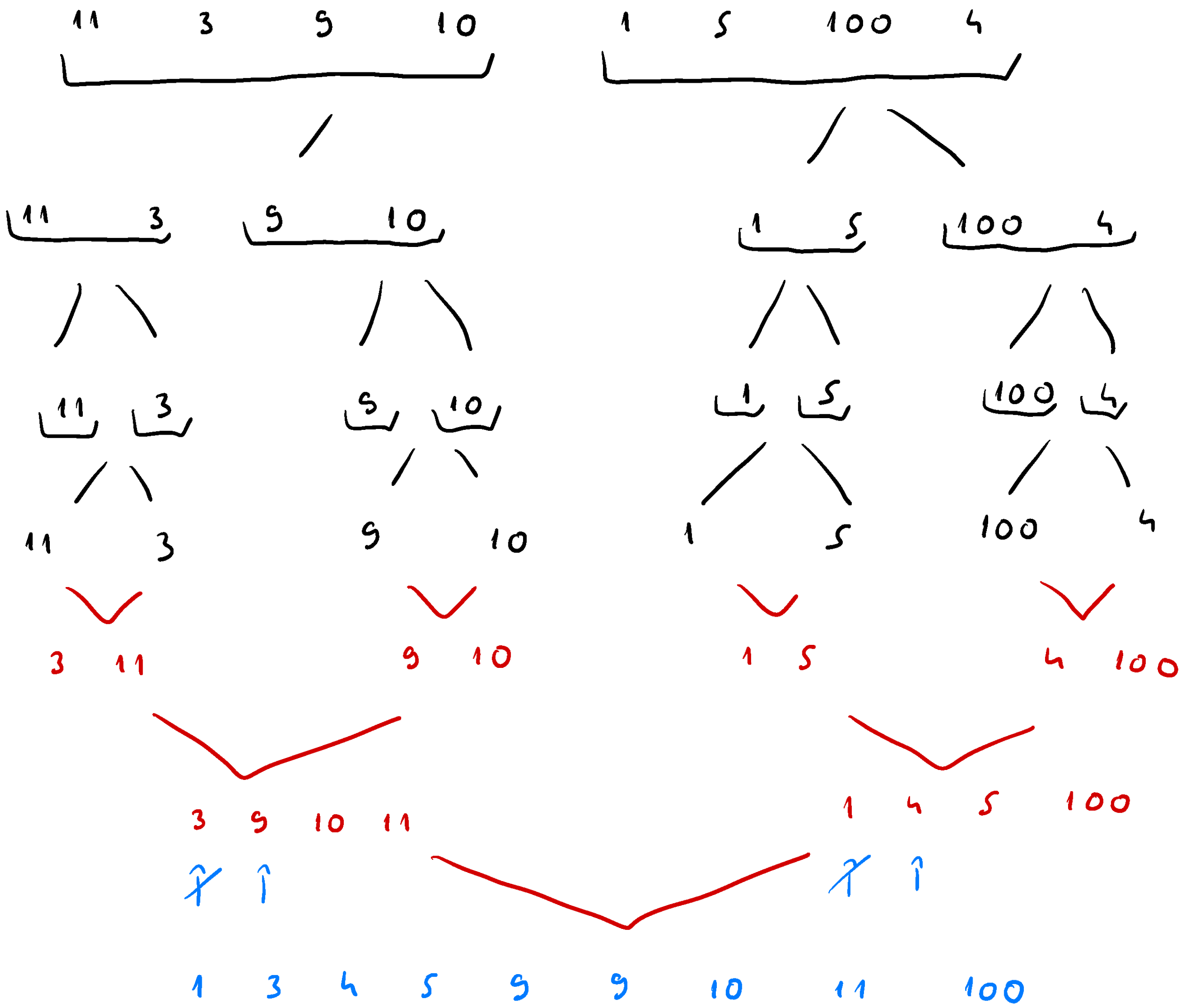
SD Curs 2

5 Mar 2025

1. Merge Sort
2. Recurențe
   - metoda substituției
   - arbore de recurențe
