Seminar 3

ex 1

a) Arătați că $\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$

b) Studiați convergența seriei $\sum\limits_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right) \cdot x^n, \ x > 0$

(Crit. comp. lim, $y_n = \frac{1}{n^2} \cdot x^n$)

ex 2

Studiați convergența (natura) seriilor :

a) $\sum\limits_{n=1}^{\infty} \frac{1}{2^n + 3^n}$ (Crit comp. cu ineg, $y_n = \frac{1}{3^n}$)

b) $\sum\limits_{n=1}^{\infty} \frac{7 \cdot 13 \cdot 19 \cdot \ldots \cdot (6n+1)}{8 \cdot 13 \cdot 18 \cdot \ldots \cdot (5n+3)} \cdot x^n, \ x > 0$

(Crit raportului => Crit. Abel - Duhamel)

c) $\sum\limits_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$ (Crit. condensării)

d) $\sum\limits_{n=1}^{\infty} \frac{a^n + n}{3^n + n^3}, \ a > 0$

e) $\sum\limits_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n(n+1)}\right)}{\left(\cos \frac{1}{n}\right) \cdot \left(\cos \frac{1}{n+1}\right)}$

ex 1

a) Arătați că $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$

b) Studiați convergența seriei $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right) \cdot x^n, \quad x > 0$

Sol:

a) $\lim_{x \to 0} \frac{1 - \cos x}{x^2} \quad \left(\frac{0}{0}\right)$

$\overset{L'H}{=} \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$ (→ 1)

b) Fie $x_n = \left(1 - \cos \frac{1}{n}\right) \cdot x^n$

$y_n = x^n \cdot \frac{1}{n^2}$

Folosim Criteriul de comparație cu limită

$$\lim_{n \to \infty} \frac{x_n}{y_n} = \lim_{n \to \infty} \frac{\left(1 - \cos \frac{1}{n}\right) \cdot \cancel{x^n}}{\frac{1}{n^2} \cdot \cancel{x^n}}$$

$$= \lim_{n \to \infty} \frac{1 - \cos \frac{1}{n}}{\left(\frac{1}{n}\right)^2} \overset{a)}{=} \frac{1}{2} \in (0, +\infty)$$

Deci $\sum x_n \sim \sum y_n$

Studiem convergența seriei $\sum y_n \left(\sum_{n=1}^{\infty} x^n \cdot \frac{1}{n^2}, \quad x > 0 \right)$

Aplicăm Criteriul raportului

$$\lim_{n \to \infty} \frac{y_{n+1}}{y_n} = \lim_{n \to \infty} \frac{x^{\cancel{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\cancel{x^n}} = \lim_{n \to \infty} x \cdot \frac{n^2}{(n+1)^2} = x$$

1) Dacă $x < 1$ (i.e. $x \in (0,1)$), atunci $\sum y_n$ conv

2) Dacă $x > 1$ (i.e. $x \in (1, +\infty)$), atunci $\sum y_n$ div

3) Dacă $x = 1$, atunci acest criteriu nu decide

Pentru $x = 1$, $y_n = \frac{1}{n^2}$, $\forall n \in \mathbb{N}^*$

$\sum y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, convergentă

(serie armonică gen, $\alpha = 2$)

Am obținut $\sum x_n$ → conv, $x \in (0, 1]$; → div, $x \in (1, \infty)$

ex 2

Studiați convergența (natura) seriilor:

a) $\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$

Sol:

Fie $x_n = \frac{1}{2^n + 3^n}$, $\forall n \in \mathbb{N}^*$

$y_n = \frac{1}{3^n}$, $\forall n \in \mathbb{N}^*$

$x_n < y_n$

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ conv

(serie geometrică gen. cu $q = \frac{1}{3}$)

Conform criteriul de comp cu ineg. avem că

$\sum x_n$ conv

b) $\sum_{n=1}^{\infty} \frac{7 \cdot 13 \cdot 19 \cdot \ldots \cdot (6n+1)}{8 \cdot 13 \cdot 18 \cdot \ldots \cdot (5n+3)} \cdot x^n, \quad x > 0$

Sol:

Fie $x_n = \frac{7 \cdot 13 \cdot 19 \cdot \ldots \cdot (6n+1)}{8 \cdot 13 \cdot 18 \cdot \ldots \cdot (5n+3)} \cdot x^n, \quad x > 0$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{\frac{7 \cdot 13 \cdot 19 \cdot \ldots \cdot (6n+1)}{8 \cdot 13 \cdot 18 \cdot \ldots \cdot (5n+3)}} \cdot \frac{(6n+7)}{(5n+8)} \cdot \cancel{x^{n+1}}}{\cancel{\frac{7 \cdot 13 \cdot 19 \cdot \ldots \cdot (6n+1)}{8 \cdot 13 \cdot 18 \cdot \ldots \cdot (5n+3)}} \cdot \cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{6n+7}{5n+8} \cdot x = \frac{6}{5} x$$

Conform Crit raportului avem:

1) dacă $\frac{6}{5} x < 1$ (i.e. $x \in (0, \frac{5}{6})$), $\sum x_n$ conv

2) dacă $\frac{6}{5} x > 1$ (i.e. $x \in (\frac{5}{6}, +\infty)$), $\sum x_n$ div

3) dacă $x = \frac{5}{6}$ acest crit. nu decide

Fie $x = \frac{5}{6}$

$$x_n = \frac{7 \cdot 13 \cdot 19 \cdot \ldots \cdot (6n+1)}{8 \cdot 13 \cdot 18 \cdot \ldots \cdot (5n+3)} \cdot x^n$$

