Baire Category Theorem for Locally Compact Spaces

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Definition 1. A subset $A \subset X$ of a topological space is said to be *nowhere dense* in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \emptyset$.

This definition is equivalent to the standard one found in all text-books: A is nowhere dense in X iff the interior of the closure of A in X is empty: $\operatorname{Int}(\overline{A}) = \emptyset$.

> Reason: Let us assume that $A \subset X$ is nowhere dense according to our definition. We claim that $\operatorname{Int}(\overline{A}) = \emptyset$. Suppose not, that is, there exists a nonempty open set $U \subset \operatorname{Int}(\overline{A})$. This means that every point of U is a limit point of A. In particular, if $V \subset U$ is a nonempty open set, then every $x \in V$ is a limit point of A and hence $V \cap A \neq \emptyset$. Then A cannot be nowhere dense according to our definition.
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> Conversely, let us assume that $\operatorname{Int}(\overline{A}) = \emptyset$. Let U be a nonempty open set. If every nonempty open subset V of U meets A nontrivially, then every point of U is a limit point of A and hence $U \subset \overline{A}$. Hence $\operatorname{Int}\overline{A}$ cannot be empty, a contradiction.

I prefer the first one. It gives us a better geometric intuition as it uses only the primitive concept of topology.

Example 2. Let V be any proper vector subspace of $\mathbb{R}^n$. Then V is nowhere dense in $\mathbb{R}^n$. This is a typical example of a nowhere dense set.

More generally, let X be a normed linear space. Let V be any proper vector subspace of X. Then V is nowhere dense in X. (So are its translates, since translation is a homeomorphism. In particular, any line in $\mathbb{R}^2$ is a nowhere dense set.)

> Reason: Suppose not. Then there exists an open ball $B(p,r) \subset V$. Since $B(p,r) = p + B(0,r)$ and V is vector subspace, we conclude that $B(0,r) = B(p,r) - p \subset V$. Given any nonzero vector $x \in X$, the vector $y = \frac{r}{2\|x\|}x \in B(0,r) \subset V$. Since x is a scalar multiple of y and V is a vector subspace, it follows that $x \in V$. Since $x \in X$ is an arbitrary nonzero vector, we conclude that $V = X$. This contradicts our assumption that V is proper vector subspace.

We shall give the formulation of Baire category theorem in a form which will be more useful than the one which uses the notion of category.

Theorem 3 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space.*

(1) *Let U_n be open dense subsets of X, for $n \in \mathbb{N}$. Then $\cap_n U_n$ is dense in X.*

(2) *Let F_n be nonempty closed subsets of X such that $X = \cup_n F_n$. Then at least one of F_n's has nonempty interior. In other words, a complete metric space cannot be a countable union of nowhere dense closed subsets.*

Proof. We first observe that both the statements are equivalent. For, G is open and dense iff its complement $F := X \setminus G$ is closed and nowhere dense.

> Reason: Let G be open and dense. Then $F := X \setminus G$ is closed. If $U := \mathrm{Int}\,(F)$ is not empty, then $U \cap G = \emptyset$. This contradicts the density of G.
>
> Conversely, if F is closed and nowhere dense, then $G := X \setminus F$ is open. If G is not dense, then there exists a nonempty open set U such that $U \cap G = \emptyset$. Hence $U \subset F$. But then $\mathrm{Int}\,(F) \supset U$ and hence $\mathrm{Int}\,(F)$ s not empty, a contradiction.

Hence any one of them follows from the other by taking complements. So, we confine ourselves to proving the first.

Let $U := \cap_n U_n$. We have to prove that U is dense in X. Let $x \in X$ and $r > 0$ be given. We need to show that $B(x, r) \cap U \neq \emptyset$. Since U_1 is dense and $B(x, r)$ is open there exists $x_1 \in B(x, r) \cap U_1$. Since $B(x, r) \cap U_1$ is open, there exists r_1 such that $0 < r_1 < 1/2$ and $B[x_1, r_1] \subset B(x, r) \cap U_1$. We repeat this argument for the open set $B(x_1, r_1)$ and the dense set U_2 to get $x_2 \in B(x_1, r_1) \cap U_2$. Again, we can find r_2 such that $0 < r_2 < 2^{-2}$ and $B[x_2, r_2] \subset B(x_1, r_1) \cap U_2$. Proceeding this way, we get for each $n \in \mathbb{N}$, x_n and r_n with the properties

$$B[x_n, r_n] \subset B(x_{n-1}, r_{n-1}) \cap U_n \text{ and } 0 < r_n < 2^{-n}.$$

Clearly, the sequence (x_n) is Cauchy: if $m \leq n$,

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \leq \sum_{k=m}^{n} 2^{-k}.$$

Since $\sum_k 2^{-k}$ is convergent, it follows that (x_n) is Cauchy. (Or, $x_n \in B[x_k, r_k]$ for all $n \geq k$ and hence $d(x_m, x_n) \leq 2r_k$ for $m, n \geq k$.)

