# Algebră

Notă seminar : 2 lucrări 1,5 p fiecare (1 h)

maxim 2 p din 3 p

18.11 (sem 7 din seminarele 1-6)

13.01 (ult. sem din sem 7 -...)

# Seminar 1

## Mulțimi și funcții

ex 1

Det. $A = \{ a \in \mathbb{Z} \mid \frac{2a+1}{a+1} \in \mathbb{Z} \}$

ex 2

Det $B = \{ x \mid x \in \mathbb{R}$ și $x = \frac{a+1}{2a+1}, a \in \mathbb{R} \setminus \{-\frac{1}{2}\} \}$

ex 3

Det $(3\mathbb{N}+2) \cap (5\mathbb{N}+1) = 15\mathbb{N}+11$

ex 4

Det $A, B$ știind că

1) $A \cup B = \{1, 2, 3, 4, 5\}$

2) $A \setminus B = \{1, 3\}$

3) $A \cap B \not\subset \{3, 4, 5\}$

ex 1

**Sol:**

Cond. existență : $a \neq -1$

$$\frac{2a+1}{a+1} = \frac{2a+2-1}{a+1} = 2 - \frac{1}{a+1}$$

$$\left.\begin{array}{l} \dfrac{1}{a+1} \in \mathbb{Z} \\ a+1 \in \mathbb{Z} \end{array}\right| \Rightarrow \quad a+1 \mid 1$$

$$a+1 \in \{-1, 1\}$$

$$a \in \{0, -2\} \quad \Rightarrow \quad A \subseteq \{-2, 0\}$$

Evident pt $a = -2$, $a = 0$ $\Rightarrow$ $\dfrac{2a+1}{a+1} \in \mathbb{Z}$

$$\Rightarrow \quad A \supseteq \{-2, 0\}$$

$$\Rightarrow \quad A = \{-2, 0\}$$

**b)** Det. $A = \left\{ a \in \mathbb{Q} \;\middle|\; \dfrac{2a+1}{a+1} \in \mathbb{Z} \right\}$

Cond. existență: $a \neq -1$

$$\left.\underbrace{\frac{2a+1}{a+1} = 2}_{\in \mathbb{Z}} - \frac{1}{a+1}\right| \Leftrightarrow \quad \frac{1}{a+1} \in \mathbb{Z}$$

$$\Updownarrow$$

$$\frac{1}{a+1} = k \in \mathbb{Z}$$

$$\Leftrightarrow \quad a + 1 = \frac{1}{k} \quad \Rightarrow \quad a = \frac{1}{k} - 1 \quad (\neq -1 \quad \forall k \in \mathbb{Z}^*)$$

$$\Rightarrow \quad A \; \overset{=}{\cancel{\subseteq}} \; \left\{ \frac{1-k}{k} \;\middle|\; k \in \mathbb{Z}^* \right\}$$

$$\boxed{\begin{array}{l} \text{Evident pt } \quad a = \dfrac{1-k}{k}, \ \forall k \in \mathbb{Z}^* \ \ldots \Rightarrow \ \underline{\qquad} \in \mathbb{Z} \\ \hspace{8cm} \Rightarrow A \supseteq \ldots \end{array}}$$

$\square$

ex 2

Det $B = \{ x \mid x \in \mathbb{R}$ și $x = \frac{a+1}{2a+1}$, $a \in \mathbb{R} \setminus \{-\frac{1}{2}\} \}$

**Sol**:

$$x = \frac{a+1}{2a+1} \quad \forall a \in \mathbb{R} \setminus \{-\tfrac{1}{2}\}$$

$$x = \frac{a+1+a-a}{2a+1} \iff x = \frac{2a+1-a}{2a+1}$$

$$\iff x = \frac{2a+1}{2a+1} - \frac{a}{2a+1}$$

$$\iff x = 1 - \frac{a}{2a+1} \quad \forall a \in \mathbb{R} \setminus \{-\tfrac{1}{2}\}$$

