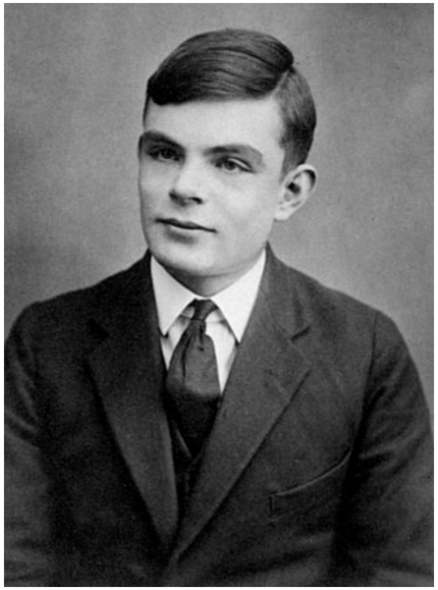
Turing Machines



Alan M. Turing (1912-1954)

- English mathematician, computer scientist
- providing the concepts of algorithm and computation with the Turing machine, which can be considered a model of a computer.

- During the 2nd World War, Turing devised a technique for breaking German ciphers machine (Enigma).
- Turing was prosecuted in 1952 for homosexual acts.
- He accepted chemical treatment, as an alternative to prison. However, it was unsuccessful.
- In 1954, he committed suicide by cyanide poisoning.
- In 2017, UK retroactively apologize for an outlawed homosexual acts.

The Language Hierarchy

$a^n b^n c^n$?

ww ?

Context-Free Languages

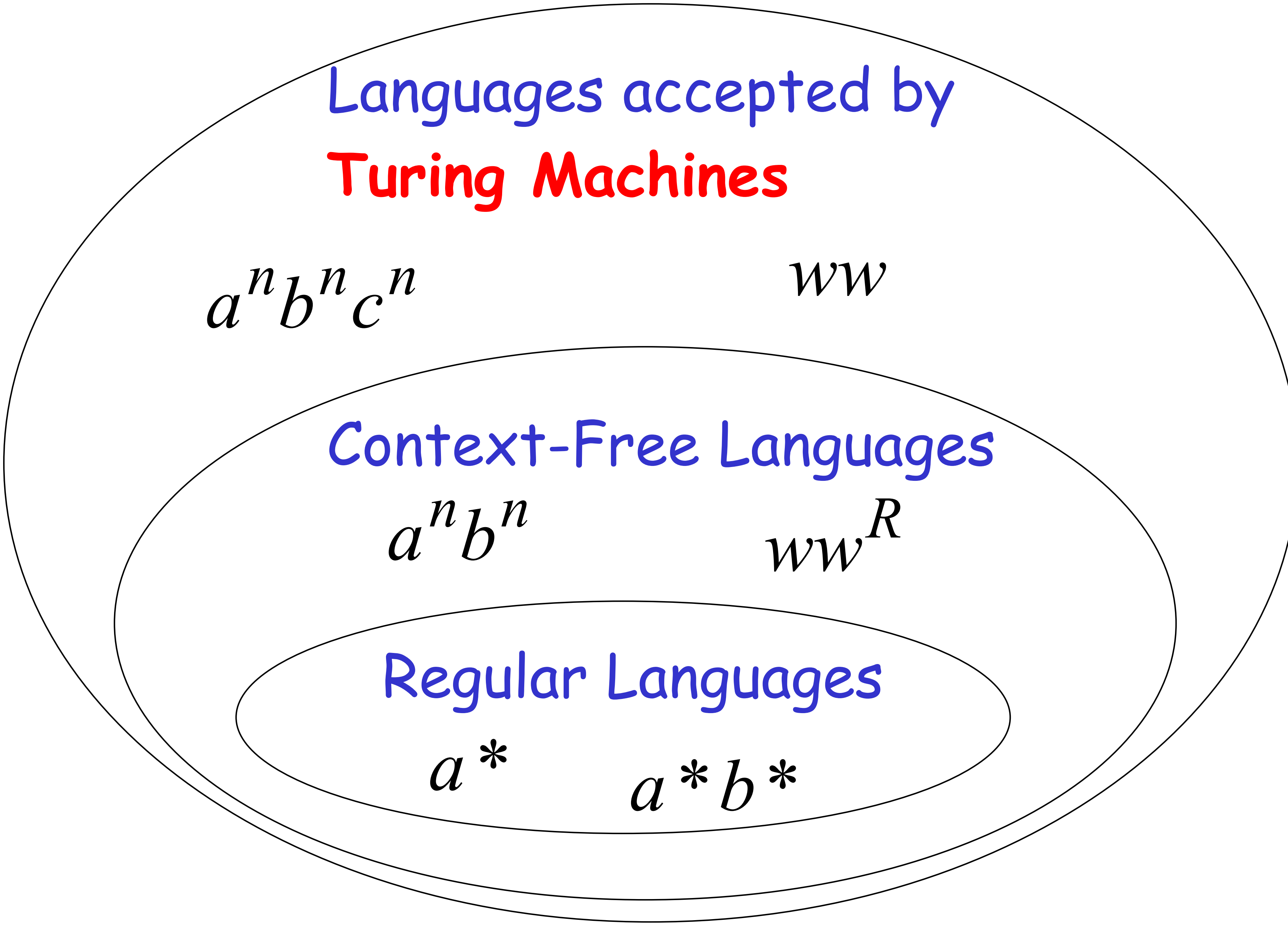
$a^n b^n$

ww^R

Regular Languages

a^*

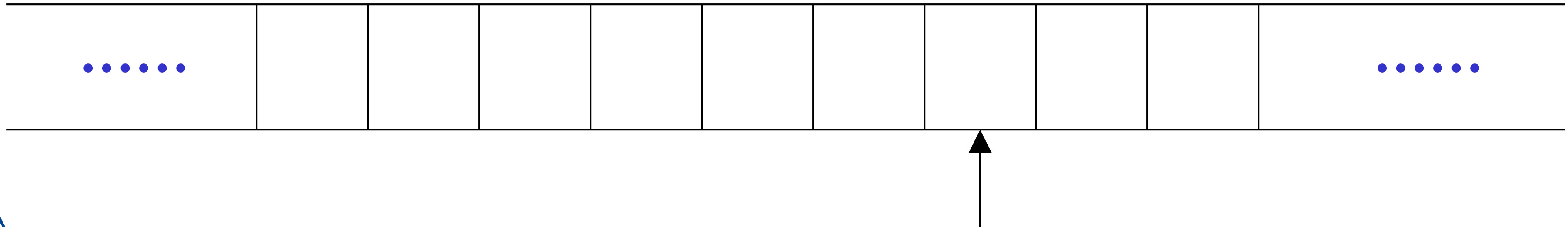
$a^* b^*$


Languages accepted by
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
$a^n b^n$
ww^R
Regular Languages
a^*
$a^* b^*$

A Turing Machine

มี 2 ส่วน

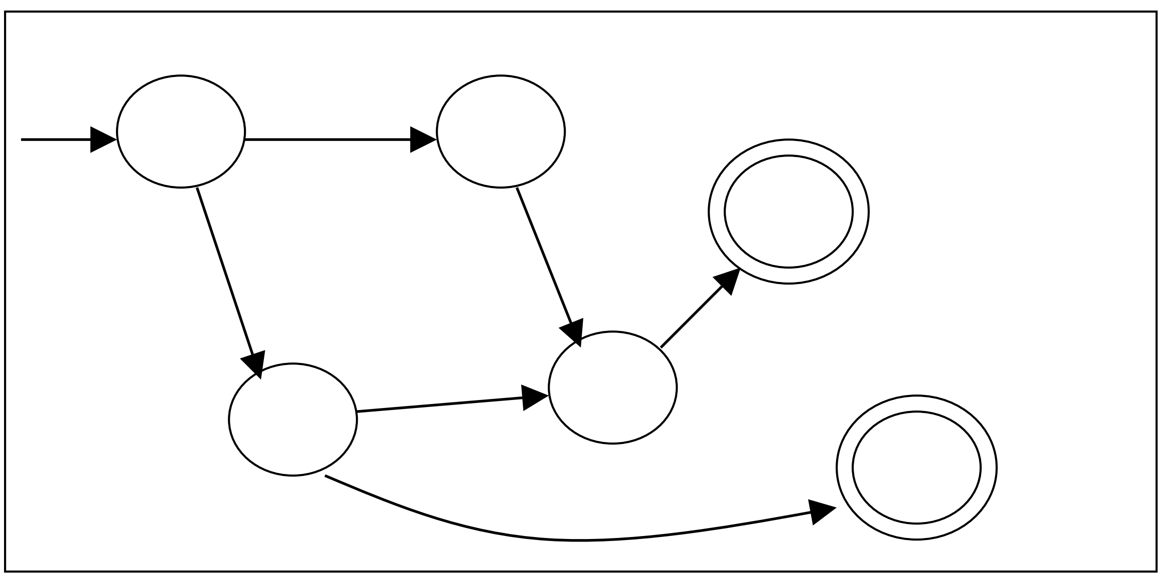
Tape



Read-Write head

อ่าน เขียนได้

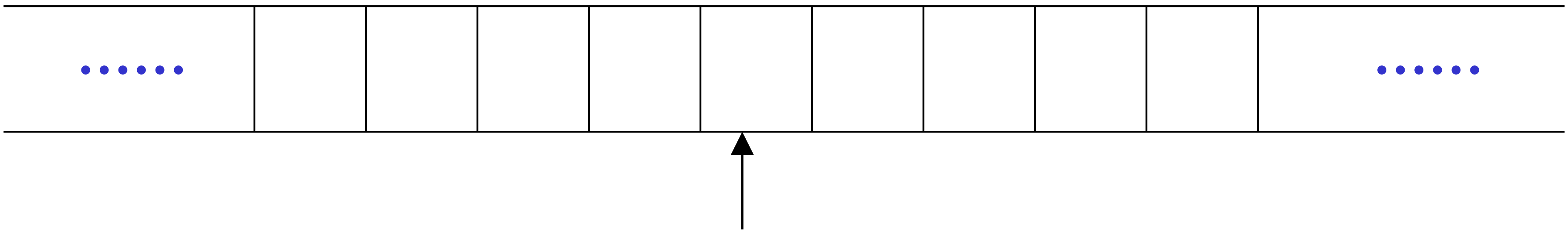
Control Unit



The Tape

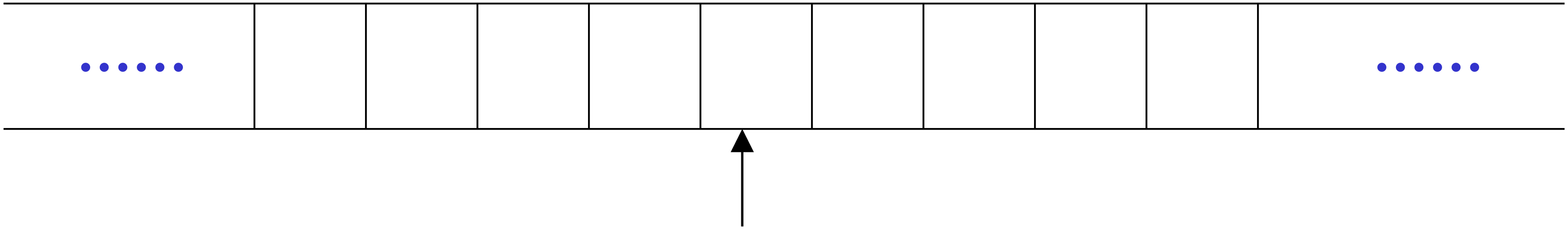
ยาวไม่จำกัด (ซ้ายก็ได้ - ขวาก็ได้)

No boundaries -- infinite length



Read-Write head

The head moves Left or Right



Read-Write head

ขั้นตอน 1 time step

The head at each time step:

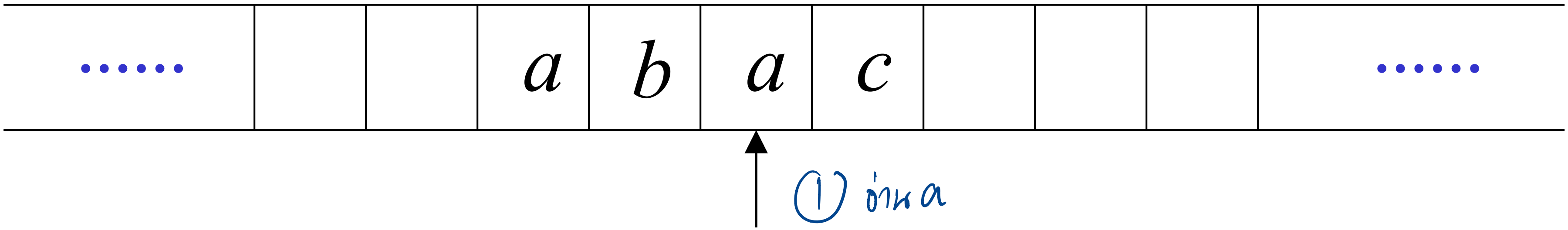
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

เลือกอย่างใดอย่างหนึ่ง

เคลื่อนได้ทีละ 1 ช่อง

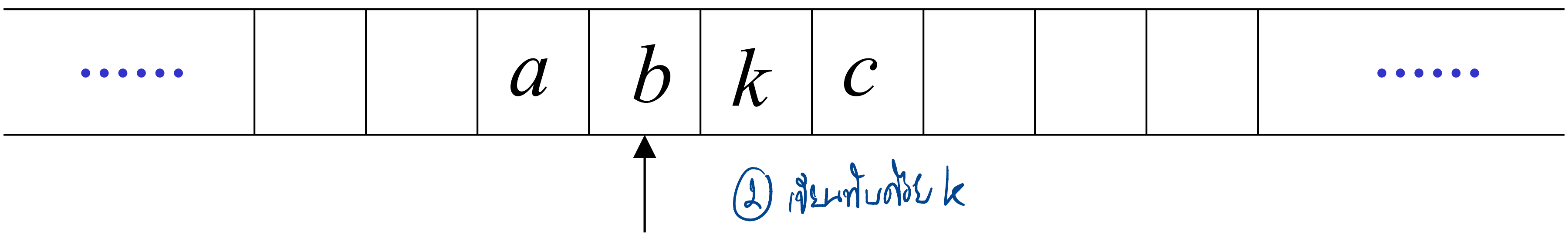
Example:

Time 0



① อ่าน a

Time 1

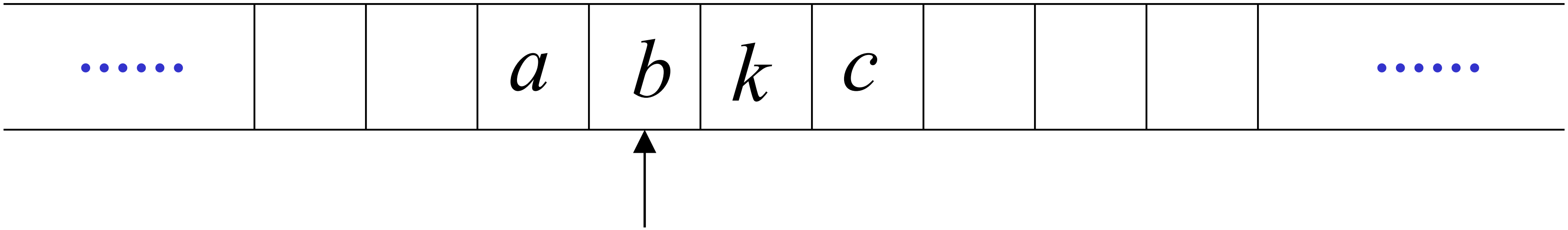


② เขียนทับด้วย k

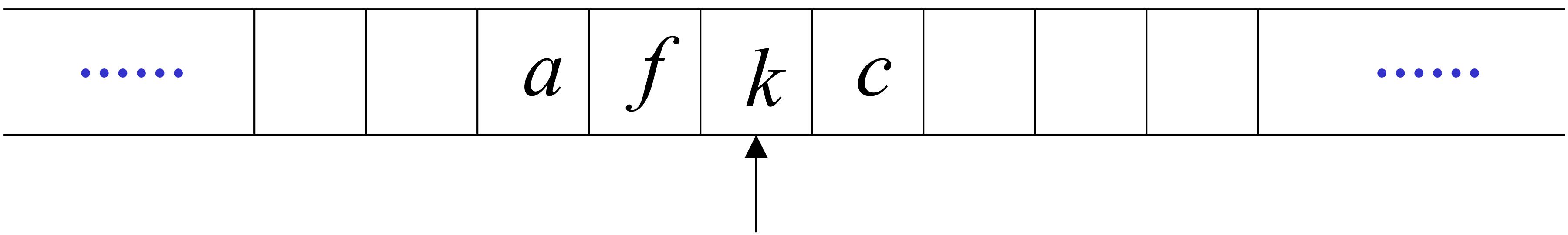
③ เลื่อนหัวอ่านไปทางซ้าย 1 ตน.

1. Reads a
2. Writes k
3. Moves Left

Time 1

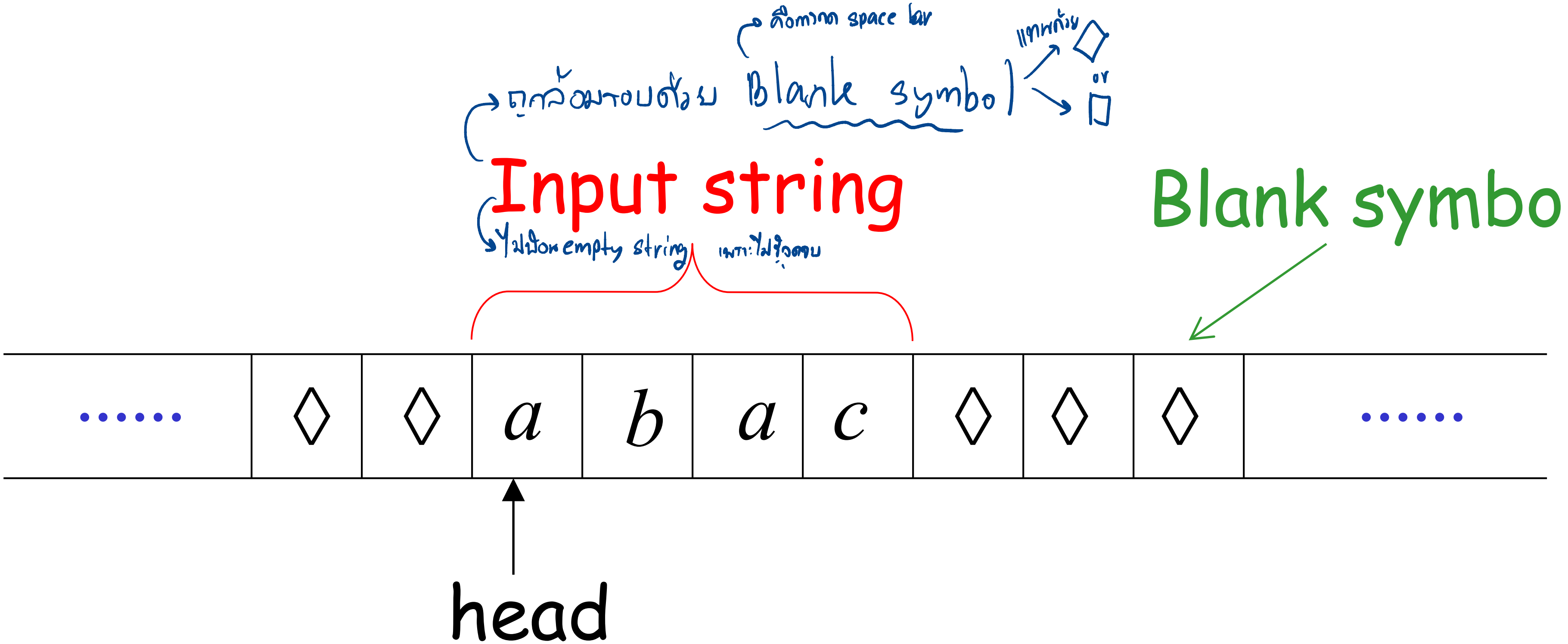


Time 2

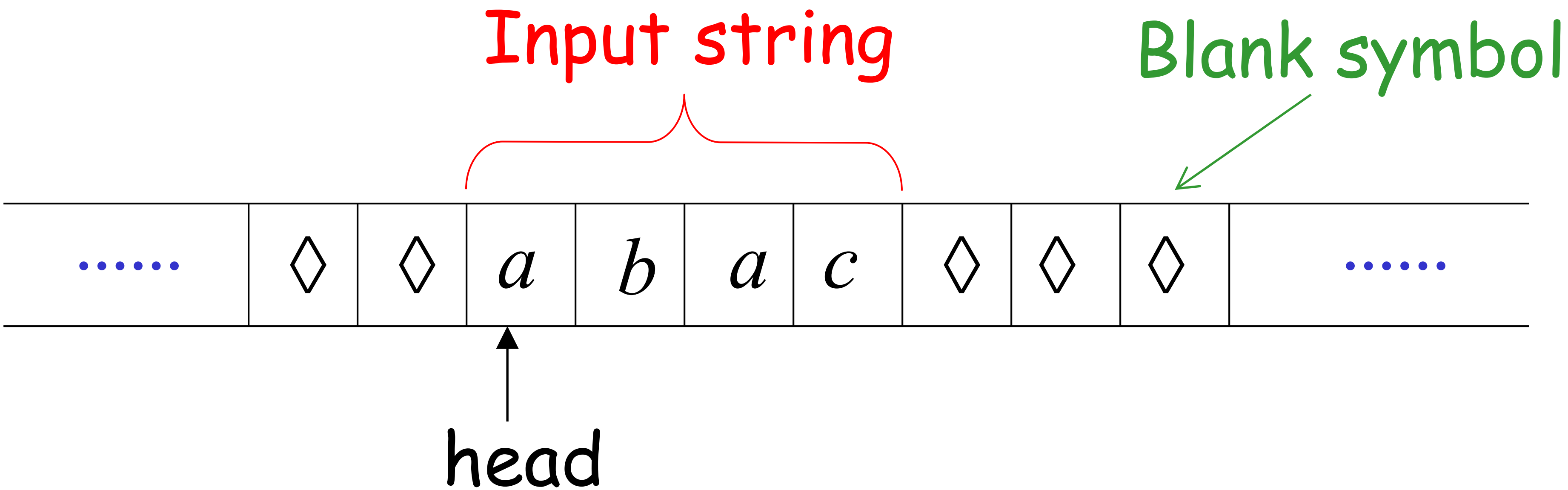


1. Reads b
2. Writes f
3. Moves Right

The Input String

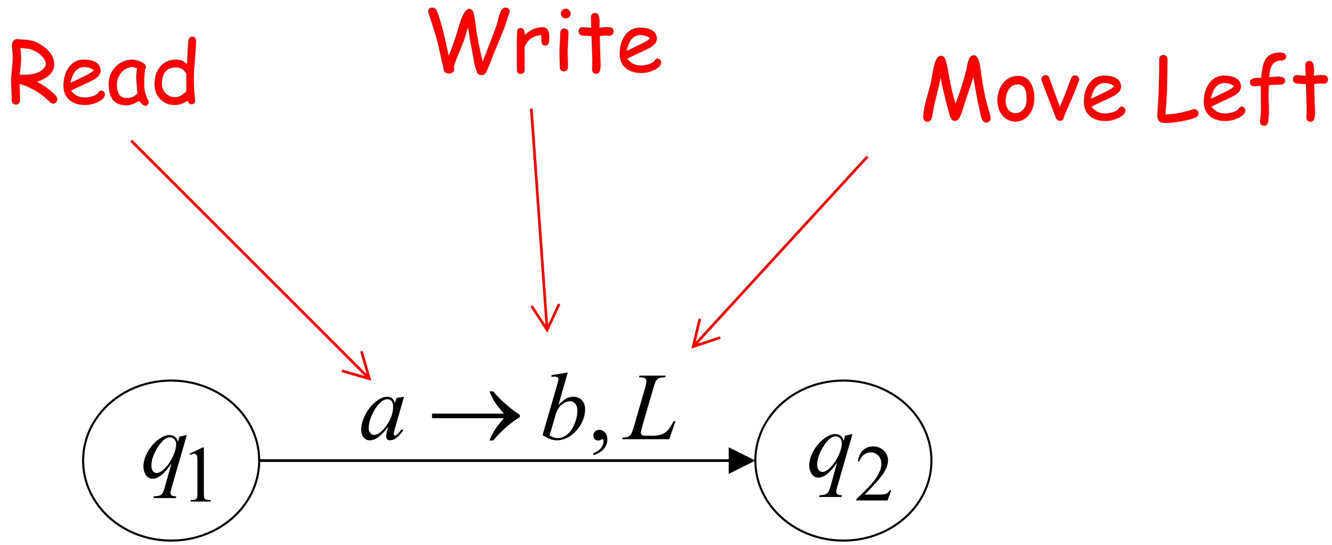


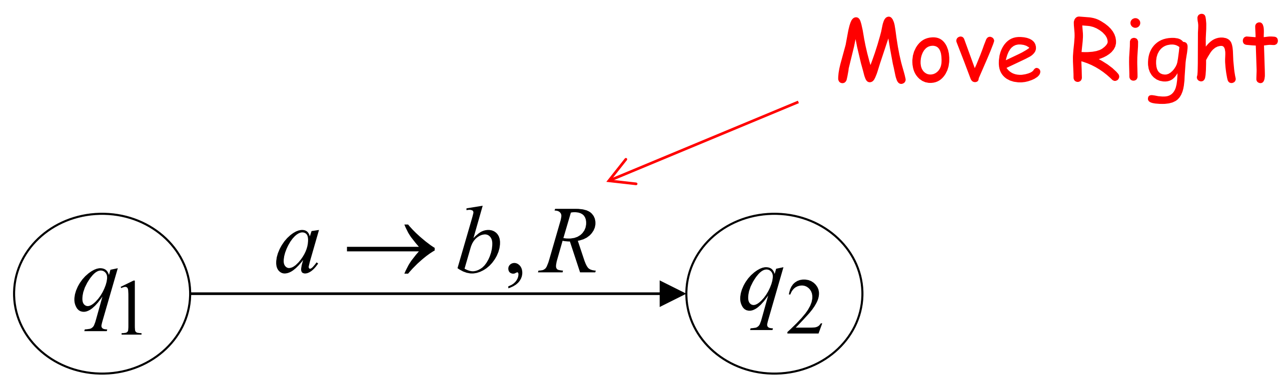
Head starts at the leftmost position
of the input string