Since X is complete, there exists $x_0 \in X$ such that $x_n \to x_0$. Since x_0 is the limit of the sequence $(x_n)_{n \geq k}$ in the closed set $B[x_k, r_k]$, we deduce that $x_0 \in B[x_k, r_k] \subset B(x_{k-1}, r_{k-1}) \cap U_k$ for all k. In particular, $x_0 \in B(x, r) \cap U_k$ for all k. □

Remark 4. The importance of our formulation is this. The first statement tells us of a typical way in which Baire category can be used. Imagine that we are on the look-out for an element $x \in X$ with some specific properties. Further assume that the sets of elements which have properties "arbitrarily close" to the one desired are dense open sets in X. Then the result says that there exists at least one element with the desired property. Thus the first formulation is useful when we are interested in the existence problems. This vague way of remembering is well-illustrated in some of the applications below. See especially the existence of everywhere continuous nowhere differentiable function.

The second formulation says that X cannot be a countable union of "hollow" sets. A typical application: $\mathbb{R}^n$ cannot be the union of a countable collection of lower dimensional subspaces. Another instance: a complete normed linear space cannot be countable dimensional. See Proposition 5 below. More importantly, it might lead us to an open set when the space is written as a countable union of closed sets.

Proposition 5. *Let X be an infinite dimensional complete normed linear space. Then X cannot be countable dimensional.*

Proof. Let, if possible, $\{e_n : n \in \mathbb{N}\}$ be a (Hamel/algebraic) basis of X. This means that any vector $x \in X$ is a finite linear combination of e_n's. If we let F_n stand for the vector subspace spanned by $\{e_k | 1 \leq k \leq n\}$, then F_n is finite dimensional and we have $X = \cup_n F_n$. It is well-known that all norms on a finite dimensional vector space are equivalent so that any finite dimensional normed linear space is necessarily complete. In view of this we conclude that each F_n is closed in X. Since F_n is finite dimensional, F_n is a proper vector subspace of X. By Example 2, F_n is nowhere dense for each n. We have thus shown that the complete metric space X is the union of the countable family $\{F_n\}$ of nowhere dense closed sets, a contradiction. □

Example 6. There exists no metric d_1 on $\mathbb{Q}$ such that a set is open in d_1-topology iff it is open in the standard topology and such that the metric space $(\mathbb{Q}, d_1)$ is complete. Hint: $\mathbb{Q} = \cup_{r\in\mathbb{Q}}\{r\}$ is a countable union of nowhere dense closed sets.

Ex. 7. An amusing exercise: Let (x_n) be any sequence of real numbers. Show that the set $\{x \in \mathbb{R} : x \neq x_n, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Hence conclude that $\mathbb{R}$ is uncountable.

Show that $\mathbb{Q}$ cannot be written as the intersection of a countable family of open subsets of $\mathbb{R}$.

The next result is a beautiful application of Baire's theorem which uses both the versions! To put it in perspective, recall that the pointwise limit of a sequence of continuous functions need not be continuous while the uniform limit is. However, the poinwise limit cannot be too wild.

Theorem 8. *Let X be a complete metric space. Let $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions. Assume that there exists a function $f \colon X \to \mathbb{R}$ such that $f_n(x)$ converges to $f(x)$ for each $x \in X$. Then there exists a dense subset D of X such that each point of D is a point of continuity of f.*

Proof. Fix $\varepsilon > 0$. Define, for each $k \in \mathbb{N}$,

$$E_k(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq 1/k, \text{ for all } m, n \geq n\}.$$

Then we claim that $E_k(\varepsilon)$ is closed for each k.

Reason: Fix $m, n \geq k$. Then the set $E_k^{m,n}(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq 1/k\}$ is a closed subset of X, since $|f_n - f_m|$ is continuous. Now, since $E_k(\varepsilon) = \cap_{m,n\geq k} E_k^{m,n}(\varepsilon)$, the claim follows.

It is easy to show that $X = \cup_k E_k(\varepsilon)$.

Reason: Let $x_0 \in X$. Since $f_n(x_0) \to f(x_0)$, the sequence $(f_n(x_0))$ is Cauchy. Hence for the given $\varepsilon > 0$, there exists k_0 such that for $m, n \geq k_0$, we have $|f_m(x_0) - f_n(x_0)| \leq 1/k_0$. Hence we conclude that $x_0 \in E_{k_0}(\varepsilon)$.

Since $X$is a complete metric space, at least one of $E_k(\varepsilon)$ should have nonempty interior. Let $U_\varepsilon := \cup_k \mathrm{Int}\,(E_k(\varepsilon))$. Then U_ε is a nonempty open subset of X.

