A Heston implementation

A Thesis
Presented to
The Division of Applied Mathematics
Fundação Getulio Vargas

In Partial Fulfillment
of the Requirements for the Degree
M.Sc. of Mathematics

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May 2017

Approved for the Division
(Mathematics)

Advisor F. Name

Acknowledgements

I want to thank a few people.

Preface

This is an example of a thesis setup to use the reed thesis document class (for LaTeX) and the R bookdown package, in general.

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Abstract

The preface pretty much says it all.

Second paragraph of abstract starts here.

Dedication

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Chapter 1

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Chapter 2

Literature Review

This chapter presents the concepts of stochastic calculus, from the historic conception of how it first arose through the basic principles and applications in finance. More precisely, we address the classical Black-Scholes model and its limitations and the Heston model. This model is also well known, it introduces the concept of stochastic volatility which brings us closer to reality.

2.1 Stochastic Calculus

Stochastic calculus arises from stochastic processes and allows the creation of a theory of integration where both the integrand and integrator terms are stochastic processes. Stochastic calculus, also known as, Itô calculus due to the name of it's creator, the Japanese mathematician Kiyosi Itô in the 1940s and 1950s is used for modelling financial options and in another wide variety of fields [1]. In this chapter we present the historical contexts in which the tools and models used arise, but our focus is introducing the concepts and notations that will be further used in our work.

2.1.1 Brownian Motion

The Brownian motion is the name given to the irregular motion observed in the motion of pollen particles suspended in fluid resulting from particle collision with atoms or molecules. It is named after Robert Brown, the first to have observed the movement in 1828. He noted two characteristic in the pollen movement [1]:

- the path of a givern particle is very irregular, having a tangent at no point
- the motion of two distinct particles appear to be independent

The first quantitative works in brownian motion come from an interest in stock price fluctuation by Bachelier in 1900. Albert Einstein also leaned over the subject and in 1905 derived the transition density for Brownian motion from molecular-kinectic theory of heat [1,2].

In 1923, the Wiener process was coined in honor of Norbert Wiener mathematical proof of existence of the brownian motion and stating its properties as follows [3]:

- $W_0 = 0$
- The change in W, given by $\Delta W = W_{t+1} - W_t$, is normally distributed with mean Wero and standard deviation $\sqrt{\Delta t}$, meaning that $\Delta W = \epsilon\sqrt{\Delta t}$, where ϵ is $N(0,1)$.
- If the increment Δt_1 does not overlap with the time increment Δt_2, then ΔW_1 and ΔW_2 are independent.
- The process is continuous, meaning that there are no jumps in the process.
- The process is a Markov process. This means that the conditional expectation of W_{t+1} given its entire history is equal to the conditional expectation of W_{t+1} given today's information. This can be written as: $E[W_{t+1}|W_1, ..., W_t] = E[W_{t+1}|W_t]$.
- Consider the time interval $[0,t]$ with n equally spaced intervals given by $t_i = \frac{it}{n}$. Then the paths of the Brownian motion have unbounded variation, this means that they are not differentiable and go towards infinity as n increases. The quadratic variation is given by $\sum_{i=1}^{n} (Z_{t_i} - Z_{t_{i-1}})^2 \to t$, meaning that when n increases it stays constant at t.

2.1.2 Itô's Lemma

Let X_t be a real-valued stochastic process that satisfies [4–6]:

$$X_t = X_0 + \int_0^t \mu_t dt + \int_0^t \sigma_t dW_t \tag{2.1}$$

for some μ_t, σ_t and $t \in [0,T]$. This equation is often rewritten in its differential stochastic form:

$$dX_t = \mu_t dt + \sigma_t dW_t \tag{2.2}$$

for $0 \leq t \leq T$.

Theorem

Assume that X_t has a stochastic differential given by:

$$dX_t = \mu_t dt + \sigma_t dW_t \tag{2.3}$$

for μ_t, σ_t and $t \in [0,T]$. Assume $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous.

$$Y_t := u(X_t, t)$$

Then Y has the following stochastic differential:

$$dY_t = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2 dt$$
$$= \left(\frac{\partial u}{\partial t} + \mu_t \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2\right)dt + \sigma_t \frac{\partial u}{\partial x}dW_t \tag{2.4}$$

where the argument of u, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ above is (X_t, t) .

