Seminar 8

ex 1

Stud. convergența simplă și uniformă pentru urm. șiruri de funcții:

a) $f_n : [\frac{1}{2}, 1] \to \mathbb{R}$, $f_n(x) = \frac{(1+x)^n}{e^{2nx}}$ $\forall n \in \mathbb{N}^*$

(C. u. $\Rightarrow$ Th. Dini)

b) $f_n : [\frac{1}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $f_n(x) = (\cos x)^n$ $\forall n \in \mathbb{N}^*$

(C. u. $\Rightarrow$ Th. Polya)

ex 2

Stud conv simplă și uniformă pentru $(f_n)_n$ și $(f_n')_n$ unde

$f_n : [0, \pi] \to \mathbb{R}$, $f_n(x) = \frac{\cos nx}{n}$ $\forall n \in \mathbb{N}^*$

ex 3

Arătați că seria de funcții $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^4}$ converge uniform

(Th. Weierstrass)

ex 4

Det. mulțimea de convergență pt. urm. serii de puteri:

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot x^n$

b) $\sum_{n=1}^{\infty} \frac{n! \cdot x^n}{(a+1)\dots(a+n)}$, $a>1$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot (x+3)^n$

d) $\sum_{n=1}^{\infty} \frac{2^n}{2n+1} \cdot (x-2)^n$

ex 1

Stud. convergența simplă și uniformă pentru urm. șiruri de funcții:

a) $f_n : [\frac{1}{2}, 1] \to \mathbb{R}$, $f_n(x) = \dfrac{(1+x)^n}{e^{2nx}}$ $\forall n \in \mathbb{N}^*$

Sol:

C. s.

Fie $x \in [\frac{1}{2}, 1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{(1+x)^n}{e^{2nx}} = \lim_{n\to\infty} \left(\frac{1+x}{e^{2x}}\right)^n$$

$$e^{2x} \geq e^{2\cdot\frac{1}{2}} = e > 2 > x+1 \quad \forall x \in [\tfrac{1}{2}, 1]$$

$$\Rightarrow 0 < \frac{x+1}{e^{2x}} < 1$$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$$

$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, unde $f : [\frac{1}{2}, 1] \to \mathbb{R}$, $f(x) = 0$

C. u.

1) $[\frac{1}{2}, 1]$ compactă

2) f_n, f cont, $\forall n \in \mathbb{N}^*$

3) $0 < \dfrac{1+x}{e^{2x}} < 1 \Rightarrow \underbrace{\left(\dfrac{1+x}{e^{2x}}\right)^n}_{f_n} > \underbrace{\left(\dfrac{1+x}{e^{2x}}\right)^{n+1}}_{f_{n+1}}$

$\Rightarrow f_n$ s. desc.

4) $f_n \xrightarrow[n\to\infty]{s} f$

Conform Teoremei lui Dini avem că $f_n \xrightarrow[n\to\infty]{u} f$

b) $f_n : [\frac{1}{2}, \frac{\pi}{2}] \to \mathbb{R}, \quad f_n(x) = (\cos x)^n \quad \forall n \in \mathbb{N}^*$

Sol:

c. s.

Fie $x \in [\frac{1}{2}, \frac{\pi}{2}]$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (\cos x)^n$

$0 \le \cos x \le 1, \quad \forall x \in [0, \frac{\pi}{2}]$

$[\frac{1}{2}, \frac{\pi}{2}] \subset [0, \frac{\pi}{2}] \quad \Rightarrow \lim_{n\to\infty} (\cos x)^n = 0$

$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \quad f : [\frac{1}{2}, \frac{\pi}{2}] \to \mathbb{R}, \quad f(x) = 0$

c. u.

1) $f_n, f : [\frac{1}{2}, \frac{\pi}{2}] \to \mathbb{R}, \quad \forall n \in \mathbb{N}^*$

2) f cont

3) $\underset{[\frac{1}{2}, \frac{\pi}{2}]}{\overset{x}{\cap}} \longrightarrow \underset{[0,1]}{\overset{\cos x}{\cap}}$ (strict) descrescatoare

$\Rightarrow f_n$ (strict) desc.