Remark: the input string is never empty

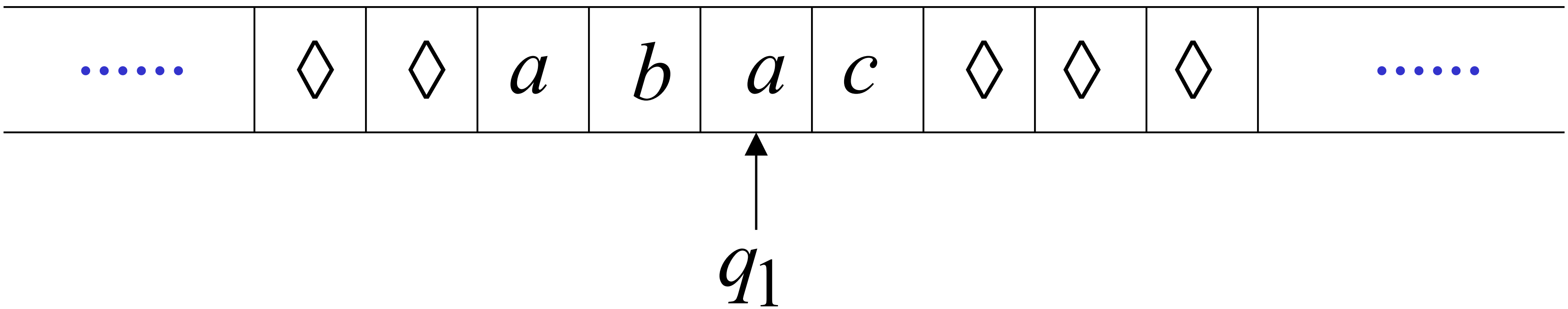
States & Transitions



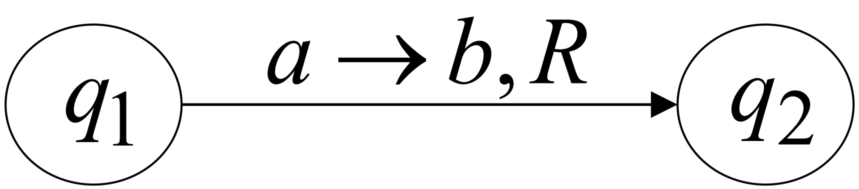


Example:

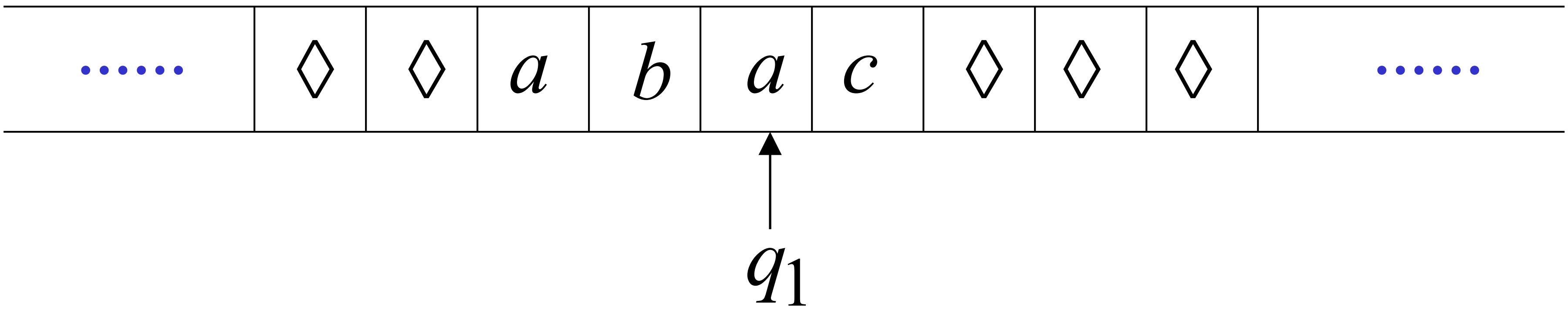
Time 1



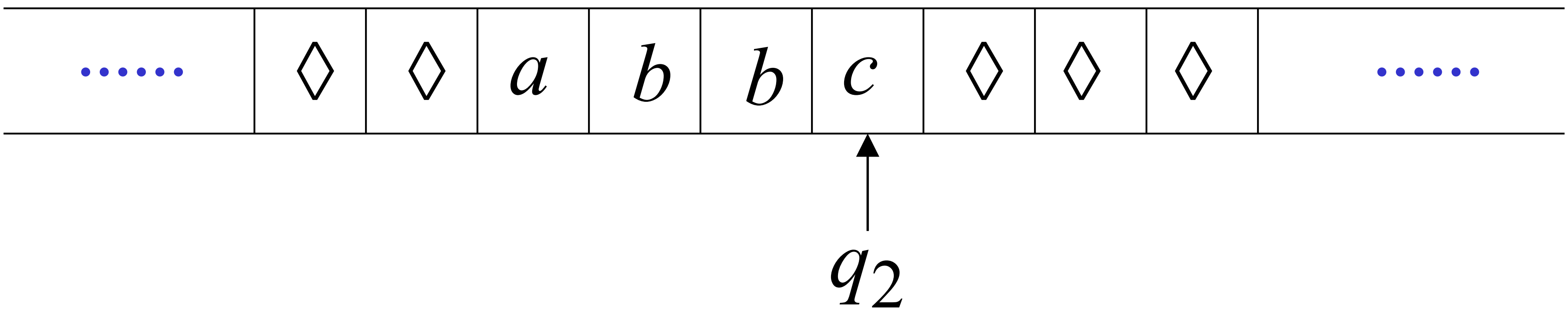
current state

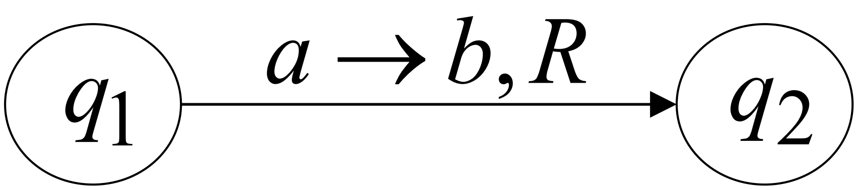


Time 1



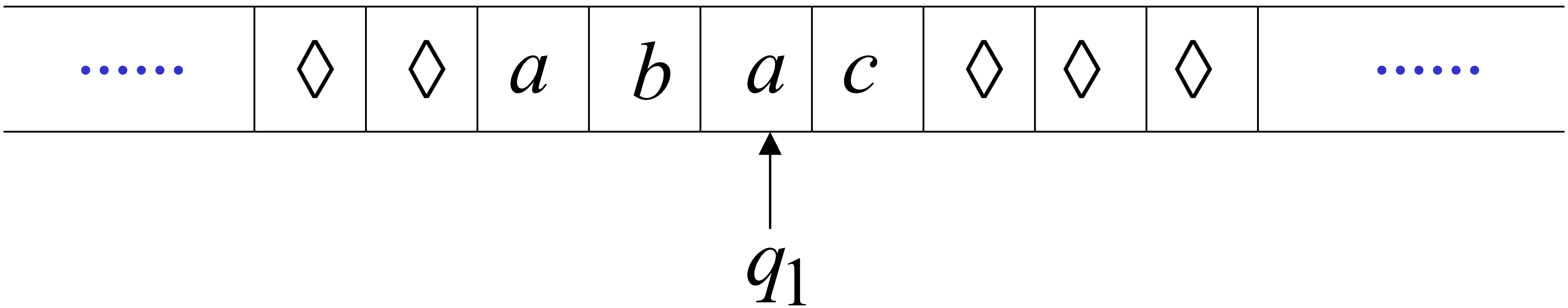
Time 2



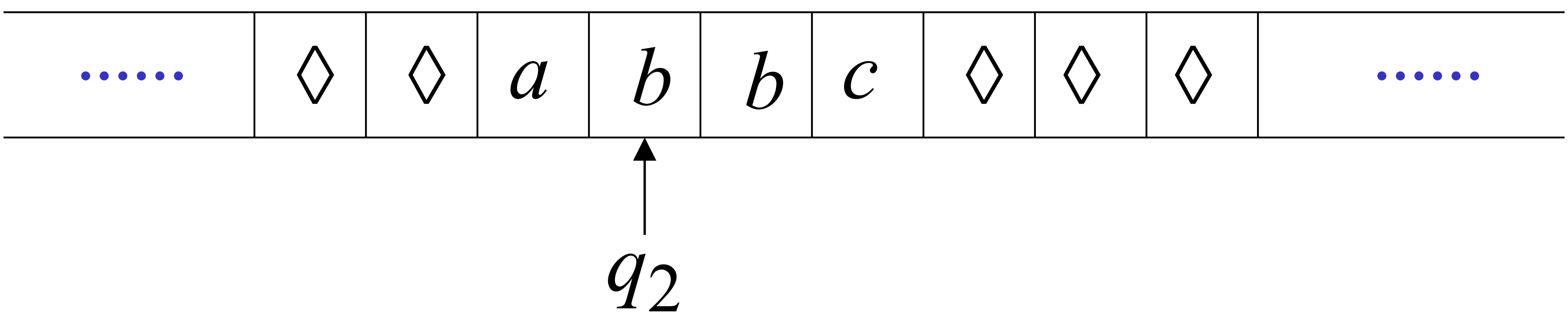


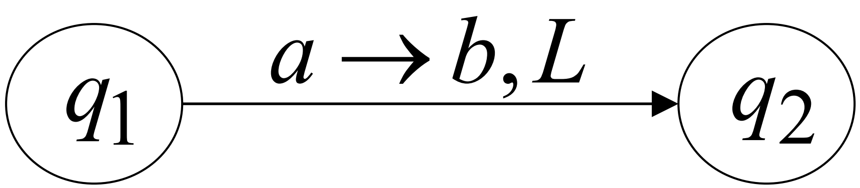
Example:

Time 1



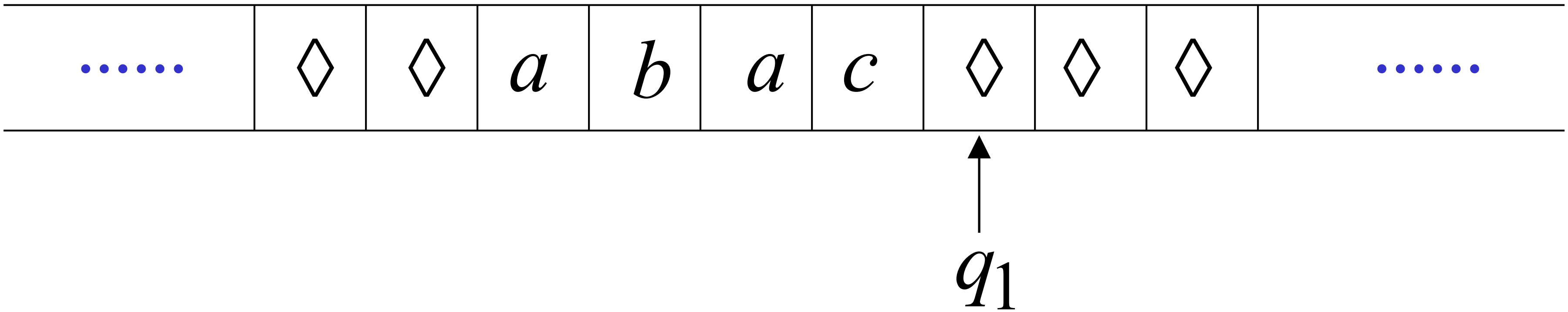
Time 2



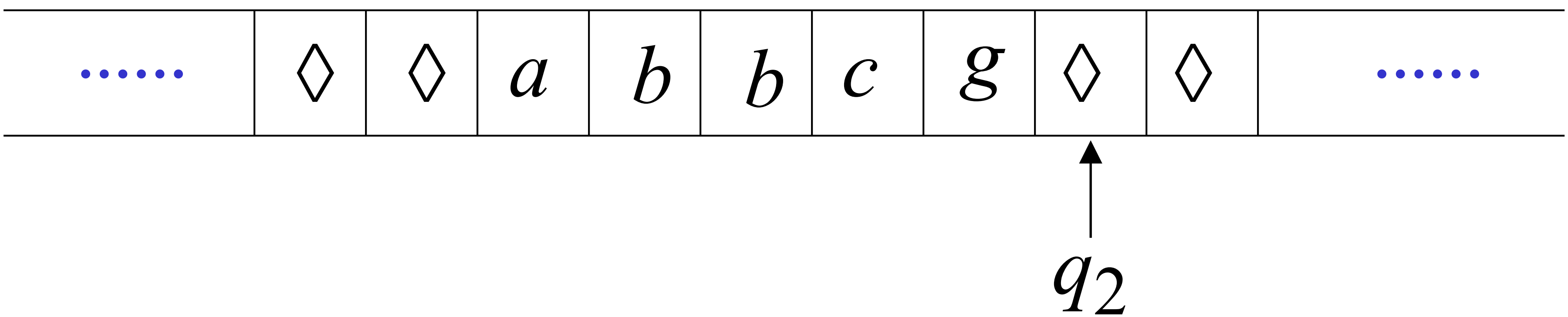


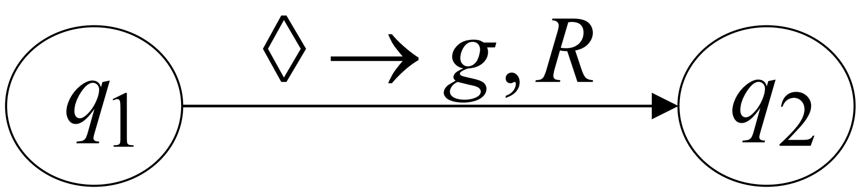
Example:

Time 1



Time 2



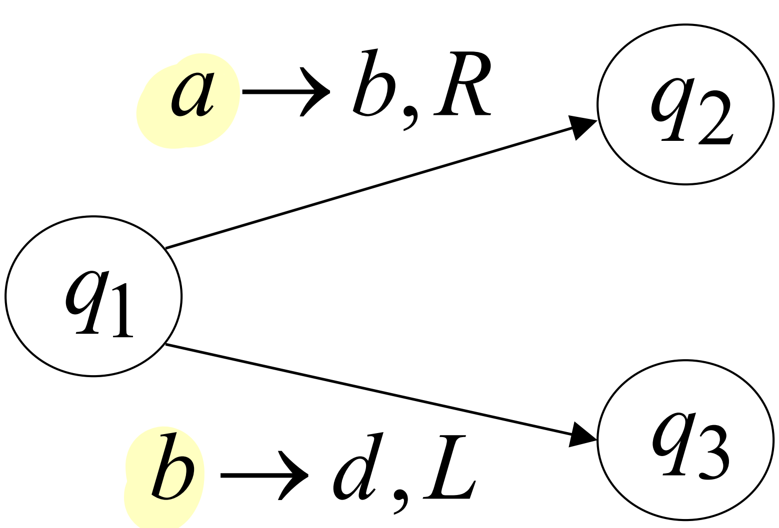


Determinism = non Determinism

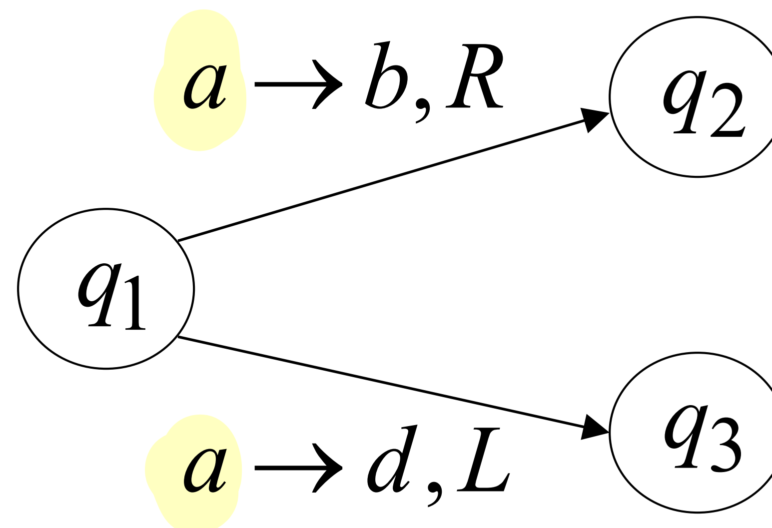
Turing Machines are deterministic

transition ต้องเป็นอะไรที่คาดเดาได้

Allowed



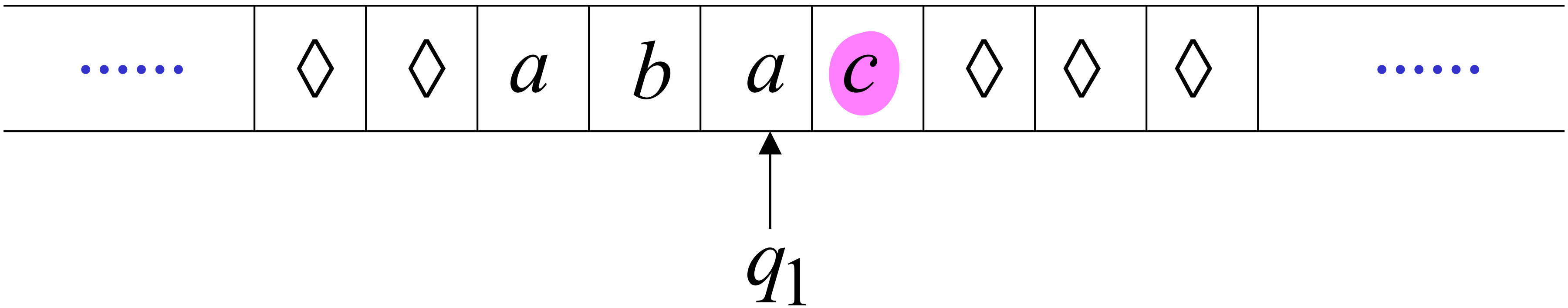
Not Allowed

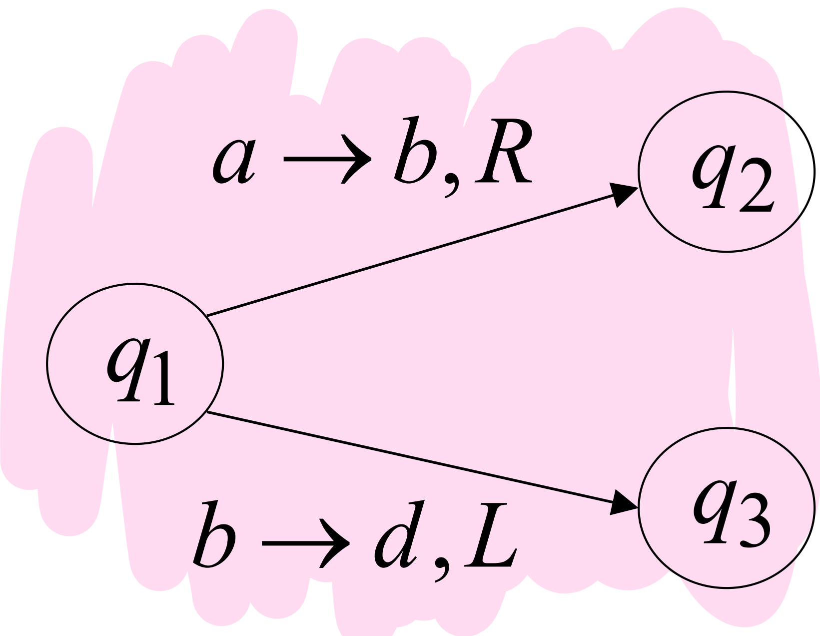


No lambda transitions allowed

Partial Transition Function

Example: transition ไม่ต้องมีทุก input (ตามที่เห็นไปแล้ว)





Allowed:

No transition for input symbol c

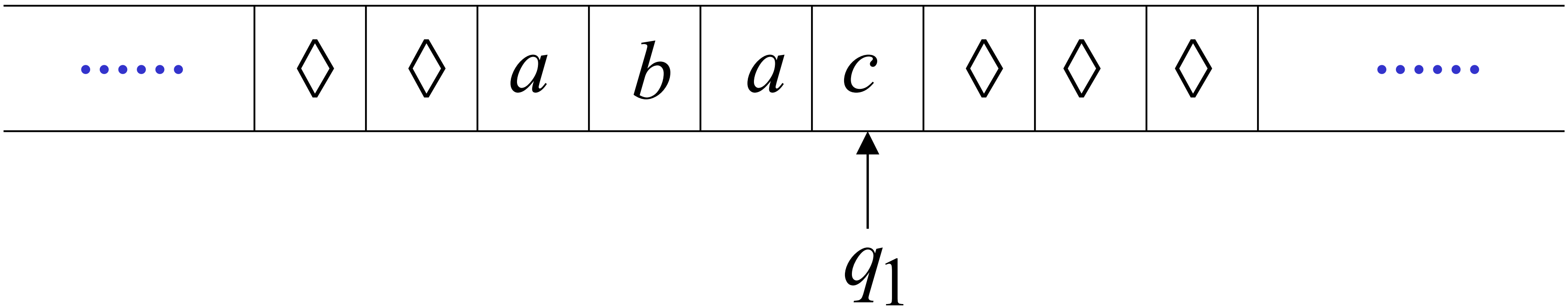
Halting

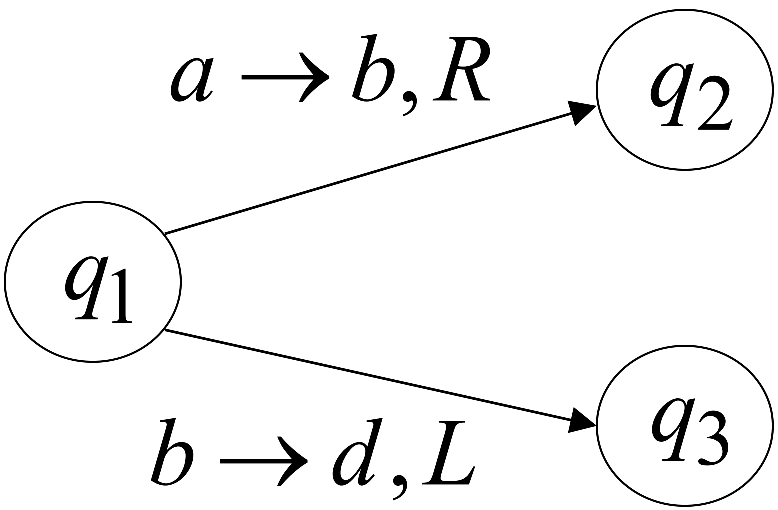
การหยุดการทำงาน

The machine ***halts*** if there are no possible transitions to follow

เมื่อไม่มี transition ที่เป็นไปได้

Example:

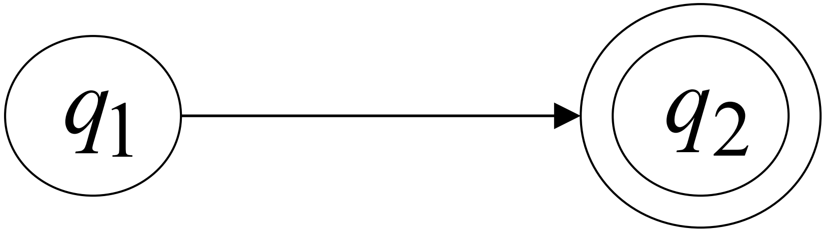




No possible transition

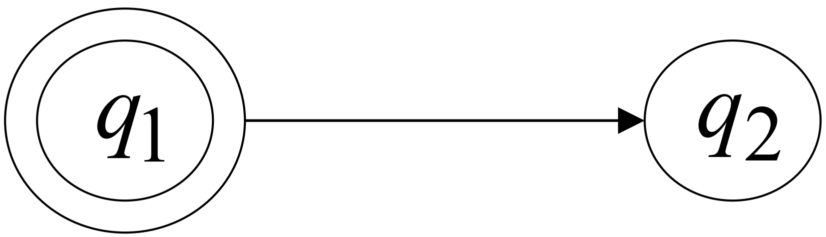
HALT!!!

Final States



Allowed

เข้า ◎ ต้องหยุดเลย ห้ามออก



Not Allowed

- Final states have no outgoing transitions

- In a final state the machine halts

Acceptance เงื่อนไข

เข้า final state พร้อมหยุดมั้ย

Accept Input → If machine halts in a final state

Reject Input → If machine halts ① in ที่ a non-final state

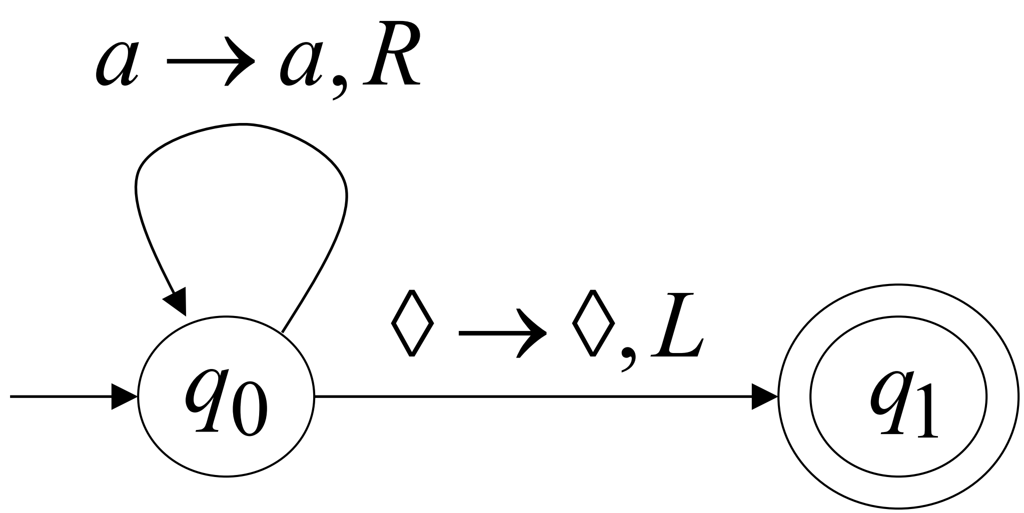
or

If machine enters an ติด *infinite loop* ②

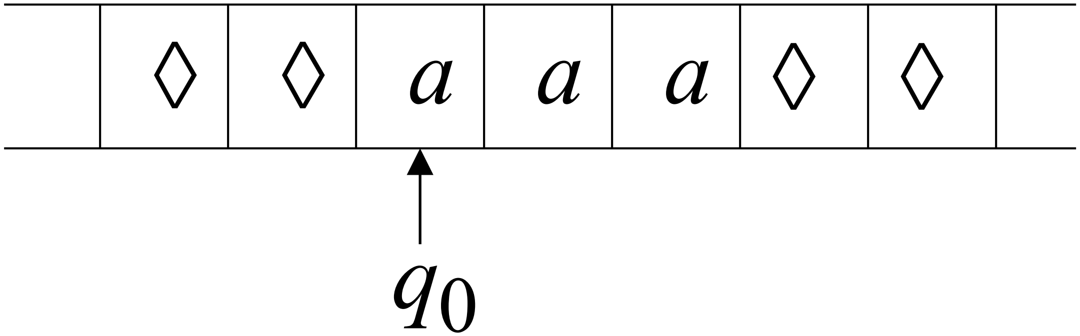
Turing Machine Example

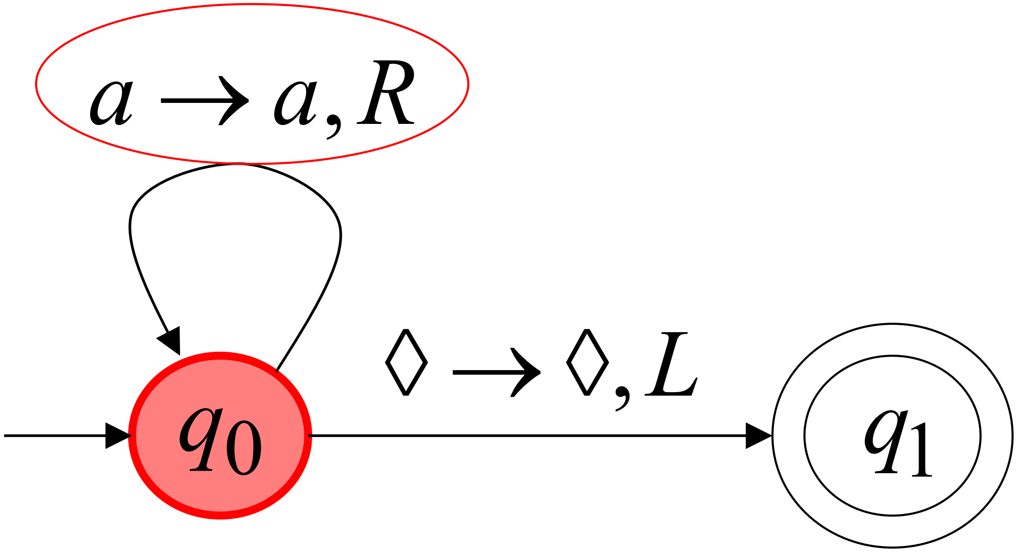
A Turing machine that accepts the language:

