Seminar 5

ex 1

Fie $n \in \mathbb{N}^*$ și $d_2 \overset{not}{=} d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$d(x, y) = \sqrt{\sum_{n=1}^{\infty} (x_i - y_i)^2}$$

Arătați că d_2 este metrică pe $\mathbb{R}^n$

ex 4

Faceți analiza top a mulțimii $A \subset \mathbb{R}$, unde

a) $A = (0, 1) \cup \{2\}$

b) $A = \mathbb{N}$

c) $A = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

ex 1

Fie $n \in \mathbb{N}^*$ și $d_2 \overset{not}{=} d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$d(x, y) = \sqrt{\sum_{n=1}^{\infty} (x_i - y_i)^2}$$

Arătați că d_2 este metrică pe $\mathbb{R}^n$

Sol:

Fie $x, y, z \in \mathbb{R}^n$

1) $d(x, y) = \sqrt{\sum_{n=1}^{\infty} (x_i - y_i)^2} \geq 0$ (evident)

2) $d(x, y) = 0 \iff \sqrt{\sum_{n=1}^{\infty} (x_i - y_i)^2} = 0 \iff$

$\iff \sum_{n=1}^{\infty} (x_i - y_i)^2 = 0 \iff (x_i - y_i)^2 = 0 \iff$

$\iff x_i - y_i = 0 \iff x_i = y_i \ \forall i = \overline{1, n} \iff x = y$

3) $d(x, y) = \sqrt{\sum_{n=1}^{\infty} (x_i - y_i)^2} = \sqrt{\sum_{n=1}^{\infty} [-(y_i - x_i)]^2}$

$= \sqrt{\sum_{n=1}^{\infty} (y_i - x_i)^2} = d(y, x)$

4) Arătam că $d(x, z) \leq d(x, y) + d(y, z)$

Folosim inegalitatea Cauchy - Buniakowski - Schwarz

Pt. orice $n \in \mathbb{N}^*$ și orice $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

avem

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \cdot \left(\sum_{n=1}^{\infty} b_i^2 \right)$$

$$\Updownarrow$$

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$$

$$d(x,y) = \sqrt{\sum (x_i - y_i)^2} = \sqrt{\sum (x_i - y_i + y_i - z_i)}$$

$$= \sqrt{\sum_{i=1}^{n} [(x_i - y_i)^2 + (y_i - z_i)^2 + 2(x_i - y_i)(y_i - z_i))}$$

$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{i=1}^{n} (y_i - z_i) + 2 \sum (x_i - y_i)(y_i - z_i)} \leq$$

$$\underset{\text{C.B.S.}}{\leq} \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{i=1}^{n} (y_i - z_i) + 2 \left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \cdot \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \right)} =$$

$$\Rightarrow \sum_{i=1}^{n} (x_i - y_i)(y_i - z_i) \leq \left| \sum_{i=1}^{n} (x_i - y_i)(y_i - z_i) \right| \leq$$

$$\leq \left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \right) \left(\sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \right)$$

$$= \sqrt{\sum (x_i - y_i)^2} + \sqrt{\sum (x_i - y_i)^2}$$

$$= d(x,y) + d(y,z)$$

Deci d este metrică pe $\mathbb{R}^n$

ex 4

Faceți analiza top a mulțimii $A \subset \mathbb{R}$, unde

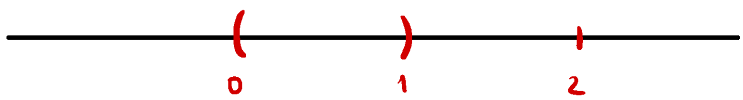
a) $A = (0,1) \cup \{2\}$

b) $A = \mathbb{N}$

c) $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$

Sol:

a) $A = (0,1) \cup \{2\}$



1) $\mathring{A} = ?$ (puncte interioare)