4) $f_n \xrightarrow[n\to\infty]{s} f$

Conform Teoremei lui Polya avem că

$f_n \xrightarrow[n\to\infty]{u} f$

ex 2

Stud conv simplă și uniformă pentru

$(f_n)_n$ și $(f_n')_n$ unde

a) $f_n : [0, \pi] \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n}$ $\forall n \in \mathbb{N}^*$

Sol:

pt $(f_n)_n$

C. s.

Fie $x \in [0, \pi]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\cos nx}{n}$$

$$-1 \le \cos nx \le 1, \quad \forall x \in [0, \pi] \quad \forall n \in \mathbb{N}^*$$

$$-\frac{1}{n} \le \frac{\cos nx}{n} \le \frac{1}{n}$$

$$\xrightarrow{n\to\infty} 0$$

Conform Crit. Cleștelui avem că $\lim\limits_{n\to\infty} f_n(x) = 0$,

Deci $f_n \xrightarrow[n\to\infty]{s} f$, $f : [0, \pi] \to \mathbb{R}$, $f(x) = 0$

C. u.

$$\sup_{x\in[0,\pi]} \left(| f_n(x) - f(x) | \right) = \sup_{x\in[0,\pi]} \frac{|\cos nx|}{n} \le \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{u} f$$

pt $(f_n')_n$

$$f_n'(x) = \left(\frac{\cos nx}{n}\right)' = \frac{1}{n} \cdot n \cdot (-\sin nx) = -\sin nx$$

$\forall x \in [0, \pi], \ \forall n \in \mathbb{N}^*$

c.s.

Alegem $x = \frac{\pi}{2}$

$$f_{4k}\left(\frac{\pi}{2}\right) = -\sin\left(4k \cdot \frac{\pi}{2}\right) = -\sin 2k\pi = 0 \xrightarrow[k \to \infty]{} 0$$

$$f_{4k+1}\left(\frac{\pi}{2}\right) = -\sin\left(4k+1 \cdot \frac{\pi}{2}\right) = -\sin\left(2k\pi + \frac{\pi}{2}\right) = -1 \xrightarrow[k \to \infty]{} -1$$

$$0 \neq -1 \Rightarrow \nexists \lim_{n \to \infty} f_n'\left(\frac{\pi}{2}\right)$$

Deci $(f_n')_n$ nu converge simplu

c.u.

$(f_n')_n$ nu conv s. deci $(f_n')_n$ nu conv unif.

b) $f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \frac{\operatorname{arctg} nx}{n} \quad \forall n \in \mathbb{N}^*$

pt $(f_n)_n$

c.u.

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\operatorname{arctg} nx}{n} = 0$$

$$\Rightarrow f_n \xrightarrow[n \to \infty]{s} f, \quad f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 0$$

c.s.

$$\sup_{x \in \mathbb{R}} \left(|f_n(x) - f(x)|\right) = \sup_{x \in \mathbb{R}} \frac{|\operatorname{arctg} nx|}{n} - 0$$

$$\lim_{x \to \infty} \left|\frac{\operatorname{arctg} nx}{n}\right| = \frac{\pi}{n} \xrightarrow[n \to \infty]{} 0 \Rightarrow f_n \xrightarrow[n \to \infty]{u} s$$

ex 3

Arătați că seria de funcții $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^4}$ converge uniform

Sol:

Fie $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \operatorname{arctg} \frac{2x}{x^2+n^4}$, $\forall n \in \mathbb{N}^*$

$$\frac{x^2+n^4}{2} \geq \sqrt{x^2 \cdot n^4}$$

$$\iff \frac{x^2+n^4}{2} \geq |x| \cdot n^2 \quad \Big| : x^2+n^2$$

$$\iff \frac{1}{2} \geq \frac{|x|}{x^2+n^4} \cdot n^2 \quad \Big| : n^2$$

$$\frac{1}{2n^2} \geq \frac{|x|}{x^2+n^2}$$

$$\frac{1}{n^2} \geq \frac{2|x|}{x^2+n^2} \iff \left| \frac{2x}{x^2+n^4} \right| \leq \frac{1}{n^2}$$

$$\left. \begin{array}{l} -\frac{1}{n^2} \leq \frac{2x}{x^2+n^4} \leq \frac{1}{n^2} \\ \text{arctg strict cresc} \end{array} \right\} \Rightarrow$$

$$-\operatorname{arctg} \frac{1}{n^2} \leq \operatorname{arctg} \frac{2x}{x^2+n^4} \leq \operatorname{arctg} \frac{1}{n^2} \iff$$

$$\iff \left| \operatorname{arctg} \frac{2x}{1+x^2} \right| \leq \operatorname{arctg} \frac{1}{n^2}$$

$$|f_n(x)| \leq a_n \qquad (1)$$

Fie $a_n = \operatorname{arctg} \frac{1}{n^2}$, $\forall n \in \mathbb{N}^*$

Aratam că $\sum_{n=1}^{\infty} a_n$ conv

Avem $\lim_{x \to 0} \frac{\operatorname{arctg} x}{x} = 1$

Fie $b_n = \frac{1}{n^2}$, $\forall n \in \mathbb{N}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\operatorname{arctg} \frac{1}{n^2}}{\frac{1}{n^2}} = 1 \in (0, \infty)$$

