# Financial Vulnerability and Monetary Policy

Tobias Adrian
Fernando Duarte

**FRBNY**
*Staff*
REPORTS

**Financial Vulnerability and Monetary Policy**
Tobias Adrian and Fernando Duarte
*Federal Reserve Bank of New York Staff Reports*, no. 804
December 2016; revised September 2017
JEL classification: G10, G12, E52

**Abstract**

We present a microfounded New Keynesian model that features financial vulnerabilities. Financial intermediaries' occasionally binding value-at-risk constraints give rise to variation in the pricing of risk that generates time-varying risk in the conditional mean and volatility of the output gap. The conditional mean and volatility are negatively related: during times of easy financial conditions, growth tends to be high, and risk tends to be low. Monetary policy affects output directly through the investment-savings curve, and indirectly through the pricing of risk that relates to the tightness of the value-at-risk constraint. The optimal monetary policy rule always depends on financial vulnerabilities in addition to the output gap, inflation, and the natural rate of interest. We show that a classic Taylor rule exacerbates deviations of the output gap from its target value of zero relative to an optimal interest rate rule that includes vulnerability. Simulations show that optimal policy significantly increases welfare relative to a classic Taylor rule. Alternative policy paths using historical examples illustrate the usefulness of the proposed policy rule.

Key words: monetary policy, macro-finance, financial stability

# 1 Introduction

FOMC statements mention financial conditions with increasing frequency (see Peek, Rosengren, and Tootell (2015)). Additionally, the notion of downside risks to growth has become more prevalent in the speeches of monetary policy makers (see Yellen (2016b)). In this paper, we explore the link between financial conditions and downside risks to GDP.

When macro-financial linkages are generated by financial vulnerabilities, they can generate downside risks to growth. Bernanke and Gertler (1989) and Bernanke and Blinder (1992) argue for the credit channel of monetary policy where financial frictions shift credit demand or supply curves, thus generating an amplification mechanism that is transmitted via changes in the pricing of risk. Changes in the pricing of risk are generally caused by deeper frictions linked to leverage in the financial and non-financial sectors and the degree of maturity transformation (see Brunnermeier, Eisenbach, and Sannikov (2013)) or Adrian, Covitz, and Liang (2015) for surveys on the role of vulnerabilities for macro-financial linkages). Financial vulnerabilities can cause changes in downside risks to GDP, and the pricing of risk tends to reflect financial vulnerabilities.

Recent literature on monetary policy has started to explore the role of financial conditions and vulnerabilities in monetary policy settings (see Adrian and Shin (2010), Borio and Zhu (2012)). Curdia and Woodford (2010) and Gambacorta and Signoretti (2014) have argued for a Taylor rule that is augmented with financial variables. In this paper, we explicitly study the link of financial conditions to downside risks of GDP, and ask to what extent optimal monetary policy should take such downside risks into account.

We present a parsimonious macroeconomic framework for incorporating financial vulnerabilities in monetary policy. Our starting point is the standard New Keynesian model of Woodford (2003) and Galí (2008). Households have risk averse utility over differentiated products and supply labor to an intermediate goods producing sector. Intermediate goods have a constant returns to scale technology with exogenous productivity and labor as only input. These intermediate goods producing firms maximize profits subject to a demand curve for differentiated products and Calvo style price stickiness. Their output is sold to the final goods producers in a monopolistically competitive way. The profits of intermediate goods producers are distributed as dividends to shareholders. Without loss of generality, as the Modigliani-Miller theorem holds, these intermediate goods producing firms are fully equity financed. The final goods sector is perfectly competitive,

uses intermediate goods as only input and produces zero profits.

The point of departure from the standard NK model is the existence of banks. Households cannot directly invest in the shares of the intermediate goods producing sector, as it is assumed that all financing is intermediated by the banks. There is a continuum of identical banks that issue riskless deposits that pay the risk free rate of return. The deposits, as well as the risky equity of banks, are owned by households. Banks maximize profits by investing in all available risky assets in the economy. The banks' portfolio selection problem is subject to a value at risk (VaR) constraint on their net worth (i.e. on bank equity). Banks do not consume and distribute their profits to their shareholders. Relative to earlier NK models with banks, the VaR is the main difference (see, for example, Gertler and Karadi (2011) and Curdia and Woodford (2010)).

The only source of risk in the economy are shocks to the time preference rate of banks. These shocks capture differences in beliefs or effective risk aversion between banks and the other agents in the economy. Because there is only one source of risk, and bank equity and bank deposits trade continuously, markets are complete, even from the point of view of the household that cannot invest in the goods producers' stock. Using martingale techniques from continuous-time financial economics, we can solve for the equilibrium in closed form. Importantly, we solve for the full stochastic equilibrium, which is characterized by conditional means and conditional volatilities as a function of state variables. For analytical tractability, we linearize first and second moments of the continuous-time stochastic processes. This is a novel approach relative to standard first order approximations, as it preserves the equilibrium conditions that are imposed on second moments. The model and solution technique thus lends themselves to study the risk return tradeoff of monetary policy.

The linearized solution can be represented as a parsimonious four-equation reduced form model. Relative to the standard NK model, the IS curve (which is derived from the Euler equation) features time-varying risk premia. Risk premia, in turn, depend on the vulnerability of aggregate economic activity. Vulnerability is defined as the VaR of the output gap. The evolution of vulnerability is the third equation. Finally, there is the stochastic process that determines risk.

Our modeling approach is motivated by the empirical evidence that financial conditions forecast tail risks. Estrella and Hardouvelis (1991) and Estrella and Mishkin (1998) show that the term spread, an indicator of the pricing of interest rate risk, forecasts recessions. Gilchrist and Zakrajšek (2012), López-Salido, Stein, and Zakrajšek (2016), and Krishnamurthy and Muir (2016) find that credit spreads forecast downside risks to

GDP growth. More generally, Adrian, Boyarchenko, and Giannone (2016) document that financial conditions are strong forecasters of downside risks to GDP growth. Deteriorating financial conditions give rise to an increase in the conditional volatility of GDP and a decline in the conditional mean of GDP. Moreover, the upper quantiles of GDP growth are more or less constant, while lower quantiles are varying sharply. Hence the unconditional distribution of GDP is highly skewed to the left as a function of financial conditions. In this paper, we verify that the same patterns that Adrian, Boyarchenko, and Giannone (2016) found for GDP are also present in the output gap.

The four equation reduced form NK model captures these dynamics of the conditional output gap distribution. The model gives rise to a relationship between the conditional mean and the conditional volatility of output that generates the empirical features of the conditional output distribution, namely, the negative correlations between mean and volatility that Adrian, Boyarchenko, and Giannone (2016) document. As a result, the model features strongly time varying downside risk that is a function of financial conditions, while upside risks are more or less constant. Monetary policy impacts not only conditional means through the standard intertemporal substitution channel, but also conditional volatilities by altering the tightness of the VaR constraint of banks, which in turn affects risk premia for the entire aggregate economy.

The central bank is assumed to minimize a standard loss function with squared deviations of the output gap and inflation from target as arguments. We can solve for the optimal policy rules in closed form using dynamic programming, as our new approximation technique preserves the tractability of the linear-quadratic framework while retaining the crucial non-linear dynamics.

Our optimal monetary policy rule can be cast in the language of a flexible inflation targeting framework, such as the one in Svensson (1999), Rudebusch and Svensson (1999), Svensson (2002), and Giannoni and Woodford (2012). Relative to the standard New Keynesian model, there are two important differences. First, vulnerability enters the optimal rule, not only because it forecasts the distribution of the output gap, but also because it determines the current efficacy in the transmission of monetary policy through the risk-taking channel. Second, the coefficients in the linear optimal targeting criterion rule that trade off deviations in output and inflation from their desired levels, depend on the parameters that govern vulnerability.

Optimal monetary policy can also be expressed as an augmented Taylor rule. The nominal interest rate not only depends on inflation and output, but also on financial vulnerability. The optimal coefficients on output and inflation are different from the

standard ones because they take the parameters that govern vulnerability into account.

Our model captures the intuition that in recent years monetary policy has explicitly taken into account and influenced financial conditions.[1] A deterioration of financial conditions corresponds to an increase in tail risk, as conditional GDP volatility rises, while the conditional growth forecast deteriorates. As a result of such an increase in financial vulnerability, i.e. an increase in the downside risk to GDP growth, monetary policy is relatively easier than under the classic Taylor rule. This results in a concurrent lowering of vulnerability, and hence in less severe left skewness of GDP.

Our model also features a volatility paradox. GDP vulnerability is a mean reverting process: times of easy financial conditions associated with low downside risks tend to be followed by the buildup of tail risks in the medium term. This intertemporal tradeoff is an important consideration in the setting of optimal policy.

While we do not explictly consider macroprudential policy, we show in reduced form that the stance of prucential policy can change the tradeoffs faced by monetatry policy. When prudential policy is appropriately designed, vulnerabilities are mitigated, improving the tradeoffs for the monetary policy authority. Perfect macroprudential policy would eliminate the need for monetary policy to condition on vulnerability.

We also study an extension with a zero lower bound on nominal interest rates. The zero lower bound implies a flexible inflation targeting rule like the one previously discussed when interest rates are away from the bound, and a forward guidance rule that also depends on vulnerability when the zero lower bound is reached. Therefore, the New Keynesian model with financial vulnerability can be extended to settings with a zero lower bound.

The remainder of the paper is organized as follows. Section 2 provides the motivation for our model from the existing empirical and theoretical literature on financial stability in a macroeconomic context. Section 3 presents the model. The solution ot the model is presented in Section 4. Section 5 derives the optimal monetary policy rule in the reduced form. Section 6 concludes.

# 2   Financial Vulnerability

Financial vulnerability refers to the presence of amplification mechanisms that are linked to leverage or maturity transformation. Financial vulnerabiltiy is reflected in the pricing

---

[1]Dudley (2015, 2017); Yellen (2016a).

of risk. When financial vulnerability is large, small shocks can have severe aggregate macroeconomic consequences (see Adrian, Covitz, and Liang (2015) or Aikman, Kiley, Lee, Palumbo, and Warusawitharana (2015)). We follow Adrian, Boyarchenko, and Giannone (2016) by constructing a measure of financial vulnerability from the National Financial Conditions Index (NFCI) of the Federal Reserve Bank of Chicago. That index aggregates 105 financial market, money market, credit supply, and shadow bank indicators to compute a single index using the filtering methodology of Stock and Watson (1998). Adrian, Boyarchenko, and Giannone (2016) show that the conditional GDP distribution features strong downside risk as a function of financial conditions. We reproduce the main results of Adrian, Boyarchenko, and Giannone (2016) here using a conditionally heteroskedastic model to estimate the conditional first and second moments of GDP gap

$$y_t = \gamma_0^y + \gamma_1^y y_{t-1} + \gamma_2^y \pi_{t-1} + \gamma_3^y x_{t-1} + \sigma_t^y \varepsilon_t^y \tag{1}$$

$$\ln\left(\sigma_t^y\right) = \delta_0^y + \delta_1^y x_{t-1} \tag{2}$$

where $\varepsilon_t^y \sim N(0,1)$, $x_t$ denotes the NFCI financial conditions index, and $y_t$ is the GDP gap. Mean GDP gap also depends on the lagged quarterly core PCE inflation rate $\pi$ and on the lagged GDP gap. In addition to estimating the conditional mean and conditional volatility of the GDP gap, we also estimate an analogous equation for the inflation rate:

$$\pi_t = \gamma_0^\pi + \gamma_1^\pi y_{t-1} + \gamma_2^\pi \pi_{t-1} + \sigma_t \varepsilon_t^\pi \tag{3}$$

$$\ln\left(\sigma_t^\pi\right) = \delta_0^\pi + \delta_1^\pi \pi_{t-1} \tag{4}$$

The model is estimated via maximum likelihood.

The estimation results are in Figure 1 and Table 1. In Panel (a) of Figure 1, we present the conditional mean of GDP gap, actual GDP gap, and the 5th and 95th quantiles. The distribution is left skewed as deteriorating financial conditions are associated with an increase in conditional volatility, and at the same time a decline in the conditional mean of GDP gap (see Table 1). Due the negative correlation of mean and volatility the unconditional distribution is negatively skewed, even though the conditional distribution is conditionally Gaussian. For inflation, financial conditions aren't significant for either the conditional mean equation or the conditional volatility. However, the volatility of inflation scales in the level of inflation. Hence the conditional mean and the conditional volatility are positively correlated. Importantly, financial conditions play no role for

Figure 1: **Estimated Conditional Distribution of One Quarter Ahead GDP Gap and PCE Inflation**. The figure reports estimates from equations (1), (2), (3), and (4). Panel (a) shows the actual GDP gap, the conditional mean of GDP gap, and the 5th and 95th quantile. Panel (b) shows the actual PCE inflation, the conditional mean of inflation, and the 5th and 95th quantile.
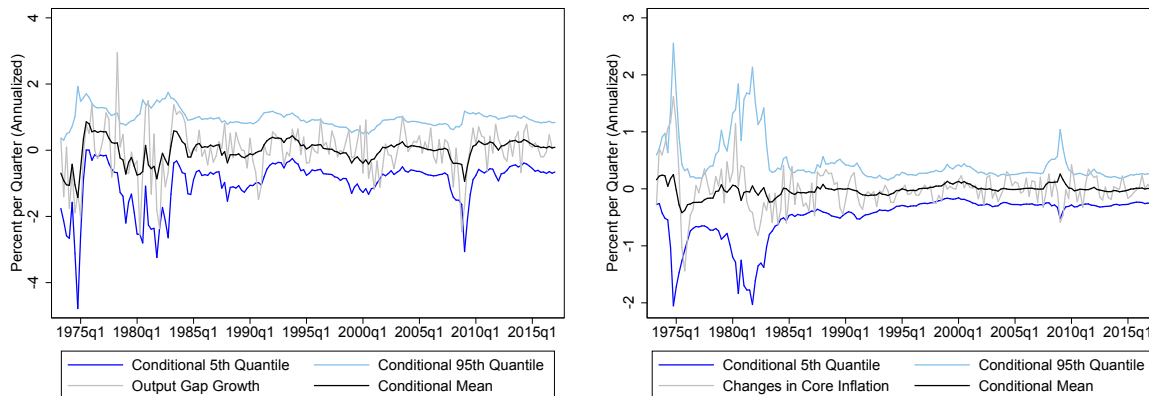


Table 1: **GDP Gap and Inflation Conditional Mean and Volatility Estimates**

| VARIABLES | (1) GDP | (2) HET |
|---|---|---|
| Financial Conditions (lag) | -1.715*** | 0.551*** |
| | [-5.096] | [3.765] |
| GDP Gap (lag) | -0.000356 | |
| | [-1.510] | |
| Inflation Rate (lag) | 0.00277 | |
| | [0.0842] | |
| Constant | 6.213*** | 1.785*** |
| | [11.02] | [21.69] |
| | | |
| Observations | 173 | 173 |

*** p<0.01, ** p<0.05, * p<0.1

inflation dynamics.

The estimates for GDP have the unusual property that the shift in the conditional mean and volatility of GDP offset each other in such a way that the 95th quantile is close to constant. In contrast, the 5th quantile strongly varies as a function of financial conditions. Importantly, this property only arises when the GDP distribution is estimated as a function of financial conditions—real economic indicators do not contain significant information for the tail of the GDP distribution. This is shown more generally by Adrian, Boyarchenko, and Giannone (2016). This property is visualized in Figure

Figure 2: **Estimated Conditional Mean and Conditional Volatility of One Quarter Ahead GDP Gap and PCE Inflation**. The figure reports estimates from equations (1), (2), (3), and (4). Panel (a) plots the GDP gap mean against the GDP gap volatility, panel (b) plots PCE inflation mean against PCE inflation volatility.
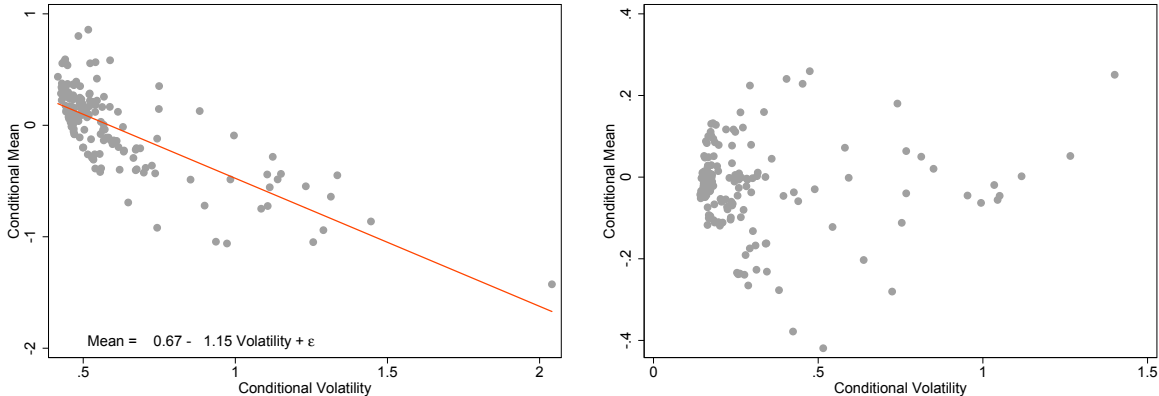


2, which scatters the conditional mean against the conditional volatility for the GDP gap and PCE inflation. For the GDP gap, mean and volatility are strongly negatively correlated, but there is no correlatoin for PCE inflation.

These results suggest that GDP vulnerability is related to the time varying left tail of the GDP distribution, as a function of financial conditions. In this paper, we define GDP vulnerability $V_t$ as the value at risk of the GDP gap

$$V_t = \mathcal{N}^{-1}(p)\mathbb{V}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\sqrt{\tau} - \mathbb{E}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\tau. \tag{5}$$

$\mathcal{F}_t$ denotes the filtration generated by the underlying stochastic processes, $\mathbb{V}$ denotes the volatility operator (the square root of the instantaneous variance of $dy_t$), and $\mathbb{E}$ denotes the expectations operator. The expectation is multiplied by the horizon of the value at risk, $\tau$, while the volatility is multiplied by the square root of $\tau$. $\mathcal{N}^{-1}(p)$ denotes the inverse cumulative Gaussian distribution function with probability p. As vulnerability measures the left tail of the GDP gap distribution, $p$ is small, and therefore $\mathcal{N}^{-1}(p)$ is negative. For example, $\mathbb{N}^{-1}(5\%) = -1.96$. To save notation, we will denote $\alpha = -\mathcal{N}^{-1}(p)$.

We will derive a fully microfounded NK model in the next sections. To foreshadow where we will end up, we present the reduced form model here, which consists of the

following four equations:

$$dy_t = \gamma^{-1}\left(R_t - r_t + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{6}$$

$$V_t = -\alpha\mathbb{V}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\sqrt{\tau} - \mathbb{E}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\tau \tag{7}$$

$$ds_t = \kappa_s\left(\bar{s} - s_t\right)dt + \sigma_s dZ_t \tag{8}$$

$$d\pi_t = \left(\beta\pi_t - \kappa y_t\right)dt \tag{9}$$

Equation (6) is the Euler equation (or IS curve) of a standard NK model augmented with a risk premium. The risk premium has drift $\gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)$ and volatility $\xi\left(V_t - s_t\right)$. Note that both drift and volatility are proportional to $V_t - s_t$, where $V_t$ is the VaR of GDP as defined in (7) and $s_t$ is a state variable defined in (8). Importantly, vulnerability is *endogenous* to the stochastic evolution of GDP as a function of shocks to the risk premium. The shock $dZ_t$ is a standard Brownian motion. We can interpret $V_t$ as conditional volatility of GDP, and $s_t$ as a mean reverting shock to volatility. Equation (9) is a standard Phillips curve.

