ปั๊มทุกกรณี

หลุดหมด → L ไม่ใช่ CFL

ยังมี w ∈ L → แสดงว่าเงื่อนไข str ผิด / เป็น CFL

The Pumping Lemma for Context-Free Languages

* เราแค่แน่ใจว่า ภาษานี้ที่ไม่ใช่ CFL

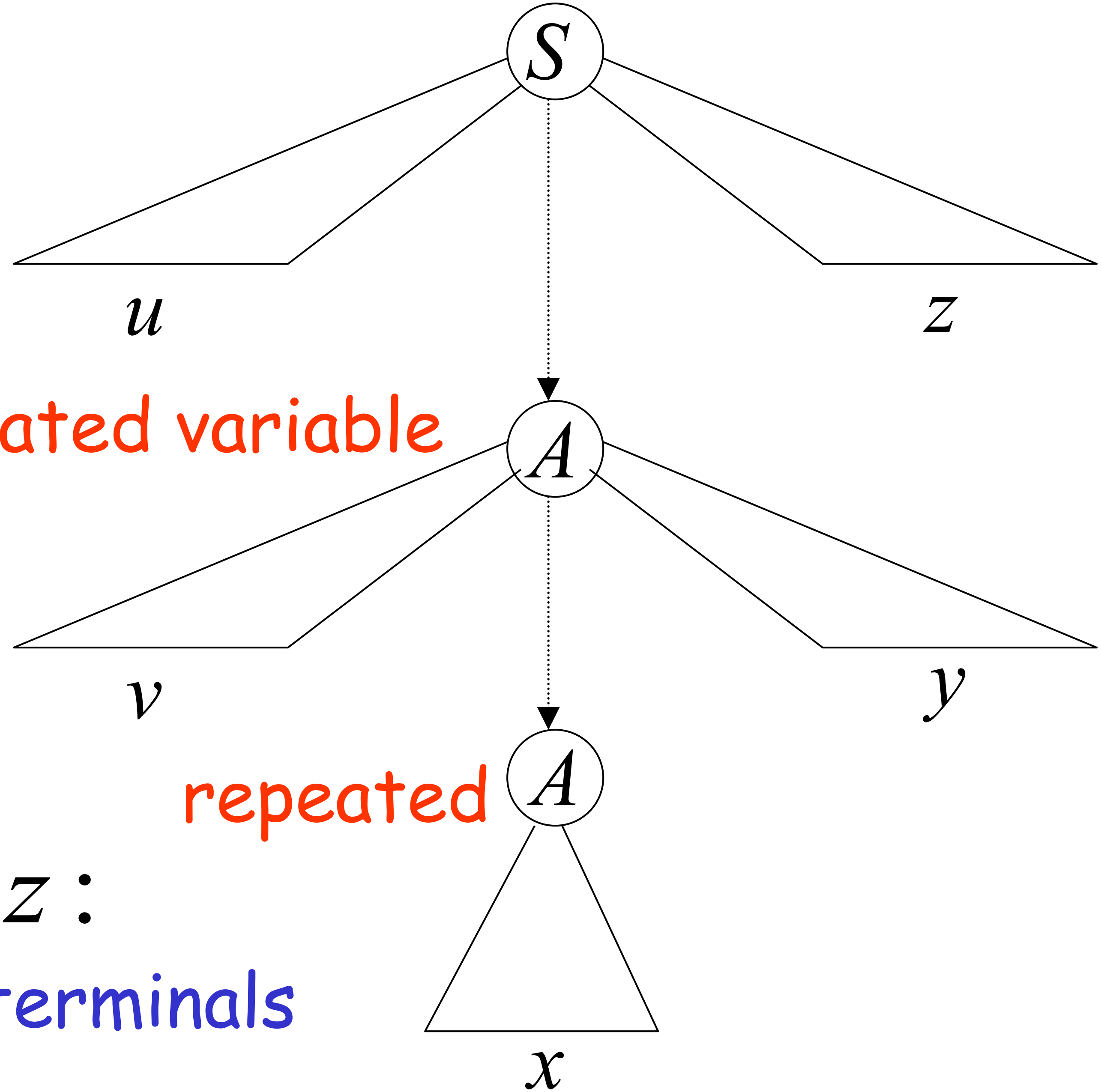
เพราะ pumping Lemma คือ เครื่องมือสำหรับพิสูจน์ ความผิดว่าไม่ใช่จริงๆ