Let $U_n := U_{1/n}$. We claim that each U_n is dense in X.

Reason: It is enough if we show that every closed ball $B := B[x, r]$ meets U_n nontrivially. (Why?)

Reason: To show a set A is dense in a metric space, it suffices to show that $A \cap B(x, r) \neq \emptyset$ for any $x \in X$and $r > 0$. Assume that $A \cap B[z, \rho] \neq \emptyset$ for any $z \in X$ and $\rho > 0$. Then given any $B(x, r)$, we may take $z = x$ and $\rho = r/2$. Then $\emptyset \neq A \cap B[x, \rho] \subset A \cap B(x, r)$.

Observe that the closed set (and hence a complete metric space) B is the union of a countable family of closed sets: $B = \cup_n (B \cap E_k(1/n))$. By Baire, at least one of them has nonempty interior, say, $\mathrm{Int}\,(B \cap E_k(1/n)) \neq \emptyset$. Since $\mathrm{Int}\,(B \cap E_k(1/n)) \subset B \cap \mathrm{Int}\, E_k(1/n)$, it follows that $B[x, r] \cap U_n \neq \emptyset$ and hence the claim is proved.

Let $D := \cap_n U_n$. By Baire, D is dense in X. We claim that every $x \in D$ is a point of continuity of f.

Reason: Fix $p \in D$. Let $\varepsilon > 0$ be given. Choose $N \gg 0$ such that $1/N < \varepsilon$. Since $p \in D$, $p \in U_N$ and hence there exists $k \in \mathbb{N}$ such that $p \in \mathrm{Int}\,(E_k(1/N)$. By continuity of f_k at p, there exists an open neighbourhood V of p contained in $\mathrm{Int}\, E_k(1/N)$ such that

$$|f_k(x) - f_k(p)| < \varepsilon, \text{ for all } x \in V. \tag{1}$$

For $x \in V$, since $V \subset E_k(1/N)$, by the definition of $E_k(\varepsilon)$'s, we have

$$|f_m(x) - f_k(x)| \leq 1/N, \text{ for all } m \geq k. \tag{2}$$

Letting $m \to \infty$ in the above equation, we obtain

$$|f(x) - f_k(x)| \leq 1/N, \text{ for all } x \in V. \tag{3}$$

We are now ready for the kill. We claim that $|f(x) - f(p)| < 3\varepsilon$ for $x \in V$.

$$\begin{aligned} |f(x) - f(p)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(p)| + |f_k(p) - f(p)| \\ &\leq 1/N + \varepsilon + 1/N \\ &< 3\varepsilon. \end{aligned}$$

This shows that f is continuous at every point of D.

□

Theorem 9 (Baire category theorem for locally compact spaces)**.** *Let X be a locally compact hausdorff space. Let (U_n) be a sequence of open dense sets in X. Then $\cap_n U_n$ is dense in X.*

Proof. Let G be a nonempty open set in X. We need to prove that there exists $x \in G$ such that $x \in U_n$ for all n. The strategy is to mimic the proof in the case of metric spaces replacing open balls by the existence of open sets V such that $\overline{V}$ is compact and $x \in V \subset \overline{V} \subset U$ for any given open set U and $x \in U$ and then invoking Cantor intersection theorem for a decreasing sequence of compact sets.

Since G is a nonempty open set and U_1 is dense, there exists $x_1 \in G \cap U_1$. Since $G \cap U_1$ is open, $x \in G \cap U_1$ and X is locally compact hausdorff space, there exists an open set V_1 such that $x \in V_1$, $\overline{V}_1$ is compact and $\overline{V}_1 \subset G \cap U_1$. Assume, by way of induction, that we have chosen x_i, $V_i \ni x_i$, $\overline{V}_i$ is compact and that $x_i \in V_i \subset \overline{V}_i \subset V_{i-1} \cap U_i$, for $1 \leq i \leq n$.

Now given a nonempty open set V_n, since $V_n \cap U_{n+1}$ is nonempty, there exists $x_{n+1} \in V_n \cap U_{n+1}$. Since X is locally compact and hausdorff, there exists an open set $V_{n+1} \ni x_{n+1}$ such that $\overline{V}_{n+1}$ is compact and $x_{n+1} \in V_{n+1} \subset \overline{V}_{n+1} \subset V_n \cap U_{n+1}$. Let $K_n := \overline{V}_n$. Thus we have a decreasing sequence (K_n) of nonempty compact subsets. Hence by Cantor intersection theorem, there exists $x \in \cap_n K_n$. Since $x \in K_n = \overline{V}_n \subset U_n$, it follows that $x \in \cap U_n$. Also, $x \in K_1 \subset U$. □