$x \in \mathring{A} \iff \exists\, r > 0$ a.î. $(x-r, x+r) \subset A$

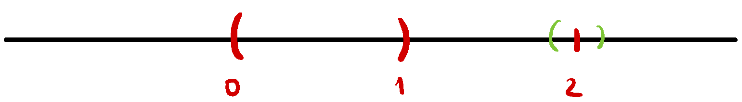
$\mathring{A} \subset A$

$$\left. \begin{array}{l} (0,1) \subset A \\ (0,1) \text{ deschisă} \end{array} \right\} \Rightarrow (0,1) \subset \mathring{A}$$

Deci $(0,1) \subset \mathring{A} \subset (0,1) \cup \{2\}$

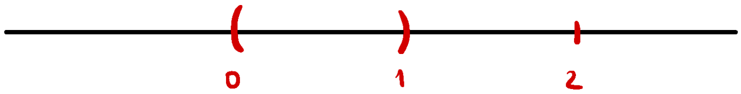
Studiem dacă $2 \in \mathring{A}$

$2 \in \mathring{A} \iff \exists\, r > 0$ a.î. $(2-r, 2+r) \subset A$



Deci $2 \notin \mathring{A}$

2) $\bar{A} = ?$ (puncte aderență)



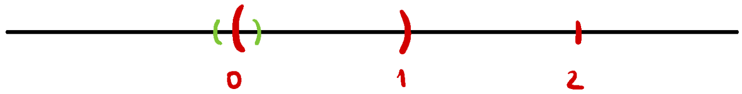
$x \in \bar{A} \iff \forall\, r > 0$, avem $(x-r, x+r) \cap A \neq \emptyset$

$A \subset \bar{A}$

$$\left.\begin{array}{l} [0,1] \cup \{2\} \supset A \\ [0,1] \cup \{2\} \text{ închisă} \end{array}\right\} \Rightarrow \bar{A} \subset [0,1] \cup \{2\}$$

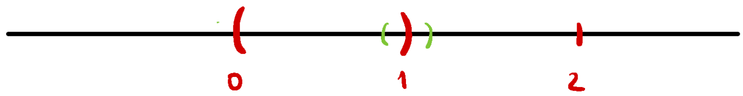
Studiem dacă $0 \in \bar{A}$ și $1 \in \bar{A}$

$0 \in \bar{A} \iff \forall\, r>0$, avem $(0-r, 0+r) \cap A \neq \emptyset$



Deci $0 \in \bar{A}$

$1 \in \bar{A} \iff \forall\, r>0$, avem $(1-r, 1+r) \cap A \neq \emptyset$

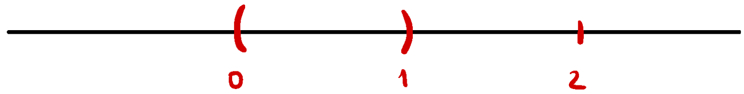


Deci $1 \in \bar{A}$

Așadar $\bar{A} = [0,1] \cup \{2\}$

3) $A' = ?$ (puncte acumulare)

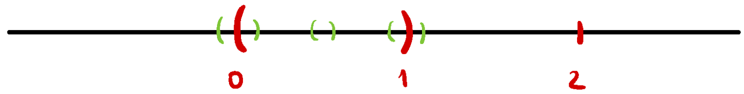
$x \in A' \iff \forall r > 0$, avem $(x-r, x+r) \cap (A \setminus \{x_0\}) \neq \emptyset$



$A' \subset \bar{A} = [0, 1] \cup \{2\}$

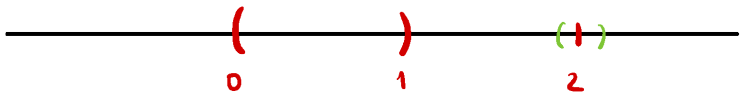
- Fie $x \in [0, 1]$

  $x \in A' \iff \forall r > 0$, avem $(x-r, x+r) \cap (A \setminus \{x_0\}) \neq \emptyset$



Deci $x \in A'$

- $2 \in A' \iff \forall r > 0$, avem $(2-r, 2+r) \cap (A \setminus \{2\}) \neq \emptyset$



