On the Numerical simulation of the Heston model

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You get pseudo-order when you seek order; you only get a measure of order and control when you embrace randomness. — Nassim Nicholas Taleb

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Abstract

In this thesis we revisit numerical methods for the simulation of the Heston model's European call. Specifically, we study the Euler, the Kahl-Jackel an two versions of the exact algorithm schemes. To perform this task, firstly we present a literature review which brings stochastic calculus, the Black-Scholes (BS) model and its limitations, the stochastic volatility methods and why they resolve the issues of the BS model, and the peculiarities of the numerical methods - convergence, discretization and stability. We provide recommendations when we acknowledge that the reader might need more specifics and might need to dive deeper into a given topic. We introduce the methods aforementioned providing all our implementations in R language within a package.

Keywords: Heston, Stochastic, Volatility, Black-Scholes, European call, R

Chapter 1

altadvisor: 'Your Other Advisor'

Chapter 2

Theoretical Framework

This chapter presents the concepts of stochastic calculus, from the historic conception of how it first arose through the basic principles and applications in finance. We address with more care the classical Black-Scholes model and its limitations and the Heston model. This model is also well known, it brings the concept of stochastic volatility in it, which presents results closer to reality.

2.1 Stochastic Calculus

Stochastic calculus arises from stochastic processes and allows the creation of a theory of integration where both the integrand and integrator terms are stochastic processes. Stochastic calculus was created by the Japanese mathematician Kiyosi Itô in the 1940s and 1950s and is used for modeling financial options and in another wide variety of fields [1]. In this chapter we present the historical contexts in which the tools and models are used, but our focus is introducing the concepts and notations that will be further used in our work.

2.1.1 Stochastic differential equation - SDE

At first, before introducing stochastic differential equation, it is helpful to start with ordinary differential equation. Let x_t denote an asset at time t so that the change in the asset at time t is given by the following deterministic differential equation:

$$\begin{aligned} dx_t &= f(t, x_t)dt \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

We now add a "noise" to this equation:

$$\begin{aligned} dx_t &= \overbrace{\underbrace{f(t, x_t)}_{\text{drift}}\, dt}^{\text{deterministic}} + \overbrace{\underbrace{g(t, x_t)}_{\text{diffusion}}\, dW_t}^{\text{random shock}} \\ x(0) &= x_0 \end{aligned} \tag{2.2}$$

This "noise" W_t is a *random* Wiener process (which will be clarified below) and x_0 is our initial value.

The $g(t, x_t)$ part of the SDE is often referred as *diffusion coefficient*. Before moving on, we must carefully define what the term *random* means and the best way to begin doing so is to precisely define a probability space:

Definition 2.1.1 (Probability Space)**.** A triple $(\Omega, \mathcal{U}, \mathcal{P})$ is called a *probability space* provided Ω is any set, $\mathcal{U}$ is a σ-algebra of subsets of Ω and $\mathcal{P}$ is a probability measure on $\mathcal{U}$.

2.1.2 Brownian Motion

The Brownian motion is the name given to the irregular motion observed in the motion of pollen particles suspended in fluid resulting from particle collision with atoms or molecules. It is named after Robert Brown, the first to have observed the movement in 1828. He noted two characteristic in the pollen movement [1]:

- the path of a given particle is very irregular, having a tangent at no point
- the motion of two distinct particles appear to be independent

The first quantitative works in Brownian motion come from an interest in stock price fluctuation by Bachelier in 1900. Albert Einstein also leaned over the subject and in 1905 derived the transition density for Brownian motion from molecular-kinetic theory of heat [1,2].

In 1923, the Wiener process was coined in honor of Norbert Wiener mathematical proof of existence of the Brownian motion and stating its properties.[1]

Definition 2.1.2 (Wiener Process)**.** Given a probability space $(\Omega, \mathcal{U}, \mathcal{P})$, a stochastic process W_t defined in this space is a *Wiener process* if it satisfies the following properties:

- $W_0 = 0$
- The change in W, given by $\Delta W = W_{t+\Delta} - W_t$, is normally distributed with mean zero and standard deviation $\sqrt{\Delta t}$, meaning that $\Delta W = \epsilon\sqrt{\Delta t}$, where ϵ is $N(0, 1)$.
- If the increment Δt_1 does not overlap with the time increment Δt_2, then ΔW_1 and ΔW_2 are independent.
- The process is continuous, meaning that there are no jumps in the path, almost surely.

Furthermore, if a process is a Wiener process, we shall have the following results by construction:

[1] More can be found on [3–5].