Det mulțimea = dublă incluziune

"$\subseteq$" Fie $x \in B \Rightarrow x = 1 - \frac{a}{2a+1}$, $a \in \mathbb{R} \setminus \{-\frac{1}{2}\}$ $\Rightarrow x \in \mathbb{R}$

$$f : \mathbb{R} \setminus \{-\tfrac{1}{2}\} \to \mathbb{R}$$

$$f(a) = \frac{a+1}{2a+1}$$

$$B = \operatorname{Im} f = \{ f(a) \mid a \in \mathbb{R} \setminus \{-\tfrac{1}{2}\} \}$$

$$x = \frac{a+1}{2a+1} \iff 2ax + x = a+1$$

$$\iff a(2x-1) = 1-x$$

$$\iff a = \frac{1-x}{2x-1} \quad \text{dacă } x \neq -\frac{1}{2}$$

Dem că $B = \mathbb{R} \setminus \{-\frac{1}{2}\}$

"$\subseteq$" Fie $x \in B \Rightarrow x \in \mathbb{R}$

$a \in \mathbb{R} \setminus \{-\frac{1}{2}\}$

$$x = \frac{1}{2} \Leftrightarrow \frac{a+1}{2a+1} = \frac{1}{2}$$

$$\Leftrightarrow 2a + 2 = 2a + 1$$

$$\Leftrightarrow 1 = 0 \quad \text{Fals}$$

$\Rightarrow x \in \mathbb{R} \setminus \{-\frac{1}{2}\}$

$\Rightarrow B \subseteq \mathbb{R} \setminus \{\frac{1}{2}\}$

"$\supseteq$" Fie $x_0 \in \mathbb{R} \setminus \{\frac{1}{2}\}$

Pt $a = \dfrac{1 - x_0}{2x_0 - 1} \Rightarrow x_0 = \dfrac{a+1}{2a+1} \in B$

$\square$

ex 3

Dat $(3\mathbb{N} + 2) \cap (5\mathbb{N} + 1) = 15\mathbb{N} + 11$

**Sol:**

Sisteme de congr
$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \dots \\ x \equiv a_n \pmod{n_n} \end{cases}$$

are sol. unică modulo $n_1 \cdot \dots \cdot n_n$

Dat $m \in \mathbb{N}^*$

descomp în fact. primi $\longrightarrow p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_h^{\alpha_h}$

A afla resturile împ lui $b$ la $m$

$\Updownarrow$ L. C. R.

a afla resturile împ lui $b$ la $p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$

$$(3\mathbb{N}+2) \cap (5\mathbb{N}+1) = 15\mathbb{N}+11$$

"$\supseteq$"

$$15h + 11 = 3(5h+3) + 2 \in 3\mathbb{N}+2$$

$$= 5(3h+2) + 1 \in 5\mathbb{N}+1$$

"$\subseteq$"

Fie $x \in (3\mathbb{N}+2) \cap (5\mathbb{N}+1)$

$$\Rightarrow \begin{cases} x \equiv 2 \pmod 3 \\ x \equiv 1 \pmod 5 \end{cases}$$

$\Rightarrow x \in 3\mathbb{N}+2$ și $5\mathbb{N}+1$

$x = 3a + 2$, $a \in \mathbb{N}$

$x = 5b + 1$, $b \in \mathbb{N}$

$\Rightarrow \quad 3a + 2 = 5b + 1$

$3a + 1 = 5b$

$a = \dfrac{5b-1}{3} \in \mathbb{N} \qquad b = \dfrac{3a+1}{5} \in \mathbb{N}$

$b \in \mathbb{N} \Downarrow$

$3 \mid 5b - 1$

$5b - 1 \equiv 0 \pmod 3 \iff$

$5b \equiv 1 \pmod 3 \iff$

$-b \equiv 1 \pmod 3 \iff$

$b \equiv 2 \pmod 3$

1) Dacă $b = 3l \Rightarrow 5b - 1 = 15l - 1$

nu e div

2) Dacă $b = 3l + 1 \Rightarrow 5b - 1 = 5(3l + 1) - 1$

$= 15l + 4$

$= 3(5l + 1) + 1$

NU e div cu 3

3) Dacă $b = 3l + 2 \Rightarrow 5b - 1 = 5(3l + 2) - 1$

$= 15l + 10 - 1$

$= 15l + 9$

$= 3(5l + 3)$

$\vdots\, 3$

$$x = 5(3l + 2) + 1$$

$$= 15l + 10 + 1 \quad , \quad l \in \mathbb{N}$$

$$\Rightarrow \quad x \in 15\mathbb{N} + 11$$

$$\Rightarrow \quad (3\mathbb{N} + 2) \cap (5\mathbb{N} + 1) \subseteq \ldots$$

$$\square$$