Seminar GEOM (12)

$u = (x, y, z)$

$\|u\| = \sqrt{x^2+y^2+z^2}$

Ex 2

Să se aducă la forma ortogonală

$Q(x) = -3x_2^2 + 4x_1x_2 + 10x_1x_3 - 4x_2x_3$

$A = \begin{pmatrix} 0 & 2 & 5 \\ 2 & -3 & -2 \\ 5 & -2 & 0 \end{pmatrix}$

$P_A(\lambda) = \begin{vmatrix} \lambda & -2 & -5 \\ -2 & \lambda+3 & 2 \\ -5 & 2 & \lambda \end{vmatrix} = \begin{vmatrix} 0 & \frac{\lambda^2+3\lambda-3}{2} & \lambda-5 \\ -2 & \lambda+3 & 2 \\ 0 & \frac{-5\lambda-4}{2} & \lambda-5 \end{vmatrix} \Rightarrow 2\begin{vmatrix} \lambda^2+3\lambda-4 & \lambda-5 \\ \frac{-5\lambda-11}{2} & \lambda-5 \end{vmatrix} = (\lambda-5)\begin{vmatrix} \lambda^2+3\lambda-4 & 1 \\ -5\lambda-11 & 1 \end{vmatrix}$

$\Rightarrow (\lambda-5)(\lambda^2+3\lambda-4+5\lambda+11) \Rightarrow \cancel{\lambda_{2,3} = -8}$

$= (\lambda-5)(\lambda^2+8\lambda+7) \Rightarrow \lambda_{2,3} = \frac{-8 \pm 6}{2} \quad \Rightarrow \sigma(A) = \{5, -1, -7\}$

$V_{-7}: \begin{cases} 2y+5z = -7x \\ 2x-3y-2z = -7y \\ 5x-2y = -7z \end{cases} \Rightarrow \begin{cases} 7x+2y+5z = 0 \\ 2x+4y-2z = 0 \\ 5x-2y+7z = 0 \end{cases} \Rightarrow \begin{cases} x+2y-z = 0 \\ 7x+2y+5z = 0 \end{cases} \Rightarrow \begin{cases} x+2y-z = 0 \\ -12y+12z = 0 \end{cases} \Rightarrow \begin{cases} y = z \\ x = -z \end{cases}$

V_x

$\Rightarrow \langle (1,-1,-1) \rangle = \langle \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \rangle$

$V_{-1}: \begin{cases} 2y+5z = -x \\ 2x-3y-2z = -y \\ 5x-2y = -z \end{cases} \Rightarrow \begin{cases} x+2y+5z = 0 \\ 2x-2y-2z = 0 \\ 5x-2y+z = 0 \end{cases} \Rightarrow \begin{cases} 0y+2z = 0 \\ x-y-z = 0 \end{cases} \Rightarrow \begin{cases} y = -2z \\ x = -z \end{cases}$

$V_{-1} = \langle (1,2,-1) \rangle$

$V_5: \begin{cases} 2y+5z = 5x \\ 2x-3y-2z = 5y \\ 5x-2y = 5z \end{cases} \Rightarrow \begin{cases} -5x+2y+5z = 0 \\ 2x-8y-2z = 0 \\ 5x-2y-5z \end{cases} \Rightarrow \begin{cases} x-4y-z = 0 \\ 5x-2y-5z = 0 \end{cases} \Rightarrow \begin{cases} -9x+9z = 0 \\ x = z \\ y = 0 \end{cases}$

$V_5 = \langle (1,0,1) \rangle$

$\Rightarrow V_5 = \langle \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \rangle$

$Q_2 = -7y_1^2 - y_2^2 + 5y_3^2$

$(x_1, x_2, x_3) = y_1 f_1 + y_2 f_2 + y_3 f_3$

$y_1 = \frac{x_1 - x_2 - x_3}{\sqrt{3}} \qquad y_3 = \frac{x_1 - x_3}{\sqrt{2}}$

$y_2 = \frac{x_1 + 2x_2 - x_3}{\sqrt{6}}$

$\det(2) = \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & -1 & 1 \end{vmatrix}$

$\Rightarrow \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -2 & -2 & 0 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} -1 & 2 \\ -2 & -2 \end{vmatrix} = 1$

Ex.