- The process is Markovian. This means that the conditional expectation of $W_{t+\Delta t}$ given its entire history is equal to the conditional expectation of $W_{t+\Delta t}$ given today's information. This can be written as: $E[W_{t+\Delta t}|W_t]$.
- Consider the time interval $[0, t]$ with n equally spaced intervals given by $t_i = \frac{it}{n}$. Then the paths of the Brownian motion have unbounded variation, they are not differentiable. The quadratic variation is given by $\sum_{i=1}^{n} (Z_{t_i} - Z_{t_{i-1}})^2 \to t$, meaning that when n increases it stays constant at t.

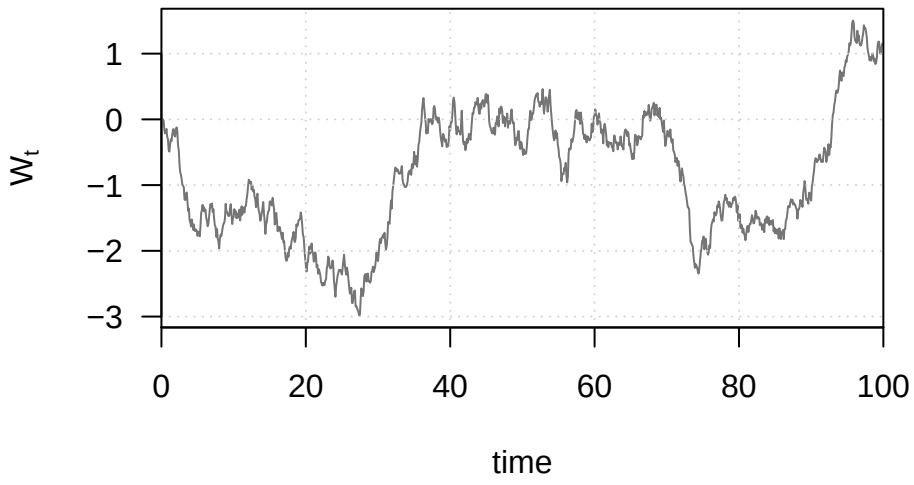

Figure 2.1: A Wiener process trajectory path example

Correlated Brownian Motions

Two independent Brownian motions that are correlated can describe a new process Z_t. Let W_1 and W_2 be these two *independent* Brownian motions and let $-1 \leq \rho \leq 1$ be a given number. For $0 \leq t \leq T$ define the new process Z_t as [1]:

$$Z_t = \rho W_{1,t} + \sqrt{1 - \rho^2} W_{2,t} \tag{2.3}$$

This equation is a linear combination of independent normals at each timestep t, so Z_t is normally distributed. It is proven that Z is a Brownian motion and that Z and $W_{1,t}$ have correlation ρ [1].

2.1.3 Itô's Integral

Formally, the SDE presented in equation (2.2) is interpreted as [4–9]:

$$x_t = x_0 + \int_0^t f(s, x_s) ds + \int_0^t g(s, x_s) dW_s \tag{2.4}$$

for some f, g and $s \in [0,t]$.

The Itô integral can, as the Riemann integral, be approximated by a finite sum [2]. Also, it has a definition as a certain limit. Itô's lemma 2.1.1 plays the same role as the fundamental theorem of calculus in allowing to evaluate integrals. It presents an extra term not encountered in the conventional calculus theorem that is due to the non-smoothness characteristics of Brownian motion paths. It is possible, though, to define the integral as the summation below:

$$Y_{\Delta t}(t) = \sum_{t_k < t} g(t_k)\Delta W_k \tag{2.5}$$

with the usual notions $t_k = k\Delta t$, and $\Delta W_k = W(t_{k+1}) - W(t_k)$. And in a more rigorous form, if the limit exists, then the Ito integral is:

$$Y(t) = \lim_{\Delta t \to 0} Y_{\Delta t}(t) \tag{2.6}$$

It is essential that the *forward difference*, that is $W(t_{k+1}) - W(t_k)$ is used rather than the backward difference $W(t_k) - W(t_{k-1})$, which would be wrong.

Lemma 2.1.1 (Itô's Lemma)**.** *Assume that S_t has a stochastic differential given by:*

$$dS_t = \mu_t dt + \sigma_t dW_t \tag{2.7}$$

for μ_t, σ_t and $t \in [0,T]$ and that $\mathbb{C}^{2,1}(\mathbb{R} \times [0,T])$. Assume $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous.

$$Y_t := u(S_t, t)$$

Then Y has the following stochastic differential:

$$\begin{aligned} dY_t &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dS_t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2 dt \\ &= \left(\frac{\partial u}{\partial t} + \mu_t\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2\right)dt + \sigma_t\frac{\partial u}{\partial x}dW_t \end{aligned} \tag{2.8}$$

where the argument of $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ above is (S_t, t) .

Equation (2.8) is the stochastic equivalent to the chain rule, also known as Itô's formula or Itô's chain rule. The proof to this theorem is based on the Taylor expansion of the function $f(S_t, t)$ [4,6]. For practical use you should write out a second-order Taylor expansion for the function to be analyzed and apply the multiplication table [1] presented in Table 2.1.