Using (6) in (5) and solving for $V_t$ gives vulnerability as a function of the interest rate

$$V_t = \frac{-\gamma^{-1}\left(R_t - r_t\right) + \alpha\xi s_t\sqrt{\tau} + \hat{\eta}\xi\left(s_t + \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\tau}{1 + \alpha\xi\sqrt{\tau} + \hat{\eta}\xi\tau}. \tag{10}$$

Vulnerability depends on the interest rate in excess of the natural rate $R_t - r_t$ and the process $s_t$. We can thus interpret $s_t$ as a shock to vulnerability. Higher interest rates make vulnerability more negative.

The sign of the dependence of $V_t$ on the interest rate $R_t$ depends on the sign of $-\left(1 + \alpha\xi\sqrt{\tau} + \hat{\eta}\xi\tau\right)$. The empirical results presented above can help us pin down the sign of these parameters. The mean-variance tradeoff for $y_t$ follows by writing $\mathbb{E}\left[dy_t|\mathcal{F}_t\right]$ and $\mathbb{V}\left[dy_t|\mathcal{F}_t\right]$ as functions of vulnerability $V_t$ and the shock to vulnerability $s_t$

$$\mathbb{E}\left[dy_t|\mathcal{F}_t\right] = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right) \tag{11}$$

$$\mathbb{V}\left[dy_t|\mathcal{F}_t\right] = \xi\left(V_t - s_t\right) \tag{12}$$

8

and then eliminating $V_t$ to get

$$\mathbb{E}\left[dy_t|\mathcal{F}_t\right] = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi}\mathbb{V}\left[dy_t|\mathcal{F}_t\right] - \left(\frac{1}{\tau}\right)s_t. \tag{13}$$

Empirically, the slope is negative and the intercept is positive, hence we need

$$-\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi} < 0 \tag{14}$$

$$\left(\frac{1}{\tau}\right)\bar{s} > 0 \tag{15}$$

To calibrate the reduced form model, we set $\alpha = -1.645$, which corresponds to a VaR value of 5%. We choose a VaR horizon of one year, $\sqrt{\tau} = 1$. To match the data, we set the slope $-\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi} = -1.15$ and the intercept $\bar{s} = -0.67\tau$ which gives $\xi = 0.36$ and $\bar{s} = -0.67$. These calibrations imply that GDP vulnerability $V_t$ and interest rates $i_t$ are negatively correlated. This correlation is consistent with the empirical observation that when financial conditions deteriorate, GDP vulnerability increases, and short-term interest rates decline. Figure 3 shows a simulated path of (6), (7), (8) setting $R - r$ to zero, for simplicity. The simulation clearly features the stylized facts of Figure 1.

Figure 3: **Simulated Conditional Distribution of One Quarter Ahead GDP Growth**. The figure shows simulated conditional mean of GDP, and the 5th and 95th quantile of model (6), (7), (8).
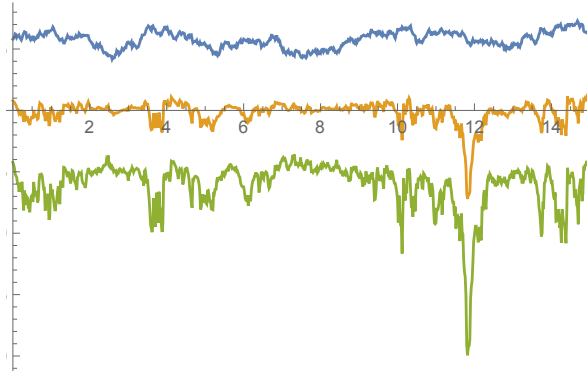


The IS curve augmented with the shocks to risk premia that depend on vulnerability lead to an additional channel for monetary policy. The traditional transmission channel is via the drift of the IS curve: higher interest rates are associated with a higher growth rate of output. This is because a higher interest rate shifts consumption from the present to the future, via increased savings. The additional channel that arises in the current setup is the impact of monetary policy on vulnerability, and hence on the volatility of

the risk premium. Monetary policy impacts total risk in the economy. This channel is sometimes called the "risk taking channel of monetary policy" (see Adrian and Shin (2010) and Borio and Zhu (2012)). When we study optimal monetary policy in the next section, this tradeoff is going to emerge prominently.

# 3 The Model

## 3.1 Physical Environment

Time is continuous. There is a continuum of mass one, identical, infinitely lived households who rank consumption streams $C_t$ and labor streams $N_t$ according to

$$\mathbb{E}_0 \int_0^\infty e^{-\beta t} \left( \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\xi}}{1+\xi} \right) dt, \tag{16}$$

where $\beta > 0$ is a time-preference parameter, $\gamma > 0$ is the coefficient of relative risk aversion (and the inverse of the intertemporal elasticity of substitution) and $\xi > 0$ is the inverse of the Frisch elasticity of labor supply. The variable $C_t$ represents a consumption index given by

$$C_t \equiv \left( \int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{17}$$

where $C_t(i)$ is the quantity of differentiated good $i \in [0, 1]$ consumed by the household at time $t$ and $\varepsilon > 1$ is the constant elasticity of substitution across different goods. Because households are identical of mass one, $C_t$ is also the aggregate consumption of the economy. The variable $N_t$ is the total labor supplied by the household to all firms and given by

$$N_t \equiv \int_0^1 N_t(i) di, \tag{18}$$

where $N_t(i)$ is the amount of labor supplied at time $t$ to the firm that produces goods of type $i$. Because households are identical of mass one, $N_t$ is also the aggregate labor supply of the economy. Output $Y_t(i)$ for each good $i$ can be produced by the following constant returns to scale technology

$$Y_t(i) = AN_t(i), \tag{19}$$

where $A$ is the constant economy-wide level of technology. The final good $Y_t$ is produced with the technology

$$Y_t = \left( \int_0^1 Y_t(i)^{1-\frac{1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{20}$$

There is no government spending and the economy is closed to imports and exports. The resource constraint of the economy is

$$C_t = Y_t. \tag{21}$$

## 3.2 First Best

The first best is obtained by choosing paths of $C_t(i)$, $N_t(i)$ and $Y_t(i)$ to maximize the utility of the representative household, given in equation (16), subject only to the structure of the economy's physical environment described by equations (17)-(21). The solution to this this central planner problem is

$$C_t(i) = C_t(j), \tag{22}$$

$$Y_t(i) = Y_t(j), \tag{23}$$

$$N_t(i) = N_t(j), \tag{24}$$

for all $i$ and $j$ and

$$N_t = A^{\frac{1-\gamma}{\gamma+\xi}}, \tag{25}$$

$$C_t = Y_t = A^{\frac{1-\gamma}{\gamma+\xi}+1}. \tag{26}$$

Note that the first best features constant aggregate levels of output, consumption and labor.

## 3.3 Market Structure

We now describe the structure of the market economy that we use to solve for the decentralized equilibrium. Unlike the standard New Keynesian model, in this setup it becomes important to spell out the financial assets available in the economy and the capital structure of firms. The only source of uncertainty is a single standard Brownian motion, $Z_t$.

Good-producing firms are exactly as in the standard New Keynesian model. There are

intermediate good producers and final good producers. The intermediate good producers finance their operations by issuing equity shares that pay out profits as dividends. An equity-only capital structure (i.e., a balance sheet with zero debt) is without loss of generality, as the Modigliani-Miller theorem holds for intermediate good producers. Final good producers operate in perfectly competitive marktes and make zero profits, so their ownership and corporate capital structure are irrelevant for equilibrium outcomes.

Households are as in the standard New Keynesian model except for two differences. First, they cannot invest in shares of the intermediate good producers (but can still invest in all other financial assets). The reason is that all financing of firms must be intermediated by the financial sector, since we assume that banks are the only agents that have the necessary information, expertise or relationships to do so.

The financial sector consists of a mass one of intermediation firms that we label banks. Banks issue equity shares that can be purchased and held by the households or by the banks themselves. Banks are also allowed to trade a real riskless bond among themselves and with the households[2]. A bond trade between a bank and a household in which the bank has a negative position can be thought of as a risk-free deposit. Banks have a risk averse objective function over total distributions (dividends, plus payouts to depositors, plus net payoffs from positions in the riskless bond, which for simplicity we henceforth refert to as dividends) that has preference shocks. The preference shocks are the driven by a standard Brownian motion, $Z_t$, and is the only shock in the economy. Banks maximize their objective by choosing a portfolio of investments in all financial assets available in the economy. The banks' portfolio selection problem is subject to a Value-at-Risk (VaR) constraint on their wealth (net worth). In summary, the liabilities of the banks are deposits and equity while their assets are positions in the bond and stocks.

There is a central bank that sets the nominal interest rate by paying interest on base money in the cashless limit, as in Woodford (2003). There is no fiscal spending, so no need for the government to issue bonds or levy taxes (so we assume it does neither). Fiscal policy is therefore "Ricardian" (in the terminology of Woodford (2001)) or "passive" (in the terminology of Leeper (1991)).

Based on the stucture just explained, the financial assets traded in the economy are a riskless bond and two types of equity shares –or stocks– that are in positive net

[2]It does not matter whether the bond is riskless in real or nominal terms since households and banks effectively trade a complete set of Arrow-Debreu securities with each other. We assume the bond is real because it slightly simplifies notation.

supply, the stocks of banks and the stocks of the intermediate good producers. Because all banks receive the same identical shock, we can group all of their equity shares into a single banking sector stock; for the same reason, we can group all intermediate good producer shares into a single intermediate good producer sector stock. Therefore, we can assume without any loss of generality that there are exactly two stocks in positive net supply, and we normalize the supply to one. The banking sector stock pays the aggregate dividends of all banks and the producer sector stock pays the aggregate dividends of all producers[3].

We index the three financial securities traded in the economy by $j \in \{0, goods, banks\}$ and write their prices, $S_{j,t}$, and dividend processes, $D_{j,t}$, in real terms. The first security, indexed by $j = 0$, is the riskless bond. It has price $S_{0,t}$ that follows

$$dS_{0,t} = S_{0,t} R_t dt, \tag{27}$$

where $R_t$ is the equilibrium real riskless interest rate. The remaining two securities are risky stocks that have equilibrium prices given by

$$\frac{dS_{j,t}}{S_{j,t}} = \alpha_{j,t} dt + \sigma_{j,t} dZ_t, \tag{28}$$

where $\alpha_{j,t}$ is the real expected return (including any dividends[4]) and $\sigma_{j,t}$ is the exposure to the standard Brownian motion $Z_t$. We can write the stock price processes in vector notation

$$dS_t = diag\left(S_t\right)\left(\alpha_t dt + \sigma_t dZ_t\right), \tag{29}$$

where $S_t$ is a $2 \times 1$ vector (that does not include the bond), $\alpha_t$ is a $2 \times 1$ vector and $\sigma_t$ is a $2 \times 1$ vector. Because $\sigma_t$ is endogenous, we do not yet know whether markets are complete in equilibrium. If either stock has $\sigma_{j,t} \neq 0$ a.s., markets are complete. Because there is a single source of uncertainty and two stocks and a bond, one of the stocks will be redundant in equilibrium, in that it can be fully replicated by a portfolio of bonds and the other stock. However, although one of the stocks can be replicated, it is only the banks that can do so, since households can only trade in the stock of good producers

---

[3]We could allow for unrestricted trade among households and banks of a complete set of Arrow-Debreu securities in zero net supply that span all risks in the economy and none of our results would change. In our setup, markets will be complete in equilibrium even without introducing these Arrow-Debreu securities, so adding any new securities would be redundant.

[4]With some abuse of language, we refer to $S_t$ as the "price" of the stock instead of using the sometimes more precise "gain process" terminology.

while the banks can trade in both stocks. It is therefore important for our results to have both stocks in the economy.

We define real expected excess returns $\mu_t$ as the $2 \times 1$ vector

$$\mu_t \equiv \alpha_t - R_t, \tag{30}$$

and the market price of risk –the risk-premium– as the process $\eta_t$ that satisfies

$$\sigma_t^T \eta_t \equiv \mu_t. \tag{31}$$

where the superscript $T$ denotes the transpose of a vector or a matrix. Equation (31) has a solution iff there is no arbitrage, which is a necessary condition for equilibrium. If markets are complete, $\eta_t$ is unique and given by

$$\eta_t = (\sigma_{j,t})^{-1} \mu_{j,t}. \tag{32}$$

for all $j$ such that $\sigma_{j,t} \neq 0$.

The market price of risk is closely related to the equivalent martingale measure $\mathbb{Q}$. Indeed, $Z_t^{\mathbb{Q}}$ defined by

$$Z_t^{\mathbb{Q}} \equiv Z_t + \eta_t, \tag{33}$$

is a standard Brownian motion under the equivalent martingale measure (the "risk-neutral measure"). Under $\mathbb{Q}$, stock prices discounted by bond prices are martigales, i.e.,

$$\mathbb{E}_t^{\mathbb{Q}} \left[ d \left( \frac{S_{j,t}}{S_{0,t}} \right) \right] = 0. \tag{34}$$

We also define the real state price density (SPD) $Q_t$ as the solution to

$$dQ_t \equiv -Q_t R_t dt - Q_t \eta_t^T dZ_t, \tag{35}$$

$$Q_0 \equiv 1. \tag{36}$$

and the nominal SPD, $Q_t^{\$}$, by

$$Q_t^{\$} \equiv Q_t P_t.$$

Under the physical measure, stock prices are given by the SPD-deflated stream of dividends

$$Q_t S_{j,t} = \mathbb{E}_t \left[ \int_t^\infty Q_s D_{j,s} ds \right].$$

14

In equilibrium, we will find that markets are complete and that $\sigma_{j,t} \neq 0$ for $j = \{goods, banks\}$. No arbitrage then requires

$$\eta_t = (\sigma_{goods,t})^{-1} \mu_{goods,t} = (\sigma_{banks,t})^{-1} \mu_{banks,t}. \tag{37}$$

## 3.4 Banks

Bank liabilities consist of wealth (equity capital or net worth), which we denote by $X_t$, and real deposits issued to households. Bank assets consist of a portfolio of the two traded stocks and bonds. Because banks can replicate one of the stocks by trading on the other stock and the bond, the portfolio choice of the bank only determines the allocation of wealth between the bond and a portfolio of the two stocks. We therefore solve the portfolio problem of banks assuming for simplicity that there is a single risky asset (a portfolio of the two stocks) instead of two risky assets. By equation (37), this portfolio of stocks can be taken to have any portfolio weights in each of the two stocks without affecting the optimal portfolio choice of banks. Therefore, $\mu_t$ and $\sigma_t$ in the portfolio choice problem of the bank described below should be interpreted as the drfit and volatility of the protfolio of stocks and not as a vector of drifts and volatilities that contain the drifts and volatilities of each stock. The actual equilibrium weights for the portfolio of two stocks that banks invest in will be determined by the market clearing condition that banks must hold the entire supply of good producers' stock.

The bank solves a standard Merton portfolio problem augmented by a Value-at-Risk constraint and preference shocks

$$\max_{\{\theta_t, \delta_t\}_{t \geq s}} \mathbb{E}_s \left[ \int_s^\infty e^{-\beta(t-s)} e^{\zeta_t} \log (\delta_t X_t) \, dt \right] \tag{38}$$

s.t.

$$\frac{dX_t}{X_t} = (R_t - \delta_t + \theta_t \mu_t) \, dt + \theta_t \sigma_t dZ_t, \tag{39}$$

$$VaR_{\tau,\alpha} (t, \theta_t, \delta_t, X_t) \leq a_V X_t, \tag{40}$$

$$d\zeta_t = -\frac{1}{2} m_t^2 dt - m_t dZ_t, \ \zeta_0 = 0, \tag{41}$$

$$dm_t = -\kappa m_t + \sigma_m dZ_t, \tag{42}$$

$$X_s \text{ given,} \tag{43}$$

where $\zeta_t$ is a preference shock, $\delta_t$ is the share of wealth distributed to the household, $\delta_t X_t$ are real dividends distributed to the household, $\theta_t$ is the share of wealth invested in

the portfolio of risky stocks, and $VaR_{\tau,\alpha}(t,\theta_t,\delta_t,X_t)$ is the value-at-risk of bank wealth over the interval $[t,t+\tau]$ at level $\alpha \in (0,1/2]$ with $\tau > 0$ and $a_V \in (0,1)$.

The dynamics of $\zeta_t$ in equations (41)-(42) imply that $e^{\zeta_t}$ is a Radon–Nikodym derivative (a change of measure). Changing to the measure defined by $e^{\zeta_t}$, the bank problem can be restated as

$$\max_{\{\theta_t,\delta_t\}_{t \geq s}} \mathbb{E}_s^{bank}\left[\int_s^{\infty} e^{-\beta(t-s)} \log\left(\delta_t X_t\right) dt\right] \tag{44}$$

$$s.t.$$

$$\frac{dX_t}{X_t} = \left(R_t - \delta_t + \theta_t\left(\mu_t - \sigma_t m_t\right)\right) dt + \theta_t \sigma_t dZ_t^m, \tag{45}$$

$$VaR_{\tau,\alpha}(t,\theta_t,\delta_t,X_t) \leq a_V X_t, \tag{46}$$

$$X_s \text{ given}, \tag{47}$$

where, under the new measure, $Z_t^m$ is a standard Brownian motion and $\mathbb{E}_s^{bank}[\cdot]$ is the expectation operator, with

$$Z_t^m = Z_t + m_t.$$

Thus, the preference shock can also be interpreted as the bank and the household having beliefs.

As is the case in practice, the portfolio manager evaluates its $VaR$ by assuming that the portfolio weights remain constant between $t$ and $t+\tau$. Let

$$Q\left(t,\theta_t,\delta_t\right) \equiv R_t - \delta_t + \theta_t\left(\mu_t - \sigma_t m_t\right) - \frac{1}{2}\left(\theta_t \sigma_t\right)^2$$

be the drift of $d\log X_t$ under the bank measure. Then, the dynamic budget constraint of the bank has a strong solution[5]

$$X_t = X_0 \exp\left\{\int_0^t Q\left(t,\theta_s,\delta_s\right) ds + \int_0^t \theta_s \sigma_s dZ_s^m\right\},$$
$$X_0 \text{ given}.$$

Projected wealth loss between $t$ and $t+\tau$ when keeping the portfolio constant at $(\theta_t,\delta_t)$ for $t \in [t,t+\tau]$ is

$$X_t - X_{t+\tau} = X_t\left[1 - \exp\left\{Q\left(t,\theta_t,\delta_t\right)\tau + \theta_t\sigma_t\left(Z_{t+\tau}^m - Z_t^m\right)\right\}\right]$$

---

[5]In this context, a solution is "strong" if it holds path by path.

16

Thus, the $\alpha^{th}$ percentile of the projected wealth loss, $X_t - X_{t+\tau}$, conditional on time-$t$ information is

$$X_t \left[ 1 - \exp \left\{ Q\left(t, \theta_t, \delta_t\right) \tau + \mathcal{N}^{-1}\left(\alpha\right) \left|\theta_t \sigma_t\right| \sqrt{\tau} \right\} \right]$$

Value-at-risk is then

$$VaR_{\tau,\alpha}\left(t, \theta_t, \delta_t, X_t\right) \equiv X_t \left[ 1 - \exp \left\{ Q\left(t, \theta_t, \delta_t\right) \tau + \mathcal{N}^{-1}\left(\alpha\right) \left|\theta_t \sigma_t\right| \sqrt{\tau} \right\} \right]$$

Define

$$g_V\left(t, \theta_t, \delta_t\right) \equiv -Q\left(t, \theta_t, \delta_t\right) \tau - \mathcal{N}^{-1}\left(\alpha\right) \left|\theta_t \sigma_t\right| \sqrt{\tau}$$

Then

$$VaR_{\tau,\alpha}\left(t, \theta_t, \delta_t, X_t\right) \leq X_t a_V \iff g_V\left(t, \theta_t, \delta_t\right) \leq \overline{VaR}$$

where

$$\overline{VaR} \equiv \log \frac{1}{1 - a_V}$$

The choice $\alpha \in (0, 1/2]$ guarantees that $\mathcal{N}^{-1}\left(\alpha\right) \leq 0$ and that $g_V\left(t, \theta_t, \delta_t\right)$ is convex in $(\theta_t, \delta_t)$.