$$a^*$$

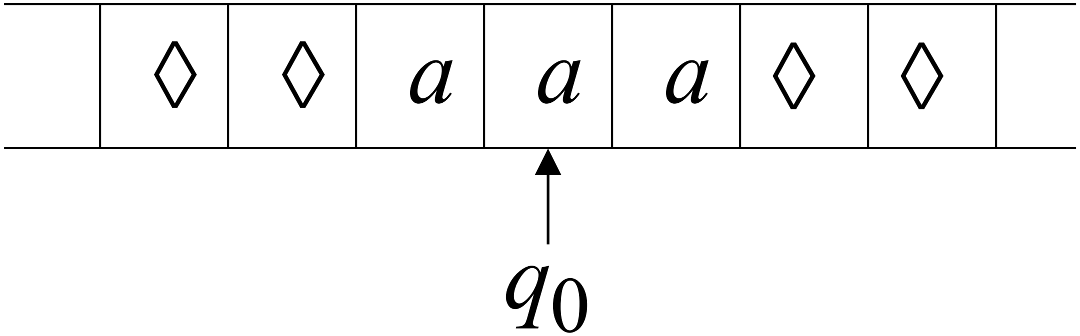


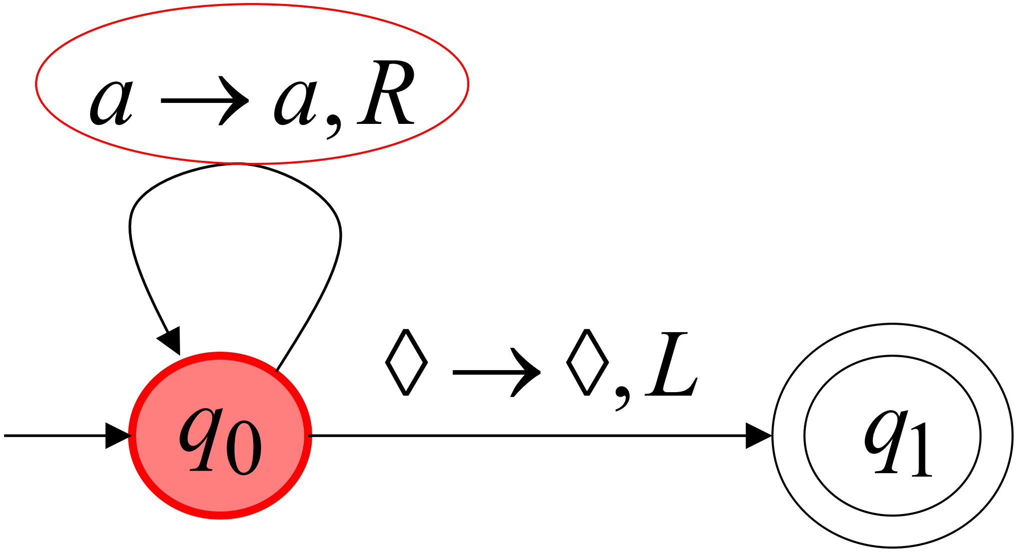
Time 0



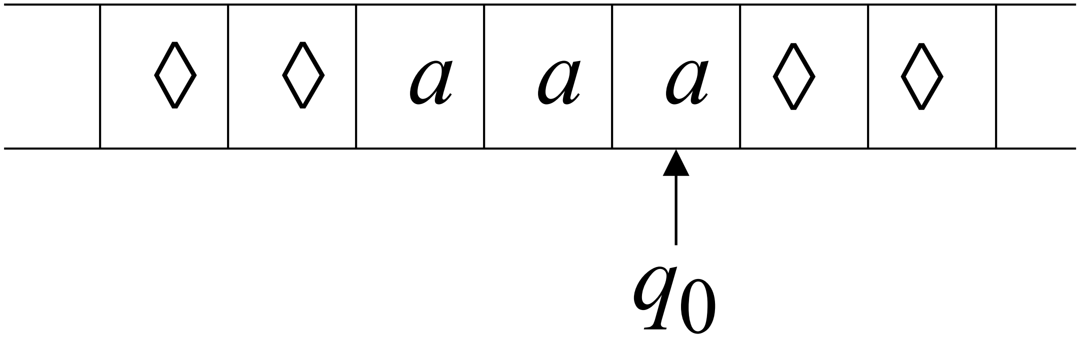


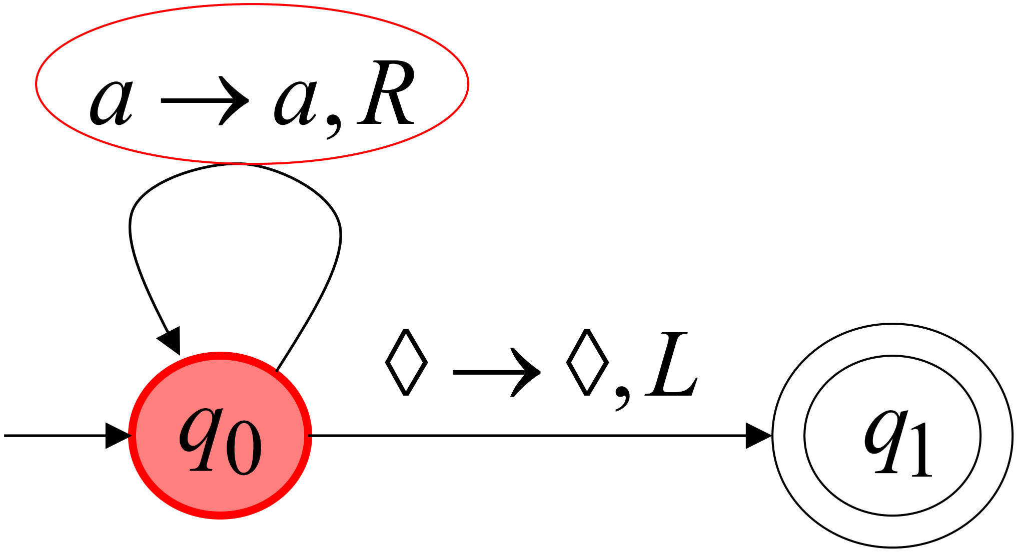
Time 1



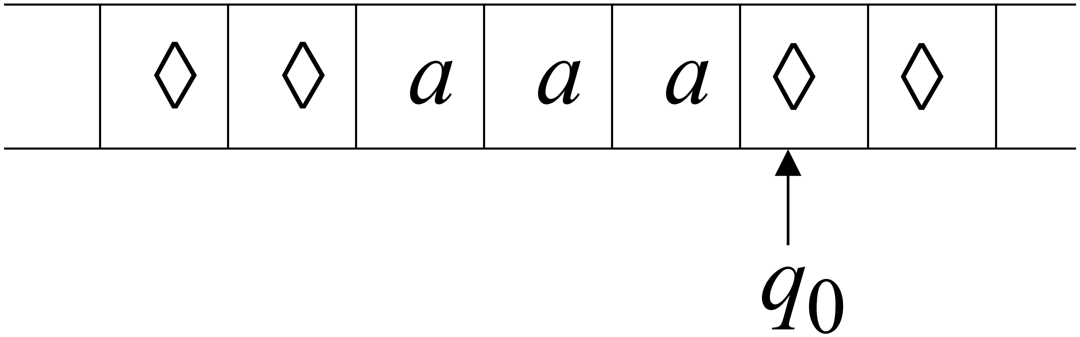


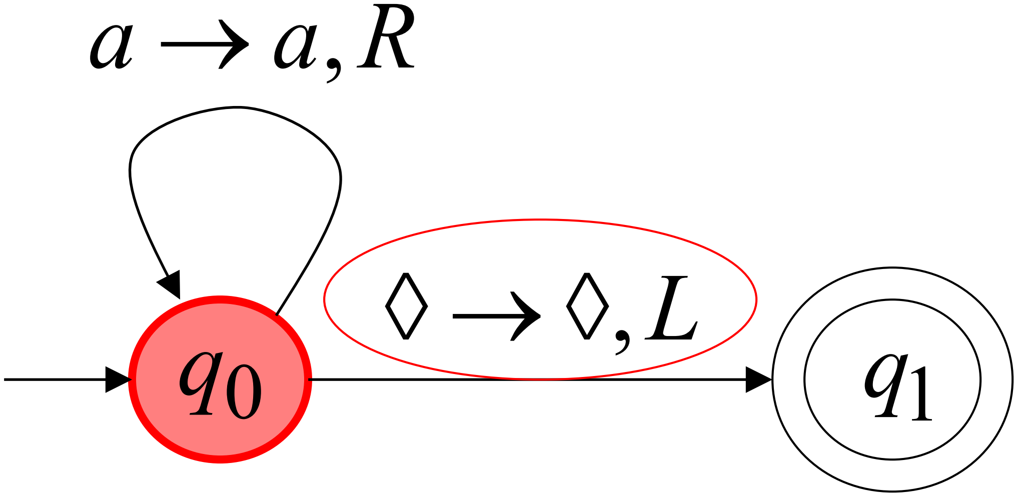
Time 2



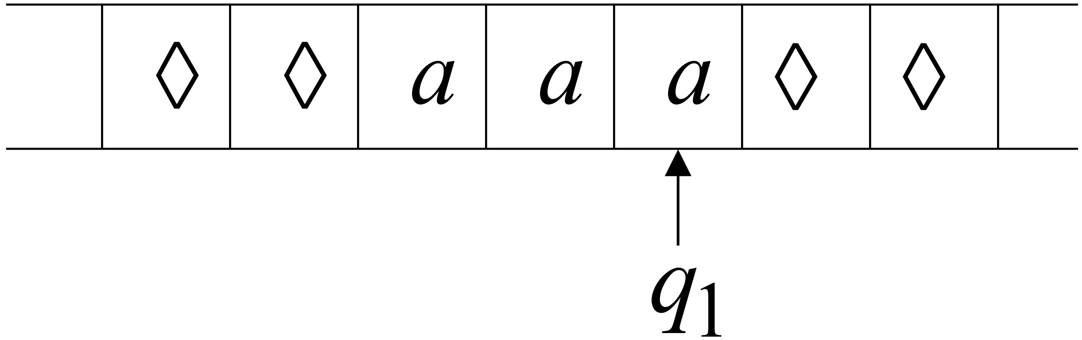


Time 3

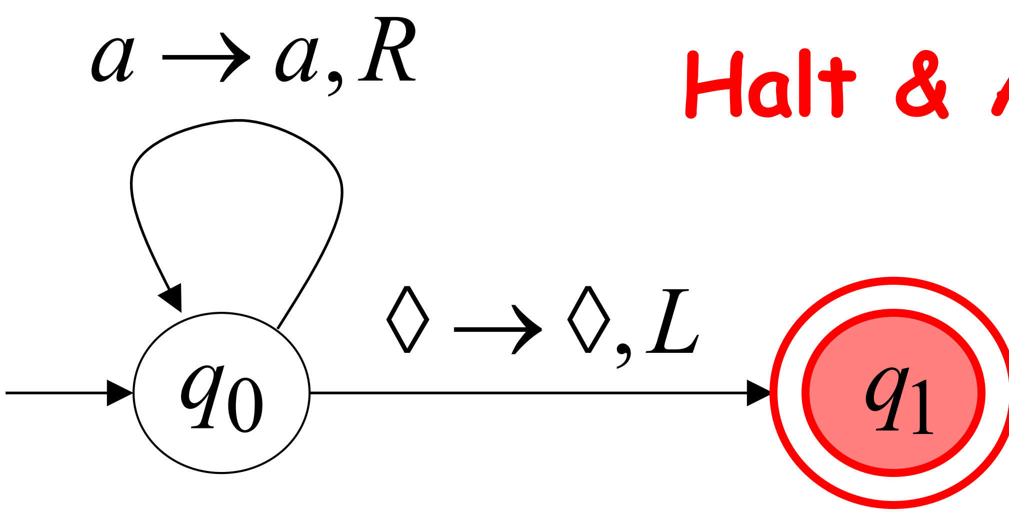




Time 4

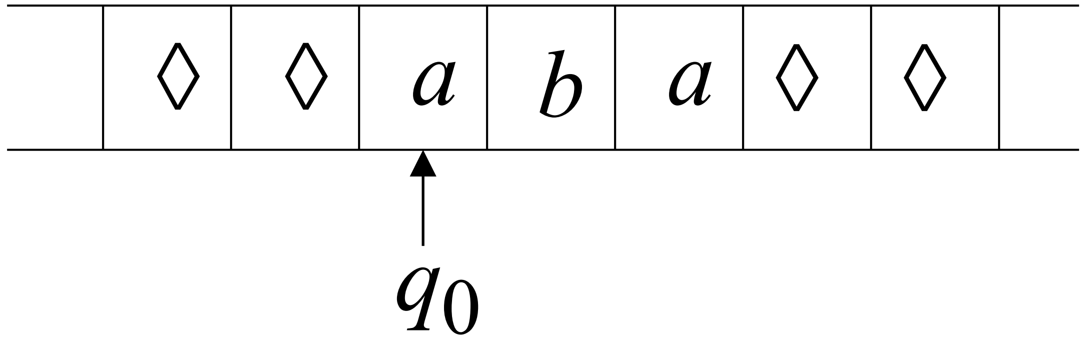


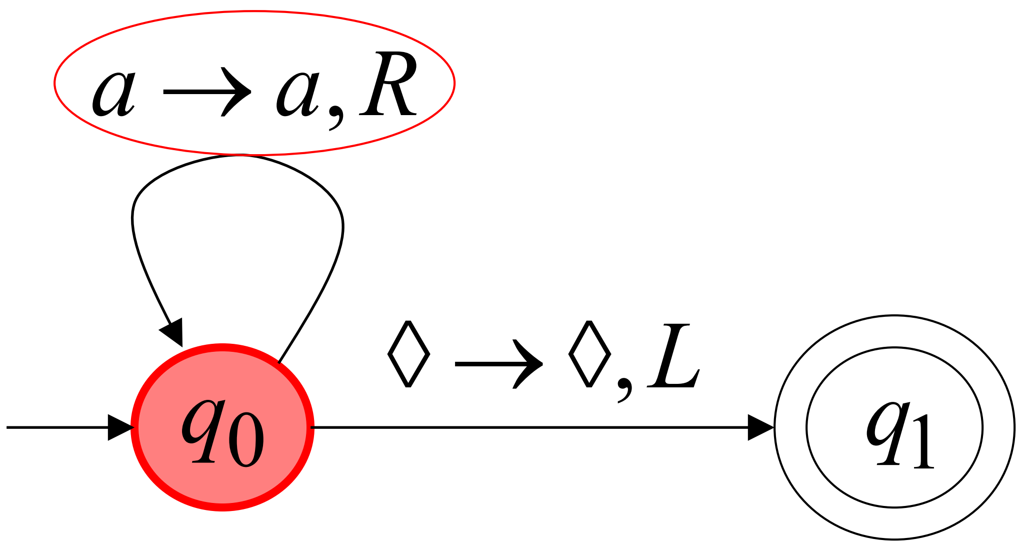
Halt & Accept



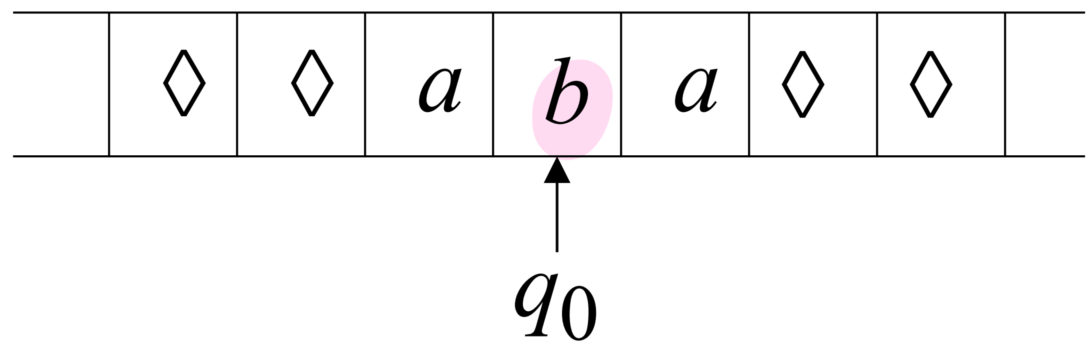
Rejection Example

Time 0





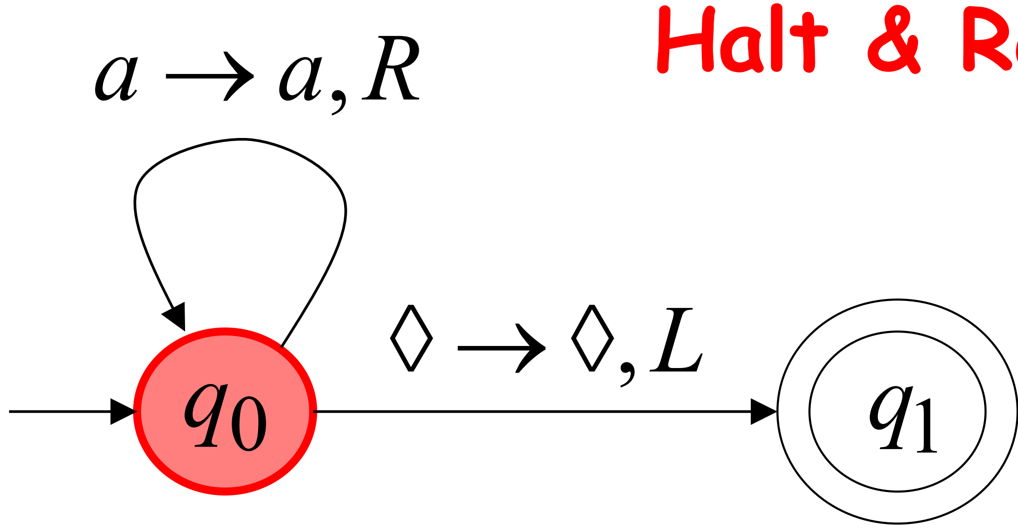
Time 1



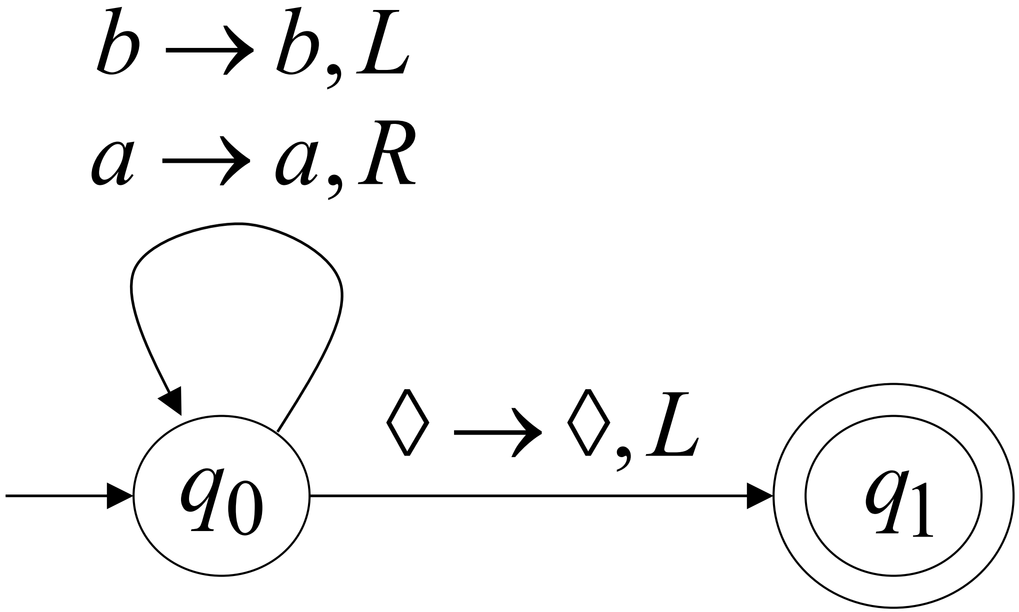
ไม่มี transition สำหรับตัว b

No possible Transition

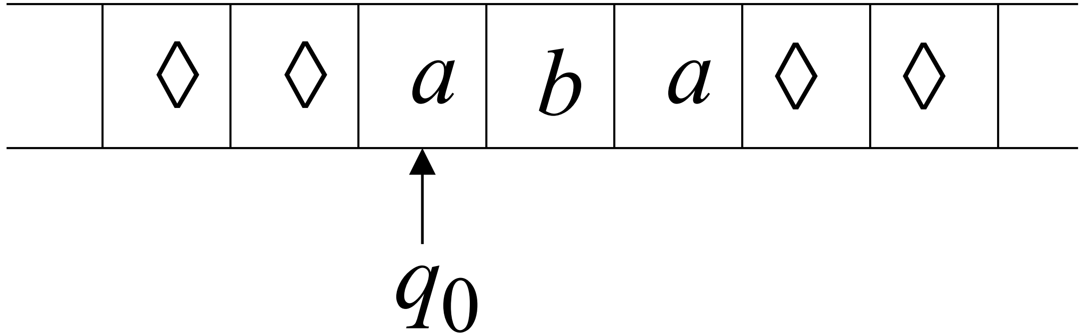
Halt & Reject

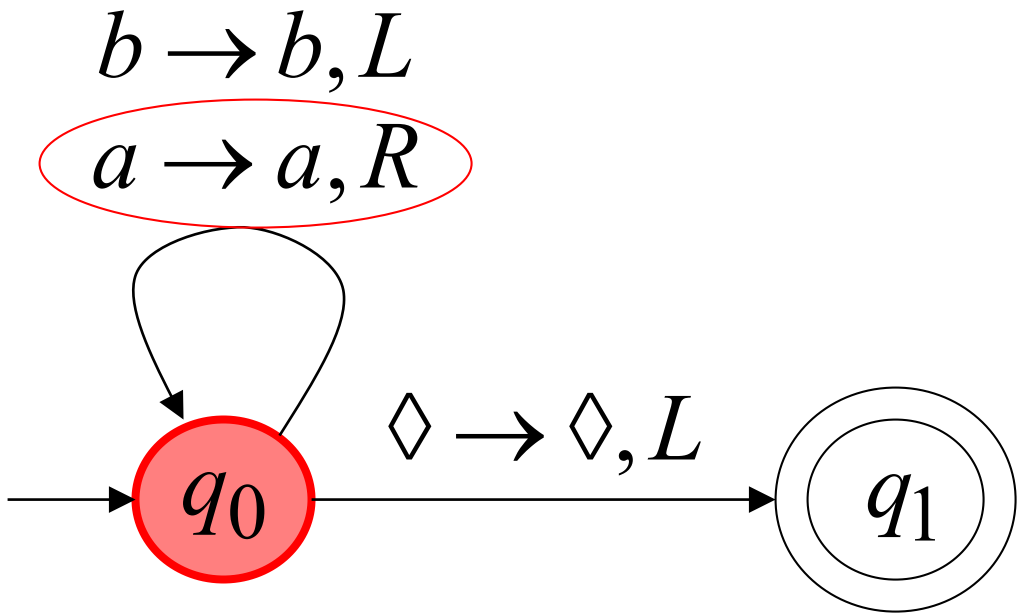


Infinite Loop Example

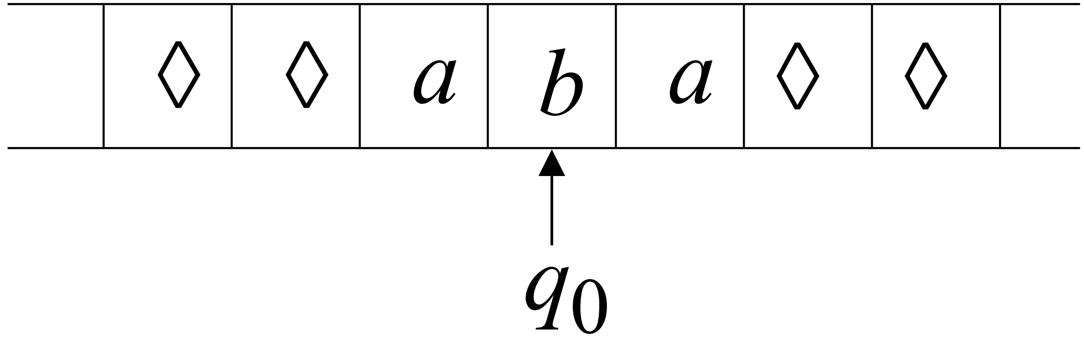


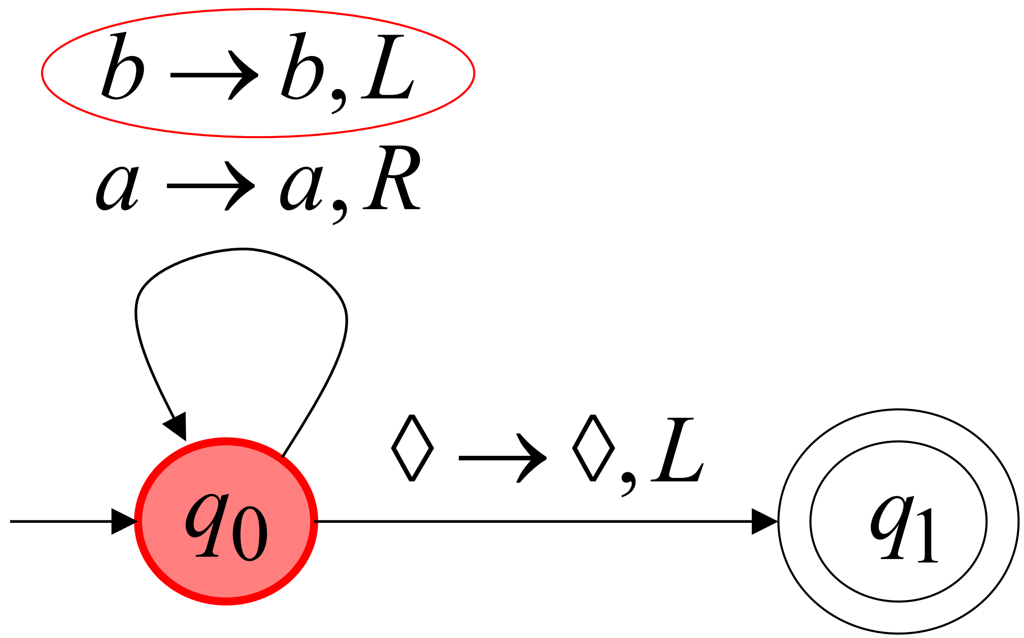
