Seminar 2

ex 1

Determinați $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și precizați dacă $(\exists) \lim\limits_{n\to\infty} x_n$ unde:

a) $x_n = 1 + 2\cdot(-1)^{n+1} + 3\cdot(-1)^{\frac{n(n+1)}{2}}$, $\forall\, n \in \mathbb{N}^*$

b) $x_n = \left(1+\frac{1}{n}\right)^n \sin\frac{n\pi}{3}$, $\forall\, n \in \mathbb{N}^*$

c) $x_n = \dfrac{n\cdot \cos\frac{n\pi}{2}}{n^2+1}$, $\forall\, n \in \mathbb{N}$

ex 2

Determinați suma seriei $\sum\limits_{n=1}^{\infty} \frac{n}{(n+1)!}$ și precizați dacă este convergentă

ex 3

Studiați convergența (natura) urm. serii:

a) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n-1}}{n^2}$ (Crit. comp. ineg, $y_n = \frac{\sqrt{n}}{n^2}$, conv)

b) $\sum\limits_{n=1}^{\infty} \dfrac{a^n}{\sqrt[n]{n}}$, $a>0$ (Crit. raportului; Crit. nec. div)

c) $\sum\limits_{n=1}^{\infty} \left(\dfrac{an^2+3n+2}{2n^2+n+1}\right)^n$, $a>0$ (Crit. radicalului, Crit. nec. div)

d) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$ (Crit. comp. limita, $y_n = \frac{\sqrt{n^2}}{\sqrt{n^3}}$)

ex 1

Determinați $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și precizați

dacă $(\exists) \lim\limits_{n\to\infty} x_n$ unde:

a) $x_n = 1 + 2\cdot(-1)^{n+1} + 3\cdot(-1)^{\frac{n(n+1)}{2}}$, $\forall\, n \in \mathbb{N}^*$

Sol:

$$x_{4k} = 1 + 2\cdot(-1)^{4k+1} + 3\cdot(-1)^{\frac{\cancel{4}^{2}k(4k+1)}{\cancel{2}}}$$

$$= 1 - 2 + 3 = 2 \xrightarrow[k\to\infty]{} 2$$

$$x_{4k+1} = 1 + 2\cdot(-1)^{4k+2} + 3\cdot(-1)^{\frac{(4k+1)\cdot(4k+2)}{2}}$$

$$= 1 + 2 - 3 = 0 \xrightarrow[k\to\infty]{} 0$$

$$x_{4k+2} = 1 + 2\cdot(-1)^{4k+3} + 3\cdot(-1)^{\frac{(4k+2)(4k+3)}{2}}$$

$$= 1 - 2 - 3 = -4 \xrightarrow[k\to\infty]{} -4$$

$$x_{4k+3} = 1 + 2\cdot(-1)^{4k+4} + 3\cdot(-1)^{\frac{(4k+3)(4k+4)}{2}}$$

$$= 1 + 2 + 3 = 6 \xrightarrow[k\to\infty]{} 6$$

$$\mathbb{N} = 4\mathbb{N} \cup (4\mathbb{N}+1) \cup (4\mathbb{N}+2) \cup (4\mathbb{N}+3)$$

$$L((x_n)_n) = \{-4, 0, 2, 6\}$$

$$\left.\begin{array}{l} \underline{\lim}\, x_n = -4 \\ \overline{\lim}\, x_n = 6 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \underline{\lim}\, x_n \neq \overline{\lim}\, x_n \Rightarrow \nexists \lim_{n\to\infty} x_n \quad \square$$

b) $x_n = \left(1+\frac{1}{n}\right)^n \sin\frac{n\pi}{3}, \quad \forall n \in \mathbb{N}^*$

<u>Sol</u>:

$$x_{6k} = \left(1+\frac{1}{6k}\right)^{6k} \cdot \sin\frac{6k\pi}{3} = \left(1+\frac{1}{6k}\right)^{6k} \cdot \sin 2\pi$$

$$= \xrightarrow[k\to\infty]{} e \cdot 0 = 0$$

$$x_{6k+1} = \left(1+\frac{1}{6k+1}\right)^{6k+1} \cdot \sin\frac{(6k+1)\pi}{3}$$

$$= \left(1+\frac{1}{6k+1}\right)^{6k+1} \cdot \sin\left(2k\pi+\frac{\pi}{3}\right) \xrightarrow[k\to\infty]{} e\cdot\frac{\sqrt{3}}{2}$$

