Range Minimum Queries

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?

31	41	59	26	53	58	97	93

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

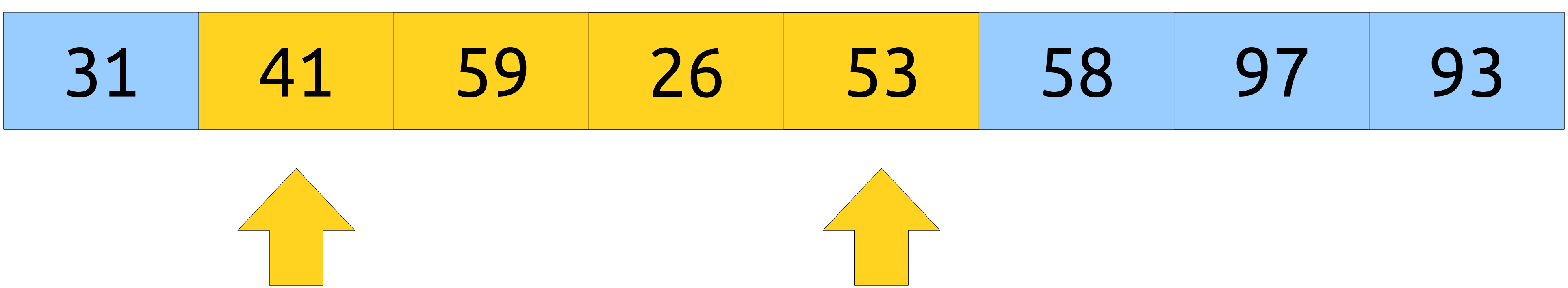
  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?

31	41	59	26	53	58	97	93

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

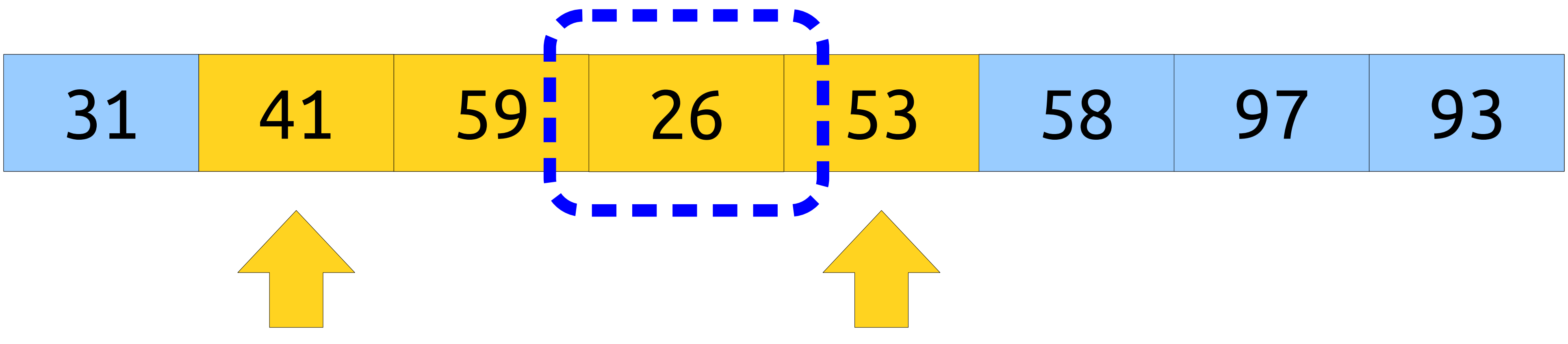
  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

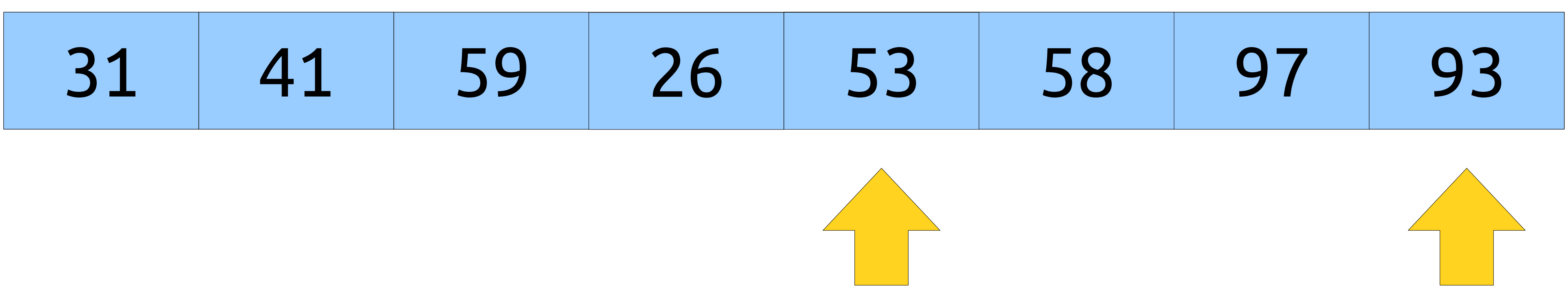
  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[$i + 1$], …, A[$j - 1$], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

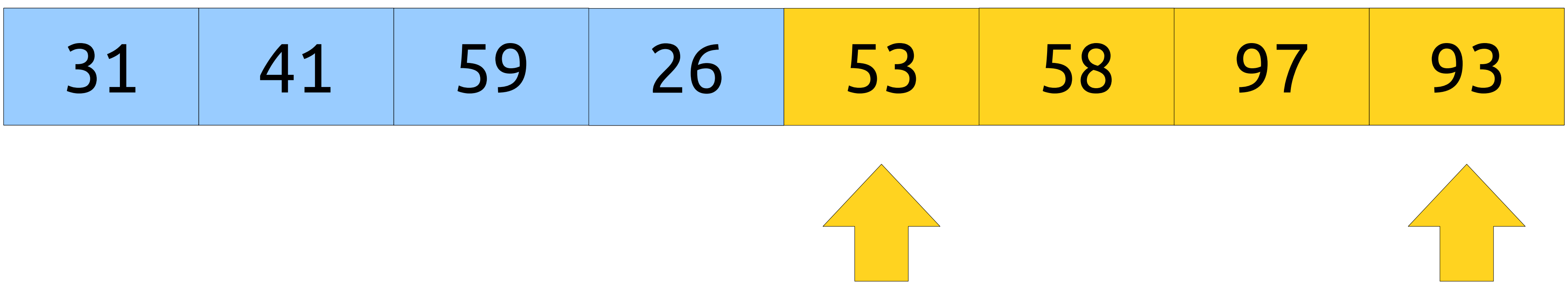
  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

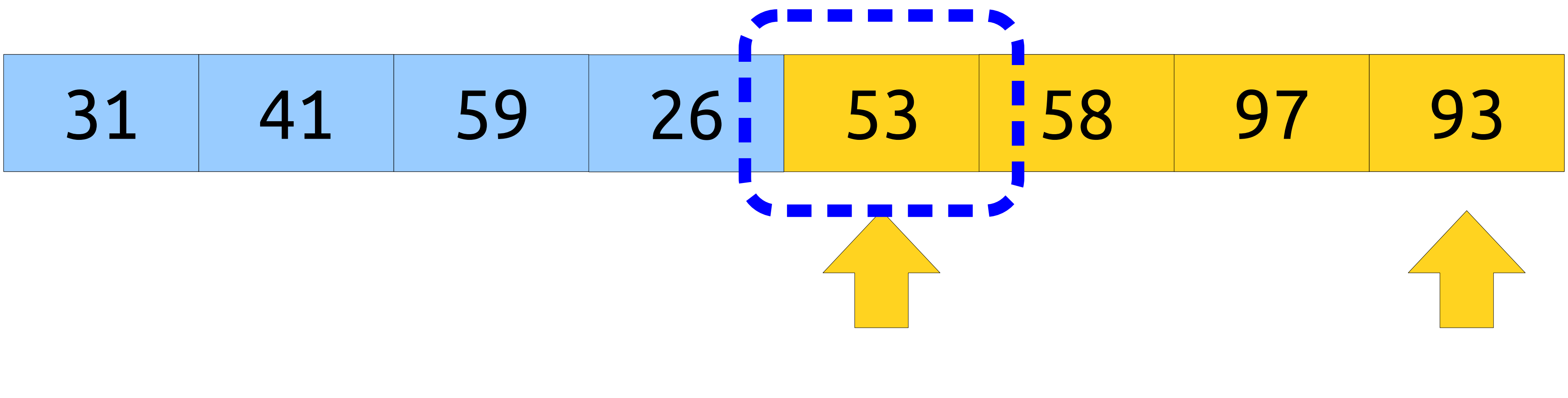
  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

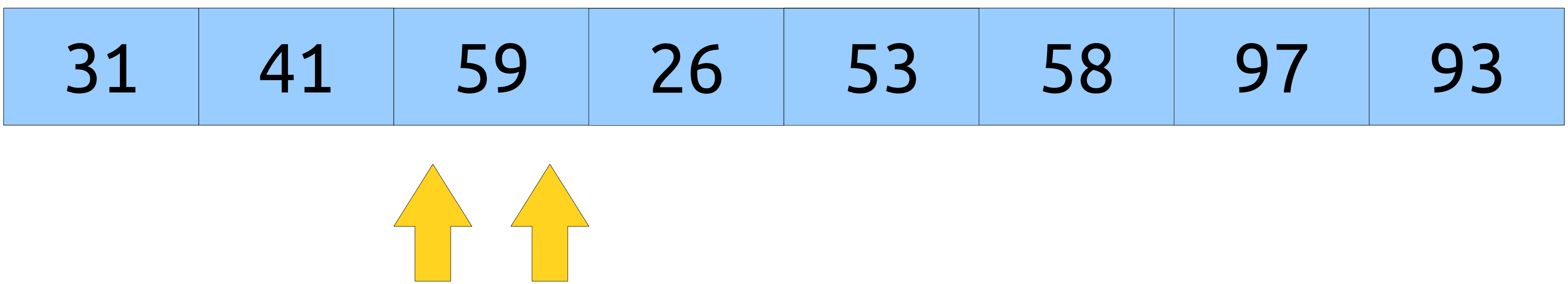
  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

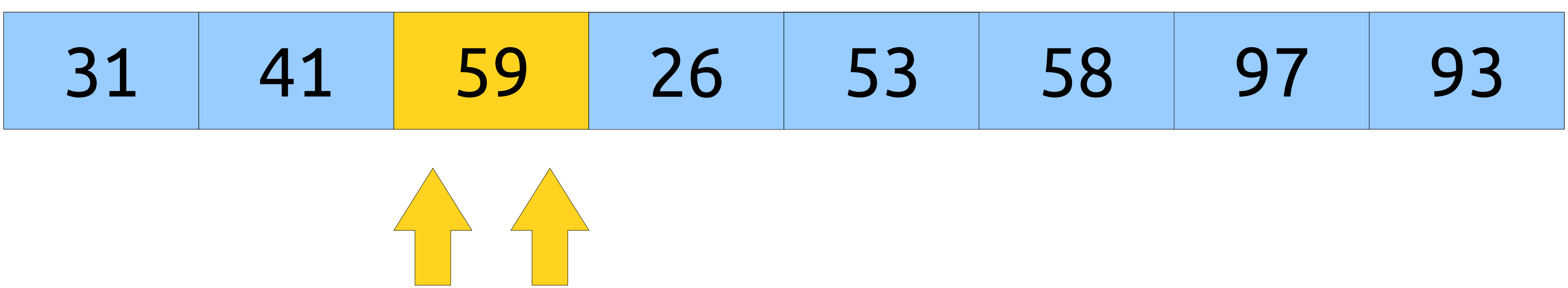
  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

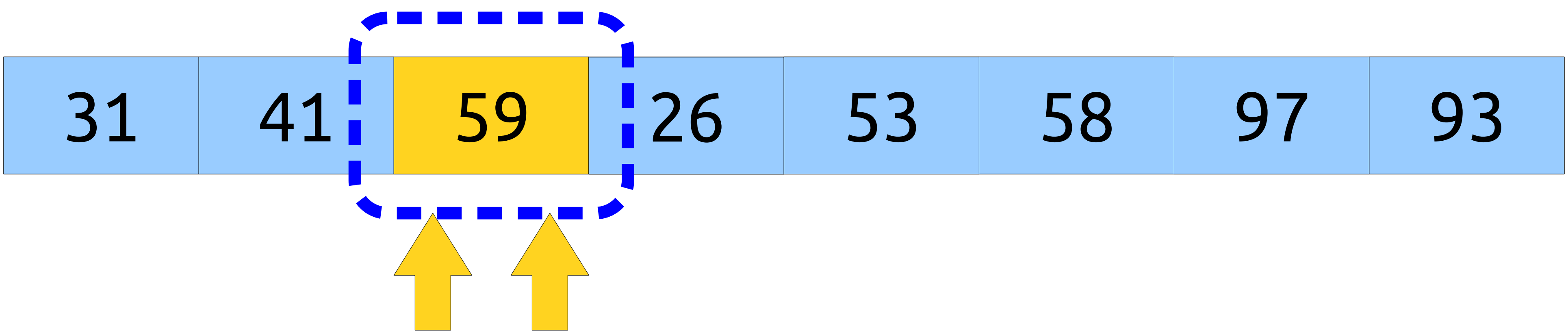
Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of $A[i], A[i + 1], \ldots, A[j - 1], A[j]$?

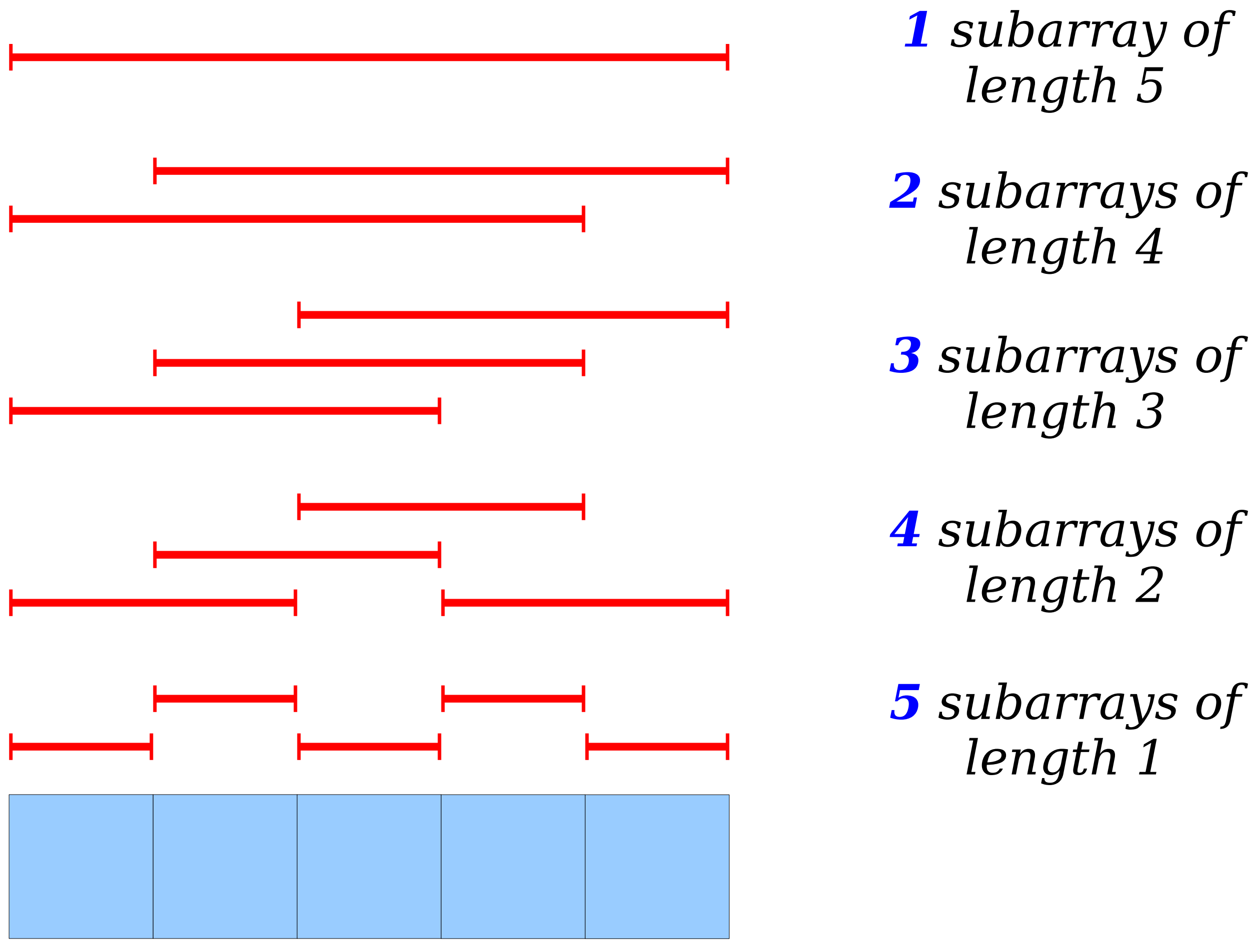
- Notation: We'll denote a range minimum query in array A between indices i and j as $\mathbf{RMQ_A(\mathit{i}, \mathit{j})}$.
- For simplicity, let's assume 0-indexing.

A Trivial Solution

- There's a simple O(n)-time algorithm for evaluating $\text{RMQ}_A(i, j)$: just iterate across the elements between i and j, inclusive, and take the minimum!
- So… why is this problem at all algorithmically interesting?
- Suppose that the array A is fixed in advance and you're told that we're going to make multiple queries on it.
- Can we do better than the naïve algorithm?

An Observation

- In an array of length n, there are only $\Theta(n^2)$ distinct possible queries.
- Why?



A Different Approach

- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.

16	18	33	98
0	1	2	3

A Different Approach

- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.

16	18	33	98
0	1	2	3

	0	1	2	3
0				
1				
2				
3				

A Different Approach

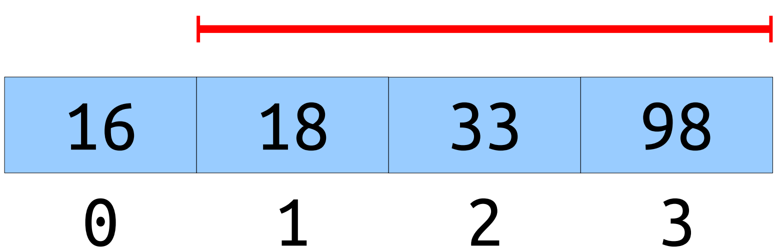
- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.

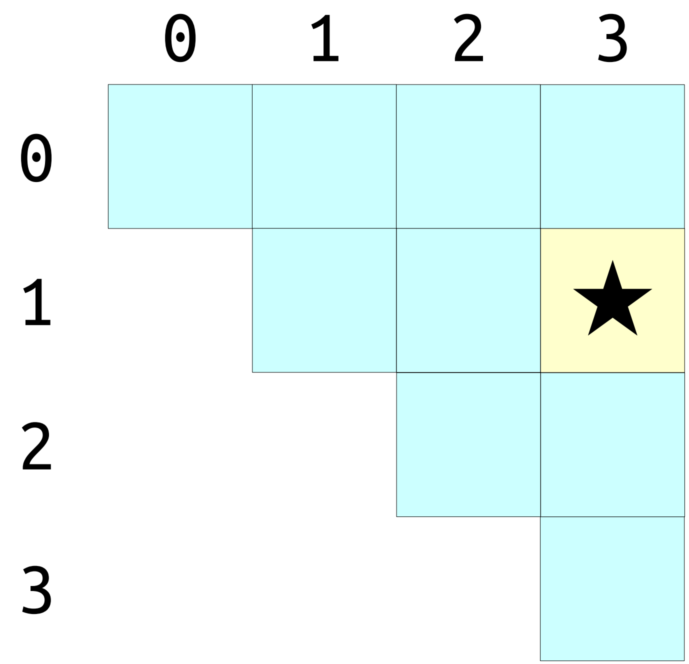
16	18	33	98
0	1	2	3

	0	1	2	3
0				
1				★
2				
3				

A Different Approach

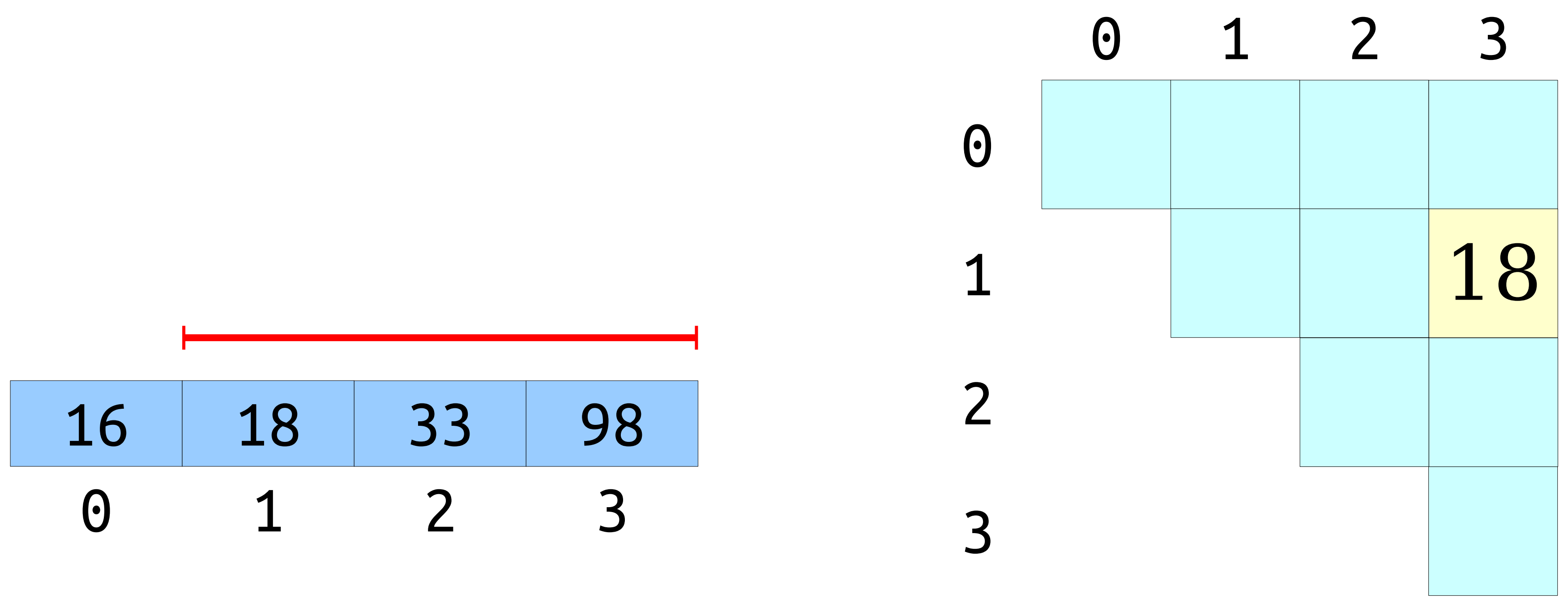
- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.





A Different Approach

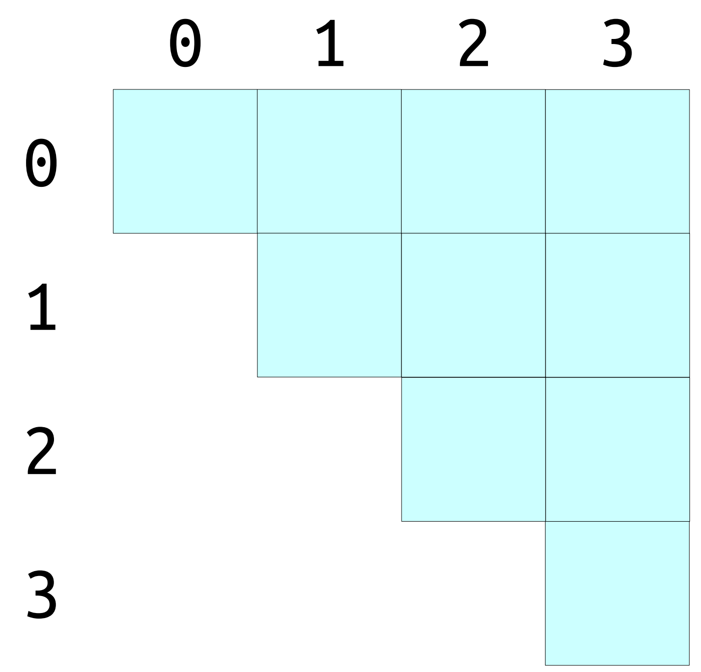
- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.



A Different Approach

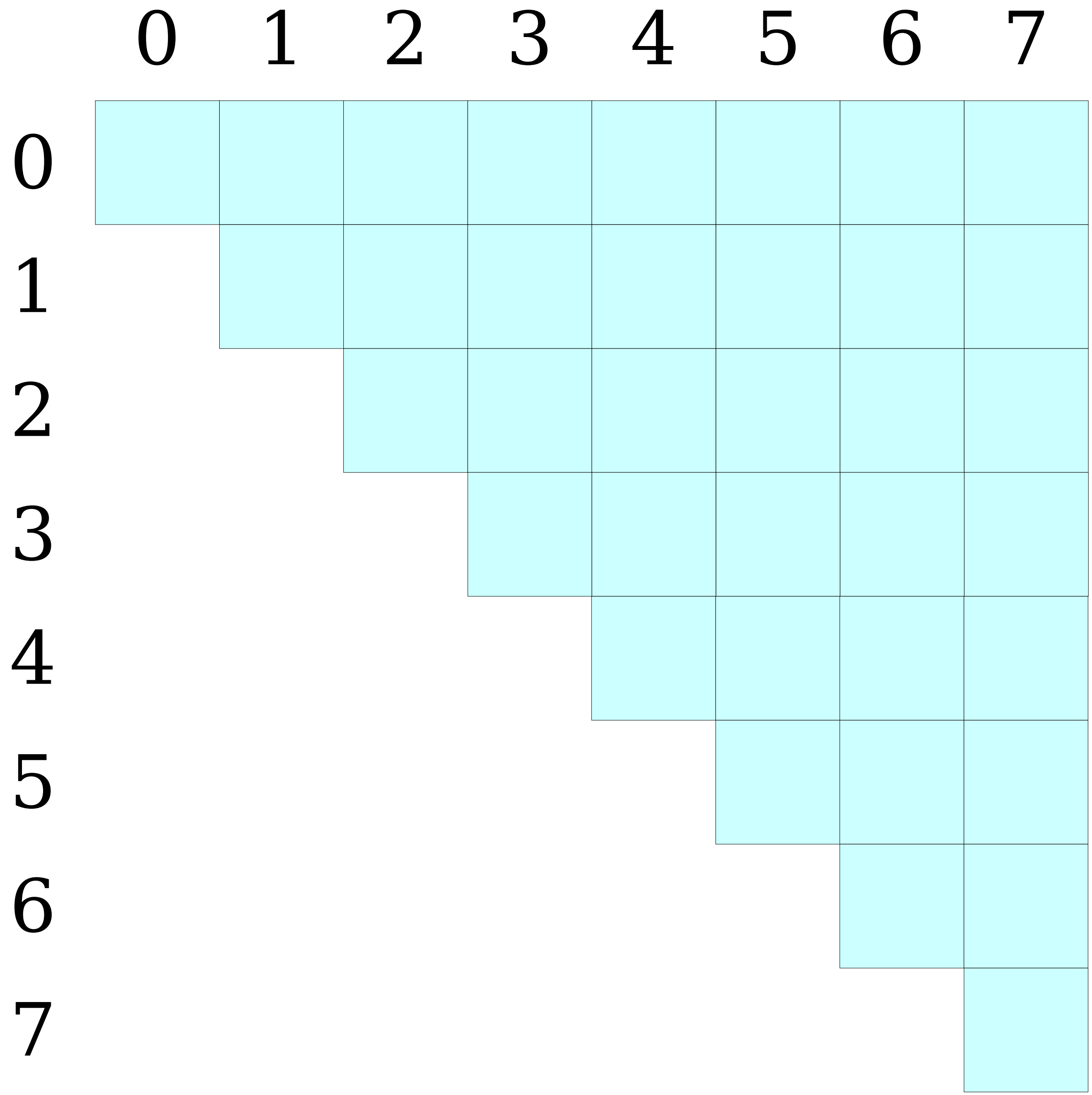
- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.

