Numerical methods for stochastic volatility models:

Heston model

A Thesis
Presented to
The Division of Applied Mathematics
Getulio Vargas Foundation

In Partial Fulfillment
of the Requirements for the Degree
M.Sc. of Mathematics

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Aug 15, 2017

Approved for the Division
(Mathematics)

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Acknowledgements

I want to thank a few people. Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin. - John von Neumann You get pseudo-order when you seek order; you only get a measure of order and control when you embrace randomness. — Nassim Nicholas Taleb

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Abstract

The preface pretty much says it all.

Second paragraph of abstract starts here.

Dedication

You can have a dedication here if you wish.

Chapter 1

Introduction

The french mathematician Louis Bachelier was the trail-blazer that brought Brownian motion, previously restricted to the field of botanics where it was firstly observed, to the financial framework. He modeled the stock prices as a brownian motion with drift. In 1973, Black and Scholes [1] designed a model based on the geometric brownian motion to price options.

Options are derivatives that give their bearers the rights to buy or sell a specific asset in a future date and with a predetermined price. They are, by design, affected by small variations in the underlying assets' components, for example, the variance.

The Black-Scholes model was once the standard way of option pricing, but was replaced by more modern models that are now prevalent. One of the main drawbacks of the Black-Scholes model is the strong assumption that the stock returns' volatility is constant. Thus, the implied model's volatility results in a flat surface when plotted against the option's strike price and maturity. Real world implied volatility varies with the strike price and maturity, forming what is called the 'volatility smile'.

The Heston model is an extension of the Black-Scholes model that tackles this volatility issue replacing the constant volatility with a stochastic process. There are many models that stochastically model volatility, but the Heston has valuable characteristics suach as presenting an analytical solution to the option pricing and also having a computationally simple implementation when compared to more sophisticated competitors.

This thesis is divided into five chapters, the first being this introduction. Following, we have a literature review that mainly addresses stochastic calculus, the Black-Scholes models and the Heston model. Thereafter we present the different Heston model known implementations and the implementation improvement we experiment. Chapter 4 brings the results of what was introduced in the previous chapter for different model calibrations.

Chapter 2

Literature Review

This chapter presents the concepts of stochastic calculus, from the historic conception of how it first arose through the basic principles and applications in finance. We address with more care the classical Black-Scholes model and its limitations and the Heston model. This model is also well known, it brings the concept of stochastic volatility in it, which brings its results closer to reality.

2.1 Stochastic Calculus

Stochastic calculus arises from stochastic processes and allows the creation of a theory of integration where both the integrand and integrator terms are stochastic processes. Stochastic calculus was created by the Japanese mathematician Kiyosi Itô [1] in the 1940s and 1950s and is used for modeling financial options and in another wide variety of fields [2]. In this chapter we present the historical contexts in which the tools and models used arise, but our focus is introducing the concepts and notations that will be further used in our work.

2.1.1 The Stochastic differential equation - SDE

At first, before introducing stochastic differential equation, it is helpful to start with ordinary differential equation. Let $x(t) = x_t$ denote the population at time t so that the change in the population at time t is given by the following deterministic differential equation:

$$\begin{aligned} dx_t &= f(t, x_t)dt \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

---

[1] There is another important stochastic integral, called the *Stratonovich Integral* that unlike the Itô's integral, respects the conventional calculus chain rule. Also, the integral is evaluated at the interval's midpoint, instead of its left extreme. A Stratonovich integral can be expressed as an Itô integral and vice versa.

We now add a “noise” to this equation:

$$dx_t = \overbrace{\underbrace{f(t, x_t)}_{\text{drift}}\, dt}^{\text{deterministic}} + \overbrace{\underbrace{g(t, x_t)}_{\text{diffusion}}\, dW_t}^{\text{stochastic}} \tag{2.2}$$
$$x(0) = x_0$$

This “noise” dW_t is a *random* Wiener process time derivative (which will be clarified below) and X_0 is our initial value.

The $g(t, x_t)$ part of the SDE is often referred as a *diffusion process.* These processes generally have a continuous paths. Before moving on, we must carefully define what the term *random*, means and the best way to begin doing so is to precisely define a probability space:

Definition 2.1.1. (Probability Space) A triple $(\Omega, \mathcal{U}, \mathcal{P})$ is called a *probability space* provided Ω is any set, $\mathcal{U}$ is a σ-algebra of subsets of Ω and $\mathcal{P}$ is a probability measure on $\mathcal{U}$.