A1 Fie $L \subset \mathbb{R}^3$, $L = <\{(1,1,1), (1,-1,1)\}>$ Det. pr. lui $(1,2,3)$ pe L
$\qquad\qquad e_1 \qquad\qquad e_2$

Pas 1

Det. o bază ortonormată pe L

$f_1 = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$

$f_2 = \left(\frac{1}{\sqrt3}, -\frac{2}{\sqrt6}, \frac{1}{\sqrt2}\right)$

$\overline{f_2} = -<f_1, e_1> f_2 + e_2 = -\frac{1}{\sqrt3}\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) + (1,-1,1) = \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right) + (1,-1,1) = \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right)$

$\|\overline{f_2}\| = \sqrt{\frac{8+16}{9}} = \sqrt{\frac{24}{9}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt2}{\sqrt3}$

$f_2 = \frac{\sqrt3}{2\sqrt3}\left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right)$

Pas 2

$x = a_1 f_1 + a_2 f_2 + x_2$

$a_1 = <x, f_1> = 2\sqrt3$

$a_2 = <x, f_2> = 0$

$x_1 = (2,2,2)$

$x_2 = (-1,0,1)$

$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$

$14 = 12 + 2$ (A)

$L = <f_1, \dots, f_k>$, $B = \{f_1, \dots, f_k\}$ ORTONORMATĂ

$x \in L$

$x = a_1 f_1 + \dots + a_k f_k + x_0$, cu $x_0 \in L^*$

$a_j = <x, f_j>$ s.n. coef. FOURIER ai lui x în rap. cu B

$\|x_1\| = a_1^2 + \dots + a_k^2$

$a_1^2 + \dots + a_k^2 \le \|x\|$ – INEG lui BESEL

$Q(x)$ formă pătratică pe $\mathbb{R}^n$

$Q(x) = <Ax, x>$ cu $A \in \mathcal{M}_n(\mathbb{R})$ A simetrică $A =: [Q]$

B' cu mat de trecere la Q $[Q]_{B'} = {}^t A Q$

Fie $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = A \cdot x$

$[f] = A$, $[f]_{B'} = Q^{-1} A Q$

$[g]_{B'} = [f]_{B'} \iff Q \in O(n) \iff B$ bz. ortonormată

$A \in M_n(\mathbb{R})$ $\sigma(\lambda) = \{\lambda_1, \ldots, \lambda_n\}$ diagonalizab.

$\lambda_i \neq \lambda_j$, $V_{\lambda_1} \perp V_{\lambda_2}$, $\langle Ax, y \rangle = \langle x, Ay \rangle$ simetric

$(\exists)$ B' ORTONORMATĂ a.î. $[f]_{B'} = [g]_{B'} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} = \text{diag}\, \lambda$

Clasificarea AP. ORTOGONALE

$\boxed{\langle fx, fy \rangle = \langle x, y \rangle}$ -ortogonale

$f: \mathbb{R}^3 \to \mathbb{R}^3$, grad (Pol caract.) $= 3$

$\sigma(f) \ni \lambda$

$\lambda \in \sigma(f)$, fie $x \in V_\lambda$, $\|x\| = 1$, dar $f \in O(3)$ $y \in H \Rightarrow f(y) \in H$

$\left.\begin{array}{l} \|fx\|^2 = \langle fx, fx \rangle = \lambda^2 \|x\| = \lambda^2 \\ = \|x\| = 1 \end{array}\right\} \Rightarrow \lambda^2 = 1$

$H \leq$ Fie $H = x^\perp$

$\langle x, y \rangle = 0$, $(\forall)$ $y \in H$

$\langle fy, \lambda x \rangle = \langle fy, fx \rangle = 0 \Rightarrow \langle fx, y \rangle = 0$
$= \langle fx, fy \rangle = \langle y, x \rangle = 0$

Cazuri:

$\boxed{I_1}$ $\lambda = 1$, $\det f = 1$

$B = \{x, y_1, y_2\}$ bz. orto. a.î. $\{y_1, y_2\} \in H$

$[f]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & [f_H]_{\{y_1, y_2\}} & \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$ $\boxed{\text{rotație de axă } X}$

$\boxed{I_2}$ $\lambda = 1$, $\det f = -1$, $\Rightarrow \det(f_H) = -1 \Rightarrow$

f_H simetrie $\Rightarrow (\exists)$ $\{y_1', y_2'\}$ vect. propr.

a.î. $f y_1' = y_1'$, $f y_2' = -y_2'$, $B' = \{x, y_1', y_2'\}$

$[f]_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\boxed{\text{sim. ortog. față de un plan}}$

$\boxed{II_1}$

$\lambda = -1$, $\det f = 1$

$\Rightarrow \det(f_H) = -1 \Rightarrow B' = \{x, y_1', y_2'\}$ a.î.