Time 0


◊ ◊ a b a ◊ ◊
q_0

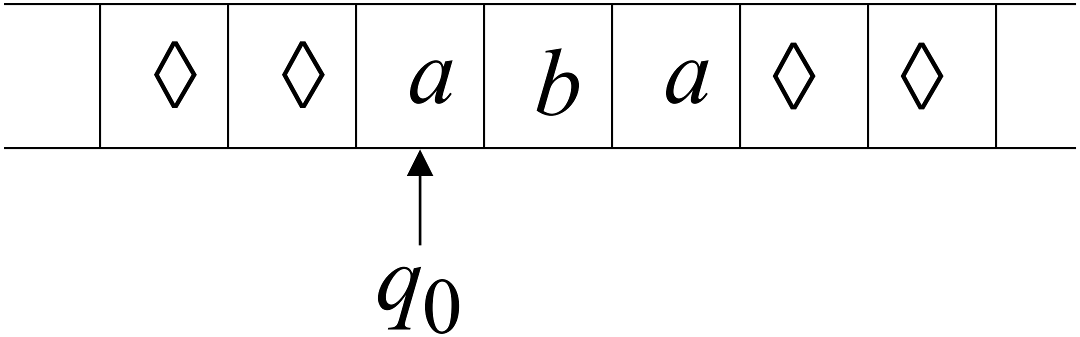

$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Time 1

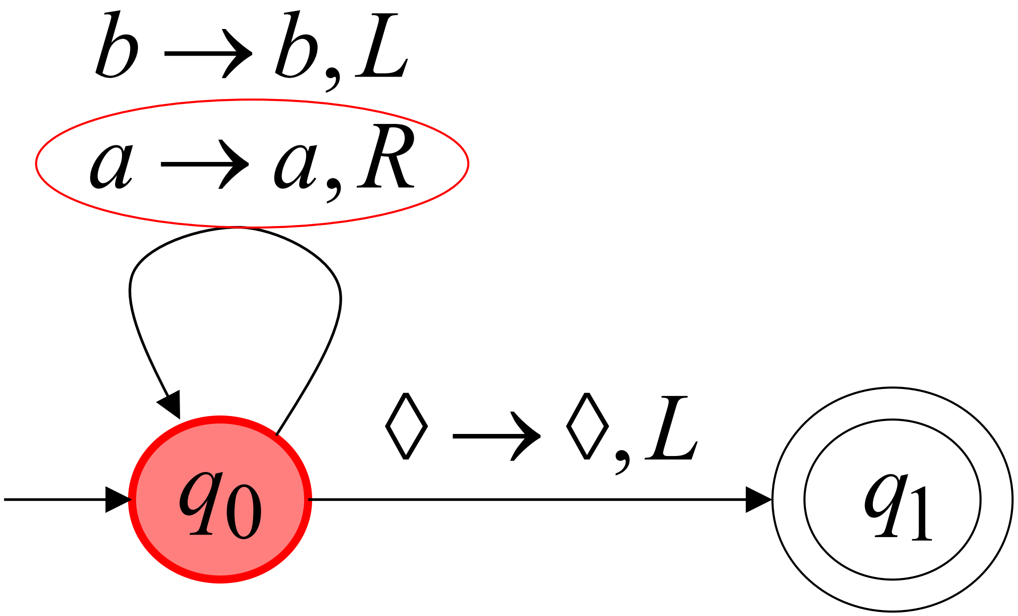

◊ ◊ a b a ◊ ◊
q_0


$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Time 2

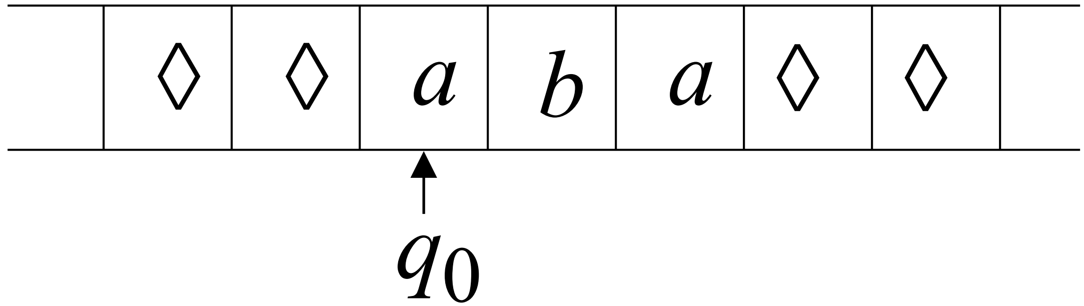


◊
◊
a
b
a
◊
◊
q_0

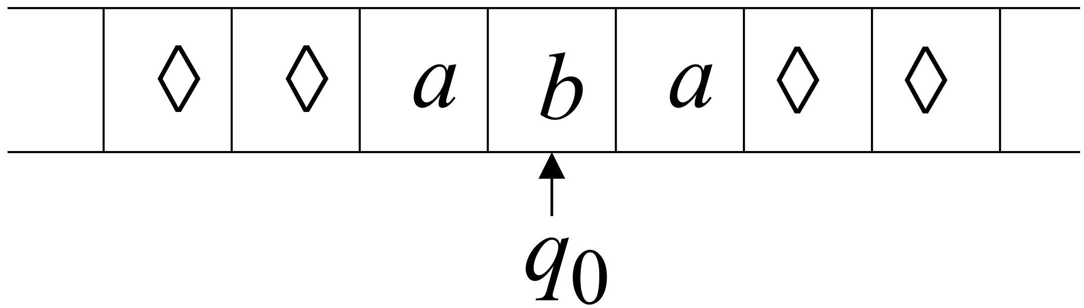


$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

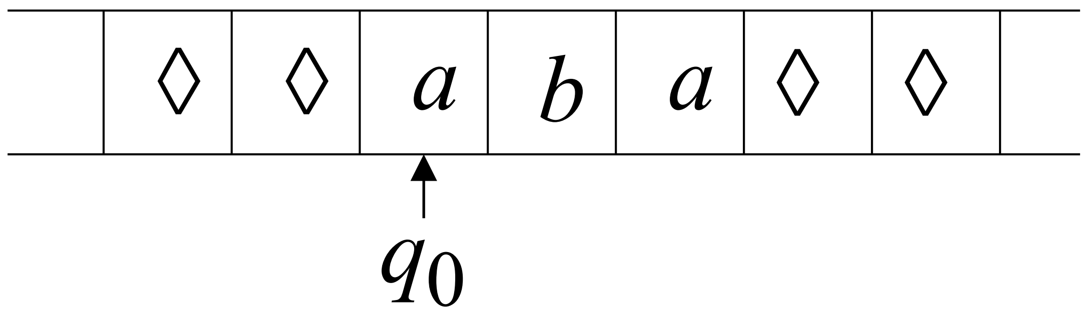
Time 2



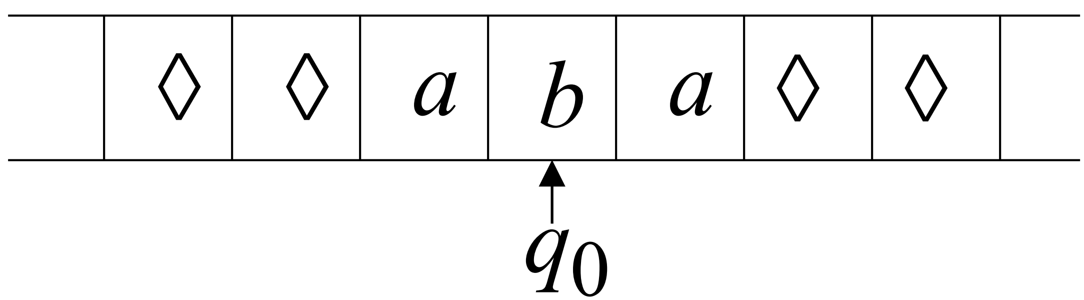
Time 3



Time 4



Time 5



Infinite loop

Because of the infinite loop:

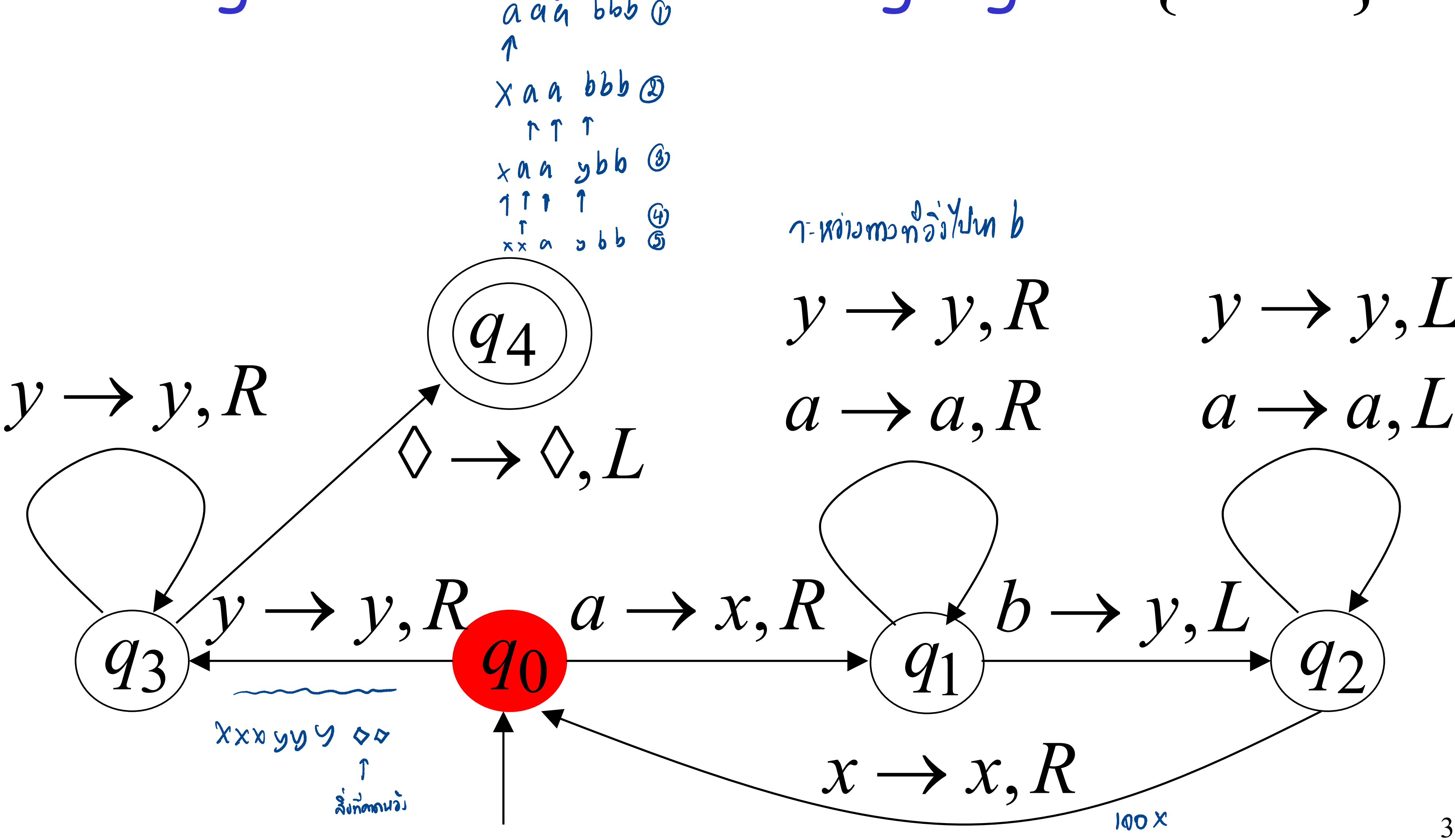
- The final state cannot be reached

- The machine never halts

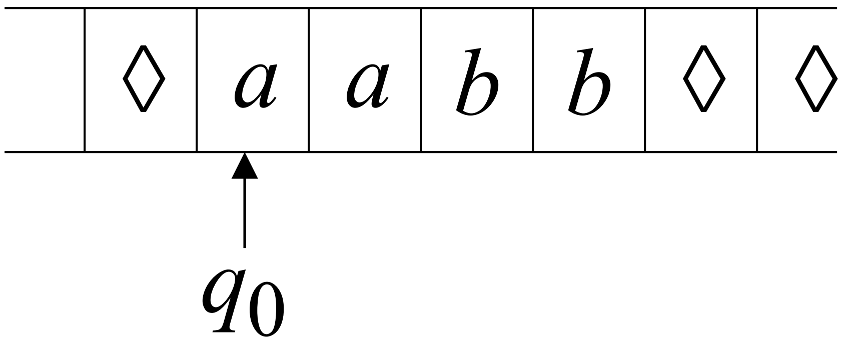
- The input is not accepted

Another Turing Machine Example

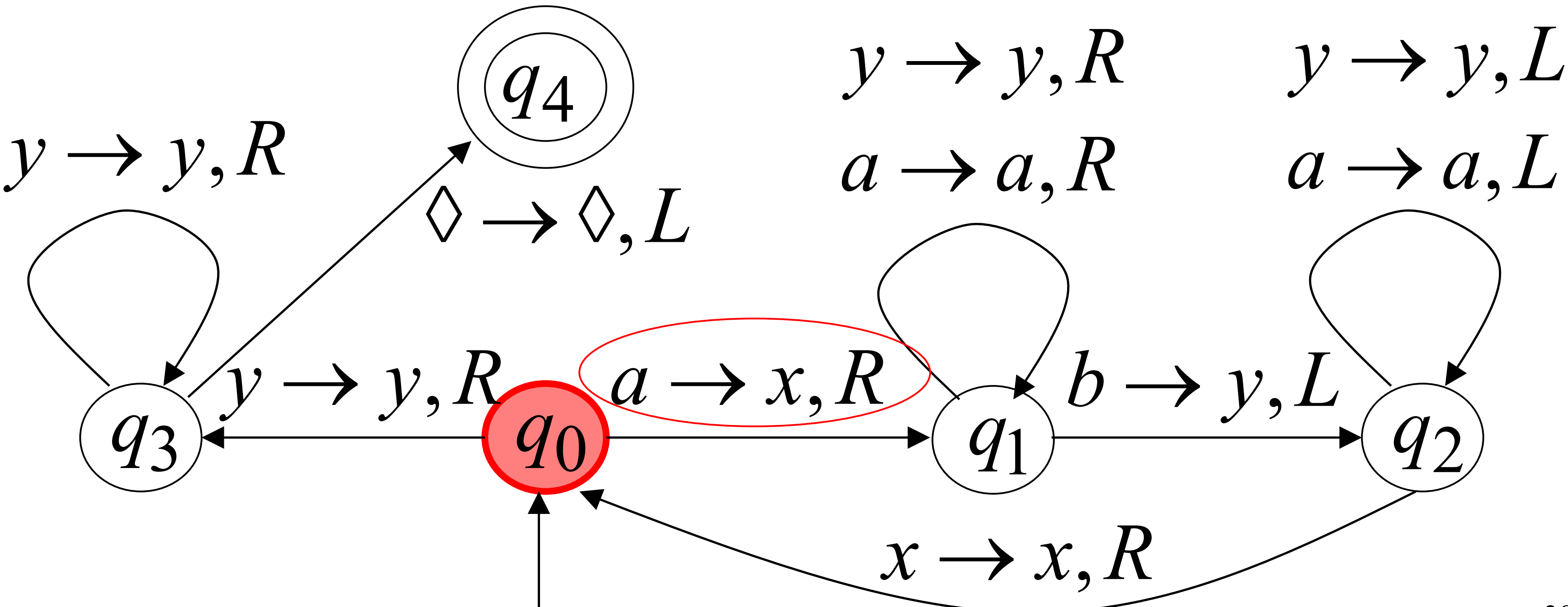
Turing machine for the language $\{a^n b^n\}$