2.1.2 Brownian Motion

The Brownian motion is the name given to the irregular motion observed in the motion of pollen particles suspended in fluid resulting from particle collision with atoms or molecules. It is named after Robert Brown, the first to have observed the movement in 1828. He noted two characteristic in the pollen movement [2]:

- the path of a given particle is very irregular, having a tangent at no point
- the motion of two distinct particles appear to be independent

The first quantitative works in brownian motion come from an interest in stock price fluctuation by Bachelier in 1900. Albert Einstein also leaned over the subject and in 1905 derived the transition density for Brownian motion from molecular-kinetic theory of heat [2,3].

In 1923, the Wiener process was coined in honor of Norbert Wiener mathematical proof of existence of the brownian motion and stating its properties.[2]

Definition 2.1.2. (Wiener Process) Given a probability space $(\Omega, \mathcal{U}, \mathcal{P})$, a stochastic process W_t defined in this space is a *wiener process* if it satisfies the following properties:

- $W_0 = 0$
- The change in W, given by $\Delta W = W_{t+1} - W_t$, is normally distributed with mean zero and standard deviation $\sqrt{\Delta t}$, meaning that $\Delta W = \epsilon\sqrt{\Delta t}$, where ϵ is $N(0, 1)$.
- If the increment Δt_1 does not overlap with the time increment Δt_2, then ΔW_1 and ΔW_2 are independent.

---

[2] More can be found on [4–6].

- The process is continuous, meaning that there are no jumps in the process.
- The process is a Markov process. This means that the conditional expectation of W_{t+1} given its entire history is equal to the conditional expectation of W_{t+1} given today's information. This can be written as: $E[W_{t+1}|W_1, ..., W_t] = E[W_{t+1}|W_t]$.
- Consider the time interval $[0, t]$ with n equally spaced intervals given by $t_i = \frac{it}{n}$. Then the paths of the Brownian motion have unbounded variation, this means that they are not differentiable and go towards infinity as n increases. The quadratic variation is given by $\sum_{i=1}^{n} (Z_{t_i} - Z_{t_{i-1}})^2 \to t$, meaning that when n increases it stays constant at t.

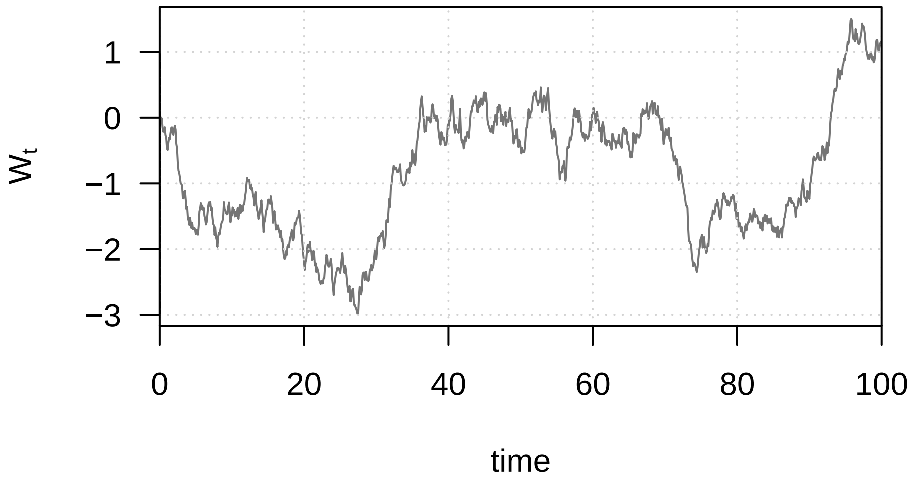

Figure 2.1: A Wiener process trajectory path example

Correlated Brownian Motions

Two independent brownian motions that are correlated can describe a new process Z_t. Let W_1 and W_2 be these two *independent* Brownian motions and let $-1 \leq \rho \leq 1$ be a given number. For $0 \leq t \leq T$ define the new process Z_t as [2]:

$$Z_t = \rho W_{1,t} + \sqrt{1 - \rho^2} W_{2,t} \tag{2.3}$$

This equation is a linear combination of independent normals at each timestep t, so Z_t is normally distributed. It is proven that Z is a Brownian motion and that Z and $W_{1,t}$ are correlated [2].

