Seminar 7

ex 1

Stud. uniform continuitatea funcțiilor

a) $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$

b) $f : [1, 2) \to \mathbb{R}$, $f(x) = \frac{1}{x}$

c) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$

ex 2

Fie $a \geq 0$ și $f : (a, +\infty) \to \mathbb{R}$, $f(x) = \ln x$

Arătați că f este u.c $\Leftrightarrow a > 0$

ex 4

Stud. convergența simplă și uniformă pentru urm. șiruri de funcții:

a) $f_n : [0, +\infty) \to \mathbb{R}$, $f_n(x) = \frac{x}{x+n}$, $\forall n \in \mathbb{N}^*$

b) $f_n : [2, 3] \to \mathbb{R}$, $f_n(x) = \frac{x}{x+n}$, $\forall n \in \mathbb{N}^*$

c) $f_n : [0, +\infty) \to \mathbb{R}$, $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $\forall n \in \mathbb{N}^*$

d) $f_n : [0, +\infty) \to \mathbb{R}$, $f_n(x) = \frac{n}{n+x}$

e) $f_n : (0, 1] \to \mathbb{R}$, $f_n(x) = x^n$

ex 1

Stud. uniform continuitatea funcțiilor

a) $f : [0, \infty) \to \mathbb{R}, \quad f(x) = \sqrt{x}$

b) $f : [1, 2) \to \mathbb{R}, \quad f(x) = \frac{1}{x}$

c) $f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{1}{x}$

Sol:

a) $f : [0, \infty) \to \mathbb{R}, \quad f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \forall x \in (0, \infty)$$

$$|f'(x)| = \left| \frac{1}{2\sqrt{x}} \right| = \frac{1}{2\sqrt{x}} \leq \frac{1}{2} \quad \forall x \in [1, \infty)$$

$f|_{[1,\infty)}$ u.c

$$\left.\begin{array}{l} f|_{[0,1]} \text{ cont} \\ [0,1] \text{ compactă} \end{array}\right\} \Rightarrow f|_{[0,1]} \text{ u.c.}$$

$$\left.\begin{array}{l} f|_{[0,1]} \text{ u.c.} \\ f|_{[1,\infty)} \text{ u.c} \end{array}\right\} \Rightarrow f \text{ u.c.}$$

$\square$

b) $f : [1, 2) \to \mathbb{R}, \quad f(x) = \frac{1}{x}$

$$f'(x) = -\frac{1}{x^2}, \quad \forall x \in [1, 2)$$

$$|f'(x)| = \left| -\frac{1}{x^2} \right| = \frac{1}{x^2} \leq 1 \quad \forall x \in [1, 2)$$

$\Rightarrow f$ u.c

$\square$

c) $f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{1}{x}$

Alegem $(x_n)_n \subset (0, \infty), \quad x_n = \frac{1}{n}, \quad \forall\, n \in \mathbb{N}^*$

$(y_n)_n \subset (0, \infty), \quad y_n = \frac{1}{2n}, \quad \forall\, n \in \mathbb{N}^*$

$$\lim_{n\to\infty} (x_n - y_n) = \lim_{n\to\infty} \frac{1}{n} - \frac{1}{2n} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

$$\lim_{n\to\infty} (f(x_n) - f(y_n)) = \lim_{n\to\infty} \frac{1}{x_n} - \frac{1}{y_n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{\frac{1}{n}} - \frac{1}{\frac{1}{2n}} \right) = \lim_{n\to\infty} n - 2n = -\infty \neq 0$$

Deci f nu e u.c. $\square$

ex 2

Fie $a \geq 0$ și $f : (a, +\infty) \to \mathbb{R}, \quad f(x) = \ln x$

Arătați că f este u.c $\Leftrightarrow a > 0$

Sol:

„ $\Leftarrow$ "

Pp. $a > 0$.

Vrem să arătam că f este u.c.