Equation (2.4) is the stochastic equivalent to the chain rule, also known as Itô's formula or Itô's chain rule. The proof to this theorem is based on the Taylor expansion of the function $f(X_t, t)$ [4,5]. For practical uses you should write out a second-order Taylor expansion for the function to be analyzed and apply the 2.1 multiplication table [1].

Table 2.1: Box calculus

	dt	dW_t
dt	0	0
dW_t	0	dt

the **??** multiplication table.
??

```
par(mar = c(4, 4, 0.1, 0.1))
plot(pressure, pch = 19, type = "b")
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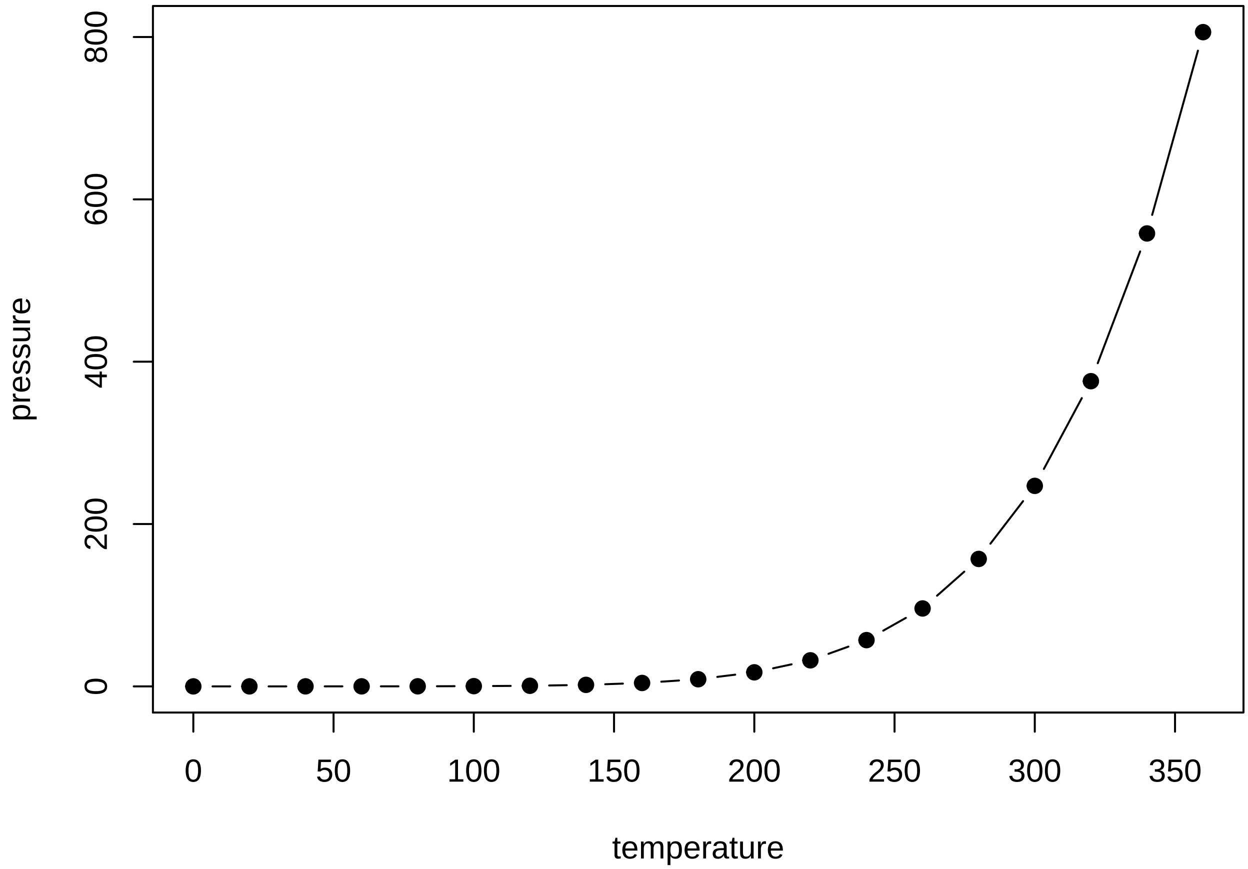

2.2 Black-Scholes Model

- Model

2.2.1 Derivative Contracts

European Call and Put

2.2.2 Limitations

2.3 Heston Model

Chapter 3

Mathematics and Science

Chapter 4

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Conclusion

Chapter 5

The First Appendix

References

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