Conform Crit de comp. cu limită avem că $\sum a_n \sim \sum b_n$

$\sum b_n = \sum \frac{1}{n^2}$ conv (serie arm. gen. $\alpha = 2$)

$\Rightarrow$ $\sum a_n$ conv

Conform Teoremei lui Weierstrass avem că $\sum f_n$ conv. uniform

ex 4

Det. mulțimea de convergență pt.

urm. serii de puteri:

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot x^n$

Sol: Fie $a_n = \frac{(-1)^n}{n \cdot 2^n}$, $\forall n \in \mathbb{N}^*$

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim \sqrt[n]{\left|\frac{(-1)^n}{n \cdot 2^n}\right|} = \lim_{n \to \infty} \sqrt[n]{\frac{1}{n \cdot 2^n}}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt[n]{n} \cdot \sqrt[n]{2^n}} = \frac{1}{2}$$

1

$$R = \frac{1}{\frac{1}{2}} = 2$$

Fie M mulțimea de conv a seriei din enunț.

$(-R, R) \subset M \subset [R, R]$

$(-2, 2) \subset M \subset [2, 2]$

Studiem dacă $-2 \in M$ și $2 \in M$

Dacă $x = 2$, seria devine

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot \cancel{2^n}} \cdot \cancel{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv (Crit. lui Leibniz)

Deci $2 \in M$

Dacă $x = -2$, seria devine

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot (-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\cancel{2^n}} \cdot \frac{1}{n} \cdot (-1)^n \cdot \cancel{2^n}$$

$$= \sum_{n=1}^{\infty} (-1)^{2n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{div (seria arm. gen, } \alpha = 1)$$

Deci $-2 \notin M$

Așadar, $M = (-2, 2]$ $\square$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n! \cdot x^n}{(a+1)\dots(a+n)}$, $a > 1$

Sol:

Fie $a_n = \dfrac{n!}{(a+1)\cdot\dots\cdot(a+n)}$

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{\dfrac{(n+1)\cancel{!}}{\cancel{(a+1)}\cdot\dots\cdot\cancel{(a+n)}\cdot(a+n+1)}}{\dfrac{\cancel{n!}}{\cancel{(a+1)}\cdot\dots\cdot\cancel{(a+n)}}}$$

$$= \lim_{n \to \infty} \frac{n+1}{a+n+1} = 1$$

$R = \dfrac{1}{1} = 1$

Fie M mulțimea de conv. a seriei de puteri din enunț

$(-1, 1) \subset M \subset [-1, 1]$

Studiem dacă $-1 \in M$ și $1 \in M$

Dacă $x = 1$, seria devine

$$\sum_{n=1}^{\infty} \frac{n! \cdot 1^n}{(a+1)\dots(a+n)} = \sum_{n=1}^{\infty} \frac{n!}{(a+1)\dots(a+n)}$$

Fie $x_n = \dfrac{n!}{(a+1)\dots(a+n)}$

$$\lim_{n\to\infty} n \cdot \left(\frac{x_n}{x_{n+1}} - 1 \right)$$

$$= \lim_{n\to\infty} n \cdot \left(\frac{a+n+1}{n+1} - 1 \right) = \lim_{n\to\infty} n \left(\frac{a + \cancel{n} + \cancel{1} - \cancel{n} - \cancel{1}}{n+1} \right)$$

$$= \lim_{n\to\infty} \frac{n \cdot a}{n+1} = a > 1 \in (0, \infty)$$

Conform Crit Raabe - Duhamel, avem că $\sum x_n$ este conv

Deci $1 \in M$

Dacă $x = -1$, seria devine

$$\sum_{n=1}^{\infty} \frac{n!}{(a+1)\dots(a+n)} \cdot (-1)^n$$

Fie $y_n = \dfrac{n!}{(a+1)\dots(a+n)} \cdot (-1)^n \quad \forall n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} |y_n| = \sum_{n=1}^{\infty} \frac{n!}{(a+1)\dots(a+n)} \quad \text{conv}$$