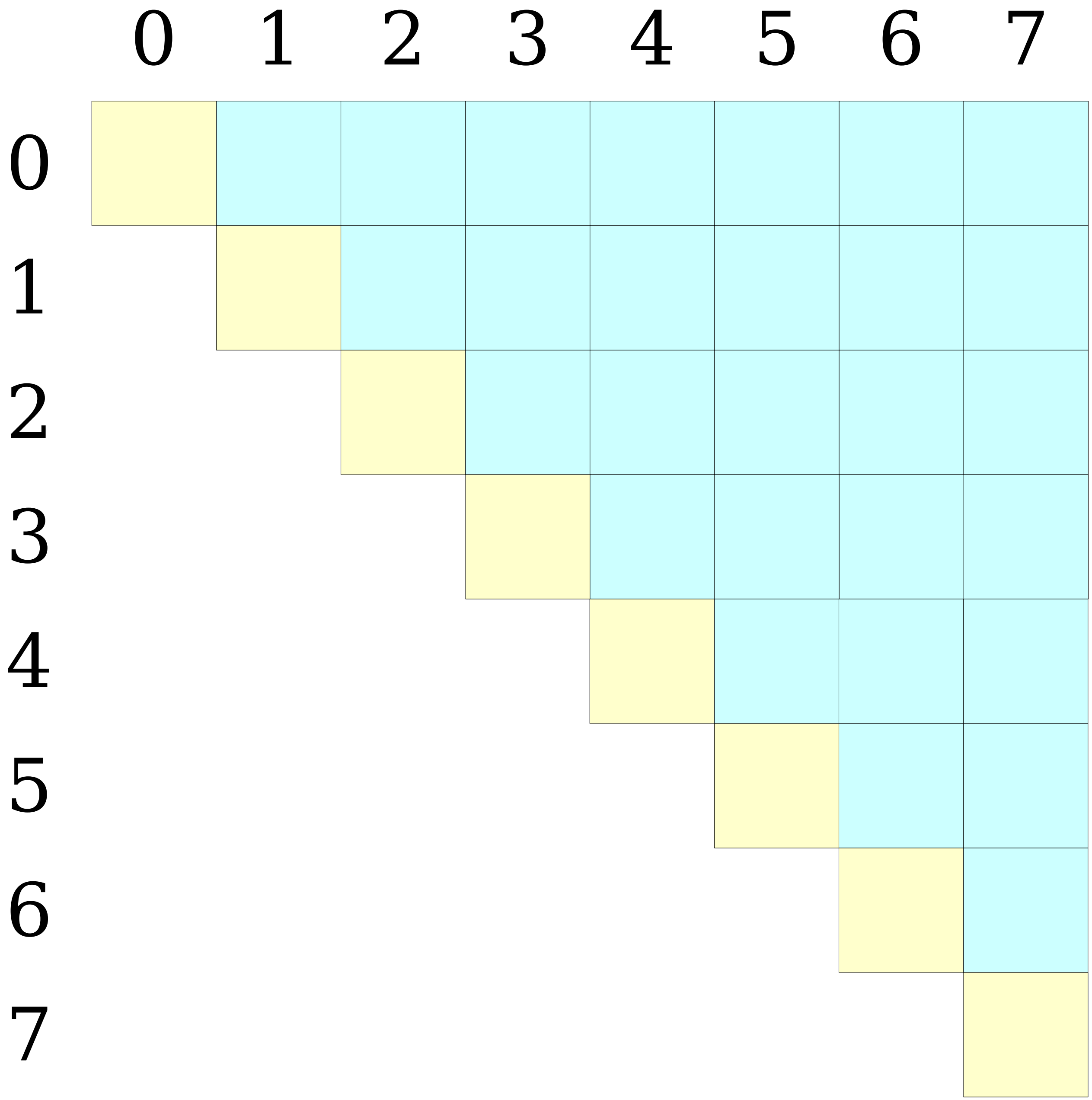
16	18	33	98
0	1	2	3

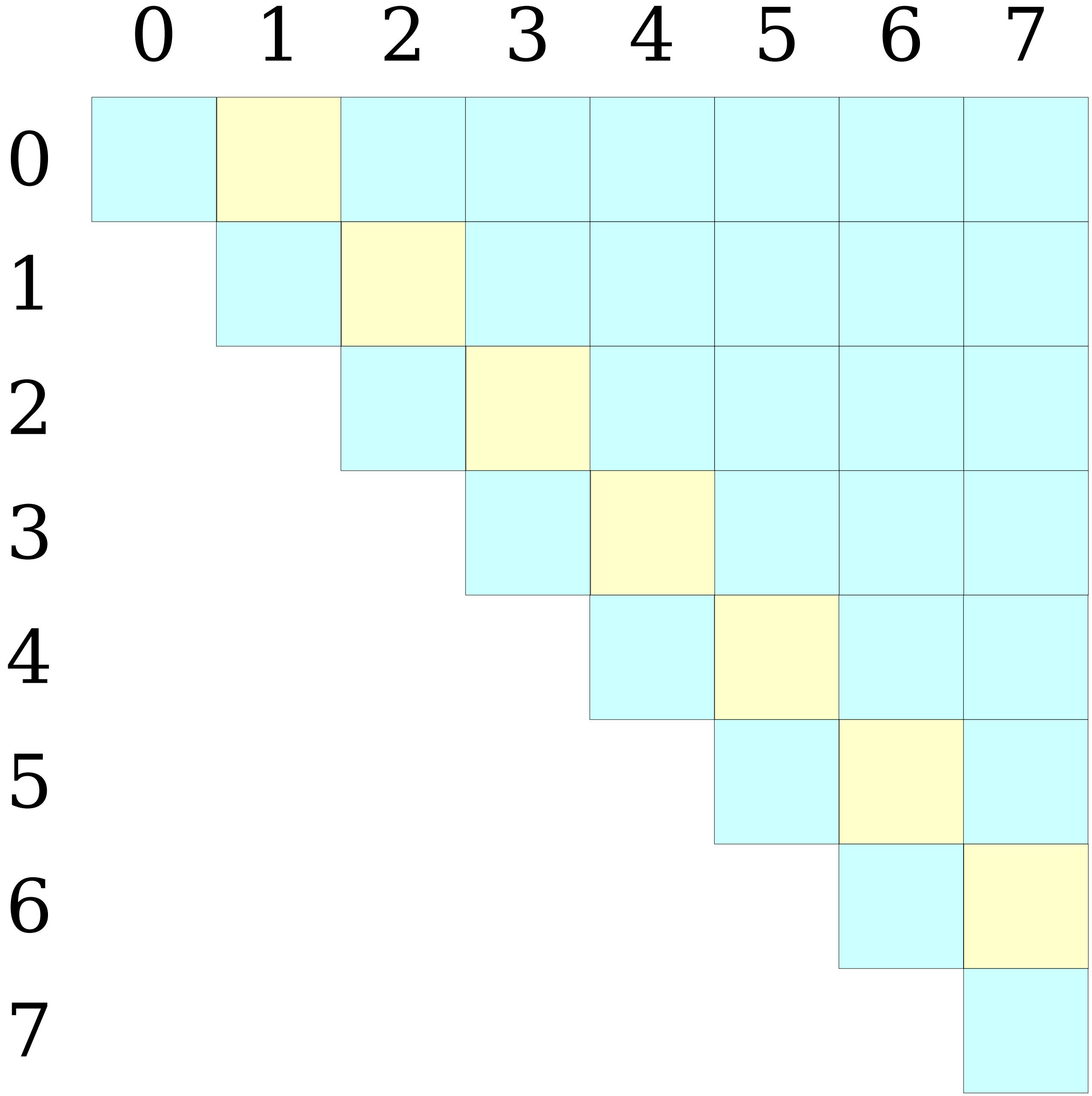


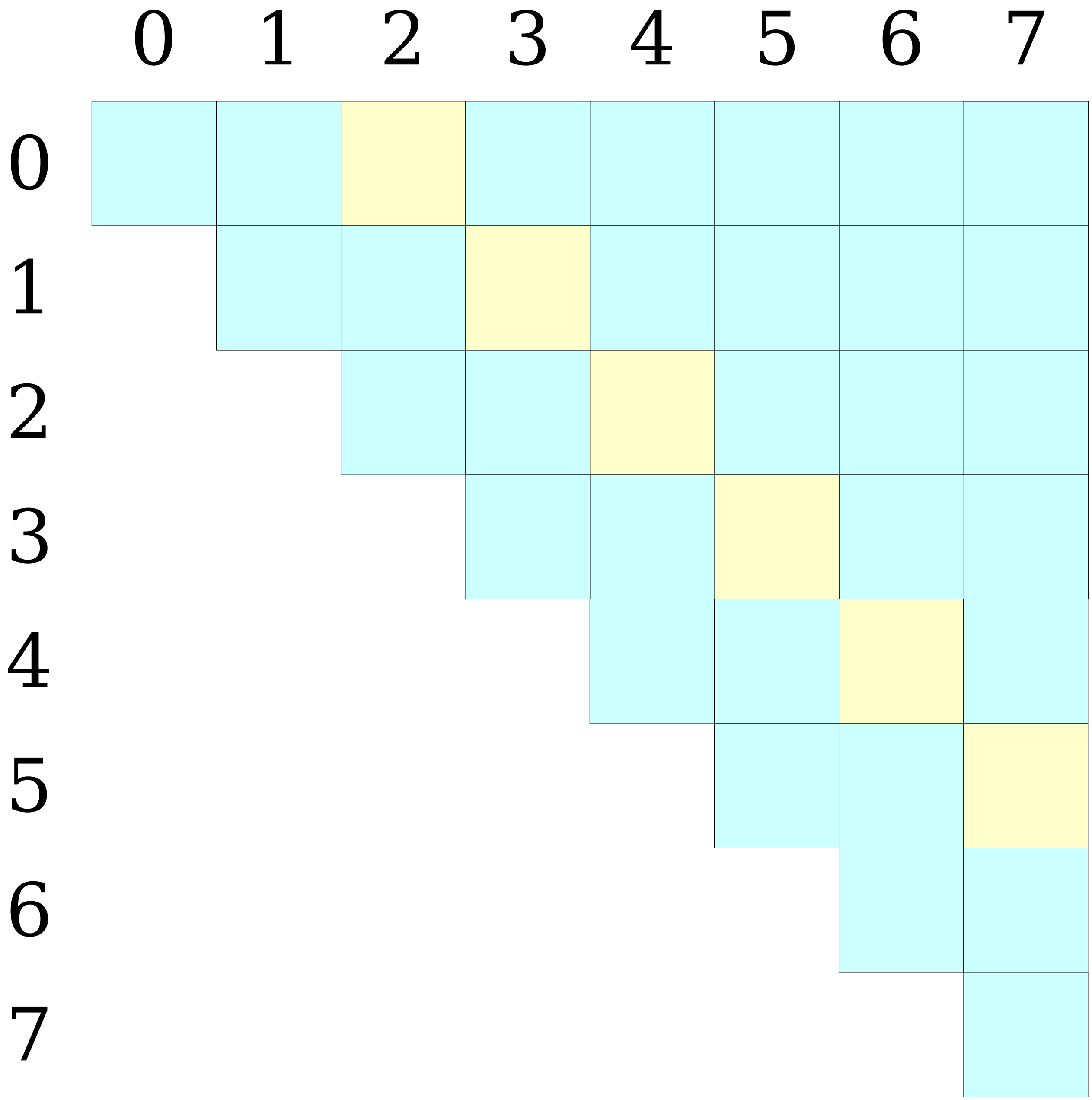
Building the Table

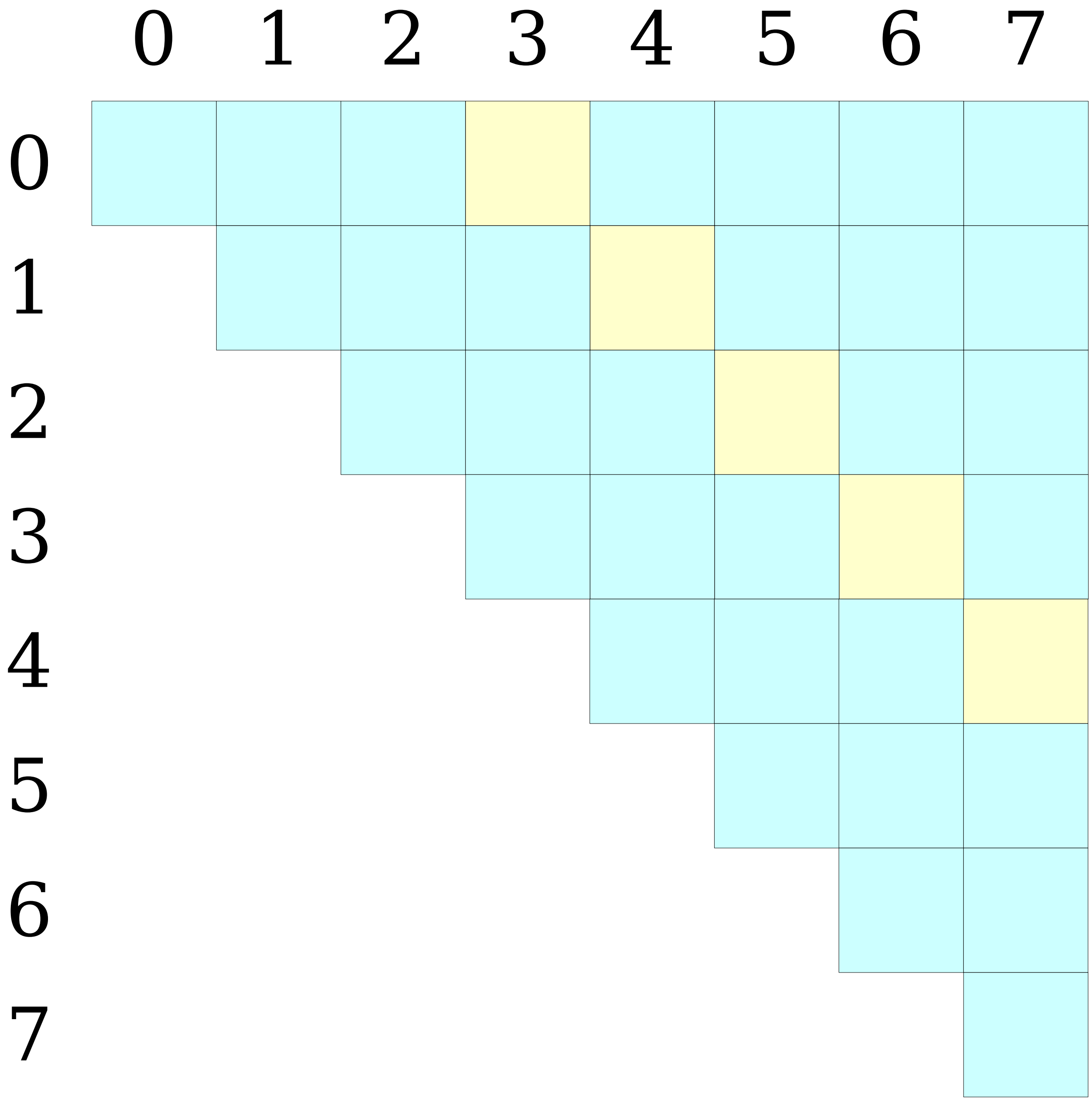
- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
  - Number of entries: $\Theta(n^2)$.
  - Time to evaluate each entry: $O(n)$.
  - Time required: $O(n^3)$.
- The runtime is $O(n^3)$ using this approach. Is it also $\Theta(n^3)$?