Aplicăm Criteriul Raabe - Duhamel

$$\lim_{n\to\infty} n \cdot \left(\frac{x_n}{x_{n+1}} - 1 \right) = \lim_{n\to\infty} n \cdot \left(\frac{5n+8}{6n+7} \cdot \frac{6}{5} - 1 \right)$$

$$= \lim_{n\to\infty} n \cdot \left(\frac{30n + 48 - 30n - 35}{30n + 35} \right) = \lim_{n\to\infty} \frac{13n}{30n+35} = \frac{13}{30} < 1$$

Deci $\sum x_n$ e div $\square$

c) $\sum\limits_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$

Sol:

$x_n = \frac{1}{n \cdot \ln n}, \quad \forall n \geq 2$

x_n descrescator

Aplicam Criteriul condensării

$\sum x_n \sim \sum 2^n \cdot x_{2^n}$

Studiem convergența serii $\sum 2^n \cdot x_{2^n}$

$\sum\limits_{n=2}^{\infty} 2^n \cdot x_{2^n} = \sum\limits_{n=2}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \cdot \ln(2^n)}$

$= \sum\limits_{n=2}^{\infty} \frac{1}{n \ln 2}$, (deoarece $\sum \frac{1}{n}$ este div,
serie arm. gen., $\alpha = 1$)

Deci $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ div $\square$

d) $\sum\limits_{n=1}^{\infty} \frac{a^n + n}{3^n + n^3}, \quad a > 0$

Sol: Fie $x_n = \frac{a^n + n}{3^n + n^3}$

$y_n = \frac{a^n}{3^n + n^3}$

$z_n = \frac{n}{3^n + n^3}$

Fie $t_n = \frac{n}{n^3}$

$z_n \leq t_n$

$\sum t_n = \sum \frac{1}{n^2}$, conv (serie arm. gen, $\alpha = 2$)

Conform Crit de comp. cu ineg. avem că $\sum z_n$ conv

Deci $\sum x_n \sim \sum y_n$

Studiem conv seriei $\sum \frac{a^n}{3^n + n^3}$

Fie $u_n = \frac{a^n}{3^n}$, $\forall n \in \mathbb{N}$

Aplicăm Criteriul de comparație cu limita

$$\lim_{n\to\infty} \frac{y_n}{u_n} = \lim_{n\to\infty} \frac{\cancel{a^n}}{3^n + n^3} \cdot \frac{3^n}{\cancel{a^n}} = \lim_{n\to\infty} \frac{3^n}{3^n + n^3}$$

$$= \lim \frac{\cancel{3^n}(1)}{\cancel{3^n}\left(1 + \frac{n^3}{3^n}\right)}$$

$$= \lim \frac{1}{1 + \frac{n^3}{3^n}}$$

$$\lim_{n\to\infty} \frac{n^3}{3^n}$$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^3}{3 n^3} = \frac{1}{3} < 1$$

Conform. Crit raportului pt. șiruri cu termeni strict pozitivi

avem ca $\lim_{n\to\infty} \frac{n^3}{3} = 0$

$$= \lim_{n\to\infty} \frac{1}{1+0} = 1 \in (0, \infty)$$

Deci $\sum y_n \sim \sum u_n$

Stud. conv. seriei $\sum u_n$

$$\sum u_n = \sum \left(\frac{a}{3}\right)^n \quad \begin{array}{l} \nearrow \text{ conv, dacă } a \in (0, 3) \\ \searrow \text{ div, dacă } a \in [3, \infty) \end{array}$$

(serie geom., $q = \frac{a}{3}$)

Aradar $\sum x_n \quad \begin{array}{l} \nearrow \text{ conv, dacă } a \in (0, 3) \\ \searrow \text{ div, dacă } a \in [3, \infty) \end{array}$

e) $$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n(n+1)}\right)}{\left(\cos\frac{1}{n}\right)\cdot\left(\cos\frac{1}{n+1}\right)}$$

Sol:

$$x_n = \frac{\sin\left(\frac{1}{n(n+1)}\right)}{\left(\cos\frac{1}{n}\right)\cdot\left(\cos\frac{1}{n+1}\right)} = \frac{\sin\left(\frac{1}{n} - \frac{1}{n+1}\right)}{\left(\cos\frac{1}{n}\right)\cdot\left(\cos\frac{1}{n+1}\right)}$$

$$= \frac{\sin\frac{1}{n}\cos\frac{1}{n+1} - \sin\frac{1}{n+1}\cos\frac{1}{n}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}}$$

$$= \frac{\sin\frac{1}{n}}{\cos\frac{1}{n}} - \frac{\sin\frac{1}{n+1}}{\cos\frac{1}{n+1}} = \operatorname{tg}\frac{1}{n} - \operatorname{tg}\frac{1}{n+1}$$

$$s_n = x_1 + x_2 + \dots + x_n$$

$$= \operatorname{tg} 1 - \cancel{\operatorname{tg} \frac{1}{2}} + \cancel{\operatorname{tg} \frac{1}{2}} - \cancel{\operatorname{tg} \frac{1}{3}} + \dots + \cancel{\operatorname{tg} \frac{1}{n}} - \operatorname{tg} \frac{1}{n+1}$$

$$= \operatorname{tg} 1 - \operatorname{tg} \left(\frac{1}{n+1} \right)$$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(\operatorname{tg} 1 - \operatorname{tg} \left(\frac{1}{n+1} \right) \right) = \operatorname{tg} 1 - \operatorname{tg} 0 = \operatorname{tg} 1$$

Deci $\sum_{n=1}^{\infty} x_n = \operatorname{tg} 1$, prin urmare $\sum x_n$ conv