

# Numerical methods for stochastic volatility models: Heston model

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I want to thank a few people.



# Table of Contents

<b>Chapter 1: altadvisor: ‘Your Other Advisor’</b>	<b>1</b>
<b>Chapter 2: Literature Review</b>	<b>3</b>
2.1 Stochastic Calculus	3
2.1.1 The Stochastic differential equation - SDE	3
2.1.2 Brownian Motion	4
2.1.3 Itô’s Integral	6
2.2 Black-Scholes Model	7
2.2.1 Basics	7
2.2.2 The model	9
2.2.3 Limitations	10
2.3 Stochastic Volatility models	10
2.3.1 Other models (PRESENT MODELS PREVIOUSLY USED)	10
2.3.2 Cox-Ingersoll-Ross model	10
2.3.3 Heston Model	10
2.4 Numerical Methods	11
2.4.1 Convergence ??	11
2.4.2 Euler-Maruyama	11
2.4.3 Milstein	11
2.4.4 Stability	11
<b>Chapter 3: The Heston Model Implementation</b>	<b>13</b>
<b>Chapter 4: This chunk ensures that the thesisdown package is</b>	<b>15</b>
<b>Chapter 5: Conclusion</b>	<b>17</b>
<b>Chapter 6: The First Appendix</b>	<b>19</b>
<b>References</b>	<b>21</b>





# List of Tables

2.1	Box calculus . . . . .	7
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# List of Figures

2.1	A Wiener process trajectory path example . . . . .	5
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# Abstract

The preface pretty much says it all.

Second paragraph of abstract starts here.



# Dedication

You can have a dedication here if you wish.





# Chapter 1

**altadvisor: ‘Your Other Advisor’**



# Chapter 2

## Literature Review

This chapter presents the concepts of stochastic calculus, from the historic conception of how it first arose through the basic principles and applications in finance. We address with more care the classical Black-Scholes model and its limitations and the Heston model. This model is also well known, it brings the concept of stochastic volatility in it, which brings its results closer to reality.

### 2.1 Stochastic Calculus

Stochastic calculus arises from stochastic processes and allows the creation of a theory of integration where both the integrand and integrator terms are stochastic processes. Stochastic calculus was created by the Japanese mathematician Kiyosi Itô <sup>1</sup> in the 1940s and 1950s and is used for modeling financial options and in another wide variety of fields [1]. In this chapter we present the historical contexts in which the tools and models used arise, but our focus is introducing the concepts and notations that will be further used in our work.

#### 2.1.1 The Stochastic differential equation - SDE

At first, before introducing stochastic differential equation, it is helpful to start with ordinary differential equation. Let  $x(t) = x_t$  denote the population at time  $t$  so that the change in the population at time  $t$  is given by the following deterministic differential equation:

$$\begin{aligned} dx_t &= f(t, x_t)dt \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

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<sup>1</sup>There is another important stochastic integral, called the *Stratonovich Integral* that unlike the Itô's integral, respects the conventional calculus chain rule. Also, the integral is evaluated at the interval's midpoint, instead of its left extreme. A Stratonovich integral can be expressed as an Itô integral and vice versa.

We now add a “noise” to this equation:

$$dx_t = \underbrace{f(t, x_t) dt}_{\text{drift}} + \underbrace{g(t, x_t) dW_t}_{\text{diffusion}} \quad (2.2)$$

$$x(0) = x_0$$

This “noise”  $dW_t$  is a *random* Wiener process time derivative (which will be clarified below) and  $X_0$  is our initial value.

The  $g(t, x_t)$  part of the SDE is often referred as a *diffusion process*. These processes generally have a continuous paths. Before moving on, we must carefully define what the term *random*, means and the best way to begin doing so is to precisely define a probability space:

**Definition 2.1.1.** (Probability Space) A triple  $(\Omega, \mathcal{U}, \mathcal{P})$  is called a *probability space* provided  $\Omega$  is any set,  $\mathcal{U}$  is a  $\sigma$ -algebra of subsets of  $\Omega$  and  $\mathcal{P}$  is a probability measure on  $\mathcal{U}$ .