In Appendix A we show that the banks' problem can be simplified to a non-stochastic one. To maximize

$$\mathbb{E}_0^{bank} \int_0^\infty e^{-\beta t} \log\left(\delta_t X_t\right) dt$$

over the constrained set, it suffices to maximize

$$h\left(t, \theta_t, \delta_t\right) \equiv \log\left(\delta\right) + \frac{1}{\beta} Q\left(t, \theta_t, \delta_t\right)$$

pathwise over the constrained set. For a fixed path and a fixed time $t$, the bank then solves

$$\max_{\theta_t, f_t} h\left(t, \theta_t, \delta_t\right)$$
$$s.t.$$
$$g_V\left(t, \theta_t, \delta_t\right) \leq \overline{VaR} \tag{48}$$

The function $h\left(t, \theta_t, \delta_t\right)$ is concave in $(\theta_t, \delta_t)$ and maximized over $(\theta_t, \delta_t)$ by

$$\delta_{M,t} = \beta$$
$$\theta_t = \theta_{M,t}$$

17

when the $VaR$ constraint is not binding, where we derive $\theta_{M,t}, \delta_{M,t}$ using the FOC

$$
\begin{aligned}
[f_t] \quad : \quad & 0 = \frac{\partial}{\partial \delta_t} h\left(t, \theta_t, \delta_t\right) \\
: \quad & 0 = \frac{1}{\delta_{M,t}} - \frac{1}{\beta} \\
: \quad & \delta_{M,t} \equiv \beta \\
[\theta_t] \quad : \quad & 0 = \frac{\partial}{\partial \theta_t} h\left(t, \theta_t, \delta_t\right) \\
: \quad & 0 = \frac{1}{\beta}\left(\mu_t - \sigma_t m_t - \sigma_t^2 \theta_{M,t}\right) \\
: \quad & \theta_{M,t} \equiv \frac{1}{\sigma_t}\left(\frac{\mu_t}{\sigma_t} - m_t\right)
\end{aligned}
$$

The solution $(\theta_{M,t}, \delta_{M,t})$ when the $VaR$ constraint is not binding coincides with the standard Merton portfolio solution for an agent that does not face a $VaR$ constraint. Using the definition of the market price of risk $\eta_t$ in equation (31), we can also write

$$
\theta_{M,t} \equiv (\sigma_t)^{-1}\left(\eta_t - m_t\right)
$$

As just derived, if $(\theta_{M,t}, \delta_{M,t})$ satisfy

$$
g_V\left(t, \theta_{M,t}, \delta_{M,t}\right) \leq \overline{VaR}
$$

then $(\theta_t^*, \delta_t^*) = (\theta_{M,t}, \delta_{M,t})$ is the solution to the bank's problem with the $VaR$ constraint (and the $VaR$ constraint does not bind). Otherwise, because the constraint set is compact and convex, and the objective is continuous, there will be a unique solution $(\theta_t^*, \delta_t^*)$. Moreover, $(\theta_t^*, \delta_t^*)$ must be such that the $VaR$ holds with equality.

In Appendix C we derive the optimal portfolio of the bank, which is given by

$$
\begin{aligned}
\theta_t \quad = \quad & \min\left\{1, \max\left\{0, \varphi_t\right\}\right\} \theta_{M,t} & (49) \\
\delta_t \quad = \quad & u\left(t, \min\left\{1, \varphi_t\right\}\right) f_{M,t} 1_{\{\varphi_t > 0\}} & (50) \\
& + \left(R_t + \frac{1}{\tau}\log\frac{1}{1 - a_V}\right) 1_{\{\varphi_t \leq 0\}} \\
\varphi_t \text{ such that:} \quad & g_V\left(t, \varphi_t \theta_{M,t}, u\left(t, \varphi_t\right) f_{M,t}\right) = \overline{VaR} & (51)
\end{aligned}
$$

18

where we omit the asterisks for ease of notation and

$$u(t, z) \equiv 1 + \frac{\sqrt{\tau} \, |\theta_{M,t} \sigma_t|}{\mathcal{N}^{-1}(\alpha)} (1 - z)$$

We see that under the bank measure, the $VaR$ constraint does not distort the composition of the portfolio, as $\theta_t$ is a multiple of mean-variance efficient portfolio, $\theta_{M,t}$. Under the physical measure, the $VaR$ constraint does distort the conditional composition of the portfolio but not its unconditional composition. In other words, under the physical measure, the bank invests in the mean-variance efficient portfolio on average, but not necessarily at any given point in time.

On the other hand, under both the bank and the physical measures, the $VaR$ constraint changes the amount invested in the mean-variance efficient portfolio. Instead of $\theta_t = \theta_{M,t}$, as would obtain without the $VaR$ constraint, we now get $\theta_t = \theta_{M,t}/\gamma_t$ where

$$\begin{aligned}
\gamma_t &= \frac{1}{\min\{1, \max\{0, \varphi_t\}\}} \in [1, \infty) \\
&= \begin{cases} \infty & , \text{ if } \varphi_t \in (-\infty, 0] \\ \frac{1}{\varphi_t} & , \text{ if } \varphi_t \in (0, 1] \\ 1 & , \text{ if } \varphi_t \in (1, \infty) \end{cases}
\end{aligned}$$

Thus, the $VaR$ constraint makes the agent behave as an agent with a time-varying effective risk aversion of $\gamma_t$ that is higher than its true risk aversion of 1.

We can find an explicit expression for $\varphi_t$, as shown in Appendix D

$$\varphi_t = 1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau} \, |\eta_t - m_t|} \pm \sqrt{2(R_t - \delta_t)\tau + 2\overline{VaR} + |\eta_t - m_t|^2 \tau^2 \left(1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau} \, |\eta_t - m_t|}\right)^2}$$

Evaluating the left-hand side of equation (48) at the optimal policies of the bank in

equations (147)-(149) gives

$$
\begin{aligned}
g_V\left(t, \theta_t, \delta_t, \nu\right) = & -\left(R_t - \delta_t + \theta_t \mu_t - \theta_t \sigma_t m_t - \frac{1}{2}\left(\theta_t \sigma_t\right)^2\right) \tau \\
& - \mathcal{N}^{-1}\left(\alpha\right)\left|\theta_t \sigma_t\right| \sqrt{\tau} \\
= & -\left(R_t - \delta_t\right) \tau \\
& -\left(\min\left\{1, \max\left\{0, \varphi_t\right\}\right\} - \frac{1}{2}\min\left\{1, \max\left\{0, \varphi_t\right\}\right\}^2\right)\left(\eta_t - m_t\right)^2 \tau \\
& - \mathcal{N}^{-1}\left(\alpha\right) \min\left\{1, \max\left\{0, \varphi_t\right\}\right\}\left|\eta_t - m_t\right| \sqrt{\tau}
\end{aligned}
\tag{52}
$$

Of course, equation (52) evaluated at any $\varphi_t \in [0,1]$ gives $g_V\left(t, \theta_t, \delta_t\right) = \overline{VaR}$.

Finally, we note that the Lagrange multiplier $\lambda$ for the $VaR$ constraint is

$$
\lambda = \frac{1}{\tau}\left(\frac{1}{\delta_t} - \frac{1}{\beta}\right)
$$

so the Lagrange multiplier for the original problem under the bank's probability measure is

$$
\begin{aligned}
\lambda_{VaR, t, m} & = \lambda e^{-\beta t} \\
& = \frac{1}{\tau}\left(\frac{1}{\delta_t} - \frac{1}{\beta}\right) e^{-\beta t}
\end{aligned}
$$

and under the physical measure is

$$
\begin{aligned}
\lambda_{VaR, t} & = \lambda e^{-\beta t} e^{\zeta_t} \\
& = \frac{1}{\tau}\left(\frac{1}{\delta_t} - \frac{1}{\beta}\right) e^{-\beta t} e^{\zeta_t}
\end{aligned}
$$

Note that since

$$
\delta_t \leq \beta
$$

we have

$$
\lambda_{VaR, t} \geq 0
$$

## 3.5  Market Completeness and Banks' SPD

So far, we have not used the fact that markets are complete. We can use market completeness to recover the bank's SPD. The bank problem under the physical measure is

given by equations (38)-(43). Complete markets imply that the dynamic budget constraints of the bank in equation (39) is equivalent to the static budget constraint

$$X_0 = \mathbb{E}_0 \left[ \int_0^\infty Q_t \delta_t X_t dt \right] \tag{53}$$

where the banks take the SPD $Q_t$ as given. The Lagrangian for the bank's problem is then

$$\mathcal{L} = \mathbb{E}_0 \left[ \int_0^\infty e^{-\beta t} e^{\zeta_t} \log\left(\delta_t X_t\right) dt \right] + \lambda_{bc} \left( X_0 - \mathbb{E}_0 \left[ \int_0^\infty Q_t \delta_t X_t dt \right] \right)$$
$$+ \int_0^\infty \lambda_{VaR,t} \left( g_V\left(t, \theta_t, \delta_t\right) - \overline{VaR} \right) dt$$

where $\lambda_{bc} > 0$ is a number but $\lambda_{VaR,t} > 0$ is a function of time since we have one $VaR$ constraint for each $t$. The FOC for an interior solution are

$$[F_t] \quad : \quad 0 = \frac{e^{-\beta t} e^{\zeta_t}}{\delta_t X_t} - \lambda_{bc} Q_t + \lambda_{VaR,t} \frac{\tau}{X_t} \tag{54}$$

$$[\theta_t] \quad : \quad 0 = \frac{e^{-\beta t} e^{\zeta_t}}{\beta} \frac{\partial Q\left(t, \theta_t, \delta_t\right)}{\partial \theta_t} + \lambda_{VaR,t} \frac{\partial g_V\left(t, \theta_t, \delta_t\right)}{\partial \theta_t} \tag{55}$$

Re-arranging (54) gives the SPD of the bank

$$Q_t \quad = \quad \frac{e^{-\beta t} e^{\zeta_t}}{\lambda_{bc} \delta_t X_t} + \frac{\lambda_{VaR,t}}{\lambda_{bc}} \frac{\tau}{X_t}$$

$$= \quad \frac{e^{-\beta t} e^{\zeta_t}}{\lambda_{bc}} \frac{1}{\delta_t X_t} + \frac{\lambda_{VaR,t} \tau}{\lambda_{bc}} \frac{1}{X_t} \tag{56}$$

$$= \quad \frac{1}{\lambda_{bc} X_t} \left[ e^{-\beta t} e^{\zeta_t} \frac{1}{\delta_t} + \lambda_{VaR,t} \tau \right] \tag{57}$$

The term $\lambda_{VaR,t} \tau / \lambda_{bc} X_t$ is the marginal value of relaxing the $VaR$ constraint, and the term $e^{-\beta t} e^{\zeta_t} / \lambda_{bc} f_t X_t$ is the marginal value of issuing dividends.

Using $\lambda_{VaR,t}$ from equation (53)

$$\lambda_{VaR,t} = \frac{e^{-\beta t} e^{\zeta_t}}{\tau} \left( \frac{1}{\delta_t} - \frac{1}{\beta} \right)$$

gives

$$Q_t = \frac{e^{-\beta t} e^{\zeta_t}}{\lambda_{bc} X_t} \left( \frac{2}{\delta_t} - \frac{1}{\beta} \right) \tag{58}$$

The multiplier $\lambda_{bc}$ can be found from noting that we must have $Q_0 = 1$,

$$\lambda_{bc} = \frac{2}{X_0} \left( \frac{1}{\delta_0} - \frac{1}{\beta} \right)$$

or from the budget constraint (53).

## 3.6 Households

### 3.6.1 Setup

The representative household maximizes utility subject to its budget constraint, a portfolio constraint on its holdings of stocks of goods producers, and a solvency constraint (transversality condition). For time $s$, the household problem is

$$\max_{\{C_t(i), N_t, \omega_t\}_{t \geq s}} \mathbb{E}_s \left\{ \int_s^\infty e^{-\beta(t-s)} \left( \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\xi}}{1+\xi} \right) dt \right\}, \tag{59}$$

subject to

$$d\left(P_t F_t\right) \leq W_t N_t dt + \omega_{banks,t} d\left(P_t S_{banks,t}\right) + \omega_{0,t} d\left(P_t S_{0,t}\right) - P_t C_t dt \text{ for all } t \geq s, \tag{60}$$

$$\lim_{t \to \infty} \mathbb{E}_s \left[ Q_t F_t \right] = 0, \tag{61}$$

$$F_s \text{ given}, \tag{62}$$

and the definition of the aggregator for consumption in equation (17). The household maximizes utility by choosing the path $\{C_t(i), N_t, \omega_t\}_{t \geq s}$ of consumption of good $i$, $C_t(i)$, supply of labor, $N_t$, its position (number of shares) $\omega_{banks,t}$ in the stock of banks and its position $\omega_{0,t}$ in bonds. The variable $F_t$ is the household's real financial wealth at time $t$, $W_t$ is the nominal wage paid for labor supplied to the firms in an integrated competitive market[6], $S_{banks,t}$ is the real price of the stock of banks, $S_{0,t}$ is the real price of riskless bonds, $P_t$ is the aggregate price level and $Q_t$ is the real SPD.

The dynamic flow budget constraint in equation (60) states that changes in the household's nominal financial wealth must be less than or equal to nominal labor income plus the nominal payoff on financial assets (which can be negative), minus nominal consumption expenditures. Note that households are not allowed to invest in intermediate good producers' stock, so the term in the budget constraint that would correspond to

---

[6]It follows that all firms pay the same wage for homogeneous labor. This means that the household picks $N_t(i) = N_t$ for all $i$ and thus we can simplify the household problem by optimizing directly over $N_t$ instead of over $N_t(i)$.

this investment, $\omega_{goods,t}d\left(P_tS_{goods,t}\right)$, is absent. Alternatively, we could have included the term in the household's budget constraint together with the additional constraint that $\omega_{goods,t} = 0$. Despite the household being restricted in its investments, it still faces complete markets. The reason is that it can freely trade in one risky stock –the banks' stock– that fully spans the single risk in the economy, $Z_t$.

The solution to the households' problem, as shown in Appendix E, is characterized by the two familiar conditions

$$C_t^\gamma N_t^\xi \;=\; \frac{W_t}{P_t} \tag{63}$$

$$\frac{Q_t}{Q_s} \;=\; e^{-\beta(t-s)}\left(\frac{C_t}{C_s}\right)^{-\gamma} \tag{64}$$

Equation (63) is the intra-temporal first order condition of the household that determines labor supply given the real wage. Equation (64) is the Euler equation, the intertemporal first order conditoin that determines consumption given the SPD.

## 3.7    Good-Producing Firms

The setup for good-producing firms is the continuous time analog of the standard textbook version of the New Keynesian model. Hence, we relegate the details to Appendix and show here only the most essential elements that are required to characterize the equilibrium of the economy.

### 3.7.1    Final Good Sector

Firms in the final good sector produce a homogeneous good, $Y_t$, using intermediate goods, $Y_t\left(i\right)$, of different varieties $i \in [0,1]$. There is continuum of competitive final good producers of measure one. The production functions for all final good producers are identical and given by

$$Y_t = \left(\int_0^1 Y_t\left(i\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{65}$$

where $\varepsilon > 1$ is the constant elasticity of substitution for differentiated goods (and taken to be equal to the elasticity of substitution across goods for consumers). The production function has constant returns to scale and diminishing marginal product.

The representative firm chooses inputs $\{Y_t(i)\}_{i \in [0,1]}$ to maximize real profits

$$Y_t - \frac{1}{P_t} \int_0^1 P_t(i) Y_t(i) \, di$$

where $(1/P_t) \int_0^1 P_t(i) Y_t(i) \, di$ are real costs and $Y_t$ is real total revenue. Because final good producers are competitive, they take $P_t(i)$ and $P_t$ as given. Because of constant returns and competition, the size of any one final goods firm is indeterminate. However, their input demand is determined by the following cost minimization problem

$$\min_{Y_t(i)} \int_0^1 P_t(i) Y_t(i) \, di$$
$$s.t$$
$$Y_t \leq \left( \int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

The cost minimization yields a demand for intermediate good $i$ that is homogeneous of degree one in total final output

$$Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t \tag{66}$$

where $\varepsilon$ turns out to be the elasticity of demand.

### 3.7.2 Intermediate Goods Sector

There is continuum of mass one of monopolistically competitive firms indexed by $i \in [0,1]$. Each firm faces a demand curve given by equation (66). Firms use labor $N_t(i)$ to produce output according to the technology

$$Y_t(i) = A N_t(i) \tag{67}$$

Labor is hired in a competitive market with perfect mobility.

Firms set prices according to Calvo staggered pricing. The probability density of receiving the signal to change prices after an amount of time $h$ has elapsed is independent of the last time the firm received the signal and across firms, and given by

$$\delta e^{-\delta h},$$

where $\delta > 0$ is the Calvo parameter. Hence, the probability of not having received a signal between $t$ and $\tau$ is

$$
\begin{aligned}
1 - \int_t^\tau \delta e^{-\delta(s-t)} ds &= e^{-\delta(\tau-t)} \\
&\approx 1 - \delta (\tau - t)
\end{aligned}
$$

Firms that are able to adjust the price choose the price optimally. These firms maximize expected real discounted profits subject to their production technology (67), the demand curve (66) and the constraint on the frequency of price adjustment. Firms that cannot change their price adjust output to meet demand at the pre-established price. Both types of firms choose inputs to minimize costs, given output demand.

The first-order condition of firms imply that inflation, $\pi_t$, is given by

$$
\pi_t = \frac{\delta}{1-\varepsilon} \left( \left( \frac{x_{2,t}}{x_{1,t}} \right)^{1-\varepsilon} - 1 \right)
$$

where $x_{1,t}$ and $x_{2,t}$ are weighted averages of future expected output and marginal costs, defined by

$$
\begin{aligned}
x_{1,t} &\equiv \mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left( \frac{P_t}{P_s} \right)^{1-\varepsilon} \frac{Y_s}{Y_t} ds \\
x_{2,t} &\equiv \mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left( \frac{P_t}{P_s} \right)^{-\varepsilon} \frac{Y_s}{Y_t} \frac{\varepsilon}{\varepsilon-1} MC_s ds
\end{aligned}
$$

and $MC_t = W_t/AP_t$. Real profits for the producer of intermediate good $i$ are

$$
D_{t,goods}(i) = \frac{P_t(i) Y_t(i)}{P_t} - MC_t Y_t(i) \tag{68}
$$

Summing across $i$ gives the aggregate profits for the sector, which are the dividends paid by the stock of the goods producing sector:

$$
\begin{aligned}
D_{t,goods} &= \int_0^1 D_{t,goods}(i) \, di \tag{69} \\
&= \frac{1}{P_t} \int_0^1 P_t(i) Y_t(i) \, di - MC_t \int_0^1 Y_t(i) \, di \tag{70}
\end{aligned}
$$

# 4 Equilibrium

An equilibrium is a collection

$$\mathcal{E} = \{C_t, C_t(i), Y_t, Y_t(i), N_t, F_t, \omega_t, W_t, P_t, Q_t, D_t, \theta_t, X_t, \delta_t, S_t, \eta_t, \sigma_t, R_t\}_{t \geq 0}$$

such that households, firms and banks optimize, and markets for labor, intermediate goods, the final good and all financial assets clear. We now collect the first-order conditions and market clearing conditions that determine an equilibrium, and then combine them to have an explicit characterization of the equilibrium.