[2] There is another important stochastic integral, called the *Stratonovich Integral* that unlike the Itô's integral, respects the conventional calculus chain rule. Also, the integral is evaluated at the interval's midpoint, instead of its left extreme. A Stratonovich integral can be expressed as an Itô integral and vice versa.

	dt	dW_t
dt	0	0
dW_t	0	dt

Table 2.1: Box calculus

Itô's Integral Properties

Let f, $g \in \mathcal{V}$ and let $0 \leq t_0 < u < T$. Then

(i) $\int_{t_0}^{T} f dB_t = \int_{t_0}^{u} f dB_t + \int_{u}^{T} f dB_t$

(ii) $\int_{t_0}^{T} (\alpha f + \beta g) dB_t = \alpha \int_{t_0}^{T} f dB_t + \beta \int_{t_0}^{T} g dB_t$

(iii) $\mathbb{E}\left[\int_{t_0}^{T} f dB_t\right] = 0$

(iv) $\mathbb{E}\left[\left(\int_0^t H_s \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, ds\right]$ (Isometry)

(v) $\mathbb{E}\left[\int_{t_0}^{T} f dB_t \mid \mathcal{F}_s\right] = \int_{t_0}^{s} f dB_t, \qquad for\ s < T.$ (Martingale[3])

2.2 Black-Scholes Model

2.2.1 Basics

The Black-Scholes (B-S) model arises from the need to price european options in the derivative markets. Derivatives are financial instruments traded in the market, stock exchange or over-the-counter (OTC) market, whose values depend on the values of an underlying asset. [10–12]

- A call option is a derivative that gives its bearer the right, but not the obligation, to purchase a specific asset by a fixed price before or on a given date.
- A put option is a derivative that gives its bearer the right, but not the obligation, to sell a specific asset by a fixed price before or on a given date.

The trading price of the option is called the option *premium* and the asset from which the option derives is called the *underlying asset*. This asset may be interest rates, exchange rates, stock exchanges indices, commodities or stocks. The fixed price in

[3] A martingale is a stochastic process with certain characteristics. The main one is that the expected value in time $t + \Delta t$ for X is the X value in t. This means there are no winning strategies when we are dealing with martingales (unlike when we play poker, for example). A Wiener process is a martingale.

contract in which the underlying asset might to be bought or sold is the *strick price*. The option expiration date is called the *maturity*. [10,12]

There are two major different option types: European and American. The difference between these two is that the bearer of the first may exercise it only at the end of its life, at its maturity while the latter can be exercised at any given time until its maturity. [10,13]

Geometric Brownian Motion

A stochastic process S_t is a Geometric Brownian Motion[4] if it is described by the solution of the following stochastic differential equation [1,6,14].

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.9}$$

for given constants $\mu \in \mathbb{R}$ and $\sigma > 0$. Also, the assumed initial value is positive, $S_0 > 0$.

Figure 2.2 shows the GBM,[5] which is quite often applied to model the dynamics of some financial assets because of its properties [15]. Equation (2.10) shows the formula to generate a GBM and we provide proof of this solution in appendix 6[6]

$$S_t = S_0 \times exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad t > 0 \tag{2.10}$$

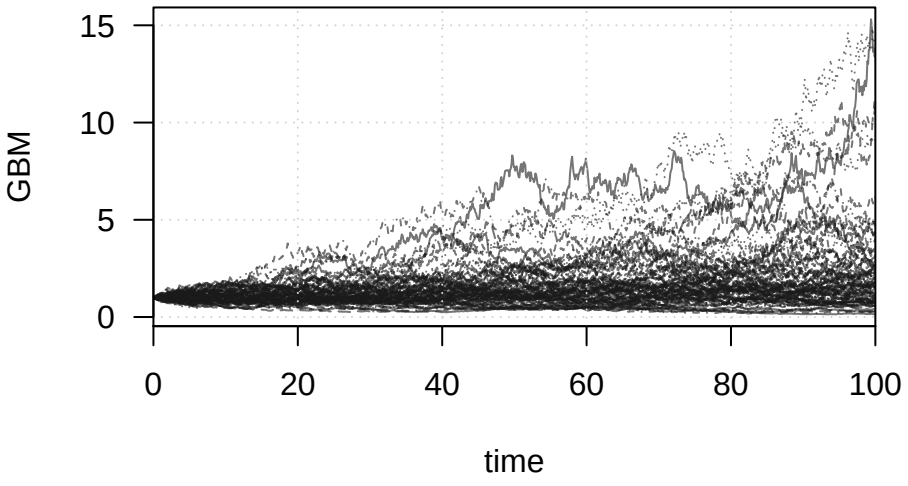

Figure 2.2: A GBM trajectory path example

[4]There is an Arithmetic Brownian Motion: $dS_t = \mu dt + \sigma dB_t$. More information can be obtained in [1].