$[f]_{B'} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\boxed{\text{sim. față de o dr.}}$

$\boxed{II_2}$ $\lambda = -1$, $\det f = -1 \Rightarrow$

$\det(f_H) = 1$

$[f]_B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$ $\boxed{\text{o rotație și o sim. față de un plan}}$

3. Să se det. rotația lui $\frac{\pi}{6}$ în jurul axei $<(1,0,-1)>$

$<(1,0,-1)> = <\left(\frac{1}{\sqrt2},0,-\frac{1}{\sqrt2}\right)>$ $\quad \|<1,0,1>\| = \sqrt{1^2+0^2+1^2}\ \sqrt2 \Rightarrow$

$H: x-z=0$

$H = <(0,1,0),(1,0,1)> \Rightarrow <(0,1,0),\left(\frac{1}{\sqrt2},0,\frac{1}{\sqrt2}\right)>$

$B = \begin{vmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{vmatrix}$ $\quad B = \left\{\left(\frac{1}{\sqrt2},0,-\frac{1}{\sqrt2}\right),(0,1,0),\left(\frac{1}{\sqrt2},0,\frac{1}{\sqrt3}\right)\right\}$ $\quad [R]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & -\frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix}$

$\begin{matrix} 1 & 0 & 0 \\ 0 & \cos & \sin \\ 0 & -\sin & \cos \end{matrix}$

dar $[R]_B = A^*[R]A \Rightarrow [R] = A[R]_B A^*$

$$[R] = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} = \begin{pmatrix} \frac12+\frac{1}{2\sqrt2} & -\frac12 & -\frac12+\frac{1}{2\sqrt2} \\ \frac12 & \frac{1}{\sqrt2} & \frac12 \\ -\frac12+\frac{1}{2\sqrt2} & -\frac12 & \frac12+\frac{1}{2\sqrt2} \end{pmatrix}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad c_1 \qquad c_2 \qquad c_3$

$\|c_1\| = \frac14+\frac18+\cancel{\frac{1}{2\sqrt2}}+\frac14+\frac14+\frac18 - \frac{1}{2\sqrt2} = \frac34+\frac14 = 1$

$\|c_2\| = 1$

$\|c_3\| \cancel{= \frac14+\frac18}$

$\langle c_1, c_2\rangle = -\cancel{\frac14}-\frac{1}{4\sqrt2}+\frac{1}{2\sqrt2}+\cancel{\frac14}-\frac{1}{4\sqrt2} = -\frac{1}{2\sqrt3}+\frac{1}{2\sqrt3} = 0$

$\langle c_1, c_3\rangle = \frac18-\cancel{\frac14}+\cancel{\frac14}+\frac18-\frac14 = 0$

4. Det. sim. ortog. față de $H: x-z=0$

$H^\perp = <\left(\frac{1}{\sqrt2},0,-\frac{1}{\sqrt2}\right)>$

$H = <\left\{(0,1,0),\left(\frac{1}{\sqrt2},0,\frac{1}{\sqrt2}\right)\right\}>$

$B = \left\{(0,1,0),\left(\frac{1}{\sqrt2},0,\frac{1}{\sqrt2}\right),\left(\frac{1}{\sqrt2},0,-\frac{1}{\sqrt2}\right)\right\}$

$[S]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ - axele rămân pe loc $\qquad [S] = \begin{pmatrix} 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \end{pmatrix}$

$S(x,y,z) = (z,y,x)$ $\qquad = \begin{pmatrix} 0 & \frac12 & -\frac12 \\ 1 & 0 & 0 \\ 0 & \frac12 & \frac{1}{\sqrt2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$