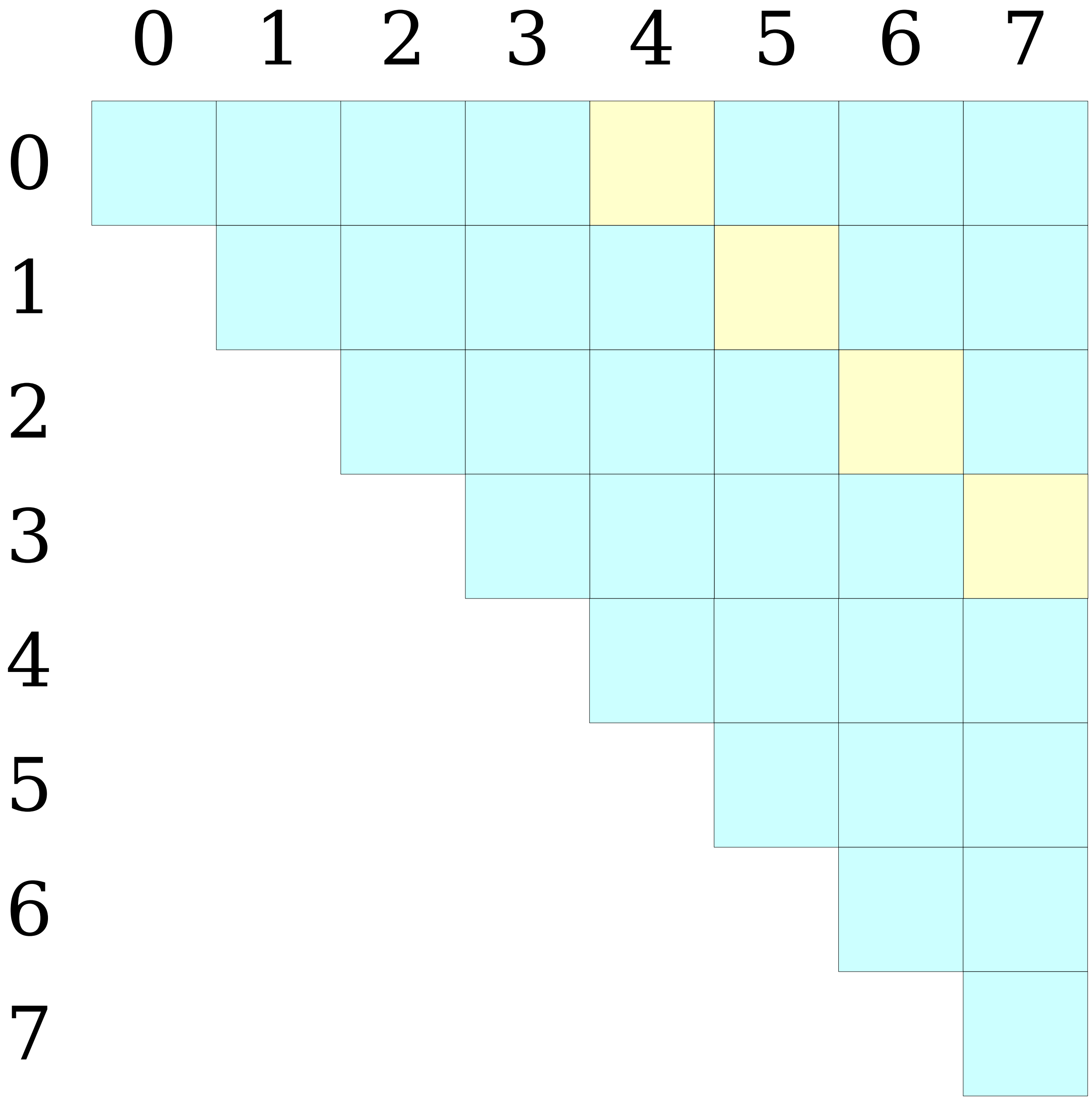

0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7

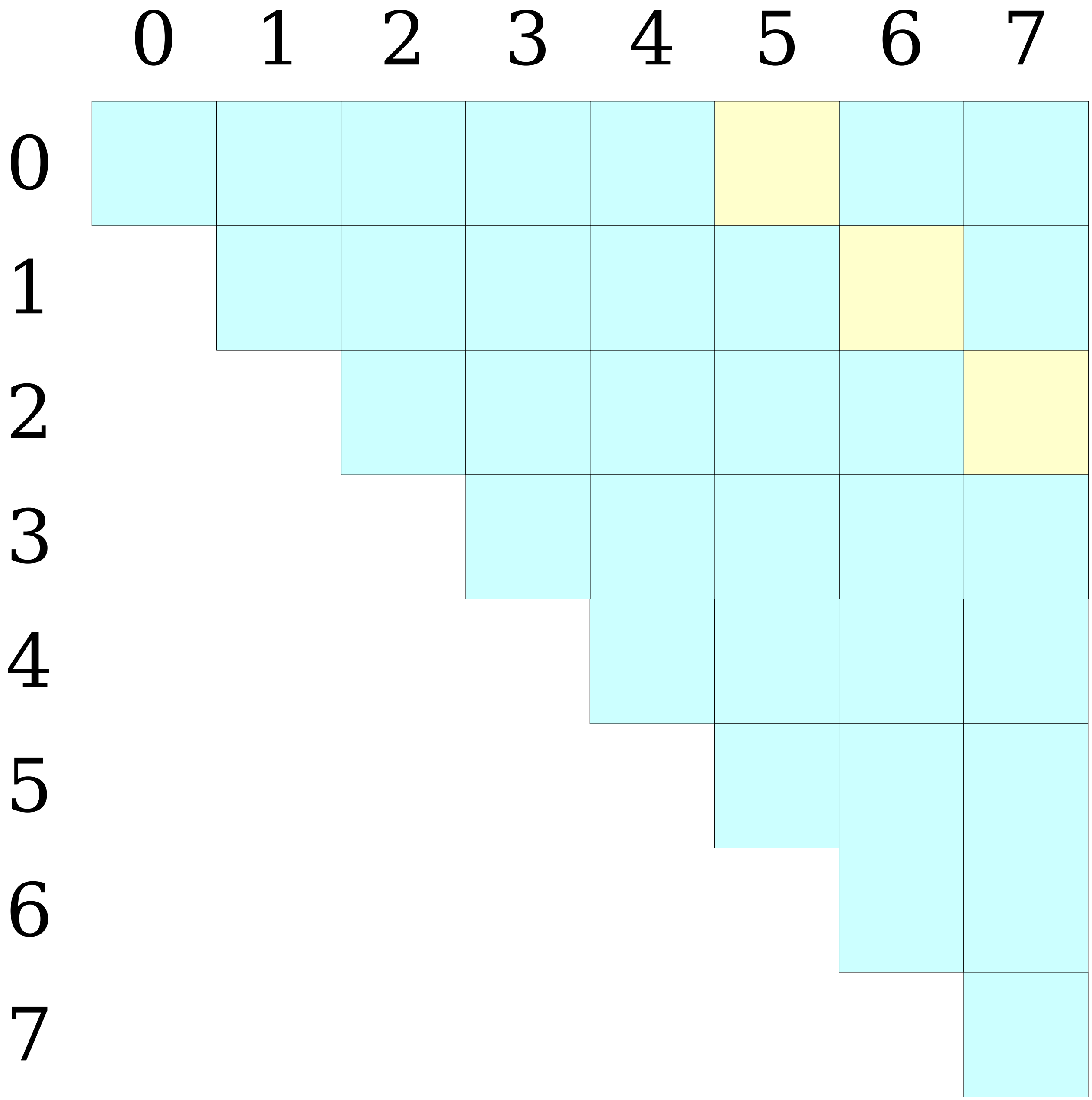

0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7

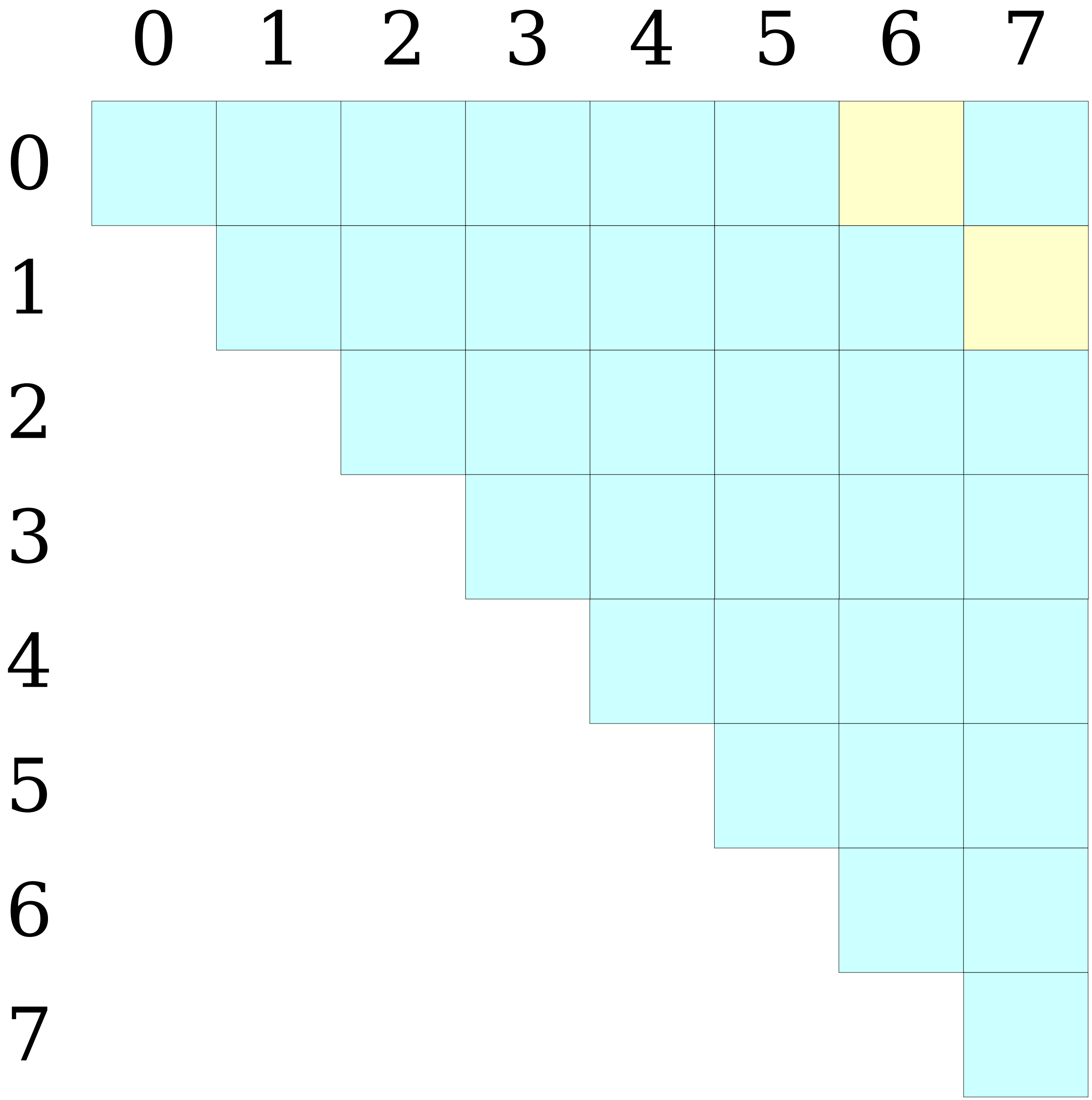

0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7

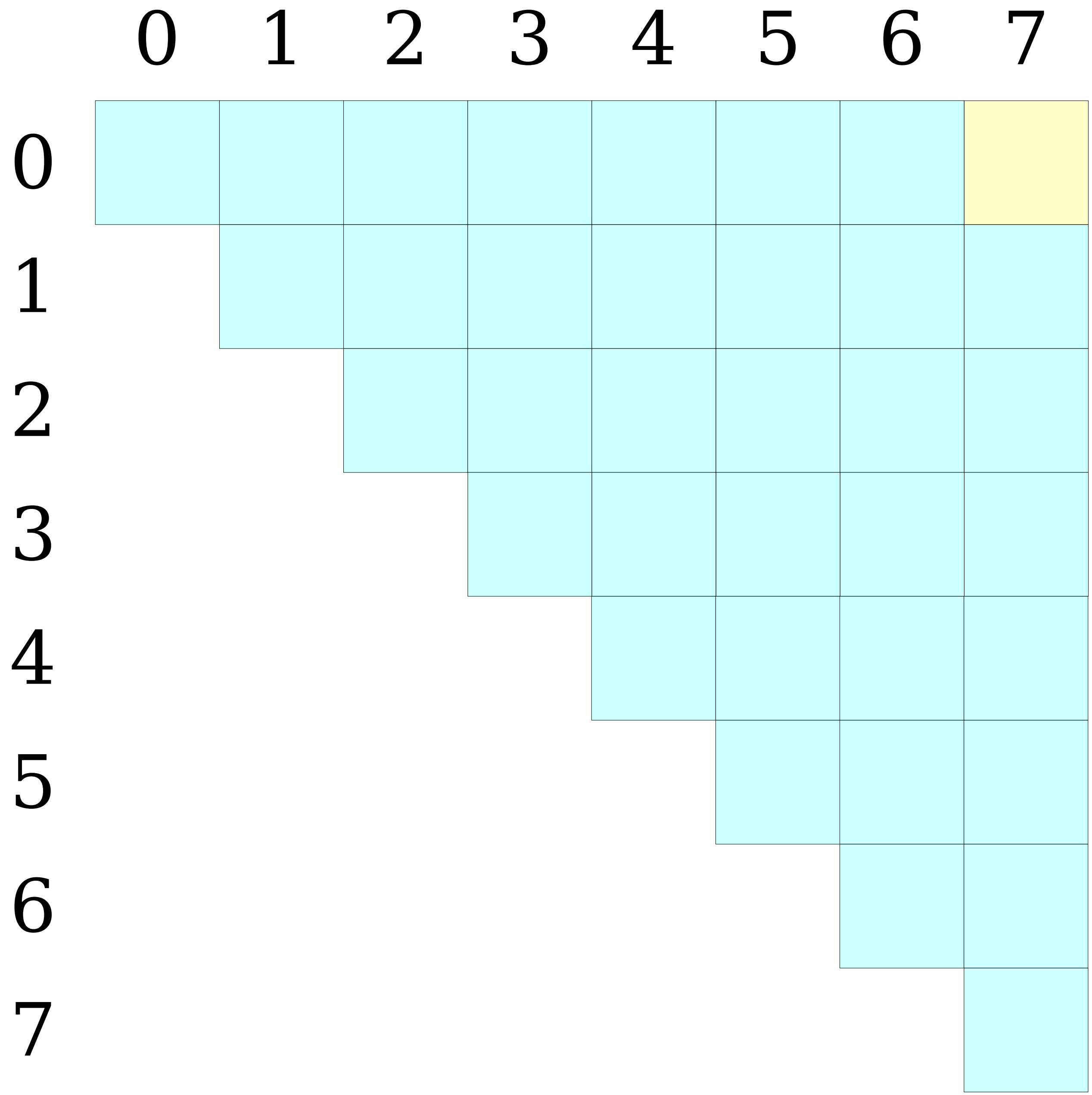

0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7

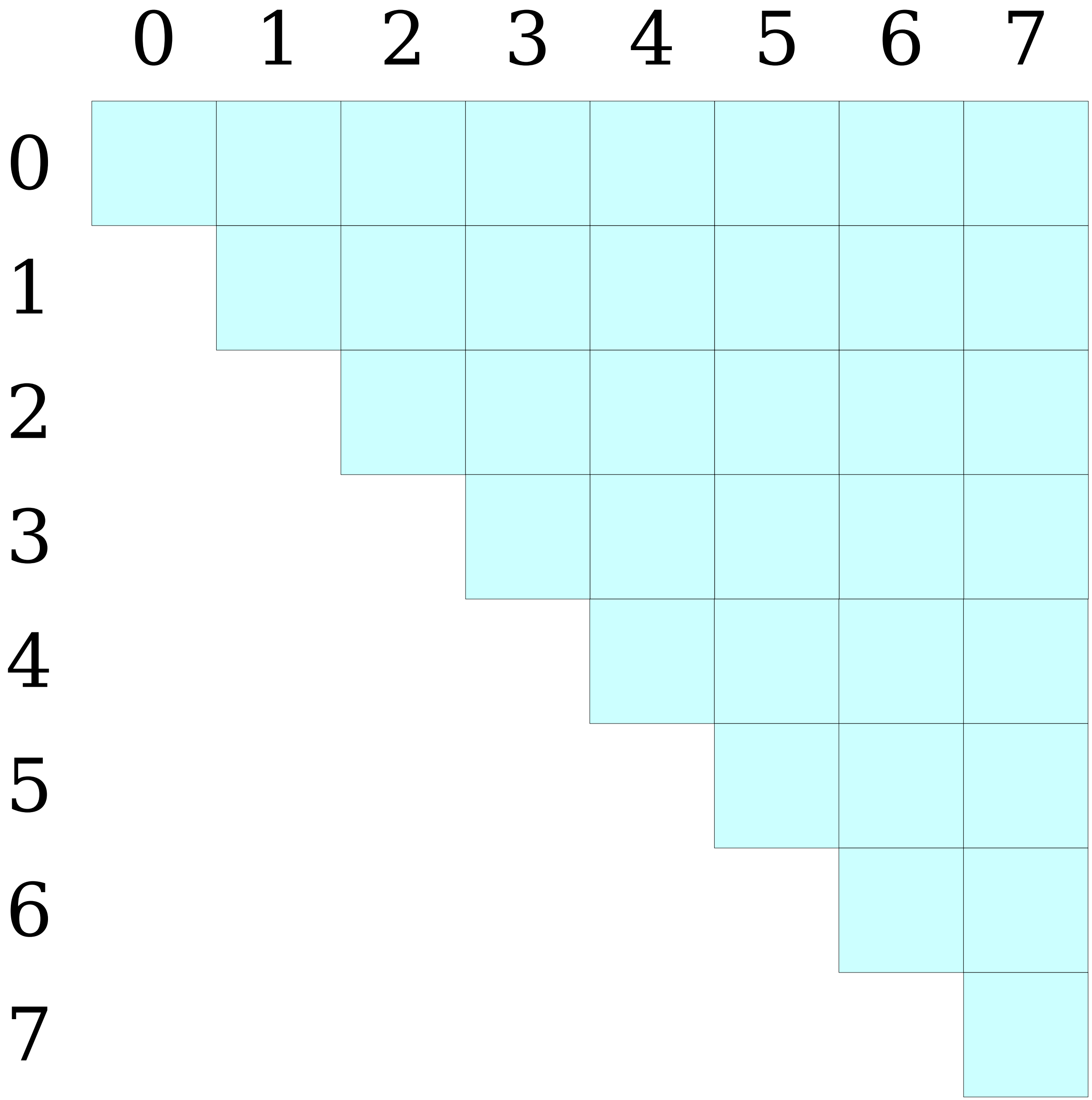

0
1
2
3
4
5
6
7
0
1
2
3
4
5
6
7

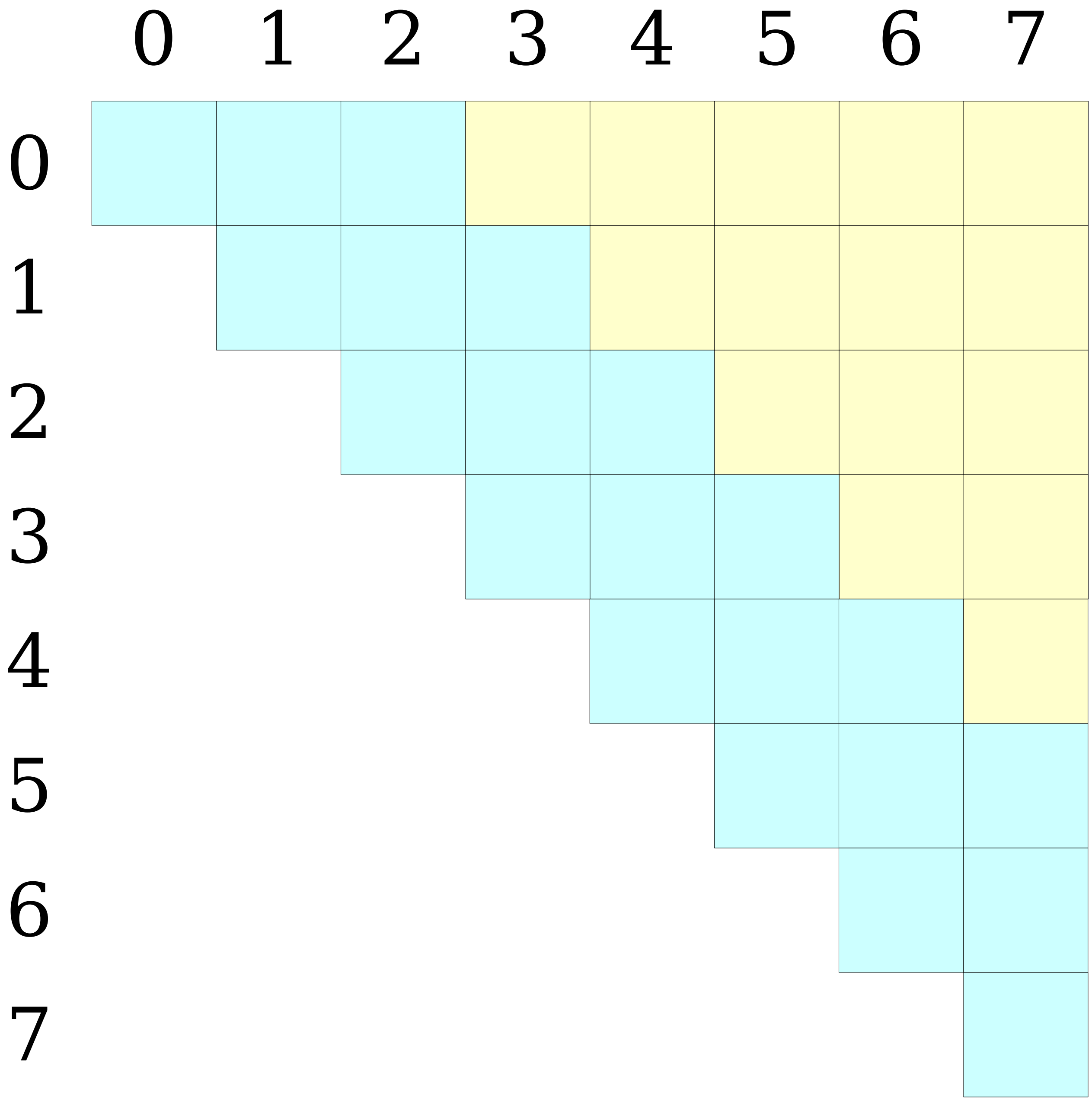

0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7

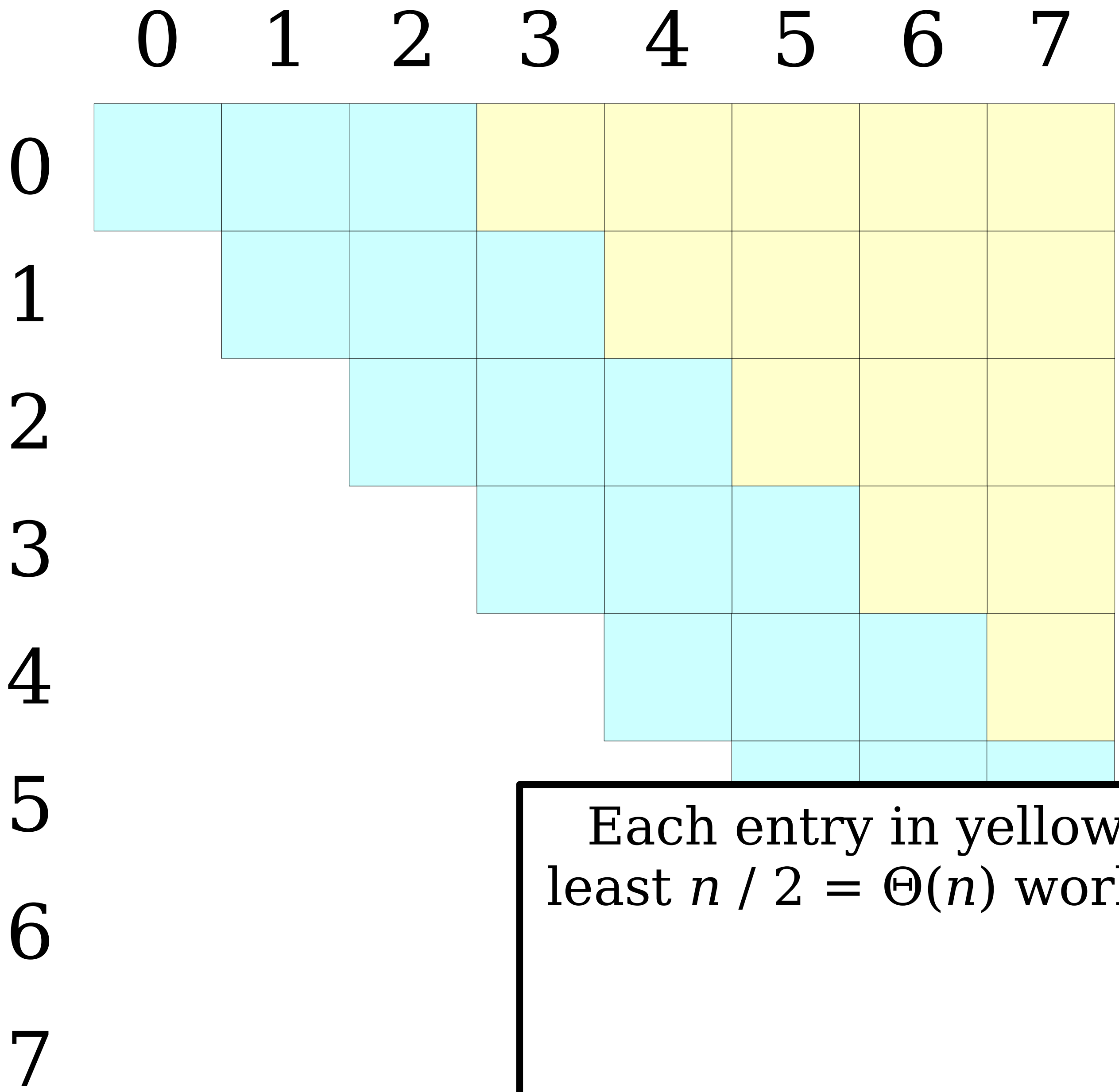

0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7


0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7

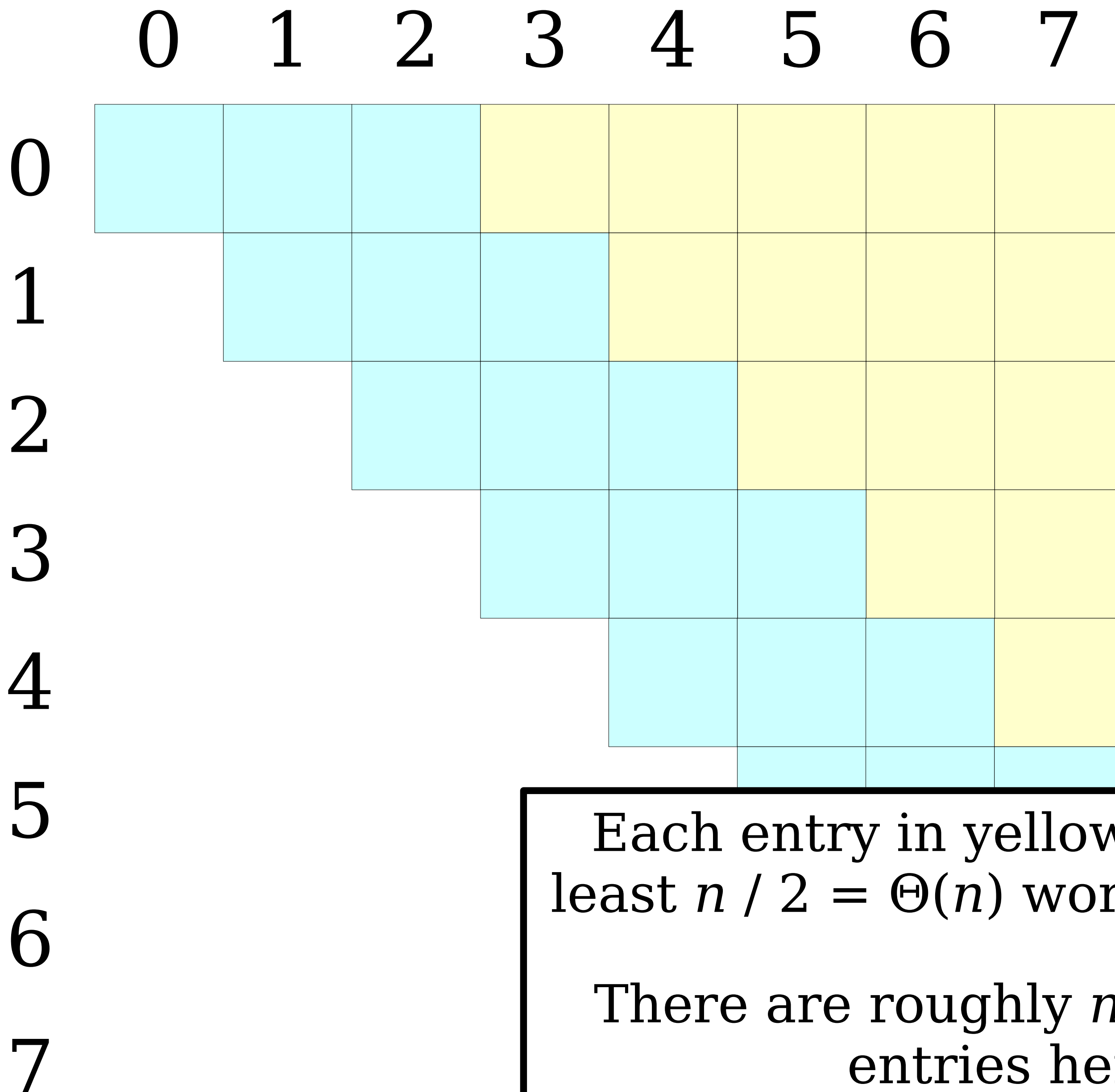

0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7


0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7


0 1 2 3 4 5 6 7
0
1
2
3
4
5
6
7

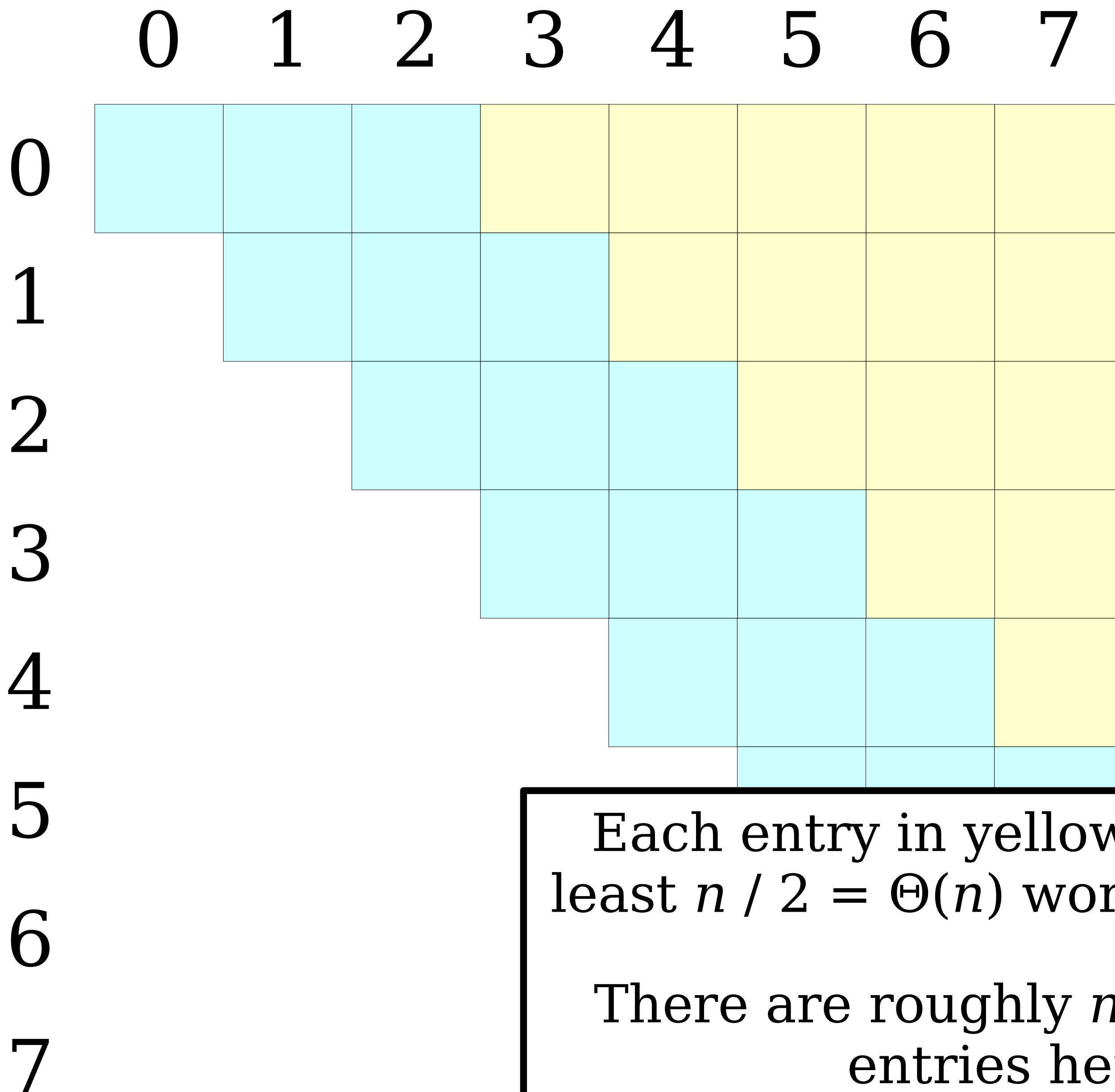


Each entry in yellow requires at least n / 2 = $\Theta(n)$ work to evaluate.



Each entry in yellow requires at least n / 2 = $\Theta(n)$ work to evaluate.

There are roughly n^2 / 8 = $\Theta(n^2)$ entries here.



Each entry in yellow requires at least n / 2 = $\Theta(n)$ work to evaluate.

There are roughly n^2 / 8 = $\Theta(n^2)$ entries here.

Total work required: $\Theta(n^3)$

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0				
1				
2				
3				

A Different Approach

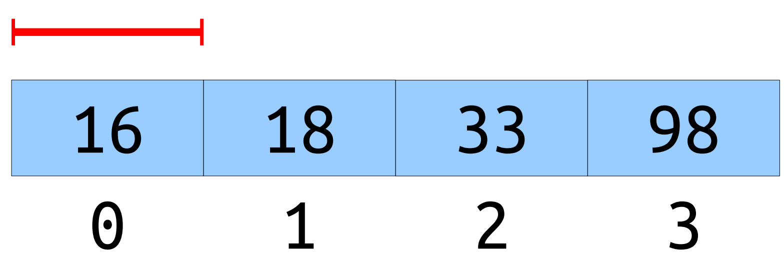
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

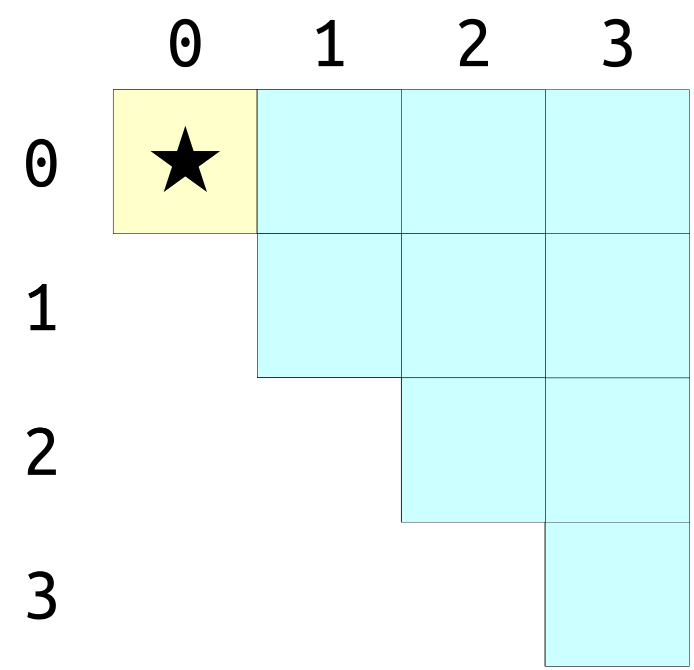
16	18	33	98
0	1	2	3

	0	1	2	3
0	★			
1				
2				
3				

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.





A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1				
2				
3				

A Different Approach

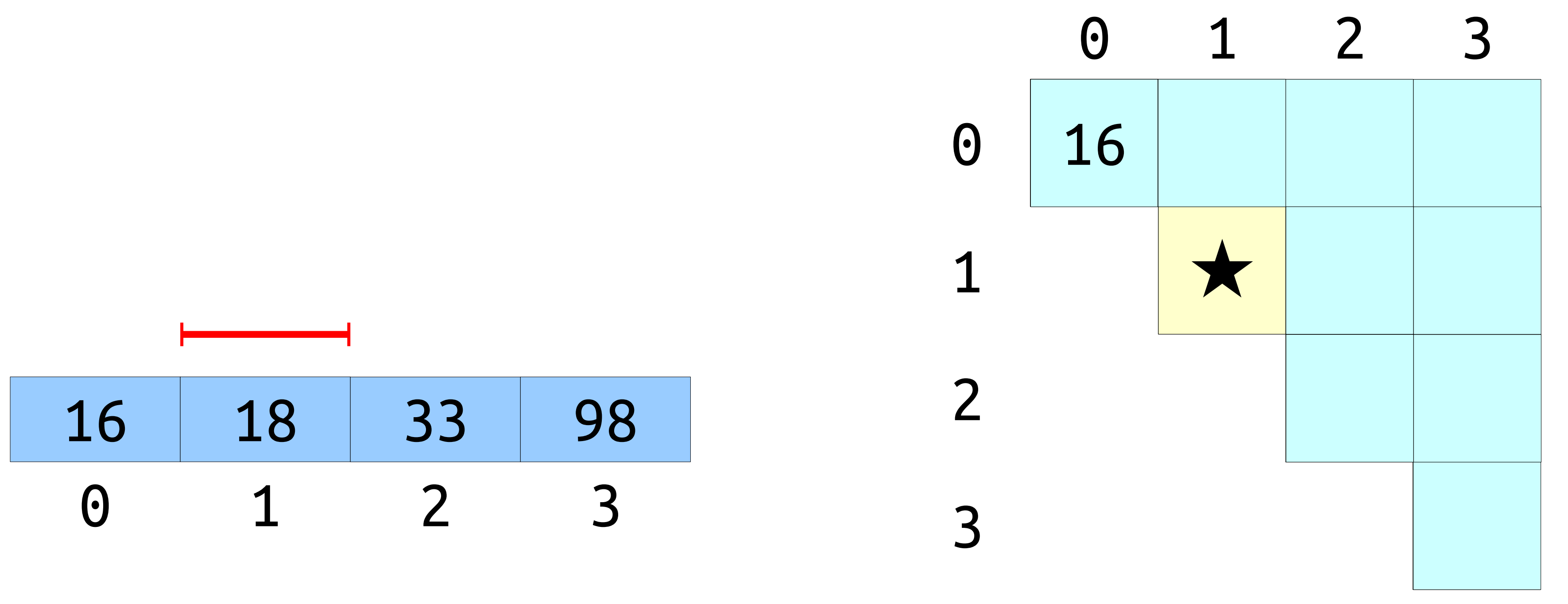
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		★		
2				
3				

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



A Different Approach

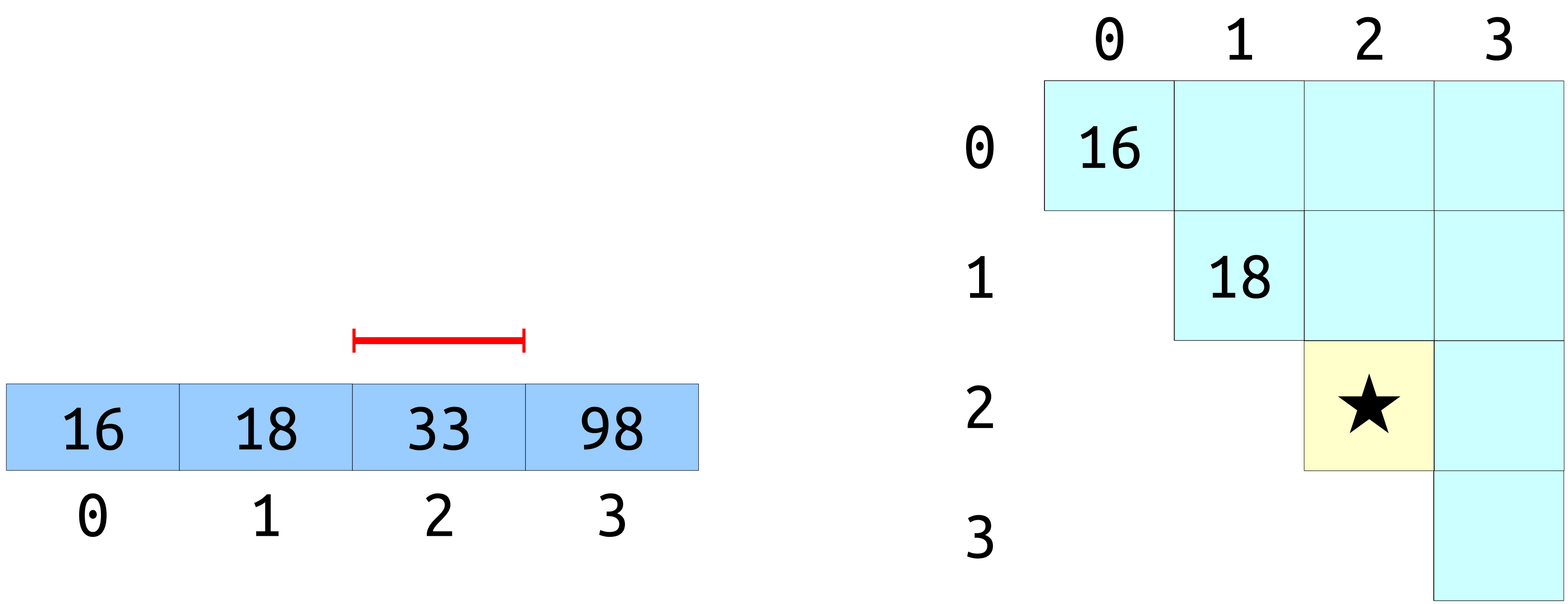
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		18		
2				
3				

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



A Different Approach

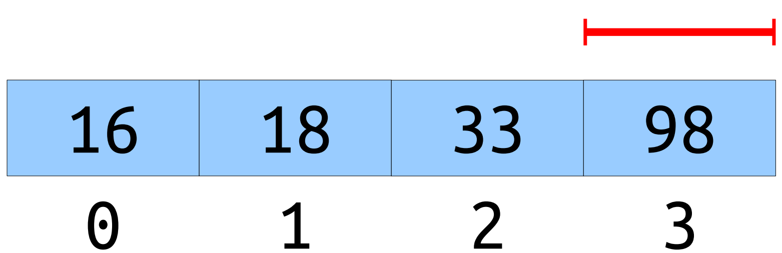
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		18		
2			33	
3				

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16			
1		18		
2			33	
3				★

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		18		
2			33	
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		18		
2			33	
3				98