Household optimization gives

$$\text{Labor supply: } N_t = \left( \frac{W_t}{P_t} C_t^{-\gamma} \right)^{\frac{1}{\xi}} \tag{71}$$

$$\text{Intertemporal consumption: } Q_t = e^{-\beta t} C_t^{-\gamma} \tag{72}$$

$$\text{Budget constraint: } F_t = E_t \left[ \int_t^\infty \frac{Q_s}{Q_t} \left( C_s - \frac{W_s}{P_s} N_s \right) ds \right] \tag{73}$$

$$\text{Demand for financial assets: } \frac{dF_t}{F_t} = \left( R_t + \omega_t \mu_t - \frac{C_t}{F_t} + \frac{W_t}{P_t} \frac{N_t}{F_t} \right) dt + \omega_t \sigma_t dZ_t \tag{74}$$

$$\tag{75}$$

Optimization for the final good producers gives

$$\text{Demand of intermediate goods:} Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t \tag{76}$$

$$\text{Supply of final goods:} Y_t = \left( \int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{77}$$

Intermediate good producers' optimization for inputs of prodcution is characterized by

$$\text{Labor demand: } N_t(i) = \frac{Y_t(i)}{A} \tag{78}$$

$$\text{Supply of goods: } Y_t(i) = A N_t(i) \tag{79}$$

$$\text{Profits/Dividends: } D_{goods,t}(i) = \frac{P_t(i) Y_t(i)}{P_t} - MC_t Y_t(i) \tag{80}$$

. Intermediate good producers' price setting decision is given by

$$\pi_t = \frac{\vartheta}{1-\varepsilon} \left( \Pi_t^{1-\varepsilon} - 1 \right) \tag{81}$$

$$\Pi_t = \frac{x_{2,t}}{x_{1,t}} \tag{82}$$

$$dx_{1,t} = (x_{1,t} + 1) \, \vartheta dt + (1-\varepsilon) \, x_{1,t} \pi_t dt + x_{1,t} Q_t Y_t d \left( \frac{1}{Q_t Y_t} \right) \tag{83}$$

$$dx_{2,t} = \left( x_{2,t} + \frac{MC_t}{MC} \right) \vartheta dt - \varepsilon x_{2,t} \pi_t dt + x_{2,t} Q_t Y_t d \left( \frac{1}{Q_t Y_t} \right) \tag{84}$$

Optimization for banks is given by

$$\text{Dividends: } \delta_t = u \left( t, \min\{1, \varphi_t\} \right) f_{M,t} 1_{\{\varphi_t > 0\}} + \left( R_t + \frac{1}{\tau} \overline{VaR} \right) 1_{\{\varphi_t \leq 0\}} \tag{85}$$

$$\text{Optimal portfolio: } \theta_t = \min\{1, \max\{0, \varphi_t\}\} \, \theta_{M,t} \tag{86}$$

$$\text{Wealth accumulation: } \frac{dX_t}{X_t} = (R_t - \delta_t + \theta_t \mu_t) \, dt + \theta_t \sigma_t dZ_t \tag{87}$$

where

$$f_{M,t} = \beta \tag{88}$$

$$\theta_{M,t} = (\sigma_t)^{-1} (\eta_t - m_t) \tag{89}$$

are the dividends and portfolio positions of an unconstrained but otherwise identical bank, and $\varphi_t$ is such that the $VaR$ constraint holds with equality

$$\varphi_t = 1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau} |\eta_t - m_t|} \pm \sqrt{2 (R_t - \delta_t) \tau + 2\overline{VaR} + |\eta_t - m_t|^2 \tau^2 \left( 1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau} |\eta_t - m_t|} \right)^2} \tag{90}$$

Market clearing conditions for labor and goods are

$$\text{Intermediate goods: } \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t = A N_t(i) \tag{91}$$

$$\text{Final goods: } C_t = Y_t \tag{92}$$

$$\text{Labor: } \left( \frac{W_t}{P_t} C_t^{-\gamma} \right)^{\frac{1}{\xi}} = \frac{Y_t(i)}{A} \tag{93}$$

and market clearing conditions for financial assets are

$$\text{Banks' stock:} \quad \frac{F_t \omega_{banks,t}}{S_{banks,t}} + \frac{X_t \theta_{banks,t}}{S_{banks,t}} = 1 \tag{94}$$

$$\text{Good producers' stock:} \quad \frac{X_t \theta_{goods,t}}{S_{goods,t}} = 1 \tag{95}$$

$$\text{Bond:} \quad \frac{F_t \omega_{0,t}}{S_{0,t}} + \frac{X_t \theta_{0,t}}{S_{0,t}} = 0 \tag{96}$$

Since banks must hold the entire stock of good producers, $\varphi_t > 0$.

Aggregation of output gives

$$\int_0^1 Y_t(i)\, di = v_t Y_t = A N_t$$

where

$$v_t \equiv \int_0^1 \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} di.$$

gives the aggregate efficiency loss due to price distortions. Aggregation of dividends across good producers then gives

$$D_{goods,t} = (1 - MC_t v_t)\, Y_t$$

Finally, the central bank sets nominal interest $i_t$, which are linked to real interest rates and inflation by $i_t = R_t + \pi_t$.

## 4.1 Equilibrium characterization

Combining labor demand, labor supply and clearing of labor markets gives real wages and labor

$$\frac{W_t}{P_t} = \left( \frac{v_t}{A} \right)^{\xi} Y_t^{\xi+\gamma} \tag{97}$$

$$N_t = \frac{v_t Y_t}{A} \tag{98}$$

Equations (97) and (98) in turn imply that

$$MC_t = \frac{1}{A}\left(\frac{v_t}{A}\right)^{\xi} Y_t^{\xi+\gamma} \tag{99}$$

$$D_{goods,t} = \left(1 - \left(\frac{v_t}{A}\right)^{1+\xi} Y_t^{\xi+\gamma}\right) Y_t \tag{100}$$

Defining the natural rate of output $Y_t^n$, the natural rate of interes, $r_t$, and the output gap $y_t$, as

$$Y_t^n \equiv v_t^{-\frac{\xi}{\xi+\gamma}} A^{\frac{1+\xi}{\xi+\gamma}} MC^{\frac{1}{\xi+\gamma}}$$

$$y_t \equiv \log Y_t - \log Y_t^n$$

$$r_t \equiv \beta - \frac{\varphi\sigma}{\varphi+\sigma}\frac{E_t\left[d\log v_t\right]}{dt}$$

equations (169), (72) and (92) give the dynamic IS equation

$$dy_t = \frac{1}{\gamma}\left(R_t - r_t + \frac{1}{2}\eta_t^2\right) dt + \frac{\eta_t}{\gamma} dZ_t \tag{101}$$

Because the household and the bank trade in complete markets, their SPD must agree,

$$e^{-\beta t} Y_t^{-\gamma} = \frac{e^{-\beta t} e^{\zeta_t}}{\lambda_{bc} X_t}\left(\frac{2}{\delta_t} - \frac{1}{\beta}\right)$$

Taking derivatives,

$$-\gamma d\log Y_t = d\zeta_t - d\log X_t + d\log\left(\frac{2}{\delta_t} - \frac{1}{\beta}\right) \tag{102}$$

Matching drift and stochastic parts of the left and right-hand sides of (102) gives

$$\begin{aligned}
d\delta_t &= \left(\delta_t - \beta + \frac{1}{2}\left(\eta_t^2 - m_t^2\right) - \theta_t\mu_t + \frac{1}{2}\left(\theta_t\sigma_t\right)^2\right)\left(\frac{2\left(2\beta - \delta_t\right)}{\beta}\right) dt \\
&\quad + \left(\eta_t - m_t - \theta_t\sigma_t\right)^2 \left(\frac{1}{2} - \frac{\delta_t\left(12\beta^2 - \delta_t^2 - 2\beta\delta_t\right)}{16\beta^3}\right) dt \\
&\quad + \left(\eta_t - m_t - \theta_t\sigma_t\right)\left(\frac{2\beta - \delta_t}{2\beta}\right) dZ_t
\end{aligned} \tag{103}$$

Using Ito's lemma, equations (85), (90) and (103) give

$$d\varphi_t = G\left(\varphi_t, R_t\right) dt + S\left(\varphi_t, R_t\right) dZ_t \tag{104}$$

for two functions $G$, $S$. Any three of the four equations (85), (90), (103), (104) characterize the optimal decision of the banks.

Using the definition of vulnerability in equation (5) and the dynamic IS in equation (108), we obtain

$$V_t = -\frac{1}{\gamma}\left(R_t - r_t + \frac{1}{2}\eta_t^2\right)\tau - \mathcal{N}^{-1}\left(\alpha\right)\frac{\eta_t}{\gamma}\sqrt{\tau} \tag{105}$$

Solving (105) for $R_t$ gives

$$R_t\left(V_t, \eta_t\right) = -\frac{1}{2}\eta_t^2 - \frac{\mathcal{N}^{-1}\left(\alpha\right)}{\sqrt{\tau}}\eta_t - \frac{\gamma V_t}{\tau} + r_t \tag{106}$$

Assuming[7] $\eta_t - m_t > 0$, plugging (106) into (90) and using (85), we can solve for $\eta_t$ as a function of $\varphi_t$, $m_t$ and $V_t$

$$\eta_t = \eta\left(\varphi_t, m_t, V_t\right) \tag{107}$$

where the function $\eta$ is given by

$$
\begin{aligned}
\eta\left(\varphi_t, m_t, V_t\right) &= -\frac{1}{2A}\left(B - \sqrt{-4AC + B^2}\right) \\
A &= -\frac{1}{2}\tau\left(\varphi_t - 1\right)^2 \\
B &= \tau\varphi_t^2 m_t + \left(-2\tau\right)\varphi_t m_t + \left(N\sqrt{\tau} - \frac{1}{N}\beta\tau^{\frac{3}{2}}\right)\varphi_t - \tau\left(\frac{N}{\sqrt{\tau}} - \frac{1}{N}\beta\sqrt{\tau}\right) \\
C &= \overline{VaR} + \left(-\frac{1}{2}\tau\right)\varphi_t^2 m_t^2 + \tau\varphi_t m_t^2 + \left(\frac{1}{N}\beta\tau^{\frac{3}{2}} - N\sqrt{\tau}\right)\varphi_t m_t \\
&\quad + \left(-\frac{1}{N}\beta\tau^{\frac{3}{2}}\right)m_t + \left(-\gamma\right)V_t + \tau\left(r_t - \beta\right)
\end{aligned}
$$

---

[7]We later linearize the model around a steady-state with $\eta_t - m_t > 0$ so that small perturbations always preserve the sign of $\eta_t - m_t$. With that in mind, we simplify at this point by assuming $\eta_t - m_t > 0$ to simplify exposition. Solving the non-linear version of the model allowing for $\eta_t - m_t < 0$ is straightforward.

The characterization of the equilibrium is then given by the following equations:

$$dy_t = \frac{1}{\gamma}\left(i_t - \pi_t - r_t + \frac{1}{2}\eta_t^2\right)dt + \frac{\eta_t}{\gamma}dZ_t \tag{108}$$

$$\frac{d\pi_t}{dt} = ((1-\varepsilon)\pi_t + \vartheta)\left[\left(\frac{\varepsilon}{\varepsilon-1}\frac{Y_t^{\xi+\gamma}}{A}\left(\frac{v_t}{A}\right)^\xi\left(\frac{1-\varepsilon}{\vartheta}\pi_t+1\right)^{\frac{1}{\varepsilon-1}}-1\right)\vartheta e_{1,t}\right] \tag{109}$$

$$- ((1-\varepsilon)\pi_t + \vartheta)\pi_t \tag{110}$$

$$d\varphi_t = G_\varphi(\varphi_t, R_t)\,dt + S_\varphi(\varphi_t, R_t)\,dZ_t \tag{111}$$

$$i_t = f(y_t, \pi_t, \varphi_t) \tag{112}$$

$$\eta_t = \eta(\varphi_t, m_t, V_t) \tag{113}$$

$$V_t = -\alpha\mathbb{V}_t\left[\frac{dy_t}{dt}\right]\sqrt{\tau} - \mathbb{E}_t\left[\frac{dy_t}{dt}\right]\tau \tag{114}$$

Equation (108) is the dynamic IS equation, the demand block of the model. Equation (110) gives inflation dynamics, the supply side of the model. The inflation dynamics depend on the present discounted value of nominal output for firms that can reset their price, $x_{1,t}$, with dynamics given by $e_{1,t} \equiv x_{1,t}^{-1}$

$$\frac{de_{1,t}}{e_{1,t}} = -\left[(1+e_{1,t})\vartheta + (1-\varepsilon)\pi_t + \beta + \frac{(\gamma-1)}{2\gamma^2}\eta_t^2 + \frac{(\gamma-1)}{\gamma}(R_t-\beta)\right]dt \tag{115}$$

$$- \frac{(\gamma-1)\eta_t}{\gamma}dZ_t \tag{116}$$

Inflation dynamics also depend on and the output losses due to inefficient price dispersion, $v_t$, with dynamics

$$dv_t = \vartheta\left(\frac{1-\varepsilon}{\vartheta}\pi_t+1\right)^{-\frac{\varepsilon}{1-\varepsilon}}dt + (\varepsilon\pi_t - \vartheta)v_t dt \tag{117}$$

Equation (111) corresponds to the financial sector block of the model and gives the dynamics of the tightness of the VaR constraint of the bank, $\varphi_t$. Equation (112) is the monetary policy rule for the central bank. Equation (113) connects the household and bank behavior through the market price of risk, vulnerability and the tightness of the VaR constraint.

## 4.2 Deterministic Steady State

Variables without their time subscript denote their values in a deterministic steady state. In a deterministic steady state, we have

$$
\begin{aligned}
v &= 1 \\
P &= 1 \\
\pi &= 0 \\
MC &= 1 - \frac{1}{\varepsilon} \\
Y &= Y^n = A^{\frac{1+\varphi}{\varphi+\sigma}} \left(1 - \frac{1}{\varepsilon}\right)^{\frac{1}{\varphi+\sigma}} \\
\frac{W}{P} &= A\left(1 - \frac{1}{\varepsilon}\right) \\
N &= \frac{Y}{A} \\
D_{goods} &= \frac{1}{\varepsilon}\left(1 - \frac{1}{\varepsilon}\right)^{\frac{1}{\varphi+\sigma}} A^{\frac{1+\varphi}{\varphi+\sigma}} \\
r &= \beta \\
R &= i = r = \beta \\
s &= 0 \\
\eta &= 0 \\
\delta &= \beta \\
\theta &= 0 \\
g_V(t,\theta,\delta) &= 0
\end{aligned}
$$

## 4.3 Linearized Version

We linearize $\eta\left(\varphi_t, V_t\right), \eta\left(\varphi_t, V_t\right)^2, G_\varphi\left(\varphi_t, R_t\right), S_\varphi\left(\varphi_t, R_t\right)$ in

$$
\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(R_t - r_t + \frac{1}{2}\eta_t^2\right) dt + \frac{\eta_t}{\gamma} dZ_t \\
d\varphi_t &= G_\varphi\left(\varphi_t, R_t\right) dt + S_\varphi\left(\varphi_t, R_t\right) dZ_t \\
\eta_t &= \eta\left(\varphi_t, V_t\right) \\
V_t &= V\left(\eta_t, R_t\right)
\end{aligned}
$$

around the deterministic steady state

$$
\begin{aligned}
\eta_t &= \Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t \\
\eta_t^2 &= -\hat{\eta}^2 + 2\hat{\eta}\left(\Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t\right) \\
d\varphi_t &= \left(\Upsilon_0 + \Upsilon_r R_t + \Upsilon_\varphi \varphi_t + \Upsilon_m m_t\right) dt + \left(\Psi_0 + \Psi_r R_t + \Psi_\varphi \varphi_t + \Psi_m m_t\right) dZ_t
\end{aligned}
$$

where $\hat{\eta}$ is the point around which $\hat{\eta}$ is linearized, i.e. $\hat{\eta} = \eta\left(\hat{\varphi}, \hat{V}, \hat{m}\right)$ where the Taylor expansion of $\eta$ was performed on $(\varphi_t, V_t, m_t)$ around $\left(\hat{\varphi}, \hat{V}, \hat{m}\right)$. Also note that to first order, $v_t = 1$ and thus the natural rate $r_t$ is constant at $r$.

Using the linearizations, we get

$$
\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(R_t - r + \frac{1}{2}\eta_t^2\right) dt + \frac{\eta_t}{\gamma} dZ_t \\
d\varphi_t &= \left(\Upsilon_0 + \Upsilon_r R_t + \Upsilon_\varphi \varphi_t + \Upsilon_m m_t\right) dt \\
&\quad + \left(\Psi_0 + \Psi_r R_t + \Psi_\varphi \varphi_t + \Psi_m m_t\right) dZ_t \\
\eta_t &= \Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t \\
\eta_t^2 &= -\hat{\eta}^2 + 2\hat{\eta}\left(\Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t\right) \\
V_t &= V\left(\eta_t, R_t\right)
\end{aligned}
$$

Rearrange

$$
\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(R_t - r + \frac{1}{2}\eta_t^2\right) dt + \frac{\eta_t}{\gamma} dZ_t \\
dy_t &= \frac{1}{\gamma}\left(R_t - r + \frac{1}{2}\left(-\hat{\eta}^2 + 2\hat{\eta}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right)\right)\right) dt \\
&\quad + \frac{1}{\gamma}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right) dZ_t \\
dy_t &= \frac{1}{\gamma}\left(R_t - r + \hat{\eta}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t - \frac{1}{2}\frac{\hat{\eta}^2}{\hat{\eta}\Phi_v}\right)\right) dt \\
&\quad + \frac{1}{\gamma}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right) dZ_t
\end{aligned}
$$

Define

$$\varphi_t \equiv -\frac{\Phi_\varphi}{\Phi_v}\varphi_t$$

$$s_t \equiv -\left(\frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right)$$

$$\overline{s} \equiv -\frac{\Phi_0}{\Phi_s}$$

$$\xi \equiv \frac{\Phi_v}{\gamma}$$

$$\alpha \equiv N^{-1}(\alpha)$$

$$d(rp_t) \equiv \hat\eta\xi\left(V_t - \chi_t - s_t - \frac{1}{2}\frac{\hat\eta}{\xi\gamma}\right)dt$$
$$+\xi(V_t - \chi_t - s_t)\,dZ_t$$

and

$$\left(-\frac{\Phi_\varphi\Upsilon_0}{\Phi_v}\right) \equiv a_0, \left(-\frac{\Phi_\varphi\Upsilon_r}{\Phi_v}\right) \equiv a_r, \left(-\frac{\Phi_\varphi\Upsilon_m}{\Phi_v}\right) \equiv a_s, \Upsilon_\varphi \equiv a_\varphi$$

$$\left(-\frac{\Phi_\varphi\Psi_0}{\Phi_v}\right) \equiv b_0, \left(-\frac{\Phi_\varphi\Psi_r}{\Phi_v}\right) \equiv b_r, \left(-\frac{\Phi_\varphi\Psi_m}{\Phi_v}\right) \equiv b_s, \Psi_\varphi \equiv a_\varphi$$

Then we get

$$dy_t = \frac{1}{\gamma}(R_t - r)\,dt + d(rp_t)$$

$$d(rp_t) = \hat\eta\xi\left(V_t - \varphi_t - s_t - \frac{1}{2}\frac{\hat\eta}{\xi\gamma}\right)dt + \xi(V_t - \varphi_t - s_t)\,dZ_t$$

$$V_t = -\frac{1}{dt}\mathbb{E}_t[dy_t]\tau - \alpha\mathbb{V}_t[dy_t]\sqrt{\tau}$$

$$d\varphi_t = (a_0 + \Upsilon_\varphi\varphi_t + a_r R_t + a_s s_t)\,dt + (b_0 + b_\varphi\varphi_t + b_r R_t + b_s s_t)\,dZ_t$$

$$ds_t = -\kappa(s_t - \overline{s}) + \sigma_s dZ_t$$

## 4.4   No Direct Feedback from Monetary Policy to $\varphi_t$

We consider the simpler case of $\varphi_t = 0$ and fixed prices. The case with $\varphi_t = 0$ corresponds to monetary policy not affecting the bank's VaR constraint directly, but only through general equilibrium (discount rate) effects. We can analyze the case in which $\varphi_t \neq 0$ and a Phillips curve is present in the same way that we analyze the simpler case; the control problem for monetary policy is still linear-quadratic even in the general case.

