GAL Seminar 6

ex 1

a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = ?$ Vandermonde

Vectori proprii și valori proprii

ex 2

Să se calculeze valorile proprii și vectorii proprii

a) $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $T(x,y) = (2x, 2y) = 2(x,y)$

b) $A = \begin{pmatrix} 3+i & -1 \\ 2i & 1-i \end{pmatrix}$

c) $A \in \mathcal{M}_2(\mathbb{R})$, $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

d) $A = \begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$

ex 3

$T: \mathbb{R}^3 \to \mathbb{R}^3$ a.î. $M = A$

$B = \{(1,1,1), (1,1,0), (1,0,1)\}$ bază în $\mathbb{R}^3$

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

matrice diagonalizată

ex 1

a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = ?$

Vandermonde

Sol:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \xrightarrow[C_3 - C_1]{C_2 - C_1} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c^2-a^2) - (c-a)(b^2-a^2)$$

$$= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a)$$

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b) \quad \square$$

Vectori proprii și valori proprii

$T : L \to L$ (aplicație / operator liniar)

x vector propriu dacă: $\begin{cases} x \neq 0 \\ T(x) = \lambda \cdot x \end{cases}$

unic

λ se n. valoare proprie a lui x

$\sigma(T) = \{$ valorile proprii ale lui $T \}$

$\lambda \in \sigma(T)$: $L_x = \{ x \in L \mid Tx => x \}$

$\sigma(T) = \{ \lambda \in K \mid \det(\lambda\, id\ T) = 0 \}$

↳ polinomul caracteristic

ex 2

Să se calculeze valorile proprii și vectorii proprii

a)

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad T(x,y) = (2x, 2y) = 2(x,y)$$

Sol:

$\sigma(A) = 2$

$T\vec{v} = 2 \cdot \vec{v}$; $\vec{v} = (x,y)$

$V_2 = \mathbb{R}^2$

$\square$

b)

$$A = \begin{pmatrix} 3+i & -1 \\ 2i & 1-i \end{pmatrix}$$

$$P(\lambda) = \begin{vmatrix} \lambda - (3+i) & 1 \\ -2i & \lambda - (1-i) \end{vmatrix}$$

$= [\lambda - (3+i)][\lambda - (1-i)] + 2i$

$= \lambda^2 - \lambda + \cancel{i\lambda} - 3\lambda + 3 + 3i - \cancel{i\lambda} + i - i^2 + 2i$

$= \lambda^2 - 4\lambda + 4 - 2i + 2i$

$= \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 \Rightarrow \lambda = 2$

$$V_2 = \begin{cases} (3+i)x - y = 2x \\ 2i\,x + (1-i)\,y = 2y \end{cases}$$

$$\Leftrightarrow \begin{cases} (1+i)x - y = 0 \\ 2i\,x + (-1-i)\,y = 0 \end{cases}$$

$$\Leftrightarrow \quad y = (1+i)\,x$$

$$V_2 = \langle 1, 1+i \rangle \qquad \square$$

c) $A \in \mathcal{M}_2(\mathbb{R})$, $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$+$

Sol:

$$P_A(\lambda) = \begin{vmatrix} \lambda - \cos\theta & -\sin\theta \\ \sin\theta & \lambda - \cos\theta \end{vmatrix}$$

$$= (\lambda - \cos\theta)^2 - \sin\theta \cdot (-\sin\theta)$$

$$= \lambda^2 - 2\lambda\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}$$

$$= \lambda^2 - 2\lambda\cos\theta + 1$$

$$\Delta = 4\cos^2\theta - 4 = 4(\cos^2\theta - 1) \leq 0$$

$$\theta \neq 0, \pi \ ; \quad \Delta < 0 \implies \sigma(A) = \emptyset$$

d)

$$A = \begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

+

$$P_A(\lambda) = \begin{vmatrix} \lambda-4 & 1 & 2 \\ -2 & \lambda-1 & 2 \\ -1 & 1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-4)(\lambda^2-2\lambda+1) - 2 - 4 + 2(\lambda-1) - 2(\lambda-4) + 2(\lambda-1)$$

$$= \lambda^3 - 2\lambda^2 + \underline{\lambda} - 4\lambda^2 + \underline{8\lambda} - 4 - \cancel{6} + \cancel{2\lambda} - \cancel{2} - \cancel{2\lambda} + \cancel{8} + \underline{2\lambda} - 2$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

$$= \lambda^3 - 3\lambda^2 - 3\lambda^2 + 9\lambda + 2\lambda - 6$$

$$= \lambda^2(\lambda-3) - 3\lambda(\lambda-3) + 2(\lambda-3)$$

$$= (\lambda-3)(\lambda^2 - 3\lambda + 2)$$

$$= (\lambda-3)(\lambda-2)(\lambda-1)$$

$$\Rightarrow \sigma(A) = \{1, 2, 3\}$$

$\lambda = 1$

$$\begin{cases} 4x - y - 2z = 1 \cdot x \\ 2x + y - 2z = 1 \cdot y \\ x - y + z = 1 \cdot z \end{cases}$$

$$\begin{cases} 3x - y - 2z = 0 \\ 2x - 2z = 0 \quad = x = z \qquad \Rightarrow \quad x = y = z \\ x - y = 0 \quad \Rightarrow x = y \end{cases}$$

$\lambda = 2$

$$\begin{cases} 4x - y - 2z = 2 \cdot x \\ 2x + y - 2z = 2 \cdot y \\ x - y + z = 2 \cdot z \end{cases} \qquad \begin{cases} 2x - 2y - 2z = 0 \\ 2x - y - 2z = 0 \\ x - y - z = 0 \end{cases} \quad =$$

$$\begin{cases} x - y - z = 0 \\ 2x - y - 2z = 0 \end{cases} \quad (-)$$

$$x - z = 0 \Rightarrow x = z$$

$$y = 0$$

$\lambda = 3$

$$\begin{cases} 4x - y - 2z = 3 \cdot x \\ 2x + y - 2z = 3 \cdot y \\ x - y + z = 3 \cdot z \end{cases} \qquad \begin{cases} x - y - 2z = 0 \\ 2x - 2y - z = 0 \\ x - y - 2z = 0 \end{cases} \quad =$$

$$\begin{cases} x - y - 2z = 0 & | \cdot -2 \\ 2x - 2y - z = 0 \end{cases} \quad (+)$$

$$3z = 0 \Rightarrow z = 0 \Rightarrow x - y = 0$$

$$x = y$$

ex 3

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$B = \{(1,1,1), (1,1,0), (1,0,1)\}$ bază în $\mathbb{R}^3$

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

matrice diagonalizată

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$