Drepte și plane în $\mathbb{R}^3$

Drepte

$A = (x_1, y_1, z_1)$

$B = (x_2, y_2, z_2)$

$AB : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = u.$

$AB : \begin{cases} x = x_1 + lt \\ y = y_1 + mt \\ z = z_1 + nt \end{cases}$ unde $\begin{matrix} l = (x_2 - x_1) \\ m = y_2 - y_1 \\ n = z_2 - z_1 \end{matrix}$

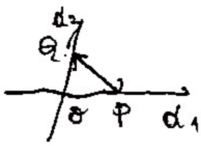
$\overrightarrow{AB} = (l, m, n) = \mathrm{dir}(d) \neq (0,0,0)$

$\mathrm{dir}(d) = \overrightarrow{AB}$

$d_i : \frac{x - x_i}{l_i} = \frac{y - y_i}{m_i} = \frac{z - z_i}{n_i}, \ i = \overline{1,2}$

a) $d_1 \parallel d_2 \iff (l_1, m_1, n_1) = \alpha (l_2, m_2, n_2), \ \alpha \neq 0$

b) $d_1 \cap d_2$



$\begin{cases} \overrightarrow{OP} \in \mathrm{dim}(d_1) \setminus \{0\} \\ \overrightarrow{OQ} \in \mathrm{dim}(d_2) \setminus \{0\} \end{cases}$ $\quad P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} \in \langle \{\overrightarrow{OP}, \overrightarrow{OQ}\} \rangle \iff \{\overrightarrow{PQ}, \overrightarrow{OP}, \overrightarrow{OQ}\}$ SLD

$\{\overrightarrow{OP}, \overrightarrow{OQ}\}$ SLI

$\Delta = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \Rightarrow d_1 \cap d_2 = \{\emptyset\}$

c) $D \neq 0 \Rightarrow d_1, d_2$ necoplanare

d) $d_1 \perp d_2 \iff \mathrm{dir}(d_1) \perp \mathrm{dir}(d_2)$

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

APLICAȚII

A1. Fie $P = (1,1,1)$ și $d : \frac{x+1}{2} = \frac{y-1}{3} = \frac{z}{0}$

a) Să se det. $\parallel$ prin P la d

$d : \frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{0}$

b) ~~proiecția~~ perp. (⊥) din P pe d

$\underline{Q(t)} \in d \qquad Q(t) = (2t-1, 3t+1, 0)$

$\overrightarrow{PQ}(t) = (2t-2, 3t, -1)$

$\langle \overrightarrow{PQ}(t), (2,3,0) \rangle = 0 \iff 4t - 4 + 9t = 0 \Rightarrow t = \frac{4}{13}$

b) (continuare)

$Q=\left(-\frac{5}{13},\frac{25}{13},0\right)$ $\quad PQ=\left(-\frac{18}{13},\frac{12}{13},-1\right)$

$PQ: \frac{x-1}{-\frac{18}{13}}=\frac{y-1}{\frac{12}{13}}=\frac{z-1}{-1} \iff PQ: \frac{x-1}{-18}=\frac{y-1}{12}=\frac{z-1}{13}$

c) calc $d(P,d)$

$d(P,d)=|PQ|=\|PQ\|=\frac{1}{13}\sqrt{324+144+169}=\frac{1}{13}\sqrt{468+169}=\frac{\sqrt{637}}{13}=\frac{7\sqrt{13}}{13}=\frac{7}{\sqrt{13}}$

$\begin{array}{r|l} 637 & 7 \\ 91 & 7 \\ 13 & 13 \end{array}$ $\quad 7\sqrt{13}$

Plane

$\pi: ax+by+cz+d=0 \quad ((a,b,c)\neq(0,0,0))$

$P\in\pi \quad P=(x_0,y_0,z_0) \quad \pi: a(x-x_0)+b(y-y_0)+c(z-z_0)$

$\pi=\{(x,y,z)\mid(x-x_0,y-y_0,z-z_0)\perp(a,b,c)\}$

$Dir(\pi)=(a,b,c)^{\perp}$, unde (a,b,c) s.n. parametrii normali (director)

$\pi_i: a(x-x_i)+b(y-y_i)+c(z-z_i)=0$

a) $\pi_1 \parallel \pi_2 \iff$ are $(a_1,b_1,c_1)=\lambda(a_2,c_2,b_2)$ $\lambda\neq0$

b) $\pi_1\perp\pi_2 \iff a_1a_2+b_1b_2+c_1c_2=0$

Fie

dr. $d: \frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ a) $d\parallel\pi_1 \iff dir(d)\in Dir(\pi)\iff(a,b,c)\perp(l,m,n)\iff al+bm+cn=0$

b) $d\subset\pi_1\iff d\parallel\pi_1$ și $a_1(x_1-x_0)+b_1(y_1-x_0)+c_1(z_1-z_0)=0$

c) $d\perp\pi_1\iff dir(d)\parallel(a,b,c)\iff(l,m,n)=r(a,b,c)$

APLICAȚII

A_2 Fie $P=(1,1,1)$, $\pi: x-y+2z=3$

a) Det. proiecția pe dir P pe π

$d: \frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-1}{+2}=t \Rightarrow Q=(t+1,-t+1,2t+1)\in\pi$

b) det $Q=d\cap\pi$

$t+1-(-t+1)+2(2t+1)=3 \iff 6t=-2\ldots t=+\frac{1}{6} \Rightarrow Q=\left(\frac{7}{6},\frac{5}{6},\frac{8}{6}\right)$

c) det $d(P,\pi)$

$d(P,\pi)=|P\pi|=\sqrt{t^2+t^2+4t^2}=|t|\sqrt{6}=\frac{1}{6}\sqrt{6}=\frac{1}{\sqrt{6}}$