## 2.1.2 Brownian Motion

The Brownian motion is the name given to the irregular motion observed in the motion of pollen particles suspended in fluid resulting from particle collision with atoms or molecules. It is named after Robert Brown, the first to have observed the movement in 1828. He noted two characteristic in the pollen movement [1]:

- the path of a given particle is very irregular, having a tangent at no point
- the motion of two distinct particles appear to be independent

The first quantitative works in brownian motion come from an interest in stock price fluctuation by Bachelier in 1900. Albert Einstein also leaned over the subject and in 1905 derived the transition density for Brownian motion from molecular-kinetic theory of heat [1,2].

In 1923, the Wiener process was coined in honor of Norbert Wiener mathematical proof of existence of the brownian motion and stating its properties.<sup>2</sup>

**Definition 2.1.2.** (Wiener Process) Given a probability space  $(\Omega, \mathcal{U}, \mathcal{P})$ , a stochastic process  $W_t$  defined in this space is a *wiener process* if it satisfies the following properties:

- $W_0 = 0$
- The change in  $W$ , given by  $\Delta W = W_{t+1} - W_t$ , is normally distributed with mean zero and standard deviation  $\sqrt{\Delta t}$ , meaning that  $\Delta W = \epsilon\sqrt{\Delta t}$ , where  $\epsilon$  is  $N(0, 1)$ .
- If the increment  $\Delta t_1$  does not overlap with the time increment  $\Delta t_2$ , then  $\Delta W_1$  and  $\Delta W_2$  are independent.

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<sup>2</sup>More can be found on [3-5].

- The process is continuous, meaning that there are no jumps in the process.
- The process is a Markov process. This means that the conditional expectation of  $W_{t+1}$  given its entire history is equal to the conditional expectation of  $W_{t+1}$  given today's information. This can be written as:  $E[W_{t+1}|W_1, \dots, W_t] = E[W_{t+1}|W_t]$ .
- Consider the time interval  $[0, t]$  with  $n$  equally spaced intervals given by  $t_i = \frac{it}{n}$ . Then the paths of the Brownian motion have unbounded variation, this means that they are not differentiable and go towards infinity as  $n$  increases. The quadratic variation is given by  $\sum_{i=1}^n (Z_{t_i} - Z_{t_{i-1}})^2 \rightarrow t$ , meaning that when  $n$  increases it stays constant at  $t$ .

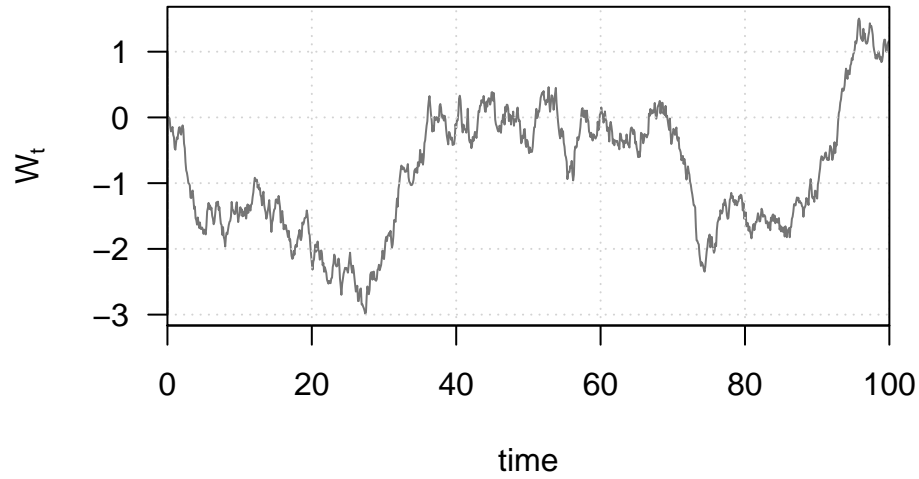


Figure 2.1: A Wiener process trajectory path example

### Correlated Brownian Motions

Two independent brownian motions that are correlated can describe a new process  $Z_t$ . Let  $W_1$  and  $W_2$  be these two *independent* Brownian motions and let  $-1 \leq \rho \leq 1$  be a given number. For  $0 \leq t \leq T$  define the new process  $Z_t$  as [1]:

$$Z_t = \rho W_{1,t} + \sqrt{1 - \rho^2} W_{2,t} \quad (2.3)$$

This equation is a linear combination of independent normals at each timestep  $t$ , so  $Z_t$  is normally distributed. It is proven that  $Z$  is a Brownian motion and that  $Z$  and  $W_{1,t}$  are correlated [1].

### 2.1.3 Itô's Integral

Formally, the SDE presented in equation 2.4 only exists because we can rewrite it in the form [4–9]:

$$x_t = x_0 + \int_0^t f(s, x_s) ds + \int_0^t g(s, x_s) dW_s \quad (2.4)$$

for some  $f(s, x_s)$ ,  $g(s, x_s)$  and  $s \in [0, t]$ .

The Itô integral can, as the Riemann integral, be approximated by a finite sum. Also, it has a definition as a certain limit. Itô's lemma 2.1.1 plays the same role as the fundamental theorem of calculus in allowing to evaluate integrals. It is the formal definition and presents an extra term not encountered in the conventional calculus theorem that is due to the non-smoothness characteristics of Brownian motion paths. It is possible, though, to define the integral in a less rigorous way:

$$Y_{\Delta t}(t) \approx \sum_{t_k < t} g(t_k) \Delta W_k \quad (2.5)$$

with the usual notions  $t_k = k\Delta t$ , and  $\Delta W_k = W(t_{k+1}) - W(t_k)$ . If the limit exists, then the Ito integral is:

$$Y(t) = \lim_{\Delta t \rightarrow 0} Y_{\Delta t}(t) \quad (2.6)$$

It is essential that the *forward difference* is used rather than the backward difference, which would be **wrong**.

**Theorem 2.1.1** (Itô's Lemma). *Assume that  $S_t$  has a stochastic differential given by:*

$$dS_t = \mu_t dt + \sigma_t dW_t \quad (2.7)$$

*for  $\mu_t$ ,  $\sigma_t$  and  $t \in [0, T]$ . Assume  $u : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$  is continuous and that  $\frac{\partial u}{\partial t}$ ,  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial^2 u}{\partial x^2}$  exist and are continuous.*

$$Y_t := u(S_t, t)$$

*Then  $Y$  has the following stochastic differential:*

$$\begin{aligned} dY_t &= \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dS_t + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma_t^2 dt \\ &= \left( \frac{\partial u}{\partial t} + \mu_t \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma_t^2 \right) dt + \sigma_t \frac{\partial u}{\partial x} dW_t \end{aligned} \quad (2.8)$$

*where the argument of  $u$ ,  $\frac{\partial u}{\partial x}$  and  $\frac{\partial^2 u}{\partial x^2}$  above is  $(S_t, t)$ .*

Equation (2.8) is the stochastic equivalent to the chain rule, also known as Itô's formula or Itô's chain rule. The proof to this theorem is based on the Taylor expansion of the function  $f(S_t, t)$  [4,6]. For practical uses you should write out a second-order Taylor expansion for the function to be analyzed and apply the 2.1 multiplication table [1].