[5]Also known as exponential Brownian motion.

[6]An intuitive proof can be found at [16].

The Black-Scholes model provides analytical solution to the price of a European call at time t and can be described as follows [3,10,11]:

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \tag{2.11}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{2.12}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{2.13}$$

Where:

- S_t is the spot price of the underlying asset at time t
- r is the risk free rate (generally an annual rate)[7]
- σ is the volatility of returns of the underlying asset [8]
- $N(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution
- K is the strike price
- $T - t$ is the time to maturity

Also, the stock price path is a Geometric Brownian Motion as previously stated, and is under the risk-neutral measure with the following dynamics [3,17]:

$$dS_t = (r - q)S_t d_t + \sigma S_t dW_t \tag{2.14}$$

Here W_t is a Wiener process [10,17], r is the risk free rate and q is the dividend yield[9] and t denotes the current point in time.

2.2.2 Limitations

Although the Black-Scholes is very popular and the *de facto* standard in the market there are implications to the B-S model assumptions that affect the results and that are unrealistic. The main assumption that does not hold up is the deterministic (constant) volatility, that can more accurately be described as a stochastic process since we observe that small moves usually are followed by small moves and large moves by large moves. [3,11]

Other assumptions that are critical to the B-S model and are not always observed in practice refer to the asset's continuity through time (no jumps), being allowed to perform continuous hedge without transactions costs and normal (Gaussian) returns.

Most models focus on the volatility problem because transaction costs often implies rises in volatility and that fat-tails (abnormal) returns can be simulated by stochastic volatility and market or volatility jumps.

[7] Assumed to be constant.

[8] See footnote 1.

[9] r and q are assumed to be constant.

2.3 Stochastic Volatility models

Introducing stochastic volatility to models brings complexity, but enables modeling some features observed in reality that are crucial, like the randomic market volatility effects, skewness (market returns are more realistically modeled) and volatility smile[10] (see Figure 2.3). This kind of model is applied highly succesfully in foreign exchange and credit markets.

Definition 2.3.1 (Volatility Smile)**.** Volatility smiles are implied volatility patterns that arise in pricing financial options. In particular for a given expiration, options whose strike price differs substantially from the underlying asset's price command higher prices (and thus implied volatilities) than what is suggested by standard option pricing models. These options are said to be either deep in-the-money or out-of-the-money.

Furthermore, stochastic volatility models use statistical methods as foundations to price and forecast options' behaviors and the underlying's security volatility is arbitrary. The Heston, the 3/2 and other models, like the GARCH[11] and SABR,[12] are considered standard smile models.

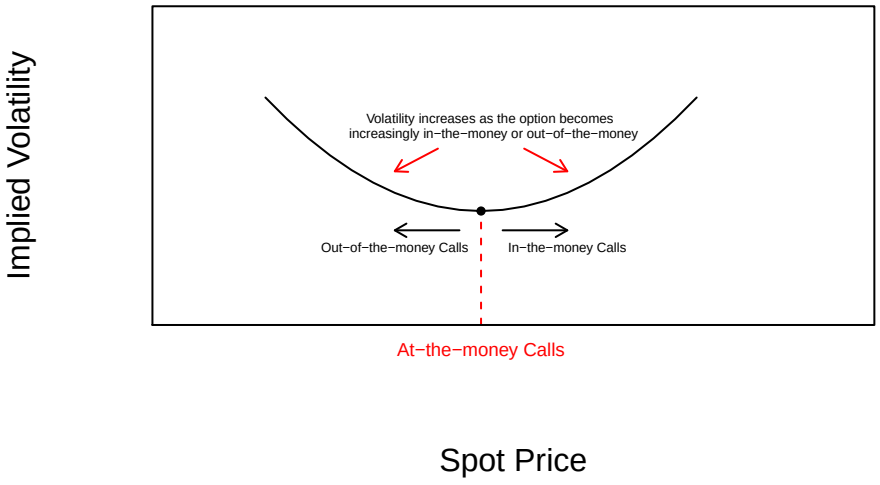

Figure 2.3: Volatility Smile

[10] The name derives from the concave shape of the graph, which resembles a smile.
[11] generalized autoregressive conditional heteroscedasticity. [18]
[12] stochastic alpha, beta, rho. [20]

2.3.1 Cox-Ingersoll-Ross model

The Cox-Ingersoll-Ross (CIR) model is a well-known short-rate model that describes the interest rate movements driven by one source of market risk. The dynamics are described as follows [21,22]:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dB_t \tag{2.15}$$

Where, r_t is the short rate interest described by parameters κ the speed of mean reversion, θ the long-run mean variance and σ the volatility of the variance process.

