n = \frac{5^n \cdot (2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot \ldots \cdot(5+\sqrt{n})} \cdot \quad \forall\, n \in \mathbb{N}^*$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n\to\infty} \cdot \frac{\dfrac{5^{\cancel{n+1}} \cdot (2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n}) \cdot (2+\sqrt{n+1})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot \ldots \cdot(5+\sqrt{n})\cdot(5+\sqrt{n+1})}}{\dfrac{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot \ldots \cdot(5+\sqrt{n})}{\cancel{5^n} \cdot (2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}}$$

$$= \lim_{n\to\infty} \left(5 \cdot \frac{2+\sqrt{n+1}}{5+\sqrt{n+1}}\right) = 5$$

$$R = \frac{1}{5}$$

Fie N mulțimea de conv a seriei de puteri $\displaystyle\sum_{n=1}^{\infty} a_n \cdot y^n$

Avem $\left(-\frac{1}{5}, \frac{1}{5}\right) \cup N \subset \left[\frac{1}{5}, \frac{1}{5}\right]$

Studiem dacă $-\frac{1}{5} \in N$ și $\frac{1}{5} \in N$

Daca $y = \frac{1}{5}$, seria devine

$$\sum_{n=1}^{\infty} \cancel{5^n} \cdot \frac{(2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot\ldots\cdot(5+\sqrt{n})} \cdot \frac{1}{\cancel{5^n}}$$

$$= \sum_{n=1}^{\infty} \frac{(2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot\ldots\cdot(5+\sqrt{n})}$$

Fie $x_n = \frac{(2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot\ldots\cdot(5+\sqrt{n})}$ $\forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\cdot\left(\frac{5+\sqrt{n+1}}{2+\sqrt{n+1}} - 1\right)$$

$$= \lim_{n\to\infty} \frac{3n}{2+\sqrt{n+1}} = \infty > 1$$

Conform Crit Abel-Duhamel avem că $\sum_{n=1}^{\infty} x_n$ conv

Deci $\frac{1}{5} \in N$

Daca $y = -\frac{1}{5}$, seria devine

$$\sum_{n=1}^{\infty} \cancel{5^n} \cdot \frac{(2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot\ldots\cdot(5+\sqrt{n})} \cdot \frac{(-1)^n}{\cancel{5^n}}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot\ldots\cdot(5+\sqrt{n})}$$

Fie $y_n = (-1)^n \cdot \frac{(2+\sqrt{1})\cdot(2+\sqrt{2})\cdot \ldots \cdot(2+\sqrt{n})}{(5+\sqrt{1})\cdot(5+\sqrt{2})\cdot\ldots\cdot(5+\sqrt{n})}$

$\sum |y_n|$ conv (vezi mai sus)

$\Rightarrow \sum y_n$ absolut conv și deci $\sum y_n$ conv

Deci $-\frac{1}{5} \in N$

Am arătat $N = \left[-\frac{1}{5}, \frac{1}{5}\right]$

Fie M mulțimea de convergență a seriei de puteri din enunț

$$y \in N \iff -\frac{1}{5} \le \underset{\substack{\| \\ x+1}}{y} \le \frac{1}{5} \iff -\frac{1}{5} \le x+1 \le \frac{1}{5} \quad \Big| -1$$

$$-\frac{6}{5} \le x \le -\frac{4}{5}$$

Deci $M = \left[-\frac{6}{5}, \frac{4}{5}\right]$

$\square$