2.1.3 Itô's Integral

Formally, the SDE presented in equation 2.4 only exists because we can rewrite it in the form [5–10]:

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)dW_s \tag{2.4}$$

for some $f(s, x_s)$, $g(s, x_s)$ and $s \in [0, t]$.

The Itô integral can, as the Riemann integral, be approximated by a finite sum. Also, it has a definition as a certain limit. Itô's lemma 2.1.1 plays the same role as the fundamental theorem of calculus in allowing to evaluate integrals. It is the formal definition and presents an extra term not encountered in the conventional calculus theorem that is due to the non-smoothness characteristics of Brownian motion paths. It is possible, though, to define the integral in a less rigorous way:

$$Y_{\Delta t}(t) \approx \sum_{t_k < t} g(t_k)\Delta W_k \tag{2.5}$$

with the usual notions $t_k = k\Delta t$, and $\Delta W_k = W(t_{k+1}) - W(t_k))$. If the limit exists, then the Ito integral is:

$$Y(t) = \lim_{\Delta t \to 0} Y_{\Delta t}(t) \tag{2.6}$$

It is essential that the *forward difference* is used rather than the backward difference, which would be **wrong**.

Theorem 2.1.1 (Itô's Lemma)**.** *Assume that S_t has a stochastic differential given by:*

$$dS_t = \mu_t dt + \sigma_t dW_t \tag{2.7}$$

for μ_t, σ_t and $t \in [0, T]$. Assume $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous.

$$Y_t := u(S_t, t)$$

Then Y has the following stochastic differential:

$$\begin{aligned} dY_t &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dS_t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2 dt \\ &= \left(\frac{\partial u}{\partial t} + \mu_t \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma_t^2\right) dt + \sigma_t \frac{\partial u}{\partial x} dW_t \end{aligned} \tag{2.8}$$

where the argument of u, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ above is (S_t, t) .

Equation (2.8) is the stochastic equivalent to the chain rule, also known as Itô's formula or Itô's chain rule. The proof to this theorem is based on the Taylor expansion of the function $f(S_t, t)$ [5,7]. For practical uses you should write out a second-order Taylor expansion for the function to be analyzed and apply the 2.1 multiplication table [2].

Table 2.1: Box calculus

	dt	dW_t
dt	0	0
dW_t	0	dt

Theorem 2.1.2 (Itô's Integral Properties)**.** *Let* $f,\ g \in \mathcal{V}$ *and let* $0 \leq t_0 < u < T$. *Then*

(i) $\int_{t_0}^{T} f dB_t = \int_{t_0}^{u} f dB_t + \int_{u}^{T} f dB_t$

(ii) $\int_{t_0}^{T} (\alpha f + \beta g) dB_t = \alpha \int_{t_0}^{T} f dB_t + \int_{t_0}^{T} \beta g dB_t$

(iii) $\mathbb{E}\left[\int_{t_0}^{T} f dB_t\right] = 0$

(iv) $\mathbb{E}\left[\left(\int_0^t H_s\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2\, ds\right]$ *(Isometry)*

(v) $\mathbb{E}\left[\int_{t_0}^{T} f dB_t \mid \mathcal{F}_s\right] = \int_{t_0}^{s} f dB_t, \qquad for\ s < T.$ *(Martingale[3])*

2.2 Black-Scholes Model

2.2.1 Basics

The Black-Scholes (B-S) model arises from the need to price european options in the derivative markets. Derivatives are financial instruments traded in the market, stock exchange or over-the-counter (OTC) market, whose values depend on the values of an underlying asset. [1,11,12]

- A call option is a derivative that gives its bearer the right, but not the obligation, to purchase a specific asset by a fixed price before or on a given date.
- A put option is a derivative that gives its bearer the right, but not the obligation, to sell a specific asset by a fixed price before or on a given date.

