Numerical methods for the Heston model

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Acknowledgements

Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin. - John von Neumann

You get pseudo-order when you seek order; you only get a measure of order and control when you embrace randomness. — Nassim Nicholas Taleb

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Abstract

The preface pretty much says it all.

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Dedication

You can have a dedication here if you wish.

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altadvisor: ‘Your Other Advisor’

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Chapter 3

The Heston Model Implementation

In section **??** we presented Heston's SDE system in one of its structures. Another common way [1,3,10] to write down the system is using the property presented in subsection **??** as in equation (3.1).

$$\begin{aligned} dS_t &= \mu S_t dt + \rho\sqrt{V_t} dB_t + \sqrt{1-\rho^2}\sqrt{V_t} S_t dW_t \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t} dB_t \end{aligned} \tag{3.1}$$

3.1 Characteristic Function

The Heston model characteristic function is firstly presented in the 1993 Steven Heston's paper [8] and is described below [5]:

$$f(S_t, V_t, t) = e^{A(T-t)+B(T-t)S_t+C(T-t)V_t+i\phi S_t} \tag{3.2}$$

If we let $\tau = T - t$, then the explicit form of the Heston characteristic function is:

$$\begin{aligned} f(i\phi) &= e^{A(\tau)+B(\tau)S_t+C(\tau)V_t+i\phi S_t} \\ A(\tau) &= ri\phi\tau + \frac{\kappa\theta}{\sigma^2}\left[-(\rho\sigma i\phi - \kappa - M)\tau - 2\ln\left(\frac{1-Ne^{M\tau}}{1-N}\right)\right] \\ B(\tau) &= 0 \\ C(\tau) &= \frac{(e^{M\tau}-1)(\rho\sigma i\phi - \kappa - M)}{\sigma^2(1-Ne^{M\tau})} \end{aligned}$$

Where:

$$\begin{aligned} M &= \sqrt{(\rho\sigma i\phi - \kappa)^2 + \sigma^2(i\phi + \phi^2)} \\ N &= \frac{\rho\sigma i\phi - \kappa - M}{\rho\sigma i\phi - \kappa + M} \end{aligned}$$

This function is the driving force behind the following formula, that calculates the fair valur of a European call option at time t, given a strike price K, that expires at time T [5]:

$$\begin{aligned} C =& \frac{1}{2}S(t) + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty \Re\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi}\right]d\phi \\ &- Ke^{-r(T-t)}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right) \end{aligned} \tag{3.3}$$

3.2 Euler Scheme

Given the fact that the underlying asset is temporal dependent upon the solution of the SDE's volatility, we simulate the volatility's path before the asset's. If the Black-Scholes model enabled using Ito's Lemma directly for solving S_t, this equation system requires numerical methods. We present here the Euler Scheme - Full Truncation algorithm (and compare to other similar schemes) [3] along with some insights on how it was implemented in R. The Euler discretization brings approximation paths to stock prices and variance processes. If we set $t_0 = 0 < t_1 < \cdots < t_M = T$ as partitions of a time interval of M equal segments of lenght δt, we have the following discretization for the stock price:

$$S_{t+1} = S_t + rS_t + \sqrt{V_t}S_tZ_s \tag{3.4}$$

And for the variance process:

$$V_{t+1} = f_1(V_t) + \kappa(\theta - f_2(V_t)) + \sigma\sqrt{f_3(V_t)}Z_v \tag{3.5}$$

Z_s being a standard normal random variable, i.e. $N \sim (0,1)$, we set Z_t and Z_v as two independent standard normal random variables and Z_s and Z_v having correlation ρ. This means we can write $Z_s = \rho Z_v + \sqrt{1-\rho^2}Z_t$.

The immediate observable problem in the proposed discretization scheme is that V can become negative with non-zero probability making the computation of $\sqrt{V_t}$ impossible [1]. There are several proposed fixes that can be used as you can see below:

Table 3.1: Truncation schemes

Scheme	$f_1(V_t)$	$f_2(V_t)$	$f_3(V_t)$
Reflection	$\mid V \mid$	$\mid V \mid$	$\mid V \mid$
Partial Truncation	V	V	V^+
Full Truncation	V	V^+	V^+

Where $V^+ = \max(V, 0)$ and $\mid V \mid$ is the absolute value of V.