Formula:

$\pi: ax+by+cz+d=0 \quad P=(x_0,y_0,z_0)$

$$\boxed{d(P,\pi) = \frac{|ax_0+by_0+cz_0|}{\sqrt{a^2+b^2+c^2}}}$$

Exemplu (pt. A_2)

$$d(P,\pi) = \frac{|1-1+2-3|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

A3

Fie $P=(1,1,1)$, $d: \frac{x-2}{3} = \frac{y+1}{1} = \frac{z-2}{0}$

a) det. // la d (d')

$d': \frac{x-1}{3} = \frac{y-1}{1} = \frac{z-1}{0}$

b) det. pl. perpend. $\pi \ni P$, $d \subset \pi$

$a(x-1)+b(y-1)+c(z-1)=0$

$d \parallel \pi \quad 3a+b=0$

$d \subset \pi \quad a(2-1)+b(-1-1)+c(2-1)=0 \Rightarrow \begin{cases} a-2b+c=0 \\ 3a+b=0 \end{cases} \Rightarrow \begin{cases} a-2b+c=0 \\ 7b-3c=0 \end{cases} \Rightarrow b=\frac{3c}{7}$

$a-\frac{6c}{7}+c=0 \Rightarrow$

$a=-\frac{c}{7}$

$\Rightarrow \pi: -\frac{c}{7}(x-1)+\frac{3c}{7}(y-1)+c(z-1)=0 \quad (c\neq 0)$

$\Rightarrow \pi: -(x-1)+3(y-1)+7(z-1)=0$

$\pi: -x+3y+7z-9=0 \Rightarrow \pi: -x+3y+7z=9$

c) Să se det. $d \vee d'$ (planul gen. de d și d')

$d \vee d': a(x-1)+b(y-1)+c(z-1)=0$

$\left.\begin{cases} 3a+b \\ a-2b+c=0 \end{cases}\right\} \Rightarrow P \vee d = d \vee d'$

A4

Fie $d_1: \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{0}$, $d_2: \frac{x+1}{1} = \frac{y-2}{0} = \frac{z}{1}$

a) Ar. că d_1 și d_2 necop. $d_1 \cap d_2 = \{0\}$ (i.e. det $\neq 0$) $\quad P_1=(1,-1,0) \in d_1$

$\quad P_2=(-1,2,0) \in d_2$

$$\Rightarrow \Delta = \begin{vmatrix} 2 & -3 & 0 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} -d_1(2,0,1) \\ -d_2 \\ =12\neq 0 \end{matrix}$$

b) det planul $\pi \supset d_1$, $\pi \parallel d_2$

$a(x-1)+b(y+1)+cz=0$

$\begin{cases} 2a+3b=0 \\ a+c=0 \end{cases} \Rightarrow \begin{cases} a+c=0 \\ 3b=2c \end{cases} \Rightarrow \begin{cases} b=\frac{2c}{3} \\ a=-c \end{cases} \Rightarrow \pi: 3(x-1)+2(y+1)-z=0$

$\Rightarrow 3x+2y-z=1$

c) dist. ⊥ (perpend) comună și $d(d_1,d_2)$ $(\exists)!$ și e UNICĂ $\Leftrightarrow d_1 \cap d_2 \neq \{\emptyset\} \Rightarrow$ coplanare)

$P(\underline{s}) \in d_1$, $Q(t) \in d_2$ a.î. $\overrightarrow{P(\underline{s})Q(t)} \perp d_1, d_2$

$P(\underline{s}) = (2\underline{s}+1, 3\underline{s}-1, 0)$

$Q(t) - (t-1, 2, t)$

$\overrightarrow{P(\underline{s})Q(t)} = (t-2\underline{s}-2, 3-3\underline{s}, t)$

$2t-4\underline{s}-4-9\underline{s}+9=0 \Leftrightarrow 2t-13\underline{s}=-5$

$\overrightarrow{PQ} = \left(-\frac{18}{11}, \frac{12}{11}, \frac{18}{11}\right)$

$\Rightarrow$ sist $\begin{cases} t-\underline{s}=1 \\ 2t-13\underline{s}=-5 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} -11\underline{s}=-7 \\ \underline{s}=\frac{7}{11} \end{cases}$

$t=\frac{18}{11}$

$d(d_1,d_2) = \|\overrightarrow{PQ}\| = \frac{1}{11}\sqrt{18^2+12^2+18^2} = \frac{1}{11}\sqrt{18^2\cdot 2+12^2} = \frac{1}{11}\sqrt{2\cdot 324+144} = \frac{1}{11}\sqrt{648+144}$

$= \frac{1}{11}\sqrt{792} = \frac{1}{11}\sqrt{11\cdot 72} = \frac{6\sqrt{2}}{\sqrt{11}}$

$P = \left(\frac{14}{11}+1, \frac{21-11}{11}, 0\right) = \left(\frac{25}{11}, \frac{10}{11}, 0\right)$, $Q = \left(\frac{18-11}{11}, 2, \frac{18}{11}\right) = \left(\frac{7}{11}, 2, \frac{18}{11}\right)$

$PQ: \frac{x-\frac{25}{11}}{-3} = \frac{y-\frac{10}{11}}{2} = \frac{z}{3}$

(2) (-3) (0)