Derivation tree of string w

$S \to uAz$

$A \to vAy$

$A \to x$



Last repeated variable

repeated

$$w = uvxyz$$

$$u, v, x, y, z:$$

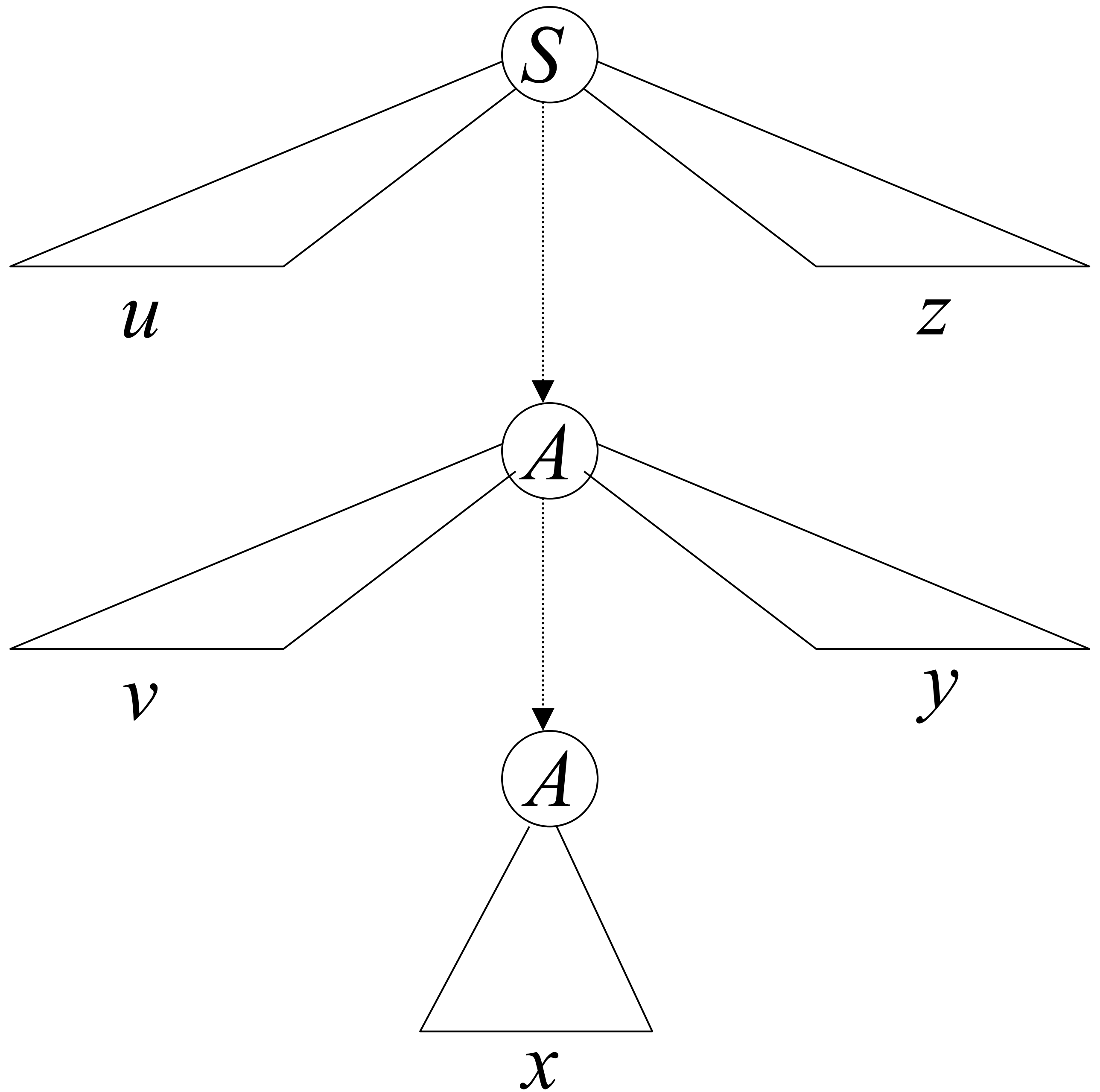
Strings of terminals

Possible derivations:

$$S \overset{*}{\Rightarrow} uAz$$

$$A \overset{*}{\Rightarrow} vAy$$

$$A \overset{*}{\Rightarrow} x$$



We know:

$$S \stackrel{*}{\Rightarrow} uAz \qquad A \stackrel{*}{\Rightarrow} vAy \qquad A \stackrel{*}{\Rightarrow} x$$

