Numerical methods for stochastic volatility models:

Heston model

A Thesis
Presented to
The Division of Applied Mathematics
Getulio Vargas Foundation

In Partial Fulfillment
of the Requirements for the Degree
M.Sc. of Mathematics

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agosto 06, 2017

Approved for the Division
(Mathematics)

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Acknowledgements

I want to thank a few people.

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Abstract

The preface pretty much says it all.

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Dedication

You can have a dedication here if you wish.

Chapter 1

altadvisor: ‘Your Other Advisor’

Chapter 2

Literature Review

This chapter presents the concepts of stochastic calculus, from the historic conception of how it first arose through the basic principles and applications in finance. More precisely, we address the classical Black-Scholes model and its limitations and the Heston model. This model is also well known, it introduces the concept of stochastic volatility which brings us closer to reality.

2.1 Stochastic Calculus

Stochastic calculus arises from stochastic processes and allows the creation of a theory of integration where both the integrand and integrator terms are stochastic processes. Stochastic calculus was created by the Japanese mathematician Kiyosi Itô~[1] in the 1940s and 1950s and is used for modelling financial options and in another wide variety of fields [1]. In this chapter we present the historical contexts in which the tools and models used arise, but our focus is introducing the concepts and notations that will be further used in our work.

2.1.1 Brownian Motion

The Brownian motion is the name given to the irregular motion observed in the motion of pollen particles suspended in fluid resulting from particle collision with atoms or molecules. It is named after Robert Brown, the first to have observed the movement in 1828. He noted two characteristic in the pollen movement [1]:

- the path of a given particle is very irregular, having a tangent at no point
- the motion of two distinct particles appear to be independent

The first quantitative works in brownian motion come from an interest in stock price fluctuation by Bachelier in 1900. Albert Einstein also leaned over the subject and

[1] There is another important stochastic integral, called the *Stratonovich Integral* that unlike the Itô's integral, respects the conventional calculus chain rule. Also, the integral is evaluated at the interval's midpoint, instead of its left extreme.

in 1905 derived the transition density for Brownian motion from molecular-kinetic theory of heat [1,2].

In 1923, the Wiener process was coined in honor of Norbert Wiener mathematical proof of existence of the brownian motion and stating its properties as follows [3]:

- $W_0 = 0$
- The change in W, given by $\Delta W = W_{t+1} - W_t$, is normally distributed with mean zero and standard deviation $\sqrt{\Delta t}$, meaning that $\Delta W = \epsilon\sqrt{\Delta t}$, where ϵ is $N(0, 1)$.
- If the increment Δt_1 does not overlap with the time increment Δt_2, then ΔW_1 and ΔW_2 are independent.
- The process is continuous, meaning that there are no jumps in the process.
- The process is a Markov process. This means that the conditional expectation of W_{t+1} given its entire history is equal to the conditional expectation of W_{t+1} given today's information. This can be written as: $E[W_{t+1}|W_1, ..., W_t] = E[W_{t+1}|W_t]$.
- Consider the time interval $[0, t]$ with n equally spaced intervals given by $t_i = \frac{it}{n}$. Then the paths of the Brownian motion have unbounded variation, this means that they are not differentiable and go towards infinity as n increases. The quadratic variation is given by $\sum_{i=1}^{n} (Z_{t_i} - Z_{t_{i-1}})^2 \to t$, meaning that when n increases it stays constant at t. 2.1

