Seminar 4

ex 1

Studiați convergența (natura) urm. serii

a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}} \cdot x^n$, $x > 0$ (Crit. raportului)

b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $x \in (-1, 1)$ (Absolut convergentă

pt. $\sum |x_n|$ Crit. comp. ineg., $y_n = \frac{1}{n^2}$)

c) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\lambda}}$, $x \in \mathbb{R}$, $\lambda > 0$

(Crit. Abel Dirichlet I, $x_n = \frac{1}{n^{\lambda}}$, $y_n = \cos nx$)

d) $\sum_{n=1}^{\infty} \frac{\cos n \cdot \cos \frac{1}{n}}{n}$

(Crit. Abel Dirichlet II, $x_n = \cos \frac{1}{n}$, $y_n = \frac{\cos n}{n}$)

e) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n} + 1}{n}$

(Crit. Leibniz pt $x_n = \frac{(-1)^n \cdot \sqrt{n}}{n}$)

ex 2

Fie $n \in \mathbb{N}^*$, $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_1(\underset{(x_1, \dots, x_n)}{\underset{\shortparallel}{x}}, \underset{(y_1, \dots, y_n)}{\underset{\shortparallel}{y}}) = |x_1 - y_1| + \dots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|$$

a) Arătați ca d_1 este metrică pe $\mathbb{R}^n$

b) Fie $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in \mathbb{R}^n$

și $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Arătați că $\lim\limits_{k \to \infty} x^k \overset{d_1}{=} x$ dacă și numai

dacă $\lim\limits_{k \to \infty} x_i^k = x_i$, $\forall\, i = \overline{1, n}$

ex 1

Studiați convergența (natura) urm. serii

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}} \cdot x^n$, $x > 0$

Sol:

Fie $x_n = \frac{1}{n\sqrt{n+1}} \cdot x^n$, $x > 0$

Aplicam criteriul raportului

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{x^{\cancel{n+1}}}{(n+1)\sqrt{n+2}} \cdot \frac{n\sqrt{n+1}}{\cancel{x^n}} = x$$

1) Dacă $x < 1$ (i.e. $x \in (0,1)$), $\sum x_n$ conv

2) Dacă $x > 1$ (i.e. $x \in (1,\infty)$), $\sum x_n$ div

3) Dacă $x = 1$, atunci crit. nu decide

Pt $x = 1$, $x_n = \frac{1}{n\sqrt{n+1}}$

Fie $y_n = \frac{1}{n\sqrt{n}}$, $\forall n \in \mathbb{N}^*$

$y_n \geq x_n$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \text{ convergentă}$$

(serie arm. gen, $\alpha = \frac{3}{2}$)

Conform crit de comp. cu ineg avem că $\sum x_n$ conv

$\square$

b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $x \in (-1, 1)$

Sol:

Fie $x_n = \frac{x^n}{n^2}$, $\forall n \in \mathbb{N}^*$

Studiem convergența seriei $\sum_{n=1}^{\infty} |x_n|$

$|x_n| = \left| \frac{x^n}{n^2} \right| = \frac{|x|^n}{n^2}$, $\forall n \in \mathbb{N}^*$

Fie $y_n = \frac{1}{n^2}$, $\forall n \in \mathbb{N}^*$

$|y_n| > |x_n|$

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, conv (serie arm. gen, $\alpha = 2$)

Conform Crit. de comp. cu ineg. avem că $\sum |x_n|$ e conv.

Prin urmare $\sum x_n$ e absolut conv

Așadar $\sum x_n$ conv

c) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\lambda}}$, $x \in \mathbb{R}$, $\lambda > 0$

Sol:

Aplicam Crit. Abel - Dirichlet (I)

Fie $x_n = \frac{1}{n^{\lambda}}$, $\forall n \in \mathbb{N}^*$

$(x_n)_n$ descrescător și $\lim_{n \to \infty} x_n = 0$ (1)

Fie $y_n = \cos nx, \quad \forall n \in \mathbb{N}^*$

Arătăm că $\exists M > 0$ a.î. $\forall n \in \mathbb{N}^*$ avem $|y_1 + y_2 + \dots + y_n| \leq M$

Fie $n \in \mathbb{N}^*$

$|y_1 + y_2 + \dots + y_n| = |\cos x + \cos 2x + \dots + \cos nx|$

M de mai sus NU poate depinde de n, dar poate depinde de x.

Fie $z = \cos x + i \sin x$

$z^2 = \cos 2x + i \sin 2x$

$z^3 = \cos 3x + i \sin 3x$

...

