Seminar 14

R inel

ideal, inel factor

morf de inele

Inel

$$\begin{cases} (R, +) \text{ grup abelian} \\ (R, \cdot) \text{ monoid} \\ a(b+c) = ab + ac \\ (b+c)\cdot a = b\cdot a + c \cdot a \end{cases}$$

$a \cdot b = b \cdot a \quad \forall a, b \Rightarrow R$ inel com.

De acum înainte R e inel com

$(R, +, \cdot)$ inel com

$$\emptyset \neq I \trianglelefteq R \iff \begin{cases} 1) \ (I, +) \leq (R, +) \quad \left(\iff \forall a, b \in I \quad a - b \in I\right) \\ 2) \ \forall a \in I \quad \forall r \in R \Rightarrow a \cdot r \in I \end{cases}$$

$R/_I \to$ inel factor (modulo I)

$$\hat{a} = \hat{b} \iff a - b \in I$$

$$R/_I = \{ \hat{a} \mid a \in R \}$$

$(R/_I, +, \cdot)$ $\quad \hat{a} + \hat{b} \overset{def}{=} \widehat{a+b}$

$\quad \hat{a} \cdot \hat{b} \overset{def}{=} \widehat{a \cdot b}$

$f : (R, +, \cdot) \to (S, +, \cdot)$ morfism de inel

$$f(a+b) = f(a) + f(b) \quad \forall a, b \in R$$

$$f(a \cdot b) = f(a) \cdot f(b)$$

$$f(1_R) = 1_S$$

$\mathbb{Q}[\sqrt{5}] = \{ a + b\sqrt{5} \mid a, b \in \mathbb{Q} \}$

$(\mathbb{Q}[\sqrt{5}], +, \cdot) \to$ corp

ex 1

Det morfismele de inele de la $\mathbb{Q}$ in $\mathbb{Q}$, $\mathbb{Q}[\sqrt{5}]$ in $\mathbb{Q}[\sqrt{5}]$

Aratati că $f : \mathbb{Z}[i] \to \mathbb{Z}_2 \quad f(a+bi) = \widehat{a+b}$ e morfism de inele

ex 2

Det nr. de el. ale inelului factor $\mathbb{Z}[i]/_{(3)}$

ex 3

$$a_n X^n + a_{n-1} X^{n-1} + \dots + a_0, \quad a_n \neq 0$$

Fie R inel com, $f(x) \in R[X]$ având coef. dominant inversabil în R ($a_n \in U(R)$)

$$R[x]/_{f(x)} = \{ b_{n-1} x^{n-1} + \dots + b_1 x + b_0 \mid b_0, \dots, b_{n-1} \in R \}$$

($\{ b_{n-1} x^{n-1} + \dots + b_1 x + b_0 \mid b_0, \dots, b_{n-1} \in R \}$ e un S.C.R. pt $R[x]/_{(f(x))}$)

ex 1

$f : \mathbb{Q} \to \mathbb{Q}$ morfism de inele $\Rightarrow f = 1_{\mathbb{Q}}$

Separam în întregi și fracționare

Din definiția morfismului avem $f(1) = 1$

$\underline{\mathbb{Z}}$ $n \in \mathbb{N}$ Inducție după n $f(n) = n \cdot f(1) = n$

$$f(n) = f(1 + \dots + 1) = f(1) + \dots + f(1)$$

$$f(0) = f(0+0) = 2f(0) \Rightarrow f(0) = 0$$

$n \in \mathbb{Z}$

$n < 0 \Rightarrow -n > 0$

$$f(n + (-n)) \overset{f}{\underset{\text{morf}}{=}} f(n) + f(-n)$$

$\shortparallel$

$$f(0) = 0$$

$$0 = f(n) + (-n) \Rightarrow f(n) = n \quad \forall n < 0$$

$$\Rightarrow f(n) = n \in \mathbb{Z}$$

Fie $\frac{m}{n} \in \mathbb{Q}$ $\left(\begin{array}{l} m, n \in \mathbb{Z} \\ (m,n) = 1 \end{array}, \; n \neq 0 \right)$

$$f(\underbrace{\tfrac{1}{n} + \tfrac{1}{n} + \dots + \tfrac{1}{n}}_{n \text{ ori}}) \underset{\text{morf}}{\overset{f}{=}} f(\tfrac{1}{n}) + \dots + f(\tfrac{1}{n})$$