Time 0

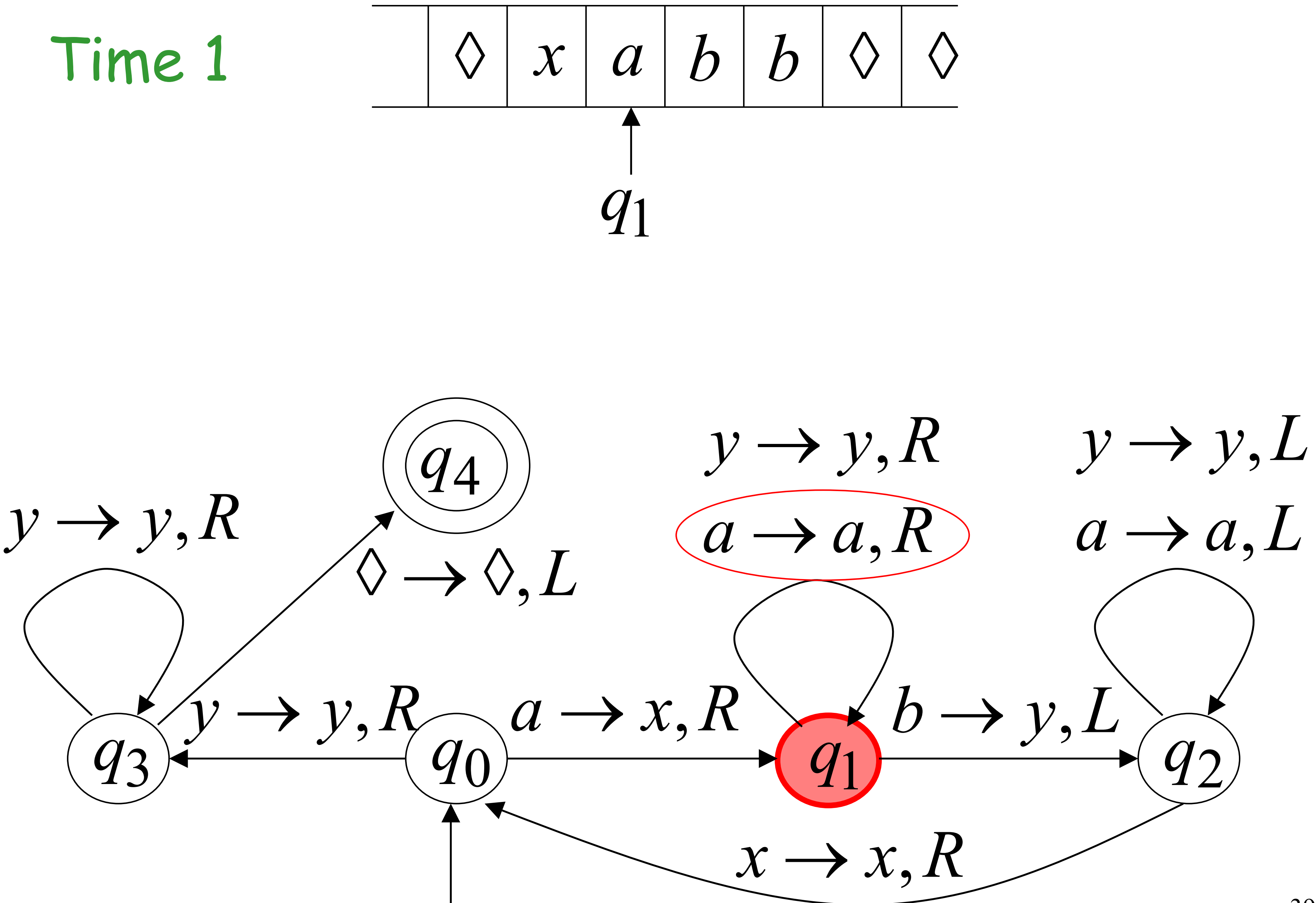


◊
a
a
b
b
◊
◊
q_0

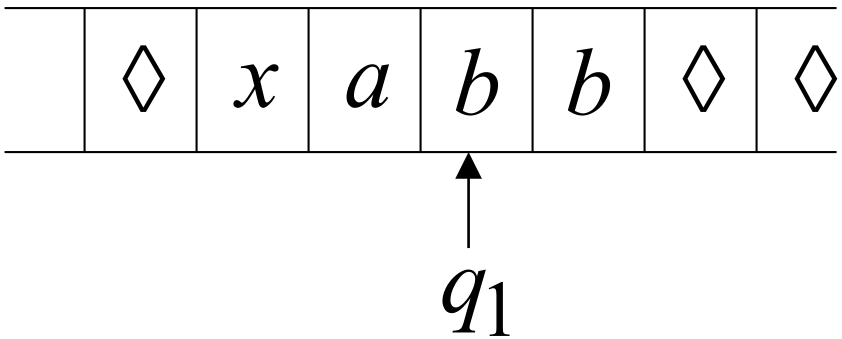


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

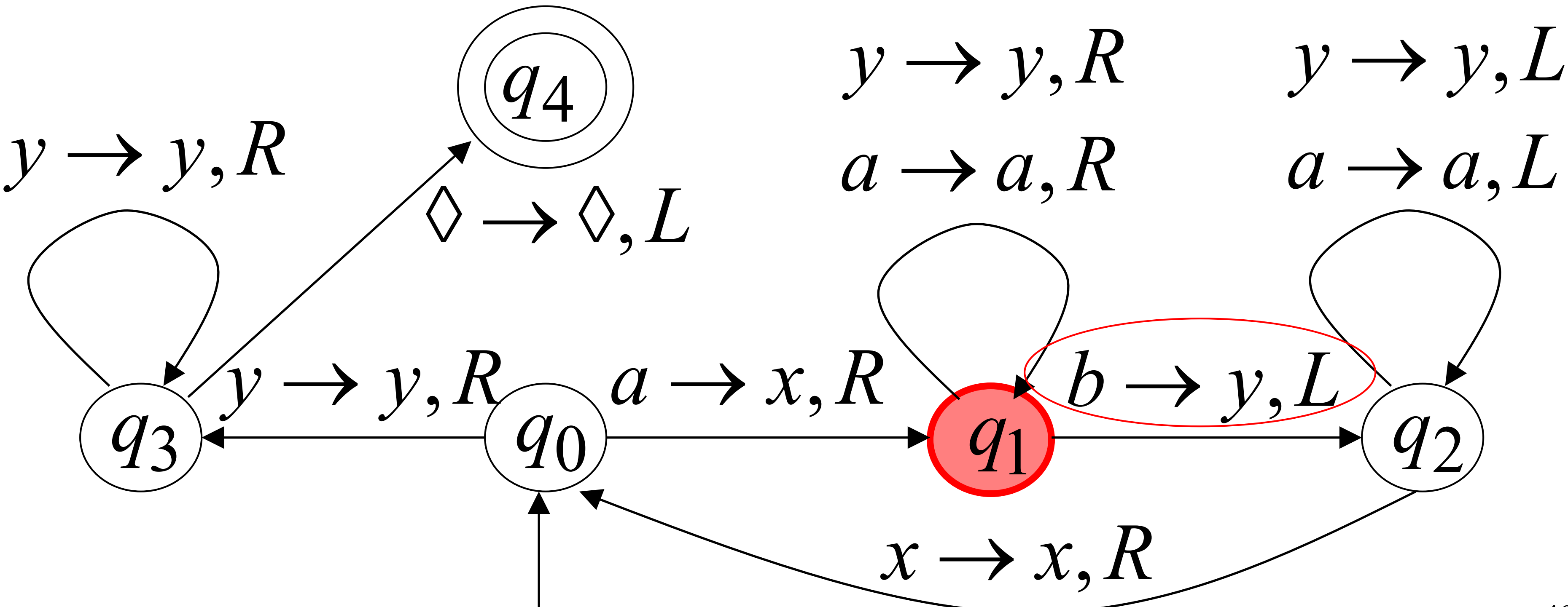
Time 1


◊
x
a
b
b
◊
◊
q_1
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 2

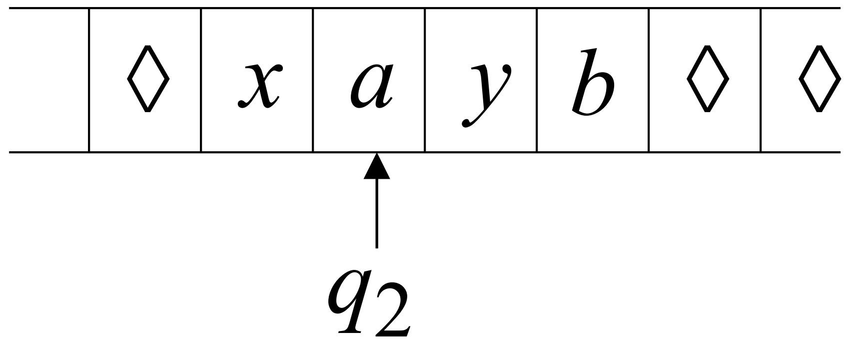


◊
x
a
b
b
◊
◊
q_1

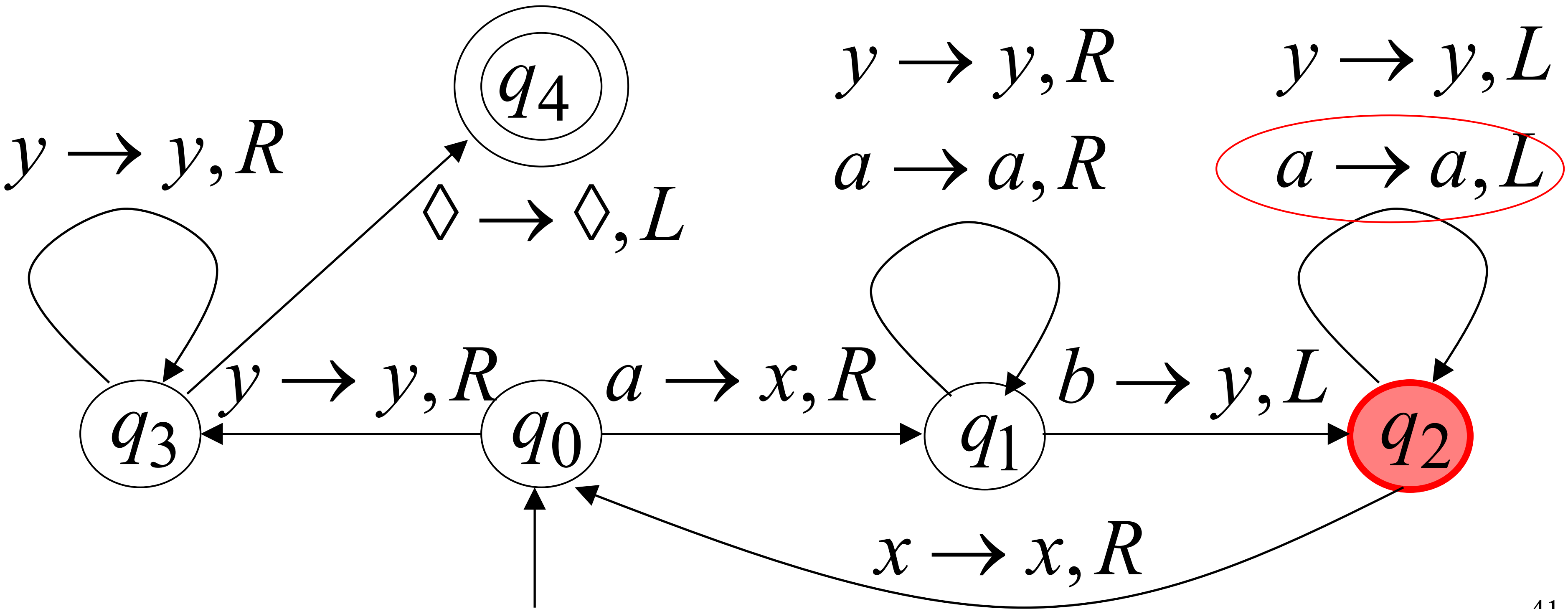


q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 3

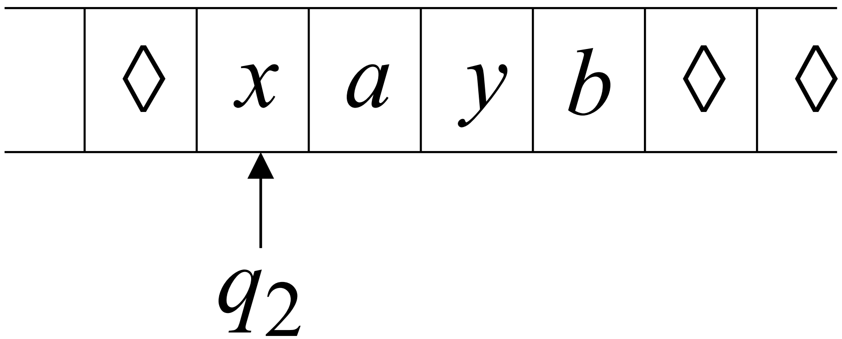


◊
x
a
y
b
◊
◊
q_2

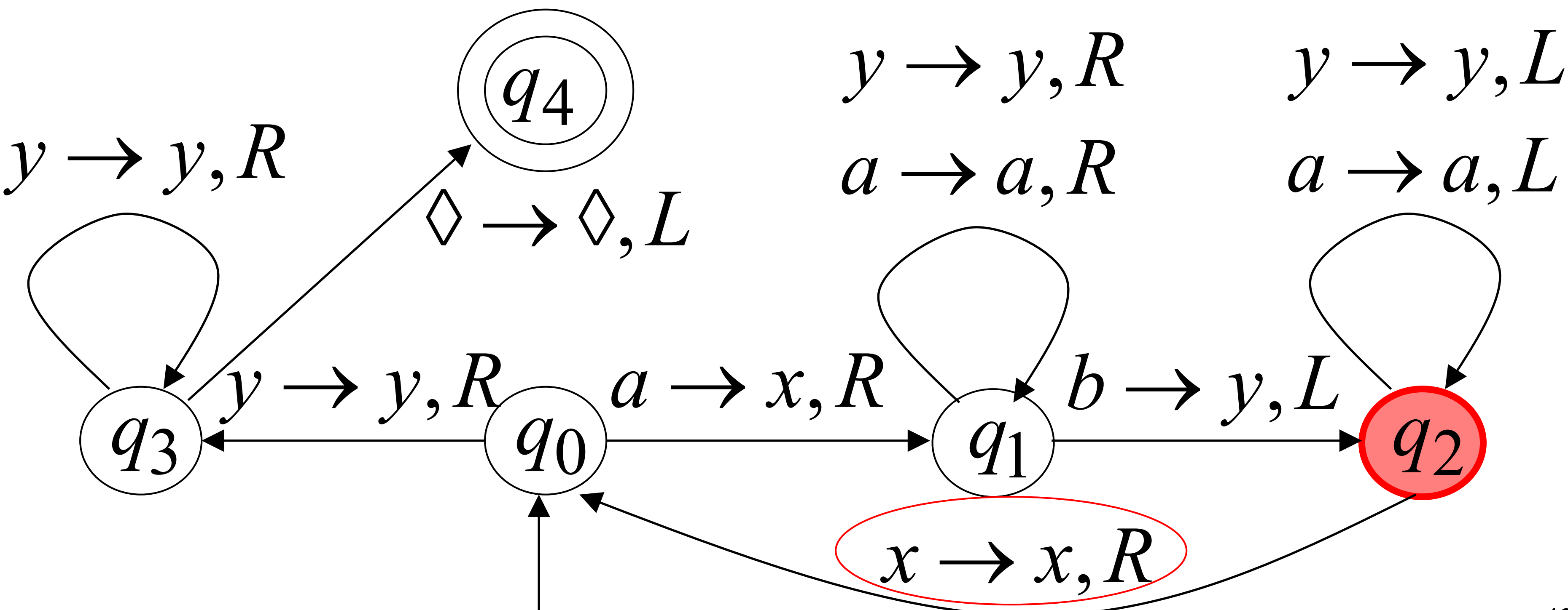


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 4

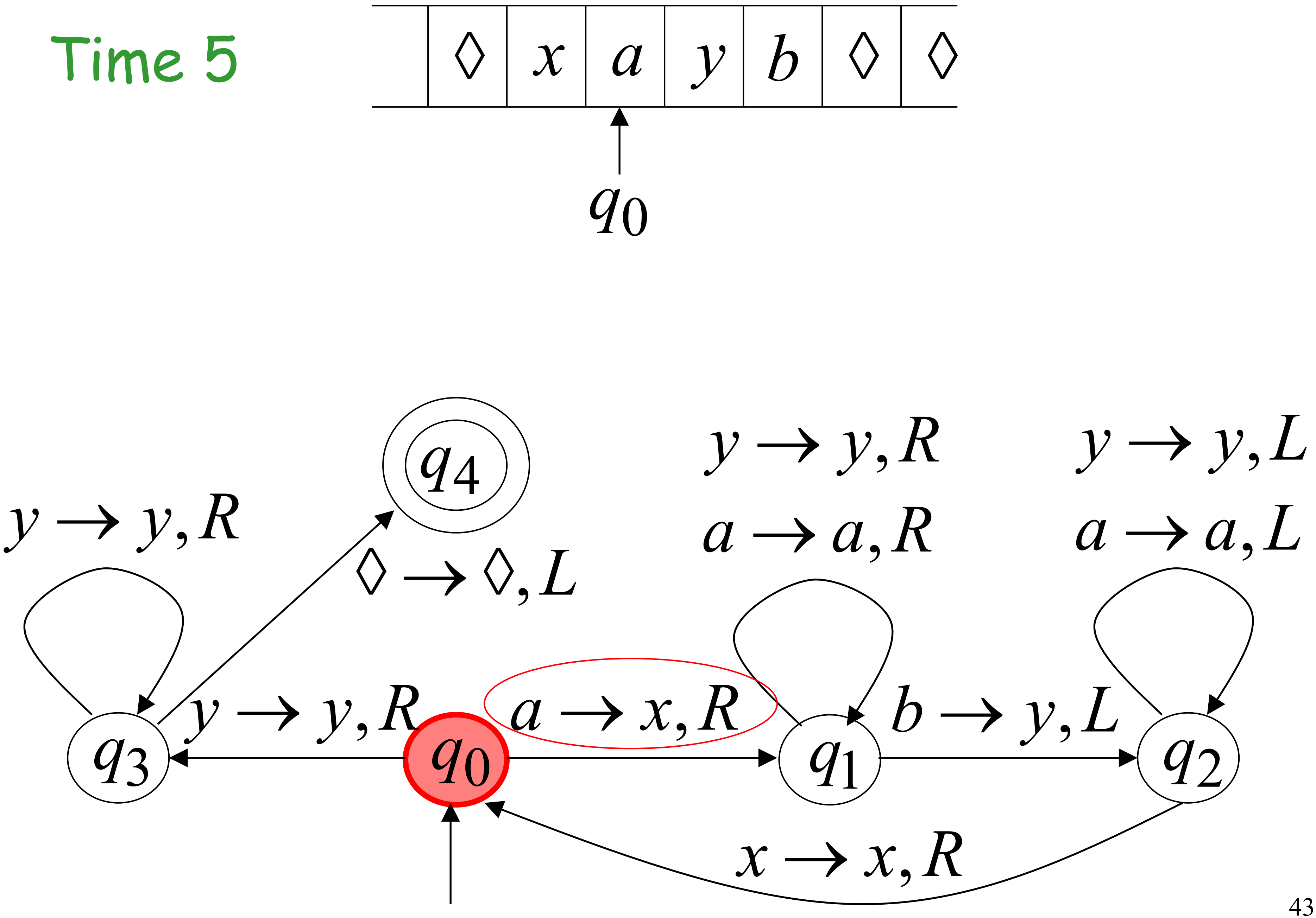


◊
x
a
y
b
◊
◊
q_2

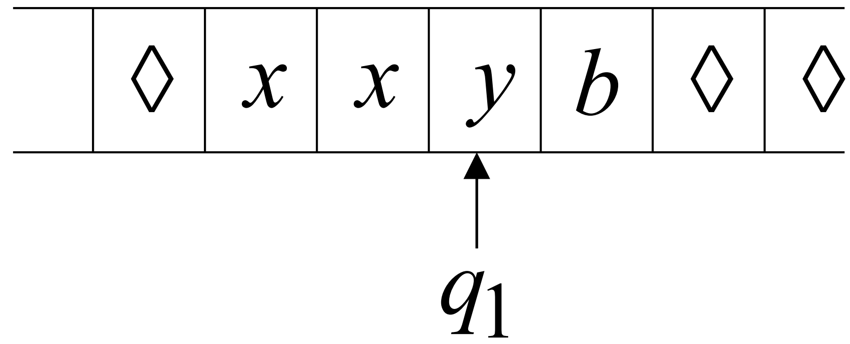


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

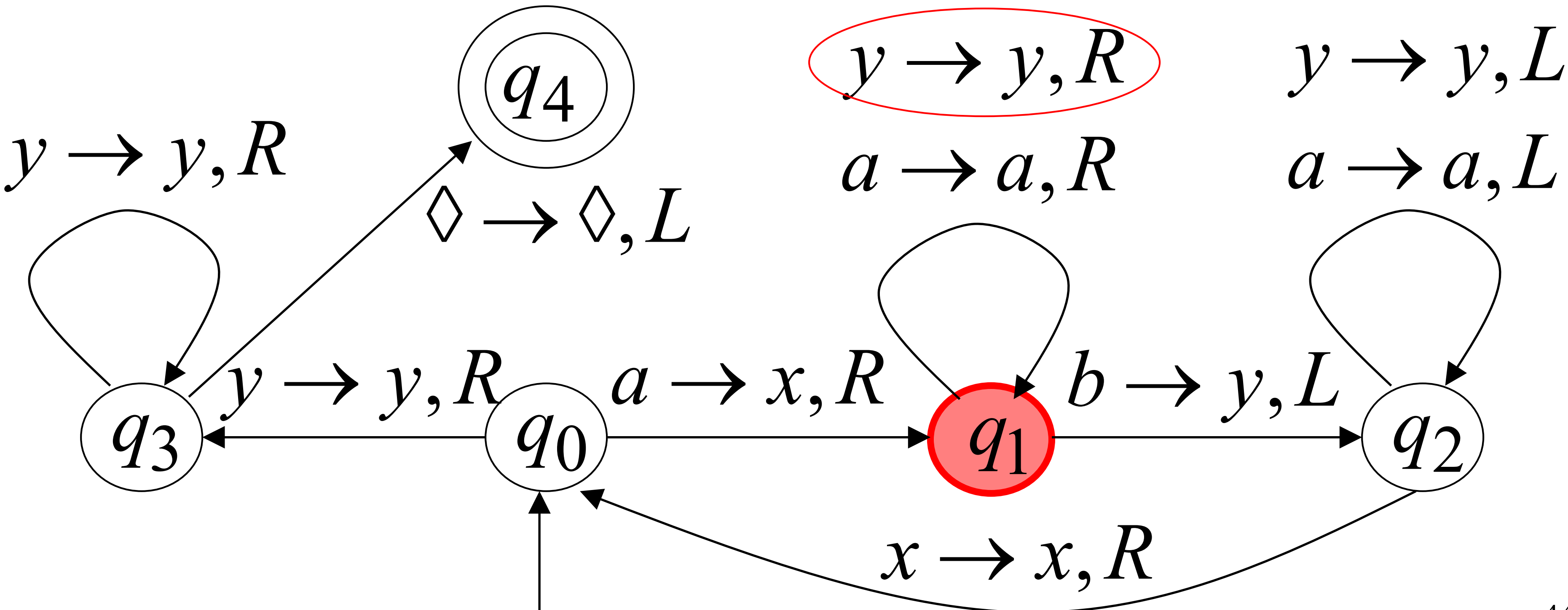
Time 5


$\Diamond$
x
a
y
b
$\Diamond$
$\Diamond$
q_0
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 6

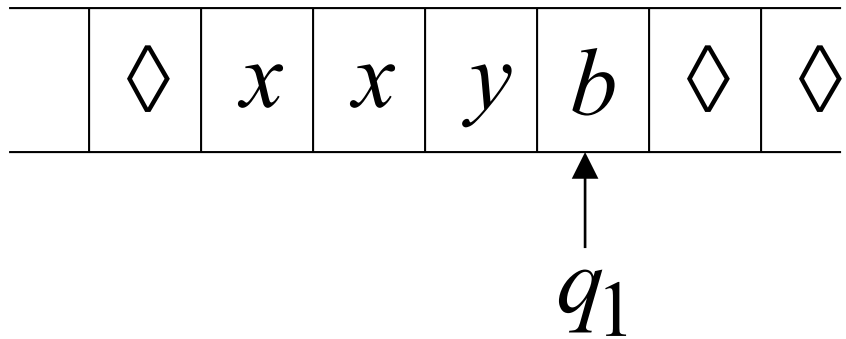


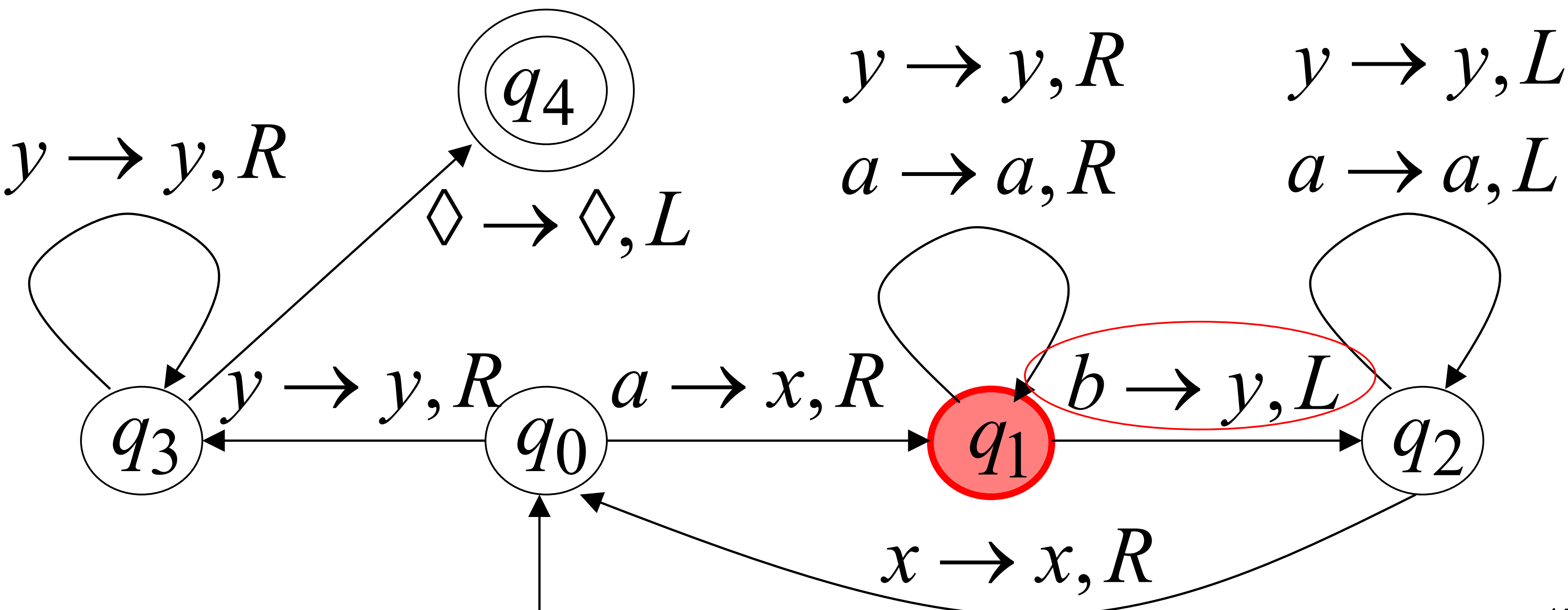
◊
x
x
y
b
◊
◊
q_1



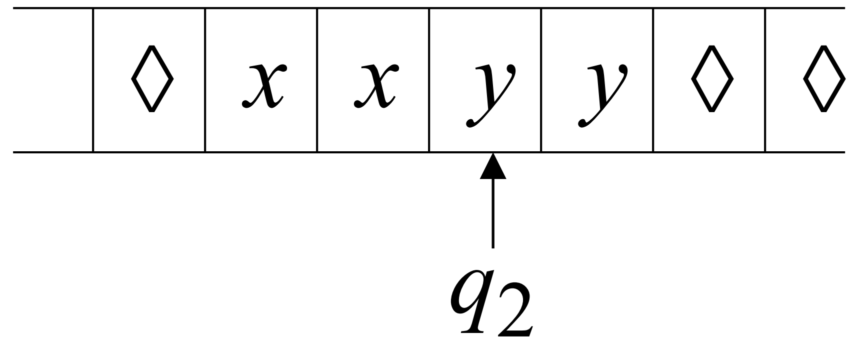
q_4
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

