ON LATTICE COMPLEMENTS

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Let $(L, \leqq)$ be a distributive lattice with first element 0 and last element 1. If a, b in L have complements, then these must be unique, and the De Morgan laws provide complements for $a \vee b$ and $a \wedge b$. We show that the converse statement holds under weaker conditions.

THEOREM 1. *If $(L, \leqq)$ is a modular lattice with* 0 *and* 1 *and if a, b in L are such that $a \vee b$ and $a \wedge b$ have (not necessarily unique) complements, then a and b have complements.*

Proof. We indicate by $=^*$ the steps at which the modular condition is used. Choosing any complements $(a \vee b)'$ and $(a \wedge b)'$ of $a \vee b$ and $a \wedge b$ respectively, let

$$x = (a \vee b)' \vee [(a \wedge b)' \wedge b].$$

Then

$$\begin{aligned} a \vee x &= a \vee (a \wedge b) \vee (a \vee b)' \vee [(a \wedge b)' \wedge b] \\ &=^* a \vee (a \vee b)' \vee \{[(a \wedge b) \vee (a \wedge b)'] \wedge b\} = a \vee (a \vee b)' \vee b = 1, \end{aligned}$$

and

$$\begin{aligned} a \wedge x &= a \wedge (a \vee b) \wedge \{(a \vee b)' \vee [(a \wedge b)' \wedge b]\} \\ &=^* a \wedge \{0 \vee [(a \wedge b)' \wedge b]\} = a \wedge (a \wedge b)' \wedge b = 0, \end{aligned}$$

so x is a complement of a. Similarly $y = (a \vee b)' \vee [(a \wedge b)' \wedge a]$ is a complement of b.

This provides an extension of the De Morgan laws to modular lattices.

THEOREM 2. *If $(L, \leqq)$ is a modular lattice with* 0 *and* 1, *if a, b in L have unique complements a', b' respectively, and if $a \vee b$ and $a \wedge b$ have complements, then $a' \vee b'$ is a complement of $a \wedge b$ and $a' \wedge b'$ is a complement of $a \vee b$.*

Proof. Form x and y as in the above proof. By dualization, $\bar{x} = (a \wedge b)' \wedge [(a \vee b)' \vee b]$ and $\bar{y} = (a \wedge b)' \wedge [(a \vee b)' \vee a]$ are also complements of a and b respectively. Then since a' and b' are unique, $a' = x = \bar{x}$ and $b' = y = \bar{y}$. Now

$$\begin{aligned} a' \vee b' \vee (a \wedge b) &= x \vee y \vee (a \wedge b) \\ &= (a \vee b)' \vee [(a \wedge b)' \wedge b] \vee [(a \wedge b)' \wedge a] \vee (a \wedge b) \\ &=^* (a \vee b)' \vee \{b \wedge [(a \wedge b)' \vee (a \wedge b)]\} \vee \{a \wedge [(a \wedge b)' \vee (a \wedge b)]\} \\ &= (a \vee b)' \vee a \vee b = 1, \end{aligned}$$

and

$$\begin{aligned} (a' \vee b') \wedge (a \wedge b) &= (\bar{x} \vee \bar{y}) \wedge (a \wedge b) \\ &= \{[(a \wedge b)' \wedge [(a \vee b)' \vee b]] \vee [(a \wedge b)' \wedge [(a \vee b)' \vee a]]\} \wedge (a \wedge b) \\ &=^* (a \wedge b)' \wedge \{(a \vee b)' \vee b \vee [(a \wedge b)' \wedge [(a \vee b)' \vee a]]\} \wedge (a \wedge b) = 0, \end{aligned}$$

so $a' \vee b'$ is a complement of $a \wedge b$. Dually $a' \wedge b'$ is a complement of $a \vee b$.

Figure 1 shows a modular lattice of order seven which contains uniquely complemented elements a and b such that $a \vee b$ has no complement. Contrary to Exercise 4, p. 153 of **[1]**, there are no lattices of order six or less, modular or otherwise, in which the complemented elements fail to form a sublattice.

If $(L, \leqq)$ is a *complemented* modular lattice, then, by a theorem of J. von Neumann [**1**, p. 124], the uniquely complemented elements constitute the center of the lattice, which in any lattice with 0 and 1 is a sublattice [**1**, p. 27]. But [**1**, p. 120] in a complemented modular

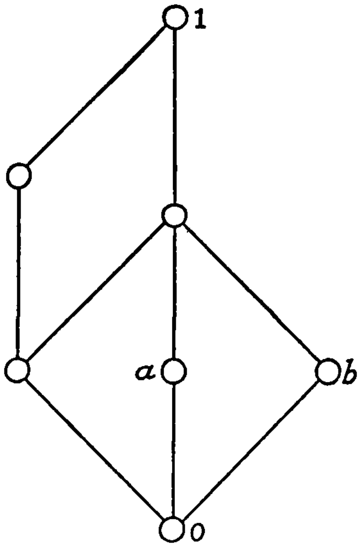

FIG. 1.

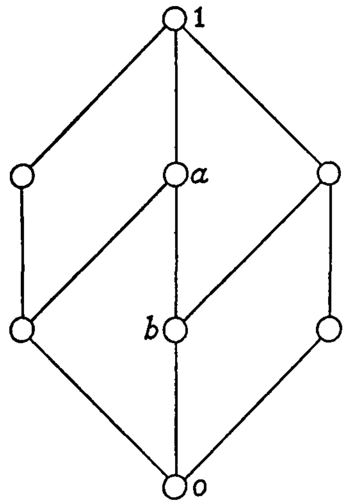

FIG. 2.

lattice every central element is neutral, and hence [**1**, p. 28] distributes with every pair of elements of the lattice. Thus in a complemented modular lattice those elements which have unique complements form a Boolean algebra, and hence satisfy the De Morgan laws. Some questions in this direction remain unanswered. For example, if $(L, \leqq)$ is modular, if a and b have unique complements in L, and if $a \vee b$ and $a \wedge b$ have complements, are these complements necessarily unique?

Finally, Dilworth [**2**] has shown, without giving an example, the existence of non-distributive lattices in which each element has a unique complement. Such lattices are non-modular by what we have said above. It seems doubtful that the De Morgan laws would hold in all such lattices. Figure 2 shows a complemented, though not uniquely complemented, non-modular lattice containing elements a and b such that a, b, $a \vee b$ and $a \wedge b$ have unique complements, but both De Morgan laws fail.

REFERENCES

1. G. Birkhoff, *Lattice theory* (American Mathematical Society Colloquium Publications, Vol. 25; 2nd edition, 1948).

2. R. P. Dilworth, Lattices with unique complements, *Trans. Amer. Math. Soc.* **57** (1945), 123–154.

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