MATH520 Homework 1

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Exercise 1.5

Suppose that you are shown four cards, laid out in a row. Each card has a letter on one side and a number on the other. On the visible side of the cards are printed the symbols

S 8 3 A

Determine which cards should you turn over to decide if the following rule is true or false: "If there is a vowel on one side of the card, then there is an even number on the other side."

Solution:

If a card has a vowel we should turn it over to check that the other side has an even number. Consonants are not mentioned in the rule so what is on the other side of a consonant is irrelevant. Odd numbers should be turned over to make sure there is not a vowel on the other side and even numbers are OK no matter what is on the other side so no need to turn them over.

Answer: 3 and A.

Exercise 2.10

Use exercise 2.9 to show that the norm $||\cdot||$ is a uniformly continuous function.

Solution:

Let $\varepsilon > 0$, then take $\delta = \varepsilon$, using exercise 2.9 we get that

$$|||x|| - ||y||| \leq ||x - y||,$$

then $||x - y|| < \varepsilon \implies |||x|| - ||y||| < \varepsilon$. Notice that δ depends only on ε, it doesn't depend on x nor y, so $||\cdot||$ is not only continuous but uniformly continuous.

Exercise 3.6

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of the matrix $I_n - A$ are $1 - \lambda_1, \ldots, 1 - \lambda_n$.

Solution:

Let v_i be an eigenvector associated with the eigenvalue λ_i, then

$$(I_n - A)v_i = v_i - Av_i = v_i - \lambda_i v_i = (1 - \lambda_i)v_i.$$

So the eigenvalues of $I_n - A$ are indeed $(1 - \lambda_i)$.

Exercise 3.17

Consider the matrix

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

a. Is the matrix positive definite, negative definite, or indefinite?

Solution 1:

Explicitly computing the eigenvalues using Sarrus' rule to find the characteristic polynomial $(-\lambda^3 + 3\lambda + 2)$ and then finding its roots gives us:
$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 2$. So it is indefinite.

Solution 2: Use the thing in the book THIS LOOKS INTERESTING TODO.!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

Exercise 5.7

Suppose that $f(x) = o(g(x))$. Show that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that if $||x|| < \delta$, then $||f(x)|| < \varepsilon|g(x)|$.

Solution:

By definition

$$\lim_{x \to 0, x \in \Omega} \frac{||f(x)||}{|g(x)|} = 0;$$

which means that for any $\varepsilon > 0$ there is a $\delta > 0$, such that if $||x|| < \delta$ and $x \in \Omega$ then

$$\frac{||f(x)||}{|g(x)|} < \varepsilon,$$

which is equivalent to

$$||f(x)|| < \varepsilon|g(x)|,$$

so we can take the same δ given by the limit.

Exercise 5.10

Write down the Taylor series expansion of the following functions about the given points X_0. Neglect terms of order three or higher.

a. $f(x) = x_1 e^{-x_2} + x_2 + 1$, $x_0 = [1, 0]$.

Solution:

TODO
b. $f(x) = x_1^4 + 2x_1^2x_2^2 + x_2^4,\ x_0 = [1,1]^T.$
Solution:
TODO
c. $f(x) = e^{x_1 - x_2} + e^{x_1 + x_2} + x_1 + x_2 + 1,\ x_0 = [1,0]^T.$
Solution:
TODO