$$x_{6k+2} = \left(1+\frac{1}{6k+2}\right)^{6k+2} \cdot \sin\frac{(6k+2)\pi}{3} \xrightarrow[k\to\infty]{} e\cdot\sin\frac{2\pi}{3} = e\cdot\frac{\sqrt{3}}{2}$$

$$x_{6k+3} = \left(1+\frac{1}{6k+3}\right)^{6k+3} \cdot \sin(2k\pi+\pi) \xrightarrow[k\to\infty]{} e\cdot\sin\pi = 0$$

$$x_{6k+4} = \left(1+\frac{1}{6k+4}\right)^{6k+4} \sin\left(2k\pi+\frac{4}{3}\pi\right)$$

$$= \left(1+\frac{1}{6k+4}\right)^{6k+4} \sin\left(\pi+\frac{\pi}{3}\right)$$

$$= \left(1+\frac{1}{6k+4}\right)^{6k+4} \cdot \left(-\sin\frac{\pi}{3}\right) \xrightarrow[k\to\infty]{} e\cdot -\frac{\sqrt{3}}{2}$$

$$x_{6k+5} = \left(1+\frac{1}{6k+5}\right)^{6k+5} \cdot \sin\left(2k\pi+\frac{5\pi}{3}\right)$$

$$= \left(1+\frac{1}{6k+5}\right)^{6k+5} \cdot \sin\left(2\pi-\frac{\pi}{3}\right)$$

$$= \left(1+\frac{1}{6k+5}\right)^{6k+5} \cdot \left(-\sin\frac{\pi}{3}\right) \xrightarrow[k\to\infty]{} -\frac{e\sqrt{3}}{2}$$

$$\mathbb{N}^* = (6\mathbb{N}^*) \cup (6\mathbb{N}+1) \cup (6\mathbb{N}+2) \cup (6\mathbb{N}+3)$$
$$\cup (6\mathbb{N}+4) \cup (6\mathbb{N}+5)$$

$$L((x_n)_n) = \left\{-\frac{e\sqrt{3}}{2}, 0, \frac{e\sqrt{3}}{2}\right\}$$

$$\left.\begin{aligned} \underline{\lim}\, x_n &= -\frac{e\sqrt{3}}{2} \\ \overline{\lim}\, x_n &= \frac{e\sqrt{3}}{2} \end{aligned}\right\} \quad \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow \nexists \lim_{n\to\infty} x_n$$

$\square$

c) $x_n = \dfrac{n \cdot \cos \frac{n\pi}{2}}{n^2+1}, \quad \forall n \in \mathbb{N}$

Sol:

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{n}{n^2+1} \cdot \cos \frac{n\pi}{2}$$

$$\left.\begin{array}{l} \lim\limits_{n \to \infty} \dfrac{n}{n^2+1} = 0 \\ \\ -1 \le \cos \dfrac{n\pi}{2} \le 1 \end{array}\right\} \underset{\text{„zero · mărginit”}}{\Rightarrow} \lim_{n \to \infty} \left(\frac{n}{n^2+1} \cdot \cos \frac{n\pi}{3} \right) = 0$$

$$\Rightarrow \lim_{n \to \infty} x_n = 0 \quad \Rightarrow \quad \overline{\lim}\, x_n = \underline{\lim}\, x_n = 0$$

$\square$

ex 2

Determinați suma seriei $\sum\limits_{n=1}^{\infty} \dfrac{n}{(n+1)!}$ și precizați dacă este convergentă

Sol:

$$x_n = \frac{n}{(n+1)!}, \quad \forall n \in \mathbb{N}$$

$$\begin{aligned} s_n &= x_1 + x_2 + \dots + x_n \\ &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} \\ &= \frac{2-1}{2!} \quad \frac{3-1}{3!} \quad \frac{4-1}{4!} \quad \frac{(n+1)-n}{(n+1)!} \\ &= \frac{2}{2!} - \frac{1}{2!} + \frac{3}{3!} - \frac{1}{3!} + \frac{4}{4!} - \frac{1}{4!} + \dots + \frac{n+1}{(n+1)!} - \frac{n}{(n+1)!} \\ &= 1 - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} + \cancel{\frac{1}{3!}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n!}} - \frac{n}{(n+1)!} \\ &= 1 - \frac{n}{(n+1)!}, \quad \forall n \in \mathbb{N} \end{aligned}$$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{n}{(n+1)!} \right) = 1$$