The trading price of the option is called the option *premium* and the asset from which the option derives is called the *underlying asset*. This asset may be the interest rate, exchange rates, stock exchanges rates, commodities or stocks. The fixed price in contract in which the underlying asset might to be bought or sold is the *strick price*. The option expiration date is called the *maturity*. [1,12]

There are two major different option types: the European and the American. The difference between these two is that the bearer of the first may exercise it only at the end of its life, at its maturity while the latter can be exercised at any given time until its maturity. [1,13]

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[3] A martingale is a stochastic process with certain characteristics. The main one is that the expected value in time $t+1$ for X is the X value in t. This means there are no winning strategies when we are dealing with martingales (unlike when we play poker, for example). A Wiener process is a martingale.

Definition 2.2.1. (Implicit volatility) Given all the option's parameters in a precification model and its market price, the option's volatility is called the *implicit volatility.*

Definition 2.2.2. (Intrinsic value) The intrinsic value of a call is the difference between the underlying asset price and the strike price. The put's intrinsic value operates the other way around, being the difference between the strike and the underling asset prices.

Geometric Brownian Motion

A stochastic process S_t is a geometric brownian motion[4] if its solution is described by the solution of the following stochastic differential equation [2,7,14].

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.9}$$

for given constants $\mu \in \mathbb{R}$ and $\sigma > 0$. Also, the assumed initial value is positive, $S_0 > 0$.

This process[5] (Figure 2.2) is used quite often in finance to model the dynamics of some assets because of its properties. It has independent multiplicative increments and is the process used to price options in the Black-Scholes model [15]:

$$S_t = S_0 \times exp\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t, \quad t > 0 \tag{2.10}$$

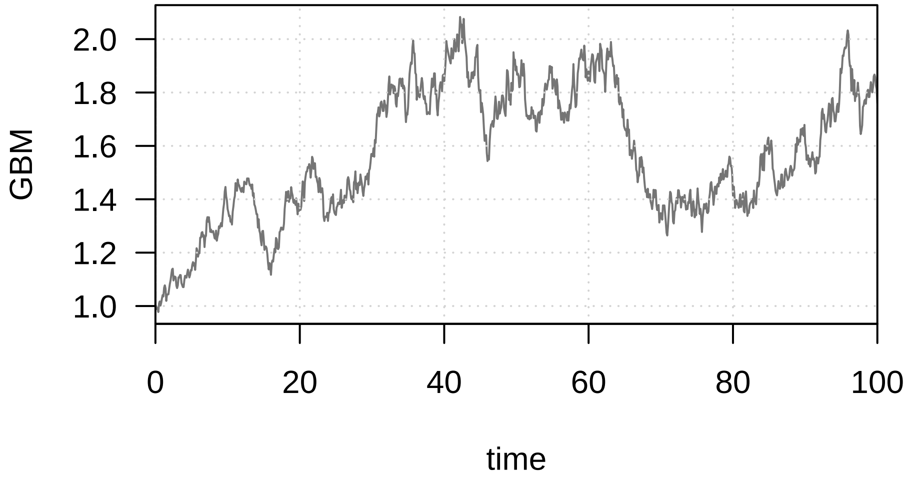

Figure 2.2: A GBM trajectory path example

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[4] There is an arithmetic brownian motion, whose equation is: $dS_t = \mu dt + \sigma dB_t$. More information can be obatined about this process looking at [2].

[5] Also known as exponential brownian motion.

2.2.2 The model

The Black-Scholes model that provides analytical solution to the price of a European call at time t can be described as follows [1,4,11]:

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \tag{2.11}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{2.12}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{2.13}$$

Where:

- S_t is the spot price of the underlying asset at time t
- r is the risk free rate (generally an annual rate)[6]
- σ is the volatility of returns of the underlying asset [7]
- $N(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution
- K is the strike price
- $T - t$ is the time to maturity

Also, the stock price path is a Geometric Brownian Motion as previously stated, and is under the risk-neutral measure with the following dynamics [4,16]:

$$dS_t = (r - q)S_t d_t + \sigma S_t dW_t \tag{2.14}$$

Where dW_t is a Wiener process [1,16], r is the risk free rate and q is the dividend yield[8] and t denotes the current point in time.