Without a Phillips curve and with $\varphi_t = 0$, the linearized equilibrium is characterized by

$$dy_t = \frac{1}{\gamma}\left(R_t - r + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{118}$$

$$V_t = -\mathbb{E}_t\left[dy_t\right]\tau - \alpha\mathbb{V}_t\left[dy_t\right]\sqrt{\tau} \tag{119}$$

$$ds_t = -\kappa\left(s_t - \bar{s}\right) + \sigma_s dZ_t \tag{120}$$

so that

$$\mathbb{E}_t\left[dy_t\right] = \frac{1}{\gamma}\left(R_t - r + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right)$$

$$\mathbb{V}_t\left[dy_t\right] = \xi\left(V_t - s_t\right)$$

Solve for $R_t$ and $V_t$ in (119) to get

$$R_t = r - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right)s_t + \frac{1}{2}\eta^2 \tag{121}$$

Plug in (121) into (118) to get

$$dy_t = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{122}$$

Use

$$\mathbb{E}_t\left[dy_t\right] = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)$$

$$\mathbb{V}_t\left[dy_t\right] = \xi\left(V_t - s_t\right)$$

and then eliminating $V_t$ to get

$$\mathbb{E}_t\left[dy_t\right] = -\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi}\mathbb{V}_t\left[dy_t\right] - \frac{1}{\tau}s_t \tag{123}$$

We have thus obtained the mean-volatility line of Figure 2. Equation (123) also makes clear that the shocks $s_t$ are shifts to vulnerability that shift the mean-volatility line up and down, while all other changes in the economy involve moving along the mean-

volaitlity line. Empirically, slope is negative and intercept is positive on average

$$-\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi} \;<\; 0$$

$$-\frac{\bar{s}}{\tau} \;>\; 0$$

This implies $\bar{s} < 0$ and

$$\xi \;>\; 0 \text{ and } 1 + \alpha\sqrt{\tau} > 0$$

$$\text{or}$$

$$\xi \;<\; 0 \text{ and } 1 + \alpha\sqrt{\tau} < 0$$

To match empirical estimates, we set

$$\alpha \;=\; -1.645$$

$$\sqrt{\tau} \;=\; 1$$

To match the actual slope and intercept

$$-\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi} \;=\; -1.15$$

$$\bar{s} \;=\; -0.67\tau$$

which gives

$$\xi \;=\; 0.36$$

$$\bar{s} \;=\; -0.67$$

We identify $s_t - \bar{s}$ with the residuals of the regression of $\mathbb{V}_t\,[dy_t]$ on $\mathbb{V}_t\,[dy_t]$. The standard deviation and $AR(1)$ coefficient of these residuals then identify $\sigma_s$ and $\kappa$, respectively. Since

$$Std\left(-\frac{1}{\tau}\left(s_t - \bar{s}\right)\right) \;=\; 0.62$$

$$AR(1) \;=\; 0.12$$

36

we get, converting to annualized values

$$\kappa = -\log(0.12) = 2.12$$
$$\sigma_s = 0.31$$

# 5 Monetary Policy

## 5.1 Optimal Monetary Policy

The central bank is minimizing a quadratic loss function over the output gap and inflation

$$L(y_t, \pi_t) = \min_{i_t} \mathbb{E}_t \int_t^\infty e^{-t\beta} \left( y_t^2 + \pi_t^2 \right) dt. \tag{124}$$

subject to the dynamics of the economy (118), (119), (120). Minimizing the quadratic loss function is a standard approach in the NK literature, as Rotemberg and Woodford (1997), Rotemberg and Woodford (1999) and Woodford (2003) have shown that aggregate welfare can be approximated by such a loss function.

We focus on the case described in the last section for ease of exposition. The interest rate $R_t$ can be eliminated from the optimization problem, so that the central bank's problem can be written as

$$L(y_t, s_t) = \min_{\{V_s\}_{s=t}^\infty} \mathbb{E}_t \int_t^\infty e^{-s\beta} y_s^2 ds \tag{125}$$

$$s.t.$$

$$V_t = \frac{-\gamma^{-1}(R_t - r_t) + \alpha\xi s_t\sqrt{\tau} + \hat{\eta}\xi\left(s_t + \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\tau}{1 + \alpha\xi\sqrt{\tau} + \hat{\eta}\xi\tau} \tag{126}$$

$$dy_t = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi(V_t - s_t)dZ_t \tag{127}$$

$$ds_t = -\kappa(s_t - \bar{s}) + \sigma_s dZ_t \tag{128}$$

The central bank thus effectively picks $V_t$, which is connected to $R_t$ in a one-to-one fashion by

$$R_t = r - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right)s_t + \frac{1}{2}\eta^2$$

The Hamilton-Jacobi-Bellman (HJB) equation for the central banker's optimization is

$$
\begin{aligned}
0 \;=\; & \min_{V} \left\{ y^2 - \beta L - \frac{\partial L}{\partial y} \frac{\alpha\sqrt{\tau}\xi + 1}{\tau} \left( V - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1} s \right) + \frac{1}{2} \frac{\partial^2 L}{\partial y^2} \xi^2 (V - s)^2 \right\} \\
& - \kappa (s - \bar{s}) \frac{\partial L}{\partial s} + \frac{1}{2} \frac{\partial^2 L}{\partial s^2} \sigma_s^2
\end{aligned}
$$

Intuitively, the HJB takes into account the current value of welfare, as well as the change in welfare associated with changes in the state variables $y$ and $s$.

The first order condition is

$$
0 \;=\; -\frac{\partial L}{\partial y} \frac{\alpha\sqrt{\tau}\xi + 1}{\tau} + \frac{\partial^2 L}{\partial y^2} \xi^2 (V - s) \tag{129}
$$

$$
V \;=\; \frac{\partial L}{\partial y} \frac{\alpha\sqrt{\tau}\xi + 1}{\tau \xi^2} \left( \frac{\partial^2 L}{\partial y^2} \right)^{-1} + s \tag{130}
$$

Hence at the optimum, vulnerability is proportional to $s$, and depends on the first and second derivative of welfare with respect to output. It is also noteworthy that $\frac{\alpha\sqrt{\tau}\xi + 1}{\tau \xi^2}$, which defines the slope of output volatility with respect to expected output, appears in the FOC.

We look for a quadratic solution of the form

$$
L(y, x) = c_0 + c_1 y + c_2 y^2 + c_3 s + c_4 s^2 + c_5 ys
$$

where $c_.$ are constants.

Plugging into the HJB, and using

$$
\begin{aligned}
\frac{\partial L}{\partial y} &= c_1 + 2 c_2 y + c_5 s \\
\frac{\partial^2 L}{\partial y^2} &= 2 c_2 \\
\frac{\partial L}{\partial s} &= c_3 + 2 c_4 s + c_5 y \\
\frac{\partial^2 L}{\partial s^2} &= 2 c_4
\end{aligned}
$$

we get the following system of equations on the coefficients $c_0, ..., c_5$

$$[y^2] \quad : \quad 0 = \left(-\beta - \frac{1}{\tau^2\xi^2}\left(\alpha\sqrt{\tau}\xi + 1\right)^2\right)c_2 + 1$$

$$[ys] \quad : \quad 0 = \left(-\frac{2}{\tau}\right)c_2 + \left(-\frac{1}{\tau^2\xi^2}\left(2\alpha\sqrt{\tau}\xi + \alpha^2\tau\xi^2 + \beta\tau^2\xi^2 + 1\right)\right)c_5$$

$$[y] \quad : \quad 0 = -\frac{1}{\tau^2\xi^2}c_1\left(2\alpha\sqrt{\tau}\xi + \alpha^2\tau\xi^2 + \beta\tau^2\xi^2 + 1\right)$$

$$[s^2] \quad : \quad 0 = -\frac{1}{4\tau^2\xi^2c_2}\left(c_5^2\left(2\alpha\sqrt{\tau}\xi + \alpha^2\tau\xi^2 + 1\right) + 4\tau\xi^2c_2c_5 + 4\beta\tau^2\xi^2c_2c_4\right)$$

$$[s] \quad : \quad 0 = -\frac{1}{2\tau^2\xi^2c_2}\left(c_1c_5\left(2\alpha\sqrt{\tau}\xi + \alpha^2\tau\xi^2 + 1\right) + 2\tau\xi^2c_1c_2 + 2\beta\tau^2\xi^2c_2c_3\right)$$

$$[const] \quad : \quad 0 = -\frac{1}{4\tau^2\xi^2c_2}\left(c_1^2\left(2\alpha\sqrt{\tau}\xi + \alpha^2\tau\xi^2 + 1\right) + 4\beta\tau^2\xi^2c_0c_2 - 4\tau^2\xi^2\sigma_s^2c_2c_4\right)$$

with solution

$$c_0 \quad = \quad \frac{\tau^2\xi^4\sigma_s^2\left(\left(\alpha\sqrt{\tau}\xi + 1\right)^2 + 2\beta\tau^2\xi^2\right)}{\beta^2\left(\left(\alpha\sqrt{\tau}\xi + 1\right)^2 + \beta\tau^2\xi^2\right)^3} > 0$$

$$c_1 \quad = \quad 0$$

$$c_2 \quad = \quad \frac{\tau^2\xi^2}{\tau^2\xi^2\beta + \left(\alpha\sqrt{\tau}\xi + 1\right)^2} > 0$$

$$c_3 \quad = \quad 0$$

$$c_4 \quad = \quad \frac{\xi^4\tau^2\left(\left(\alpha\sqrt{\tau}\xi + 1\right)^2 + 2\beta\tau^2\xi^2\right)}{\beta\left(\left(\alpha\sqrt{\tau}\xi + 1\right)^2 + \beta\tau^2\xi^2\right)^3} > 0$$

$$c_5 \quad = \quad -\frac{2\tau^3\xi^4}{\left(\left(\alpha\sqrt{\tau}\xi + 1\right)^2 + \beta\tau^2\xi^2\right)^2} < 0$$

To pick the optimal initial conditions, we minimize $L$ with respect to $y_0$ taking $s_0$ as given

$$L\left(y_0, s_0\right) \quad = \quad c_0 + c_1y_0 + c_2y_0^2 + c_3s_0 + c_4s_0^2 + c_5y_0s_0$$

$$FOC \quad : \quad \frac{\partial L}{\partial y_0} = 0$$

$$SOC \quad : \quad \frac{\partial^2 L}{\partial y_0^2} > 0$$

The FOC and SOC can be solved to get

$$
\begin{aligned}
y_0^* &= -\left( \frac{c_1}{2c_2} + \frac{c_5}{2c_2} s_0 \right) \\
&= \frac{\tau \xi^2 s_0}{\left( \alpha \sqrt{\tau} \xi + 1 \right)^2 + \beta \tau^2 \xi^2} \\
c_2 &> 0
\end{aligned}
$$

The optimal policy in terms of $V_t$ is given by plugging in the optimal solution into the FOC in equation (129):

$$
V = \frac{\left( \alpha \sqrt{\tau} \xi + 1 \right)}{\tau \xi^2} y + \left( 1 - \frac{\left( \alpha \sqrt{\tau} \xi + 1 \right)}{\left( \alpha \sqrt{\tau} \xi + 1 \right)^2 + \beta \tau^2 \xi^2} \right) s \tag{131}
$$

This can be viewed as a "flexible inflation targeting rule" (see Svensson (1999), Svensson (2002) and Rudebusch and Svensson (1999)) or, more generally, as a linear optimal targeting criterion (Giannoni and Woodford (2012)). Even though vulnerability and its shocks, $V_t$ and $s_t$, are not target variables, i.e., they do not appear in the loss function equation (124), they still enter the inflation targeting rule, the first-order condition given by equation (131). There are no independent target values for $V_t$ and $s_t$ that the central bank hopes to achieve. The reason $V_t$ and $s_t$ enter the targeting rule is that they forecast the conditional mean and variance of $y_t$ even after controlling for the information already contained in the mean of $y_t$ itself (more generally, in the means of $y_t$ and $\pi_t$ when a Phillips Curve is included). This is consistent with the empirical results in Table 1 and with the findings in Adrian, Boyarchenko, and Giannone (2016), who show that financial conditions are excellent predictors of the tail of the GDP distribution in a way that non-financial variables are not. Alternatively, equation (131) can be interpreted as a traditional flexible inflation targeting rule in which the targets for inflation and/or output are time-varying and depend on $V_t$ and $s_t$. It also important to note that even if a central bank decided not to condition its actions on $V_t$ and $s_t$, the tradeoff between inflation and output –reflected in the coefficients of the rule in equation (131)– now depends on $\gamma$ and $\xi$, the parameters that dictate the strength of the mean-variance tradeoff of output.

Using the optimal solution in the process for the output gap in equation (127), we

then find that

$$
\begin{aligned}
dy_t &= -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi\left(V_t - s_t\right)dZ_t \\
&= -\left(\frac{\left(\alpha\sqrt{\tau}\xi + 1\right)^2}{\tau^2\xi^2}y_t + \frac{\beta\tau\xi^2}{\left(\alpha\sqrt{\tau}\xi + 1\right)^2 + \beta\tau^2\xi^2}s_t\right)dt \\
&\quad + \left(\frac{\left(\alpha\sqrt{\tau}\xi + 1\right)}{\tau\xi}y_t - \frac{\xi\left(\alpha\sqrt{\tau}\xi + 1\right)}{\left(\alpha\sqrt{\tau}\xi + 1\right)^2 + \beta\tau^2\xi^2}s_t\right)dZ_t
\end{aligned}
$$

Recalling that

$$
\mathbb{E}_t\left[dy_t\right] = -\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi}\mathbb{V}_t\left[dy_t\right] - \frac{1}{\tau}s_t \tag{132}
$$

And defining the slope as

$$
M \equiv -\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi}
$$

we get

$$
V = -\frac{M}{\xi}y + \left(1 + \frac{M}{\tau\xi\left(M^2 + \beta\right)}\right)s
$$

and

$$
dy_t = -\left(M^2 \times y_t + \frac{\beta/\tau}{M^2 + \beta} \times s_t\right)dt - \left(M \times y_t - \frac{M/\tau}{M^2 + \beta} \times s_t\right)dZ_t \tag{133}
$$

The last equation makes clear that the magnitude of the tradeoff between stabilizing the mean and variance of the output gap is given by the slope $M$ of the mean-volatility line in Figure 2.

We can also express monetary policy as an interest rate rule. Using the FOC for $V$, the optimal interest rate is

$$
\begin{aligned}
R_t &= r - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right)s_t + \frac{1}{2}\eta^2 \\
&= r + \frac{1}{2}\eta^2 - \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \frac{1}{\tau} + \eta\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right)s_t \\
&= \gamma M\left(\eta + \frac{\alpha}{\sqrt{\tau}}\right)\frac{M^2 + \beta}{\beta + M^2 + M/\tau\xi}y_t - \gamma\xi\left(\left(\eta + \frac{\alpha}{\sqrt{\tau}}\right)\frac{M/\tau\xi}{\beta + M^2 + M/\tau\xi} + \frac{1}{\tau}\right)V_t \\
&\quad + \left(\frac{1}{2}\eta^2 + r\right)
\end{aligned}
$$

The optimal interest rule can thus be viewed as an augmented Taylor rule. In addition

41

to the output gap $y$ and the equilibrium rate of interest $r$ (and inflation $\pi_t$ in the more general case), the level of vulnerability $V$ enters the optimal rule. As before, the coefficients on $y$ (and $\pi$ in the more general case) depend on the parameters that define vulnerability $\xi$ and $\gamma$ and thus monetary policy is different from the typical NK model without vulnerabilities not only because vulnerability enters the augmented Taylor rule directly, but also because the presence of vulnerabilities alter the optimal response of interest rates to changes in output and inflation.

## 5.2   Alternative Monetary Policy Rules

In general, the central bank might follow other monetary policy rules. We consider alternative linear rules that do not explicitly condition on vulnerability or its shocks:

$$i_t = \psi_0 + \psi_y y_t \tag{134}$$

We show that even after picking the coefficients $\psi_0, \psi_y$ in an optimal way, the rule in equation (134) implies quantitatively large welfare losses compared to the optimal monetary policy found in the last section. To find the coefficients $\psi_0, \psi_y$ that minimize welfare losses, we solve

$$\min_{\left(\psi_0, \psi_y\right)} L\left(y_0, s_0\right) \tag{135}$$

$$s.t.$$

$$dy_t = \frac{1}{\gamma}\left(i_t - r + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{136}$$

$$i_t = \psi_0 + \psi_y y_t \tag{137}$$

$$V_t = -\mathbb{E}_t\left[dy_t\right]\tau - \alpha\mathbb{V}_t\left[dy_t\right]\sqrt{\tau} \tag{138}$$

$$ds_t = -\kappa\left(s_t - \overline{s}\right) + \sigma_s dZ_t \tag{139}$$

Figure 4 shows the steady-state distribution of the output gap $y_t$ using the optimal policy rule that explicitly takes vulnerability into account (using equation (133)), and the Taylor-type rule that does not condition on vulnerability $V_t$, given by equation (134) with coefficients found by solving (135)-(139). Intuitively, shocks to vulnerability $s$ contain information about the conditional distribution of the output gap that the policy maker should take into account in setting optimal policy. For a given level of the output gap, a higher vulnerability –a larger VaR of output– calls for higher interest rates. Higher

42

interest rates induce the private sector to save more and consume less, thus shifting the conditional future distribution of $y_t$ upwards by shifting its conditional mean upwards. Given the link between the expected mean and the expected volatility of output induced by the presence of vulnerability, a higher conditional mean induces a lower volatility of $y_t$. Together, higher mean and lower volatility mean lower vulnerability – lower VaR for output. For the suboptimal Taylor rule that ignores vulnerability, interest rates remain unchanged when, for a given level of $y_t$, $V_t$ changes. Compared to the optimal rule, when $V_t$ increases but $i_t$ remains unchanged, the conditional mean of output is lower and its conditional volatility is higher. Over time, more frequent visit to states of lower mean and higher volatility create an unconditional distribution that is more negatively skewed. When instead $V_t$ decreases, the optimal rule and the suboptimal Taylor rule produce similar right tails for the unconditional distribution of output. The reason is that lower $V_t$ induces both higher mean and lower volatility of output. Therefore, even though the changes in mean and volatility of $y_t$ are different for the two different rules, the actual differences in outcomes for $y_t$ are small because the lower volatility minimizes all fluctuations.