   - Teorema Master

1. Merge Sort

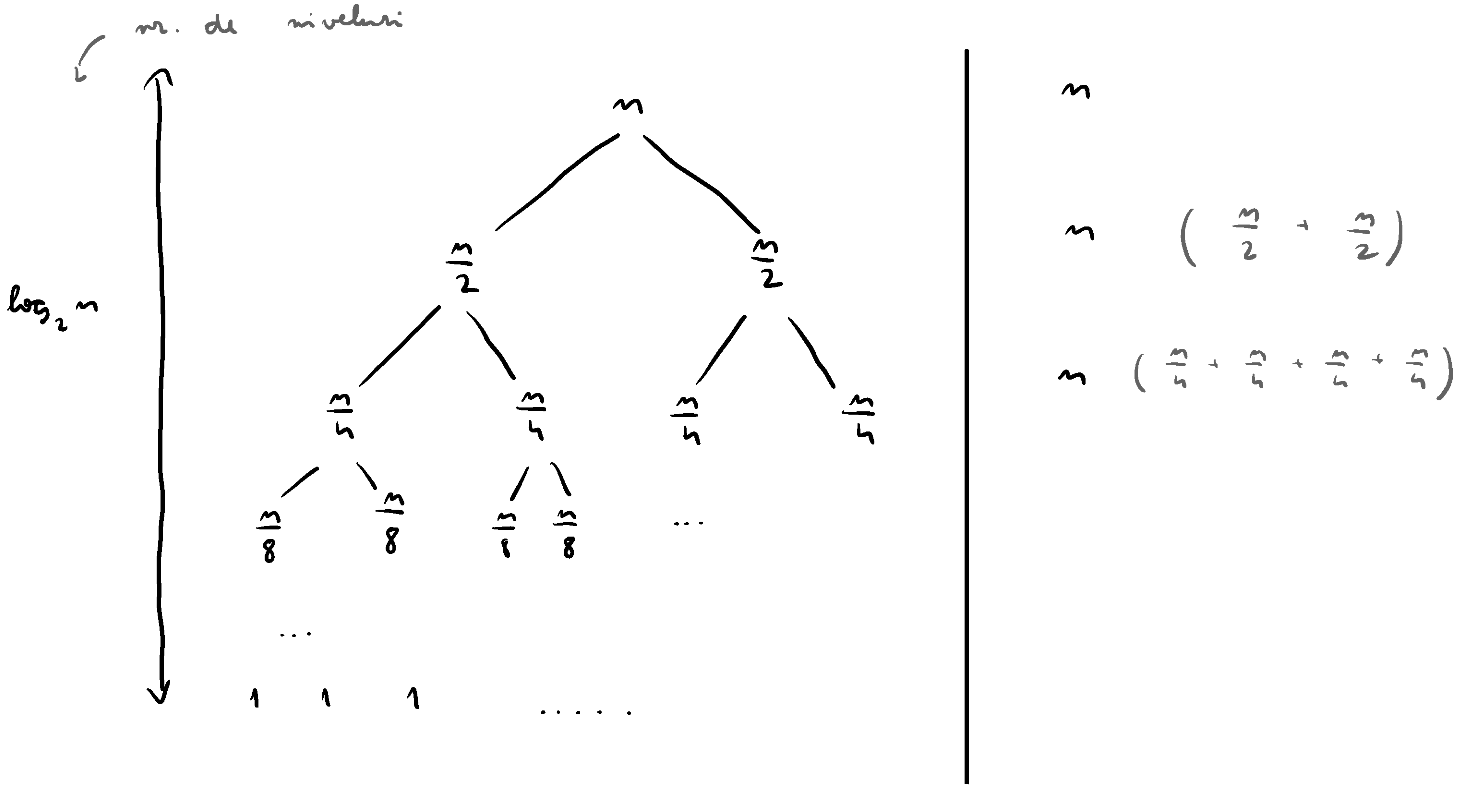


$O(n \cdot \log n)$

timpul necesar pt a interclasa
(unirea rezultatelor)