Deci $\sum\limits_{n=1}^{\infty} x_n = 1$. Prin urmare $\sum\limits_{n=1}^{\infty} x_n$ conv. $\square$

ex 3

Studiați convergența (natura) urm. serii:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n^2}$

Sol:

Fie $x_n = \dfrac{\sqrt{n-1}}{n^2}$, $\forall n \in \mathbb{N}^*$

$y_n = \dfrac{\sqrt{n}}{n^2}$, $\forall n \in \mathbb{N}$

$x_n < y_n \quad \forall n \in \mathbb{N}^*$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}, \text{ convergentă}$$

(serie armonică generalizată cu $\alpha = \frac{3}{2}$)

Conform Criteriului de comp. cu inegalități, avem că $\displaystyle\sum_{n=1}^{\infty} x_n$ e conv. $\square$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{\sqrt[n]{n}}$, $a > 0$

Sol:

Fie $x_n = \dfrac{a^n}{\sqrt[n]{n}}$, $a > 0$, $\forall n \in \mathbb{N}$

Aplicam Criteriul raportului

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{\cancel{n+1}}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{\cancel{a^n}} = \lim_{n\to\infty} a \cdot \frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}} = a$$

1

1) Dacă $a < 1$ (i.e. $a \in (0,1)$) atunci $\sum x_n$ conv

2) Dacă $a > 1$ (i.e. $a \in (1, \infty)$) atunci $\sum x_n$ div

3) Dacă $a = 1$ atunci crit. nu decide

Pentru $a = 1$, avem $x_n = \frac{1}{\sqrt[n]{n}}$, $\forall n \in \mathbb{N}^*$

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{\underbrace{\sqrt[n]{n}}_{\to 1}} = \frac{1}{1} = 1 \neq 0$$

Conform Crit. suficient de divergență, avem că $\sum x_n$ e div

Am obținut

$$\sum x_n \begin{cases} \text{conv}, & \text{pt. } a \in (0,1) \\ \text{div}, & \text{pt } a \in [1, \infty) \end{cases}$$

c) $\sum_{n=1}^{\infty} \left(\frac{a n^2 + 3n + 2}{2n^2 + n + 1} \right)^n$, $a > 0$

Sol:

Fie $x_n = \left(\frac{a n^2 + 3n + 2}{2n^2 + n + 1} \right)^n$

Aplicam Criteriul radicalului

$$\lim_{n \to \infty} \sqrt[n]{x_n} = \lim_{n \to \infty} \sqrt[n]{\left(\frac{a n^2 + 3n + 2}{2n^2 + n + 1} \right)^n} = \lim_{n \to \infty} \frac{a n^2 + 3n + 2}{2n^2 + n + 1}$$

$$= \frac{a}{2}$$

1) Dacă $\frac{a}{2} < 1$ (i.e. $a \in (0,2)$), $\sum x_n$ conv

2) Dacă $\frac{a}{2} > 1$ (i.e. $a \in (2,\infty)$), $\sum x_n$ div

3) Dacă $\frac{a}{2} = 1$ atunci acest crit. nu decide

Pentru $a = 2$, $x_n = \left(\frac{2n^2+3n+2}{2n^2+n+1} \right)^n$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(1 + \frac{2n+1}{2n^2+n+1}\right)^n$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{2n+1}{2n^2+n+1}\right)^{\frac{2n^2+n+1}{2n+1}} \right]^{\frac{2n^2+n}{2n^2+n+1}}$$

$$= e^{\lim\limits_{n\to\infty} \frac{2n^2+n}{2n^2+n+1}} = e^1 = e \neq 0$$

Conform Crit. suficient de divergență, avem că $\sum x_n$ e div

Am obținut

$$\sum x_n \begin{cases} \text{conv}, & \text{pt. } a \in (0,2) \\ \text{div}, & \text{pt } a \in [2,\infty) \end{cases}$$

d) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$

Sol:

Fie $x_n = \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$, $y_n = \frac{\sqrt{n^2}}{\sqrt{n^3}}$, $\forall n \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}} \cdot \frac{\sqrt{n^3}}{\sqrt{n^2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^5+n^3}{n^5+n^2}} = 1 \in (0,\infty)$$

Conform criteriul de comp. cu limită avem că

$\sum x_n \sim \sum y_n$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$, div

(serie armonica gen, $\alpha = \frac{1}{2}$)

Deci avem ca $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$ div