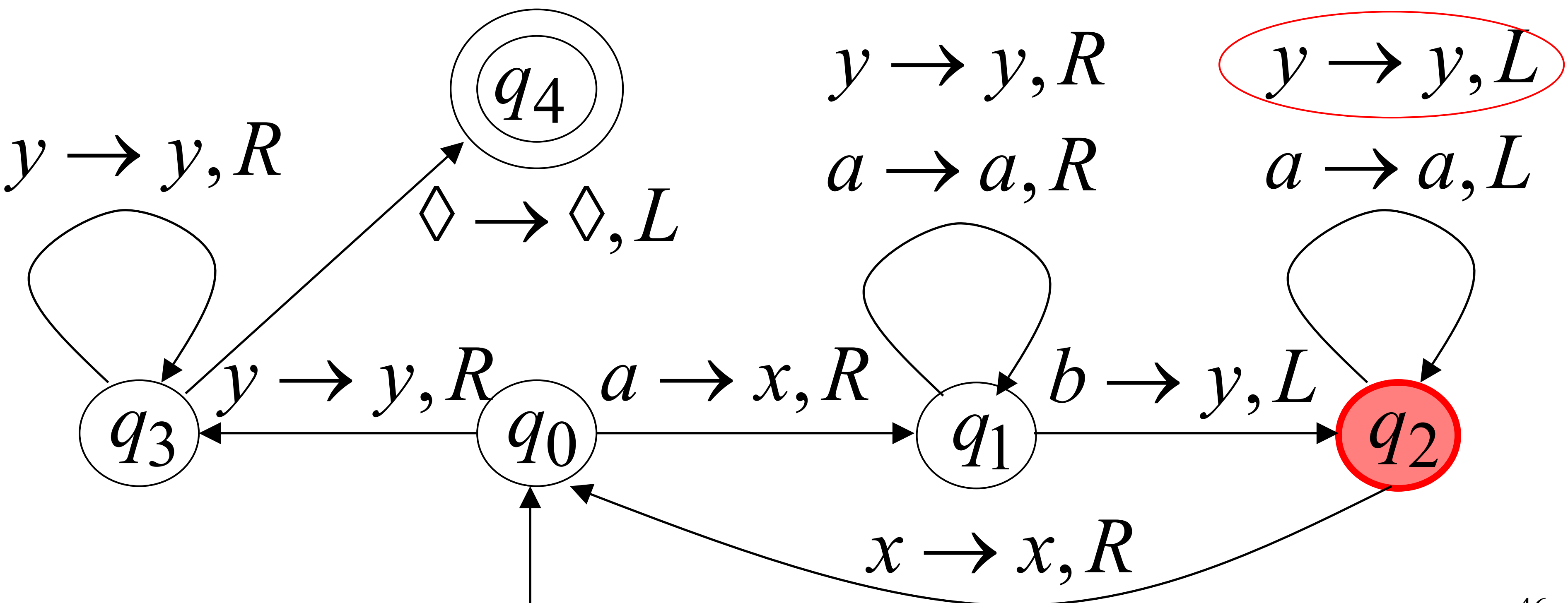
Time 7


◊
x
x
y
b
◊
◊
q_1

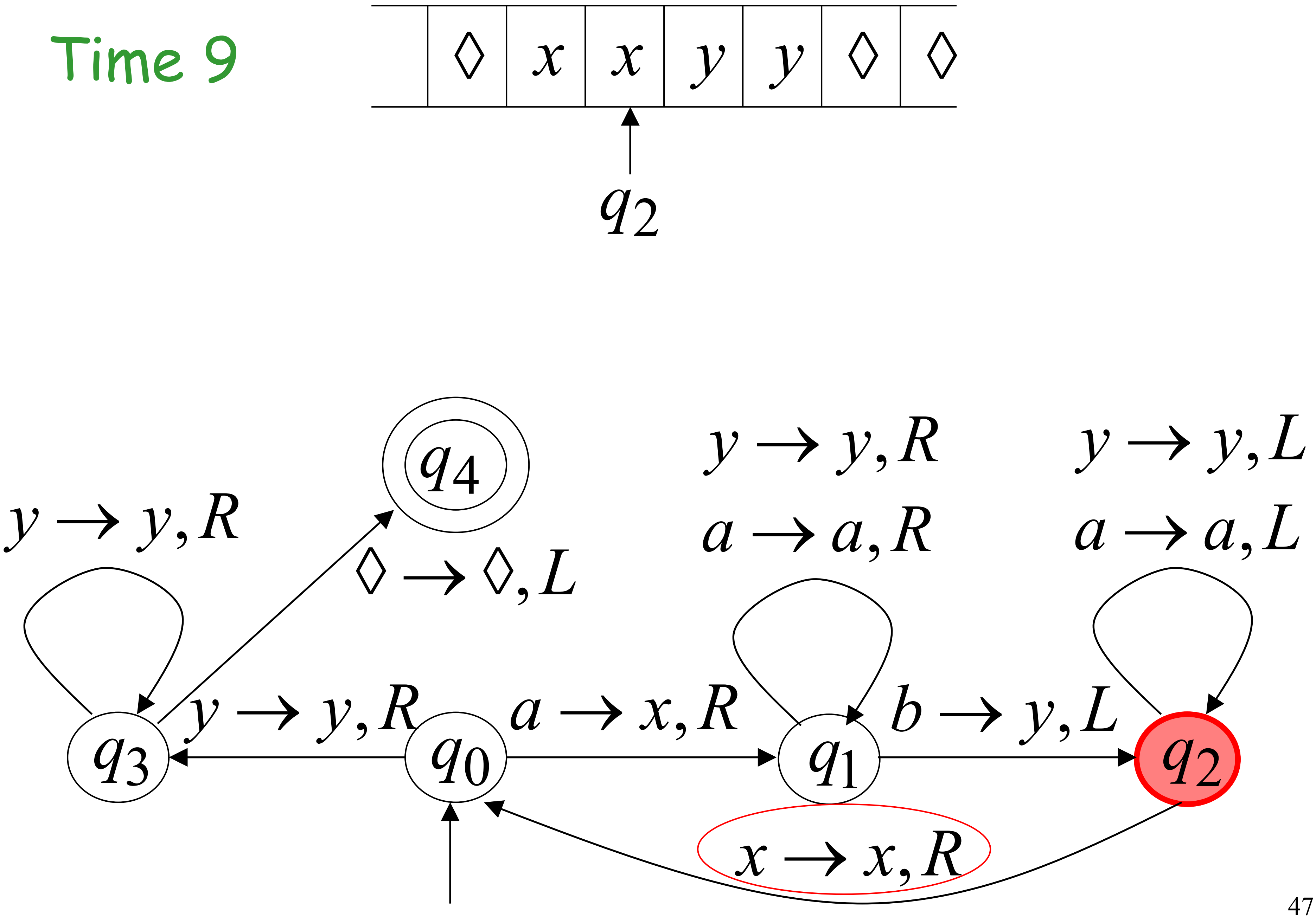

q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 8


◊ x x y y ◊ ◊
q_2

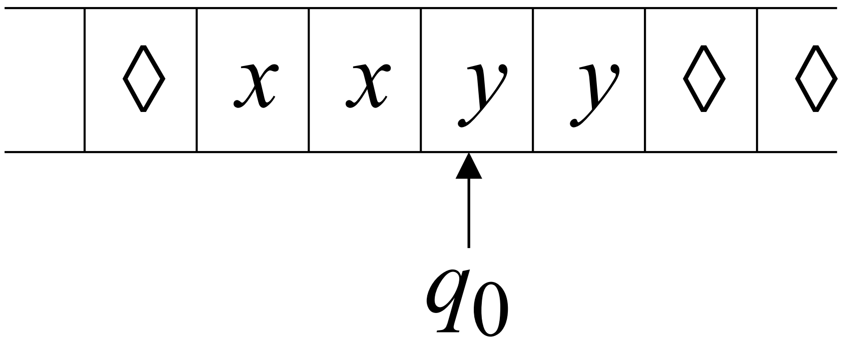

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 9

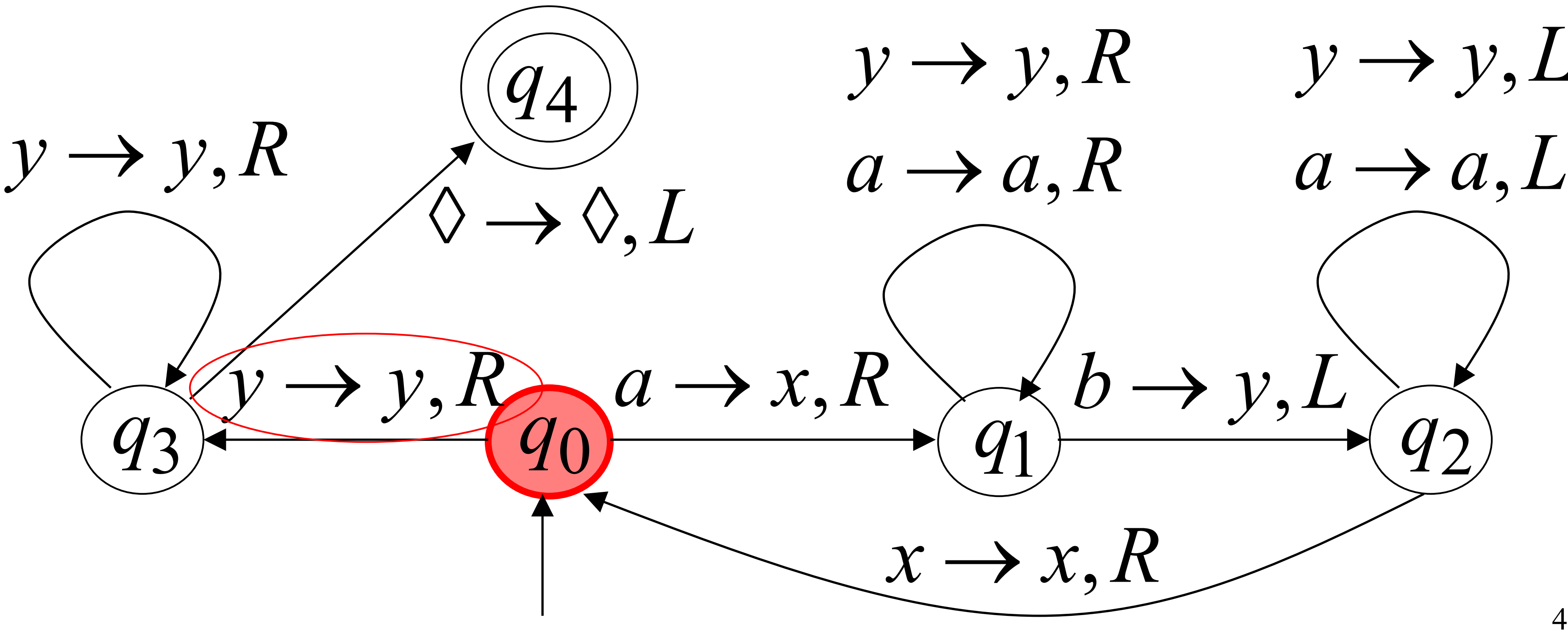


◊ x x y y ◊ ◊
q_2
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 10

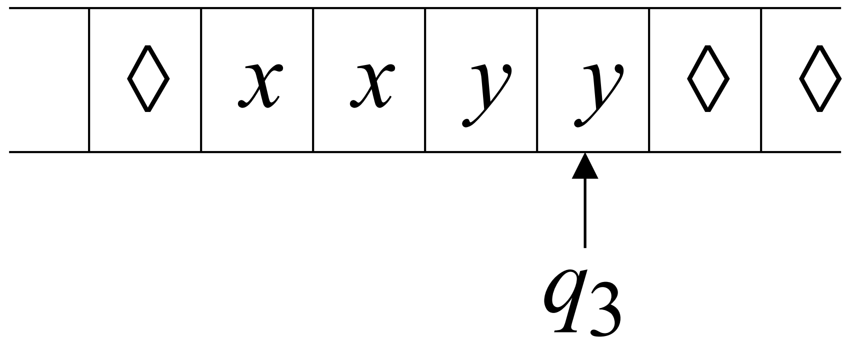


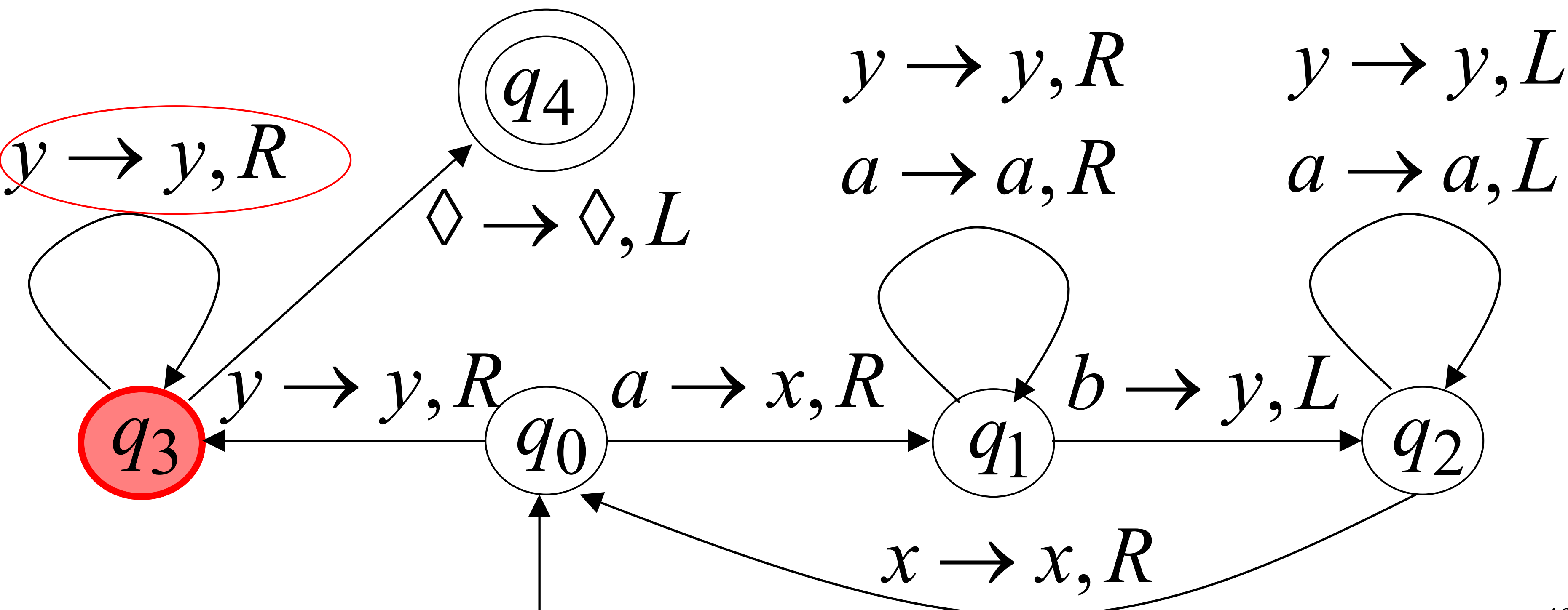
◊
x
x
y
y
◊
◊
q_0



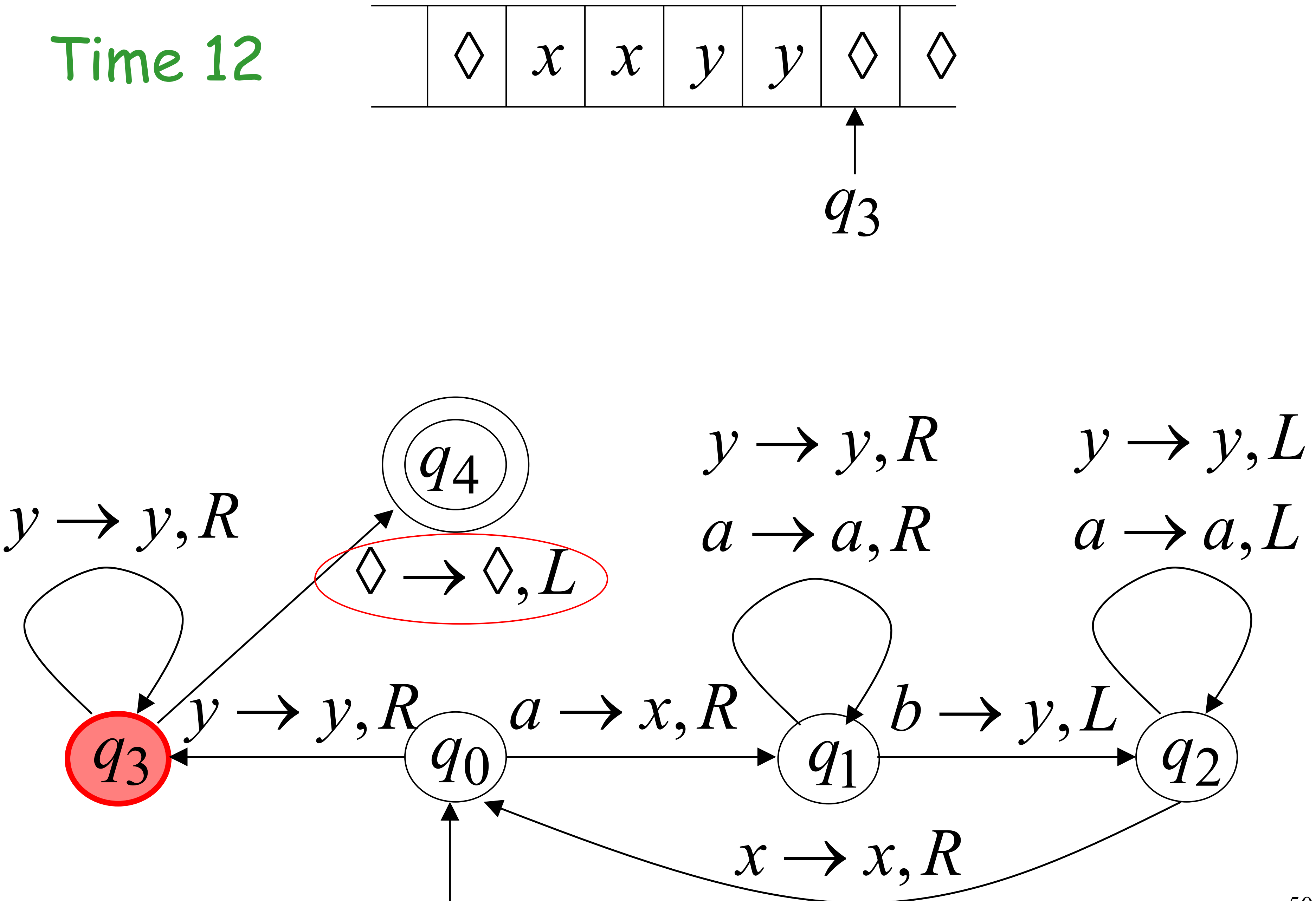
q_4
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 11


◊ x x y y ◊ ◊
q_3

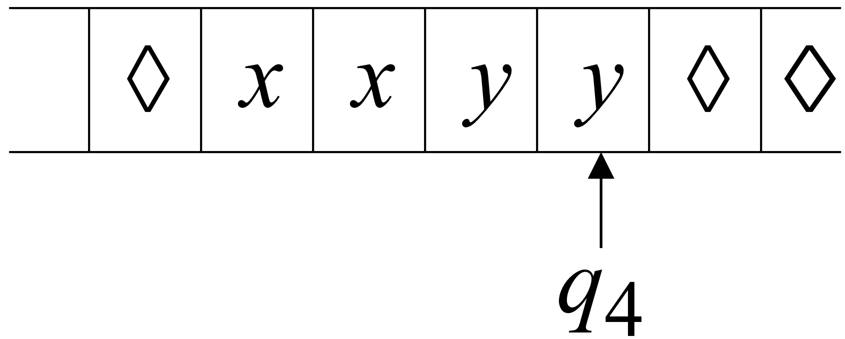

q_4
q_3
q_0
q_1
q_2
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
$x \rightarrow x, R$

Time 12

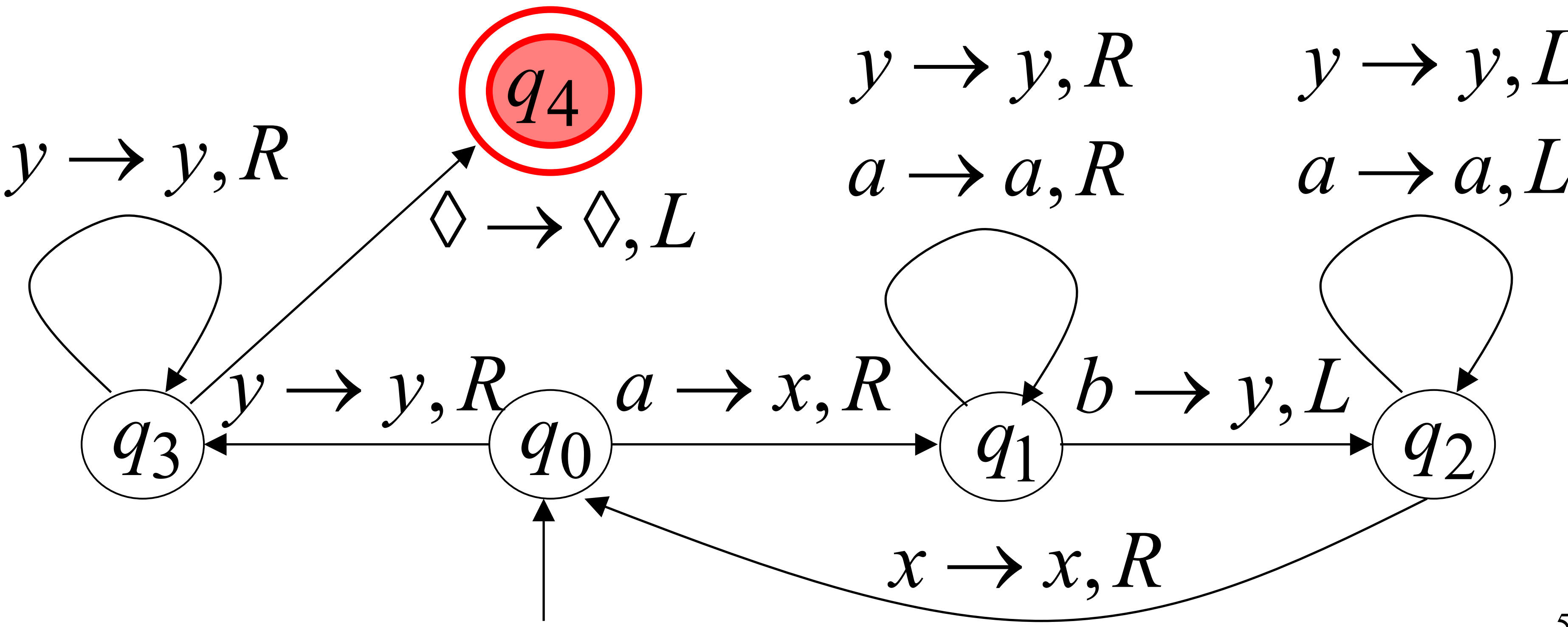


◊
x
x
y
y
◊
◊
q_3
$y \to y, R$
$\Diamond \to \Diamond, L$
q_4
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 13



Halt & Accept



Observation:

ทำภาษา แบบนี้ได้

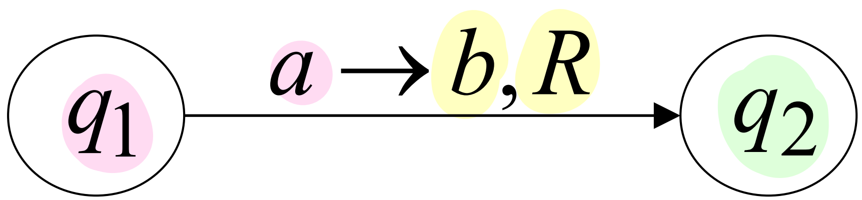
If we modify the
machine for the language $\{a^n b^n\}$

ที่ใหญ่กว่าก็ทำได้

we can easily construct
a machine for the language $\{a^n b^n c^n\}$

Formal Definitions for Turing Machines

Transition Function



state　input　　state　ตัวเปลี่ยน ตัวแทน　เลื่อน

$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function

$\delta(q_1, c) = (q_2, d, L)$

$$q_1 \xrightarrow{c \rightarrow d, L} q_2$$

$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

States

Input alphabet

Tape alphabet

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

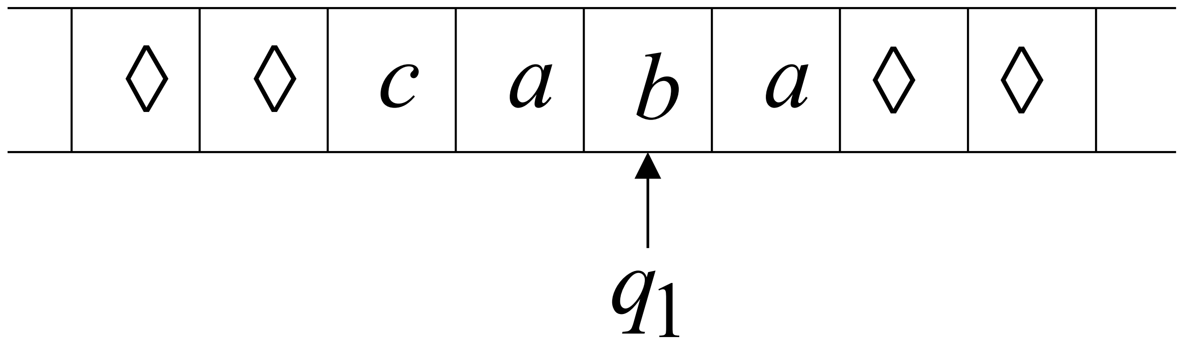
Transition function

Initial state

blank

Final states

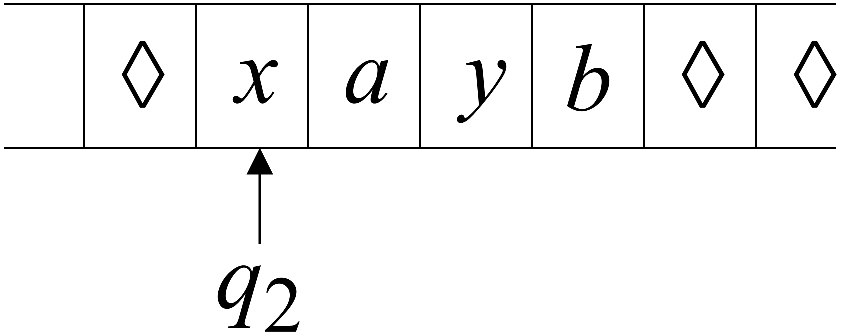
Configuration



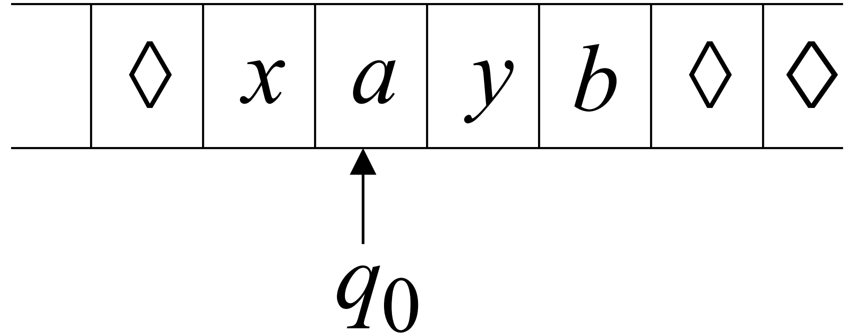
q_1 อยู่หน้าตัวไหน = มันกำลังอ่านตัวนั้น

Instantaneous description: $ca\ q_1\ ba$

Time 4

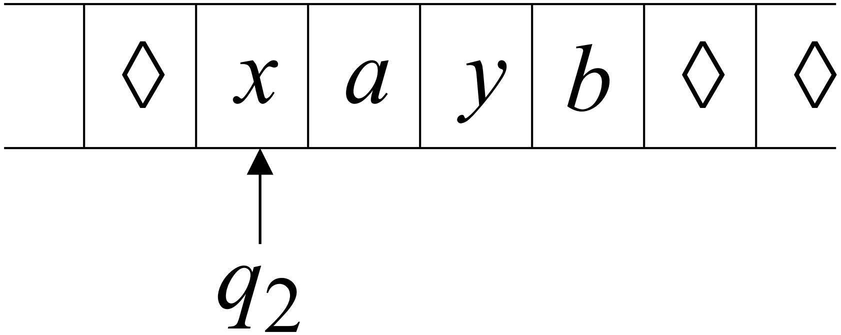


Time 5

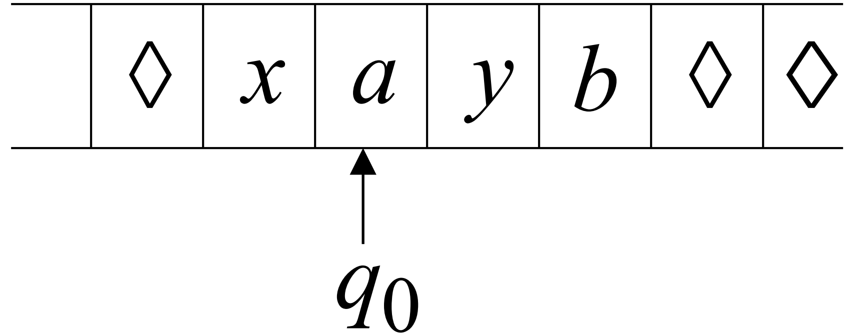


A Move: $q_2\, xayb \succ x\, q_0\, ayb$

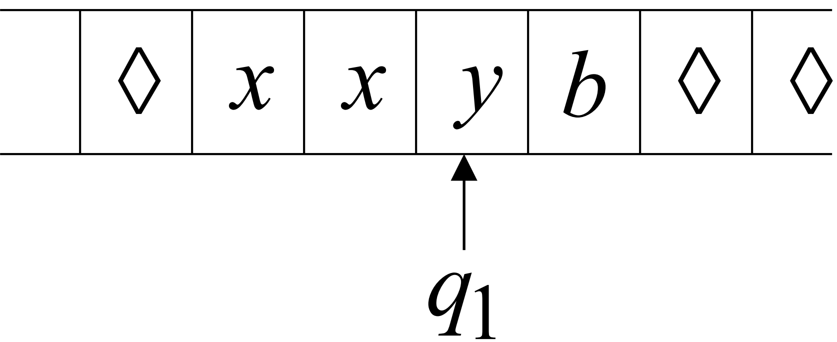
Time 4



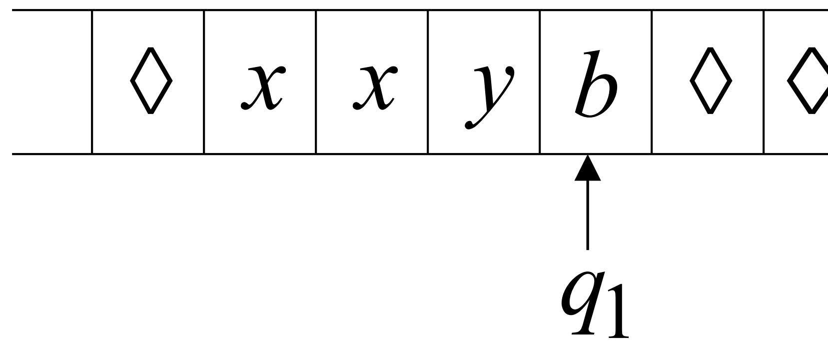
Time 5



Time 6



Time 7



$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

สนใจแค่ 2 คือ src กับ dest

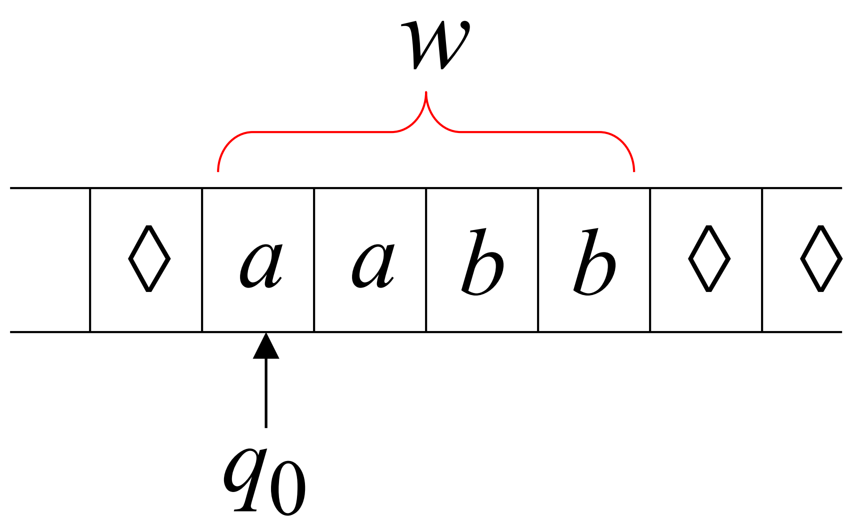
$$q_2\, xayb \succ x\, q_0\, ayb \succ xx\, q_1\, yb \succ xxy\, q_1\, b$$

q_1 อยู่หลังตัวไหน = มันกำลังอ่านตัวนั้น

Equivalent notation: $$q_2\, xayb \overset{*}{\succ} xxy\, q_1\, b$$

Initial configuration: $q_0\, w$

Input string



เริ่มต้นทำงานที่ input ตัวซ้ายสุด

The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \quad \underbrace{q_0}_{\text{Initial state}} \, w \overset{*}{\succ} x_1 \, \underbrace{q_f}_{\text{Final state}} \, x_2\}$$

Standard Turing Machine

The machine we described is the standard:

- Deterministic

ยาว

- Infinite tape in both directions

เขียน / อ่านได้

- Tape is the input/output file

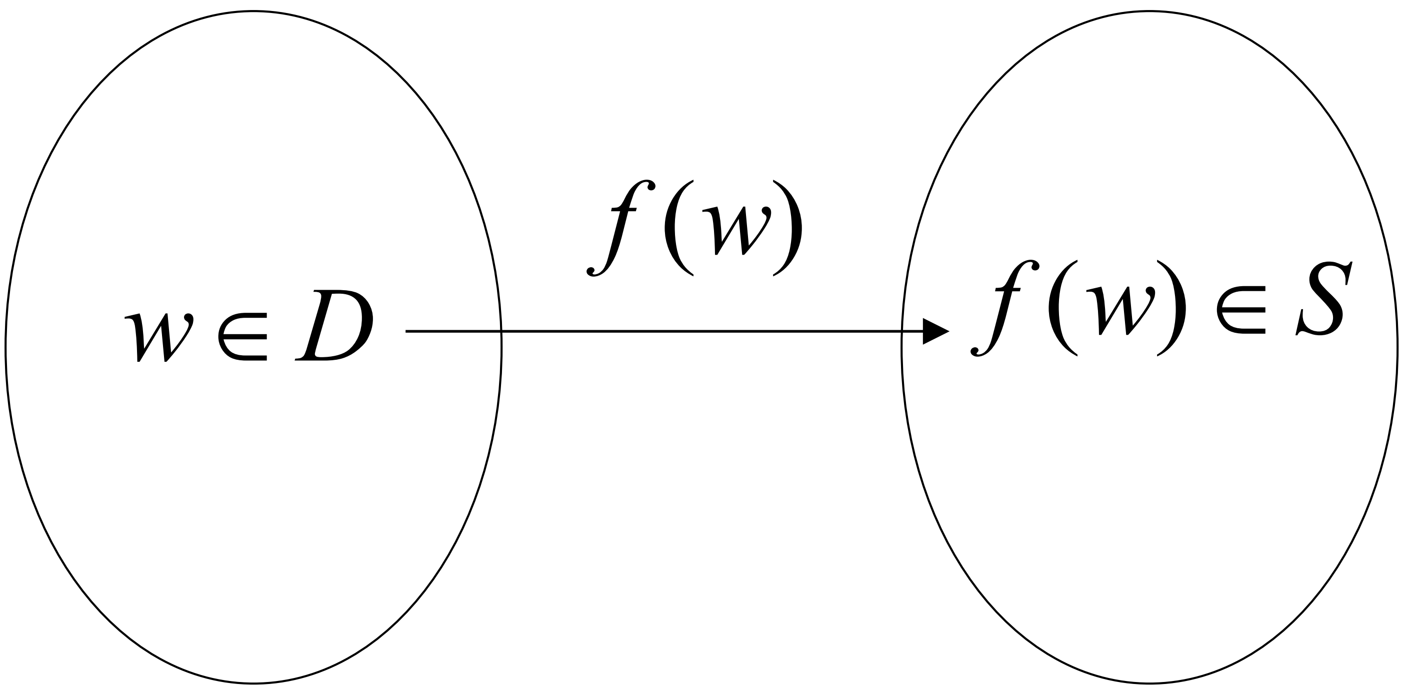
ประยุกต์ใช้กับคำนวณ func

Computing Functions with Turing Machines

A function $f(w)$ has:

Domain: D

Result Region: S



A function may have many parameters:

Example:

Addition function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5 ✗

Binary: 101 ✗

Unary: 11111 = 5

บวกลบได้แค่ unary

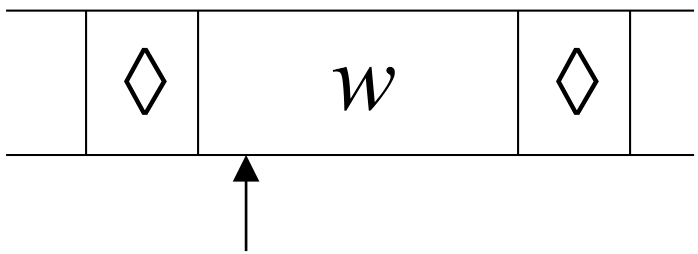
We prefer **unary** representation:

easier to manipulate with Turing machines

Definition:

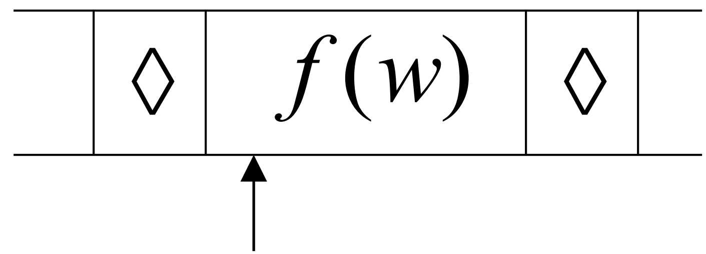
A function f is computable if there is a Turing Machine M such that:

Initial configuration



q_0 initial state

Final configuration



q_f final state

For all $w \in D$ Domain

In other words:

A function f is computable if there is a Turing Machine M such that:

$$q_0\, w \overset{*}{\succ} q_f\, f(w)$$

Initial Configuration

Final Configuration

For all $w \in D$ Domain

Example

The function $f(x, y) = x + y$ is computable

x, y are integers

Turing Machine:

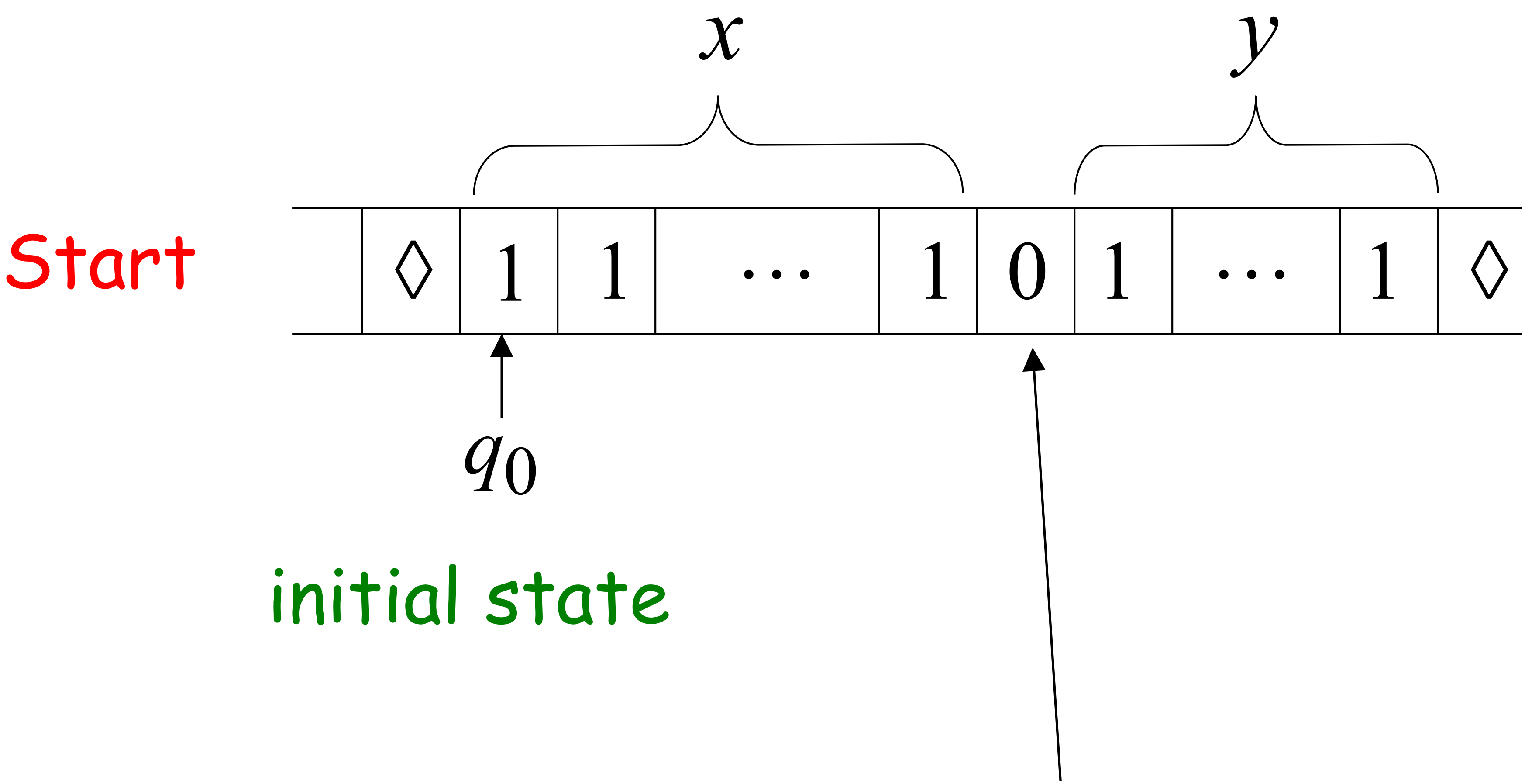
ตัวขั้น

Input string: $x0y$ unary

Output string: $xy0$ unary

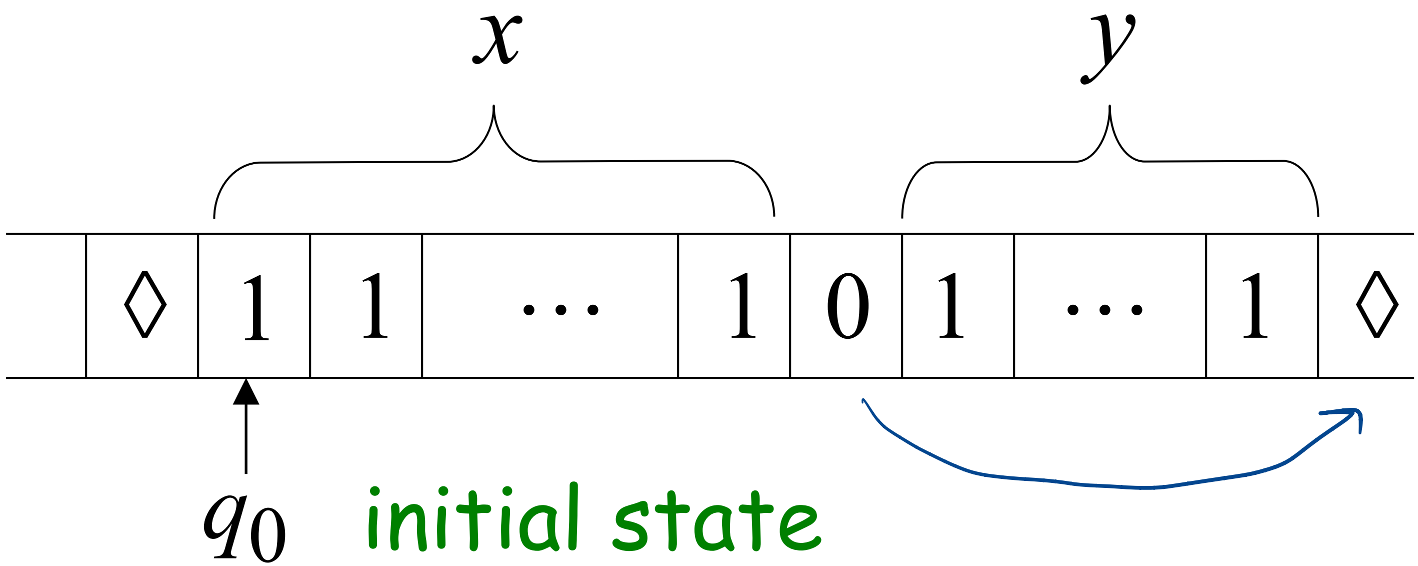
ย้าย 0 ไปข้างหลัง

อาจทำผลลัพธ์ไปทำ operation อื่น

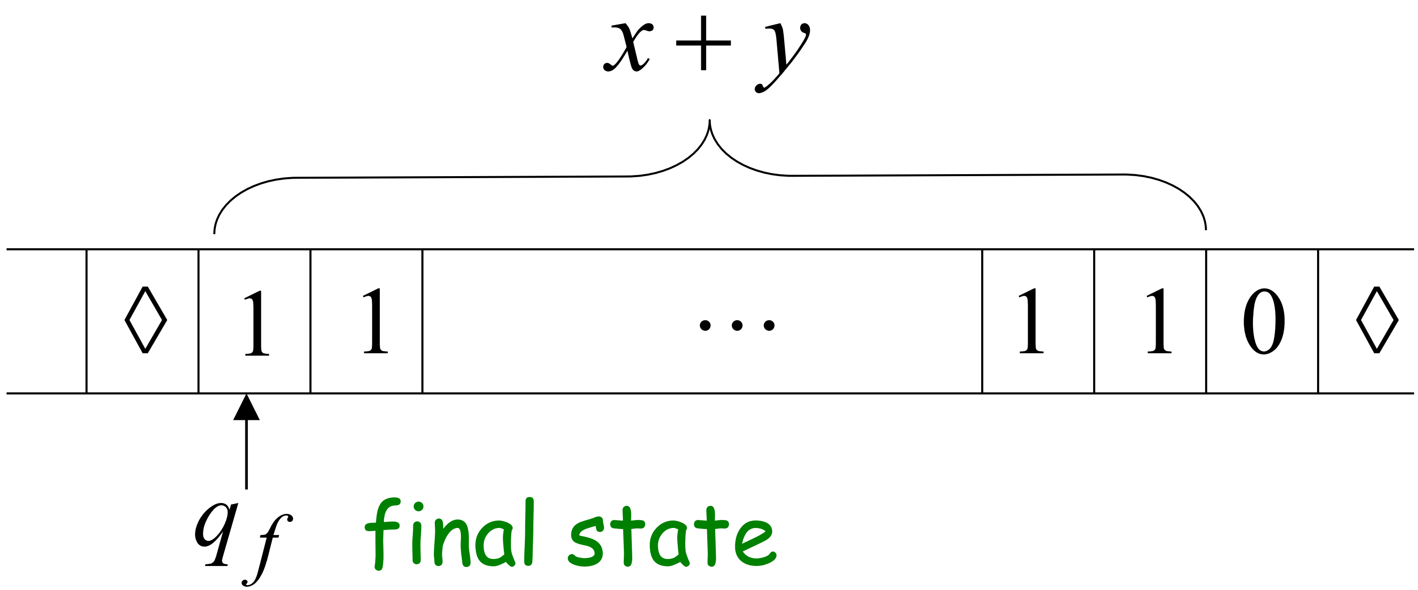


The 0 is the delimiter that separates the two numbers

Start

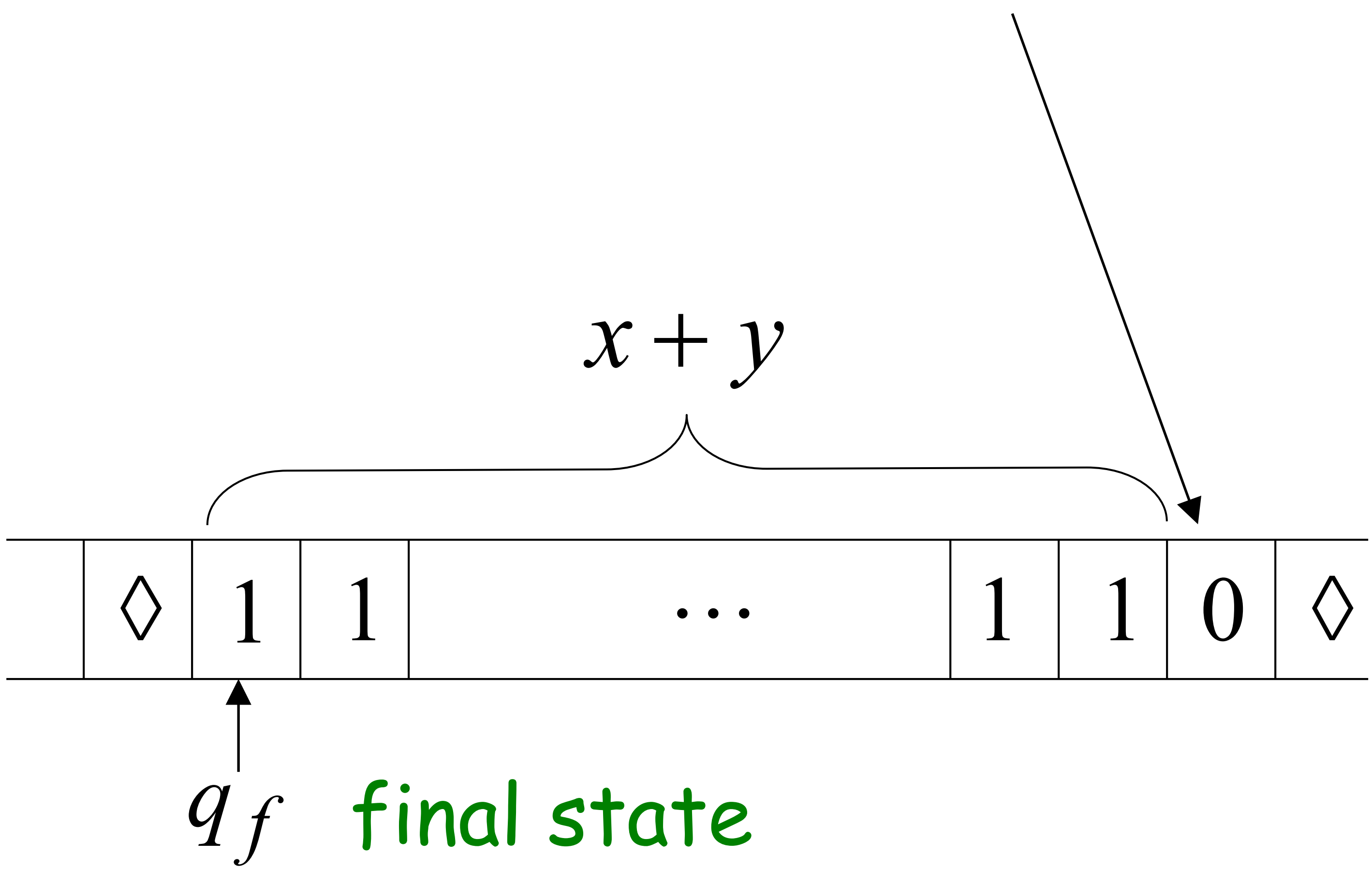


Finish



The 0 helps when we use
the result for other operations

Finish



Turing machine for function $f(x, y) = x + y$

Idea $\Diamond\ 1\ 1\ \overset{\overset{0 \to 1}{(2)}}{\cancel{0}}\ 1\ \overset{(5)\ 1 \to 0}{\Diamond}$

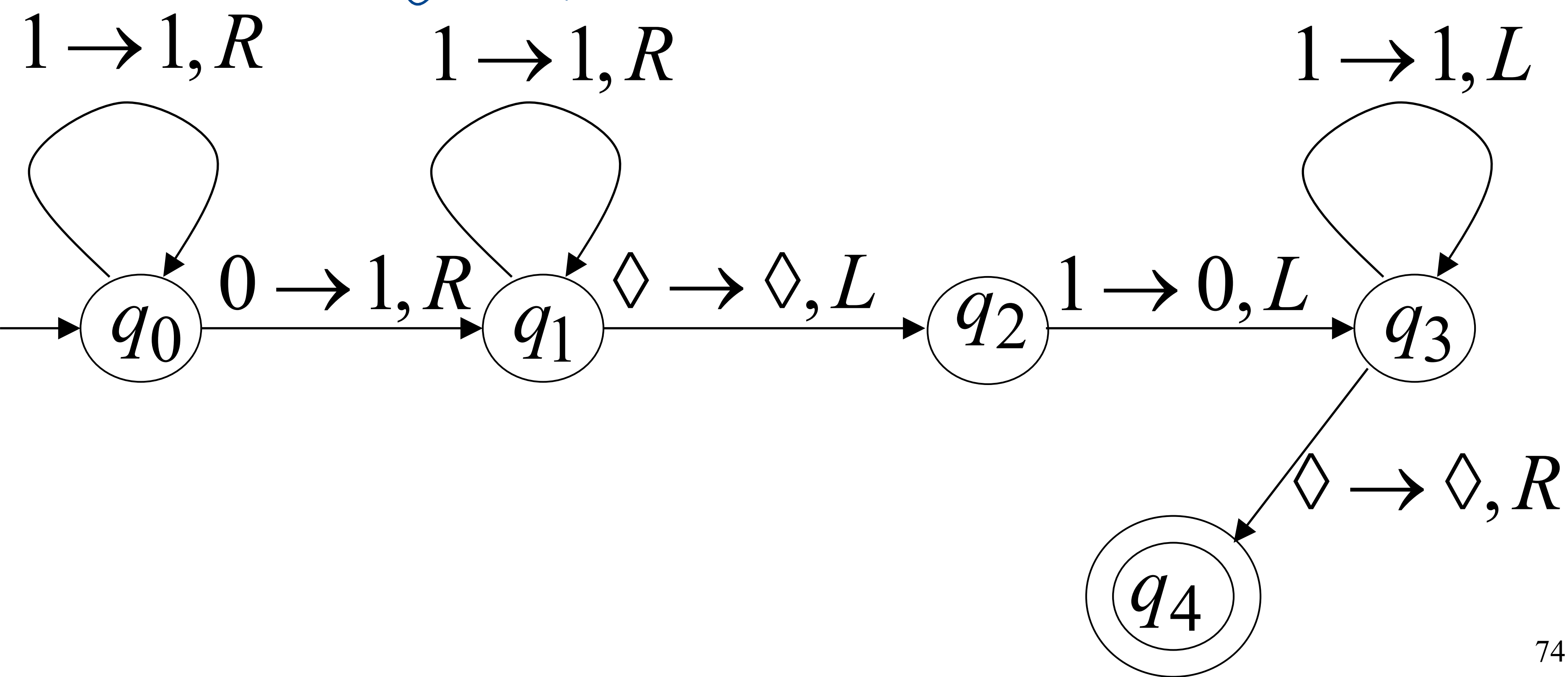
(1) $\uparrow_1 \uparrow_2 \uparrow_3$

(3) $\uparrow_1 \uparrow_2$

(4) $\uparrow_1$

(5) $\uparrow_2 \uparrow_2 \uparrow_1$

(6) $\uparrow_1$ qf

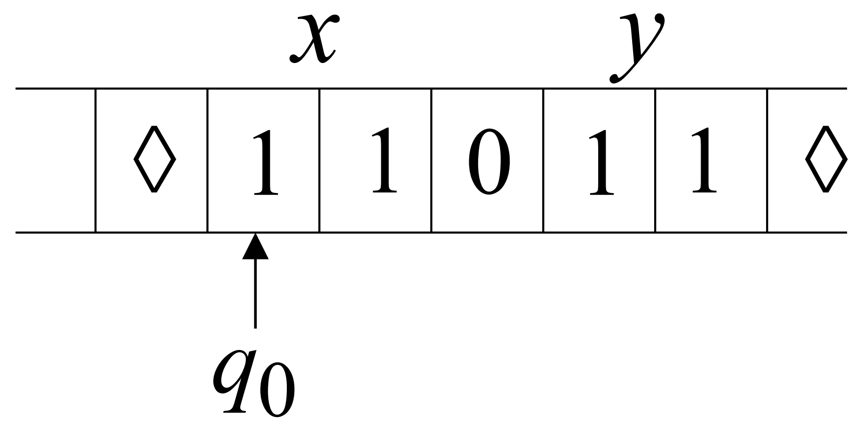


Execution Example:

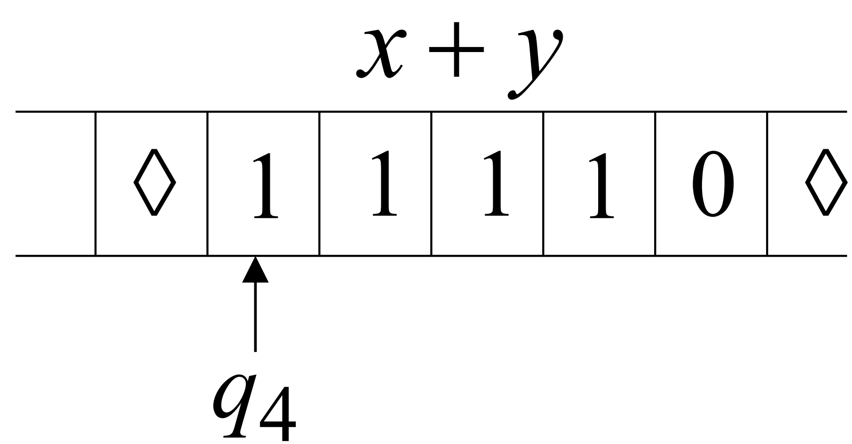
$x = 11$ (2)

$y = 11$ (2)

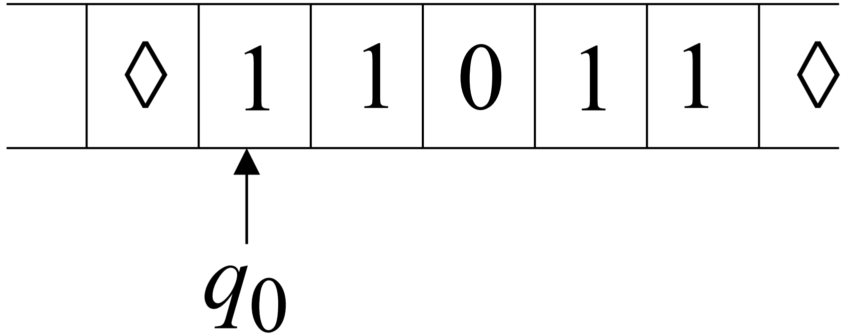
Time 0



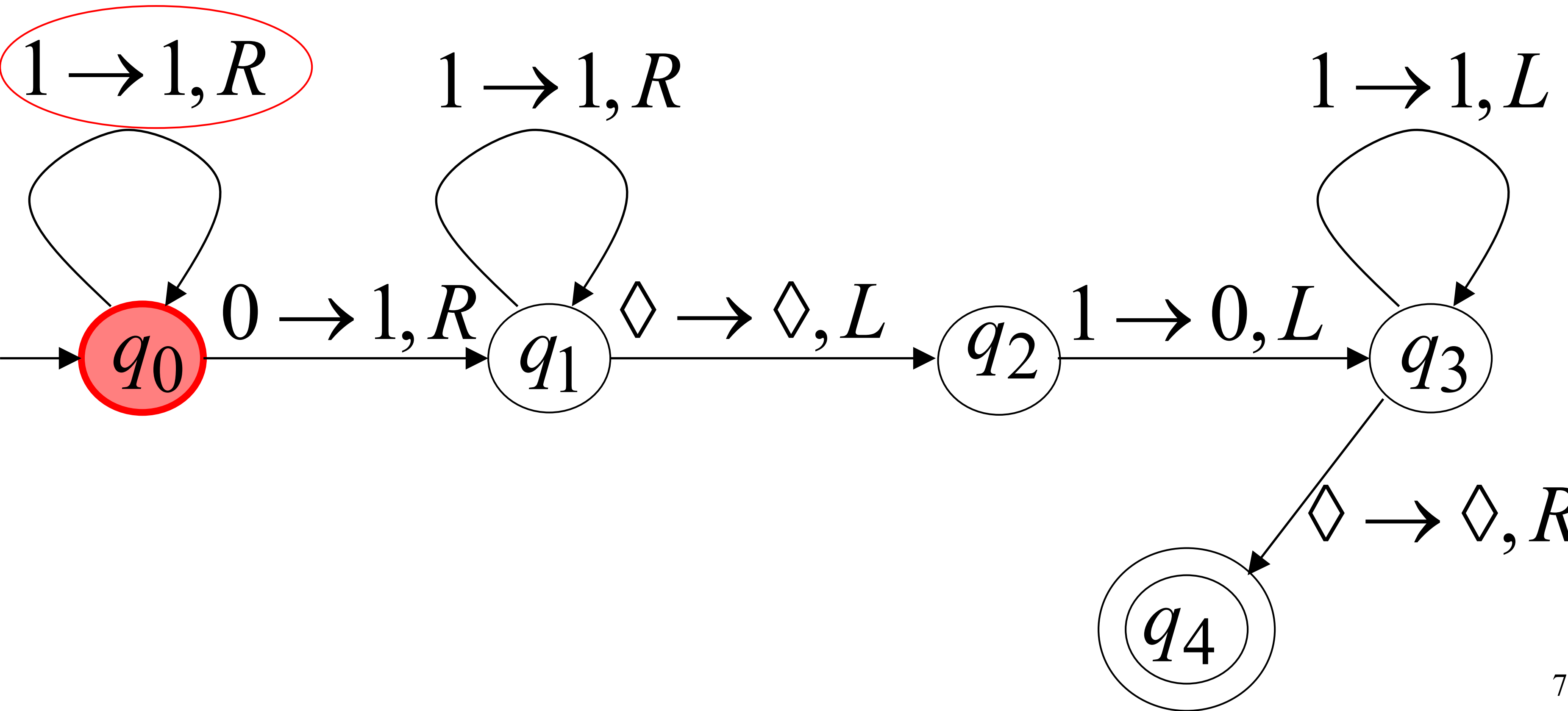
Final Result



Time 0

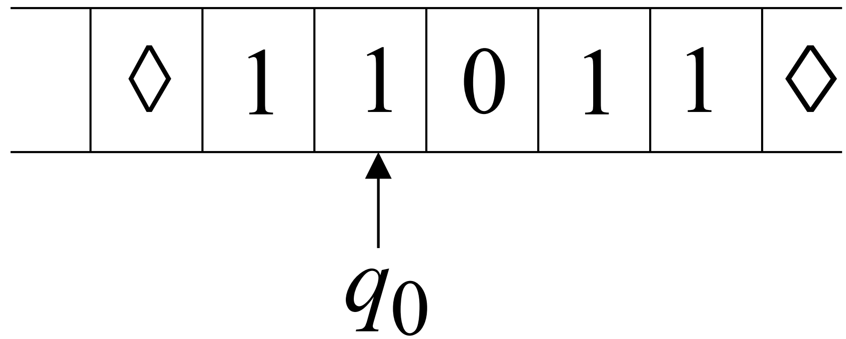


◊
1
1
0
1
1
◊
q_0

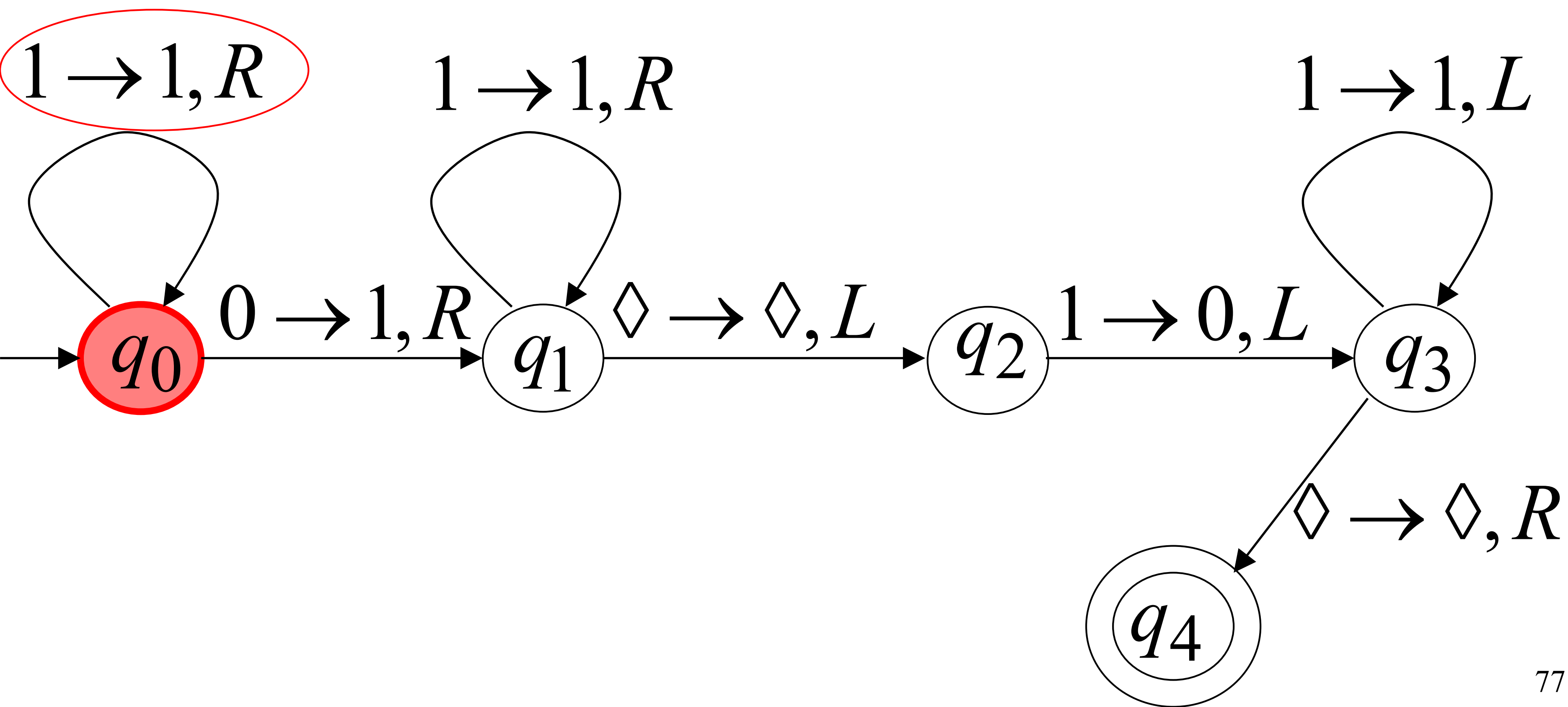


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\lozenge \rightarrow \lozenge, L$
$1 \rightarrow 0, L$
$\lozenge \rightarrow \lozenge, R$
q_0
q_1
q_2
q_3
q_4

