Seminar 5

11 Nov 2024

$f: A \to B \qquad |A| = n \qquad |B| = m$

4) Det. nr. funct. surj., dacă există

(P.I.E)

$|\{ f: A \to B \mid f \text{ funcții surj} \}|$

Mulțimea $\mathcal{M} = \{ f: A \to B \mid f \text{ fct. surj} \} \subset \underbrace{\{ f: A \to B \mid f \text{ funcție} \}}_{= \mathcal{N}}$

$|\mathcal{N}| = m^n$

$|\mathcal{M}| = |\mathcal{N}| - |\underbrace{\mathcal{N} \setminus \mathcal{M}}_{= \complement_{\mathcal{M}} \mathcal{N}}|$ complementara

$A = \{ a_1, a_2, \dots, a_n \}$

$B = \{ b_1, b_2, \dots, b_m \}$

Def f surj $\iff \underbrace{\operatorname{Im} f}_{= f(A)} \overset{\subseteq}{=} B$ ②

Ec $f(x) = b_i$ are cel puțin o sol

f surj $\overset{\text{def}}{\underset{\text{surj}}{\Longrightarrow}}$ $n \geq m$

(I) Dacă $n < m \Rightarrow |\mathcal{M}| = 0$

(II) $\underline{n \geq m}$.

f nu e surj $\iff \exists\, i_0$ a.î. $f(x) = b_i$ nu are nicio soluție, $i \in \{1, \dots, m\}$

$(\iff \exists\, i_0$ a.î. $f(A) \subseteq B \setminus \{b_{i_0}\})$

$x \in B_1 \cup \dots \cup B_m \iff \exists\, i_0$ a.î. $x \in B_{i_0}$

$i = \overline{1, m}$ $\quad \mathcal{M}_i = \{ f : A \to B \setminus \{b_{i_0}\} \mid f$ funcție $\}$

f nu e surj $\iff \exists\, i_0$ a.î. $f \in \mathcal{M}_{i_0}$

$\iff f \in \mathcal{M}_1 \cup \dots \cup \mathcal{M}_m$

$\mathcal{M}_i = \{ f : A \to B \mid f$ fct. $f(A) \subseteq B \setminus \{b_{i_0}\} \}$

$|\mathcal{N} \setminus \mathcal{M}| = |\mathcal{M}_1 \cup \dots \mathcal{M}_m|$

$$|\mathcal{M}_1 \cup \dots \mathcal{M}_m| \overset{\text{P.I.E}}{=} \sum_{i=1}^{m} |\mathcal{M}_i| - \sum_{1 \leq i_1 < i_2 \leq m} |\mathcal{M}_{i_1} \cap \mathcal{M}_{i_2}| + \dots$$

$$+ (-1)^m |\mathcal{M}_1 \cap \mathcal{M}_2 \dots \mathcal{M}_m|$$

$$|M_i| = (m-1)^n \quad \forall i$$

$$M_{i_1} \cap M_{i_2} = \{ f : A \to B \mid f \text{ a.î. } f(A) \subseteq B \setminus \{b_{i_1}\} \text{ și } f(A) \subseteq B \setminus \{b_{i_2}\} \}$$

$$\Updownarrow$$

$$f(A) \subseteq (B \setminus \{b_{i_1}\}) \cap (B \setminus \{b_{i_2}\}) = B \setminus \{b_{i_1}, b_{i_2}\}$$

$$|M_1 \cap \ldots M_m| = \phi$$

$$|\mathcal{N} - M| = C_m^1 (m-1)^n - C_m^2 (m-2)^n + \ldots + (-1)^{m-2} C_m^{m-1} \underbrace{(m-(m-1))^n}_{1^n} + 0$$

$$|\mathcal{N}| = m^n - C_m^1 (m-1)^n - C_m^2 (m-2)^n + \ldots +$$

$(\exists)$ $f : A \to B$ inj $\iff$ $|A| \leq |B|$

(surj $\iff$ $|B| \leq |A|$)

$A \sim B$ (s.n. echipotente sau cardinal echivalente) $\iff$ $\exists$ $f : A \to B$ bijecție

$|A| \overset{not}{=} |B|$

Def A s.n. numărabilă dacă $|A| = |\mathbb{N}|$; altfel dacă A e infinită și nu e numărabilă s.n. nenumarabilă

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ sunt cardinal echivalente 2 câte 2 ?