$f'(x) = \frac{1}{x}, \quad \forall\, x \in (a, +\infty)$

$|f'(x)| = \left|\frac{1}{x}\right| = \frac{1}{x}$

$x \in (a, \infty) \Rightarrow x > a \Rightarrow \frac{1}{x} < \frac{1}{a}$

$\Rightarrow |f'(x)| < \frac{1}{a}, \quad \forall\, x \in (a, \infty)$

$\Rightarrow f$ este u.c. pe $(a, +\infty)$

„ $\Rightarrow$ "

Pp. că f este u.c. pe (a, b)

Arătăm că $a > 0$.

Pp. prin absurd că $a \leq 0$. Cum $a \geq 0$ (cerință)

rezultă că $a = 0$.

Alegem $(x_n)_n \subset (0, \infty)$, $x_n = \frac{1}{n}$
$(y_n)_n \subset (0, \infty)$ $y_n = \frac{1}{2n}$ $\forall n \in \mathbb{N}^*$

$$\lim_{n \to \infty} (x_n - y_n) = \lim_{n \to \infty} \frac{1}{n} - \frac{1}{2n} = 0$$

$$\lim_{n \to \infty} (f(x_n) - f(y_n)) = \lim_{n \to \infty} \ln\left(\frac{1}{n}\right) - \ln\left(\frac{1}{2n}\right)$$

$$= \lim_{n \to \infty} \ln\left(\frac{\frac{1}{\cancel{n}}}{\frac{1}{2\cancel{n}}}\right) = \ln 2 \neq 0$$

$\Rightarrow$ f nu este u.c. (contradicție)

Rămâne că $a > 0$ $\square$

ex 4

Stud. convergența simplă și uniformă pentru urm. șiruri de funcții:

a) $f_n : [0, +\infty) \to \mathbb{R}, \quad f_n(x) = \frac{x}{x+n}, \quad \forall n \in \mathbb{N}^*$

b) $f_n : [2, 3] \to \mathbb{R}, \quad f_n(x) = \frac{x}{x+n}, \quad \forall n \in \mathbb{N}^*$

c) $f_n : [0, +\infty) \to \mathbb{R}, \quad f_n(x) = \sqrt{x^2 + \frac{1}{n}}, \quad \forall n \in \mathbb{N}^*$

d) $f_n : [0, +\infty) \to \mathbb{R}, \quad f_n(x) = \frac{n}{n+x}$

e) $f_n : (0, 1] \to \mathbb{R}, \quad f_n(x) = x^n$

Sol:

a) $f_n : [0, +\infty) \to \mathbb{R}, \quad f_n(x) = \frac{x}{x+n}, \quad \forall n \in \mathbb{N}^*$

Convergența simplă

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f \quad \text{unde} \quad f : [0, \infty) \to \mathbb{R}, \quad f(x) = 0$$

Convergența uniformă

$$\sup_{x\in[0,\infty)} \left(|f_n(x) - f(x)| \right)$$

$$= \sup_{x\in[0,\infty)} \left(\left| \frac{x}{x+n} - 0 \right| \right) = \sup_{x\in[0,\infty)} \left| \frac{x}{x+n} \right|$$

$$= \sup_{x\in[0,\infty)} \frac{x}{x+n} \underset{x=n}{\geq} \frac{n}{n+n} = \frac{\cancel{n}}{2\cancel{n}} = \frac{1}{2} \;\not\xrightarrow[n\to\infty]{}\; 0$$

$$\Rightarrow f_n \;\not\xrightarrow[n\to\infty]{u}\; f \quad \square$$

b) $f_n : [2,3] \to \mathbb{R}, \quad f_n(x) = \frac{x}{x+n}, \ \forall n \in \mathbb{N}^*$

C. s.

Fie $x \in [2,3]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \quad f : [2,3] \to \mathbb{R}, \quad f(x) = 0$$

C. u

$$\sup_{x\in[2,3]} |f_n(x) - f(x)| = \sup_{x\in[2,3]} \left|\frac{x}{x+n}\right| = \sup_{x\in[2,3]} \frac{x}{x+n}$$