This string is also generated:

$$S \stackrel{*}{\Rightarrow} uAz \stackrel{*}{\Rightarrow} uxz$$

$$uv^0xy^0z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvxyz$$

The original $w = uv^1xy^1z$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow} uvvxyyz$$

$$uv^2xy^2z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \Rightarrow uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} uvvvxyyyz$$

$$uv^3xy^3z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} \ldots$$

$$\overset{*}{\Rightarrow} uvvv \cdots vAy \cdots yyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvv \cdots vxy \cdots yyyz$$

$$uv^i xy^i z$$

Therefore, any string of the form

$$uv^i xy^i z \qquad\qquad i \geq 0$$

is generated by the grammar G

Therefore,

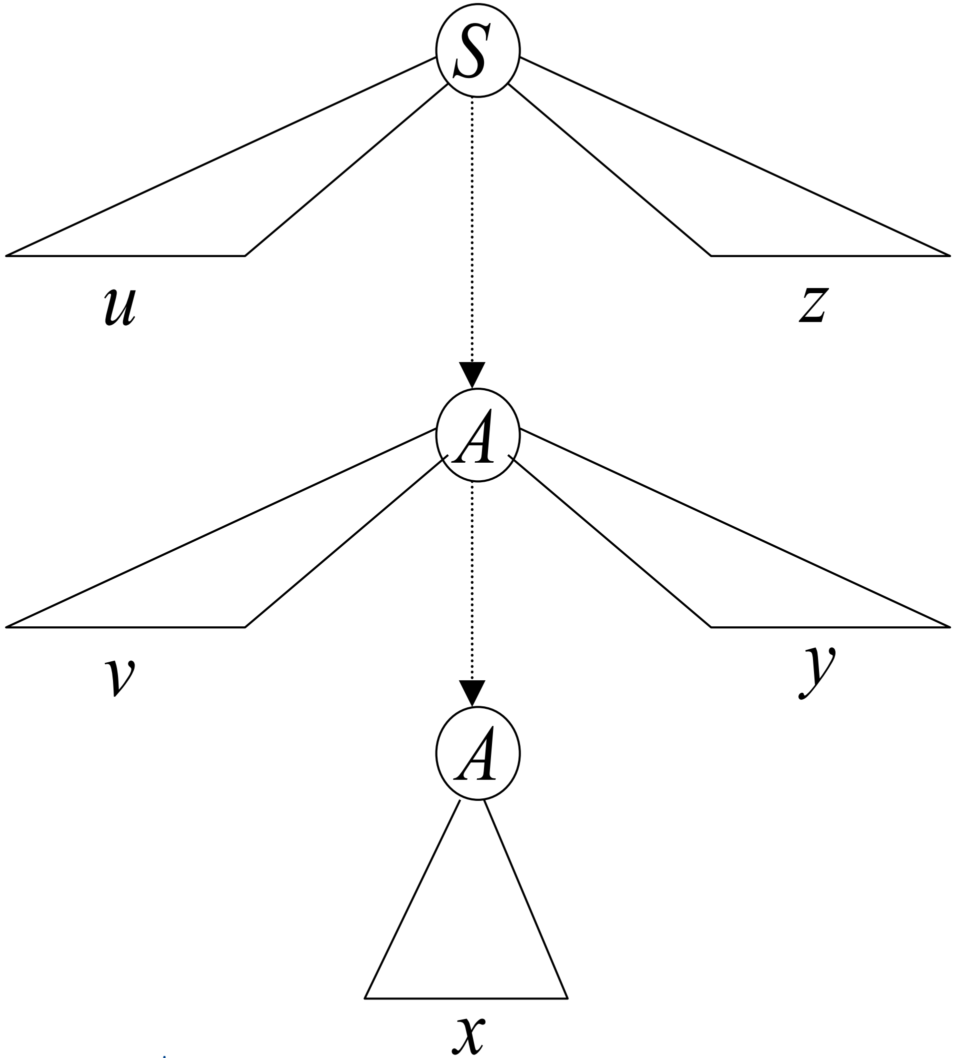
knowing that $uvxyz \in L(G)$

we also know that $uv^i xy^i z \in L(G)$

$$L(G) = L - \{\lambda\}$$

$$\Downarrow$$

$$uv^i xy^i z \in L$$



G = CFG

G : $S \rightarrow uAz$

$A \rightarrow vAy$

$A \rightarrow x$

แสดงว่ามันเป็น CFL ← เป็น L(G)