This model has been widely used to describe the dynamics of the short rate interest because it has some fundamental features like intuitive parametrization, nonnegativity and pricing formulas. Besides, it takes account of anticipations, risk aversion, investment alternatives and preferences about consumption timing and allows for detailed predictions about how changes in a wide range of underlying variables affect the term structure[21]. Furthermore, this equation constitutes one of the two Heston model equations with the volatility taking the short rate interest place.

2.3.2 Heston Model

Heston model was introduced in 1993 by Steven Heston to solve the deterministic volatility problems. It was designed to analize bond and currency options and it introduced the following equations, which represent the dynamics of the stock price and the variance processes under the risk-neutral measure [22,23]:

$$dS_t = \mu S_t dt + \sqrt{V_t}S_t dW_t^* \tag{2.16}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dB_t \tag{2.17}$$

The second equation, as described in Section 2.3.1, is the CIR model equation. The first equation states the asset price process. μ is the asset's rate of return, $dW_{t,1}$ and $dW_{t,2}$ are two correlated wiener processes with correlation coefficient of ρ.

2.4 Numerical Methods

Numerical methods are tools that are often applied to solve stochastic differential equations because most of these do not have explicit solution. This means that we are not able to solve these equations using symbolic computation. Although we are unable to find an analitical solution, when facing real problems, the approximation given by a numerical method is often sufficient. Alongside the analytical issue, the need to calculate the SDE's trajectory through time is the main reason why studying numerical methods is so important. An implementation of a numerical method is called a numerical algorithm.

We will simulate sample paths of time discrete approximations implemented in the R programming language [24] that we base on a finite discretization of a time interval $[t_0, T]$. We shall generate approximate values of the sample path for each step contained in the discretized interval [25].

In the fixed step methods, the distance between two contiguous points is $d_i = t_i - t_{i-1} = \frac{T-t_0}{N} \quad i \leq N \in \mathbb{N}$. N being the time interval partition number.

According to Kloeden [25], in the stochastic realm, simulated sample paths can be statistically analysed to find how good an approximation is compared to an exact solution. Moreover, the computational costs such as time and memory increases polynomially with the problem's dimension, which is good, and it is possible to apply variance reduction methods that allow a considerable decrease in the required sample size.

2.4.1 Convergence

As soon as we talk about numerical methods we are required to approach the topic of approximations and how to handle them. Methods efficiency receive the name of *convergence order*. In the SDE domain there are two main methods of convergence, that are classified according to their criteria. Firstly, we present the *strong order of convergence*. A method is said to have strong convergence δ to Y if a time discretized Y_δ of a continous-time process Y, with δ being the maximum time increment of the discretization, and for any fixed time horizon T holds true that [15]:

$$\mathbb{E} \mid Y_\delta(T) - Y(T) \mid \leq C\delta^\gamma, \ \ \forall \delta < \delta_0$$

with $\delta_0 > 0$ and $\mathcal{C}$ a constant not depending on δ. Strong convergence addresses the problem of solutions' trajectories. For specific conditions, the Euler method has strong convergence order $\gamma = \frac{1}{2}$. Furthermore, there is the *weak order of convergence*:

$$\mid \mathbb{E}p(Y_n) - \mathbb{E}p(Y(\tau)) \mid \leq C\Delta t^\gamma$$

Strong and weak convergence are not mutually exclusive [15]. That means that a method with a given strong order of convergence might have a higher weak order of convergence too. This is the case for the Euler scheme, with a strong order of convergence of 1/2 and a weak order of 1 (under some conditions). For a more detailed and rigorous explanation of convergence we recommend consulting [26].

It is worth noting that, altough schemes have a given convergence order, it is not unusual that they behave better than their order for some SDEs specifications.

2.4.2 Discretization

We know that convergence is an important feature to a numerical method and studies have found not all time discrete possible approximations of an SDE converge in a useful sense to the solution process as the step size adopted tends toward zero [27,28]. Moreover, particularly for SDEs, some of the more rapidly convergent methods available for ordinary differential equations (ODE) do not work, such as higher order Runge-Kutta methods.[13]

One of the methods that do work for ODEs and SDEs is the Euler method, named after the Swiss mathematician Leonhard Euler. Figure 2.4 shows an example of an

[13]The euler method is the simplest Runge-Kutta method.

implementation for the Newton's cooling law with timestep of 2 seconds compared to its analytical solution. This method (*a.k.a.* forward Euler method) is a first-order numerical procedure. It is the most basic explicit method[14] for numerical integration.