Teorema Cantor - Bernstein

$$|A| = |B| \Leftrightarrow |A| \le |B| \text{ și } |B| \le |A|$$

$$\left(\exists f: A \to B \text{ bij} \Leftrightarrow \begin{array}{l} \exists f_1 : A \to B \text{ inj} \\ \exists f_2 : B \to A \text{ inj} \end{array} \right)$$

ex 1

Să se arate că $\underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{2024}$ e în bijecție cu $\mathbb{N}$

Sol:

Aplic Cantor - Bernstein

$f_1 : \mathbb{N} \to \underbrace{\mathbb{N} \times \ldots \mathbb{N}}_{2024}$

$f_1(n) = (n, \underbrace{1, \ldots, 1}_{2023})$ $\quad f_1$ e inj $\Rightarrow |\mathbb{N}| \le |\mathbb{N} \times \ldots \mathbb{N}|$

$f_2 : \underbrace{\mathbb{N} \times \ldots \mathbb{N}}_{2024} \to \mathbb{N}$ $\quad f_2$ inj ?

$f_2(n_1, n_2, \ldots, n_{2024}) = \ldots$

$$f_2(n_1, n_2, \ldots, n_{2024}) = f_2(m_1, m_2, \ldots, m_{2024}) \to \begin{array}{l} n_1 = m_1 \\ n_2 = m_2 \\ \ldots \\ n_{2024} = m_{2024} \end{array}$$

Euclid : şirul nr. prime este infinit

$\Rightarrow \exists\ p_1, \dots, p_{2024}$ nr prime diferite două câte două

$$f_2(n_1, n_2, \dots, n_{2024}) = p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_{2024}^{n_{2024}}$$

f inj deoarece desc. în fact în primi e unică

ex 2

Arătați că oricare 2 dintre urm. mulțimi sunt echipotente :

$(-\infty, a],\ (b, +\infty),\ [c, d),\ [c, d],\ (c, d],$
$(c, d),\ (0, 1),\ \mathbb{R},\ \mathbb{R}_+,\ \mathcal{P}(n),\ \mathbb{C},\ \mathbb{R}_+^*$

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ex

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|$$

Fapt $(\forall)$ A mulțime

$$|A| < |\mathcal{P}(A)|$$

$\mathbb{R} \xhookrightarrow{f} \mathbb{C}$

$x \longmapsto f(x) = x$ f inj $\Rightarrow |\mathbb{R}| \leq |\mathbb{C}|$

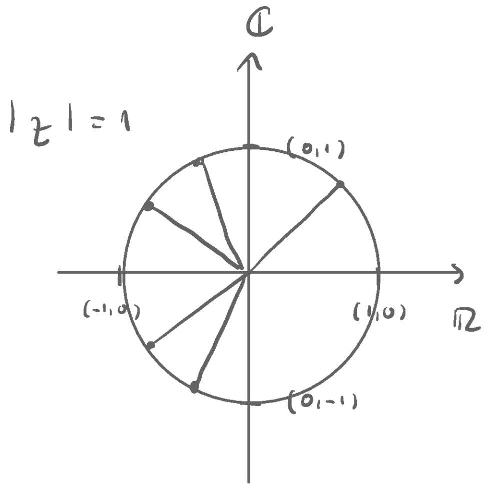
$$\mathbb{C} \xrightarrow{g} \mathbb{R}$$

$$g(a+bi) =$$

Temă !

$$f:\mathbb{R} \to \mathbb{R}_+^*$$

$$f(a) = e^a$$



$$|\mathbb{N}| = |\mathbb{Z}|$$

$$\begin{array}{ccccccccc} m & \ldots, & -2, & -1, & 0, & 1, & 2, & \ldots & n \\ a_{2\cdot m-1} & & a_{2\cdot 2-1} & a_{2\cdot 1-1} & a_{2\cdot 0} & a_{2\cdot 1} & a_{2\cdot 2} & & a_{2\cdot n} \end{array}$$

$$f: \mathbb{N} \xrightarrow{\sim} A$$

$$A = \{ \underset{a_0}{f(0)}, \underset{a_1}{f(1)}, \underset{a}{\ldots}, \underset{a_n}{f(n)} \}$$

$f: \mathbb{N} \to \mathbb{Z}$, $\qquad g: \mathbb{Z} \to \mathbb{N}$

$f(n)$

$$g(k) = \begin{cases} 2k, & k \geq 0 \\ 2\cdot(-k)-1, & k < 0 \end{cases}$$

bij $\Rightarrow |\mathbb{Z}| = |\mathbb{N}|$

$|\mathbb{N}| = |\mathbb{Q}|$

analog pt $|\mathbb{N}| = |\mathbb{Z}|$

Dacă arătăm $|\mathbb{Q}_+| = |\mathbb{N}| \Rightarrow |\mathbb{Q}| = |\mathbb{N}|$