A Different Approach

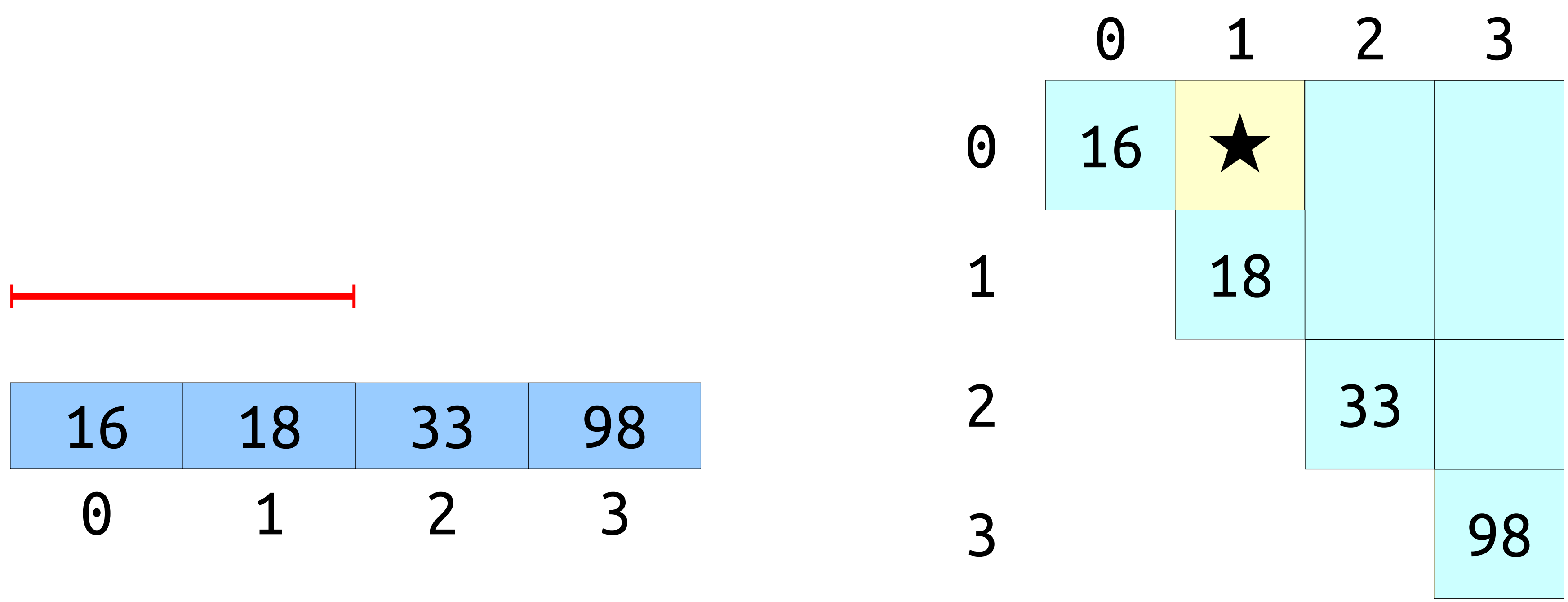
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	★		
1		18		
2			33	
3				98

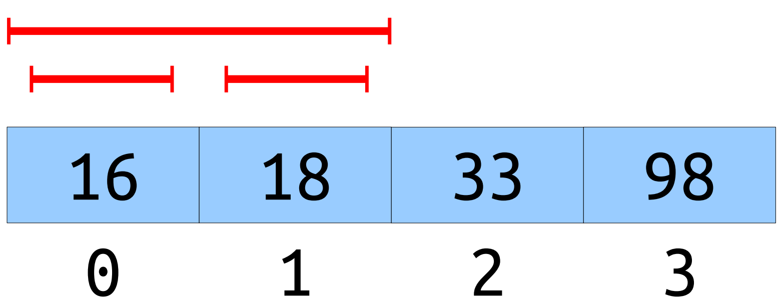
A Different Approach

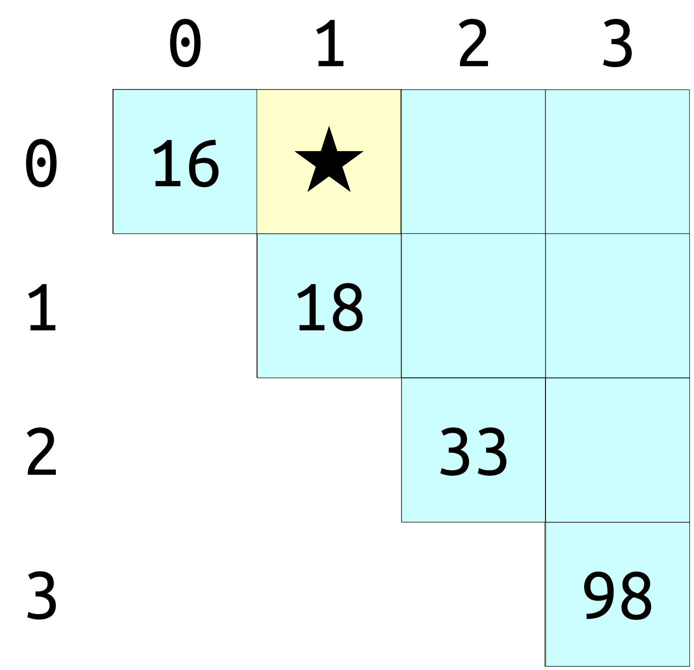
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



A Different Approach

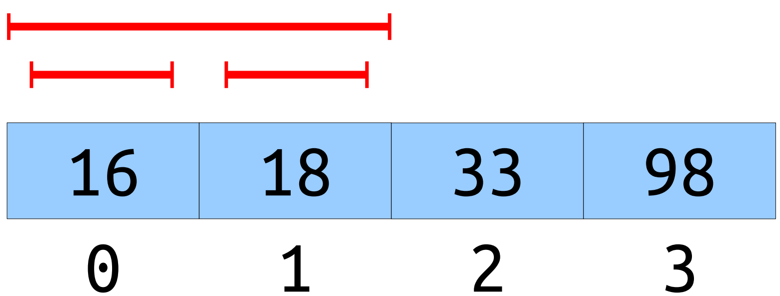
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.





A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	★		
1		18		
2			33	
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16		
1		18		
2			33	
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16		
1		18	★	
2			33	
3				98

A Different Approach

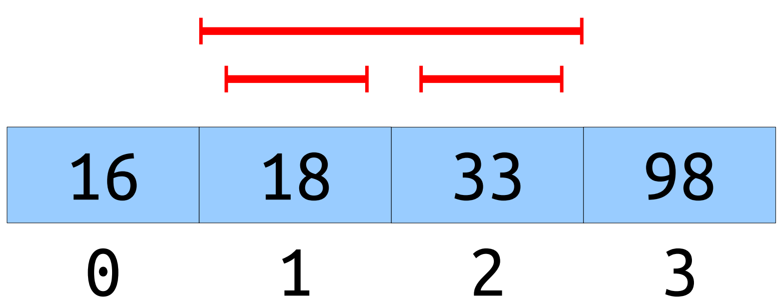
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16		
1		18	★	
2			33	
3				98

A Different Approach

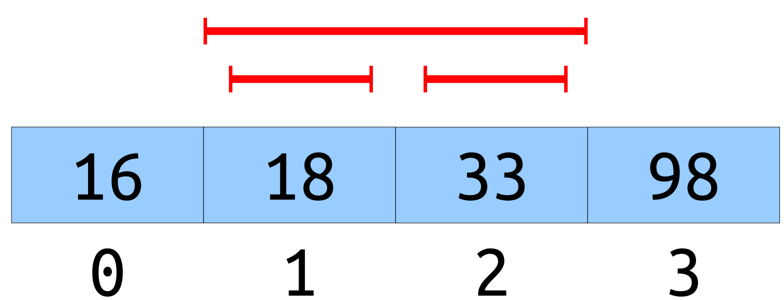
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16		
1		18	★	
2			33	
3				98

A Different Approach

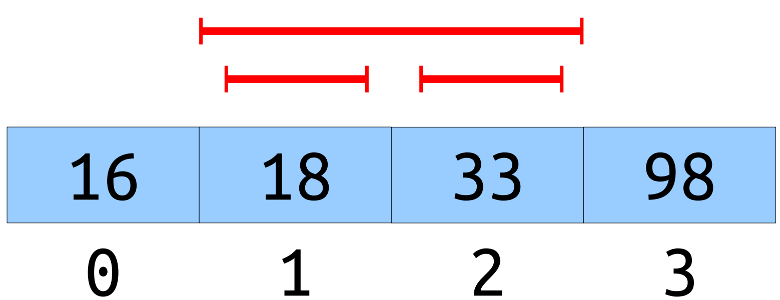
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16		
1		18	★	
2			33	
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16		
1		18	18	
2			33	
3				98

A Different Approach

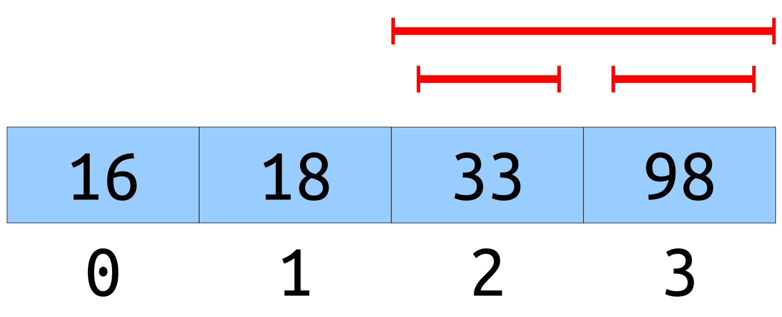
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16		
1		18	18	
2			33	★
3				98

A Different Approach

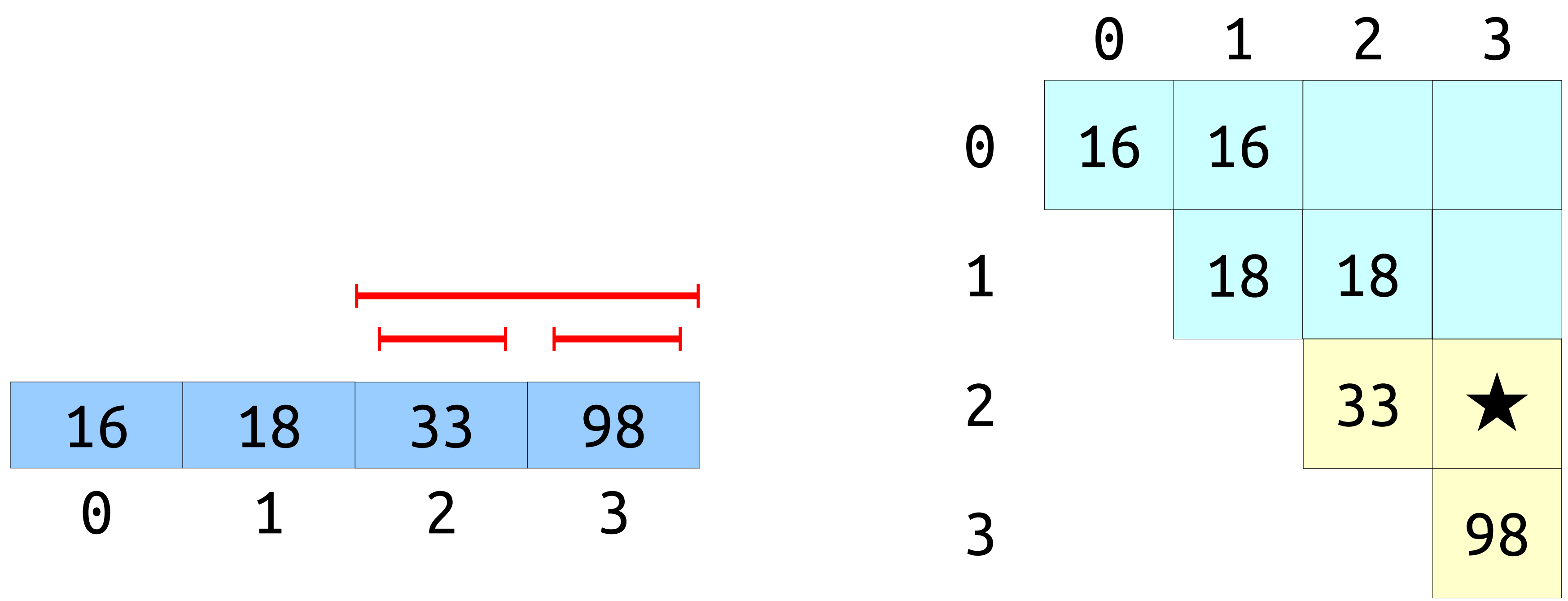
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16		
1		18	18	
2			33	★
3				98

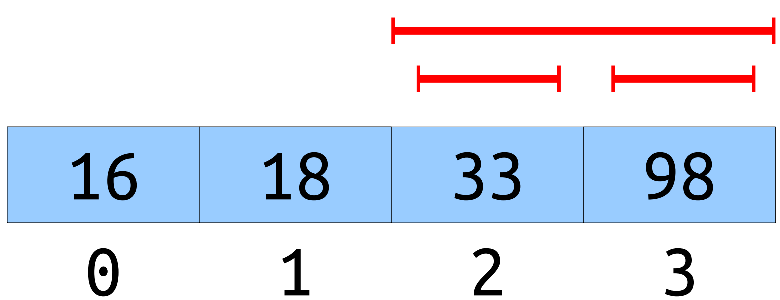
A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16		
1		18	18	
2			33	33
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16		
1		18	18	
2			33	33
3				98

A Different Approach

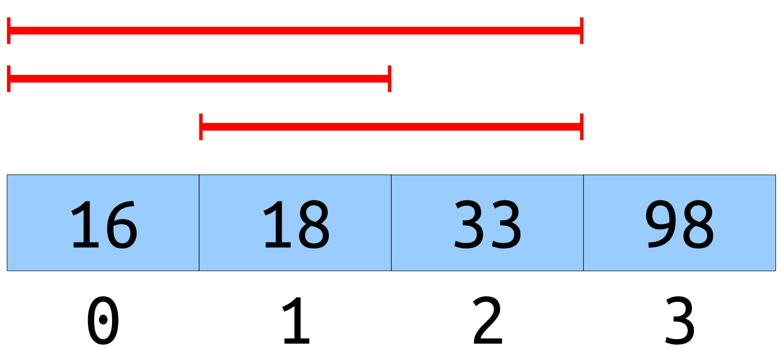
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16	★	
1		18	18	
2			33	33
3				98

A Different Approach

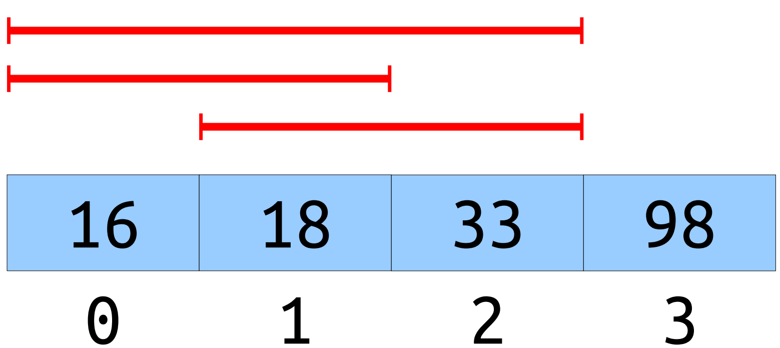
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	★	
1		18	18	
2			33	33
3				98

A Different Approach

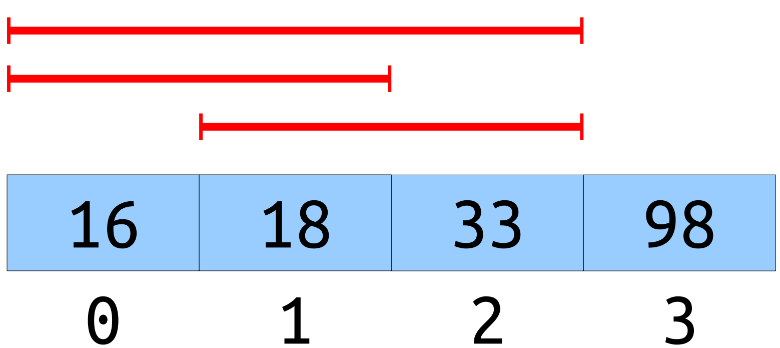
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	★	
1		18	18	
2			33	33
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	16	
1		18	18	
2			33	33
3				98

A Different Approach

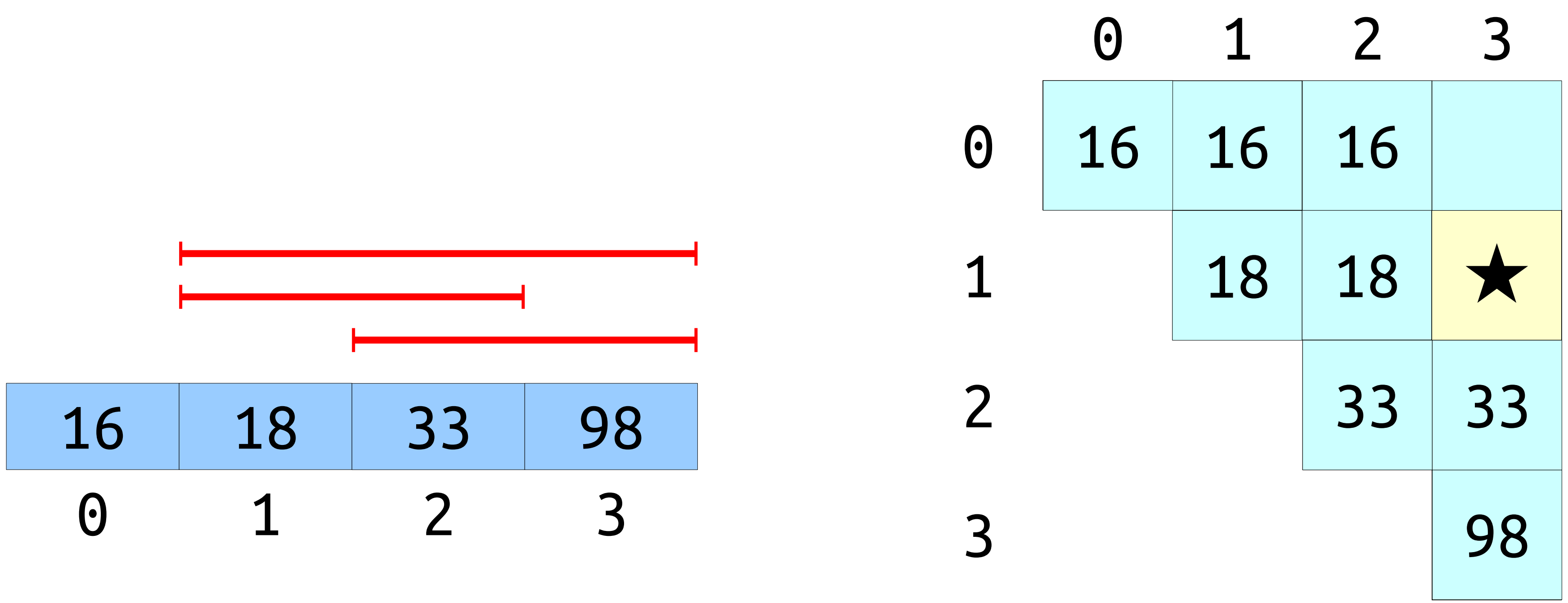
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16	16	
1		18	18	★
2			33	33
3				98

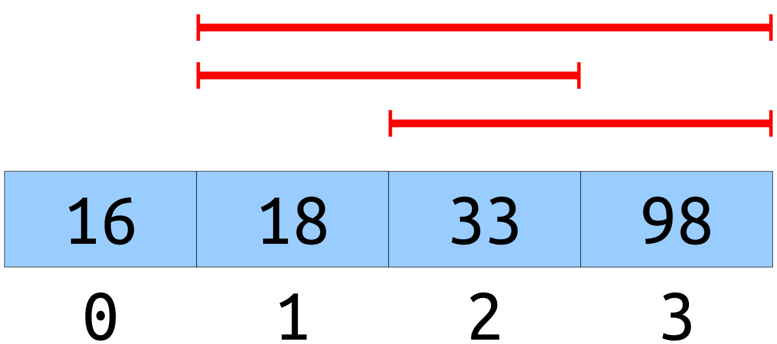
A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



A Different Approach

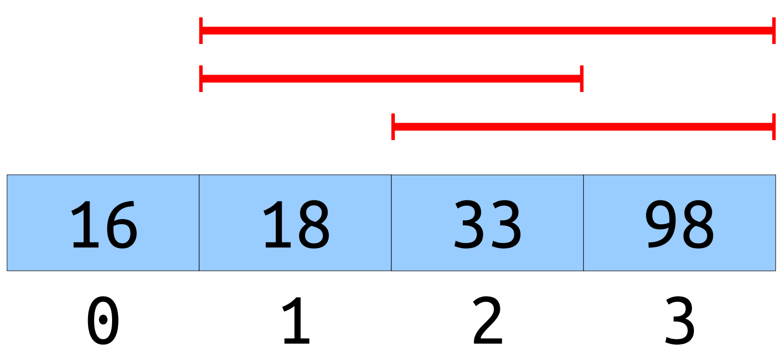
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	16	
1		18	18	★
2			33	33
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	16	
1		18	18	18
2			33	33
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16	16	
1		18	18	18
2			33	33
3				98

A Different Approach

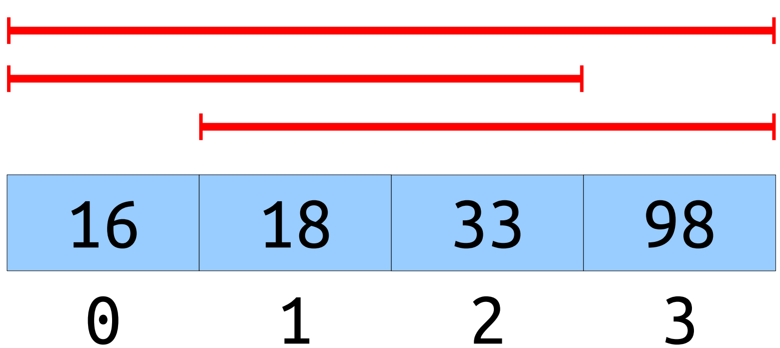
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16	16	★
1		18	18	18
2			33	33
3				98

A Different Approach

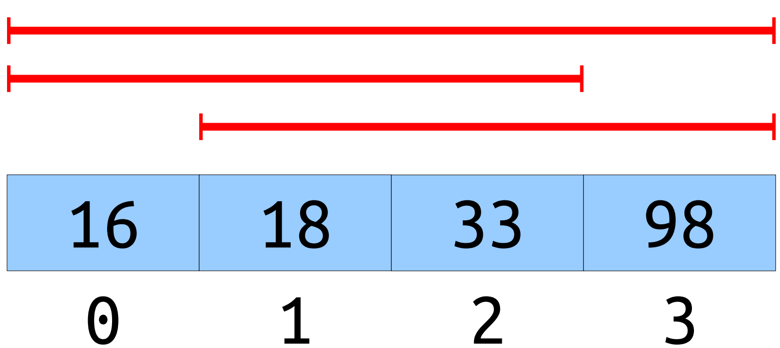
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	16	★
1		18	18	18
2			33	33
3				98

A Different Approach

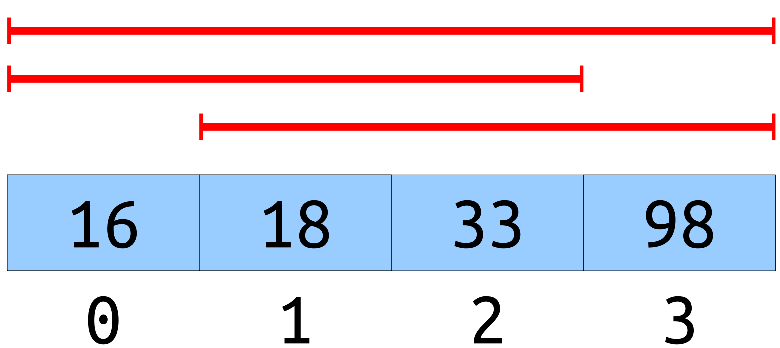
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	16	★
1		18	18	18
2			33	33
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16	16	16
1		18	18	18
2			33	33
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16	16	16
1		18	18	18
2			33	33
3				98

Some Notation

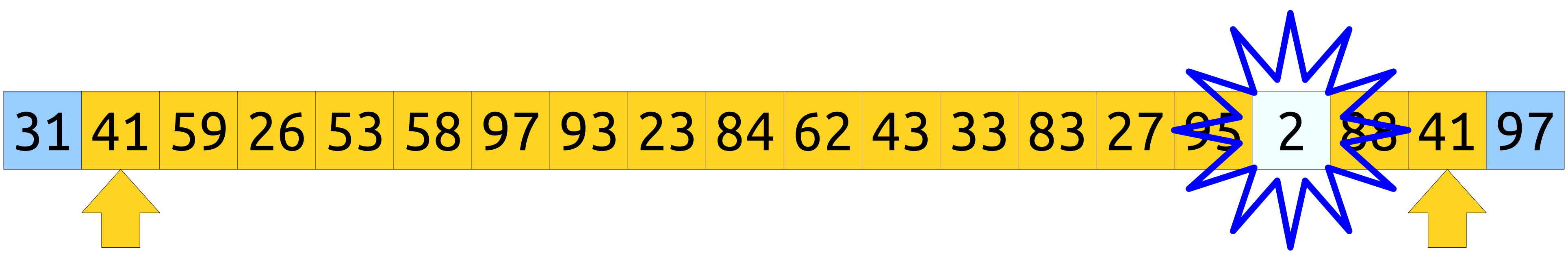
- We'll say that an RMQ data structure has time complexity **$\langle p(n), q(n) \rangle$** if
  - preprocessing takes time at most $p(n)$ and
  - queries take time at most $q(n)$.
- We now have two RMQ data structures:
  - $\langle O(1), O(n) \rangle$ with no preprocessing.
  - $\langle O(n^2), O(1) \rangle$ with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- ***Question:*** *Is there a "golden mean" between these extremes?*

Another Approach: ***Block Decomposition***

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97


31 41 59 26 53 58 97 93 23 84 62 43 33 83 27 95 2 88 41 97

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

A Block-Based Approach

$\lceil \sqrt{n} \rceil$

- Split the input into O(n / b) blocks of some "block size" b.

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 4.

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 4.
- Compute the minimum value in each block.

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

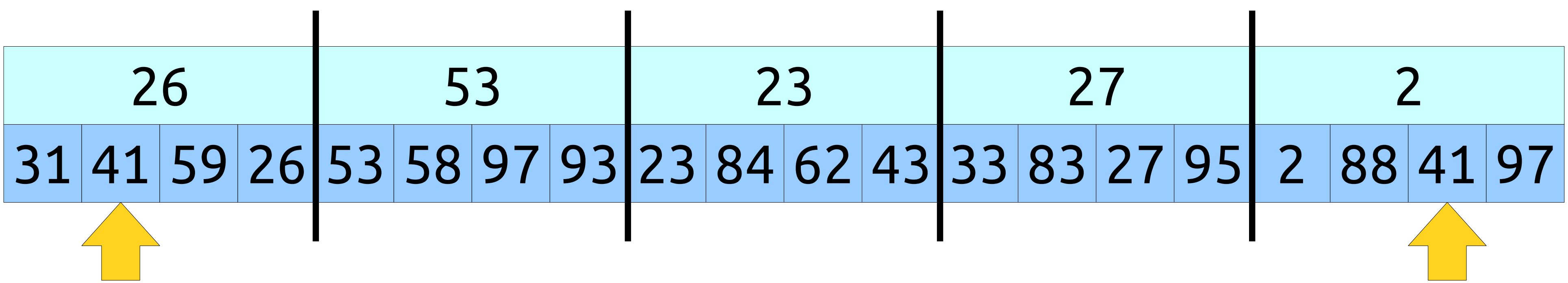
A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 4.
- Compute the minimum value in each block.

<table>
<tr><td colspan="4">26</td><td colspan="4">53</td><td colspan="4">23</td><td colspan="4">27</td><td colspan="4">2</td></tr>
<tr><td>31</td><td>41</td><td>59</td><td>26</td><td>53</td><td>58</td><td>97</td><td>93</td><td>23</td><td>84</td><td>62</td><td>43</td><td>33</td><td>83</td><td>27</td><td>95</td><td>2</td><td>88</td><td>41</td><td>97</td></tr>
</table>

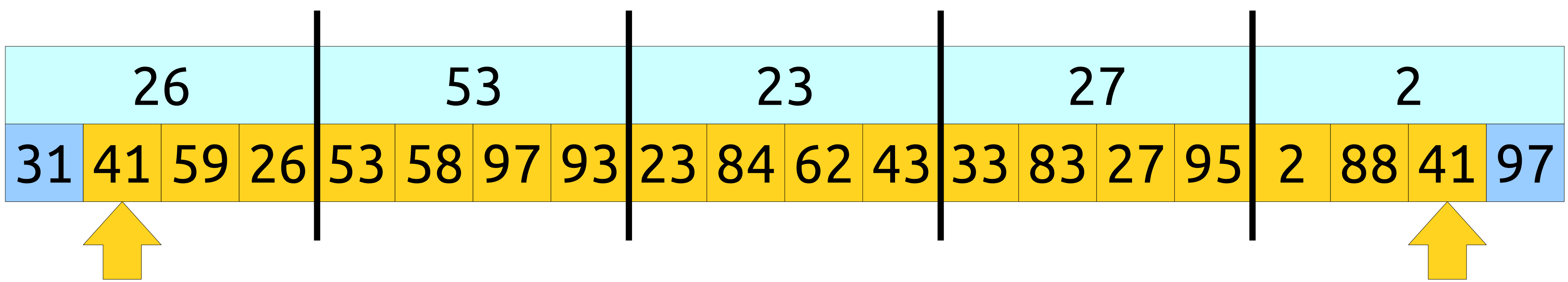
A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 4.
- Compute the minimum value in each block.



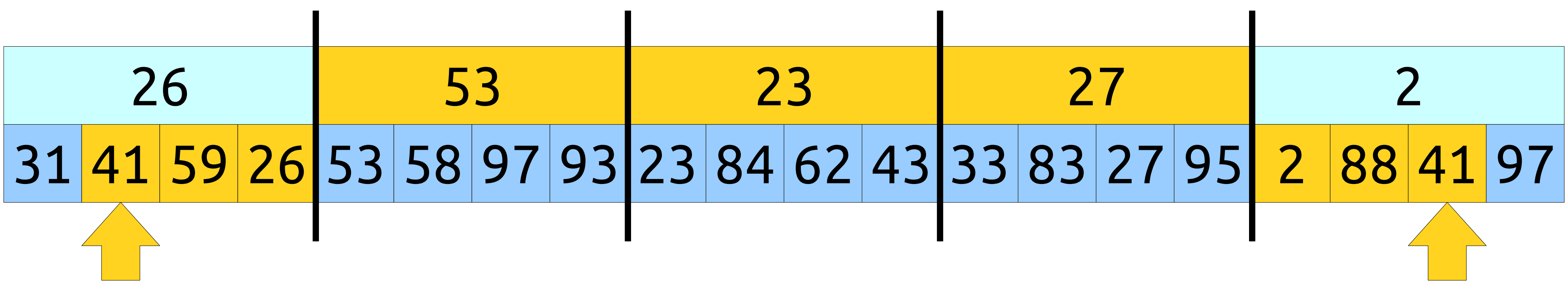
A Block-Based Approach

- Split the input into $O(n / b)$ blocks of some "block size" b.
    - Here, $b = 4$.
- Compute the minimum value in each block.