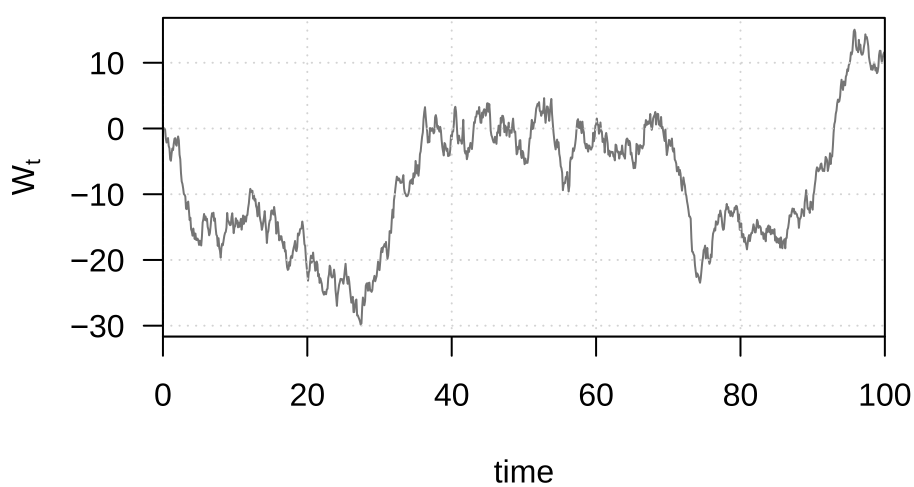

Figure 2.1: A Wiener trajectory path example

2.1.2 Correlated Brownian Motions

Two independent brownian motions that are correlated can describe a new process Z_t. Let W_1 and W_2 be these two *independent* Brownian motions and let $-1 \leq \rho \leq 1$ be a given number. For $0 \leq t \leq T$ define the new process Z_t as [1]:

$$Z_t = \rho W_{1,t} + \sqrt{1-\rho^2} W_{2,t} \tag{2.1}$$

This equation is a linear combination of independent normals at each timestep t, so Z_t is normally distributed. It is proven that Z is a Brownian motion and that Z and $W_{1,t}$ are correlated [1].

2.1.3 Arithmetic Brownian Motion

The arithmetic brownian motion is defined in literature as being a random process (S) defined as follows [1]:

$$dS_t = \mu dt + \sigma dB_t \tag{2.2}$$

or in integral form:

$$\int_{t=0}^{T} dS_t = \int_{t=0}^{T} \mu dt + \int_{t=0}^{T} \sigma dB_t \tag{2.3}$$

Where, μ and σ are known and constant with $\sigma > 0$. In this process, both the drift μ and the diffusion σ coefficient are constant. The expected value of this process is the sum of the initial value and the drift times the elapsed period $(S_0 + \mu T)$. The variance is described by $\sigma^2 T$

2.1.4 Geometric Brownian Motion

A stochastic process S_t is a geometric brownian motion if its solution is described by the solution of the following stochastic differential equation [4,5].

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.4}$$

for given constants $\mu \in \mathbb{R}$ and $\sigma > 0$. Also, the assumed initial value is positive, $S_0 > 0$.

This process is used quite often in finance to model the dynamics of some assets becaus of its properties. It has independent multiplicative increments and its solution is presented below[6]:

$$S_t = S_0 \times exp\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t, \quad t > 0 \tag{2.5}$$

2.1.5 Itô's Calculus

Let X_t be a real-valued stochastic process that satisfies [4,7,8]:

$$S_t = S_0 + \int_0^t \mu_t dt + \int_0^t \sigma_t dW_t \tag{2.6}$$

for some μ_t, σ_t and $t \in [0, T]$. This equation is often rewritten in its differential stochastic form:

$$dS_t = \mu_t dt + \sigma_t dW_t \tag{2.7}$$

for $0 \leq t \leq T$.

Theorem 2.1.1 (Itô's Lemma)**.** *Assume that S_t has a stochastic differential given by:*

$$dS_t = \mu_t dt + \sigma_t dW_t \tag{2.8}$$

for μ_t, σ_t and $t \in [0, T]$. Assume $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous.

$$Y_t := u(S_t, t)$$

Then Y has the following stochastic differential:

$$\begin{aligned} dY_t &= \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dS_t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2 dt \\ &= \left(\frac{\partial u}{\partial t} + \mu_t \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2\right) dt + \sigma_t \frac{\partial u}{\partial x} dW_t \end{aligned} \tag{2.9}$$

where the argument of u, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ above is (S_t, t) .

