162 8 66 || Seminar GEOMETRIE (13) (hipersfere, parabole, elipse, hipercuadrice)

$F(x) = \sum a_{ij} x_i x_j + \sum 2b_j x_j + c$, $\{x \in \mathbb{R}^4 \mid F(x) = 0\}$

A1. Să se aducă la forma canonică.

a) $5x^2 + 6y^2 + 7z^2 - 4xy + 6yz - 10x + 8y + 14z - 6 = 0$

$$\Rightarrow A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix} \quad P_A(\lambda) = \begin{pmatrix} \lambda-5 & 2 & 0 \\ 2 & \lambda-6 & -2 \\ 0 & -2 & \lambda-7 \end{pmatrix} = (\lambda-5)(\lambda-6)(\lambda-7) - 4(\lambda-7) + 4(\lambda-5)$$

$$= (\lambda^2 - 11\lambda + 30)(\lambda-7) - 8\lambda + 48$$

$$= \lambda^3 - 18\lambda^2 + 99\lambda - 162$$

încercăm $\lambda = 2$

$\lambda = 3$

	1	-16	99	-162
3	1	-15	54	0

$\lambda_{2,3} = \frac{15 \pm 3}{2}$

$$\Rightarrow (\lambda-3)(\lambda^2 - 15\lambda + 54) = (\lambda-3)(\lambda-6)(\lambda-9)$$

Caz 2

Baza de vect. proprii.

$$\begin{cases} 5x - 2y = 3x \\ -2x + 6y + 2z = 3y \\ 2y + 7z = 3z \end{cases} \Leftrightarrow \begin{cases} x - y = 0 \\ -2x + 3y + 2z = 0 \\ y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} x - y = 0 \\ y + 2z = 0 \end{cases} \Leftrightarrow \{ x = y = 2z$$

$f_1 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$$\begin{cases} 5x - 2y = 6x \\ -2x + 6y + 2z = 6y \\ 2y + 7z = 6z \end{cases} \Leftrightarrow \begin{cases} -x - 2y = 0 \\ -x + z = 0 \\ 2y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x = z \\ x = -2y \end{cases}$$

$f_2 = \left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

$$\begin{cases} 5x - 2y = 9x \\ -2x + 6y + 2z = 9y \\ 2y + 7z = 9z \end{cases} \Leftrightarrow \begin{cases} -4x - 2y = 0 \\ -2x - 3y + 2z = 0 \\ 2y - 2z = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + y = 0 \\ y - z = 0 \end{cases} \Leftrightarrow \{ y = z = -2x$$

$$\det R = \det(f_1 | f_2 | f_3) = \begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{vmatrix} = \frac{1}{27} \begin{vmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & -2 & -2 \end{vmatrix} = \frac{1}{27} \begin{vmatrix} 2 & -6 & -3 \\ 2 & -3 & -6 \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{27} \begin{vmatrix} -6 & -3 \\ -3 & -6 \end{vmatrix}$$

$(x, y, z) = x' f_1 + y' f_2 + z' f_3$

$x = \frac{2x' - 2y' + z'}{3}, y = \frac{2x' + y' - 2z'}{3}, z = \frac{-x' - 2y' - 2z'}{3} \Rightarrow 3x'^2 + 6y'^2 + 9z'^2$

$+ \frac{-20x' + 20y - 10z' + 16x' + 8y' - 16z' - 14x' - 28y' - 2z' + 6}{3} = 0$

$3x'^2 + 6y'^2 + 9z'^2 - 6x' - 18z' + 6 = 0$

Rosa

Efectuăm o translație

$3(x'^2 + 2\cancel{x'} + 1) + 6y'^2 + 9(z'^2 - 2z' + 1) - 6 = 0$

$3x''^2 + 6y''^2 + 9z''^2 = 6 \Leftrightarrow \frac{x''^2}{2} + y''^2 + \frac{3z''^2}{2} = 1$ (ELIPSOID) unde $x'' = x' - 1$, $y'' = y'$, $z'' = z' - 1$

$x' = \frac{2x + 2y - z}{3}, y' = \frac{-2x + y - 2z}{3}, z' = \frac{x - 2y - 2z}{3}$

2)