$z^n = \cos nx + i \sin nx$

Observam că $y_1 + y_2 + \dots + y_n = \operatorname{Re}(z + z^2 + \dots + z^n)$

Presupunem că $z \neq 1$, i.e. $\cos x + i \sin x \neq 1$, i.e. $\cos x \neq 1$ sau $\sin x \neq 0$, i.e. $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

Fie $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

$$z + z^2 + \dots + z^n = z \cdot \frac{z^n - 1}{z - 1} = \frac{z^{n+1} - z}{z - 1}$$

$$= \frac{(\cos x + i \sin x)^{n+1} - \cos x - i \sin x}{\cos x + i \sin x - 1}$$

$$= \frac{\cos (n+1)x + i \sin (n+1) - \cos x - i \sin x}{\cos x + i \sin x - 1}$$

$$= \frac{\cos(n+1)x - \cos x + i\,(\sin(n+1)x - \sin x)}{(\cos x - 1) + i \sin x}$$

$$\left\{ \begin{array}{l} \cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2} \\ \sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2} \end{array} \right.$$

$$\left\{ \begin{array}{l} \cos x = \cos\left(2 \cdot \frac{x}{2}\right) = 1 - 2 \sin^2 \frac{x}{2} \Rightarrow \cos x - 1 = -2 \sin^2 \frac{x}{2} \\ \sin x = \sin\left(2 \cdot \frac{x}{2}\right) = 2 \sin \frac{x}{2} \cos \frac{x}{2} \end{array} \right.$$

$$= \frac{-2 \sin \frac{n+2}{2} x \cdot \sin \frac{n}{2} + i \cdot \left(2 \cdot \cos \frac{n+2}{2} x \cdot \sin \frac{n}{2} x\right)}{-2 \sin^2 \frac{x}{2} + 2 i \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \frac{\overset{-i}{\cancel{2}} \sin \frac{n}{2} x \cdot \left(- \sin \frac{n+2}{2} + i \cos \frac{n+2}{2} x\right)}{\cancel{2} \sin \frac{x}{2} \left(- \sin \frac{x}{2} + i \cos \frac{x}{2}\right)}$$

$$= \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \frac{\cos \frac{n+2}{2} x + i \sin \frac{n+2}{2} x}{\cos \frac{x}{2} + i \sin \frac{x}{2}}$$

$$= \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \frac{\left(\cos \frac{x}{2} + i \sin \frac{x}{2}\right)^{\cancel{n+2}\ n+1}}{\cancel{\cos \frac{x}{2} + i \sin \frac{x}{2}}}$$

$$= \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \left(\cos \frac{n+1}{2} x + i \sin \frac{n+1}{2} x\right)$$

Deci $\cos x + \cos 2x + \ldots + \cos nx = \operatorname{Re}\left(z + z^2 + \ldots + z^n\right)$

$$= \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \cos \frac{n+1}{2} x$$

$$\Rightarrow y_1 + y_2 + \dots + y_n = \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \cos \frac{n+1}{2} x$$

$$\Rightarrow |y_1 + y_2 + \dots + y_n| = \left| \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \cos \frac{n+1}{2} x \right|$$

$$= \frac{|\sin \frac{n}{2} x| \cdot |\cos \frac{n+1}{2} x|}{|\sin \frac{x}{2}|} \leq \frac{1}{|\sin \frac{x}{2}|}$$

Alegem $M = \dfrac{1}{|\sin \frac{x}{2}|}$ și avem $|y_1 + y_2 + \dots + y_n| \leq M$ (2)

Conform Crit lui Abel Dirichlet (I) avem că

$\sum x_n \cdot y_n$ conv

Am lucrat doar cu $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

Fie $x \in \{2k\pi, k \in \mathbb{Z}\}$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\lambda}} = \sum_{n=1}^{\infty} \frac{1}{n^{\lambda}} \begin{cases} \text{conv}, & \lambda > 1 \\ \text{div}, & \lambda \leq 1 \end{cases}$$

$\square$

d) $$\sum_{n=1}^{\infty} \frac{\cos n \cdot \cos \frac{1}{n}}{n}$$

<u>**Sol**</u>: Fie $x_n = \cos \frac{1}{n}$

$$y_n = \frac{\cos n}{n}$$

$-1 \leq x_n \leq 1 \Rightarrow (x_n)_n$ mărginit (1)

$$\left.\begin{array}{l} \left(\frac{1}{n}\right)_n \text{ strict descrescator} \\ \\ x \longrightarrow \cos x \quad \text{s. desc.} \\ \uparrow \\ (0, \frac{\pi}{2}) \\ \\ \frac{1}{n} \in (0, 1] \Rightarrow \frac{1}{n} \subset (0, \frac{\pi}{2}) \end{array}\right\} (x_n) \text{ este strict crescator (2)}$$

Deci $(x_n)_n$ monoton și mărginit (1)

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\cos n}{n}$, conv (conform punctului c))

(2) $x=1$, $\lambda=1$

Din (1) și (2), Conform Crit. Abel-Dirichlet (II), avem că $\sum x_n \cdot y_n$ conv $\square$

e) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n} + 1}{n}$

Sol:

Fie $a_n = \frac{(-1)^n \cdot \sqrt{n} + 1}{n}$

$b_n = \frac{(-1)^n \cdot \sqrt{n}}{n}$ $\forall n \in \mathbb{N}^*$

$c_n = \frac{1}{n}$

$a_n = b_n + c_n$

$\sum c_n$ div (serie armonică gen, $\alpha = 1$) (1)