Table 2.1: Box calculus

	$dt$	$dW_t$
$dt$	0	0
$dW_t$	0	$dt$

**Theorem 2.1.2** (Itô's Integral Properties). *Let  $f, g \in \mathcal{V}$  and let  $0 \leq t_0 < u < T$ . Then*

$$(i) \int_{t_0}^T f dB_t = \int_{t_0}^u f dB_t + \int_u^T f dB_t$$

$$(ii) \int_{t_0}^T (\alpha f + \beta g) dB_t = \alpha \int_{t_0}^T f dB_t + \int_{t_0}^T \beta g dB_t$$

$$(iii) \mathbb{E} \left[ \int_{t_0}^T f dB_t \right] = 0$$

$$(iv) \mathbb{E} \left[ \left( \int_0^t H_s dB_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t H_s^2 ds \right] \text{ (Isometry)}$$

$$(v) \mathbb{E} \left[ \int_{t_0}^T f dB_t \mid \mathcal{F}_s \right] = \int_{t_0}^s f dB_t, \quad \text{for } s < T. \text{ (Martingale}^3\text{)}$$

## 2.2 Black-Scholes Model

### 2.2.1 Basics

Before introducing the Black-Schole model...

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The Black-Scholes (B-S) model arises from the need to price european options in the derivative markets. Derivatives are financial instruments traded in the market, stock exchange or over-the-counter (OTC) market, whose values depend on the values of an underlying asset. [10–12]

- A call option is a derivative that gives its bearer the right, but not the obligation, to purchase a specific asset by a fixed price before or on a given date.
- A put option is a derivative that gives its bearer the right, but not the obligation, to sell a specific asset by a fixed price before or on a given date.

The trading price of the option is called the option *premium* and the asset from which the option derives is called the *underlying asset*. This asset may be the interest rate, exchange rates, stock exchanges rates, commodities or stocks. The fixed price in

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<sup>3</sup>A martingale is a stochastic process with certain characteristics. The main one is that the expected value in time  $t + 1$  for  $X$  is the  $X$  value in  $t$ . This means there are no winning strategies when we are dealing with martingales (unlike when we play poker, for example). A Wiener process is a martingale.

contract in which the underlying asset might to be bought or sold is the *strick price*. The option expiration date is called the *maturity*. [11,12]

There are two major different option types: the European and the American. The difference between these two is that the bearer of the first may exercise it only at the end of its life, at its maturity while the latter can be exercised at any given time until its maturity. [12,13]

**Definition 2.2.1.** (Implicit volatility) Given all the option's parameters in a pre-cification model and its market price, the option's volatility is called the \*implicit volatility\*.

**Definition 2.2.2.** (Intrinsic value) The intrinsic value of a call is the difference between the underlying asset price and the strike price. The put's intrinsic value operates the other way around, being the difference between the strike and the underling asset prices.

### Arithmetic Brownian Motion

The arithmetic brownian motion is defined in literature as being a random process (S) defined as follows [1]:

$$dS_t = \mu dt + \sigma dB_t \quad (2.9)$$

or in integral form:

$$\int_{t=0}^T dS_t = \int_{t=0}^T \mu dt + \int_{t=0}^T \sigma dB_t \quad (2.10)$$

Where,  $\mu$  and  $\sigma$  are known and constant with  $\sigma > 0$ . In this process, both the drift  $\mu$  and the diffusion  $\sigma$  coefficient are constant. The expected value of this process is the sum of the initial value and the drift times the elapsed period ( $S_0 + \mu T$ ). The variance is described by  $\sigma^2 T$

### Geometric Brownian Motion

A stochastic process  $S_t$  is a geometric brownian motion if its solution is described by the solution of the following stochastic differential equation [6,14].

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (2.11)$$

for given constants  $\mu \in \mathbb{R}$  and  $\sigma > 0$ . Also, the assumed initial value is positive,  $S_0 > 0$ .

This process is used quite often in finance to model the dynamics of some assets becaus of its properties. It has independent multiplicative increments and its solution is presented below[15]:

$$S_t = S_0 \times \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t, \quad t > 0 \quad (2.12)$$



### 2.2.2 The model

The Black-Scholes model that provides analytical solution to the price of a European call at time  $t$  can be described as follows[3,10,12]:

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \quad (2.13)$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[ \ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right] \quad (2.14)$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (2.15)$$

Where:

- $S_t$  is the spot price of the underlying asset at time  $t$
- $r$  is the risk free rate (generally an annual rate)<sup>4</sup>
- $\sigma$  is the volatility of returns of the underlying asset <sup>5</sup>
- $N(\cdot)$  is the cumulative distribution function of the standard Gaussian distribution
- $K$  is the strike price
- $T - t$  is the time to maturity

Also, the stock price path is a Geometric Brownian Motion and is under the risk-neutral measure with the following dynamics [3,16]:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \quad (2.16)$$

Where  $dW_t$  is a Wiener process [12,16],  $r$  is the risk free rate and  $q$  is the dividend yield<sup>6</sup> and  $t$  denotes the current point in time.