$x^2 + 2y^2 - z^2 + 12xy - 4xz - 8yz + 14x + 16y - 12z - 3 = 0$

$A = \begin{pmatrix} 1 & 6 & -2 \\ 6 & 2 & -4 \\ -2 & -4 & -1 \end{pmatrix}$ $P_A(\lambda) = \begin{pmatrix} \lambda-1 & -6 & 2 \\ -6 & \lambda-2 & 4 \\ 2 & 4 & \lambda+1 \end{pmatrix} \Rightarrow$ GREȘIT în alegere

3) $4x^2 + y^2 + 4z^2 - 4xy - 8xz - 4yz - 12x - 12y + 6z = 0$ $A = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$

$P_A(\lambda) = \begin{pmatrix} \lambda-4 & 2 & -4 \\ 2 & \lambda-1 & 2 \\ -4 & 2 & \lambda-4 \end{pmatrix} = (\lambda^2 - 8\lambda + 16)(\lambda - 1) - 32 - 16(\lambda - 1) - 8(\lambda - 4)$
$= \lambda^3 - 9\lambda^2 + 24\lambda - 16 - 32 - 24\lambda + 4\lambda = \lambda^2(\lambda - 9)$
$\Rightarrow \lambda \in \{0, 9\}$

pt $\lambda = 9 \Rightarrow \begin{cases} 4x - 2y + 4z = 9x \\ -2x + y - 2z = 9y \\ 4x - 2y + 4z = 9z \end{cases} \Rightarrow \begin{cases} -5x - 2y + 4z = 0 \\ -2x - 8y - 2z = 0 \\ 4x - 2y - 5z = 0 \end{cases} \Rightarrow \begin{cases} \Rightarrow x + 4y + z = 0 \\ 18y + 8z = 0 \\ -18y - 9z = 0 \end{cases} \Leftrightarrow \begin{cases} z = -2y \\ x = -2y \end{cases}$

$\lambda = 0 \Leftrightarrow -2x + y - 2z = 0$

$v_0 = \langle\{e_2, e_3\}\rangle = \langle\{(1, 2, 0), (0, 2, 1)\}\rangle$

$f_2 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$

$f_3 = -\langle f_1, e_3\rangle f_2 + e_3 = -\frac{4}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) + (0, 2, 1) = \left(-\frac{4}{5}, \frac{2}{5}, 1\right)$

$$\|f_3\| = \sqrt{\frac{16}{25} + \ldots} \quad \Rightarrow f_1 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), f_2 = \left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}, 0\right), f_3 = \left(-\frac{4}{3\sqrt5}, \frac{2}{3\sqrt5}, \frac{5}{3\sqrt5}\right)$$

$$\Rightarrow \begin{vmatrix} \frac{2}{3} & \frac{1}{\sqrt5} & -\frac{4}{3\sqrt5} \\ -\frac{1}{3} & \frac{2}{\sqrt5} & \frac{2}{3\sqrt5} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt5} \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & \frac{1}{\sqrt5} & -\frac{9}{3\sqrt5} \\ -\frac{1}{3} & \frac{2}{\sqrt5} & \frac{2}{3\sqrt5} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt5} \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & \frac{1}{\sqrt5} & -\frac{9}{3\sqrt5} \\ -\frac{1}{3} & 0 & \frac{20}{3\sqrt5} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt5} \end{vmatrix}$$

$$\Rightarrow \frac{1}{45}\begin{vmatrix} \ldots \end{vmatrix} = 1$$

$$(x, y, z) = x' f_1 + y' f_2 + z' f_3$$

$$\Rightarrow x = \frac{2}{3}x' + \frac{y'}{\sqrt5} - \frac{4z'}{3\sqrt5}$$

$$y = -\frac{x'}{3} + \frac{2y'}{\sqrt5} + \frac{2z'}{3\sqrt5}$$

$$z = \frac{2x'}{3} + \frac{5z'}{3\sqrt5}$$

$$9x'^2 - 8x' - \frac{12y'}{\sqrt5} + \frac{16}{\sqrt5} + 4x' - \frac{24}{\sqrt5}y' - \frac{8z'}{\sqrt5} + 4x' + \frac{10z'}{\sqrt5} = 0$$

$$9x'^2 - \underbrace{\frac{36}{\sqrt5}y' + \frac{26z'}{\sqrt5}}_{\text{rameza m.p.p (rotatie}} = 0$$