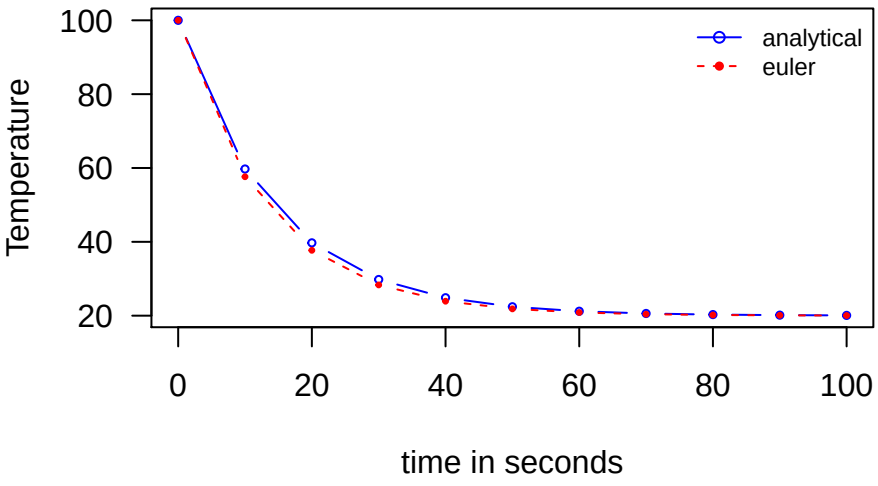

Figure 2.4: Analytical x Euler solutions

The method is first-order, as stated above, this means that the error in each step is a proportion of the square of the step size. Also, the global error at a given time is a function of the step size. We proceed to apply the Euler method to SDEs. Consider the equation:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t \tag{2.18}$$

dW_t is the Brownian motion, μ and σ are functions depending on S_t and t, over an interval $[0, T]$, and we want to discretize it as $0 = t_1 < t2 < \cdots < t_m = T$ with increments equally spaced Δt.

Integrating it from t to Δt we have the starting point for our (and any) discretization scheme:

$$S_{t+\Delta t} = S_t + \int_t^{\Delta t} \mu(S_u, u)du + \int_t^{\Delta t} \sigma(S_u, u)dW_u \tag{2.19}$$

To use the Euler discretization is the equivalent of approximating integrals using the

[14]Explicit methods calculate the state of a system at a later time from the state of the system at the current time. Mathematically we have something like $Y(t + \Delta t) = F(Y(t))$.

left-point rule as in Figure 2.5[15], we then have:

$$\begin{aligned}\int_t^{t+\Delta t} \mu(S_u,u)dW_u &\approx \mu(S_t,t)\int_t^{t+\Delta t} dW_u \\ &= \mu(S_t,t)(W_{t+\Delta t}-W_t) \\ \int_t^{t+\Delta t} \sigma(S_u,u)dW_u &\approx \sigma(S_t,t)\int_t^{t+\Delta t} dW_u = \sigma(S_t,t)(W_{t+\Delta t}-W_t) \\ &= \sigma(S_t,t)\sqrt{\Delta t}Z\end{aligned}$$

$W_{t+\Delta t}-W_t$ and $\sqrt{\Delta t}Z$ have identical distribution, Z being a standard Gaussian variable. The Euler discretization of equation (2.19) is then:

$$S_{t+\Delta t} = S_t + \mu(S_t,t)\Delta t + \sigma(S_t,t)\sqrt{\Delta t}Z \tag{2.20}$$

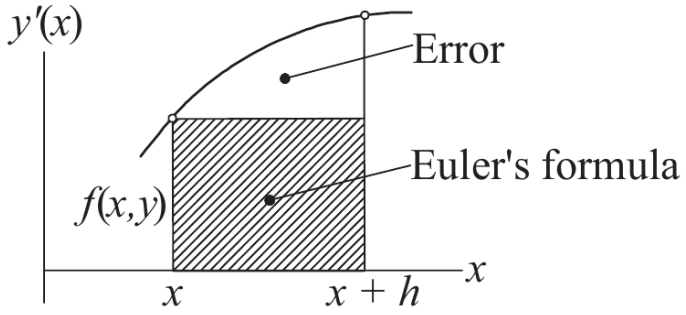

Figure 2.5: Graphical representation of Euler's formula

Source: Numerical methods in Engineering with Python 3.

Euler method - Heston model

We now proceed to apply the method to our model of interest. We retake the equations (2.16) and (2.17). We begin showing how to discretize the latter [15,26]:

$$V_{t+\Delta t} = V_t + \int_t^{t+\Delta t} \kappa(\theta - V_u)du + \int_t^{t+\Delta t} \sigma\sqrt{V_u}dB_u \tag{2.21}$$

Which discretized turns out as:

$$\begin{aligned}\int_t^{t+\Delta t} \kappa(\theta - V_u)du &\approx \kappa(\theta - V_t)\Delta t \\ \int_t^{t+\Delta t} \sigma\sqrt{V_u}dB_t &\approx \sigma\sqrt{V_t}(W_{t+\Delta t}-W_t) = \sigma\sqrt{V_t\Delta t}Z_v\end{aligned}$$