A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 4.
- Compute the minimum value in each block.



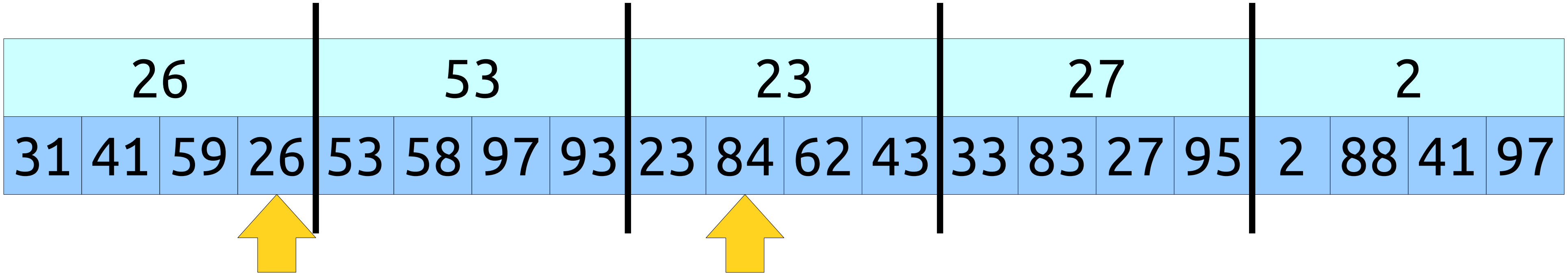
A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 4.
- Compute the minimum value in each block.

26				53				23				27				2			
31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

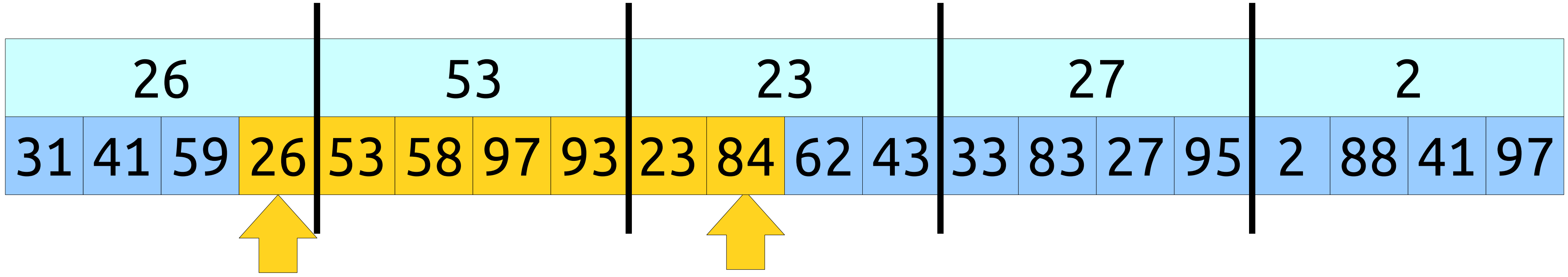
A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 4.
- Compute the minimum value in each block.



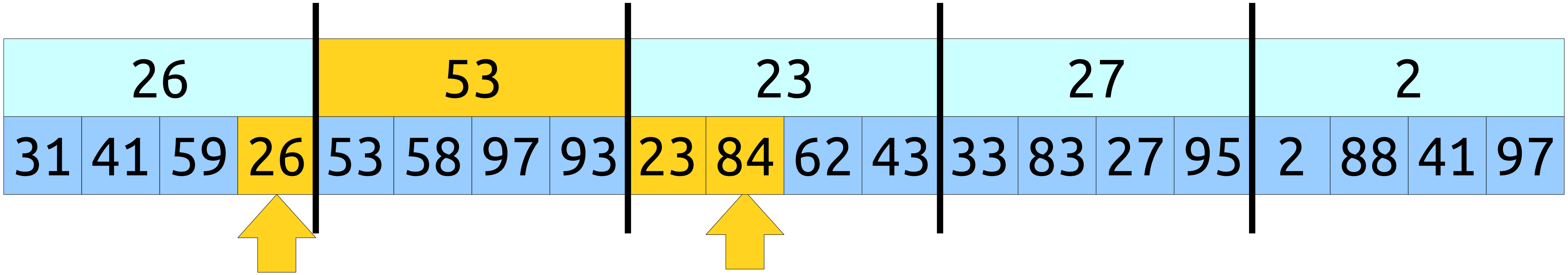
A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 4.
- Compute the minimum value in each block.



A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 4.
- Compute the minimum value in each block.



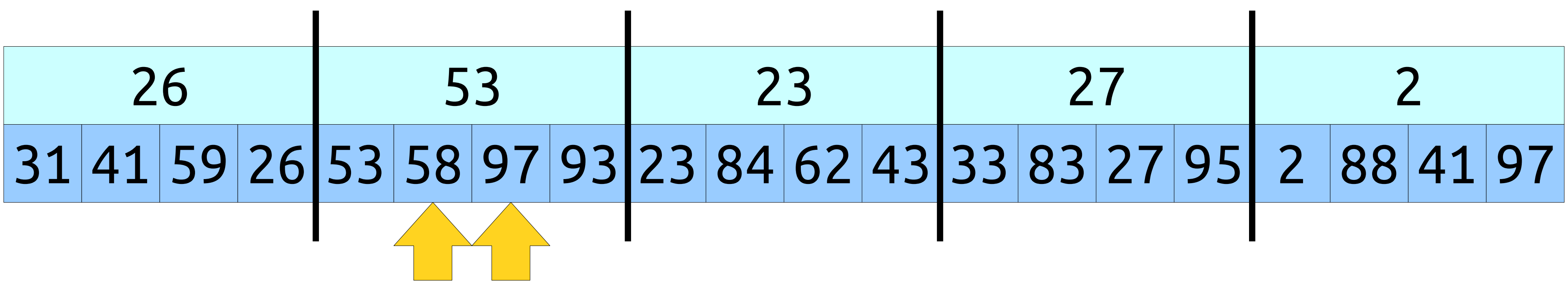
A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 4.
- Compute the minimum value in each block.

<table>
<tr><td colspan="4">26</td><td colspan="4">53</td><td colspan="4">23</td><td colspan="4">27</td><td colspan="4">2</td></tr>
<tr><td>31</td><td>41</td><td>59</td><td>26</td><td>53</td><td>58</td><td>97</td><td>93</td><td>23</td><td>84</td><td>62</td><td>43</td><td>33</td><td>83</td><td>27</td><td>95</td><td>2</td><td>88</td><td>41</td><td>97</td></tr>
</table>

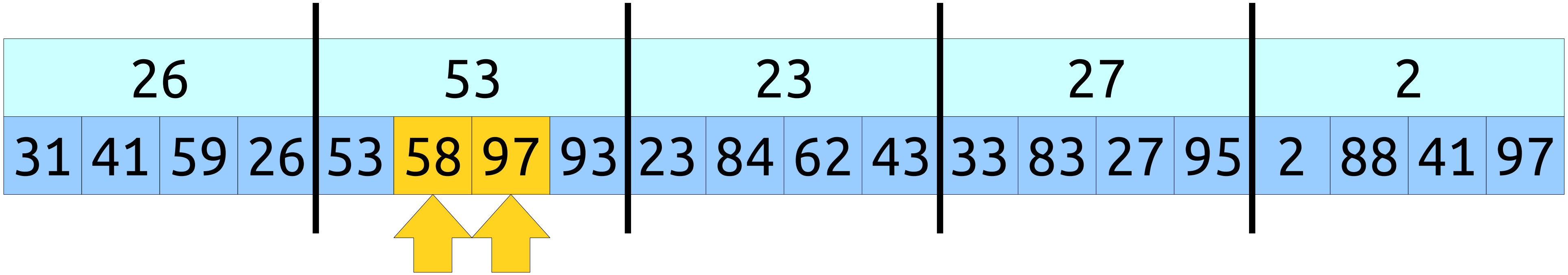
A Block-Based Approach

- Split the input into $O(n / b)$ blocks of some "block size" b.
  - Here, $b = 4$.
- Compute the minimum value in each block.



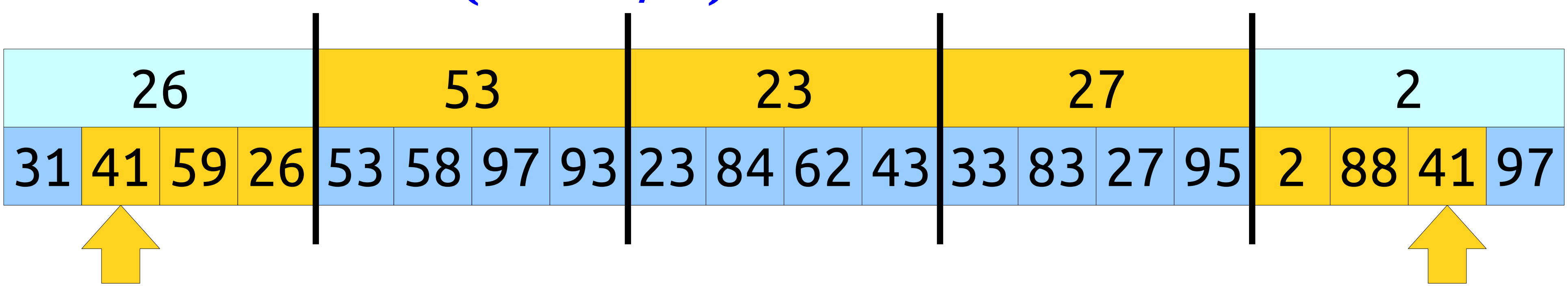
A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 4.
- Compute the minimum value in each block.



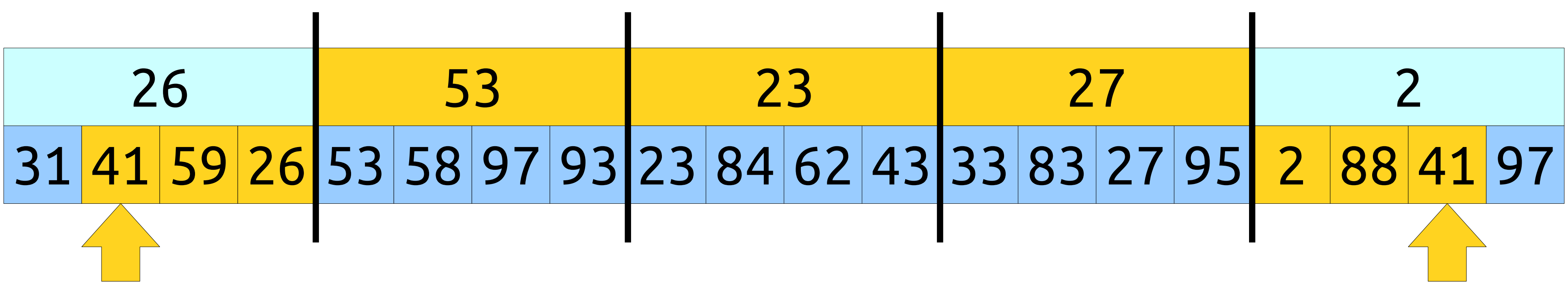
Analyzing the Approach

- Let's analyze this approach in terms of n and b.
- Preprocessing time:
  - O(b) work on O(n / b) blocks to find minima.
  - Total work: **O(n)**.
- Time to evaluate $\text{RMQ}_A(i, j)$:
  - O(1) work to find block indices (divide by block size).
  - O(b) work to scan inside i and j's blocks.
  - O(n / b) work looking at block minima between i and j.
  - Total work: **O(b + n / b)**.



Intuiting O(*b* + *n* / *b*)

- As *b* increases:
  - The ***b*** term rises (more elements to scan within each block).
  - The ***n* / *b*** term drops (fewer blocks to look at).
- As *b* decreases:
  - The ***b*** term drops (fewer elements to scan within a block).
  - The ***n* / *b*** term rises (more blocks to look at).
- Is there an optimal choice of *b* given these constraints?



Optimizing b

- What choice of b minimizes $b + n / b$?

Formulate a hypothesis!

Optimizing *b*

- What choice of b minimizes $b + n / b$?

Discuss with your neighbors!

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

Optimizing *b*

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$1-n/b^2 \;=\; 0$$

Optimizing *b*

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \end{aligned}$$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \end{aligned}$$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.

Optimizing *b*

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b)$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2})$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2})$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2}) = \mathbf{O(n^{1/2})}$

Summary of Approaches

- Three solutions so far:
  - Full preprocessing: $\langle O(n^2),\ O(1)\rangle$.
  - Block partition: $\langle O(n),\ O(n^{1/2})\rangle$.
  - No preprocessing: $\langle O(1),\ O(n)\rangle$.
- Modest preprocessing yields modest performance increases.
- ***Question:*** Can we do better?

A Second Approach: ***Sparse Tables***

An Intuition

- The $\langle O(n^2), O(1) \rangle$ solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- ***Question:*** Can we still get constant-time queries without preprocessing all possible ranges?

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

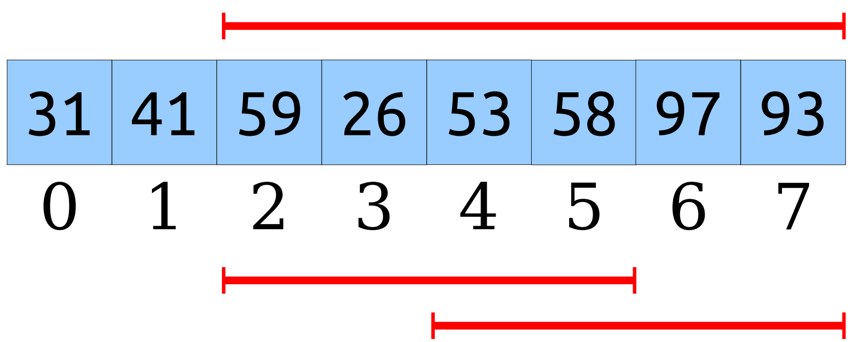
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

0	1	2	3	4	5	6	7
31	41	59	26	53	58	97	93

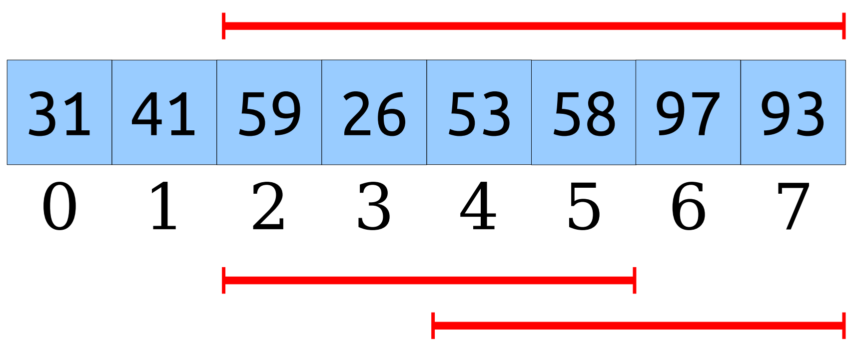
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

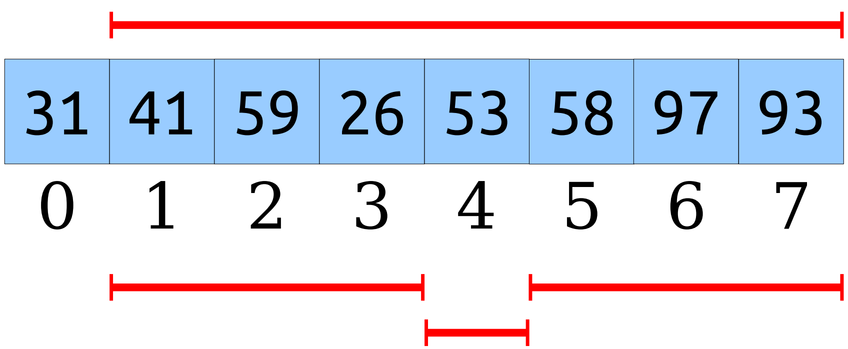
	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

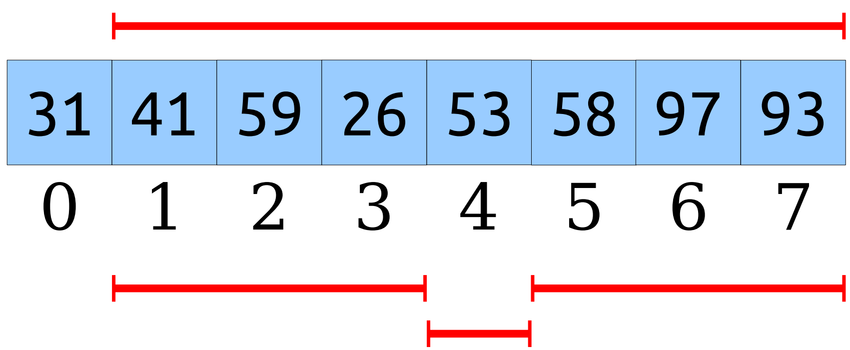
	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

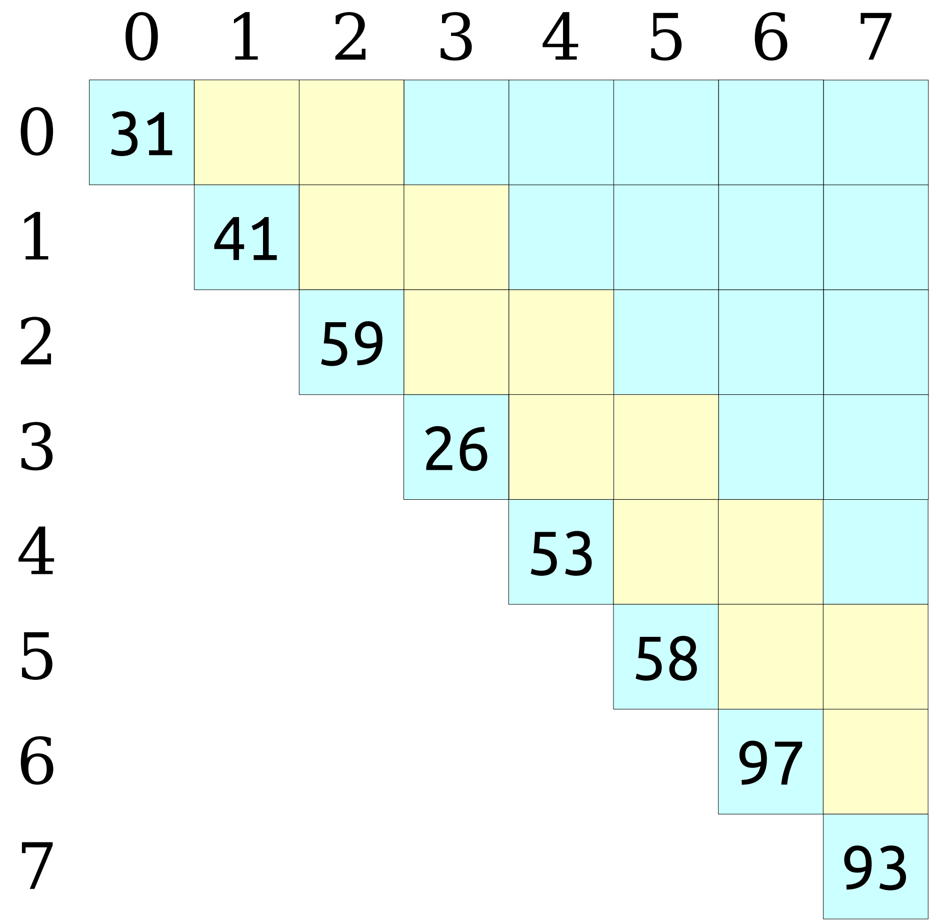
An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7



An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

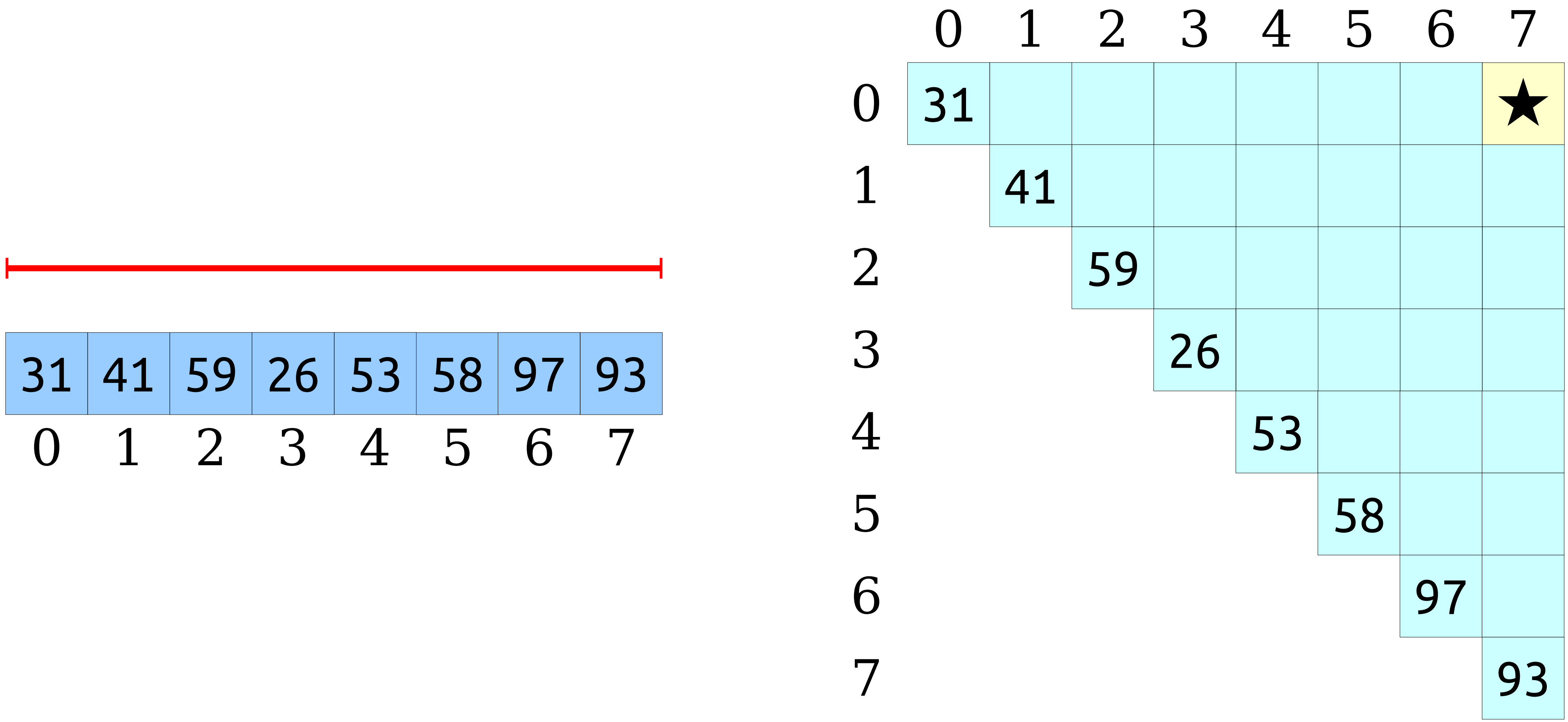
	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

An Observation

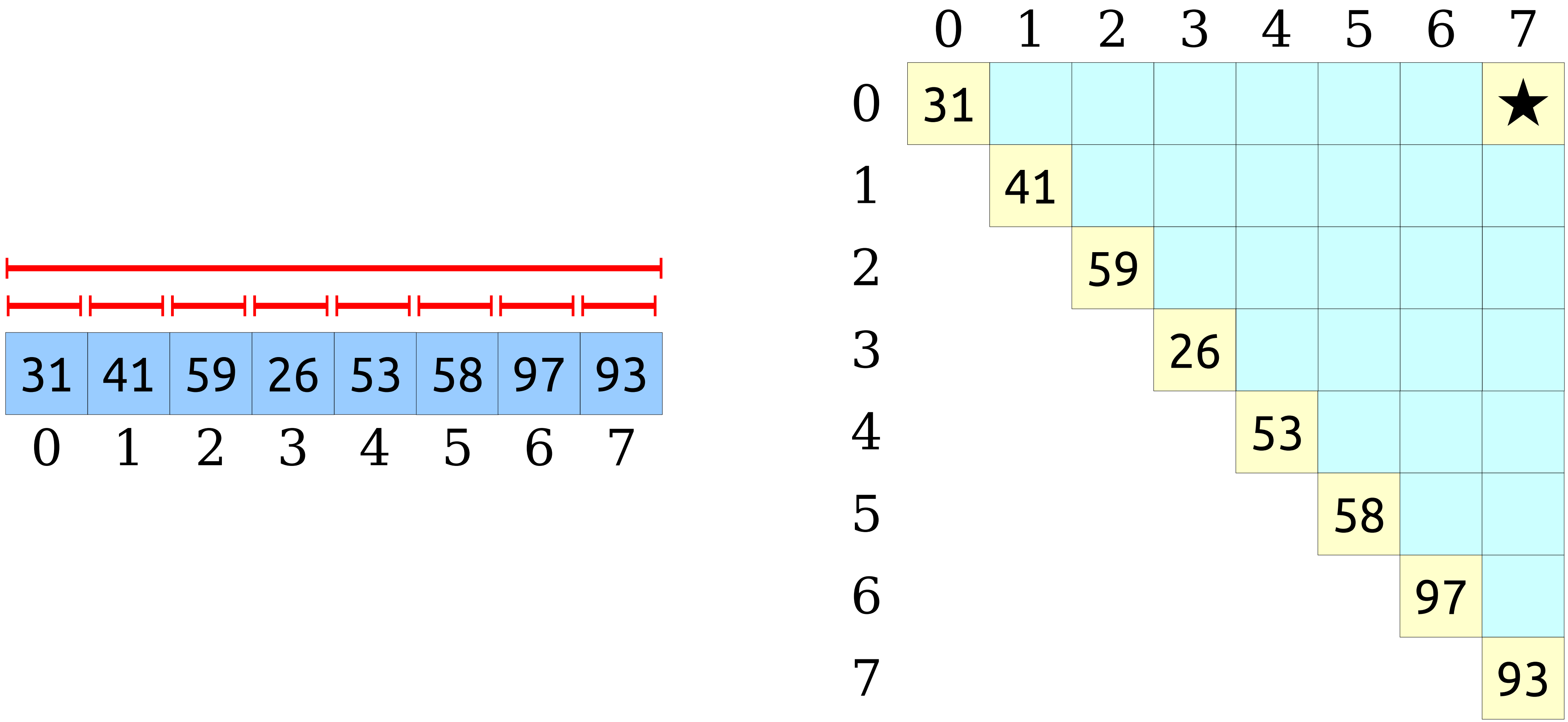
31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

An Observation



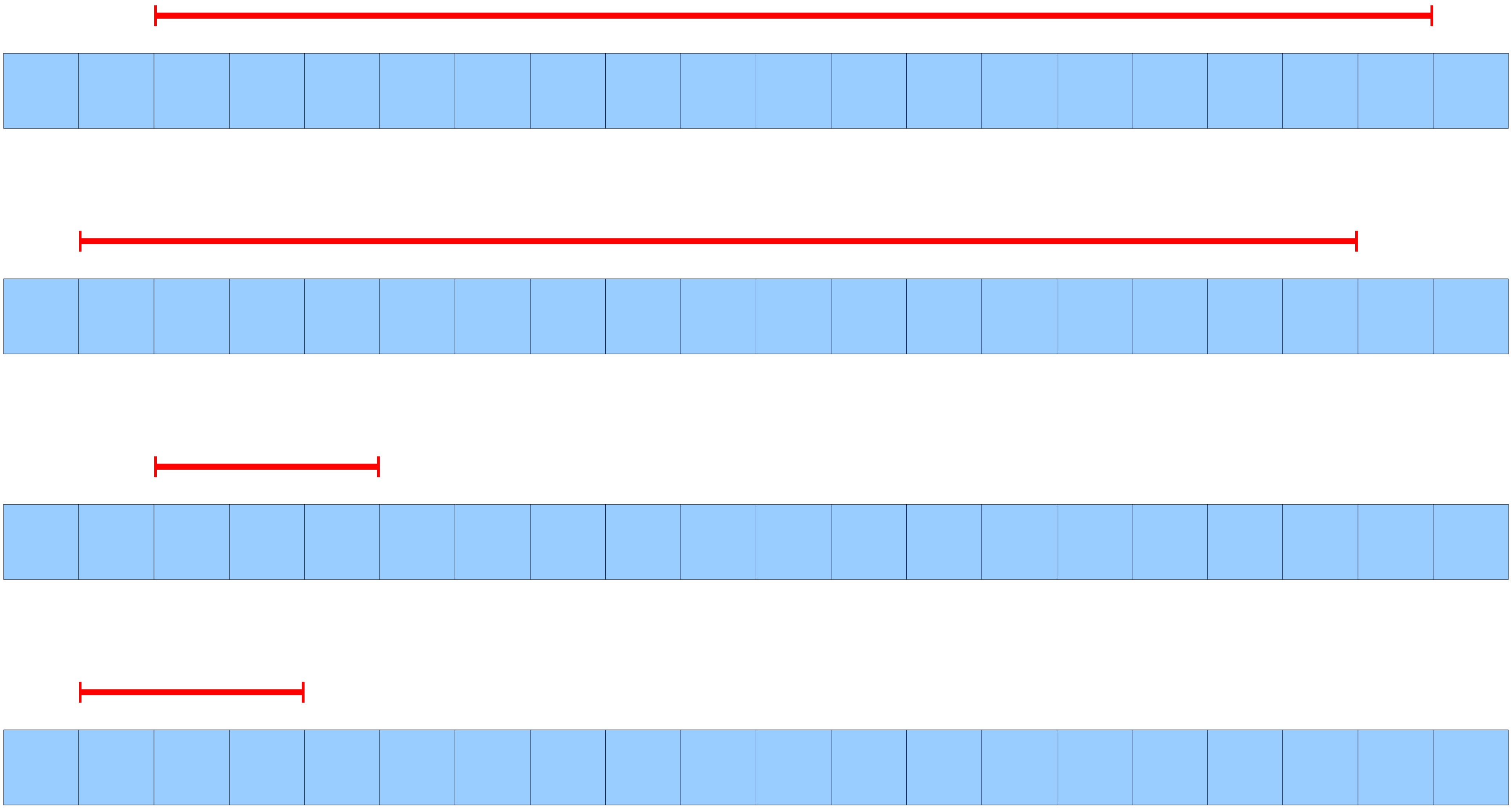
An Observation



The Intuition

- It's still possible to answer any query in time O(1) without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be O(1).
- ***Goal:*** Precompute RMQ over a set of ranges such that
  - there are $o(n^2)$ total ranges, but
  - there are enough ranges to support O(1) query times.