Fie $x_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, $\forall n \in \mathbb{N}^*$

$(x_n)_n$ strict desc. și $\lim_{n\to\infty} x_n = 0$

Conform Crit. lui Leibniz, avem că

$\sum (-1)^n \cdot x_n = \sum (-1)^n \cdot \frac{1}{\sqrt{n}}$ conv (2)

Din (1) ^ (2) avem ca $\sum a_n = \sum \frac{(-1)^n \sqrt{n} + 1}{n}$ div $\square$

ex 2

Fie $n \in \mathbb{N}^*$, $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_1(\underset{(x_1,\dots,x_n)}{x}, \underset{(y_1,\dots,y_n)}{y}) = |x_1 - y_1| + \dots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|$$

a) Arătați că d_1 este metrică pe $\mathbb{R}^n$

Sol:

Fie $x, y, z \in \mathbb{R}^n$

1) $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2| + \dots + |x_n - y_n| \geq 0$

2) $d_1(x, y) = 0 \iff |x_1 - y_1| + \dots + |x_n - y_n| = 0$

$\iff |x_1 - y_1| = \dots = |x_n - y_n| = 0 \iff$

$\iff x_i = y_i \ \forall i = \overline{1,n} \iff x = y$

3) $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |-(y_i - x_i)|$

$= \sum_{i=1}^{n} |y_i - x_i| = d_1(y, x)$

4) $d_1(x, z) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq$

$\leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d(x, y) + d(y, z)$

b) Fie $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in \mathbb{R}^n$

și $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Arătați că $\lim\limits_{k \to \infty} x^k \overset{d_1}{=} x$ dacă și numai

dacă $\lim\limits_{k \to \infty} x_i^k = x_i$, $\forall\, i = \overline{1, n}$

Sol:

„ => " Știm că $\lim\limits_{k \to +\infty} x^k \overset{d_1}{=} x$, deci știm că

$\lim\limits_{k \to +\infty} d(x^k, x) = 0$, deci știm că $\forall\, \varepsilon > 0$, $\exists\, k_\varepsilon \in \mathbb{N}$

a.î. $\forall\, k \geq k_\varepsilon$, avem $d_1(x^k, x) < \varepsilon$

Arătam că $\lim\limits_{k \to +\infty} x_i^k = x_i$, $\forall\, i = \overline{1, n}$, deci

arătăm ca $\forall\, \varepsilon > 0$, $\exists\, k_\varepsilon \in \mathbb{N}$ a.î. $\forall\, k \geq k_\varepsilon$ avem

$|x_i^k - x_i| < \varepsilon$

Fie $i \in \{1, \dots, n\}$

Fie $\varepsilon > 0$

Alegem $k_\varepsilon^i = k_\varepsilon \in \mathbb{N}$

$$\forall\, k \geq \underset{\displaystyle k_\varepsilon}{\overset{\shortparallel}{k_\varepsilon^i}}, \text{ avem } |x_i^k - x_i| \leq \sum_{j=1}^{n} |x_j^k - x_j| = d_1(x^k, x) < \varepsilon$$

Așadar, $\lim\limits_{k \to +\infty} x_i^k = x_i$, $\forall\, i = \overline{1, n}$

„ $\leq$ "

Știm că $\lim_{k \to +\infty} x_i^k = x_i$, $\forall i = \overline{1,n}$, deci știm că $\forall i = \overline{1,n}$, $\forall \varepsilon > 0$, $\exists k_\varepsilon^i \in \mathbb{N}$ a.î. $\forall k > k_\varepsilon^i$, avem $| x_i^k - x_i | < \varepsilon$

Arătam că $\lim_{k \to \infty} x^k \overset{d_1}{=} x$, deci arătam că $\lim_{k \to \infty} d_1(x^k, x) = 0$, deci arătăm că $\forall \varepsilon > 0$, $\exists k_\varepsilon \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon$ avem $| d_1(x^k, x) - 0 | < \varepsilon$

$\| $

$d_1(x^k, x) < \varepsilon$

Fie $\varepsilon > 0$

Alegem $k_\varepsilon = \max \{ k_\varepsilon^1, k_\varepsilon^2, \dots, k_\varepsilon^n \}$

Fie $k \geq k_\varepsilon$

$$d_1(x^k, x) = \underbrace{| x_1^k - x_1 |}_{< \frac{\varepsilon}{n}} + \dots + \underbrace{| x_n^k - x_n |}_{< \frac{\varepsilon}{n}} < \frac{\varepsilon}{n} \cdot n < \varepsilon$$

Deci $\lim_{k \to +\infty} d(x^k, x) = 0$ (i.e. $\lim_{k \to +\infty} x^k \overset{d_1}{=} x$)

$\square$

<u>Obs</u>

Din punctul b) al exc. anterior, deducem că în spațiul metric $(\mathbb{R}^n, d_1)$, convergența șirurilor este echivalentă cu cea a componentelor, aceasta din urmă fiind convergentă pe $\mathbb{R}$