2.2.3 Limitations

Although the Black-Scholes is very popular and the *de facto* standard in the market there are implications to the B-S model assumptions that affect the results and that are unrealistic. The main assumption that does not hold up is the deterministic (constant) volatility, that can more accurately be described as a stochastic process since we observe that small moves usually are followed by small moves and large moves by large moves. [4,11]

Other assumptions that are critical to the B-S model and are not always observed in practice refer to the asset's continuity through time (no jumps), being allowed to perform continuous hedge without transactions costs and normal (Gaussian) returns.

Most models focus on the volatility problem because transaction costs often translate to rises in volatility and fat-tails (abnormal) returns can be simulated by stochastic volatility and market or volatility jumps.

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[6] Assumed to be constant.

[7] See footnote 1.

[8] r and q are assumed to be constant.

2.3 Stochastic Volatility models

Introducing stochastic volatility to models brings complexity, but enables modeling some features observed in reality that are crucial like the randomic market volatility effects, skewness (market returns are more realistically modeled) and volatility smile. This kind of model is applied highly succesfully in foreign exchange and credit markets.

2.3.1 Cox-Ingersoll-Ross model

The Cox-Ingersoll-Ross (CIR) model is a well-known short-rate model that describes the interest rate movements driven by one source of market risk. The dynamics are described as follows[17,18]:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dB_t \tag{2.15}$$

Where, r_t is the short rate interest described by parameters k - the speed of mean reversion, θ - the long-run mean variance and σ - the volatility of the variance process.

This model has been widely used to describe the dynamics of the short rate interest because it has some fundamental features like intuitive parametrization, nonnegativity and pricing formulas. Besides, it takes account of anticipations, risk aversion, investment alternatives and preferences about consumption timing and allows for detailed predictions about how changes in a wide range of underlying variables affect the term structure[17]. Furthermore, this equation constitutes one of the two Heston model equations with the volatility taking the short rate interest place.

2.3.2 Heston Model

Heston model solves the deterministic volatility problems introducing the following equations, which represents the dynamics of the stock price and the variance processes under the risk-neutral measure [18,19]:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^* \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t}dB_t \end{aligned} \tag{2.16}$$

The second equation, as described in Section 2.3.1, is the CIR model equation. The first equation states the asset price process. μ is the asset's rate of return, $dW_{t,1}$ and $dW_{t,2}$ are two correlated wiener processes with correlation coefficient of ρ. Because, of the model specifications and what we presentend in section 2.1.2, we can rewrite the first equation as in Broadie and Kaya [20]:

$$\begin{aligned} dS_t &= \mu S_t dt + \rho\sqrt{V_t}dB_t + \sqrt{1-\rho^2}\sqrt{V_t}S_t dW_t \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t}dB_t \end{aligned} \tag{2.17}$$

2.3.3 Other Models

Ornstein-Uhlenbeck

The Ornstein-Uhlenbeck is the earliest recorded SDE. Named after Leonard Ornstein and George Eugene Uhlenbeck, it is a stochastic process that describes the acceleration of a pollen particle in a liquid subject to bombardments by molecules [2]. As we can observe in equation @ref(ou_eq), x_t represents the one dimension velocity of the particle, thus dx_t is the *change* in velocity, in other words, its acceleration. The $-\theta x_t$ component slows down the acceleration and is to be understood as frictional force. Besides, we add a noise W_t with intensity σ that models the random bombardment by the molecules.

$$dx_t = -\theta x_t dt + \sigma dW_t \tag{2.18}$$

$$\tag{2.19}$$

With θ and σ being positive constants. Expressing in terms of x_t we get:

$$x_t = e^{-\theta t} \times \left[x_0 + \sigma \int_{t=0}^{T} e^{\theta t} dW_s \right] . \tag{2.20}$$

Langevin

The Langevin equation describes a system that consists of the molecular bombardment of a speck of dust on a water surface. We know that the intensity of the bombardement does not depend on the state variables [21].

$$m \frac{dv}{dt} = -\zeta v + \delta F(t) \tag{2.21}$$

m is the mass of the particle, v it's velocity, $-\zeta v$ is the frictional force, which is proportional to the velocity, and $\delta F(t)$ is a *fluctuating* force (random) to the frictional force.

2.4 Numerical Methods

Numerical methods are tools that are often applied to solve stochastic differential equations because most of these do not have explicit solution. This means that we are not able to solve these equations using symbolic computation. Although we are unable to find an analitical solution, when facing real problems, the approximation given by a numerical method is often sufficient. Alongside the analytical issue, the need to calculate the SDE's trajectory through time is the main reason why studying numerical methods is so important. An implementation of a numerical method is called a numerical algorithm.