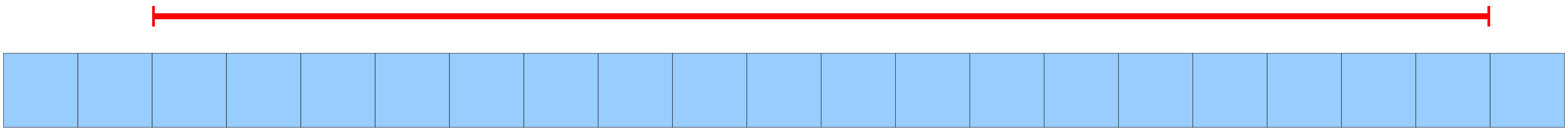
Some Observations



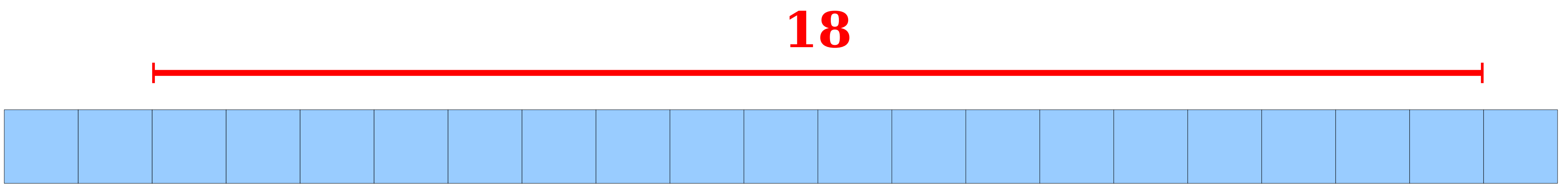
The Approach

- For each index i, compute RMQ for ranges starting at i of size 1, 2, 4, 8, 16, …, 2^k as long as they fit in the array.
  - Gives both large and small ranges starting at any point in the array.
  - Only O(log n) ranges computed for each array element.
  - Total number of ranges: O(n log n).
- ***Claim:*** Any range in the array can be formed as the union of two of these ranges.

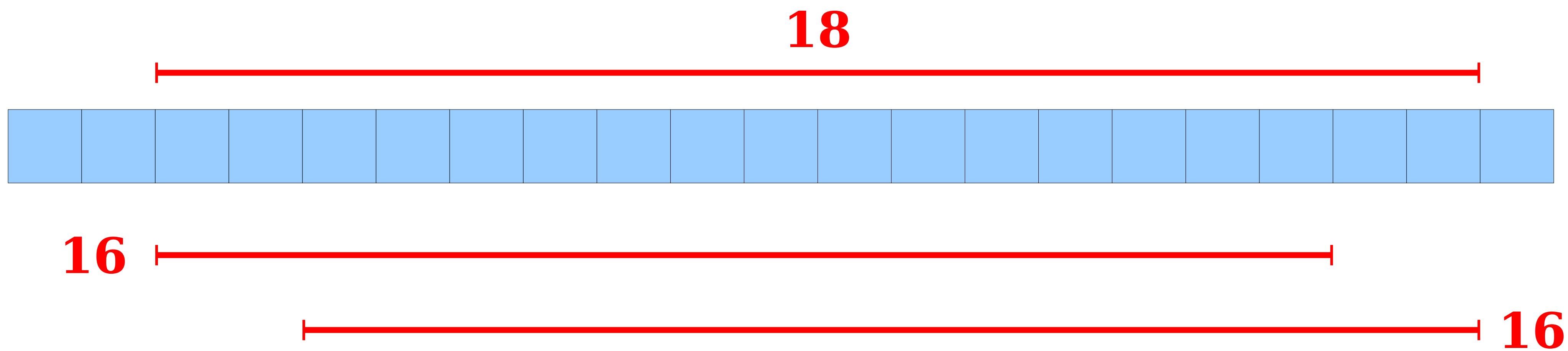
Creating Ranges



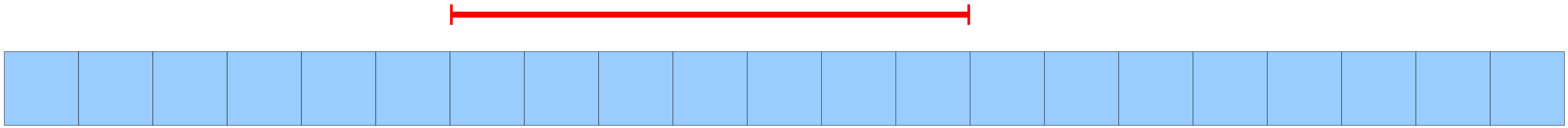
Creating Ranges



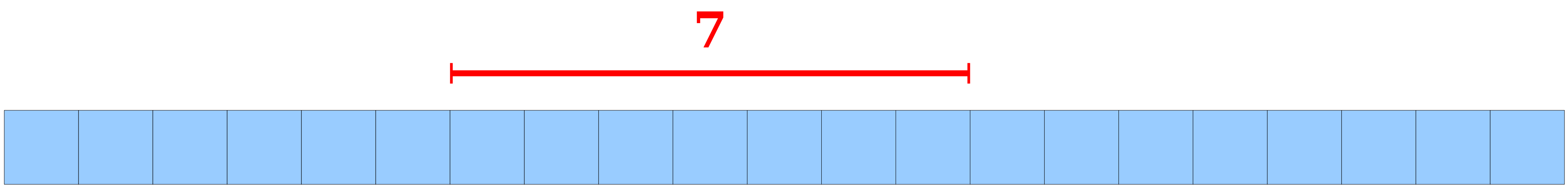
Creating Ranges



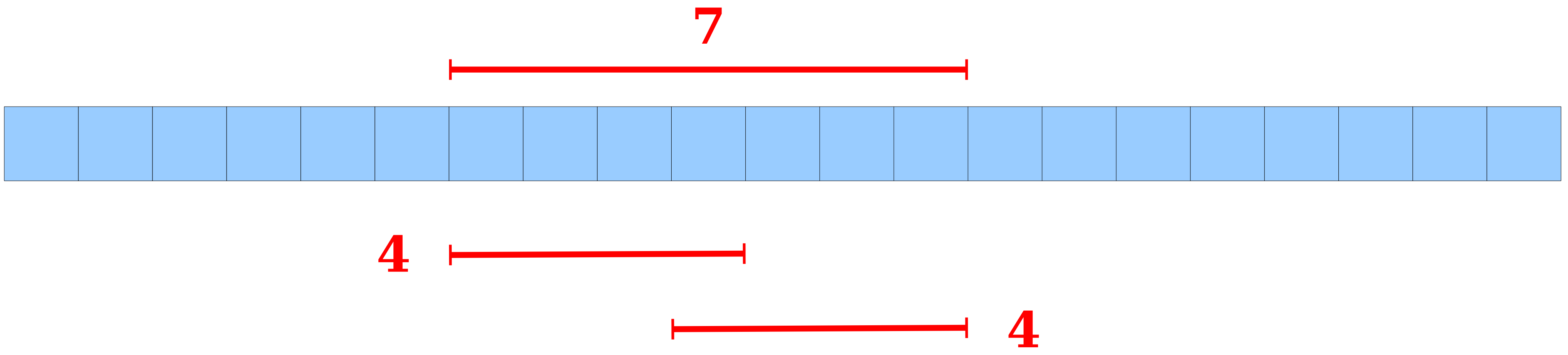
Creating Ranges



Creating Ranges



Creating Ranges



Doing a Query

- To answer $\mathrm{RMQ}_A(i, j)$:
  - Find the largest k such that $2^k \leq j - i + 1$.
    - With the right preprocessing, this can be done in time O(1); you'll figure out how in an upcoming assignment.
  - The range $[i, j]$ can be formed as the overlap of the ranges $[i, i + 2^k - 1]$ and $[j - 2^k + 1, j]$.
  - Each range can be looked up in time O(1).
  - Total time: **O(1)**.

Precomputing the Ranges

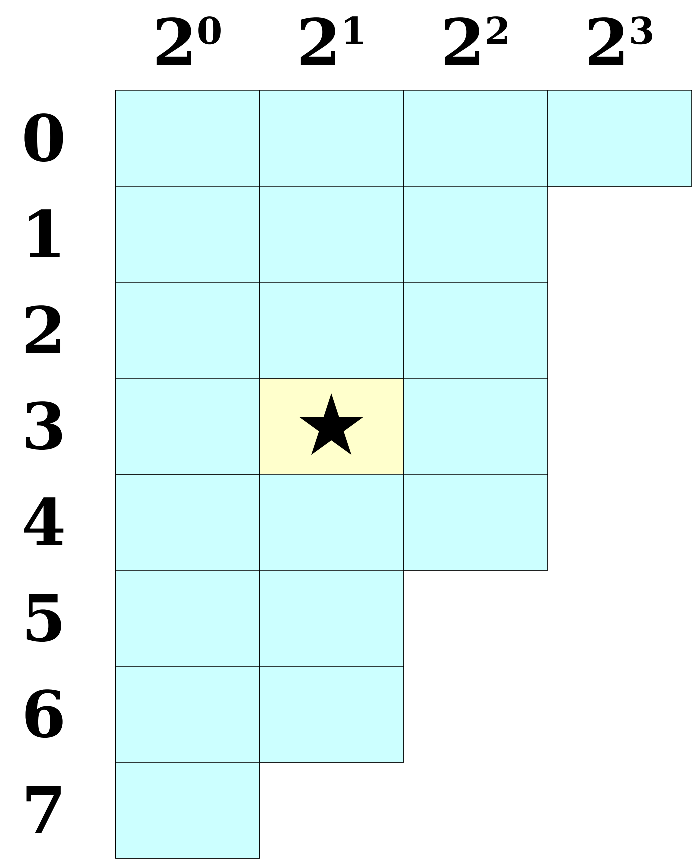
- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0				
1				
2				
3				
4				
5				
6				
7				

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

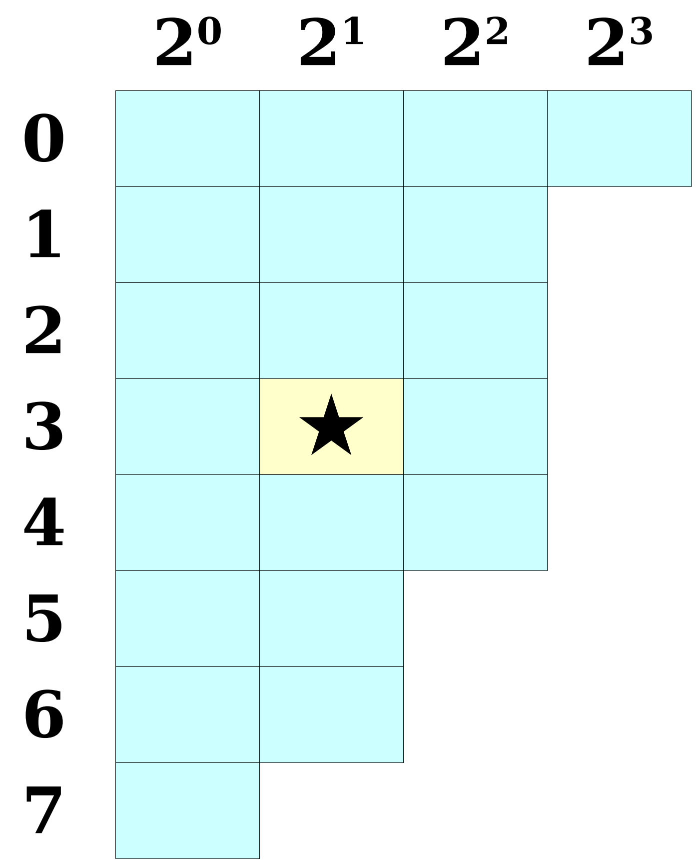
- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

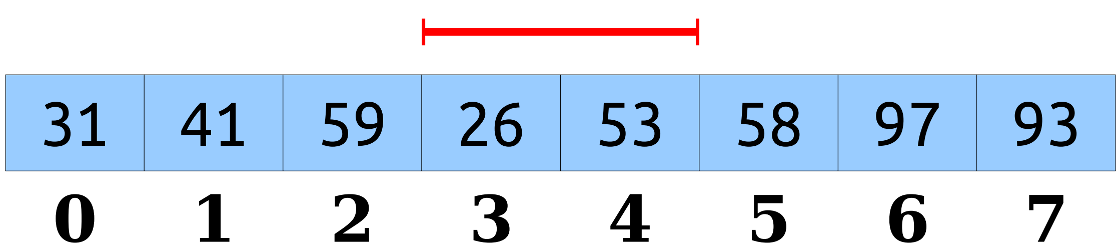


31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).





Precomputing the Ranges

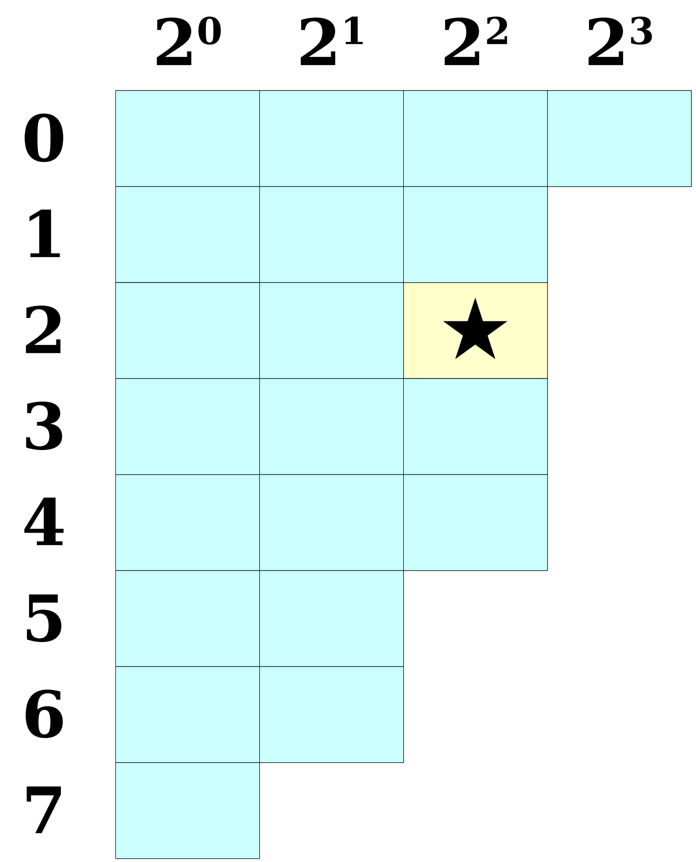
- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0				
1				
2			★	
3				
4				
5				
6				
7				

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

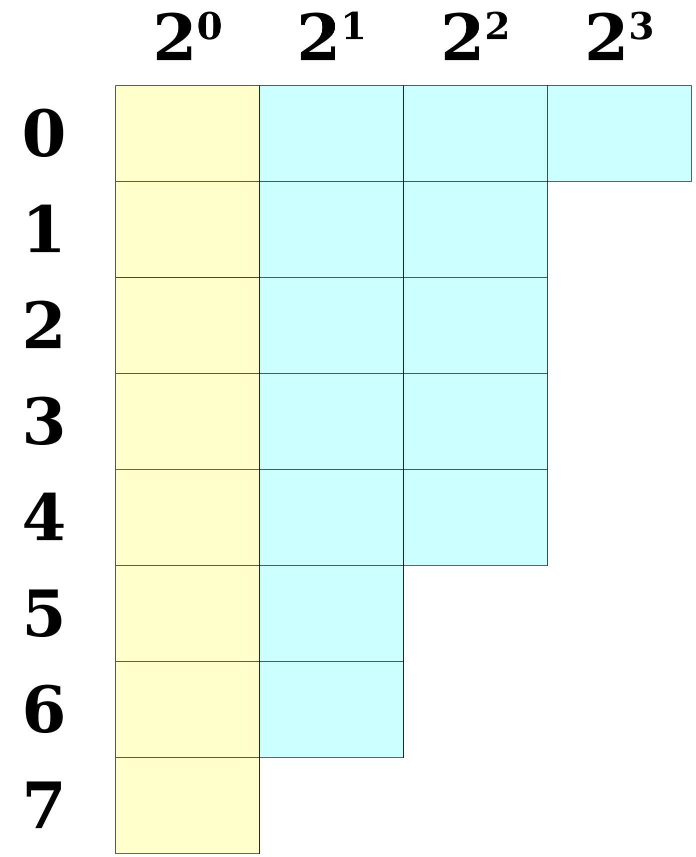


31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

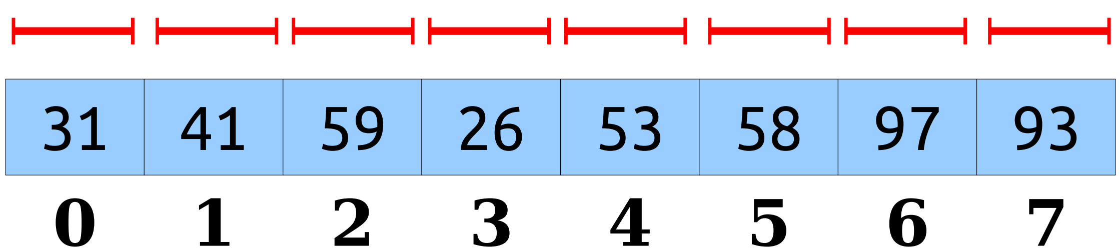
- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

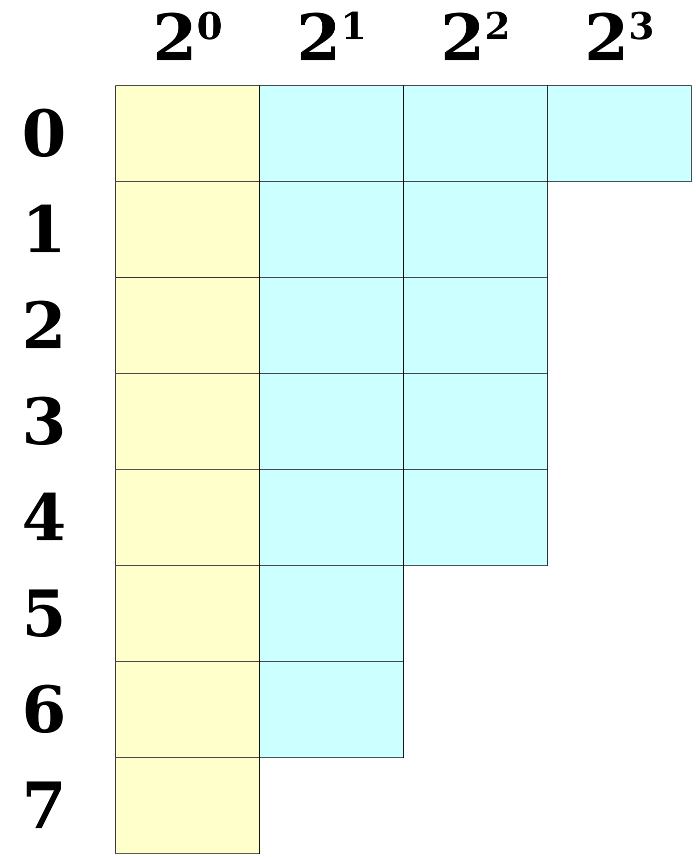
31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**



Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

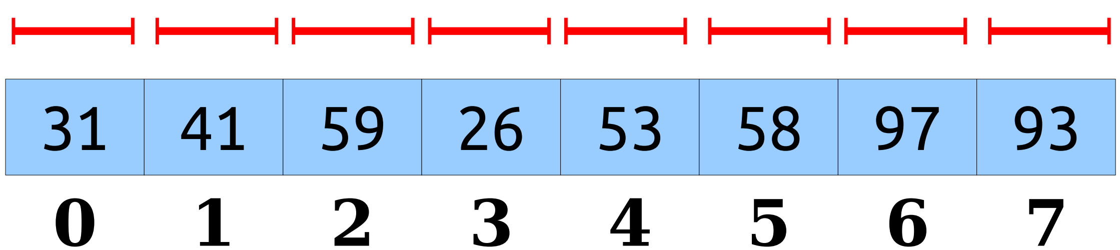




Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31			
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			



Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
0	31			
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	★		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
0	31	★		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

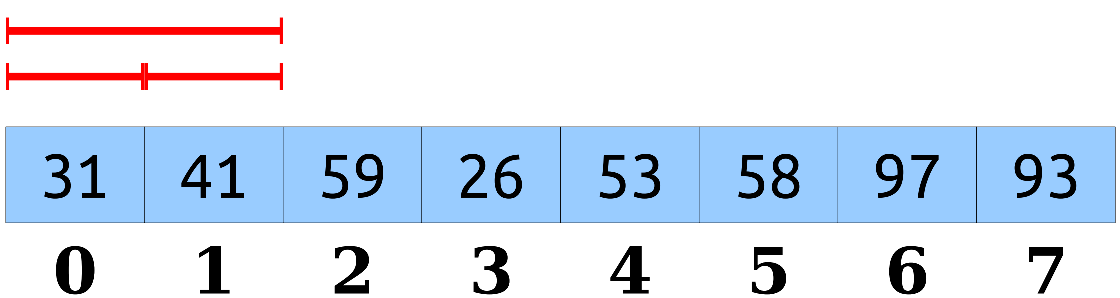
- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
0	31	★		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).



	2^0	2^1	2^2	2^3
0	31	★		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

0	1	2	3	4	5	6	7
31	41	59	26	53	58	97	93

	2^0	2^1	2^2	2^3
0	31	31		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

0	1	2	3	4	5	6	7
31	41	59	26	53	58	97	93

	2^0	2^1	2^2	2^3
0	31	31		
1	41	★		
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31		
1	41	★		
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31		
1	41	★		
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31		
1	41	★		
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31		
1	41	41		
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

	2^0	2^1	2^2	2^3
0	31	31		
1	41	41		
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

	2^0	2^1	2^2	2^3
0	31	31		
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

0	1	2	3	4	5	6	7
31	41	59	26	53	58	97	93

	2^0	2^1	2^2	2^3
0	31	31	★	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31	★	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31	★	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31	★	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31	26	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
0	31	31	26	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31	26	26
1	41	41	26	
2	59	26	26	
3	26	26	26	
4	53	53	53	
5	58	58		
6	97	93		
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Sparse Tables

- This data structure is called a ***sparse table***.
- It gives an **⟨O(n log n), O(1)⟩** solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

The Story So Far

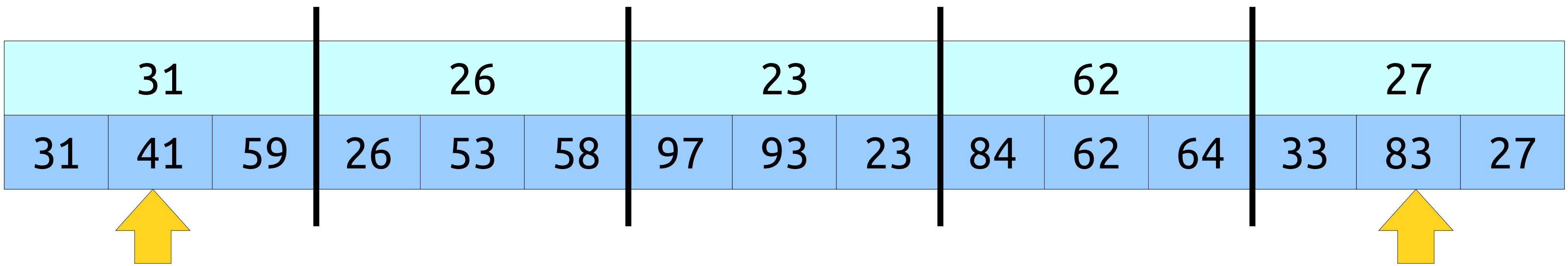
- We now have the following solutions for RMQ:
  - Precompute all: $\langle O(n^2), O(1)\rangle$.
  - Sparse table: $\langle O(n \log n), O(1)\rangle$.
  - Blocking: $\langle O(n), O(n^{1/2})\rangle$.
  - Precompute none: $\langle O(1), O(n)\rangle$.
- ***Can we do better?***

A Third Approach: ***Hybrid Strategies***

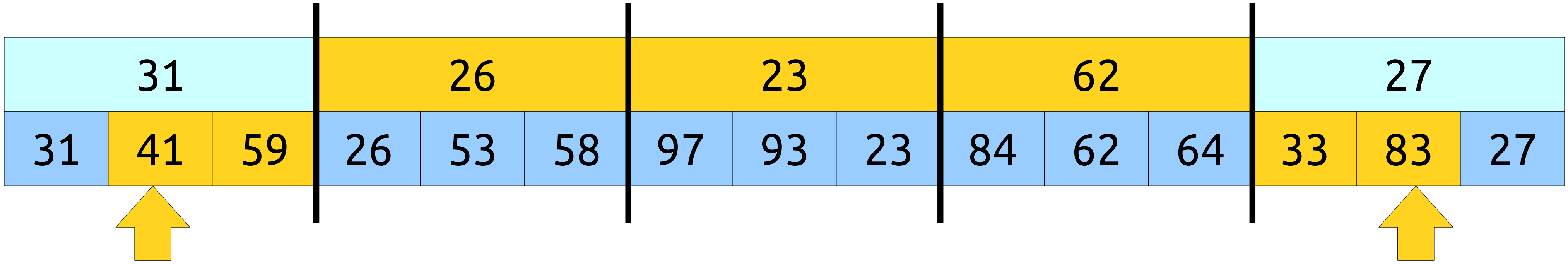
Blocking Revisited

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

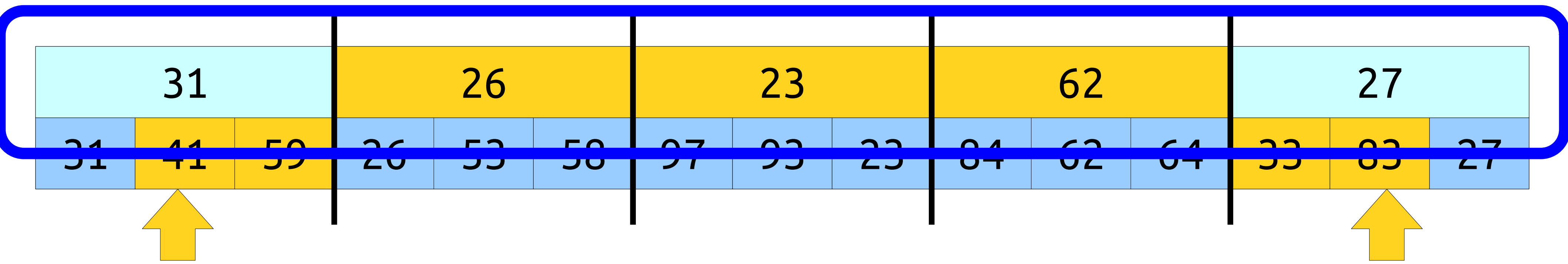
Blocking Revisited



Blocking Revisited

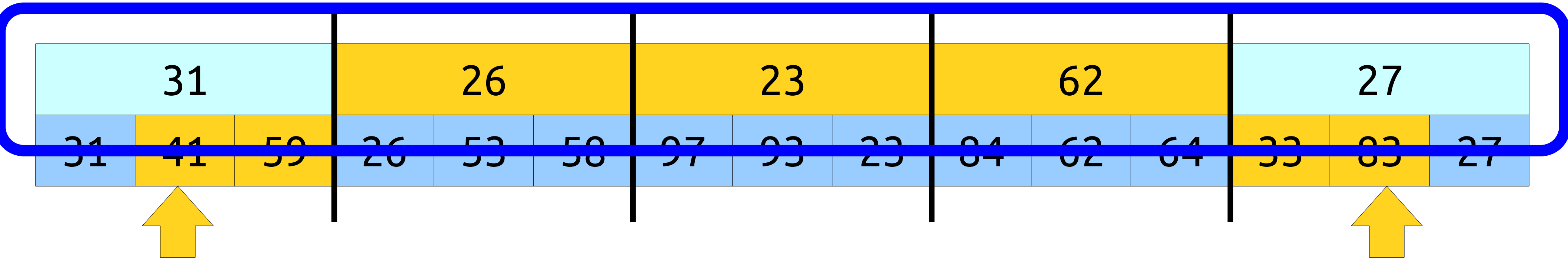


Blocking Revisited

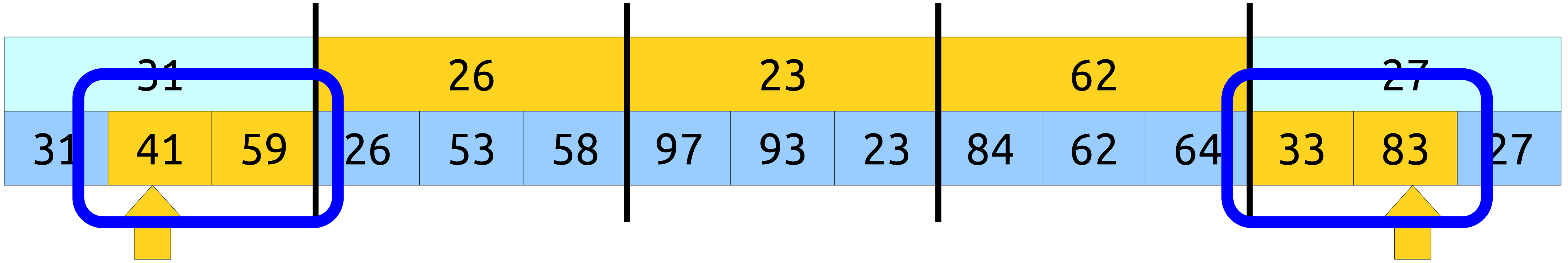


Blocking Revisited

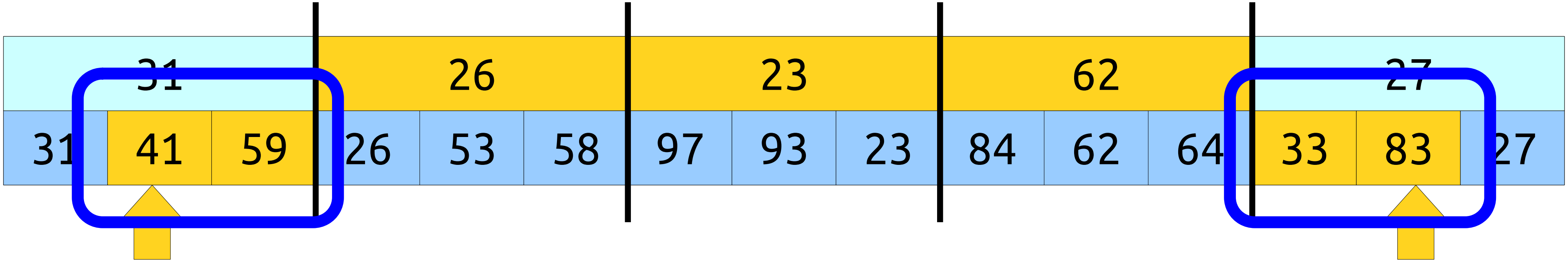
This is just RMQ on the block minima!