$T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$

1. Arbore de recurență

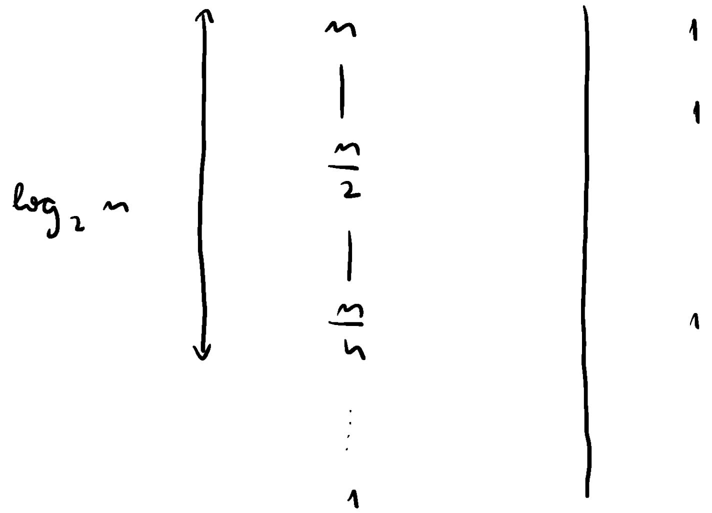


ex 2

comparația cu mijlocul

$$T(n) = T\left(\frac{n}{2}\right) + 1 = O(\log n)$$

(căutarea binară)



ex 3

$$T(n) = T\left(\frac{n}{2}\right) + n$$

$$\begin{array}{c|c}
\begin{array}{c} n \\ | \\ \frac{n}{2} \\ | \\ \frac{n}{4} \\ | \\ \dots \\ | \\ 1 \end{array} & \begin{array}{c} n \\ \left.\begin{array}{c} \frac{n}{2} \\ \frac{n}{4} \\ \dots \\ 1 \end{array}\right\} n \end{array}
\end{array}$$

$$n + \frac{n}{2} + \frac{n}{4} + \dots + \frac{n}{2^{\log_2 n}} = n\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{\log_2 n}}\right)$$

$$= n \cdot \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)$$

$$\in n \cdot \log n$$

$$1 + 2 + 4 + 8 + \dots + n = O(n)$$

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = O(\log n)$$

2. Metoda Substituției

ex 4

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$$

Arătăm prin inducție că $T(n) \le c \cdot n \log n$

Ipoteza de inducție

$$T\left(\frac{n}{2}\right) \le c \cdot \frac{n}{2} \log_2 \frac{n}{2}$$

Din formula de recurență

$$T(n) \le \cancel{2} \cdot c \frac{n}{\cancel{2}} \log_2 \frac{n}{2} + n = c \cdot n \log_2 \frac{n}{2} + n =$$

$$= c \cdot n \log_2 n - c\, n \log_2 2 + n =$$

$$= c \cdot n \log_2 n + \underbrace{n(1-c)}_{\le 0}$$

Dacă $c \ge 1$ $\le c\, n \log_2 n$

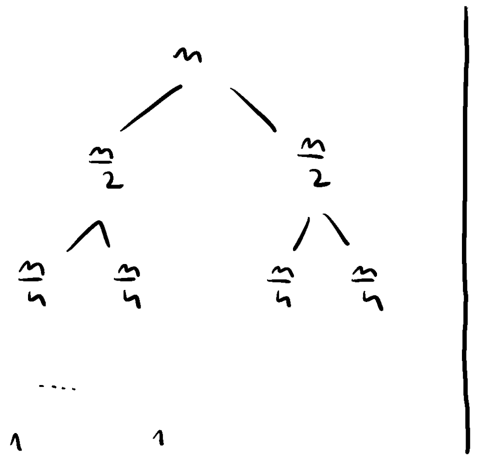
Pr. greșit că $T(n) \le c \cdot n$

$$T\left(\frac{n}{2}\right) \le c \cdot \frac{n}{2}$$

$$T(n) \le \cancel{2} \cdot c \frac{n}{\cancel{2}} + n \le c\, n + n \not\le c \cdot n$$

ex 5

$$T(n) = 2T\left(\frac{n}{2}\right) + 1 = O(n)$$



1

2

4

$$2^{\log_2 n} = n$$

$$1 + 2 + \dots + n = O(n)$$

Vrem să arătăm $T(n) \le c \cdot n$

Din ipoteza de inducție avem că

$$T\left(\frac{n}{2}\right) \le c \cdot \frac{n}{2}$$

$$T\left(\frac{n}{2}\right) \le \cancel{2} \cdot c \frac{n}{\cancel{2}} + 1 \le c \cdot n + 1 \nleq c \cdot n$$