We will simulate sample paths of time discrete approximations implemented in the R programming language [22] that we base on a finite discretization of a time interval $[t_0, T]$. We shall generate approximate values of the sample path for each step contained in the discretized interval [21].

In the fixed step methods, the distance between two contiguous points is the distance $d_i = t_i - t_{i-1} = \frac{T-t_0}{N} \quad \forall i \mid 1 \leq i \leq N \in \mathbb{N}$. N being the time interval partition number.

[21]

The simulated sample paths can then be analysed by usual statistical methods to determine how good the approximation is and in what sense it is close to the exact solution. The state variables here are not discretized as in Kushner's Markov chain approach and the structure of the SDE as provided by the drift and diffusion coefficients is used in a natural way. An advantage of considerable practical importance of this approach is that the computational costs such as time and memory required increase only polynomially with the dimension of the problem. Variance reduction methods allow a considerable decrease in the required sample size.

2.4.1 Convergence ??

2.4.2 Euler-Maruyama

2.4.3 Milstein

2.4.4 Stability

[21] Most differential equations, deterministic or stochastic, cannot be solved explicitly. Nevertheless we can often deduce alot of useful information, usually qualitative, about the behaviour of their solutions from the functional form of their coefficients. Of particular interest in applications is the long term asymptotic behaviour and sensitivity of the solutions to small changes, for example measurement errors, in the initial values. From existence and uniqueness theory we know that the solutions of a differential equation are continuous in their initial values, at least over a finite time interval. Extending this idea to an infinite time interval leads to the concept of stability.

Chapter 3

The Heston Model Implementation

3.1 Characteristic Function

The Heston model characteristic function is firstly presented in the 1993 Steven Heston's paper [18] and is described below [23]:

$$f(S_t, V_t, t) = e^{A(T-t)+B(T-t)S_t+C(T-t)V_t+i\phi S_t} \tag{3.1}$$

If we let $\tau = T - t$, then the explicit form of the Heston characteristic function is:

$$\begin{aligned}
f(i\phi) &= e^{A(\tau)+B(\tau)S_t+C(\tau)V_t+i\phi S_t} \\
A(\tau) &= ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau - 2\ln\left(\frac{1-Ne^{M\tau}}{1-N}\right)\right] \\
B(\tau) &= 0 \\
C(\tau) &= \frac{(e^{M\tau}-1)(\rho\sigma i\phi - k - M)}{\sigma^2(1-Ne^{M\tau})} \\
&\text{Where:} \\
M &= \sqrt{(\rho\sigma i\phi - k)^2 + \sigma^2(i\phi + \phi^2)} \\
N &= \frac{\rho\sigma i\phi - k - M}{\rho\sigma i\phi - k + M}
\end{aligned}$$

This function is the driving force behind the following formula, that calculates the fair valur of a European call option at time t, given a strike price K, that expires at time T [23]:

$$\begin{aligned}
C =& \frac{1}{2}S(t) + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty \Re\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi}\right]d\phi \\
&- Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right)
\end{aligned} \tag{3.2}$$

3.2 Euler Scheme - Full Truncation

We present here the Euler Scheme - Full Truncation algorithm [20] along with some insights on how it was implemented in R. The Euler discretization brings approximation paths to stock prices and variance processes. If we set $t_0 = 0 < t_1 < \cdots < t_M = T$ as partitions of a time interval of M equal segments of lenght δt, we have the following discretization for the stock price:

$$S_{t+1} = S_t + rS_t + \sqrt{V_t} S_t Z_s \tag{3.3}$$

And for the variance process:

$$V_{t+1} = V_t + k(\theta - V_t) + \sigma\sqrt{V_t} Z_v \tag{3.4}$$

Z_s being a standard normal random variable, i.e. $N\ (0,1)$, we set Z_t and Z_v as two independent standard normal random variables and Z_s and Z_v having correlation ρ. This means we can write $Z_s = \rho Z_v + \sqrt{1-\rho^2} Z_t$