Blocking Revisited



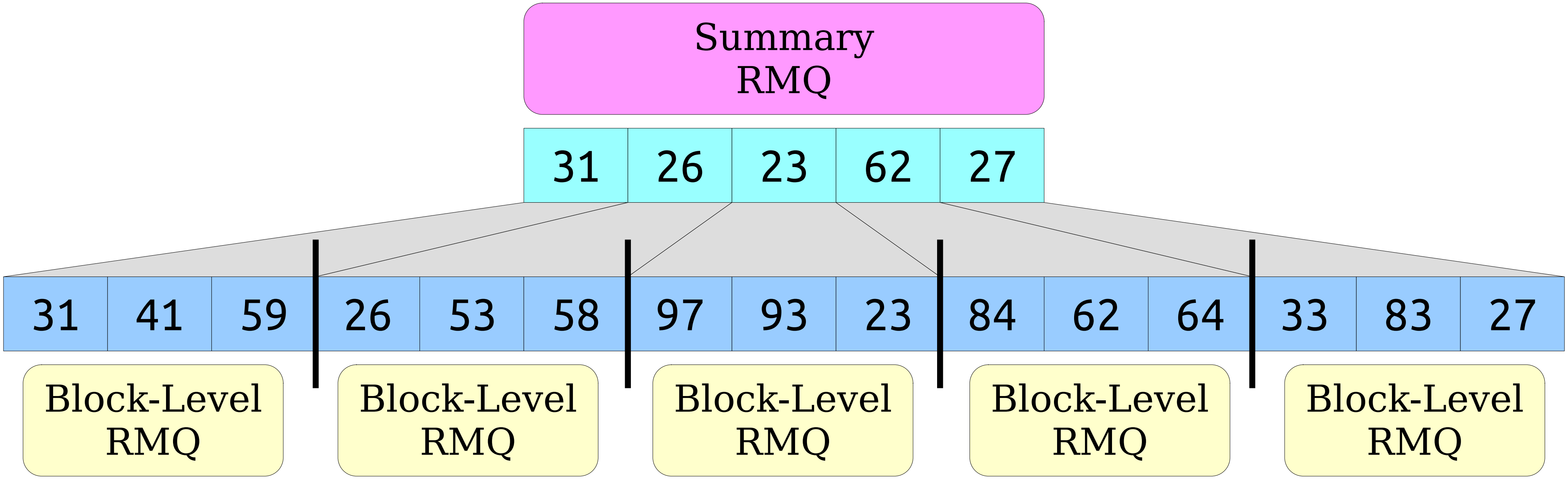
Blocking Revisited



This is just RMQ inside the blocks!

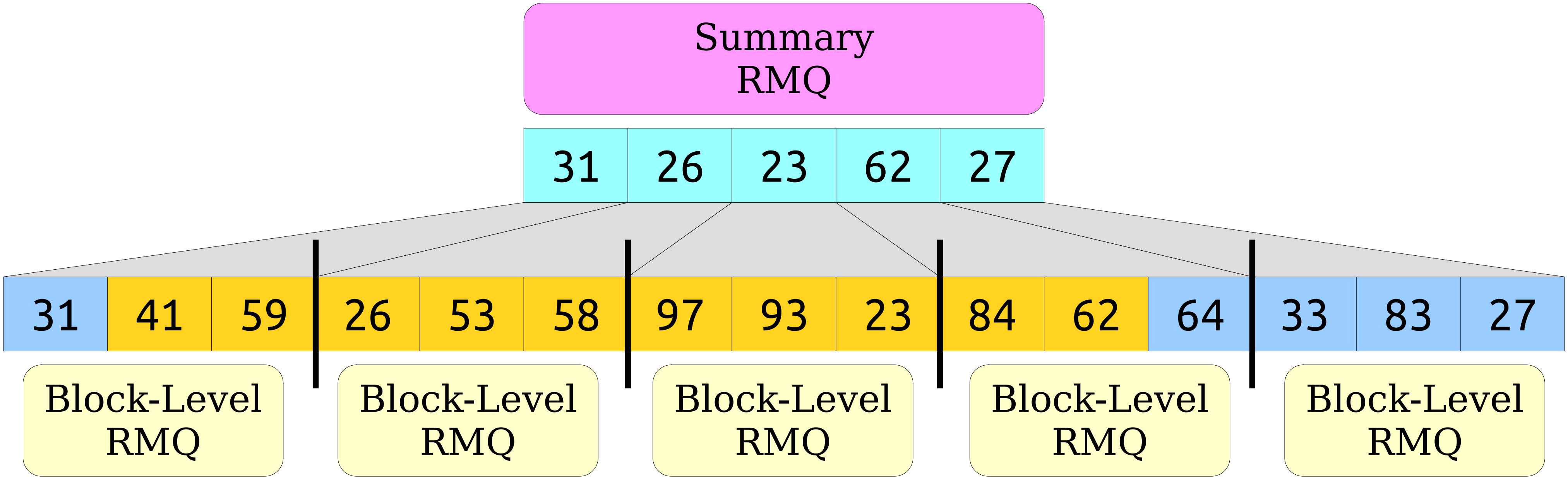
The Framework

- Split the input into blocks of size *b*.
- Form an array of the block minima.
- Construct a “summary” RMQ structure over the block minima.
- Construct “block” RMQ structures for each block.
- Aggregate the results together.



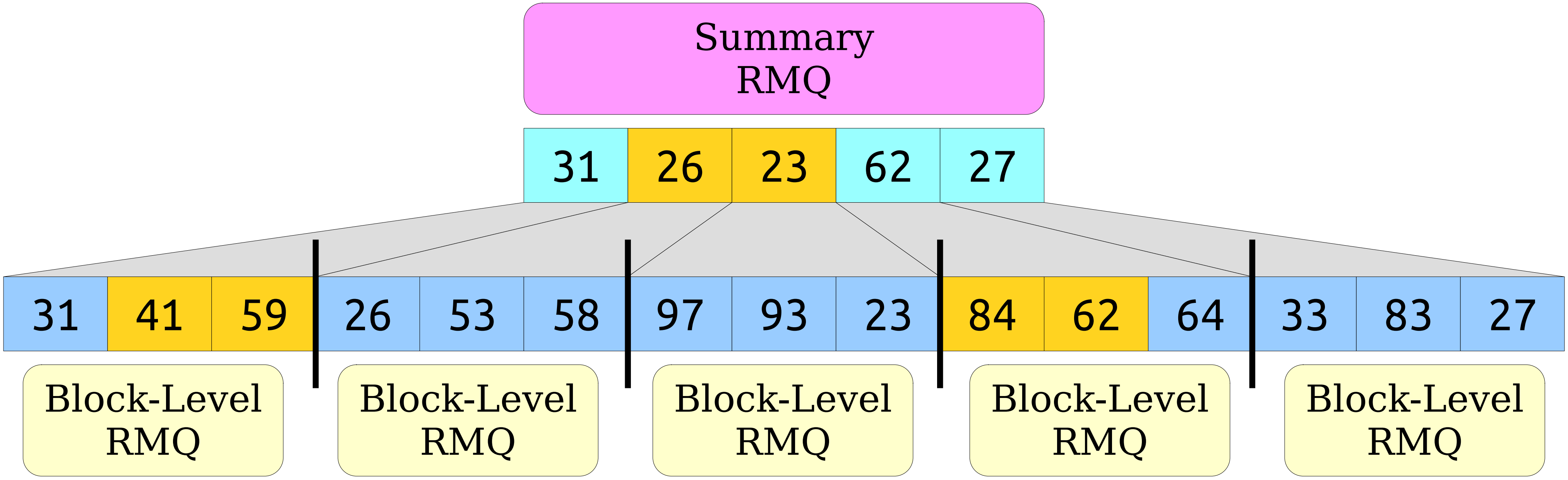
The Framework

- Split the input into blocks of size b.
- Form an array of the block minima.
- Construct a “summary” RMQ structure over the block minima.
- Construct “block” RMQ structures for each block.
- Aggregate the results together.



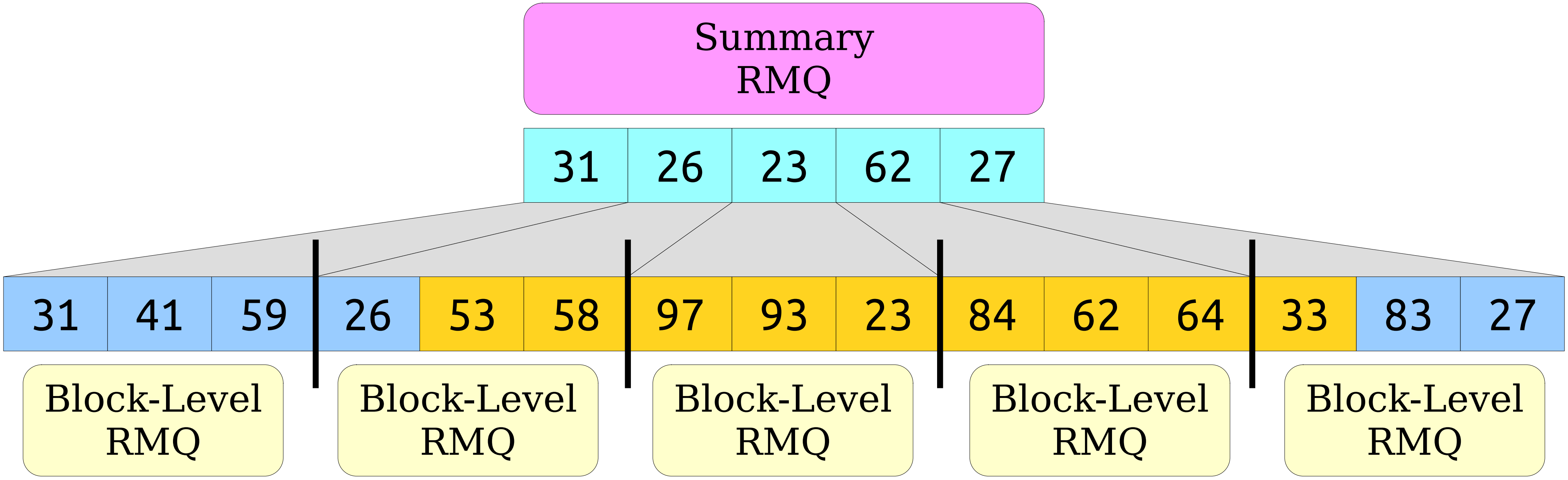
The Framework

- Split the input into blocks of size b.
- Form an array of the block minima.
- Construct a "summary" RMQ structure over the block minima.
- Construct "block" RMQ structures for each block.
- Aggregate the results together.



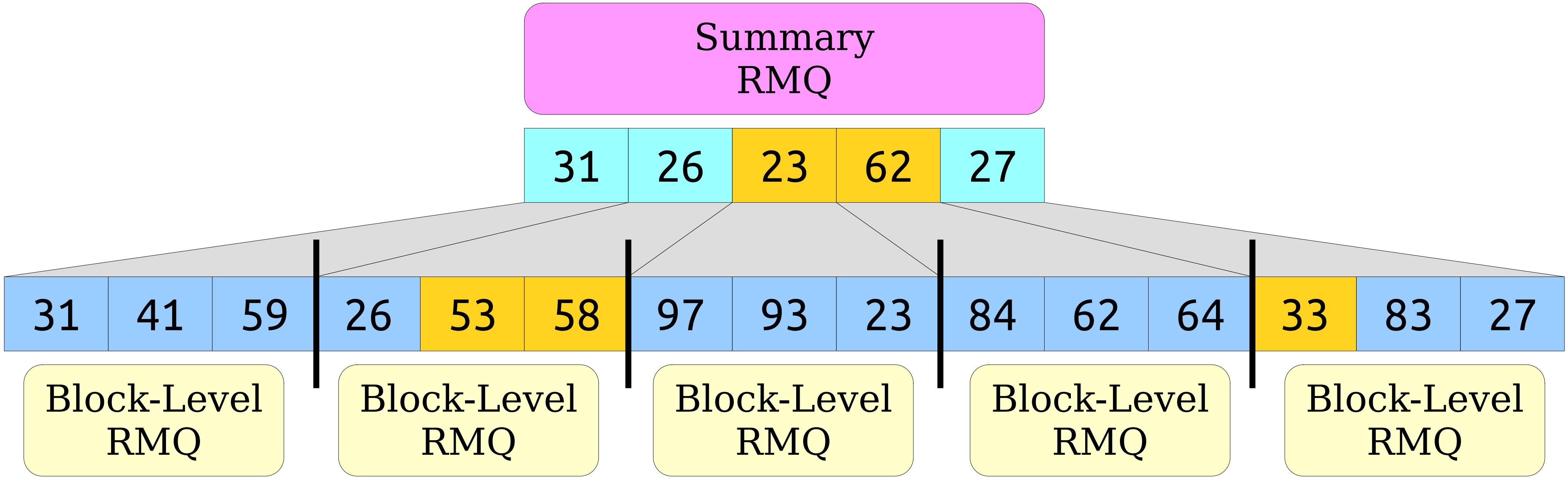
The Framework

- Split the input into blocks of size b.
- Form an array of the block minima.
- Construct a "summary" RMQ structure over the block minima.
- Construct "block" RMQ structures for each block.
- Aggregate the results together.



The Framework

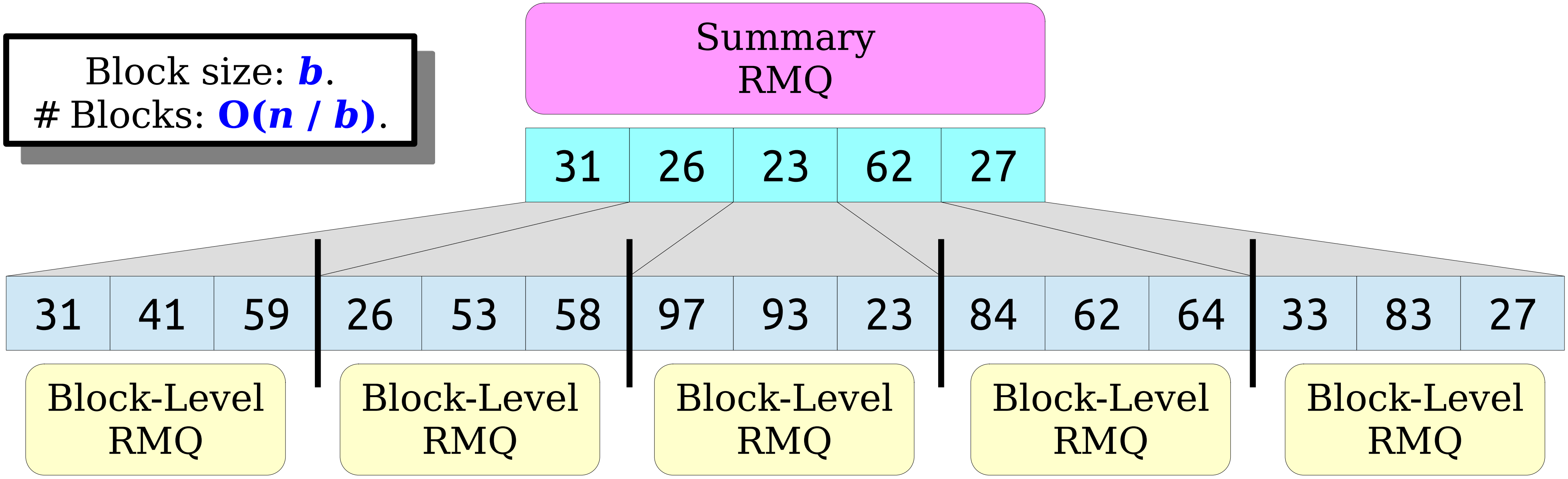
- Split the input into blocks of size b.
- Form an array of the block minima.
- Construct a “summary” RMQ structure over the block minima.
- Construct “block” RMQ structures for each block.
- Aggregate the results together.



Analyzing Efficiency

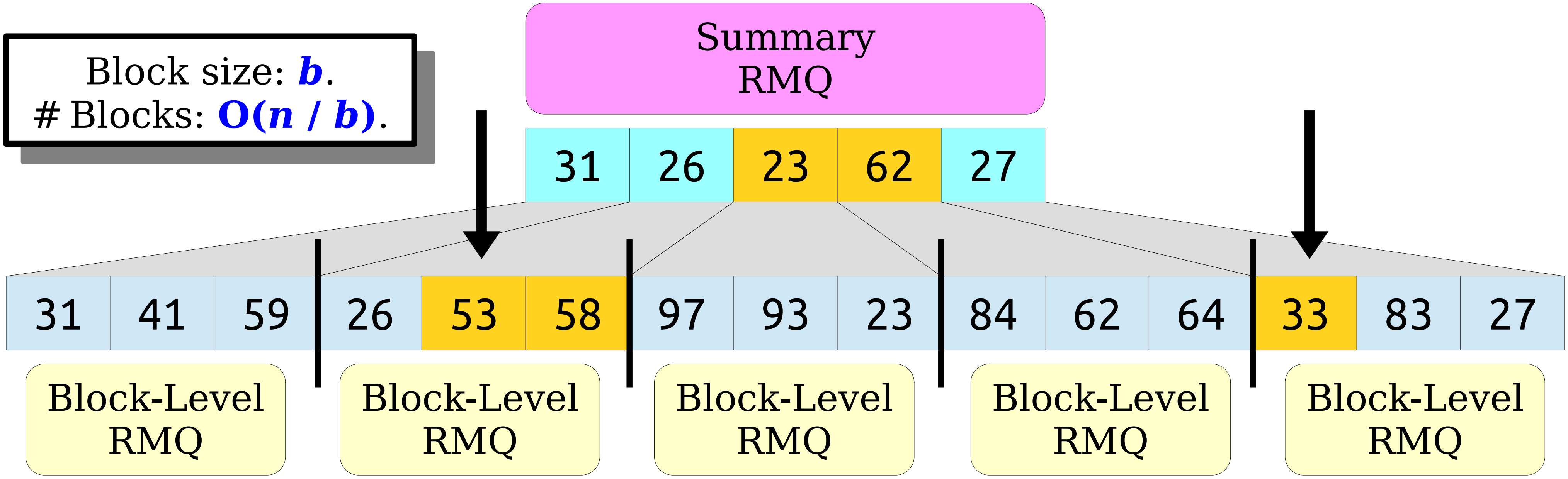
- Suppose we use a $\langle p_1(n), q_1(n)\rangle$-time RMQ for the summary RMQ and a $\langle p_2(n), q_2(n)\rangle$-time RMQ for each block, with block size b.
- What is the preprocessing time for this hybrid structure?
  - $\mathbf{O}(n)$ time to compute the minima of each block.
  - $\mathbf{O}(p_1(n / b))$ time to construct RMQ on the minima.
  - $\mathbf{O}((n / b)\ p_2(b))$ time to construct the block RMQs.
- Total construction time is $\mathbf{O}(n + p_1(n / b) + (n / b)\ p_2(b))$.

Block size: b.
\# Blocks: $\mathbf{O}(n / b)$.



Analyzing Efficiency

- Suppose we use a ⟨$p_1(n)$, $q_1(n)$⟩-time RMQ for the summary RMQ and a ⟨$p_2(n)$, $q_2(n)$⟩-time RMQ for each block, with block size b.
- What is the query time for this hybrid structure?
  - **$O(q_1(n / b))$** time to query the summary RMQ.
  - **$O(q_2(b))$** time to query the block RMQs.
- Total query time: **$O(q_1(n / b) + q_2(b))$**.



Analyzing Efficiency

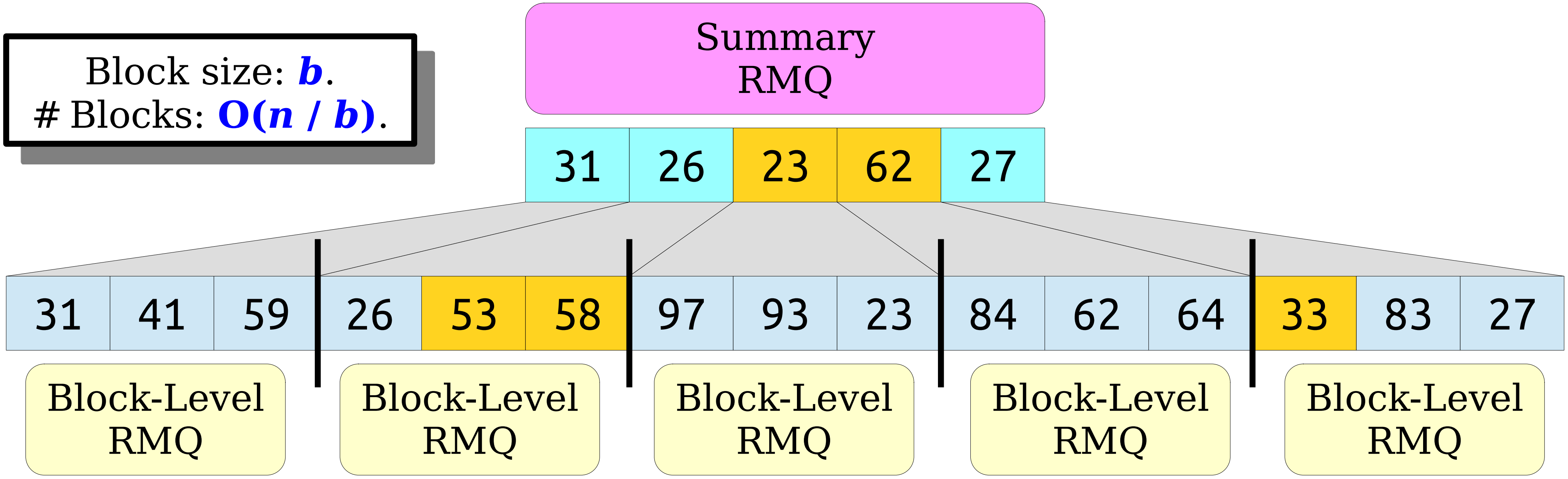
- Suppose we use a $\langle p_1(n), q_1(n)\rangle$-time RMQ for the summary RMQ and a $\langle p_2(n), q_2(n)\rangle$-time RMQ for each block, with block size b.
- Hybrid preprocessing time:

$$\mathbf{O(n + p_1(n / b) + (n / b)p_2(b))}$$

- Hybrid query time:

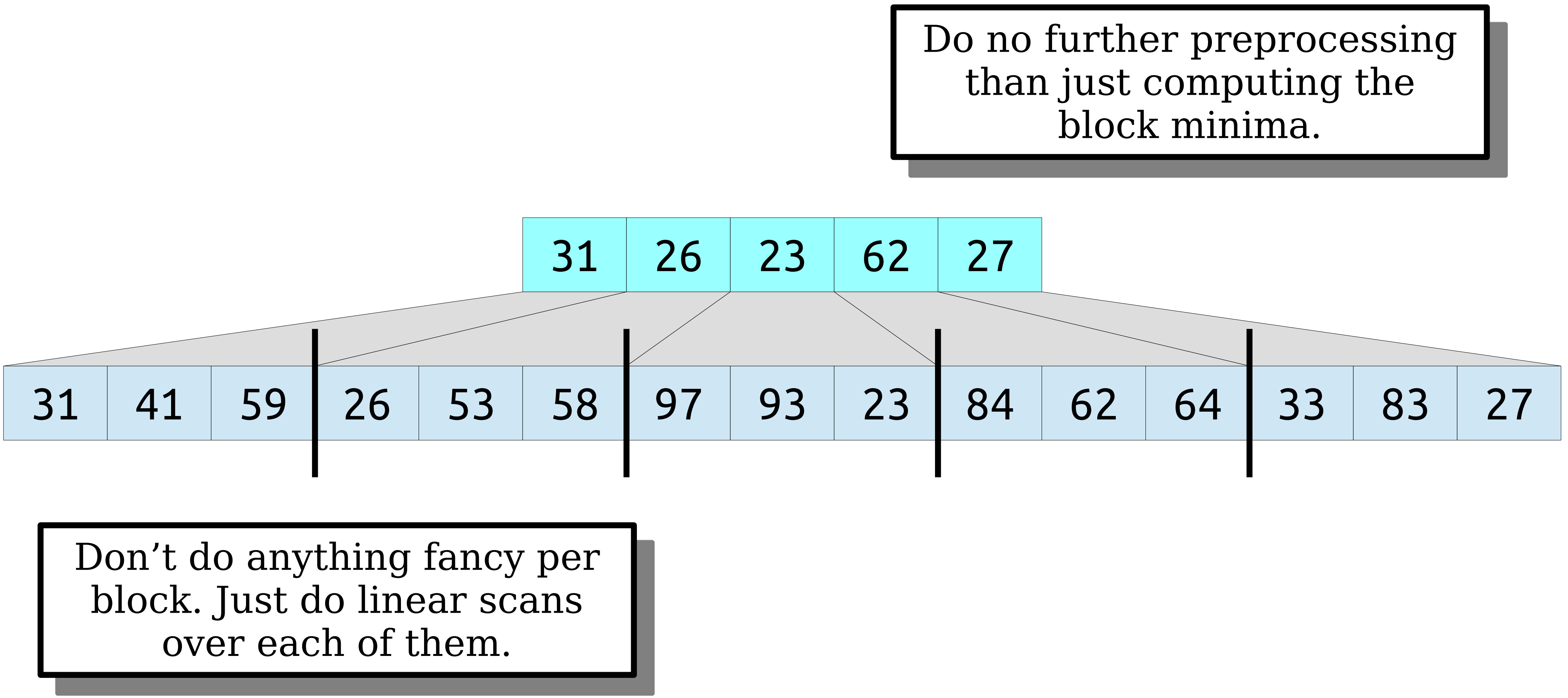
$$\mathbf{O(q_1(n / b) + q_2(b))}$$

Block size: $\boldsymbol{b}$.
\# Blocks: $\mathbf{O(n / b)}$.



A Sanity Check

- The $\langle O(n), O(n^{1/2}) \rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n) \rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.



A Sanity Check

- The $\langle O(n), O(n^{1/2})\rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n)\rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.