$$1 = f(1) = n \cdot f(\tfrac{1}{n}) \Rightarrow f(\tfrac{1}{n}) = \tfrac{1}{n}$$

$$f(\tfrac{m}{n}) = \tfrac{m}{n} \Leftarrow \tfrac{m}{n} = \underbrace{\tfrac{1}{n} + \tfrac{1}{n} + \dots + \tfrac{1}{n}}_{m \text{ ori}}$$

c)

$$f : \underset{\{a+bi \mid a,b \in \mathbb{Z}\}}{\overset{\shortparallel}{\mathbb{Z}[i]}} \to \mathbb{Z}_2, \quad f(a+bi) = \widehat{a+b} \quad (\forall)$$

3) $f(1) = f(1 + 0 \cdot i) = \widehat{1+0} = \hat{1}$

1) $f((a+bi) + (c+di)) = f((a+c) + (b+d)i) = \widehat{(a+c) + (b+d)}$

$a+bi, \; c+di \in \mathbb{Z}[i]$

$$= \widehat{(a+b) + (c+d)}$$

$$= \widehat{a+b} + \widehat{c+d}$$

$$= f(a+bi) + f(c+di)$$

$$\Rightarrow f\big((a+bi)+(c+di)\big) = f(a+bi) + f(c+di) \quad \forall$$

$a+bi \in$
$c+di \in \mathbb{Z}$

2) $$f\big((a+bi)\cdot(c+di)\big) = f\big((ac-bd) + i(ad+bc)\big) =$$

$$= \widehat{(ac-bd) + (ad+bc)}$$

$$f(a+bi)\cdot f(c+di) \overset{\text{def } f}{=} \widehat{a+b}\cdot\widehat{c+d} \underset{\text{înmulțirii factor}}{\overset{\text{def}}{=}}$$

$$\underset{\text{înm factor}}{\overset{\text{def}}{=}} \widehat{(a+b)\cdot(c+d)}$$

$$= \widehat{ac + bd + bc + ad}$$

$$\widehat{-bd} = \widehat{bd} \quad \text{în } \mathbb{Z}_2$$

b)

subinel

$$\mathbb{Q}[\sqrt{d}] = \{ a + b\sqrt{d} \mid a, b \in \mathbb{Q} \} \subseteq \mathbb{R}$$

$(\mathbb{Q}[\sqrt{5}], +, \cdot)$ corp ! Exc

Ar. că orice el. nenul e inversabil

$$0 \neq a + b\sqrt{5} \qquad = \frac{a - b\sqrt{5}}{a^2 - 5b^2} = \underbrace{\frac{a}{a^2-5b^2}}_{\in\, \mathbb{Q}} + \sqrt{5}\,\underbrace{\frac{-b}{a^2-5b}}_{\in\, \mathbb{Q}}$$

$f : \mathbb{Q}[\sqrt{5}] \longrightarrow \mathbb{Q}[\sqrt{5}]$ morfism de inele

(1) $f(q) = q \quad \forall q \in \mathbb{Q}$

$$f(\underset{\mathbb{Q}}{\underset{\cap}{a}} + \underset{\mathbb{Q}}{\underset{\cap}{b}}\sqrt{5}) \overset{f}{\underset{morf}{=}} f(a) + f(b\sqrt{5}) \overset{f}{\underset{morf}{=}} f(a) + f(b)\cdot f(\sqrt{5})$$

$$\overset{(1)}{\underset{a,b \in \mathbb{Q}}{=}} a + b + f(\sqrt{5})$$

$$\underset{f(5) = 5}{\underset{\shortparallel}{f(\sqrt{5}\cdot\sqrt{5})}} \overset{f}{\underset{morf}{=}} f(\sqrt{5})\cdot f(\sqrt{5}) \Rightarrow$$

$$\Rightarrow f(\sqrt{5})^2 = 5 \Rightarrow f(\sqrt{5}) \in \{\pm\sqrt{5}\}$$

$\Rightarrow$ Avem 2 morfisme de inele

$$f_1 = 1_{\mathbb{Q}[\sqrt{5}]} \quad (f(a+b\sqrt{5}) = a + b\sqrt{5} \quad \forall a, b \in \mathbb{Q}$$

$$f_2(a + b\sqrt{5}) = a - b\sqrt{5} \quad \forall a, b \in \mathbb{Q}$$