Getting to the Formula

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_t \tag{2.10}$$

If S were deterministic, dS_t/S_t would be the derivative of $\ln(S_t)$ with respect to S. This suggests to find an expression for the stochastic differential of $\ln(S_t)$, a function of the single random variable S_t.

$$
\begin{aligned}
df(t,S) &= \cancelto{0}{\frac{\partial f}{\partial t}dt} + \frac{\partial f}{\partial S}dS + \cancelto{0}{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + \cancelto{0}{\frac{\partial^2 f}{\partial t \partial S}dtdS} && (2.11)\\
f(t,S) &= \ln(S) && (2.12)\\
d\ln(S) &= \frac{d\ln(S)}{dS}dS + \frac{1}{2}\frac{d^2\ln(S)}{dS^2}(dS)^2 && (2.13)\\
d\ln(S) &= \frac{1}{S}(\mu S dt + \sqrt{V} S dB) + \frac{1}{2}\frac{-1}{S^2}\sqrt{V}^2 S^2 dt && (2.14)\\
d\ln(S) &= \left(\mu - \frac{1}{2}V\right)dt + \sqrt{V}dB && (2.15)\\
\int_u^T d\ln(S) &= \int_u^T \left(\mu - \frac{1}{2}V_t\right)dt + \int_u^T \sqrt{V_t}dB_t && (2.16)\\
\ln(S_t) - \ln(S_u) &= \int_u^T \left(\mu - \frac{1}{2}V_t\right)dt + \int_u^T \sqrt{V_t}dB_t && (2.17)\\
\ln\left(\frac{S_t}{S_u}\right) &= \mu(t-u) - \frac{1}{2}\int_u^T V_t dt + \int_u^T \sqrt{V_t}dB_t && (2.18)\\
\ln\left(\frac{S_t}{S_u}\right) &= \mu(t-u) - \frac{1}{2}\int_u^T V_t dt + \int_u^T \sqrt{V_t}\left(\rho dB_{1,s} + \sqrt{1-\rho^2}dB_{1,s}\right) && (2.19)\\
S_t &= S_u \times \exp\left(\mu(t-u) - \frac{1}{2}\int_u^T V_t dt + \rho\int_u^T \sqrt{V_t}dB_{1,t} + \sqrt{1-\rho^2}\int_u^T \sqrt{V_t}dB_{2,t}\right) && (2.20)
\end{aligned}
$$

Equation (2.9) is the stochastic equivalent to the chain rule, also known as Itô's formula or Itô's chain rule. The proof to this theorem is based on the Taylor expansion of the function $f(S_t, t)$ [4,7]. For practical uses you should write out a second-order Taylor expansion for the function to be analyzed and apply the 2.1 multiplication table [1].

Table 2.1: Box calculus

	dt	dW_t
dt	0	0
dW_t	0	dt

2.2 Stochastic Differential Equations (SDE's)

2.3 Black-Scholes Model

The Black-Scholes (B-S) model arises from the need to price european options in the derivative markets. Derivatives are financial instruments traded in the market, stock exchange or over-the-counter (OTC) market, whose values depend on the values of an underlying asset. [9–11]

- A call option is a derivative that gives its bearer the right, but not the obligation, to purchase a specific asset by a fixed price before or on a given date.
- A put option is a derivative that gives its bearer the right, but not the obligation,

to sell a specific asset by a fixed price before or on a given date.