For Reference

$$p_1(n) = O(1)$$
$$q_1(n) = O(n)$$

$$p_2(n) = O(1)$$
$$q_2(n) = O(n)$$

$$b = n^{1/2}$$

A Sanity Check

- The $\langle O(n), O(n^{1/2})\rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n)\rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$O(n + p_1(n / b) + (n / b)\ p_2(b))$$

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

A Sanity Check

- The $\langle O(n), O(n^{1/2})\rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n)\rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= {} & O(n + 1 + n / b)
\end{aligned}
$$

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

A Sanity Check

- The $\langle O(n), O(n^{1/2}) \rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n) \rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= \; & O(n + 1 + n / b) \\
= \; & \mathbf{O(n)}
\end{aligned}
$$

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

A Sanity Check

- The $\langle O(n), O(n^{1/2})\rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n)\rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = {} & O(n + 1 + n / b) \\ = {} & \mathbf{O(\boldsymbol{n})} \end{aligned}$$

- The query time should be

$$O(q_1(n / b) + q_2(b))$$

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

A Sanity Check

- The $\langle O(n), O(n^{1/2})\rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n)\rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ =\ & O(n + 1 + n / b) \\ =\ & \mathbf{O(n)} \end{aligned}$$

- The query time should be

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ =\ & O(n / b + b) \end{aligned}$$

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

A Sanity Check

- The $\langle O(n), O(n^{1/2}) \rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n) \rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = {} & O(n + 1 + n / b) \\ = {} & \mathbf{O(n)} \end{aligned}$$

- The query time should be

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = {} & O(n / b + b) \\ = {} & \mathbf{O(n^{1/2})} \end{aligned}$$

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

A Sanity Check

- The $\langle O(n), O(n^{1/2})\rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n)\rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$\begin{aligned} &O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ =\ &O(n + 1 + n / b) \\ =\ &\mathbf{O(n)} \end{aligned}$$

- The query time should be

$$\begin{aligned} &O(q_1(n / b) + q_2(b)) \\ =\ &O(n / b + b) \\ =\ &\mathbf{O(n^{1/2})} \end{aligned}$$

- Looks good so far!

For Reference

$$p_1(n) = O(1)$$
$$q_1(n) = O(n)$$

$$p_2(n) = O(1)$$
$$q_2(n) = O(n)$$

$$b = n^{1/2}$$

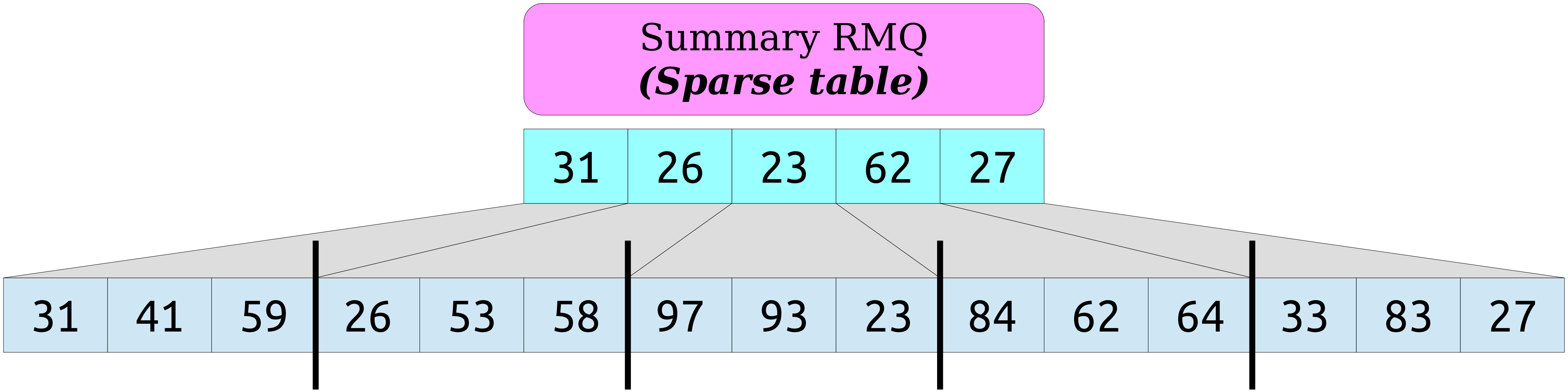
An Observation

- We can use any data structures we'd like for the summary and block RMQs.
- Suppose we use an $\langle O(n \log n), O(1) \rangle$ sparse table for the summary RMQ.
- If the block size is b, the time to construct a sparse table over the (n / b) blocks is $\mathbf{O((n / b) \log (n / b))}$.
- ***Cute trick:*** If $\mathbf{b = \Theta(\log n)}$, the time to construct a sparse table over the minima is

$$
\begin{aligned}
& O((n / \log n) \log (n / \log n)) \\
= \; & O((n / \log n) \log n) && \textit{(O is an upper bound)} \\
= \; & \mathbf{O(n)}. && \textit{(logs cancel out)}
\end{aligned}
$$

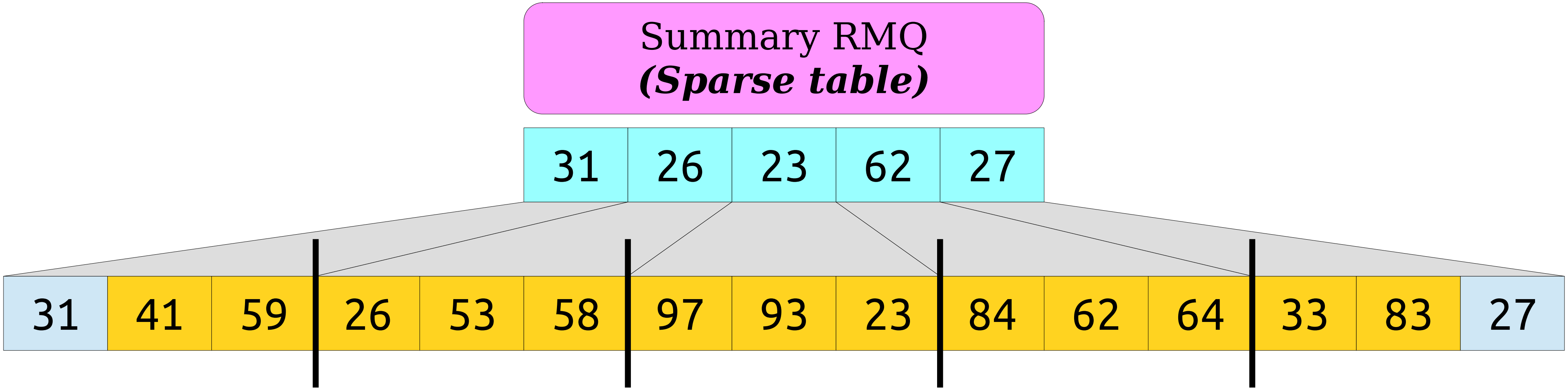
One Possible Hybrid

- Set the block size to log *n*.
- Use a sparse table for the summary RMQ.
- Use the “no preprocessing” structure for each block.



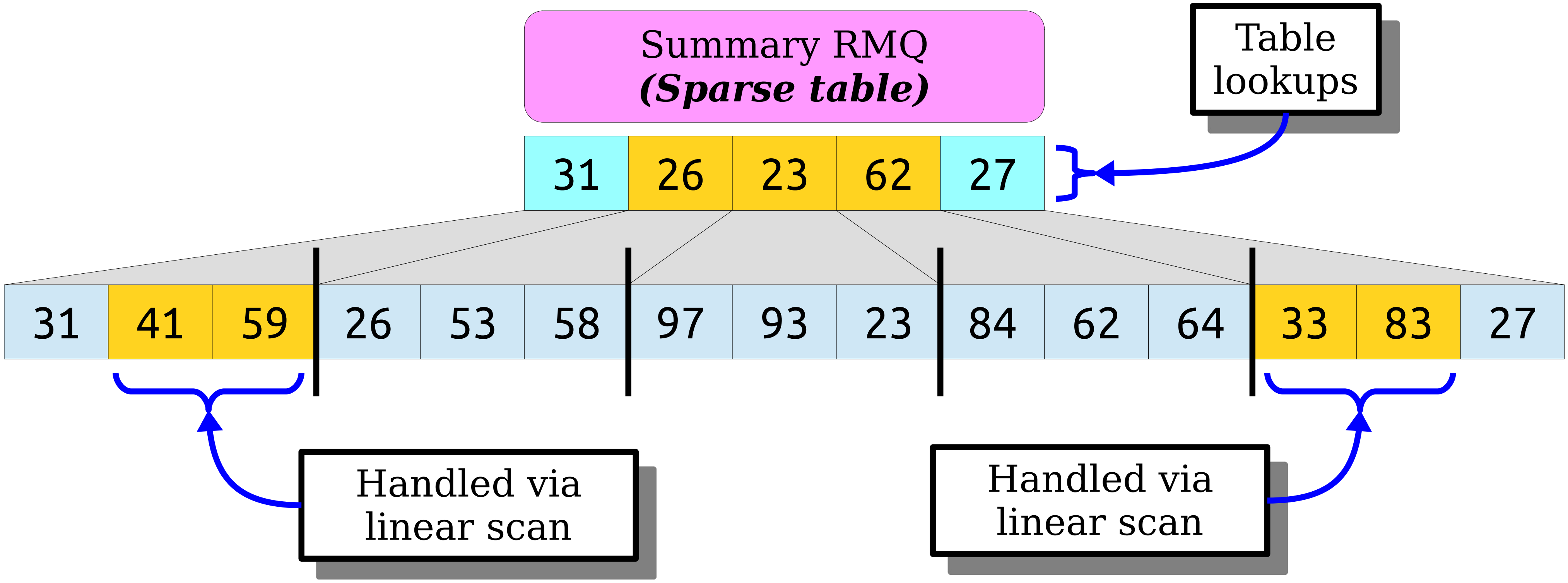
One Possible Hybrid

- Set the block size to log *n*.
- Use a sparse table for the summary RMQ.
- Use the “no preprocessing” structure for each block.



One Possible Hybrid

- Set the block size to log *n*.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.



One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$O(n + p_1(n / b) + (n / b)\ p_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$\begin{aligned} &O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = \;&O(n + n + n / b) \end{aligned}$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = {} & O(n + n + n / b) \\ = {} & \mathbf{O(\boldsymbol{n})} \end{aligned}$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = & O(n + n + n / b) \\ = & \mathbf{O(n)} \end{aligned}$$

- Query time:

$$O(q_1(n / b) + q_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the “no preprocessing” structure for each block.
- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = {} & O(n + n + n / b) \\ = {} & \mathbf{O(\boldsymbol{n})} \end{aligned}$$

- Query time:

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = {} & O(1 + b) \end{aligned}$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = \; & O(n + n + n / b) \\ = \; & \mathbf{O(n)} \end{aligned}$$

- Query time:

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = \; & O(1 + b) \\ = \; & \mathbf{O(\log n)} \end{aligned}$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = {} & O(n + n + n / b) \\ = {} & \mathbf{O(}\boldsymbol{n}\mathbf{)} \end{aligned}$$

- Query time:

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = {} & O(1 + b) \\ = {} & \mathbf{O(log}\ \boldsymbol{n}\mathbf{)} \end{aligned}$$

- An **⟨O(*n*), O(log *n*)⟩** solution!

For Reference

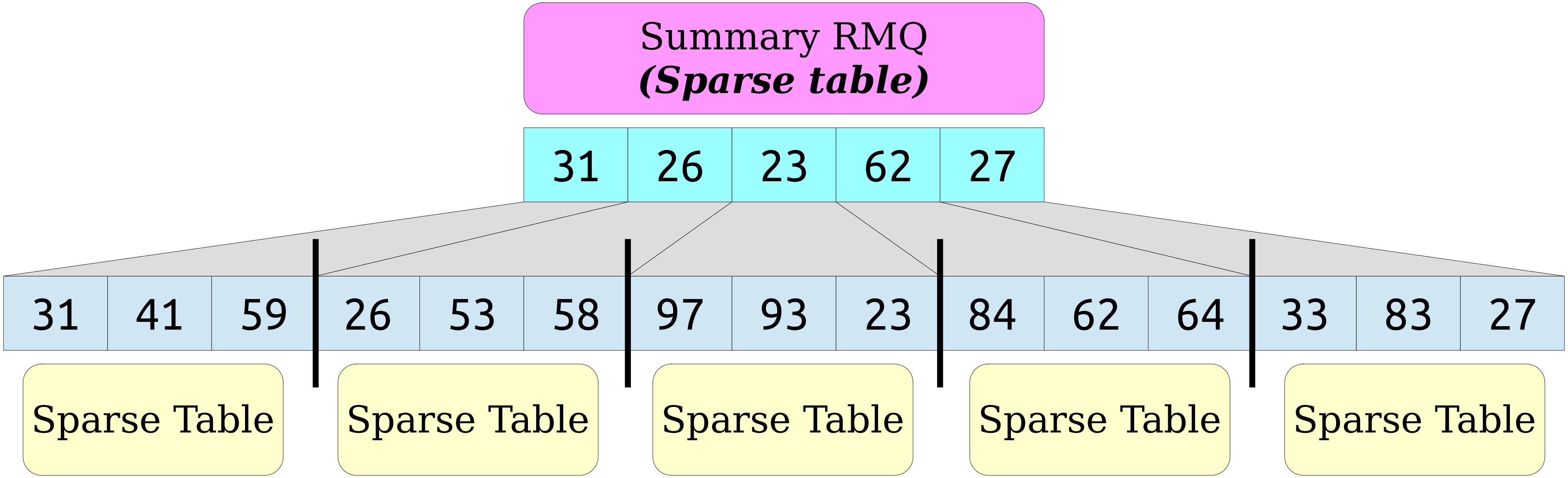
$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

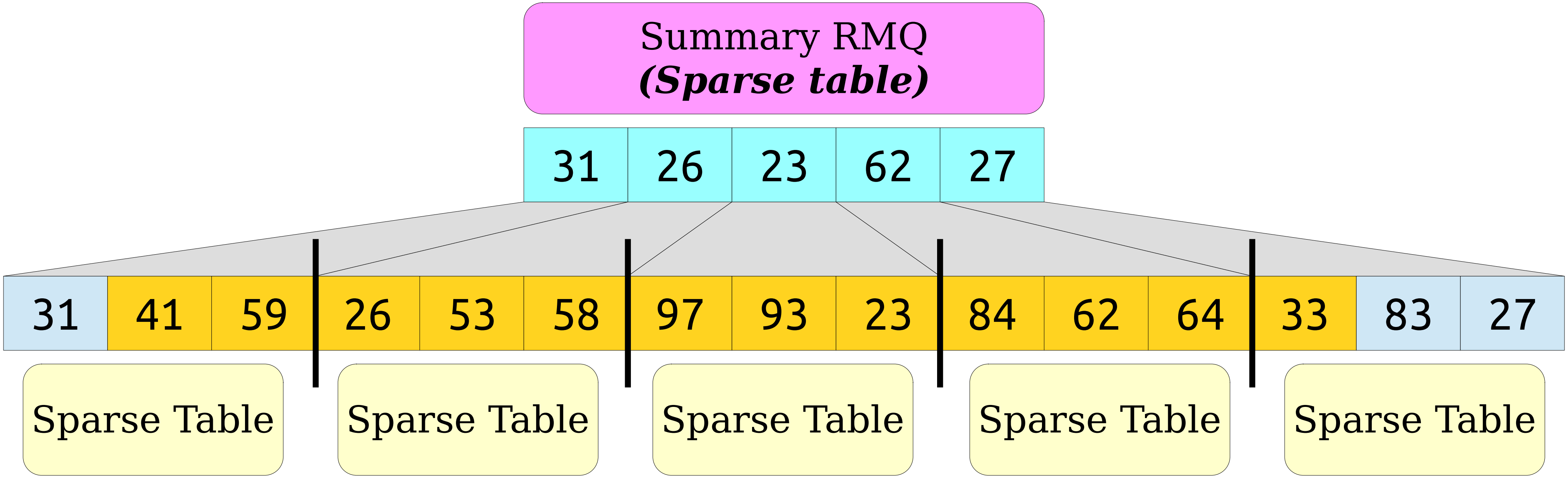
Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.



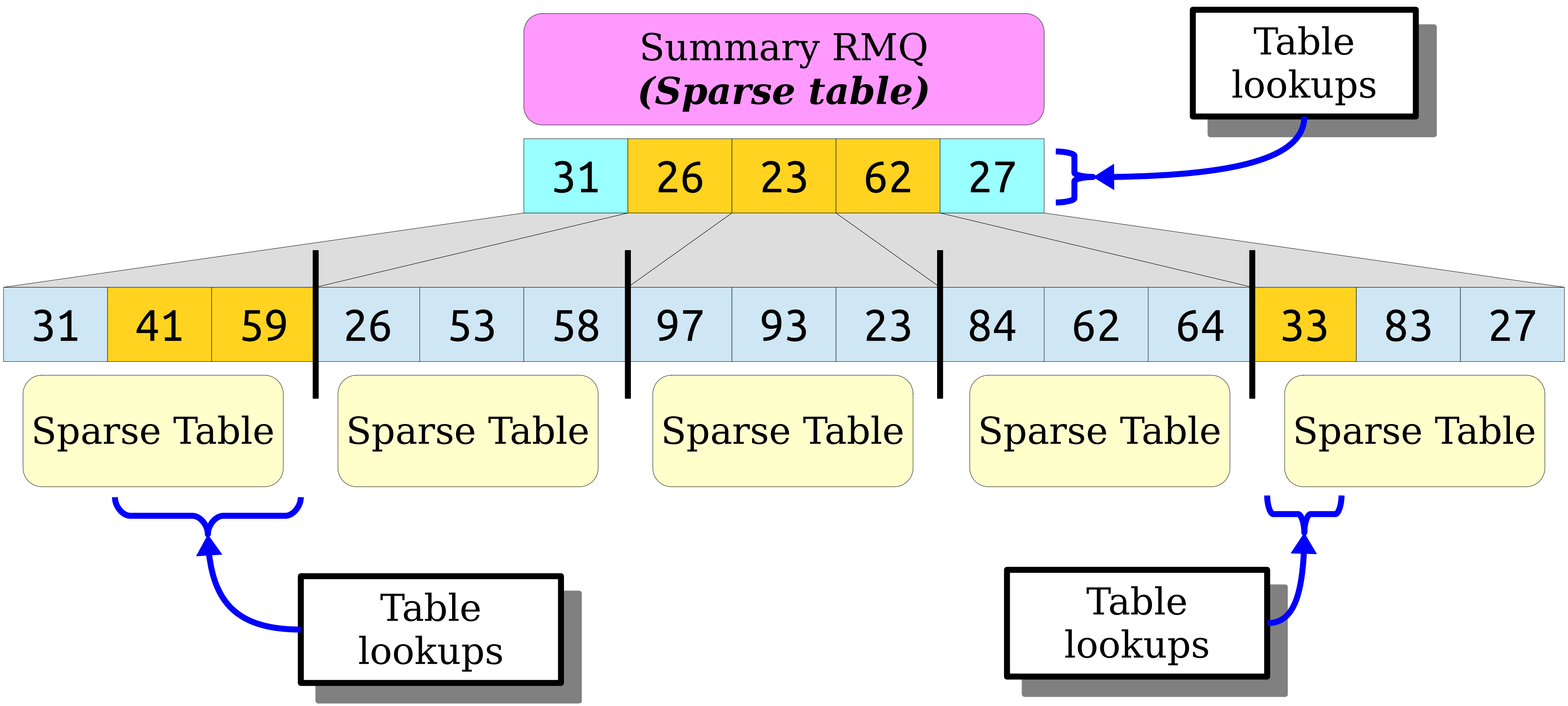
Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.



Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.



Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the $\langle O(n \log n), O(1)\rangle$ sparse table for both the summary and block RMQ structures with a block size of $\log n$.
- The preprocessing time is

$$O(n + p_1(n / b) + (n / b)\ p_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the $\langle O(n \log n), O(1)\rangle$ sparse table for both the summary and block RMQ structures with a block size of $\log n$.
- The preprocessing time is

$$\begin{aligned} &O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = \;&O(n + n + (n / b)\, b \log b) \end{aligned}$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.
- The preprocessing time is

$$
\begin{aligned}
&O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
&= O(n + n + (n / b)\, b \log b) \\
&= O(n + n \log b)
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.
- The preprocessing time is

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= \; & O(n + n + (n / b)\, b \log b) \\
= \; & O(n + n \log b) \\
= \; & \mathbf{O(n \log \log n)}
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.
- The preprocessing time is

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= {} & O(n + n + (n / b)\, b \log b) \\
= {} & O(n + n \log b) \\
= {} & \mathbf{O(n \log \log n)}
\end{aligned}
$$

- The query time is

$$O(q_1(n / b) + q_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.
- The preprocessing time is

$$
\begin{aligned}
&O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= \; &O(n + n + (n / b)\, b \log b) \\
= \; &O(n + n \log b) \\
= \; &\mathbf{O(n \log \log n)}
\end{aligned}
$$

- The query time is

$$
\begin{aligned}
&O(q_1(n / b) + q_2(b)) \\
= \; &\mathbf{O(1)}
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

Another Hybrid

- Let's suppose we use the $\langle O(n \log n), O(1)\rangle$ sparse table for both the summary and block RMQ structures with a block size of $\log n$.
- The preprocessing time is

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = \; & O(n + n + (n / b)\, b \log b) \\ = \; & O(n + n \log b) \\ = \; & \mathbf{O(n \log \log n)} \end{aligned}$$

- The query time is

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = \; & \mathbf{O(1)} \end{aligned}$$

- We have an $\mathbf{\langle O(n \log \log n), O(1)\rangle}$ solution to RMQ!

For Reference

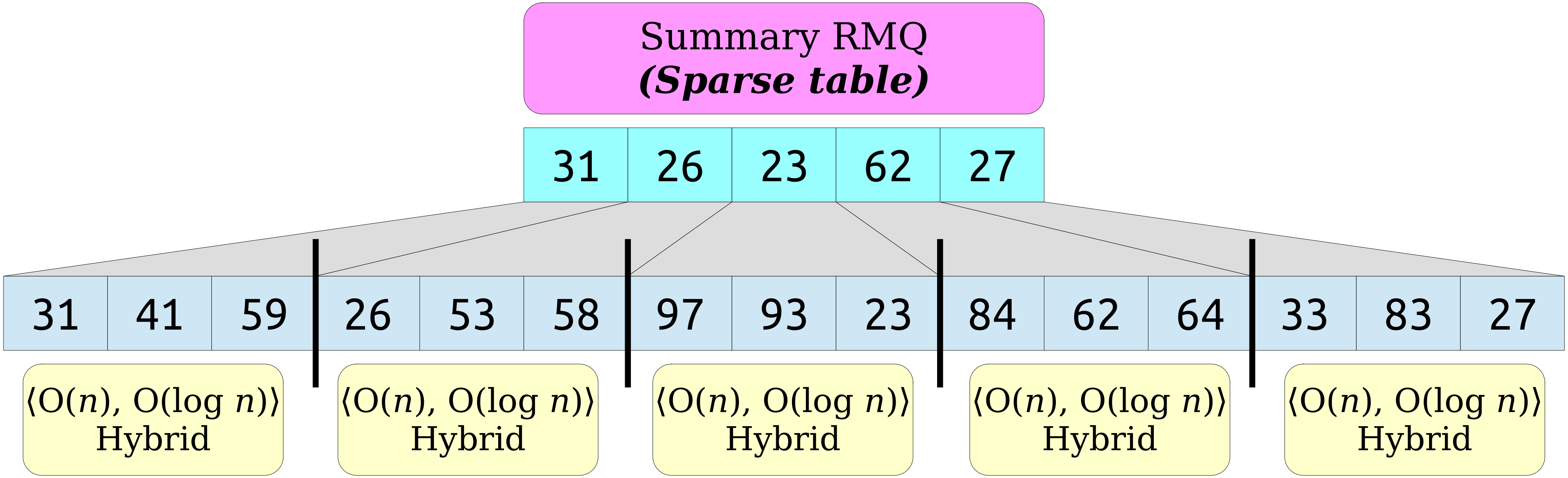
$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

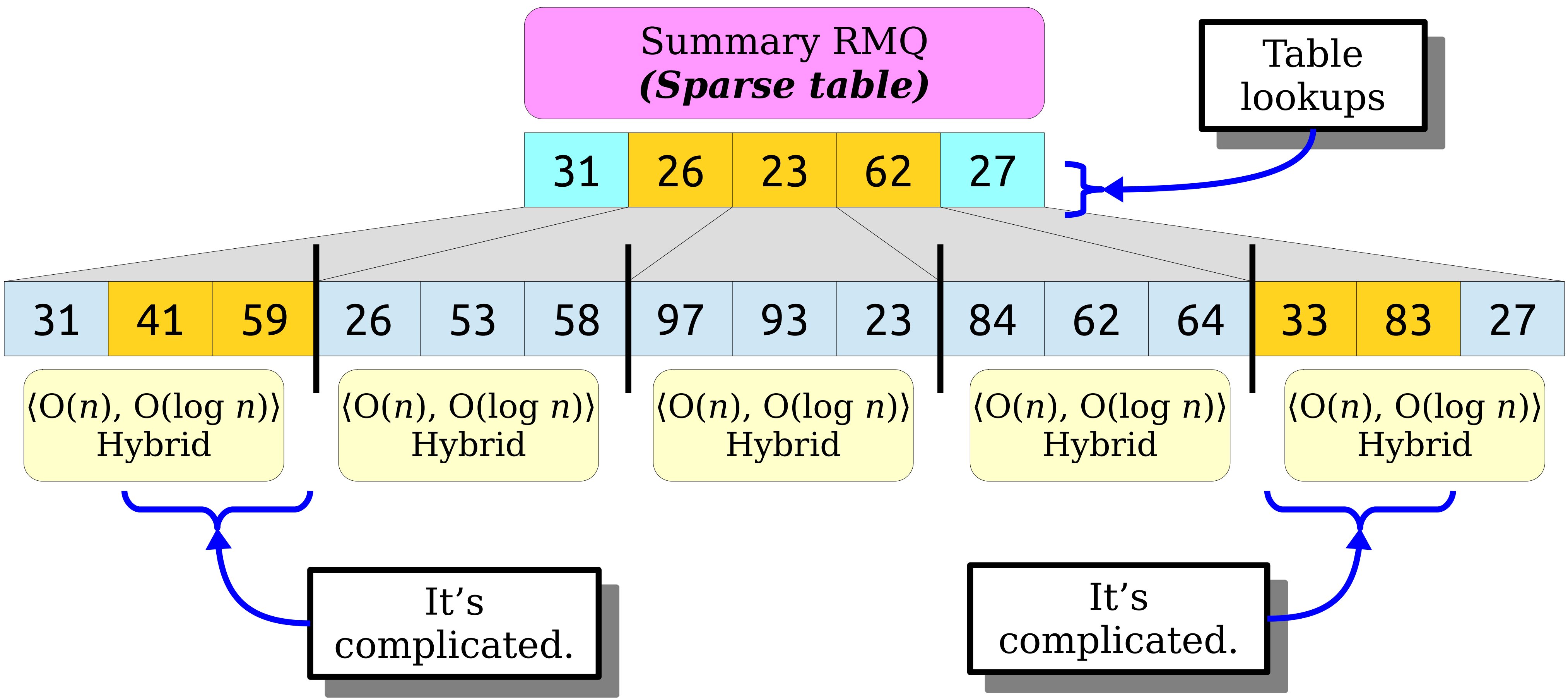
One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose b = log n.



One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose b = log n.



One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose b = log n.

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O($\log n$)⟩ solution for the block RMQs. Let's choose $b = \log n$.
- The preprocessing time is

$$O(n + p_1(n / b) + (n / b)\, p_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose $b = \log n$.
- The preprocessing time is

$$
\begin{aligned}
&O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= \; &O(n + n + (n / b)\, b)
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the $\langle O(n), O(\log n)\rangle$ solution for the block RMQs. Let's choose $b = \log n$.
- The preprocessing time is

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= \; & O(n + n + (n / b)\, b) \\
= \; & \mathbf{O(n)}
\end{aligned}
$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the $\langle O(n), O(\log n)\rangle$ solution for the block RMQs. Let's choose $b = \log n$.
- The preprocessing time is

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= {} & O(n + n + (n / b)\, b) \\
= {} & \mathbf{O(n)}
\end{aligned}
$$

- The query time is

$$O(q_1(n / b) + q_2(b))$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose $b = \log n$.
- The preprocessing time is

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\ p_2(b)) \\ = {} & O(n + n + (n / b)\ b) \\ = {} & \mathbf{O(n)} \end{aligned}$$

- The query time is

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = {} & O(1 + \log b) \end{aligned}$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose $b = \log n$.
- The preprocessing time is

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = \; & O(n + n + (n / b)\, b) \\ = \; & \mathbf{O(\boldsymbol{n})} \end{aligned}$$

- The query time is

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = \; & O(1 + \log b) \\ = \; & \mathbf{O(\log \log \boldsymbol{n})} \end{aligned}$$

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose b = log n.
- The preprocessing time is

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = \; & O(n + n + (n / b)\, b) \\ = \; & \mathbf{O(n)} \end{aligned}$$

- The query time is

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = \; & O(1 + \log b) \\ = \; & \mathbf{O(\log \log n)} \end{aligned}$$

- We have an **⟨O(n), O(log log n)⟩** solution to RMQ!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing: $\langle O(1), O(n) \rangle$
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table: $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1: $\langle O(n), O(\log n) \rangle$
  - Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Where We Stand

We've seen a bunch of RMQ structures today:

- No preprocessing: $\langle O(1), O(n) \rangle$
- Full preprocessing: $\langle O(n^2), O(1) \rangle$
- Block partition: $\langle O(n), O(n^{1/2}) \rangle$
- Sparse table: $\langle O(n \log n), O(1) \rangle$
- Hybrid 1: $\langle O(n), O(\log n) \rangle$
- Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
- Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing: $\langle O(1), O(n) \rangle$
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table: $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1: $\langle O(n), O(\log n) \rangle$
  - Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Is there an $\langle O(n), O(1) \rangle$ solution to RMQ?

Yes!

Next Time

- ***Cartesian Trees***
  - A data structure closely related to RMQ.
- ***The Method of Four Russians***
  - A technique for shaving off log factors.
- ***The Fischer-Heun Structure***
  - A clever, asymptotically optimal RMQ structure.