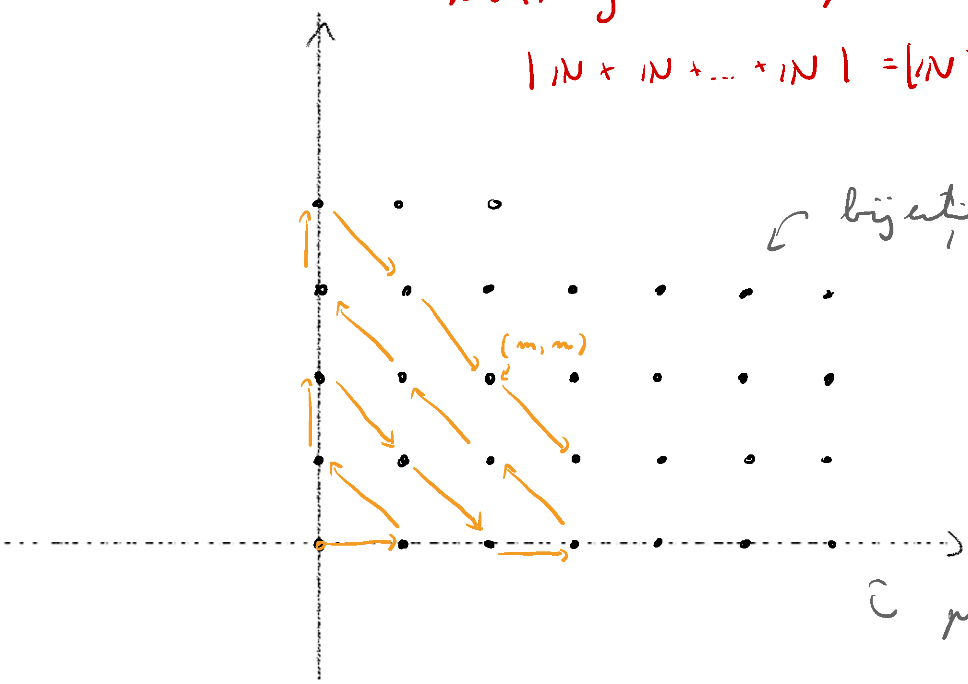
$$\mathbb{N} \xrightarrow{f} \mathbb{Q}_+$$

$$n \longmapsto f(n) = n \qquad f \text{ inj}$$

$$\begin{array}{cccccccc}
\overset{a_0}{\frac{1}{1}} & \overset{a_1}{\frac{2}{1}} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \frac{7}{1} & \dots \\
\overset{a_2}{\frac{1}{2}} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \frac{7}{2} & \dots \\
a_3\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \frac{7}{3} & \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \frac{7}{4} & \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & & & & \\
\frac{1}{6} & \frac{2}{6} & \frac{3}{6} & & & & &
\end{array}$$

Dem. geometrică pt

$|\mathbb{N} + \mathbb{N} + \dots + \mathbb{N}| = |\mathbb{N}|$



bijecție

$\subset$ puncte laticeale ?

Hotelul lui Hilbert

$|\mathbb{N}| = |\mathbb{N} \setminus A|$ A submulțime finită a lui $\mathbb{N}$

! Test

<u>Ex</u> $A = \{0, 1, 2\}$

$\mathbb{N} \setminus \{0, 1, 2\} = \{3, 4, 5\}$

$f : \mathbb{N} \setminus A \to \mathbb{N}$

$f(n) = n - 3$

Ex 2

$A = \{1, 3, 7\}$

$$\begin{array}{ccccccccccccc} \mathbb{N} \setminus A = \{ & 0, & 2, & 4, & 5, & 6, & 8, & 9, & 10, & 11 & \ldots \} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \end{array}$$

$f : \mathbb{N} \setminus A \to \mathbb{N}$

$$f(k) = \begin{cases} 0, & k = 0 \\ 1, & k = 2 \\ 4, & k = 6 \\ n - 3, & k \geq 8 \end{cases}$$