$36 - 36 = 1296$

$$\Rightarrow \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ 0 & -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$a^2 = (2^3 - 3^2)^2 + 2^3 - 13^2$$
$$= 2^2(4 \cdot 81 + 169)$$
$$= 4(324 + 1693) = 4 \qquad a = \frac{36}{\sqrt5} \quad b = \frac{26}{\sqrt5}$$

$\Rightarrow 9x''^2 = y'' \Rightarrow$ cilindru parabolic (ne apare z-ul)

A_2. $d_1: x - 2y + 1 = 0$

$d_2: 2x + y - 1 = 0$

a) $d_1 \cup d_2$

b) forma canonica

Rez.

a). $(x - 2y + 1)(2x + y - 1) = 2x^2 + xy - x - 4xy - 2y^2 + 2y + 2x + y - 1 \Rightarrow 2x^2 - 2y^2 - 3xy + x + 3y - 1 = 0$

$\Rightarrow$ not. pol. car. $A = \begin{pmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & -2 \end{pmatrix}$ $\quad P_A(\lambda) = \begin{vmatrix} 2-\lambda & \frac{3}{2} \\ \frac{3}{2} & 2+\lambda \end{vmatrix} = \left(\lambda^2 - 4\lambda - \frac{9}{4}\right) = (\lambda^2 - 4\lambda - \frac{7}{2})$

$\Rightarrow \lambda^2 - \frac{25}{4}$

$\Delta = 16 + 2 = 9$

$\lambda_{1,2} = \frac{\pm 5}{2}$

$$\begin{pmatrix} \lambda-1 & -6 & 2 \\ -6 & \lambda-2 & 4 \\ 2 & 4 & \lambda+1 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda-1 & 0 & 2 \\ -6 & \lambda-10 & 4 \\ 2 & \frac{4}{2}(\lambda+1) & \lambda+1 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda-1 & 0 & 2 \\ 3 & -\frac{\lambda}{2}+5 & -2 \\ 2 & 12(\lambda+1) & \lambda+1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda-1 & 0 & 2 \\ \lambda+2 & -\frac{\lambda}{2}+5 & 0 \\ 2 & 12(\lambda+1) & \lambda+1 \end{pmatrix} \Rightarrow (\lambda-1)\left(-\frac{\lambda}{2}+5\right)(\lambda+1)$$

$\lambda = \frac{4-\sqrt{17}}{2}$

$\lambda_1 = \frac{5}{2}$

$$\begin{cases} 2x - \frac{3}{2}y = \frac{5}{2}x \\ -\frac{3}{2}x - 2y = \frac{5}{2}y \end{cases} \Leftrightarrow \begin{cases} -12x - \frac{3}{2}y = 0 \\ -\frac{3}{2}x - \frac{9}{2}y = 0 \end{cases} \Rightarrow x = -3y$$

$\lambda_2 = -\frac{5}{2}$

$$\begin{cases} 2x - \frac{3}{2}y = -\frac{5}{2}x \\ -\frac{3}{2}x - 2y = -\frac{5}{2}y \end{cases} \Rightarrow \begin{cases} 9x - \frac{3}{2}y = 0 \\ -\frac{3}{2}x + \frac{y}{2} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 3x \end{cases}$$

$(x,y) = x' f_1 + y' f_2 \Leftrightarrow \frac{5}{2}x'^2 - \frac{5}{2}y'^2 + \frac{3x'+y' - 3x' + 9y'}{\sqrt{10}} - 1 = 0 \Leftrightarrow \frac{5}{2}x'^2 - \frac{5}{2}y'^2 + \sqrt{10}\,y' - 1 = 0$

$5x'^2 - \frac{5}{4}\left(y'^2 - \frac{2\sqrt{10}}{5}y' + \frac{2}{5}\right) = 0$

$\frac{5}{2}x'^2 - \frac{5}{4}\left(y'^2 - \frac{2\sqrt{10}}{5}y' + \frac{2}{5}\right) = 0$