Avem că $\sum y_n$ absolut conv şi deci $\sum y_n$ conv

Deci $1 \in M$

Aşadar $M = [-1, 1]$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot (x+3)^n$

Sol:

Fie $y = x+3$

Seria de puteri devine

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot y^n$$

Fie $a_n = \frac{(-1)^n}{\sqrt[3]{n}}$, $\forall n \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{\sqrt[3]{n+1}}\right| \cdot \left|\frac{\sqrt[3]{n}}{(-1)^n}\right|$$

$$= \lim_{n\to\infty} = \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} = 1$$

$R = \frac{1}{1} = 1$

Fie N multimea de conv. a seriei $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot y^n$

$(-1, 1) \subset N \subset [-1, 1]$

Studiem daca $-1 \in M$ și $1 \in M$

Daca $y = -1$, seria devine

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}} \text{ div}$$

Deci $-1 \notin N$ (seria arm. gen., $\alpha = \frac{1}{3}$)

Daca $y = 1$, seria devine

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot 1^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

Fie $x_n = \frac{1}{\sqrt[3]{n}}$, $\forall n \in \mathbb{N}^*$

$(x_n)_n$ strict desc.

$\lim_{n \to \infty} x_n = 0$

Conform Crit. lui Leibniz, avem că

$\sum_{n=1}^{\infty} (-1)^n \cdot x_n$ conv

Deci $1 \in N$

Asadar, $N = (-1, 1]$

Fie M mulțimea de conv a seriei de puteri

din enunț

$y \in N \Rightarrow -1 < y \leq 1$

$-1 < x+3 \leq 1 \quad | -3$

$-4 < x \leq -2$

Asadar $M = (-4, -2]$

d) $\sum_{n=1}^{\infty} \frac{2^n}{2n+1} \cdot (x-2)^n$

Sol:

Fie $y = x - 2$

Seria de puteri devine $\sum_{n=1}^{\infty} \frac{2^n}{2n+1} \cdot y^n$

Fie $a_n = \frac{2^n}{2n+1}$ $\forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{\cancel{n}+1}}{2(n+1)+1} \cdot \frac{2n+1}{\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{4n+2}{2n+3} = 2$$

$R = \frac{1}{2}$

Fie N mulțimea de convergență a seriei $\sum_{n=1}^{\infty} \frac{2^n}{2n+1} \cdot y^n$

$(-\frac{1}{2}, \frac{1}{2}) \subset N \subset [-\frac{1}{2}, \frac{1}{2}]$

Studiem dacă $-\frac{1}{2} \in N$ și $\frac{1}{2} \in N$

Dacă $y = \frac{1}{2}$, seria de puteri devine

$$\sum_{n=1}^{\infty} \frac{2^n}{2n+1} \cdot \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{\cancel{2^n}}{2n+1} \cdot \frac{1}{\cancel{2^n}} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

Fie $x_n = \frac{1}{2n+1}$

$y_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \in (0, \infty)$$

Conform Crit. de comp. cu limită avem că

$$\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$$

$\sum y_n = \sum \frac{1}{n}$ div (serie armonică gen, $\alpha = 1$)

Deci $\sum x_n$ div

Deci $y = \frac{1}{2} \notin N$

Dacă $y = -\frac{1}{2}$ seria devine $\sum_{n=1}^{\infty} \frac{2^n}{2n+1} \cdot \left(-\frac{1}{2}\right)^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

Fie $x_n = \frac{1}{2n+1}$, $\forall n \in \mathbb{N}$

x_n strict desc.

$\lim_{n \to \infty} x_n = 0$

Conform Crit lui Leibniz avem că $\sum (-1)^n \cdot x_n$ conv

Deci $y = -\frac{1}{2} \in N$

Prin urmare $N = [-\frac{1}{2}, \frac{1}{2})$

Fie M mulțimea de convergență a seriei

de puteri din enunț

$y \in N \Rightarrow -\frac{1}{2} \leq y < \frac{1}{2}$

$-\frac{1}{2} \leq x - 2 < \frac{1}{2} \quad | + 2$

$-\frac{1}{2} + 2 \leq x < \frac{1}{2} + 2$

$\frac{3}{2} \leq x < \frac{5}{2}$

Deci $M = [\frac{3}{2}, \frac{5}{2}]$