We chose to fix our discretization using the Full-Truncation (FT) scheme and thus, rewrite the equations as follows:

$$S_{t+1} = S_t + rS_t + \sqrt{V_t^+} S_t Z_s \tag{3.6}$$

$$V_{t+1} = V_t + \kappa(\theta - V_t^+) + \sigma\sqrt{V_t^+} Z_v \tag{3.7}$$

3.3 Kahl-Jackel

Kahl-Jackel propose a discretization method they refer to as the "IJK" method [1,10] that coupled with the implicit Milstein scheme for the variance lands the system of equations (3.8) and (3.9). It is possible to verify that this discretization always results in positive paths for V if $4\kappa\theta > \sigma^2$. Unfortunately, this inequality is rarely satisfied when we plug real market data to calibrate the parameters.

$$\begin{aligned} \ln \hat{S}(t+\Delta) = \ln \hat{S}(t) - \frac{\Delta}{4}\left(\hat{V}(t+\Delta) + \hat{V}(t)\right) + \rho\sqrt{\hat{V}(t)} Z_v \sqrt{\Delta} \\ + \frac{1}{2}\left(\sqrt{\hat{V}(t+\Delta)} + \sqrt{\hat{V}(t)}\right)\left(Z_S\sqrt{\Delta} - \rho Z_V\sqrt{\Delta}\right) + \frac{1}{4}\sigma\rho\Delta\left(Z_V^2 - 1\right) \end{aligned} \tag{3.8}$$

$$\hat{V}(t+\Delta) = \frac{\hat{V}(t) + \kappa\theta\Delta + \sigma\sqrt{\hat{V}(t)} Z_V \sqrt{\Delta} + \frac{1}{4}\sigma^2\Delta\left(Z_V^2 - 1\right)}{1 + \kappa\Delta} \tag{3.9}$$

3.4 Exact Algorithm

In 2006, Broadie-Kaya [3] propose a method that has a faster convergence rate, $\mathcal{O}\left(s^{-1/2}\right)$ than some of the more famous schemes, such as Euler's and Milstein's discretizations. They build their idea to generate an exact sample from the distribution of the terminal stock price based on numerous papers [8]. The stock price and variance are as follows:

$$S_t = S_0\, exp\left[\mu t - \frac{1}{2}\int_0^t V_s ds + \rho\int_0^t \sqrt{V_s dB_s} + \sqrt{1-\rho^2}\int_0^t \sqrt{V_s} dW_s\right] \tag{3.10}$$

The squared volatility of the variance process is:

$$V_t = V_0 + \kappa\theta t - \kappa\int_0^t V_s ds + \sigma\int_0^t \sqrt{V_s dB_s} \tag{3.11}$$

The algorithm used to generate the model consists in four steps as follows:

Step 1. Generate a sample of V_t given V_0

Step 2. Generate a sample of $\displaystyle\int_0^t V_s ds$ given V_t, V_0

Step 3. Compute $displaystyle \int_0^t \sqrt{V_s} dB_s$ given V_t, V_0 and $\int_0^t V_s ds$

Step 4. Generate a sample from the probability distribution of S_t, given $displaystyle \int_0^t \sqrt{V_s} dB_s$ and $displaystyle \int_0^t V_s ds$

3.4.1 Generate a sample of V_t given V_0

The distribution of V_t given V_0 for $0 < t$ is a noncentral chi-squared distribution [2,4]:

$$V_t = \frac{\sigma^2(1-e^{-\kappa t})}{4\kappa}\mathcal{X}^2_\delta\left(\frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}\times V_0\right)$$

where $\delta = \frac{4\theta\kappa}{\sigma^2}$ and $\mathcal{X}^2_\delta(\lambda)$ denotes a noncentral chi-squared random variable with δ degrees of freedom and λ as its noncentrality parameter.