[15]See Kiusalaas [29]

And leaves us with:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t)\Delta t + \sigma\sqrt{V_t \Delta t} Z_v \tag{2.22}$$

Z_v is a standard normal variable. To avoid problems with negative values in $\sqrt{V_t}$ we apply the *full truncation* scheme, which substitutes V_t with $V_t^+ = max(0, V_t)$.[16]

For the S_t SDE we proceed similarly:

$$S_{t+\Delta t} = S_t + \mu \int_t^{t+\Delta t} S_u du + \int_t^{t+\Delta t} \sqrt{V_u} S_u dW_u \tag{2.23}$$

Discretizing we have:

$$\begin{aligned} \int_t^{t+\Delta t} S_u du &\approx S_t \Delta t \\ \int_t^{t+\Delta t} \sqrt{V_u} S_u dW_u &\approx \sqrt{V_t} S_t (W_{t+\Delta t} - W_t) \\ &= \sqrt{V_t \Delta t} S_t Z_s \end{aligned}$$

Z_s is a standard normal variable with correlation ρ with Z_v. We have:

$$S_{t+\Delta t} = S_t + \mu S_t \Delta t + \sqrt{V_t \Delta t} S_t Z_s \tag{2.24}$$

2.4.3 Stability

Most differential equations, deterministic or stochastic, cannot be solved explicitly [25]. Hence, stability studies begin with computers and is associated with numerical methods and approximations. Convergent methods were resulting in bigger errors than what was expected that could not be only due to discretization error. Eventually, scientists discovered that this unexpected problem was caused by accumulation of successive truncation errors. Figure 2.6 retakes the cooling example previously approached to show instability due to an increase in size of the timestep and extending to 600 seconds. The top plot, with a step of 25 still converges to the real solution, but we already observe an odd behaviour since the numerical method doesn't follow the analytical solution, but instead revolver around it until convergence. The bottom plot, presents a small increase in the step $h = 29$, and this small increment is enough results in numbers completely off target.

[16] Another possible scheme (not used in this work) is the *reflection* scheme where we replace V_t with $\mid V_t \mid$

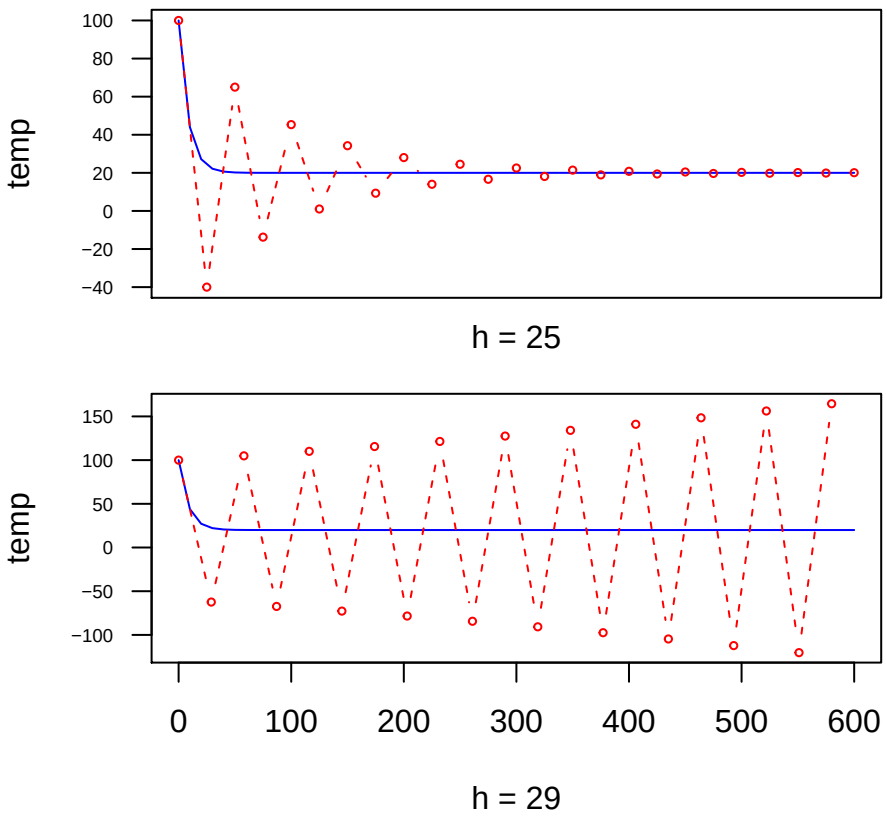

Figure 2.6: Euler's stability whith different timesteps

We know that binary machines like computers are not able to represent all the real numbers, but only a subset of them. Thus, solving these errors is not straightforward since it's not possible to eliminate *all* truncation error when using a computer and dealing with numerical solutions. When faced to an incorrect (not acceptable) solution, we have to evaluate and distinguish between two distinct situations:

i Rounding errors are considerably amplified by the algorithm. This situation is called numerical instability.

ii Small perturbations of data generate large changes in the solution. This is termed an ill-conditioned (or sensitive) problem.