Although the Black-Scholes is very popular and the *de facto* standard in the market there are implications to the B-S model assumptions that affect the results and that are unrealistic. The main assumption that does not hold up is the deterministic (constant) volatility, that can more accurately be described as a stochastic process since we observe that small moves usually are followed by small moves and large moves by large moves. [3,10]

Other assumptions that are critical to the B-S model and are not always observed in practice refer to the asset's continuity through time (no jumps), being allowed to perform continuous hedge without transactions costs and normal (Gaussian) returns.

Most models focus on the volatility problem because transaction costs often translate to rises in volatility and fat-tails (abnormal) returns can be simulated by stochastic volatility and market or volatility jumps.

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<sup>4</sup>Assumed to be constant.

<sup>5</sup>See footnote 1.

<sup>6</sup> $r$  and  $q$  are assumed to be constant.

### 2.2.3 Limitations

## 2.3 Stochastic Volatility models

Introducing stochastic volatility to models brings complexity, but enables modeling some features observed in reality that are crucial like the random market volatility effects, skewness (market returns are more realistically modeled) and volatility smile. This kind of model is applied highly successfully in foreign exchange and credit markets.

### 2.3.1 Other models (PRESENT MODELS PREVIOUSLY USED)

Langevin? OU?

### 2.3.2 Cox-Ingersoll-Ross model

The Cox-Ingersoll-Ross (CIR) model is a well-known short-rate model that describes the interest rate movements driven by one source of market risk. The dynamics are described as follows[17,18]:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dB_t \quad (2.17)$$

Where,  $r_t$  is the short rate interest described by parameters  $k$  - the speed of mean reversion,  $\theta$  - the long-run mean variance and  $\sigma$  - the volatility of the variance process.

This model has been widely used to describe the dynamics of the short rate interest because it has some fundamental features like intuitive parametrization, nonnegativity and pricing formulas. Besides, it takes account of anticipations, risk aversion, investment alternatives and preferences about consumption timing and allows for detailed predictions about how changes in a wide range of underlying variables affect the term structure[17]. Furthermore, this equation constitutes one of the two Heston model equations with the volatility taking the short rate interest place.

### 2.3.3 Heston Model

Heston model solves the deterministic volatility problems introducing the following equations, which represents the dynamics of the stock price and the variance processes under the risk-neutral measure [18,19]:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^* \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t}dB_t \end{aligned} \quad (2.18)$$

The second equation, as previously described, is the CIR model equation. The first equation states the asset price process.  $\mu$  is the asset's rate of return,  $dW_{t,1}$  and  $dW_{t,2}$  are two correlated wiener processes with correlation coefficient of  $\rho$ . Because, of the

model specifications and what we presentend in section 2.1.2, we can rewrite the first equation as in Broadie and Kaya [20]:

$$\begin{aligned}dS_t &= \mu S_t dt + \rho \sqrt{V_t} dB_t + \sqrt{1 - \rho^2} \sqrt{V_t} S_t dW_t \\dV_t &= k(\theta - V_t) dt + \sigma \sqrt{V_t} dB_t\end{aligned}\tag{2.19}$$

## 2.4 Numerical Methods

### 2.4.1 Convergence ??

### 2.4.2 Euler-Maruyama

### 2.4.3 Milstein

### 2.4.4 Stability



## **Chapter 3**

# **The Heston Model Implementation**



## Chapter 4

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## Chapter 5

## Conclusion



## Chapter 6

### The First Appendix



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