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SHORTER NOTES

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ON NOWHERE MONOTONE FUNCTIONS

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ABSTRACT. The existence of everywhere differentiable but nowhere monotone functions is established using the Baire Category Theorem, and the relatively easy fact that there are nontrivial bounded derivatives with a dense set of zeros.

Interest in everywhere differentiable, nowhere monotone functions was revived by Katznelson and Stromberg in [4] where they gave a construction of such a function which is considerably simpler than the original one due to Köpcke or the one in the book by Hobson [3, pp. 412–421]. This work was followed up by Goffman in [2] where a much shorter construction is given but which uses a deep theorem due to Zahorski. Here the existence of such functions is established using the Baire Category Theorem.

Let R denote the real line and let

$$D = \{f\colon R \to R\colon f \text{ is bounded and there is a function } F \text{ such that } F'(x) = f(x) \text{ for all } x \text{ in } R\},$$

and endow D with the metric

$$d(f,g) = \sup_{x\in R} |f(x) - g(x)|.$$

This is the metric of uniform convergence, and by a standard advanced calculus theorem, a uniform limit of a sequence of bounded derivatives is a bounded derivative. Hence D is a complete metric space. Let

$$D_0 = \{f \in D\colon \{x\colon f(x) = 0\} \text{ is dense in } R\},$$

and give to D_0 the metric of D. Then D_0 itself is complete for if $\{f_k\}$ is a sequence in D_0 converging in metric to $f \in D$, then for each k, $Z_k = \{x\colon f_k(x) = 0\}$ is a dense G_δ set and hence $Z = \bigcap_{k=1}^{\infty} Z_k$ is dense in R. But $Z \subset \{x\colon f(x) = 0\}$. Thus $f \in D_0$.

It is not hard to show that D_0 contains more than just the zero function (see [1, p. 27] or [5]). The existence of such a function and the fact that D_0 is closed

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under addition will be used below. The proof of the latter is like the completeness of D_0 only easier.

THEOREM. *Let*

$$E = \{f \in D_0\text{: there is an interval on which } f \text{ is unsigned}\}.$$

Then E is of the first category in D_0.

PROOF. Let $\{I_n\}$ be an ordering of the collection of all closed intervals having rational endpoints. Let

$$E_n = \{f \in D_0 : f(x) \geqslant 0 \text{ for all } x \in I_n\}$$

and

$$F_n = \{f \in D_0 : f(x) \leqslant 0 \text{ for all } x \in I_n\}.$$

Then clearly

$$E = \bigcup_{n=1}^{\infty} (E_n \cup F_n);$$

so it suffices to prove that E_n and F_n are closed and contain no spheres. The argument will be carried out for E_n. A similar procedure works for F_n.

That E_n is closed is immediate. To prove that E_n contains no sphere suppose $f \in D_0$ and $\varepsilon > 0$. Since $f \in D_0$ there is an $x \in I_n$ such that $f(x) = 0$. Since there are bounded derivatives having a dense set of zeros that are not identically zero, by pushing and crushing it is not hard to prove that there is a function $h \in D_0$ such that $h(x) < 0$ and $\sup_{y \in R}|h(y)| < \varepsilon$. Then $g = f + h$ belongs to D_0, $d(f, g) < \varepsilon$, and $g \notin E_n$ since $g(x) = f(x) + h(x) = h(x) < 0$ and $x \in I_n$. Thus the sphere of radius ε about f is not contained in E_n.

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