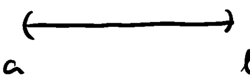
X finită

$\cup$I

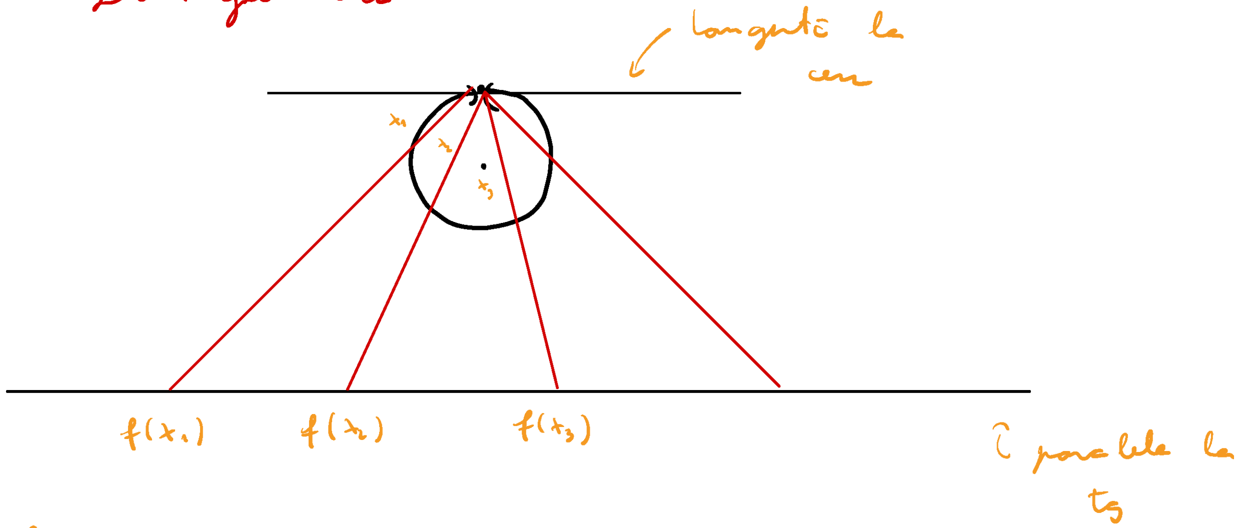
A finită

$|X \setminus A| = |X|$ ex 17

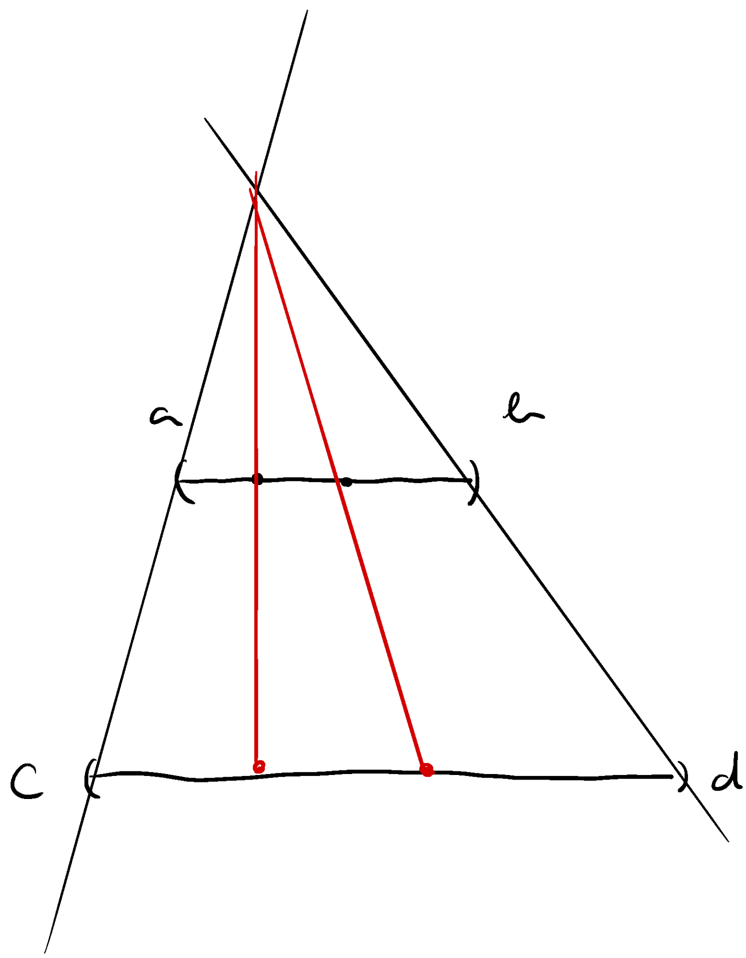
(a, b)



Dem. geometrică



$f : (a, b) \to \mathbb{R}$ bij



sunt $\parallel$
(și $\neq$)

$f : (a, b) \to (c, d)$

$f(x) = mx + n$

$f(a) = c$

$f(b) = d$