สร้าง

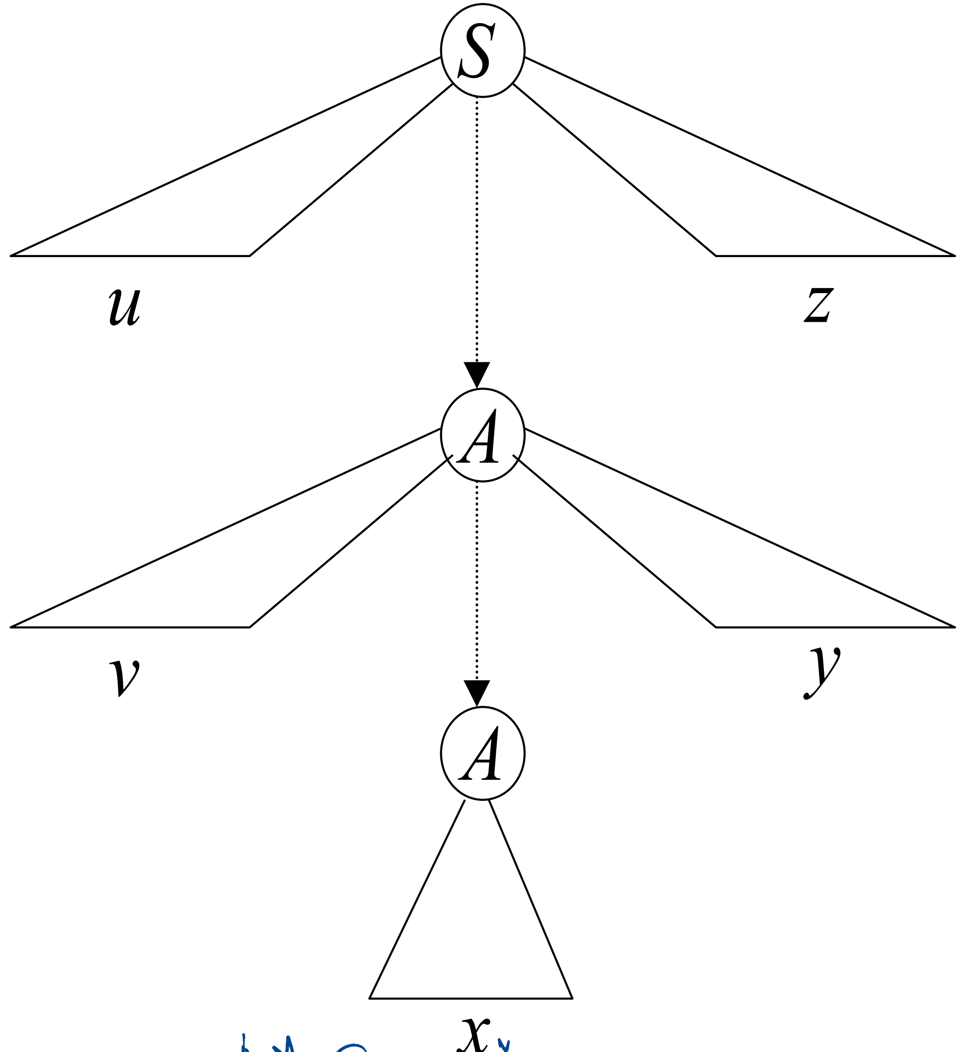
PDA

m คือ จำนวน state ของ PDA

เงื่อนไขข้อ 1

Observation: $|vxy| \leq m$

Since A is the last repeated variable



เงื่อนไข 2 ทำซ้ำ vy

Observation: $|vy| \geq 1$

Since there are no unit or λ-productions

The Pumping Lemma:

For infinite context-free language L

PDA ที่จำนวน state เท่ากับ

there exists an integer m such that

1 for any string $w \in L, \quad |w| \geq m$

2 we can write $w = uvxyz$

ต้อง

ต้อง

3 with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

4 เลือกค่า i

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Applications of The Pumping Lemma

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Theorem: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^n b^n c^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $w = a^m b^m c^m$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

We can write: $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^n b^n c^n : n \geq 0\}$$

③

$$w = a^m b^m c^m$$

vxy vxy vxy

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

หรือ
ยาว ก็ได้

① ตามที่ มีไม่เกิน m

We examine <u>**all**</u> the possible locations of string vxy in w

② พิจารณาทุก vxy ที่มีสิทธิ์จะเป็นไปได้

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within a^m

$$\underbrace{\overbrace{\underbrace{aa}_{u}\underbrace{a...aa}_{vxy}a}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}}$$

$$\underbrace{a\; bbb...bbb\; ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: v and y consist from only a