Examples of these two classes of problem can be found in [23].

Stiff equations appear very often in mathematical problems and refer to differential equations for which a numerical methods might be unstable for not small enough stepsizes [30]. A differential equation of the form $y' = f(t, y)$, if its exact solution $y(t)$ includes a term that decays exponentially to zero as t increases, but whose derivatives are greater in magnitude than the term itself. In other words, if it requires a significant

depression of the stepsize to avoid stability lost. This is a loose definition but, since we are dealing with numerical methods a proper mathematical definition isn't required. Tipically, these equations are of the form e^{-ct}, where c is a large positive constant [31]. A practical example of the stiff behavior is the following differential equation [32]:

$$y' = -100y, \quad t > 0, \quad y_0 = 1$$

Whose exact solution is $y_t = e^{-100t}$ and goes to zero as t increases. Applying Euler's method to this equation with $h = 0.1$ we stumble in the following equation

$$y_{n+1} = y_n - 100hy_n = -9y_n$$

which is wrong, since it yields an exponencially growing solution $y_n = (-9)^n$. On the other hand, if our timestep is smaller $h = 10^{-3}$, our solution using Euler's method becomes $y_n = (0.9)^n$. This solution leads to an accurate behavior regarding the exact solution, it rapidily decays to zero.

Sometimes, it is interesting to rank differential equations that are more or less stiff. Thus, people compute the quotient of the largest and the smallest eigenvalues of a linear system. They call it the equations' *stiffness ratio* and, usually the bigger the stiffness ratio, the more likely is to be stiff [30].

Stability Domain

Let's take the equation:

$$y' = \lambda y, \quad t \geq 0, \qquad y_0 = 1 \tag{2.25}$$

where $\lambda \in \mathbb{C}$ or in other terms $\lambda = \lambda_r + i\lambda_i$ and whose solution is $y + t = e^{\lambda t}$. We can rewrite this equation as a system:

$$\frac{d}{dt}\begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{pmatrix} \lambda_r & -\lambda_i \\ \lambda_i & -\lambda_r \end{pmatrix}\begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \tag{2.26}$$

The $\lim_{t\to\infty} y_t = 0$ if and only if $\mathbb{R}\lambda < 0$. The *linear stability domain* $\mathcal{D}$ is defined as the set of all numbers $\Delta\lambda \in mathbbC$ such that $\lim_{n\to\infty y+n=0}$, with $\Delta > 0$ being the stepsize. Or, as stated in Kloeden [25], the suitable values of the stepsize are expressed in terms of *region of absolute stability*, consisting of the complex numbers $\lambda\Delta$ for which an error in y_0 at t_0 will not grow in subsequent iterations of the method.

Without entering all the details, for these we recommend [25,30,32], the euler's stability domain is:

$$\mathcal{D}_{Euler} = z \in \mathbb{C} :\mid 1 + z \mid < 1$$

which represents the interior of a complex disc of unit radius and centre $z = -1$ as can be seen in Figure 2.7, on the left. The right side of the Figure 2.7 shows the stability region called A-stability.[17] If a method is A-stable, the stepsize Δ is only constraint by accuracy.

[17] Mathematically: $\mathcal{D} \subseteq \{z \in \mathbb{C} : \mathbb{R}z < 0\}$.

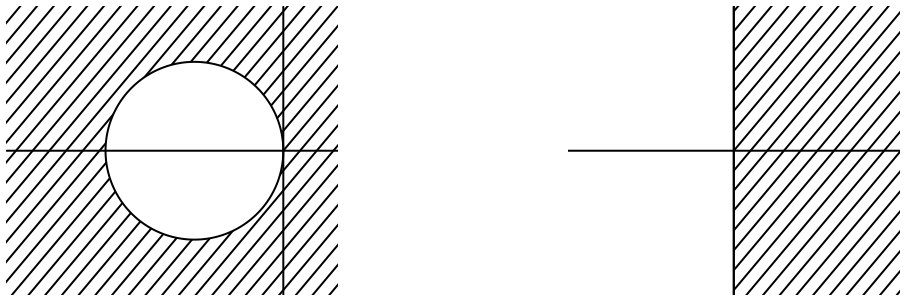

Figure 2.7: Stability domains

Thereby, we claim that stability method study is an important topic, since it enables achieving solutions that are good to stiff equations without having to overly reduce our timesteps which can be very computationally costly.

Chapter 3

The Heston Model Implementation

Chapter 4

This chunk ensures that the thesisdown package is

Chapter 5

Conclusion

Chapter 6

Black-Scholes formula

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