Deci $2 \notin A'$

Așadar $A' = [0, 1]$

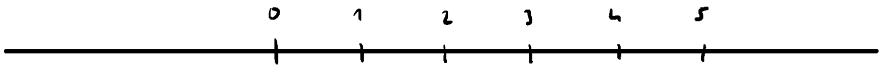
4) $\mathrm{Fr}(A) = \partial A = \bar{A} \setminus \mathring{A} = ([0,1] \cup \{2\}) \setminus (0,1) =$
$= \{0, 1, 2\}$

5) $\mathrm{Iz}(A) = {}'A = \bar{A} \setminus A' = ([0,1] \cup \{2\}) \setminus [0, 1] = \{2\}$

$\square$

b) $A = \mathbb{N}$

1) $\mathring{A} = ?$

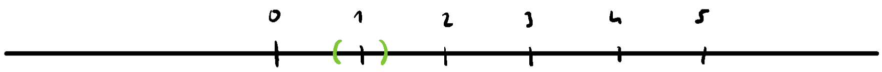


$x \in \mathring{A} \iff \exists\, r > 0$ a.î. $(x-r, x+r) \subset A$

$\mathring{A} = \emptyset$ deoarece între orice două nr. reale există o inf. de nr. raționale și o inf de nr. iraționale

2) $\bar{A} = ?$

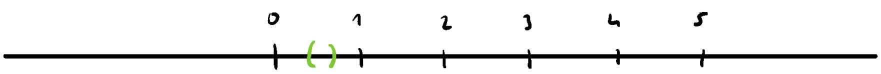
$x \in \bar{A} \iff \forall\, r > 0$, avem $(x-r, x+r) \cap A \neq \emptyset$



$A \subset \bar{A} \Rightarrow \mathbb{N} \subset \bar{A}$

Fie $x \in \mathbb{R} \setminus \mathbb{N}$

$x \in \bar{A} \iff \forall\, r > 0$, avem $(x-r, x+r) \cap A \neq \emptyset$



Deci $x \notin \bar{A}$

Așadar $\bar{A} = \mathbb{N}$

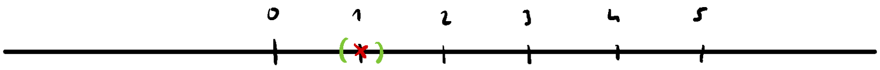
3) $A' = ?$

$x \in A' \iff \forall r > 0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq \emptyset$

$A' \subset \bar{A} = \mathbb{N}$

Fie $x \in \mathbb{N}$

$x \in A' \iff \forall r > 0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq \emptyset$



Deci $x \notin A'$

Așadar $A' = \emptyset$

4) $\mathrm{Izo}(A) = \bar{A} \setminus \mathring{A} = \mathbb{N} \setminus \emptyset = \mathbb{N}$

5) $\mathrm{Fr}(A) = \bar{A} \setminus A' = \mathbb{N} \setminus \emptyset = \mathbb{N}$

$\square$

c) $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$

1) $\mathring{A} = ?$

$x \in \mathring{A} \iff \exists r > 0$ a.î. $(x-r, x+r) \subset A$

$\mathring{A} = \emptyset$ deoarece între orice două nr. reale există o inf. de nr. raționale și o inf. de nr. iraționale

2) $\bar{A} = ?$

$x \in \bar{A} \iff \exists (x_k)_k \subset A \setminus \{x\}$ a.î. $\lim\limits_{k \to \infty} x_k = x$

Orice șir de elem. din A poate avea drept limită fie un element din A (șirurile constante) fie 0

Deci $A' = \{0\}$

3) $\bar{A} = A \cup A' = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}$

4) $Fr(A) = \bar{A} \setminus \mathring{A} = (\{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}) / \emptyset$
$= \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}$

5) $Iz(A) = \bar{A} \setminus A' = (\{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}) \setminus \{0\}$
$= \{ \frac{1}{n} \mid n \in \mathbb{N}^* \}$

$\square$