The immediate observable problem in the proposed discretization scheme is that V can become negative with non-zero probability making the computation of $\sqrt{V_t}$ impossible [24]. There are several proposed fixes that can be used, we chose the Full-Truncation (FT) and rewrite the equations as follows:

$$S_{t+1} = S_t + rS_t + \sqrt{V_t^+} S_t Z_s \tag{3.5}$$

$$V_{t+1} = V_t + k(\theta - V_t^+) + \sigma\sqrt{V_t^+} Z_v \tag{3.6}$$

Where we use the notation $V_t^+ = \max(V_t, 0)$.

```
hestoneuler <- function(S, X, r, v, theta, rho, k, sigma, t = 0,
                        dt = NULL, tau = 1, N, sensibility = 15){

    set.seed(1)

    if(is.null(dt)){ dt <- (tau-t)/1000}

    ST <- NULL
    aux <- NULL
    sqrt_dt <- sqrt(dt)

    for(i in seq(t,tau,dt)){
        # browser()
        Zv <- rnorm(N)
        Zt <- rnorm(N)
        Zs <- rho * Zv + (sqrt(1 - (rho^2)) * Zt)

        # v[v <= 0] = 0
        aux <- v
        aux[v < 0] <- 0
        sqrt_aux <- sqrt(aux)
        S <- S * (1 + r * dt + sqrt_aux * Zs * sqrt_dt)
        S[S <= 0] = 0

        v <- v + k * dt * (theta - aux) +
          sigma * sqrt_aux * Zv * sqrt_dt

        ST <- rbind(ST,S)
    }

    rm(aux, v, Zv, Zt, Zs, S)

    ST <- as.matrix(ST, ncol=N)
    # matplot(graf, type = 'l')
    # abline(h=X, lwd=2, col='red')
    media <- c()

    for (j in (X-sensibility):(X+sensibility)){

        Result <- ST[nrow(ST),] - j
        Result[Result <= 0] <- 0
        media <-c(media, mean(Result))

    }

    media <- media * exp(-r*tau)
    # plot(media, ylab = "Call price", xlab = "Strike price")

    Result <- ST[nrow(ST),] - X
    Result[Result <= 0] = 0
    call = mean(exp(-r*tau)*Result)

    lista = list('call' = call, 'Result' = Result,
                 'Spot' = ST, 'sensib' = media)
    return(lista)
}
```

3.3 Kahl-Jackel

```
Hestoncallkj <- function(S, X, r, q, v, theta, rho, k, sigma,
                         t = 0, dt = NULL, tau = 1, N,
                         sensibility = 15){

    if(is.null(dt)){ dt <- (T-t)/1000}

    v <- rep(v,N)
    theta<- rep(theta,N)
    graf <- NULL
    S <- log(S)

    for(i in seq(t,tau,dt)){
     # browser()
    Zv <- rnorm(N)
    Zt <- rnorm(N)
    Zs <- rho * Zv + sqrt(1 - rho^2) * Zt

    vt <- (v + k * theta * dt + sigma * sqrt(v) * Zv * sqrt(dt) +
              (1/4) * sigma^2 * dt * ((Zv)^2 - 1))/(1 + k * dt)

    # print(vt)

    # if(!length(vt[vt <= 0]) == 0 & length(vt[vt <= 0]) >= 1){
    #      # browser()
    vt[vt <= 0] <- v[vt <= 0] + k * dt * (theta[vt <= 0] -
                    pmax(v[vt <= 0],0)) + sigma *
                    sqrt(pmax(v[vt <= 0],0)) * Zv[vt <= 0] * sqrt(dt)
    v <- vt
    v[v<=0] <- 0

    # }

    S <- S + (r - (v+vt)/4) * dt + rho * sqrt(pmax(v,0)) *
          Zv * sqrt(dt) +
          (1/2) * (sqrt(pmax(v,0)) + sqrt(pmax(vt,0))) *
          (Zs + rho * Zv) *
          sqrt(dt) +
          ((rho * sigma * dt)/2) * ((Zv)^2 - 1)

    S[S <= 0] = 0
    graf <- rbind(graf,S)
    }

     # browser()

    graf <- as.matrix(graf, ncol=N)
    ST <- graf
    # matplot(graf, type = 'l')
    # abline(h=log(X), lwd=2, col='red')

    media <- c()

    # browser()

    for (j in (X-sensibility):(X+sensibility)){

        Result <- exp(ST[nrow(ST),]) - j
        Result[Result <= 0] <- 0
        media <-c(media, mean(Result))

    }

    media <- media * exp(-r*tau)

    # browser()

    Result <- exp(ST[nrow(ST),]) - X
    Result[Result <= 0] = 0
    call = mean(exp(-r*tau)*Result)

    lista = list('call' = call, 'Result' = Result,
                 'Spot' = graf, 'sensib' = media)
    return(lista)

}
```