Fie $f_n : [2,3] \to \mathbb{R}, \quad f_n(x) = \frac{x}{x+n}, \ \forall n \in \mathbb{N}^*$

$$f_n'(x) = \frac{x+n-x}{(x+n)^2} = \frac{n}{(x+n)^2} > 0 \quad \forall n \in \mathbb{N}^*, \ \forall x \in [2,3]$$

x	2				3
$f'(x)$		+ +	+	+ + +	
$f(x)$	$\frac{2}{2+n}$	$\nearrow$	$\nearrow$		$\frac{3}{3+n}$

$$\sup_{x\in[2,3]} \left|\frac{x}{x+n}\right| = \frac{3}{3+n} \xrightarrow[n\to\infty]{} 0$$

Prin urmare, $f_n \xrightarrow[n\to\infty]{u} f$

$\square$

c) $f_n : [0, +\infty) \to \mathbb{R}, \quad f_n(x) = \sqrt{x^2 + \frac{1}{n}}, \quad \forall n \in \mathbb{N}^*$

C. S.

Fie $x \in [0, +\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} = x$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde } f : [0, +\infty) \to \mathbb{R}, \quad f(x) = x$$

C. U.

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| \sqrt{x^2 + \frac{1}{n}} - x \right|$$

$$= \sup_{x\in[0,\infty)} \left(\frac{x^2 + \frac{1}{n} - x^2}{\sqrt{x^2 + \frac{1}{n}} + x} \right) = \sup_{x\in[0,\infty)} \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + x}$$

$$= \frac{\frac{1}{n}}{\sqrt{0^2 + \frac{1}{n}} + 0} = \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \sqrt{\frac{1}{n}} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{u} f \qquad \square$$

d) $f_n : [0, +\infty) \to \mathbb{R}, \quad f_n(x) = \frac{n}{n+x}$

C. S.

Fie $x \in [0, +\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{n+x} = 1$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde } f : [0, \infty) \to \mathbb{R}, \quad f(x) = 1$$

C. u.

$$\sup_{x\in[0,\infty)} \left(|f_n(x) - f(x)| \right) = \sup_{x\in[0,\infty)} \left(\left| \frac{n}{n+x} - 1 \right| \right)$$

$$= \sup_{x\in[0,\infty)} \left(\left| \frac{-x}{n+x} \right| \right) = \sup_{x\in[0,\infty)} \frac{x}{n+x}$$

Fie $g_n : [0, +\infty) \to \mathbb{R}$, $g_n(x) = \frac{x}{n+x}$, $\forall n \in \mathbb{N}^*$

$$g_n'(x) = \frac{n + \cancel{x} - \cancel{x}}{(n+x)^2} = \frac{n}{(n+x)^2} > 0 \quad \forall n \in \mathbb{N}^*$$

$\forall x \in [0, \infty)$

x	0						$+\infty$
$f'(x)$		+	+	+	+	+	+
$f(x)$	1		$\nearrow$	$\nearrow$	$\nearrow$		

$$\sup_{x\in[0,\infty)} \frac{x}{n+x} = \lim_{x\to\infty} \frac{x}{n+x} = 1 \neq 0$$

$$\Rightarrow f \overset{u}{\underset{n\to\infty}{\not\to}} f$$

$\square$

e) $f_n : (0, 1] \to \mathbb{R}$, $f_n(x) = x^n$

C. s.

Fie $x \in (0, 1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & x \in (0,1) \\ 1, & x = 1 \end{cases}$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde } f : (0,1] \to \mathbb{R},\ f(x) = \begin{cases} 0, & x \in (0,1) \\ 1, & x = 1 \end{cases}$$

C. u.

$$\left.\begin{array}{l} f_n \text{ continua in } 1 \\ f \text{ nu este cont in } 1 \end{array}\right\} \Rightarrow f_n \overset{u}{\underset{n\to\infty}{\not\to}} f$$