Broadie and Kaya [3] sample generating Poisson and gamma distributions as in Johnson et al. [9]. We used the built-in function in R [11] which uses this exact method for sampling.

3.4.2 Generate a sample of $\int_0^t V_s ds$ given V_t, V_0

After generating V_t, we follow the instructions in [3,9]. We use the characteristic function (3.12) to compute the probability density function $F(x)$.

$$\begin{aligned}\Phi(a) &= \mathbb{E}\left[exp\left(ia\int_0^t V_s ds \mid V_0, V_t\right)\right] \\ &= \frac{\gamma(a)e^{(-1/2)(\gamma(a)-\kappa)t}(1-e^{-\kappa t})}{\kappa(1-e^{-\gamma(a)t})} \\ &\times exp\left\{\frac{V_0+V_t}{\sigma^2}\left[\frac{\kappa(1+e^{-\kappa t})}{1-e^{-\kappa t}} - \frac{\gamma(a)(1+e^{-\gamma(a)t})}{1-e^{-\gamma(a)t}}\right]\right\} \\ &\times \frac{I_{0.5\delta-1}\left[\sqrt{V_0V_t}\frac{4\gamma(a)e^{-0.5\gamma(a)t}}{\sigma^2(1-e^{-\gamma(a)t})}\right]}{I_{0.5\delta-1}\left[\sqrt{V_0V_t}\frac{4\kappa e^{-0.5\kappa t}}{\sigma^2(1-e^{-\kappa t})}\right]}\end{aligned} \tag{3.12}$$

where $\gamma(a) = \sqrt{\kappa^2 - 2\sigma^2 ia}$, δ was previously defined and $I_v(x)$ is the modified Bessel function of the first kind.

The probability distribution function is obtained in [2,3] by Fourier inversions using Feller [6]. We use the approach in Gil-Pelaez [7], equation (3.13). We define $V(u,t)$ the random variable with the same distribution as the integral $\int_u^t V_s ds$, conditional on V_u and V_t:

$$F(x) \equiv Pr\left\{V(u,t) \le x\right\} = F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[e^{-iux}phi(u)]}{u}du \tag{3.13}$$

Im denotes the imaginary part of $e^{-iux}phi(u)$. Equation (3.13) is computed numerically and we then sample it by inversion.

Furthermore, we also introduce a simpler version for this step, that computes this integral approximation, using the solution $\int_u^t V_s ds = \frac{1}{2}(V_u + V_t)$

3.4.3 Compute $\int_0^t \sqrt{V_s} dB_s$ given V_t, V_0 and $\int_0^t V_s ds$

From equation (3.11) we are now able to compute this integral.

$$\int_0^t \sqrt{V_s dB_s} = \frac{V_t - V_0 - \kappa\theta t + \kappa \int_0^t V_s ds}{\sigma} \tag{3.14}$$

The last step of the algorithm consists of computing the conditional distribution of $logS_t$ based on the fact that the process for V_t is independent from dB_t, and the distribution of $\int_0^t \sqrt{V_s} dB_s$ is normal with mean 0 and variance $\int_0^t V_s ds$, given V_t.

$$m(u,t) = \log S_0 + \left[\mu t - \frac{1}{2}\int_0^t V_s ds + \rho \int_0^t \sqrt{V_s dB_s} + \sqrt{1-\rho^2}\int_0^t \sqrt{V_s} dW_s\right]$$

and variance

$$\sigma^2(0,t) = \left(1-\rho^2\right)\int_0^t V_s ds$$

We generate the S_t sample using a standard normal random variable Z and set:

$$S_t = e^{m(0,t)+\sigma(0,t)Z}$$

3.4.4 Limitations

The biggest limitation this scheme presents is that the second step is computationally costly. It demands the inversion of the $\displaystyle\int_0^t V_s ds \mid V_t,\, V_0$

Chapter 4

This chunk ensures that the thesisdown package is

Chapter 5

Conclusion

Chapter 6

Black-Scholes formula

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