The trading price of the option is called the option *premium* and the asset from which the option derives is called the *underlying asset*. This asset may be the interest rate, exchange rates, stock exchanges rates, commodities or stocks. The fixed price in contract in which the underlying asset might to be bought or sold is the *strick price*. The option expiration date is called the *maturity*. [10,11]

There are two major different option types: the European and the American. The difference between these two is that the bearer of the first may exercise it only at the end of its life, at its maturity while the latter can be exercised at any given time until its maturity. [11,12]

2.3.1 The model

The Black-Scholes model that provides analytical solution to the price of a European call at time t can be described as follows[3,9,11]:

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \tag{2.21}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{2.22}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{2.23}$$

Where:

- S_t is the spot price of the underlying asset at time t
- r is the risk free rate (generally an annual rate)[2]
- σ is the volatility of returns of the underlying asset [3]
- $N(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution
- K is the strike price
- $T - t$ is the time to maturity

Also, the stock price path is a Geometric Brownian Motion and is under the risk-neutral measure with the following dynamics [3,13]:

$$dS_t = (r - q)S_t d_t + \sigma S_t dW_t \tag{2.24}$$

Where dW_t is a Wiener process [11,13], r is the risk free rate and q is the dividend yield[4] and t denotes the current point in time.

Although the Black-Scholes is very popular and the *de facto* standard in the market there are implications to the B-S model assumptions that affect the results and that are unrealistic. The main assumption that does not hold up is the deterministic (constant) volatility, that can more accurately be described as a stochastic process

[2] Assumed to be constant.

[3] See footnote 1.

[4] r and q are assumed to be constant.

since we observe that small moves usually are followed by small moves and large moves by large moves. [3,9]

Other assumptions that are critical to the B-S model and are not always observed in practice refer to the asset's continuity through time (no jumps), being allowed to perform continuous hedge without transactions costs and normal (Gaussian) returns.

Most models focus on the volatility problem because transaction costs often translate to rises in volatility and fat-tails (abnormal) returns can be simulated by stochastic volatility and market or volatility jumps.

2.3.2 Limitations

2.4 Stochastic Volatility models

Introducing stochastic volatility to models brings complexity, but enables modeling some features observed in reality that are crucial like the randomic market volatility effects, skewness (market returns are more realistically modeled) and volatility smile. This kind of model is applied highly succesfully in foreign exchange and credit markets.

2.4.1 Other models (PRESENT MODELS PREVIOUSLY USED)

2.4.2 Cox-Ingersoll-Ross model

The Cox-Ingersoll-Ross (CIR) model is a well-known short-rate model that describes the interest rate movements driven by one source of market risk. The dynamics are described as follows[14,15]:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dB_t \tag{2.25}$$

Where, r_t is the short rate interest described by parameters k - the speed of mean reversion, θ - the long-run mean variance and σ - the volatility of the variance process.

This model has been widely used to describe the dynamics of the short rate interest because it has some fundamental features like intuitive parametrization, nonnegativity and pricing formulas. Besides, it takes account of anticipations, risk aversion, investment alternatives and preferences about consumption timing and allows for detailed predictions about how changes in a wide range of underlying variables affect the term structure[14]. Furthermore, this equation constitutes one of the two Heston model equations with the volatility taking the short rate interest place.

2.4.3 Heston Model

Heston model solves the deterministic volatility problems introducing the following equations, which represents the dynamics of the stock price and the variance processes under the risk-neutral measure [15,16]:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^* \\
dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t} dB_t
\end{aligned}
\tag{2.26}
$$

The second equation, as previously described, is the CIR model equation. The first equation states the asset price process. μ is the asset's rate of return, $dW_{t,1}$ and $dW_{t,2}$ are two correlated wiener processes with correlation coefficient of ρ. Because, of the model specifications and what we presentend in section 2.1.2, we can rewrite the first equation as in Broadie and Kaya [17]:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \rho\sqrt{V_t} dB_t + \sqrt{1-\rho^2}\sqrt{V_t} S_t dW_t \\
dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t} dB_t
\end{aligned}
\tag{2.27}
$$

Chapter 3

The Heston Model Implementation

Chapter 4

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Chapter 5

Conclusion

Chapter 6

The First Appendix

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