3.4 Exact Algorithm

```
hestonea <- function(S, X, r, v, theta, rho, k, sigma, t = 0,
                     dt = NULL, tau = 1, N, sensibility = 15){

    if(is.null(dt)){ dt <- (tau-t)/1000}
```

```r
    ST <- NULL

    d <- (4 * k * theta)/(sigma)^2
    c0 <- (sigma^2 * (1 - exp(-k*dt)))/(4*k)

    # browser()

    for(i in seq(t,tau,dt)){

        # sampling V
        lambda <- (4*k*exp(-k*dt)*v)/(sigma^2 * (1-exp(-k*dt)))
        vt <- c0 * rchisq(n = N, df = d, ncp = lambda)

        # Sampling int{V}
        int_v <- dt * ((1/2) * v + (1/2) * vt)

        # Sampling int{v}dw
        int_vdw <- (1/sigma) * (vt - v - k * theta * dt + k  * int_v)

        # Sampling S
        m <- log(S) + (r * dt - (1/2) * int_v + rho * int_vdw)
        std <- sqrt((1 - rho^2)) * sqrt(int_v)
        S <- exp(m + std * rnorm(N))
        v <- vt
        ST <- rbind(ST,S)
    }

    ST <- as.matrix(ST, ncol=N)
    # matplot(graf, type = 'l')
    # abline(h=X, lwd=2, col='red')
    media <- c()

    for (j in (X-sensibility):(X+sensibility)){

        Result <- ST[nrow(ST),] - j
        Result[Result <= 0] <- 0
        media <-c(media, mean(Result))

    }

    media <- media * exp(-r*tau)
    # plot(media, ylab = "Call price", xlab = "Strike price")

    Result <- ST[nrow(ST),] - X
    Result[Result <= 0] = 0
    call = mean(exp(-r*tau)*Result)

    lista = list('call' = call, 'Result' = Result,
                 'Spot' = ST, 'sensib' = media)
    return(lista)
}
```

```
require(NMOF)
setwd("../rnmethods")
source("../rnmethods/R/euler_heston.R")
```

Chapter 4

Results

Chapter 5

Conclusion

If we don't want Conclusion to have a chapter number next to it, we can add the `{-}` attribute.

More info

And here's some other random info: the first paragraph after a chapter title or section head *shouldn't be* indented, because indents are to tell the reader that you're starting a new paragraph. Since that's obvious after a chapter or section title, proper typesetting doesn't add an indent there.

Appendix A

The First Appendix

This first appendix includes all of the R chunks of code that were hidden throughout the document (using the `include = FALSE` chunk tag) to help with readibility and/or setup.

In the main Rmd file

```r
# This chunk ensures that the thesisdown package is
# installed and loaded. This thesisdown package includes
# the template files for the thesis.
if(!require(devtools))
  install.packages("devtools", repos = "http://cran.rstudio.com")
if(!require(thesisdown))
  devtools::install_github("ismayc/thesisdown")
library(thesisdown)
```

In Chapter ??:

```r
# This chunk ensures that the thesisdown package is
# installed and loaded. This thesisdown package includes
# the template files for the thesis and also two functions
# used for labeling and referencing
if(!require(devtools))
  install.packages("devtools", repos = "http://cran.rstudio.com")
if(!require(dplyr))
    install.packages("dplyr", repos = "http://cran.rstudio.com")
if(!require(ggplot2))
    install.packages("ggplot2", repos = "http://cran.rstudio.com")
if(!require(ggplot2))
    install.packages("bookdown", repos = "http://cran.rstudio.com")
if(!require(thesisdown)){
  library(devtools)
  devtools::install_github("ismayc/thesisdown")
  }
```

```r
library(thesisdown)
flights <- read.csv("data/flights.csv")
```

Appendix B

The Second Appendix, for Fun

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