$vy = a^k, \ k \geq 1$

$$\overbrace{aaa\ldots aaa}^{m}\ \overbrace{bbb\ldots bbb}^{m}\ \overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aa}_{u}\underbrace{a\ldots a}_{vxy}\underbrace{a\ bbb\ldots bbb\ ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: Repeating v and y

$$k \geq 1$$

$$\underbrace{aa}_{u}\underbrace{aaaa...aaaaa}_{v^2xy^2}\underbrace{a\,bbb...bbb\,ccc...ccc}_{z}$$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

$$\underbrace{\overbrace{a\,a\,aaaa...aaaa}^{m+k}}_{\ldots}$$

$$\underbrace{a}_{u}\,\underbrace{aaaaa...aaaaa}_{v^2xy^2}\,\underbrace{a\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

However: $uv^2xy^2z = a^{m+k}b^m c^m \notin L$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: vxy is within b^m

$$\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\,bb}_{u}\underbrace{b...bb}_{vxy}\underbrace{b\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: Similar analysis with case 1

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\;bb}_{u}\underbrace{b...bb}_{vxy}\underbrace{b\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: vxy is within c^m

$$\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa\ldots aaa\;bbb\ldots bbb\;c}_{u}\underbrace{cc\ldots c}_{vxy}\underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

อังว่าหลังจากกรณีที่ 1

Case 3: Similar analysis with case 1

$$\overbrace{aaa\ldots aaa}^{m}\ \overbrace{bbb\ldots bbb}^{m}\ \overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa\ldots aaa\ bbb\ldots bbb\ c}_{u}\ \underbrace{cc\ldots c}_{vxy}\ \underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: vxy overlaps a^m and b^m

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaa\,bbb}_{vxy}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

กรณีย่อย 1

Case 4: Possibility 1: v contains only a

y contains only b

$$\underbrace{aa}_{u}\underbrace{a\ldots aa\overbrace{a}^{}\,bb\,b}_{vxy}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

$$\overbrace{aaa\ldots aaa}^{m}\ \overbrace{bbb\ldots bbb}^{m}\ \overbrace{ccc\ldots ccc}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a k_1

y contains only b k_2

$$k_1 + k_2 \geq 1$$

$$\overbrace{aaa...aaaaaaa}^{m+k_1}\overbrace{bbbbbbb...bbb}^{m+k_2}\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaaaaaa\,bbbbbbb}_{v^2xy^2}\underbrace{...bbb\,ccc...ccc}_{z}$$

$u \rightarrow a^{m-k_1}$ $\quad v^2xy^2$ pump $\quad z\ b^{m-k_2}\ c^m$

$$L = \{a^n b^n c^n : n \ge 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$|v|$ $|y|$

$$k_1 + k_2 \ge 1$$

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\;\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaaaa\;bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\;ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 2: v contains a^{k_1} and b^{k_2}

y contains only b^{k_2}

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}...\underbrace{a\underbrace{aa\,b}_{v}b\underbrace{b}_{y}}_{vxy}\underbrace{...bbb\,ccc...ccc}_{z}$$

$u \rightarrow a^{m-k_1}$ $\qquad vxy \qquad$ z b^{m-k_2} c^m

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $k_1 + k_2 + k \geq 1$

pump i = 2 $\longrightarrow$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

Contradiction!!!

$a^3b^3c^3$ $(m=3)$

aaabbbccc

$$\underbrace{aa}_{u}\ \underbrace{ab}_{v}\ \underbrace{b}_{x}\ \underbrace{b}_{y}\ \underbrace{ccc}_{z}$$

$u = aa,\ v=ab,\ x=b,\ y=b,\ z=ccc$

$$i=2,\ uv^2xy^2z = \overset{u}{aa}\ \overset{v^2}{abab}\ \overset{x}{b}\ \overset{y^2}{bb}\ \overset{z}{ccc}$$

$$= aaababbbbccc$$

$$= a^3\ b^1\ a^1\ b^{3+1}\ c^3$$

$$a^m\ b^{k1}\ a^{k2}\ b^{m+k}\ c^m$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\;bbb}_{vxy}\underbrace{...bbb\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b

ด้วย

Similar analysis with Possibility 2

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: vxy overlaps b^m and c^m

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\;bbb...}_{u}\underbrace{bbb\;ccc...}_{vxy}\underbrace{ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

อ้างว่าคล้าย

Case 5: Similar analysis with case 4

$$\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\,bbb...}_{u}\,\underbrace{bbb\,ccc...}_{vxy}\,\underbrace{ccc}_{z}$$

There are no other cases to consider

(since $|vxy| \leq m$, string vxy cannot

overlap a^m, b^m and c^m at the same time)

ไม่เกิดกรณี overlap $a^m b^m c^m$ เพราะความยาวจะเกิน m

ดังนั้นไม่ต้องพิจารณา overlaps $a^m b^m c^m$

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free