Vrem să arătăm că $T(n) \le c \cdot n - b$

Presupunem $T\left(\frac{n}{2}\right) \le c \cdot \frac{n}{2} - b$

$$T(n) \le 2 \cdot \left(c \cdot \frac{n}{2} - b\right) + 1$$

$$c \cdot n - 2b + 1 = \underbrace{c \cdot n - b} + 1 - b \le cn - b$$

$$\forall\, b \ge 1$$

3. Teorema Master

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

$f(n)$ crește mai încet

1. $f(n) \in O(n^{\log_b a - \varepsilon})$ pt $\varepsilon > 0$

$$\Rightarrow T(n) \in \Theta(n^{\log_b a})$$

$f(n)$ și $n^{\log_b a}$ cresc la fel de repede.

2. $f(n) \in \Theta(n^{\log_b a})$

$$\Rightarrow T(n) = \Theta(n^{\log_b a} \cdot \log_2 n)$$

3. $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ pt $\varepsilon > 0$ și

$$\exists\, c < 1 \quad a\, f\left(\frac{n}{b}\right) < c \cdot f(n)$$

$$\Rightarrow T(n) = \Theta(f(n))$$

ex 6

$$T(n) = 9 \cdot T\left(\frac{n}{3}\right) + n$$

$a = 9$

$b = 3$ $\qquad n \;\square\; n^{\log_3 9} = n^2$

$f(n) = n$

$$f(n) \in O(n^{2-\varepsilon})$$

$$\Rightarrow T(n) = \Theta(n^2)$$

ex 7

$T(n) = T\left(\frac{2n}{3}\right) + 1 = \Theta(\log_2 n)$

$f(n) = 1$

$a = 1$ $\quad\quad$ $1 \quad\quad n^{\log_{\frac{3}{2}} 1} = n^0 = 1$

$b = \frac{3}{2}$ $\quad\quad$ $1 \in \Theta(1)$

ex 8

$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + n \cdot \log n$

$a = 3$

$b = 4$

$f(n) = n \cdot \log n$

$$n^{\log_b a} = n^{\log_4 3}$$

$$n \log n \in \Omega\left(n^{\log_4 3}\right)$$

Verificam că $a \cdot f\left(\frac{n}{b}\right) \le c \cdot f(n)$

pt $c < 1$

$3 \cdot \frac{n}{4} \cdot \log_2 \frac{n}{4} \le c \cdot n \log n$

Adevarat pt $c = \frac{3}{4}$ pt că

$3 \frac{n}{4} \log_2 \frac{n}{4} \le \frac{3}{4} n \log n$

$\Rightarrow T(n) \in \Theta(n \cdot \log n)$

ex 9

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

$a = b = 2$

$f(n) = n$

$n \ \square \ n^{\log_2 2} = n$

Caz 2 $\Rightarrow$ $n \in \Theta(n)$

$\Rightarrow T(n) \in \Theta(n \cdot \log n)$

ex 10

$$T(n) = T(n-1) + n \quad \in \Theta(n^2)$$

(sortarea prin inserție)

! NU merge Master

n	n
$n-1$	$n-1$
$n-2$	$n-1$
$\vdots$	1
1	

$$n + n-1 + n-2 + \dots + 1 = \frac{n(n+1)}{2}$$

$$= \frac{1}{2}(n^2 + n)$$

$$\in \Theta(n^2)$$

ex 11

$$T(n) = T\left(\frac{n}{100}\right) + T\left(\frac{99\,n}{100}\right) + n = \Theta(n \log n)$$