Numerical methods for stochastic volatility models:

Heston model

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Fernando O. Teixeira

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Hugo Alexander de la Cruz Cancino

Acknowledgements

Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin. - John von Neumann

You get pseudo-order when you seek order; you only get a measure of order and control when you embrace randomness. — Nassim Nicholas Taleb

Table of Contents

List of Tables

List of Figures

Abstract

The preface pretty much says it all.

Second paragraph of abstract starts here.

Dedication

You can have a dedication here if you wish.

Chapter 1

altadvisor: ‘Your Other Advisor’

Chapter 2

Literature Review

Chapter 3

The Heston Model Implementation

In section **??** we presented Heston's SDE system in one of its structures. Another common way [1,2,5] to write down the system is using the property presented in **??** as in equation (3.1).

$$\begin{aligned} dS_t &= \mu S_t dt + \rho\sqrt{V_t}dB_t + \sqrt{1-\rho^2}\sqrt{V_t}S_t dW_t \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t}dB_t \end{aligned} \tag{3.1}$$

3.1 Characteristic Function

The Heston model characteristic function is firstly presented in the 1993 Steven Heston's paper [4] and is described below [3]:

$$f(S_t, V_t, t) = e^{A(T-t)+B(T-t)S_t+C(T-t)V_t+i\phi S_t} \tag{3.2}$$

If we let $\tau = T - t$, then the explicit form of the Heston characteristic function is:

$$\begin{aligned} f(i\phi) &= e^{A(\tau)+B(\tau)S_t+C(\tau)V_t+i\phi S_t} \\ A(\tau) &= ri\phi\tau + \frac{\kappa\theta}{\sigma^2}\left[-(\rho\sigma i\phi - \kappa - M)\tau - 2\ln\left(\frac{1-Ne^{M\tau}}{1-N}\right)\right] \\ B(\tau) &= 0 \\ C(\tau) &= \frac{(e^{M\tau}-1)(\rho\sigma i\phi - \kappa - M)}{\sigma^2(1-Ne^{M\tau})} \end{aligned}$$

Where:

$$\begin{aligned} M &= \sqrt{(\rho\sigma i\phi - \kappa)^2 + \sigma^2(i\phi + \phi^2)} \\ N &= \frac{\rho\sigma i\phi - \kappa - M}{\rho\sigma i\phi - \kappa + M} \end{aligned}$$

This function is the driving force behind the following formula, that calculates the fair valur of a European call option at time t, given a strike price K, that expires at time T [3]:

$$
\begin{aligned}
C =& \frac{1}{2}S(t) + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty \Re\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi}\right]d\phi \\
&- Ke^{-r(T-t)}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right)
\end{aligned}
\tag{3.3}
$$

3.2 Euler Scheme - Full Truncation

We present here the Euler Scheme - Full Truncation algorithm [2] along with some insights on how it was implemented in R. The Euler discretization brings approximation paths to stock prices and variance processes. If we set $t_0 = 0 < t_1 < \cdots < t_M = T$ as partitions of a time interval of M equal segments of lenght δt, we have the following discretization for the stock price:

$$S_{t+1} = S_t + rS_t + \sqrt{V_t}S_tZ_s \tag{3.4}$$

And for the variance process:

$$V_{t+1} = V_t + \kappa(\theta - V_t) + \sigma\sqrt{V_t}Z_v \tag{3.5}$$

Z_s being a standard normal random variable, i.e. $N \sim (0,1)$, we set Z_t and Z_v as two independent standard normal random variables and Z_s and Z_v having correlation ρ. This means we can write $Z_s = \rho Z_v + \sqrt{1-\rho^2}Z_t$

The immediate observable problem in the proposed discretization scheme is that V can become negative with non-zero probability making the computation of $\sqrt{V_t}$ impossible [1]. There are several proposed fixes that can be used, we chose the Full-Truncation (FT) and rewrite the equations as follows:

$$S_{t+1} = S_t + rS_t + \sqrt{V_t^+}S_tZ_s \tag{3.6}$$

$$V_{t+1} = V_t + \kappa(\theta - V_t^+) + \sigma\sqrt{V_t^+}Z_v \tag{3.7}$$

Where we use the notation $V_t^+ = \max(V_t, 0)$.

3.3 Kahl-Jackel

Kahl-Jackel propose a discretization method they refer to as the "IJK" method [1,5] that coupled with the implicit Milstein scheme for the variance lands the system of equations (3.8) and (3.9). It is possible to verify that this discretization always results in positive paths for V if $4\kappa\theta > \sigma^2$. Unfortunately, this inequality is rarely satisfied when we plug real market data to calibrate the parameters.

$$
\begin{aligned}
\ln\hat{S}(t+\Delta) =& \ln\hat{S}(t) - \frac{\Delta}{4}\left(\hat{V}(t+\Delta)+\hat{V}(t)\right) + \rho\sqrt{\hat{V}(t)}Z_v\sqrt{\Delta} \\
&+ \frac{1}{2}\left(\sqrt{\hat{V}(t+\Delta)}+\sqrt{\hat{V}(t)}\right)\left(Z_S\sqrt{\Delta}-\rho Z_V\sqrt{\Delta}\right) + \frac{1}{4}\sigma\rho\Delta\left(Z_V^2-1\right)
\end{aligned}
\tag{3.8}
$$

$$\hat{V}(t+\Delta) = \frac{\hat{V}(t) + \kappa\theta\Delta + \sigma\sqrt{\hat{V}(t)}Z_V\sqrt{\Delta} + \frac{1}{4}\sigma^2\Delta\left(Z_V^2-1\right)}{1+\kappa\Delta} \tag{3.9}$$

3.4 Exact Algorithm

In 2006, Broadie-Kaya [2] propose a method that has a faster convergence rate, $\mathcal{O}\left(s^{-1/2}\right)$ than some of the more famous schemes, such as Euler's and Milstein's discretizations. They build their idea to generate an exact sample from the distribution of the terminal stock price based on numerous papers [4].

The algorithm used to generate the model consists in four steps as follows:

Step 1. Generate a sample of V_t given V_0

Step 2. Generate a sample of $\int_0^t V_s ds$ given V_t

Step 3. Compute $\int_0^t \sqrt{V_s} dB_s$

Step 4. teste

Chapter 4

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Chapter 5

Conclusion

Chapter 6

Placeholder

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