Figure 4: **Probability Density Functions of the Output under the Optimal Policy Rule and a Standard Taylor Rule**.   The figure shows the PDFs using the optimal policy rule and the standard Taylor rule.  The standard Taylor rule coefficients are calculated for the economy assuming that the policy maker is ignoring the presence of financial vulnerability.
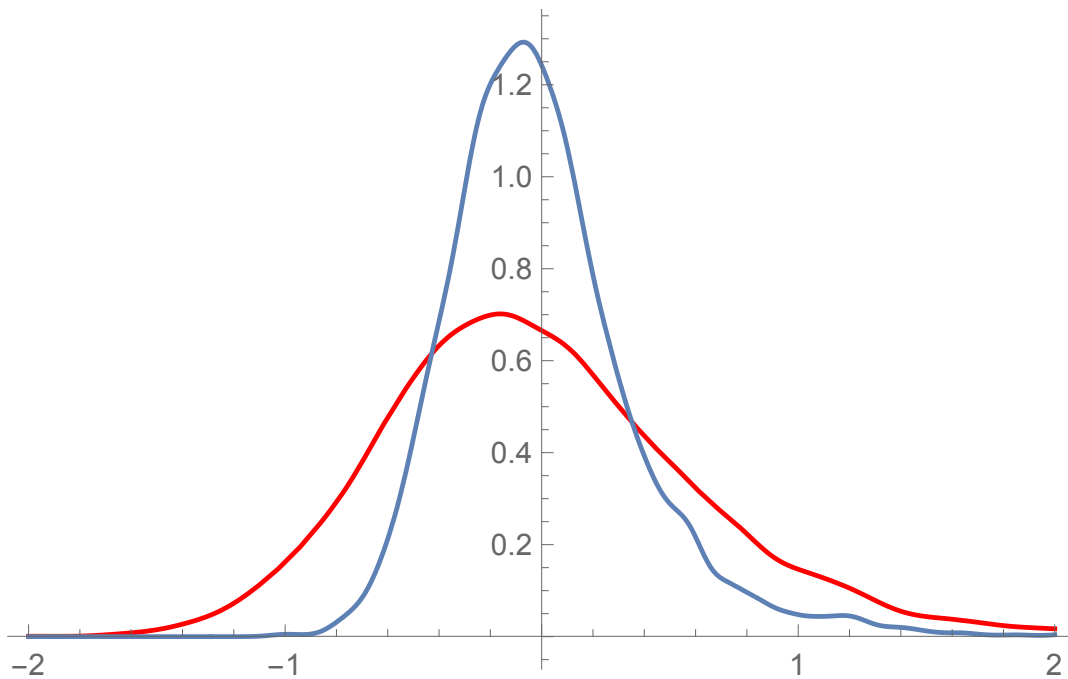
# 6 Conclusion

The degree to which financial stability considerations should be incorporated in the conduct of monetary policy has long been debated, see Adrian and Liang (2016) for an overview. In this paper, we extend the basic, two equation New Keynesian model to incorporate a notion of financial vulnerability. Shocks to risk premia impact aggregate demand via the Euler equation. The shocks to risk premia are assumed to impact the volatility of output, which is motivated from the empirical observation by Adrian, Boyarchenko, and Giannone (2016) that financial conditions forecast both the mean and the volatility of output. Importantly, our framework reproduces the stylized fact that the conditional mean and the conditional volatility of output are strongly negatively correlated, giving rise to a sharply negatively skewed unconditional output distribution. Vulnerability thus captures movements in the conditional GDP distribution that correspond to the downside risk of growth.

We further assume that the central bank minimizes the expected discounted sum of squared output gaps and squared inflation, which is standard in the literature. This is therefore a central bank that is subject to a dual mandate, without an independent financial stability objective. Despite that narrow objective function, the optimal flexible inflation targeting rule conditions on the level of vulnerability. Intuitively, all variables that provide information about the conditional distribution of GDP should be taken into account in setting optimal monetary policy. This translates into an augmented Taylor rule, where financial vulnerability—as measured by output gap tail risk as a function of financial variables—is an input into the Taylor rule. Furthermore, the magnitude of the Taylor rule coefficients on output gap and inflation depend on the parameters that determine vulnerability.

The striking result from our setup is that the central bank should **always** condition monetary policy on financial vulnerability. Relative to earlier literature that has made similar arguments (e.g. Curdia and Woodford (2010), Cúrdia and Woodford (2016) and Gambacorta and Signoretti (2014), our modeling approach is deeply rooted in empirical observations which capture macoreconomic shocks of the 2008 crisis very well. Through the negative correlation between conditional mean and conditional variance, our setup captures nonlinearity in macro dynamics in a tractable linear-quadratic setting. The implications of our results for the conduct of monetary policy are in line with the arguments or Adrian and Shin (2010) and Borio and Zhu (2012).

# References

ADRIAN, T., N. BOYARCHENKO, AND D. GIANNONE (2016): "Vulnerable Growth," *Federal Reserve Bank of New York Staff Reports*, 794.

ADRIAN, T., D. COVITZ, AND N. LIANG (2015): "Financial Stability Monitoring," *Annual Review of Financial Economics*, 7, 357–395.

ADRIAN, T., AND N. LIANG (2016): "Monetary Policy, Financial Conditions, and Financial Stability," *International Journal of Central Banking*, forthcoming.

ADRIAN, T., AND H. SHIN (2010): "Financial Intermediaries and Monetary Economics," *Handbook of Monetary Economics*, 3, 601–650.

AIKMAN, D., M. T. KILEY, S. J. LEE, M. G. PALUMBO, AND M. N. WARUSAWITHARANA (2015): "Mapping Heat in the US Financial System," *Federal Reserve Board Finance and Economics Discussion Series*, 59.

BERNANKE, B., AND A. BLINDER (1992): "The Federal Funds Rate and the Channels of Monetary Transmission," *American Economic Review*, pp. 901–921.

BERNANKE, B., AND M. GERTLER (1989): "Agency Costs, Net Worth, and Business Fluctuations," *American Economic Review*, pp. 14–31.

BORIO, C., AND H. ZHU (2012): "Capital Regulation, Risk-taking and Monetary Policy: A Missing Link in the Transmission Mechanism?," *Journal of Financial Stability*, 8(4), 236–251.

BRUNNERMEIER, M. K., T. M. EISENBACH, AND Y. SANNIKOV (2013): "Macroeconomics with Financial Frictions: A Survey," in *Advances in Economics and Econometrics: Volume 2, Applied Economics: Tenth World Congress*, vol. 50, p. 3. Cambridge University Press.

CIESLAK, A., AND A. VISSING-JORGENSEN (2017): "The Economics of the Fed Put," .

CURDIA, V., AND M. WOODFORD (2010): "Credit Spreads and Monetary Policy," *Journal of Money, Credit and Banking*, 42(1), 3–35.

CÚRDIA, V., AND M. WOODFORD (2016): "Credit Frictions and Optimal Monetary Policy," *Journal of Monetary Economics*, 84, 30–65.

DUDLEY, W. C. (2015): "Remarks at The Fed at a crossroads: Where to go next?, Brookings Institution, Washington, D.C.," `https://www.newyorkfed.org/newsevents/speeches/2015/dud151015`.

DUDLEY, W. C. (2017): "The Importance of Financial Conditions in the Conduct of Monetary Policy, Remarks at the University of South Florida Sarasota-Manatee, Sarasota, Florida," `https://www.newyorkfed.org/newsevents/speeches/2017/dud170330`.

ESTRELLA, A., AND G. A. HARDOUVELIS (1991): "The Term Structure as a Predictor of Real Economic Activity," *Journal of Finance*, 46(2), 555–576.

ESTRELLA, A., AND F. S. MISHKIN (1998): "Predicting US Recessions: Financial Variables as Leading Indicators," *Review of Economics and Statistics*, 80(1), 45–61.

GALÍ, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: an Introduction to the New Keynesian Framework and its Applications*. Princeton University Press.

Gambacorta, L., and F. M. Signoretti (2014): "Should Monetary Policy Lean against the Wind?: An Analysis based on a DSGE Model with Banking," *Journal of Economic Dynamics and Control*, 43, 146–174.

Gertler, M., and P. Karadi (2011): "A Model of Unconventional Monetary Policy," *Journal of Monetary Economics*, 58(1), 17–34.

Giannoni, M., and M. Woodford (2012): "Optimal target criteria for stabilization policy," Staff Reports 535, Federal Reserve Bank of New York.

Gilchrist, S., and E. Zakrajšek (2012): "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 102(4), 1692–1720.

Krishnamurthy, A., and T. Muir (2016): "How Credit Cycles across a Financial Crisis," *Unpublished paper, Stanford University (June)*.

Leeper, E. M. (1991): "Equilibria under 'active' and 'passive' monetary and fiscal policies," *Journal of Monetary Economics*, 27(1), 129–147.

López-Salido, D., J. C. Stein, and E. Zakrajšek (2016): "Credit-Market Sentiment and the Business Cycle," *National Bureau of Economic Research*, 21879.

Peek, J., E. S. Rosengren, and G. M. Tootell (2015): "Should US Monetary Policy Have a Ternary Mandate?," *Federal Reserve Bank of Boston 59 th Economic Conference*, Macroprudential and Monetary Policy.

Rotemberg, J., and M. Woodford (1997): "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy," *NBER Macroeconomics Annual 1997, Volume 12*, pp. 297–361.

Rotemberg, J. J., and M. Woodford (1999): "Interest Rate Rules in an Estimated Sticky Price Model," *Monetary Policy Rules*, pp. 57–126.

Rudebusch, G., and L. E. Svensson (1999): "Policy Rules for Inflation Targeting," *Monetary Policy Rules*, pp. 203–262.

Stock, J. H., and M. W. Watson (1998): "Diffusion Indexes," *National Bureau of Economic Research*, 6702.

Svensson, L. E. (1999): "Inflation Targeting as a Monetary Policy Rule," *Journal of Monetary Economics*, 43(3), 607–654.

——— (2002): "Inflation Targeting: Should It be Modeled as an Instrument Rule or a Targeting Rule?," *European Economic Review*, 46(4), 771–780.

Woodford, M. (2001): "Fiscal Requirements for Price Stability," *Journal of Money, Credit and Banking*, pp. 669–728.

——— (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

Yellen, J. (2016a): "The Federal Reserve's Monetary Policy Toolkit: Past, Present, and Future, speech at "Designing Resilient Monetary Policy Frameworks for the Future," a symposium sponsored by the Federal Reserve Bank of Kansas City, Jackson Hole, Wyoming," `https://www.federalreserve.gov/newsevents/speech/yellen20160826a.htm`.

YELLEN, J. L. (2016b): "The Outlook, Uncertainty, and Monetary Policy, Remarks at the Economic Club of New York, New York, New York," `https://www.federalreserve.gov/newsevents/speech/yellen20160329a.htm`.

# A  Reformulating the Bank's Problem

Log utility allows us to transform the bank's optimization problem into a non-stochastic problem. Indeed,

$$
\log X_t = \log X_0 + \int_0^t Q\left(s, \theta_s, f_s, \nu_s\right) ds + \int_0^t \theta_s^T \sigma_s dB_s^\nu
$$

Consider the following

$$
\int_0^\infty e^{-\beta t} \log\left(f_t X_t\right) dt \tag{140}
$$

$$
= \int_0^\infty e^{-\beta t} \log\left(X_t\right) dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt
$$

$$
= \int_0^\infty e^{-\beta t} \log\left(X_0\right) dt + \int_0^\infty e^{-\beta t} \left\{ \int_0^t Q\left(s, \theta_s, f_s, \nu_s\right) ds + \int_0^t \theta_s^T \sigma_s dB_s^\nu \right\} dt
$$

$$
+ \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt
$$

$$
= \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty \int_0^t e^{-\beta t} Q\left(s, \theta_s, f_s, \nu_s\right) ds dt
$$

$$
+ \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt \tag{141}
$$

$$
= \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty \int_s^\infty e^{-\beta t} Q\left(s, \theta_s, f_s, \nu_s\right) dt ds
$$

$$
+ \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt \tag{142}
$$

$$
= \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty Q\left(s, \theta_s, f_s, \nu_s\right) \left[ \int_s^\infty e^{-\beta t} dt \right] ds
$$

$$
+ \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt \tag{143}
$$

where the change in the order of integration follows from Fubini's theorem. We assume all the usual regularity conditions. In particular, we will need that

$$
\int_0^\infty \left\| \sigma_t^{-1} \mu_t \right\|^2 dt < \infty \tag{144}
$$

Under the regularity condition in equation the stochastic part of the bank's objective

47

function is a martingale and not just a local martingale, so

$$\mathbb{E}_0^{bank} \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt = 0$$

Therefore, taking expectations in (143) gives

$$
\begin{aligned}
\mathbb{E}_0^{bank} \int_0^\infty e^{-\beta t} \log\left(f_t X_t\right) dt &= \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \mathbb{E}_0^{bank} \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt \\
&\quad + \mathbb{E}_0^{bank} \int_0^\infty Q\left(s, \theta_s, f_s, \nu_s\right) \left[\int_s^\infty e^{-\beta t} dt\right] ds
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_0^{bank} \int_0^\infty e^{-\beta t} \log\left(f_t X_t\right) dt &= \frac{\log\left(X_0\right)}{\beta} + \mathbb{E}_0 \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt \\
&\quad + \frac{1}{\beta} \mathbb{E}_0 \int_0^\infty Q\left(s, \theta_s, f_s, \nu_s\right) e^{-\beta s} ds \\
&= \frac{\log\left(X_0\right)}{\beta} + \mathbb{E}_0 \int_0^\infty e^{-\beta t} \left(\log\left(f_t\right) + \frac{1}{\beta} Q\left(t, \theta_t, f_t, \nu_t\right)\right) dt
\end{aligned}
$$

# B    Appendix: Finding the Lagrange Multiplier for the VaR

We first compute some derivatives

$$
\begin{aligned}
\theta_t^T \mu_t &= \sum_j \theta_{j,t} \mu_{j,t} \\
\frac{\partial\left(\theta_t^T \mu_t\right)}{\partial \theta_{j,t}} &= \frac{\partial}{\partial \theta_{j,t}} \sum_{k=1}^M \theta_{k,t} \mu_{k,t} = \mu_{j,t} \\
\theta_t^T \sigma_t &= \left[\begin{array}{cccc} \sum_{k=1}^M \theta_{k,t} \sigma_{k1,t} & \sum_{k=1}^M \theta_{k,t} \sigma_{k2,t} & \dots & \sum_{k=1}^M \theta_{k,t} \sigma_{kN,t} \end{array}\right] \\
\left\|\theta_t^T \sigma_t\right\|^2 &= \sum_{n=1}^N \left(\sum_{k=1}^M \theta_{k,t} \sigma_{kn,t}\right)^2
\end{aligned}
$$

$$
\frac{\partial}{\partial \theta_{j,t}} \left\|\theta_t^T \sigma_t\right\|^2 = 2\theta_t^T \sigma_t \sigma_t^{(j)} = \left[\begin{array}{cccc} \sum_{k=1}^M \theta_{k,t} \sigma_{k1,t} & \sum_{k=1}^M \theta_{k,t} \sigma_{k2,t} & \dots & \sum_k \theta_{k,t} \sigma_{kN,t} \end{array}\right] \left[\begin{array}{c} \sigma_{j1,t} \\ \sigma_{j2,t} \\ \vdots \\ \sigma_{jN,t} \end{array}\right]
$$

where $\sigma_t^{(j)}$ is the $j^{th}$ column of $\sigma_t^T$. Putting all the vectors together, we can write

$$\frac{\partial \left( \theta_t^T \mu_t \right)}{\partial \theta_t} = \mu_t$$

$$\frac{\partial}{\partial \theta_t} \left\| \theta_t^T \sigma_t \right\|^2 = 2\sigma_t \sigma_t^T \theta_t$$

The derivative of $\left\| \theta_t^T \sigma_t \right\|$ now follows from

$$\frac{\partial}{\partial \theta_t} \left\| \theta_t^T \sigma_t \right\|^2 = 2 \left\| \theta_t^T \sigma_t \right\| \left( \frac{\partial \left\| \theta_t^T \sigma_t \right\|}{\partial \theta_t} \right) = 2\sigma_t \sigma_t^T \theta_t$$

$$\implies \frac{\partial \left\| \theta_t^T \sigma_t \right\|}{\partial \theta_t} = \frac{\sigma_t \sigma_t^T \theta_t}{\left\| \theta_t^T \sigma_t \right\|}$$

Using the above computations and the definitions

$$Q\left(t, \theta_t, f_t\right) \equiv R_t - f_t + \theta_t^T \mu_t - \frac{1}{2} \left\| \theta_t^T \sigma_t \right\|^2$$

$$g_V\left(t, \theta_t, f_t\right) \equiv -Q\left(t, \theta_t, f_t\right) \tau - N^{-1}\left(\alpha\right) \left\| \theta_t^T \sigma_t \right\| \sqrt{\tau}$$

$$h\left(t, \theta_t, f_t, \zeta_t\right) \equiv e^{-\beta t} e^{\zeta_t} \log\left(f_t\right) + \left[ \int_t^\infty e_t^{-\beta s} \mathbb{E}_t \left[ e^{\zeta_s} \right] ds \right] Q\left(t, \theta_t, f_t\right)$$

$$= e^{-\beta t} e^{\zeta_t} \log\left(f_t\right) + \frac{e^{-\beta t} e^{\zeta_t}}{\beta} Q\left(t, \theta_t, f_t\right)$$

we get

$$\frac{\partial}{\partial \theta_t} Q\left(t, \theta_t, f_t\right) = \mu_t - \sigma_t \sigma_t^T \theta_t$$

$$\frac{\partial}{\partial f_t} Q\left(t, \theta_t, f_t\right) = -1$$

$$\nabla_{f_t} g_V\left(t, \theta_t, f_t\right) = \tau$$

$$\nabla_{\theta_t} g_V\left(t, \theta_t, f_t\right) = -\left(\mu_t - \sigma_t \sigma_t^T \theta_t\right)\tau - N^{-1}\left(\alpha\right) \frac{\sigma_t \sigma_t^T \theta_t}{\left\| \theta_t^T \sigma_t \right\|} \sqrt{\tau}$$

$$\nabla_{\theta_t} h\left(t, \theta_t, f_t, \zeta_t\right) = \frac{e^{-\beta t} e^{\zeta_t}}{\beta} \left(\mu_t - \sigma_t \sigma_t^T \theta_t\right)$$

$$\nabla_{f_t} h\left(t, \theta_t, f_t, \zeta_t\right) = \frac{e^{-\beta t} e^{\zeta_t}}{f_t} - \frac{e^{-\beta t} e^{\zeta_t}}{\beta}$$

The FOC is

$$\nabla h\left(t,\theta_t,f_t,\zeta_t\right) = \lambda_{VaR}\nabla g_V\left(t,\theta_t,f_t\right)$$

i.e.

$$\nabla_{\theta_t}h\left(t,\theta_t,f_t,\zeta_t\right) = \lambda_{VaR}\nabla_{\theta_t}g_V\left(t,\theta_t,f_t\right)$$
$$\nabla_{f_t}h\left(t,\theta_t,f_t,\zeta_t\right) = \lambda_{VaR}\nabla_{f_t}g_V\left(t,\theta_t,f_t\right)$$

Using the computations above,

$$\nabla_{\theta_t}h\left(t,\theta_t,f_t,\zeta_t\right) = \lambda_{VaR}\nabla_{\theta_t}g_V\left(t,\theta_t,f_t\right)$$
$$\frac{e^{-\beta t}e^{\zeta_t}}{\beta}\left(\mu_t - \sigma_t\sigma_t^T\theta_t\right) = \lambda_{VaR}\left(-\left(\mu_t - \sigma_t\sigma_t^T\theta_t\right)\tau - N^{-1}\left(\alpha\right)\frac{\sigma_t\sigma_t^T\theta_t}{\left\|\theta_t^T\sigma_t\right\|}\sqrt{\tau}\right)$$

$$\nabla_{f_t}h\left(t,\theta_t,f_t,\zeta_t\right) = \lambda_{VaR}\nabla_{f_t}g_V\left(t,\theta_t,f_t\right)$$
$$\frac{e^{-\beta t}e^{\zeta_t}}{f_t} - \frac{e^{-\beta t}e^{\zeta_t}}{\beta} = \tau\lambda_{VaR}$$

So we have

$$\lambda_{VaR} = \frac{e^{-\beta t}e^{\zeta_t}}{\tau}\left(\frac{1}{f_t} - \frac{1}{\beta}\right)$$

Now we solve

$$\min l\left(\lambda_1,\lambda_2\right)$$
$$s.t.$$
$$g_V\left(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M}\right) = \log\frac{1}{1-a_V}$$
$$l\left(\lambda_1,\lambda_2\right) \equiv h\left(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M},\zeta_t\right)$$

$$\nabla_{\lambda_1}l\left(\lambda_1,\lambda_1\right) = \gamma\nabla_{\lambda_1}g_V\left(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M}\right)$$
$$\nabla_{\lambda_2}l\left(\lambda_1,\lambda_1\right) = \gamma\nabla_{\lambda_2}g_V\left(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M}\right)$$

$$\nabla_{\lambda_1} l\left(\lambda_1, \lambda_2\right) = \frac{\partial}{\partial \lambda_1} h\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}, \zeta_t\right)$$

$$= \frac{\partial}{\partial \lambda_1}\left(e^{-\beta t} e^{\zeta_t} \log\left(\lambda_2 f_{t,M}\right) + \left[\int_t^\infty e_t^{-\beta s} \mathbb{E}_t\left[e^{\zeta_s}\right] ds\right] Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)\right)$$

$$= \frac{e^{-\beta t} e^{\zeta_t}}{\beta} \frac{\partial}{\partial \lambda_1} Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)$$

$$= \frac{e^{-\beta t} e^{\zeta_t}}{\beta} \frac{\partial}{\partial \lambda_1}\left(R_t - \lambda_2 f_{t,M} + \left(\lambda_1 \theta_{t,M}\right)^T \mu_t - \frac{1}{2}\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|^2\right)$$

$$= \frac{e^{-\beta t} e^{\zeta_t}}{\beta}\left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right)$$

$$\nabla_{\lambda_2} l\left(\lambda_1, \lambda_2\right) = \frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}, \zeta_t\right)$$

$$= \frac{\partial}{\partial \lambda_2}\left(e^{-\beta t} e^{\zeta_t} \log\left(\lambda_2 f_{t,M}\right) + \left[\int_t^\infty e_t^{-\beta s} \mathbb{E}_t\left[e^{\zeta_s}\right] ds\right] Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)\right)$$

$$= e^{-\beta t} e^{\zeta_t} \frac{1}{\lambda_2} + \frac{e^{-\beta t} e^{\zeta_t}}{\beta} \frac{\partial}{\partial \lambda_2} Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)$$

$$= e^{-\beta t} e^{\zeta_t} \frac{1}{\lambda_2} - \frac{e^{-\beta t} e^{\zeta_t}}{\beta} f_{t,M}$$

$$\nabla_{\lambda_1} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) = \frac{\partial}{\partial \lambda_1}\left(-Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)\tau - N^{-1}\left(\alpha\right)\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}\right)$$

$$= \frac{\partial}{\partial \lambda_1}\left(\begin{array}{c} -\left(R_t - \lambda_2 f_{t,M} + \left(\lambda_1 \theta_{t,M}\right)^T \mu_t - \frac{1}{2}\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|^2\right)\tau \\ -N^{-1}\left(\alpha\right)\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau} \end{array}\right)$$

$$= -\left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right)\tau - N^{-1}\left(\alpha\right)\left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}$$

$$\nabla_{\lambda_2} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) = \frac{\partial}{\partial \lambda_2}\left(-Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)\tau - N^{-1}\left(\alpha\right)\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}\right)$$

$$= \frac{\partial}{\partial \lambda_2}\left(\begin{array}{c} -\left(R_t - \lambda_2 f_{t,M} + \left(\lambda_1 \theta_{t,M}\right)^T \mu_t - \frac{1}{2}\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|^2\right)\tau \\ -N^{-1}\left(\alpha\right)\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau} \end{array}\right)$$

$$= f_{t,M}\tau - N^{-1}\left(\alpha\right)\left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}$$

$$\nabla_{\lambda_1} l\left(\lambda_1, \lambda_1\right) = \gamma \nabla_{\lambda_1} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)$$
$$\nabla_{\lambda_2} l\left(\lambda_1, \lambda_1\right) = \gamma \nabla_{\lambda_2} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)$$

become

$$\frac{e^{-\beta t} e^{\zeta_t}}{\beta}\left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right) = -\gamma \left( \begin{array}{c} \left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right)\tau \\ +N^{-1}\left(\alpha\right)\left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau} \end{array} \right)$$

$$e^{-\beta t} e^{\zeta_t} \frac{1}{\lambda_2} - \frac{e^{-\beta t} e^{\zeta_t}}{\beta} f_{t,M} = \gamma\left(f_{t,M}\tau - N^{-1}\left(\alpha\right)\left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}\right)$$

$$\frac{\left(f_{t,M}\tau - N^{-1}\left(\alpha\right)\left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}\right)}{\beta}\left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right)$$
$$= \left(\frac{f_{t,M}}{\beta} - \frac{1}{\lambda_2}\right)\left(\left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right)\tau + N^{-1}\left(\alpha\right)\left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}\right)$$

$$\gamma = \frac{e^{-\beta t} e^{\zeta_t}}{\left(f_{t,M}\tau - N^{-1}\left(\alpha\right)\left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|\sqrt{\tau}\right)}\left(\frac{1}{\lambda_2} - \frac{f_{t,M}}{\beta}\right)$$

# C   Appendix: Solving the Banks' Problem

First, assume that $\theta_t \neq \mathbf{0}$ so that $g_V\left(t, \theta_t, f_t, \nu_t\right)$ is differentiable. Set up the Lagrangian

$$\mathcal{L} = h\left(t, \theta_t, f_t, \nu_t\right) - \lambda\left(g_V\left(t, \theta_t, f_t, \nu_t\right) - \log\frac{1}{1 - a_V}\right)$$

Direct computation (Appendix B) shows that $\nabla g_V\left(t, \theta_t, f_t, \nu_t\right) \neq \mathbf{0}$. Thus, $\lambda \neq 0$ and the FOC is

$$\nabla h\left(t, \theta_t, f_t, \nu_t\right) = \lambda \nabla g_V\left(t, \theta_t, f_t, \nu_t\right) \tag{145}$$

We compute

$$\nabla_{\theta_t} h = \frac{1}{\beta}\left(\mu_t - \sigma_t \nu_t - \sigma_t \sigma_t^T \theta_t\right)$$

$$\nabla_{\theta_t} g_V = -\left(\mu_t - \sigma_t \nu_t - \sigma_t \sigma_t^T \theta_t\right)\tau - N^{-1}\left(\alpha\right)\frac{\sigma_t \sigma_t^T \theta_t}{\left\|\theta_t^T \sigma_t\right\|}\sqrt{\tau}$$

$$\nabla_{f_t} h = \frac{1}{f_t} - \frac{1}{\beta}$$
$$\nabla_{f_t} g_V = \tau$$

so that the FOC become

$$
\begin{aligned}
\nabla_{\theta_t} h\left(t, \theta_t, f_t, \nu_t\right) &= \lambda \nabla_{\theta_t} g_V\left(t, \theta_t, f_t, \nu_t\right) \\
\frac{1}{\beta}\left(\mu_t - \sigma_t \nu_t - \sigma_t \sigma_t^T \theta_t\right) &= \lambda \begin{pmatrix} -\left(\mu_t - \sigma_t \nu_t - \sigma_t \sigma_t^T \theta_t\right) \tau \\ -N^{-1}\left(\alpha\right) \frac{\sigma_t \sigma_t^T \theta_t}{\left\|\theta_t^T \sigma_t\right\|} \sqrt{\tau} \end{pmatrix} \\
\left(1 + \beta \tau \lambda_{VaR}\right)\left(\mu_t - \sigma_t \nu_t\right) &= \left(1 + \left(\tau - \frac{\sqrt{\tau} N^{-1}\left(\alpha\right)}{\left\|\theta_t^T \sigma_t\right\|}\right) \beta \lambda_{VaR}\right) \sigma_t \sigma_t^T \theta_t
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{f_t} h\left(t, \theta_t, f_t, \nu_t\right) &= \lambda \nabla_{f_t} g_V\left(t, \theta_t, f_t, \nu_t\right) \\
\frac{1}{f_t} - \frac{1}{\beta} &= \lambda \tau \\
f_t &= \frac{\beta}{\beta \lambda \tau + 1}
\end{aligned}
$$

Writing

$$
\theta_{M,t} = \left(\sigma_t^T\right)^{-1} \sigma_t^{-1}\left(\mu_t - \sigma_t \nu_t\right)
$$

we see that $\theta_t$ is parallel to $\theta_{M,t}$ so all we need is to find $\lambda_1, \lambda_2$ that solve

$$
\begin{aligned}
\max_{\lambda_1, \lambda_2} & \; h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) \\
& s.t. \\
g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) & \leq \; \log \frac{1}{1 - a_V}
\end{aligned}
$$

Again, it can be checked that the constraint holds with equality. The Lagrangian is

$$
\mathcal{L} = h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) - \gamma \left(g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) - \log \frac{1}{1 - a_V}\right)
$$

The FOC are

$$
\begin{aligned}
\frac{\partial}{\partial \lambda_1} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) &= \gamma \frac{\partial}{\partial \lambda_1} g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) \\
\frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) &= \gamma \frac{\partial}{\partial \lambda_2} g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right)
\end{aligned}
$$

Computing the derivatives gives

$$\frac{\partial}{\partial \lambda_1} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) = \gamma \frac{\partial}{\partial \lambda_1} g_V\left(t, \theta, f, \nu\right)$$

$$\frac{1}{\beta}\left(\theta_{M,t}^T\left(\mu_t - \sigma_t \nu_t\right) - \lambda_1 \left\|\theta_{M,t}^T \sigma_t\right\|^2\right)$$

$$= \gamma\left(-\left(\theta_{M,t}^T\left(\mu_t - \sigma_t \nu_t\right) - \lambda_1 \left\|\theta_{M,t}^T \sigma_t\right\|^2\right)\tau - N^{-1}\left(\alpha\right)\left\|\theta_{M,t}^T \sigma_t\right\|\sqrt{\tau}\right)$$

and

$$\frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) = \gamma \frac{\partial}{\partial \lambda_2} g_V\left(t, \theta, f, \nu\right)$$

$$\frac{1}{\lambda_2} - \frac{f_{M,t}}{\beta} = \gamma f_{M,t}\tau$$

Eliminating $\gamma$ and using

$$\theta_{M,t}^T \sigma_t = \left(\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t \nu_t\right)\right)^T$$

$$\left\|\theta_{M,t}^T \sigma_t\right\| = \left\|\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t \nu_t\right)\right\|$$

$$\theta_{M,t}\left(\mu_t - \sigma_t \nu_t\right) = \left\|\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t \nu_t\right)\right\|^2 = \left\|\theta_{M,t}^T \sigma_t\right\|^2$$

we get

$$\lambda_2 = u\left(t, \lambda_1\right) f_{M,t}$$

$$u\left(t, z\right) \equiv \left[1 + \frac{\sqrt{\tau}\left\|\theta_{M,t}^T \sigma_t\right\|}{N^{-1}\left(\alpha\right)}\left(1 - z\right)\right] \tag{146}$$

and $\lambda_1$ is the unique number that makes the $VaR$ hold with equality

$$g_V\left(t, \lambda_1 \theta_{M,t}, u\left(t, \lambda_1\right) f_{M,t}, \nu_t\right) = \log\frac{1}{1 - a_V}$$

If $\lambda_1 \leq 0$, then there is no investment in the risky asset and

$$\theta_t = \mathbf{0}$$

$$f_t = R_t + \frac{1}{\tau}\log\frac{1}{1 - a_V}$$

Putting everything together, the optimal portfolio is then characterized by

$$\theta_t = \min\left\{1, \max\left\{0, \varphi_t\right\}\right\}\theta_{M,t} \tag{147}$$

$$f_t = u\left(t, \min\left\{1, \varphi_t\right\}\right)f_{M,t}1_{\{\varphi_t>0\}} \tag{148}$$

$$+\left(R_t + \frac{1}{\tau}\log\frac{1}{1-a_V}\right)1_{\{\varphi_t\le 0\}}$$

$$\varphi_t \text{ such that:} \qquad g_V\left(t, \varphi_t\theta_{M,t}, u\left(t, \varphi_t\right)f_{M,t}\right) = \log\frac{1}{1-a_V} \tag{149}$$

$$\theta_{M,t} = \left(\sigma_t^T\right)^{-1}\left(\sigma_t^{-1}\mu_t - \nu_t\right) \tag{150}$$

$$f_{M,t} = \beta \tag{151}$$

$$\lambda = \frac{1}{\tau}\left(\frac{1}{f_t} - \frac{1}{\beta}\right) \tag{152}$$

# D   Appendix: Solving for $\varphi$

Solving for $\varphi_t$ in (149) gives an explicit definition for $\varphi_t$. First, we express

$$\begin{aligned}
\left\|\theta_{M,t}^T\sigma_t\right\| &= \left\|\left(\left(\sigma_t^T\right)^{-1}\sigma_t^{-1}\left(\mu_t - \sigma_t\nu_t\right)\right)^T\sigma_t\right\| \\
&= \left\|\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t\nu_t\right)\right\| \\
&= \left\|\eta_t - \nu_t\right\|
\end{aligned}$$

$$\begin{aligned}
\theta_{M,t}^T\left(\mu_t - \sigma_t\nu_t\right) &= \left(\left(\sigma_t^T\right)^{-1}\sigma_t^{-1}\left(\mu_t - \sigma_t\nu_t\right)\right)^T\left(\mu_t - \sigma_t\nu_t\right) \\
&= \left[\sigma_t^{-1}\left(\mu_t - \sigma_t\nu_t\right)\right]^T\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t\nu_t\right) \\
&= \left\|\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t\nu_t\right)\right\|^2 \\
&= \left\|\eta_t - \nu_t\right\|^2
\end{aligned}$$

Then

$$\begin{aligned}
-\log\left(1 - a_V\right) &= g_V\left(t, \varphi_t\theta_{M,t}, u\left(t, \varphi_t\right)f_{M,t}\right) \\
&= -\left(R_t - f_t + \varphi_t\left\|\eta_t - \nu_t\right\|^2 - \frac{1}{2}\varphi_t^2\left\|\eta_t - \nu_t\right\|^2\right)\tau - N^{-1}\left(\alpha\right)\varphi_t\left\|\eta_t - \nu_t\right\|\sqrt{\tau} \\
&= -\left(R_t - f_t + \left\|\eta_t - \nu_t\right\|^2\left(\varphi_t - \frac{1}{2}\varphi_t^2\right)\right)\tau - N^{-1}\left(\alpha\right)\varphi_t\left\|\eta_t - \nu_t\right\|\sqrt{\tau}
\end{aligned}$$

and

$$0 = -\log(1 - a_V) + (R_t - f_t)\tau + \tau \|\eta_t - \nu_t\|^2 \varphi_t + N^{-1}(\alpha)\sqrt{\tau}\|\eta_t - \nu_t\|\varphi_t - \frac{1}{2}\tau\|\eta_t - \nu_t\|^2 \varphi_t^2$$

$$\varphi_t \equiv 1 + \frac{N^{-1}(\alpha)}{\sqrt{\tau}\|\eta_t - \nu_t\|} \pm \sqrt{2(R_t - f_t)\tau - 2\log(1 - a_V) + \|\eta_t - \nu_t\|^2 \tau^2 \left(1 + \frac{N^{-1}(\alpha)}{\sqrt{\tau}\|\eta_t - \nu_t\|}\right)^2}$$

# E    Appendix: Solving Households' Problem

We solve the problem of the household in two stages. First, we find the optimal allocation $C_t(i)$ across goods $i$ for a given level of consumption expenditures. Second, we solve the consumption/savings problem for the household who picks $C_t(i)$ optimally according to the first stage.

The Lagrangian for the first stage is

$$\mathcal{L} = \left(\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} - \mu\left(E_t - \overline{E}\right). \tag{153}$$

where

$$E_t \equiv \int_0^1 P_t(i) C_t(i) di \tag{154}$$

are total nominal consumption expenditures and $\overline{E}$ is a given constant. The associated first-order condition is

$$C_t(i)^{-\frac{1}{\varepsilon}} C_t^{\frac{1}{\varepsilon}} = \mu P_t(i) \text{ for all } i. \tag{155}$$

Therefore,

$$C_t(i) = C_t(k)\left(\frac{P_t(i)}{P_t(k)}\right)^{-\varepsilon} \tag{156}$$

for any two goods $i$ and $k$. Plugging in (156) into (154) gives

$$C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \frac{Z_t}{P_t} \tag{157}$$

where we have defined the aggregate price level as

$$P_t \equiv \left(\int_0^1 P_t(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}. \tag{158}$$

Multiplying (157) by $P_t(i)$, raising both sides by the power $1 - \frac{1}{\varepsilon}$ and integrating over $i$ gives

$$\int_0^1 P_t(i) C_t(i) di = P_t C_t, \tag{159}$$

where we have also used the definition of $C_t$, equation (17). Combining (154), (157) and

56

(159) gives

$$C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t. \tag{160}$$

Now we solve the second stage of the household maximization. We restate the optimization problem of the household in a simplified way by doing three things. First, because utility is increasing in consumption and decreasing in labor, the household's budget constraint (60) holds with equality for all $t \geq s$. Second, we replace (60) and (61) by an equivalent intertemporal budget constraint.Note that to write down the intertemporal budget constraint, it is not necessary that financial markets are complete. Third, we use the solution of the first stage so that the household now chooses $\{C_t\}_{t \geq s}$ instead of $\{C_t(i)\}_{t \geq s}$ and allocates $C_t(i)$ according to equation (160). With these changes, the problem is

$$\max_{\{C_t, N_t\}_{t \geq s}} \mathbb{E}_s \left\{ \int_s^\infty e^{-\beta(t-s)} \left[ \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\xi}}{1+\xi} \right] dt \right\} \tag{161}$$

subject to

$$Q_s F_s = \mathbb{E}_s \left[ \int_s^\infty Q_t \left( C_t - \frac{W_t}{P_t} N_t \right) dt \right] \tag{162}$$

$$F_s \text{ given} \tag{163}$$

The Lagrangian for the optimization is

$$\begin{aligned}
\mathcal{L} &= \mathbb{E}_s \left\{ \int_s^\infty e^{-\beta(t-s)} \left[ \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\xi}}{1+\xi} \right] dt \right\} + \\
&\quad -\lambda^{bc} \left[ \mathbb{E}_s \left[ \int_s^\infty Q_t \left( C_t - \frac{W_t}{P_t} N_t \right) dt \right] - Q_s F_s \right]
\end{aligned} \tag{164}$$

where $\lambda^{bc}$ is the Lagrange multiplier associated with the constraint. The first order conditions for the households problem are given by

$$[N_t] \quad : \quad -e^{-\beta(t-s)} N_t^\xi + \lambda^{bc} Q_t \frac{W_t}{P_t} = 0 \tag{165}$$

$$[C_t] \quad : \quad e^{-\beta(t-s)} C_t^{-\gamma} - \lambda^{bc} Q_t = 0 \tag{166}$$

Combining (165) and (166) to eliminate $\lambda^{bc}$ gives the intra-temporal optimality condition, which defines the labor supply curve

$$C_t^\gamma N_t^\xi = \frac{W_t}{P_t} \tag{167}$$

Using equation (166) for times $s$ and $t$ gives

$$\frac{Q_t}{Q_s} = e^{-\beta(t-s)} \left(\frac{C_t}{C_s}\right)^{-\gamma} \tag{168}$$

We identify the real and nominal state price densities

$$Q_t = e^{-\beta t} C_t^{-\gamma}, \tag{169}$$

$$Q_t^{\$} = e^{-\beta t} \frac{C_t^{-\gamma}}{P_t}, \tag{170}$$

the real stochastic discount factor

$$SDF_{t,s} \equiv \frac{Q_t}{Q_s} \tag{171}$$

and the nominal stochastic discount factor $SDF_{t,s}^{\$} \equiv Q_t^{\$}/Q_s^{\$}$.

# F    Appendix: Solving Intermediate Goods Sector Firms' Problem

There is continuum of mass one of monopolistically competitive firms owned by the households, indexed by $i \in [0,1]$. Each firm faces a demand curve given by equation (66). Firms use labor $N_t(i)$ to produce output according to the technology

$$Y_t(i) = A N_t(i) \tag{172}$$

Labor is hired in a competitive market with perfect mobility.

Firms set prices according to Calvo staggered pricing. The probability density of receiving the signal to change prices after an amount of time $h$ has elapsed is independent of the last time the firm received the signal and across firms, and given by

$$\delta e^{-\delta h},$$

where $\delta > 0$ is the Calvo parameter. Hence, the probability of not having received a signal between $t$ and $\tau$ is

$$1 - \int_t^\tau \delta e^{-\delta(s-t)} ds = e^{-\delta(\tau-t)}$$
$$\approx 1 - \delta(\tau - t)$$

Firms that are able to adjust the price choose the price optimally. These firms maximize expected real discounted profits subject to their production technology (67), the demand curve (66) and the constraint on the frequency of price adjustment. Firms that cannot

change their price adjust output to meet demand at the pre-established price. Both types of firms choose inputs to minimize costs, given output demand.

We characterize first the input choice problem conditional on output. We then characterize the optimal price adjustment and output decisions. We start by deriving input demand and marginal cost. Firm $i$ chooses $N_t(i)$ to minimize total cost, given by

$$\frac{W_t}{P_t} N_t(i) \tag{173}$$

subject to

$$A N_t(i) - Y_t(i) \geq 0 \tag{174}$$

where, as mentioned earlier, $W_t/P_t$ is the real wage. Let $MC_t$ denote the Lagrange multiplier with respect to the constraint. Note that $MC_t$ is the firm's real marginal cost (the derivative of total cost with respect to $Y_t(i)$).

The FOC with respect to $N_t(i)$ is

$$[N_t(i)] : MC_t = \frac{W_t}{A P_t} \tag{175}$$

Since the firm takes $W_t/P_t$ as given, real marginal cost is constant across firms, a result of constant returns to scale and perfect factor mobility. Equation (174) with equality gives labor demand

$$N_t(i) = \frac{Y_t(i)}{A} \tag{176}$$

We next consider optimal price setting. A firm that is allowed to change its price at time $t$ picks $P_t(i)$ to maximize

$$\mathbb{E}_t \int_t^\infty SDF_{s,t} \left( \delta e^{-\delta(s-t)} \right) \left( \frac{P_t(i)}{P_s} Y_{s|t}(i) - MC_s Y_{s|t}(i) \right) ds \tag{177}$$

subject to

$$Y_{s|t}(i) = \left( \frac{P_t(i)}{P_s} \right)^{-\varepsilon} Y_s \tag{178}$$

where $Y_{s|t}(i)$ is the demand of good $i$ at time $s$ conditional on having changed prices for the last time at time $t$. In the optimal price setting decision, the firm takes as given the paths of $SDF_{s,t}$, $P_s$, $Y_s$ and $MC_s$. Plugging equation (178) into (177) gives

$$\mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \left( \delta e^{-\delta(s-t)} \right) \left( \frac{Y_s}{P_s^{1-\varepsilon}} P_t(i)^{1-\varepsilon} - \frac{Y_s MC_s}{P_s^{-\varepsilon}} P_t(i)^{-\varepsilon} \right) ds \tag{179}$$

The FOC with respect to $P_t(i)$ is

$$[P_t(i)] : \mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \left( \delta e^{-\delta(s-t)} \right) \left( \frac{Y_s}{P_s^{1-\varepsilon}} (1-\varepsilon) P_t^*(i)^{-\varepsilon} + \varepsilon \frac{Y_s MC_s}{P_s^{-\varepsilon}} P_t^*(i)^{-\varepsilon-1} \right) ds = 0$$

59

or, rearranging,

$$\frac{P_t^*(i)}{P_t} = \frac{1}{MC} \frac{\mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{-\varepsilon} \frac{Y_s}{Y_t} MC_s ds}{\mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{1-\varepsilon} \frac{Y_s}{Y_t} ds} \tag{180}$$

where $P_t^*(i)$ is the optimal desired price and where we have defined

$$MC \equiv \left(1 - \frac{1}{\varepsilon}\right),$$

which is the steady-state level of the real marginal cost (the inverse of the steady-state gross markup). We can also write

$$P_t^*(i) = (1+\mu)\mathbb{E}_t \int_t^\infty \Upsilon_{s,t} MC_s ds$$

where

$$\Upsilon_{s,t} \equiv \frac{Q_s e^{-\delta s} P_s^{-\varepsilon} Y_s}{\mathbb{E}_t \int_t^\infty Q_s e^{-\delta s} P_s^{\varepsilon-1} Y_s ds}$$

which shows that the optimal price is a weighted average of real marginal costs times the markup (using that the nominal marginal cost $MC_s^n = P_s MC_s$, the price is also a weighted average of *nominal* marginal costs). Defining

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{1-\varepsilon} \frac{Y_s}{Y_t} ds$$

$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{-\varepsilon} \frac{Y_s}{Y_t} \frac{MC_s}{MC} ds$$

and

$$\Pi_t \equiv \frac{P_t^*}{P_t}$$

we have

$$\Pi_t = \frac{x_{2,t}}{x_{1,t}}$$

and

$$dx_{1,t} = d\left(\mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{1-\varepsilon} \frac{Y_s}{Y_t} ds\right)$$

$$= d\left(\frac{e^{\delta t}P_t^{1-\varepsilon}}{Q_tY_t}\right)\left(\frac{e^{\delta t}P_t^{1-\varepsilon}}{Q_tY_t}\right)^{-1} x_{1,t} + \delta dt \tag{181}$$

$$dx_{2,t} = d\left(\mathbb{E}_t \int_t^\infty \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{-\varepsilon} \frac{Y_s}{Y_t} \frac{MC_s}{MC} ds\right)$$

$$= d\left(\frac{e^{\delta t}P_t^{-\varepsilon}}{Q_tY_t}\right)\left(\frac{e^{\delta t}P_t^{-\varepsilon}}{Q_tY_t}\right)^{-1} x_{2,t} + \frac{\delta MC_t}{MC} dt \tag{182}$$

Note that we dropped the index $i$ from $P_t^*$ (and hence from $\Pi_t$) because the optimal price $P_t^*$ depends only on aggregate variables, so all firms that are allowed to change the price pick the same optimal price. Since the price changes are stochastically independent across firms, we have

$$P_t^{1-\varepsilon} = \int_{-\infty}^t \delta e^{-\delta(t-s)} (P_s^*)^{1-\varepsilon} ds$$

It follows that the price level is a predetermined variable at time $t$ given by the past price quotations. Differentiating with respect to time gives

$$\frac{d\left(P_t^{1-\varepsilon}\right)}{dt} = \delta (P_t^*)^{1-\varepsilon} - \delta \int_{-\infty}^t \delta e^{-\delta(t-s)} (P_s^*)^{1-\varepsilon} ds$$

$$= \delta \left[(P_t^*)^{1-\varepsilon} - P_t^{1-\varepsilon}\right] \tag{183}$$

and

$$\frac{d\left(P_t^{1-\varepsilon}\right)}{dt} = (1-\varepsilon) P_t^{-\varepsilon} \frac{dP_t}{dt} \tag{184}$$

Combining (183) and (184) gives

$$\frac{dP_t}{P_t} = \frac{\delta}{1-\varepsilon} \left(\Pi_t^{1-\varepsilon} - 1\right) dt \tag{185}$$

Defining inflation as

$$\pi_t \equiv \frac{1}{dt} \frac{dP_t}{P_t}$$

we get

$$\pi_t = \frac{\delta}{1-\varepsilon} \left(\Pi_t^{1-\varepsilon} - 1\right)$$

Using that $P_t$ is locally deterministic by equation (185) and Ito's lemma,equations (181)

and (182) become

$$dx_{1,t} = \frac{x_{1,t}Q_tY_t}{e^{\delta t}}d\left(\frac{e^{\delta t}}{Q_tY_t}\right) + \delta dt + (1-\varepsilon)\,x_{1,t}\pi_t dt \tag{186}$$

$$dx_{2,t} = \frac{x_{2,t}Q_tY_t}{e^{\delta t}}d\left(\frac{e^{\delta t}}{Q_tY_t}\right) + \frac{\delta MC_t}{MC}dt - \varepsilon x_{2,t}\pi_t dt \tag{187}$$

Therefore, the price dynamics are determined by the following four equations

$$\pi_t = \frac{\delta}{1-\varepsilon}\left(\Pi_t^{1-\varepsilon} - 1\right) \tag{188}$$

$$\Pi_t = \frac{x_{2,t}}{x_{1,t}} \tag{189}$$

$$dx_{1,t} = (x_{1,t}+1)\,\delta dt + (1-\varepsilon)\,x_{1,t}\pi_t dt + x_{1,t}Q_tY_t d\left(\frac{1}{Q_tY_t}\right) \tag{190}$$

$$dx_{2,t} = \left(x_{2,t} + \frac{MC_t}{MC}\right)\delta dt - \varepsilon x_{2,t}\pi_t dt + x_{2,t}Q_tY_t d\left(\frac{1}{Q_tY_t}\right) \tag{191}$$

where $\pi_t$, $\Pi_t$, $x_{1,t}$ and $x_{2,t}$ are all stationary.

Let us next turn to the determination of profits and dividends. The real profits for the producer of intermediate good producer $i$ is

$$D_{t,goods}\,(i) = \frac{P_t\,(i)\,Y_t\,(i)}{P_t} - MC_tY_t\,(i) \tag{192}$$

Aggregating across firms gives the aggregate profits for the sector, which are paid out as dividends to shareholders

$$D_{t,goods} = \int_0^1 D_{t,goods}\,(i)\,di \tag{193}$$

$$= \frac{1}{P_t}\int_0^1 P_t\,(i)\,Y_t\,(i)\,di - MC_t\int_0^1 Y_t\,(i)\,di \tag{194}$$

### F.0.1 Aggregation

Integrating (66) over $i$ gives

$$\int_0^1 Y_t\,(i)\,di = v_tY_t$$

where

$$v_t \equiv \int_0^1 \left(\frac{P_t\,(i)}{P_t}\right)^{-\varepsilon} di.$$

so

$$Y_t \neq \int_0^1 Y_t\,(i)\,di$$

unless all prices are identical across firms.

Itegrating (176) over $i$ gives

$$Y_t = \frac{A}{v_t} N_t \tag{195}$$

The term $1/v_t$ gives the aggregate efficiency loss due to price distortions.

Because of Calvo pricing, we have

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-s)} \left( \frac{P_s^*}{P_t} \right)^{-\varepsilon} ds$$

where recall that $P_s^*$ is the optimal price chosen by firms that can reset their price at time $s$ given that the last time they were able to change their price was at $t$. Differentiating this expression gives the dynamics of $v_t$ in terms of aggregate variables

$$
\begin{aligned}
dv_t &= d\left[ \int_{-\infty}^{t} \delta e^{-\delta(t-s)} \left( \frac{P_s^*}{P_t} \right)^{-\varepsilon} ds \right] \\
&= d\left[ e^{-\delta t} P_t^{\varepsilon} \int_{-\infty}^{t} \delta e^{\delta s} P_s^{*-\varepsilon} ds \right] \\
&= d\left[ e^{-\delta t} P_t^{\varepsilon} \right] \int_{-\infty}^{t} \delta e^{\delta s} P_s^{*-\varepsilon} ds + e^{-\delta t} P_t^{\varepsilon} d\left[ \int_{-\infty}^{t} \delta e^{\delta s} P_s^{*-\varepsilon} ds \right] \\
&= \left( \varepsilon \frac{dP_t}{P_t} - \delta dt \right) \int_{-\infty}^{t} \delta e^{-\delta(t-s)} \left( \frac{P_s^*}{P_t} \right)^{-\varepsilon} ds + \delta \left( \frac{P_t^*}{P_t} \right)^{-\varepsilon} dt
\end{aligned}
$$

so that

$$dv_t = \delta \Pi_t^{-\varepsilon} dt + (\varepsilon \pi_t - \delta) v_t dt \tag{196}$$

We can also express equation (70) in terms of aggregate variables only

$$
\begin{aligned}
D_{t,goods} &= \frac{1}{P_t} \int_0^1 P_t(i) Y_t(i) \, di - MC_t \int_0^1 Y_t(i) \, di \tag{197} \\
&= (1 - MC_t v_t) Y_t. \tag{198}
\end{aligned}
$$

# G   Appendix: Solving Stock Market Clearing

We solve for the case of two stocks (*goods*, *bank*) and a single shock from the point of view of the bank, and one stock (*bank*) from the point of view of the household. The supply of goods stocks is 1 share, i.e.

$$\frac{X_t \theta_t}{S_t} = 1 \tag{199}$$

63

The demand comes from the bank's problem

$$\theta_t = \min\{1, \max\{0, \varphi_t\}\}\,\theta_{M,t} \tag{200}$$

We will use the following

$$f_{M,t} \text{ exogenous} \tag{201}$$
$$\theta_{M,t} = \sigma_t^{-2}\mu_t \tag{202}$$
$$\eta_t = \sigma_t^{-1}\mu_t \tag{203}$$
$$dS_t = S_t\left(\mu_t + R_t\right)dt + S_t\sigma_t dB_t \tag{204}$$
$$dX_t = X_t\left(R_t - f_t + \theta_t\mu_t\right)dt + X_t\theta_t\sigma_t dB_t \tag{205}$$
$$f_t = u\left(t, \min\{1, \varphi_t\}\right)f_{M,t} \tag{206}$$

Consider first the case $\varphi_t > 1$. Then (200),(203),(205),(206) give

$$\theta_t = \sigma_t^{-2}\mu_t \tag{207}$$
$$f_t = f_{M,t} \tag{208}$$
$$dX_t = X_t\left(R_t - f_{M,t} + \sigma_t^{-2}\mu_t^2\right)dt + X_t\sigma_t^{-1}\mu_t dB_t \tag{209}$$

and (199), (204), (209) give

$$S_t = X_t\theta_t \tag{210}$$
$$dS_t = \theta_t dX_t + X_t d\theta_t + dX_t d\theta_t \tag{211}$$
$$S_t\left(\mu_t + R_t\right)dt + S_t\sigma_t dB_t = \theta_t\left(X_t\left(R_t - f_{M,t} + \sigma_t^{-2}\mu_t^2\right)dt + X_t\sigma_t^{-1}\mu_t dB_t\right) \tag{212}$$
$$+X_t d\theta_t + X_t\sigma_t^{-1}\mu_t\left(d\theta_t dB_t\right) \tag{213}$$

Matching the drift and stochastic parts of the left and hand side of equation (212) gives

$$S_t\left(\mu_t + R_t\right) = \theta_t X_t\left(R_t - f_{M,t} + \sigma_t^{-2}\mu_t^2\right) + X_t\frac{1}{dt}\mathbb{E}_t\left[d\theta_t\right] \tag{214}$$

$$+X_t\sigma_t^{-1}\mu_t\left(\frac{1}{dt}d\theta_t dB_t\right) \tag{215}$$

$$S_t\sigma_t = \theta_t X_t\sigma_t^{-1}\mu_t + X_t stoch\left(d\theta_t\right) \tag{216}$$

Using (199) and (207) in (216) gives

$$stoch\left(d\theta_t\right) = \mu_t\sigma_t^{-3}\left(\sigma_t^2 - \mu_t\right) \tag{217}$$

Using (199), (207) and (217) in (214) gives

$$\frac{1}{dt}\mathbb{E}_t\left[d\theta_t\right] = \sigma_t^{-2}\mu_t f_{M,t} \tag{218}$$

64

Equations (217) and (218) mean that

$$d\theta_t = \theta_t f_{M,t} dt + \eta_t \left(1 - \theta_t\right) dB_t \qquad (219)$$

Consider now the case of $0 < \varphi \leq 1$. Equations (200),(203),(205),(206) give

$$
\begin{aligned}
\theta_t &= \varphi_t \sigma_t^{-2} \mu_t & (220) \\
f_t &= u\left(t, \varphi_t\right) f_{M,t} & (221) \\
dX_t &= X_t \left(R_t - u\left(t, \varphi_t\right) f_{M,t} + \varphi_t \eta_t^2\right) dt + X_t \varphi_t \eta_t dB_t & (222)
\end{aligned}
$$

Equations (199), (204), (222) give

$$
\begin{aligned}
S_t &= X_t \theta_t & (223) \\
dS_t &= \theta_t dX_t + X_t d\theta_t + dX_t d\theta_t & (224) \\
S_t \left(\mu_t + R_t\right) dt + S_t \sigma_t dB_t &= \theta_t X_t \left(R_t - u\left(t, \varphi_t\right) f_{M,t} + \varphi_t \eta_t^2\right) dt & (225) \\
&\quad + \theta_t X_t \varphi_t \eta_t dB_t + X_t d\theta_t + X_t \varphi_t \eta_t \left(dB_t d\theta_t\right) & (226)
\end{aligned}
$$

Matching the drift and stochastic parts of the left and hand side of equation (225) gives

$$
\begin{aligned}
S_t \left(\mu_t + R_t\right) &= \theta_t X_t \left(R_t - u\left(t, \varphi_t\right) f_{M,t} + \varphi_t \eta_t^2\right) + X_t \frac{1}{dt} E_t \left[d\theta_t\right] & (227) \\
&\quad + X_t \varphi_t \eta_t \frac{1}{dt} \left(dB_t d\theta_t\right) & (228) \\
S_t \sigma_t dB_t &= \theta_t X_t \varphi_t \eta_t dB_t + X_t stoch\left(d\theta_t\right) & (229)
\end{aligned}
$$

Using (199), (203) and (220) in (229) gives

$$stoch\left(d\theta_t\right) = \eta_t \varphi_t \left(1 - \theta_t \varphi_t\right) \qquad (230)$$

Using (199), (203), (220) and (230) in (227) gives

$$\frac{1}{dt} E_t \left[d\theta_t\right] = \theta_t u\left(t, \varphi_t\right) f_{M,t} + \eta_t^2 \varphi_t \left(\varphi_t - 1\right) \left(\theta_t + \theta_t \varphi_t - 1\right) \qquad (231)$$

Equations (230) and (231) mean that

$$d\theta_t = \left(\eta_t^2 \varphi_t \left(\varphi_t - 1\right) \left(\theta_t \left(\varphi_t + 1\right) - 1\right) + \theta_t \varphi_t f_{M,t}\right) dt + \eta_t \varphi_t \left(1 - \theta_t \varphi_t\right) dB_t \qquad (232)$$

Note that (219) and (232) match when $\varphi_t = 1$.