Time 1

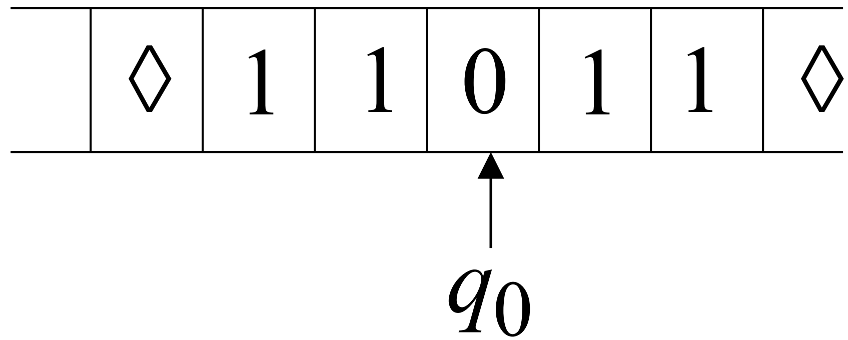


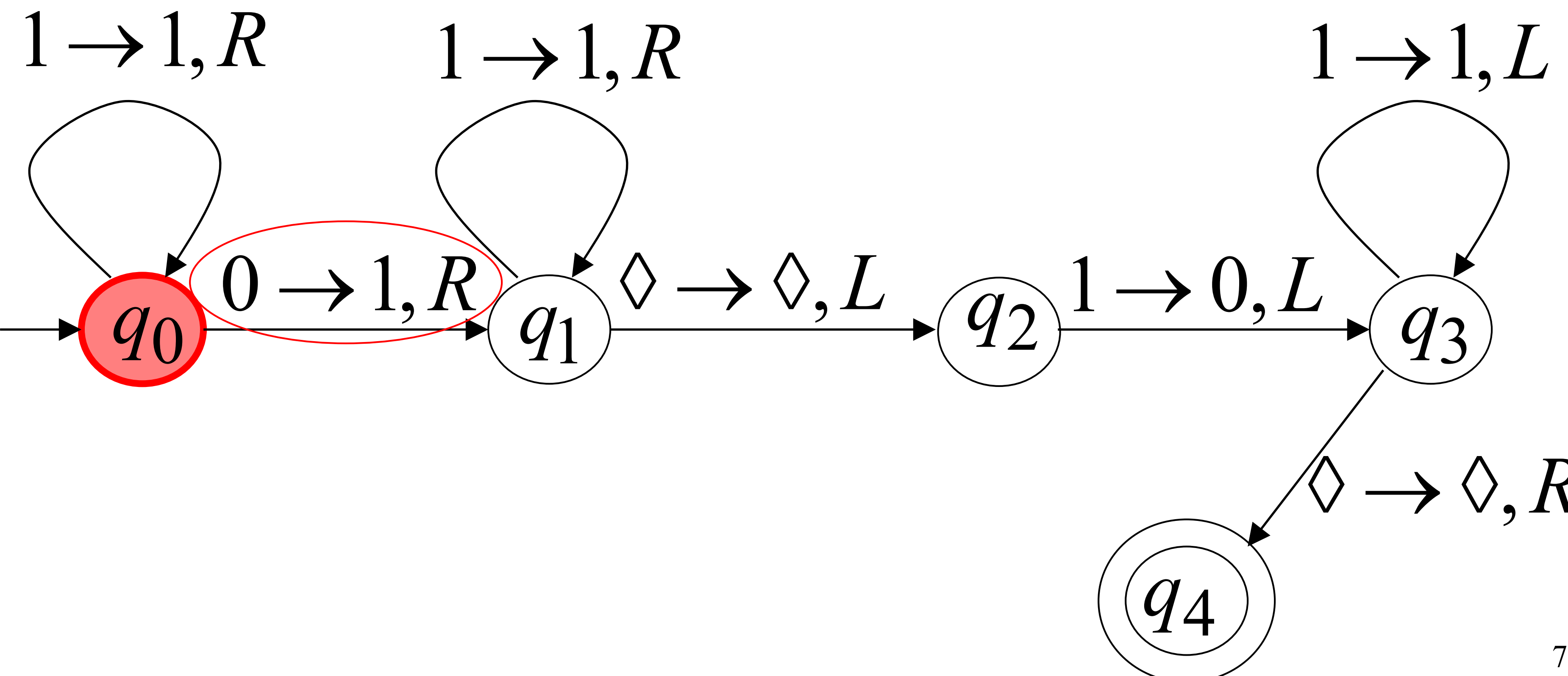
◊ 1 1 0 1 1 ◊
q_0



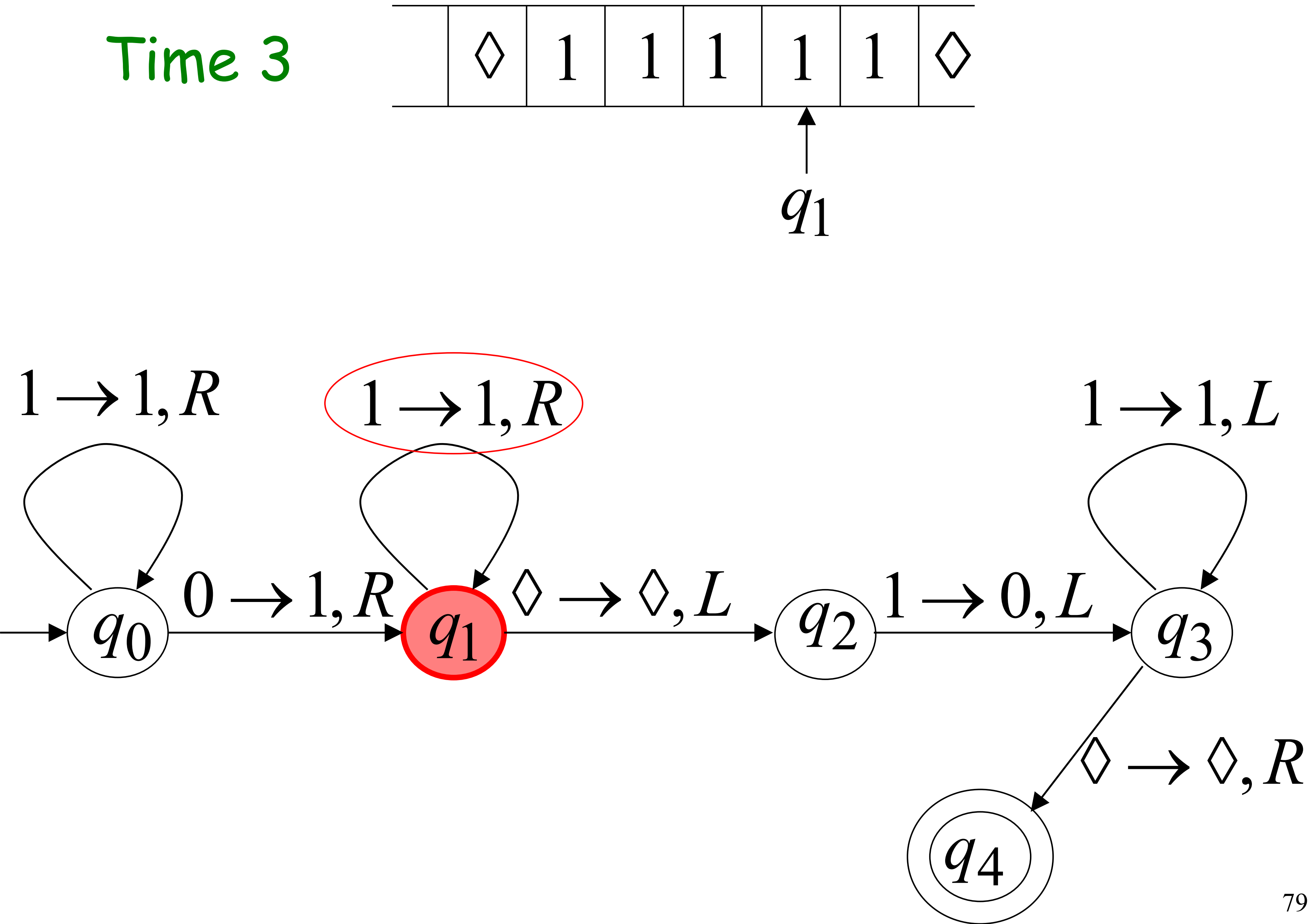
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 2


◊ 1 1 0 1 1 ◊
q_0

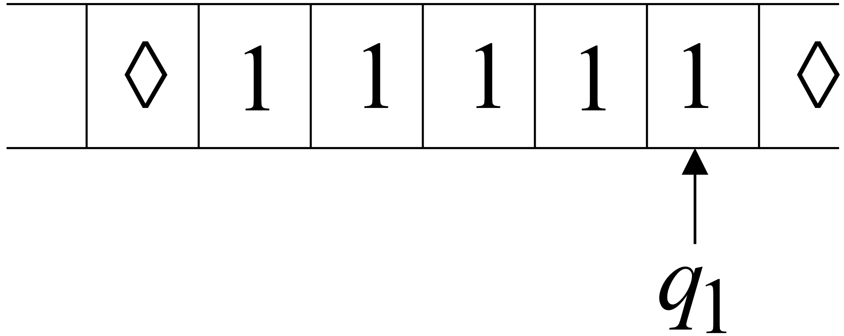

$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

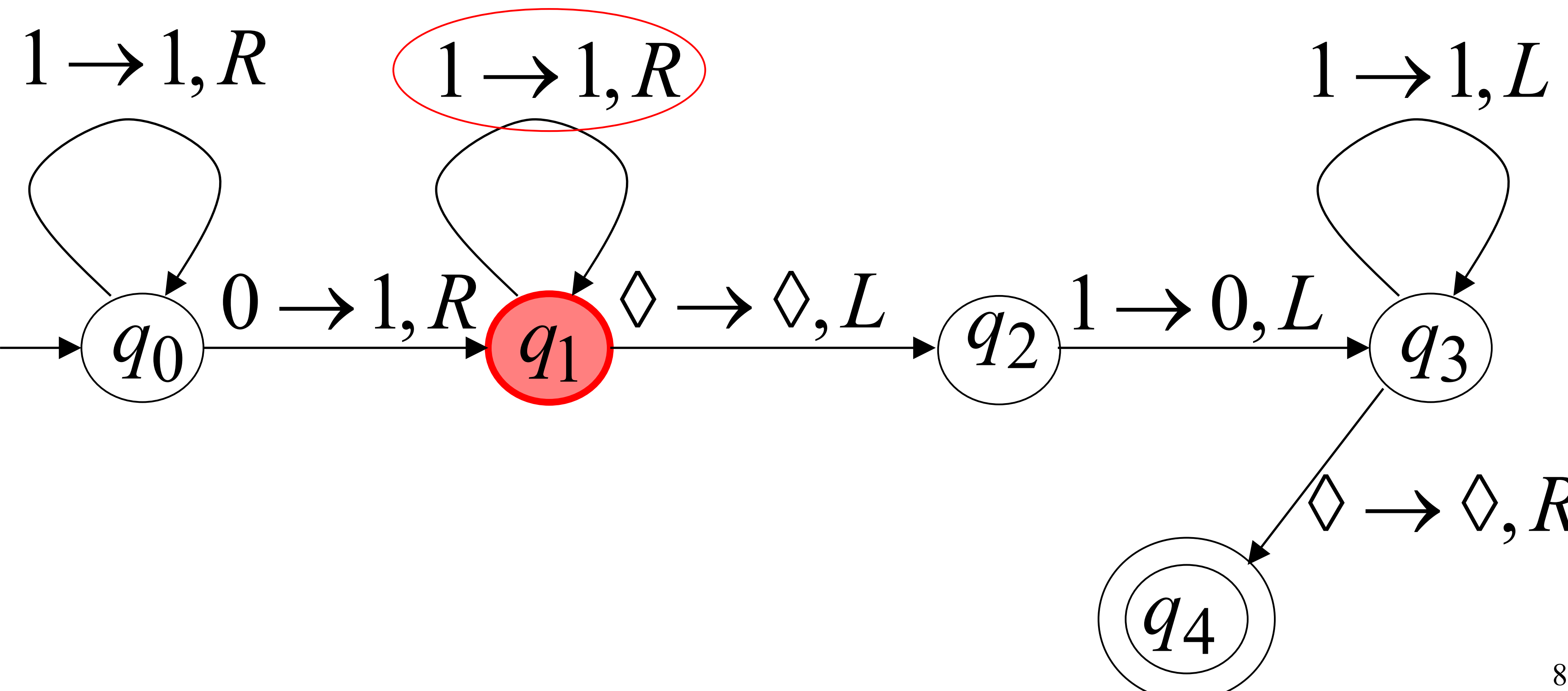
Time 3



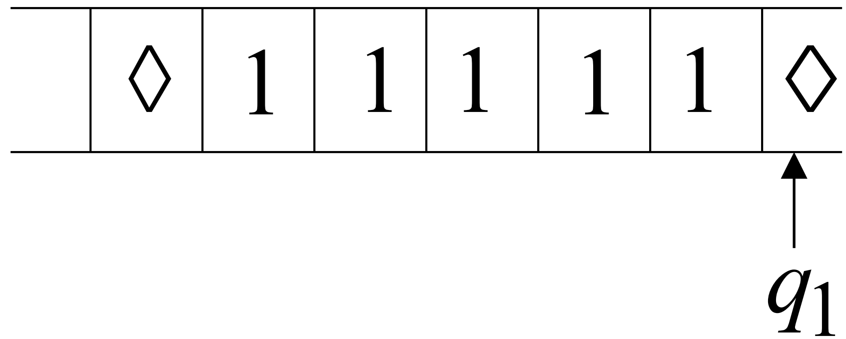
◊ 1 1 1 1 1 ◊
q_1
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 4

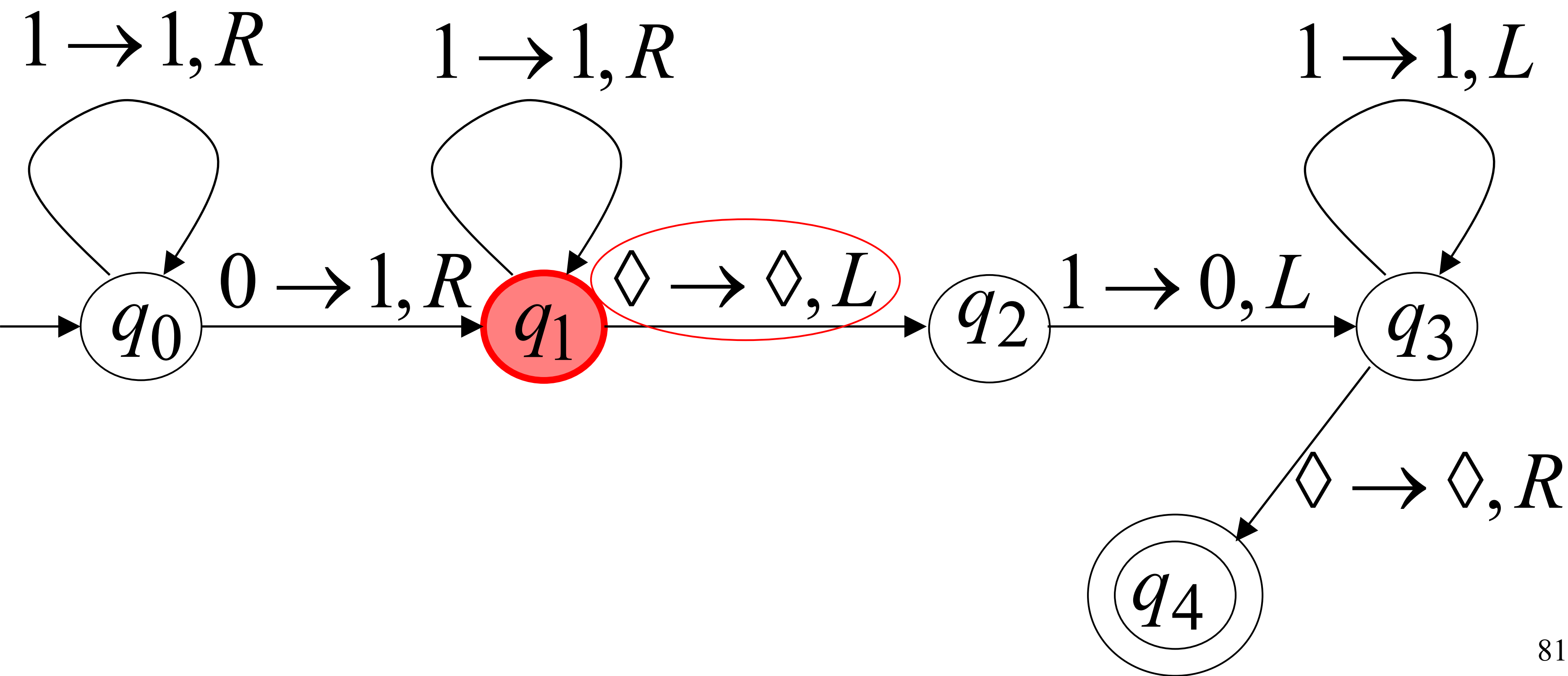

◊ 1 1 1 1 1 ◊
q_1


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 5

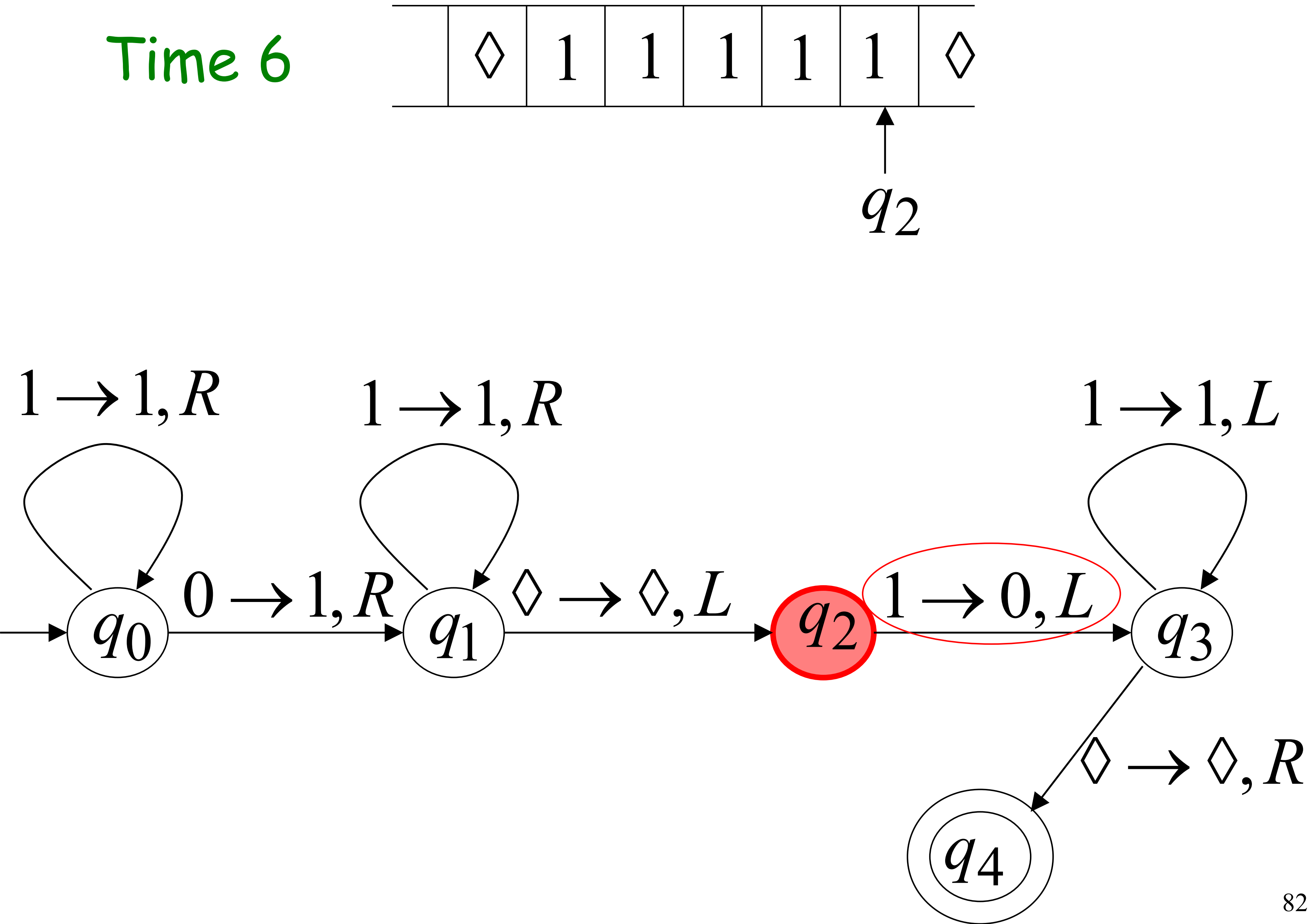


◊ 1 1 1 1 1 ◊
q_1



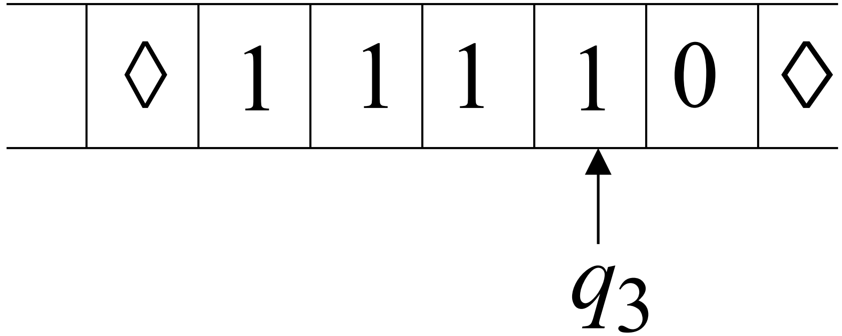
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 6

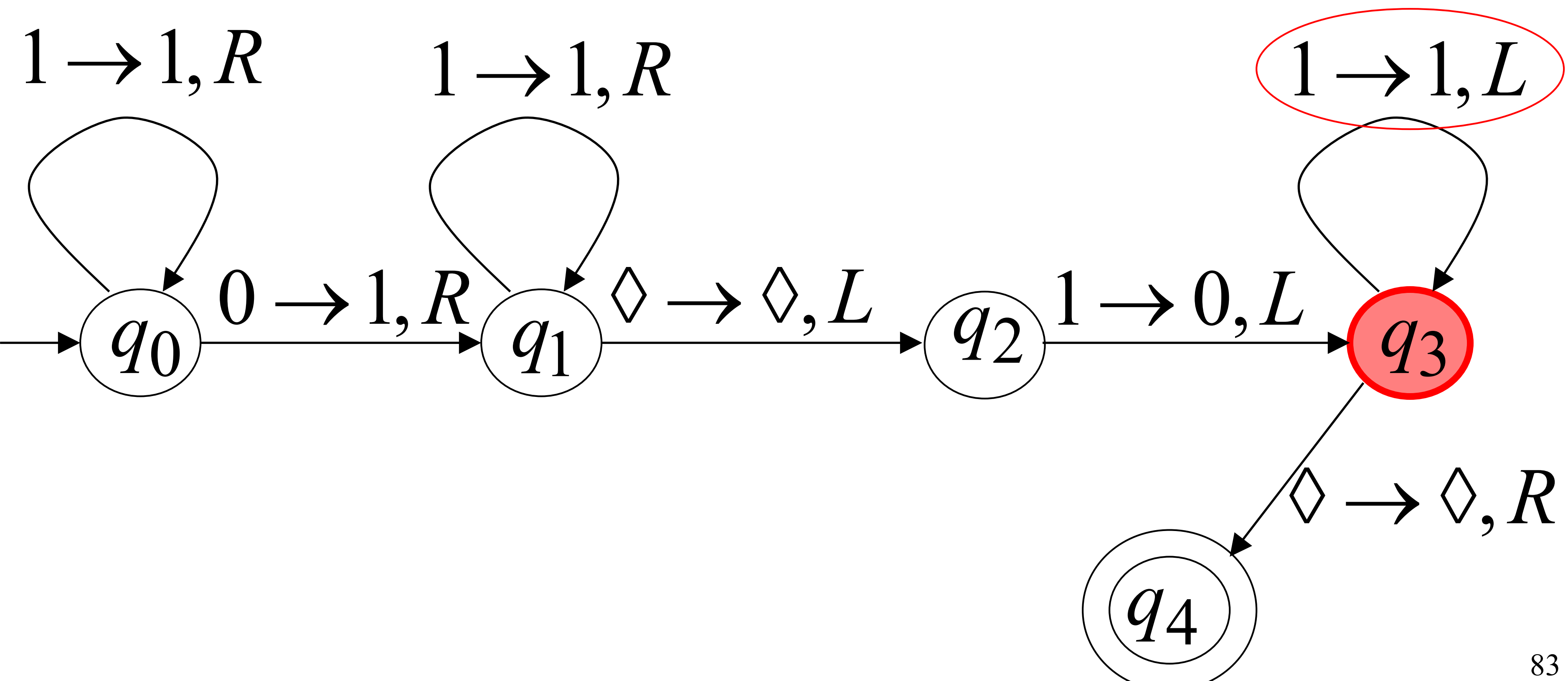


◊ 1 1 1 1 1 ◊
q_2
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4

Time 7

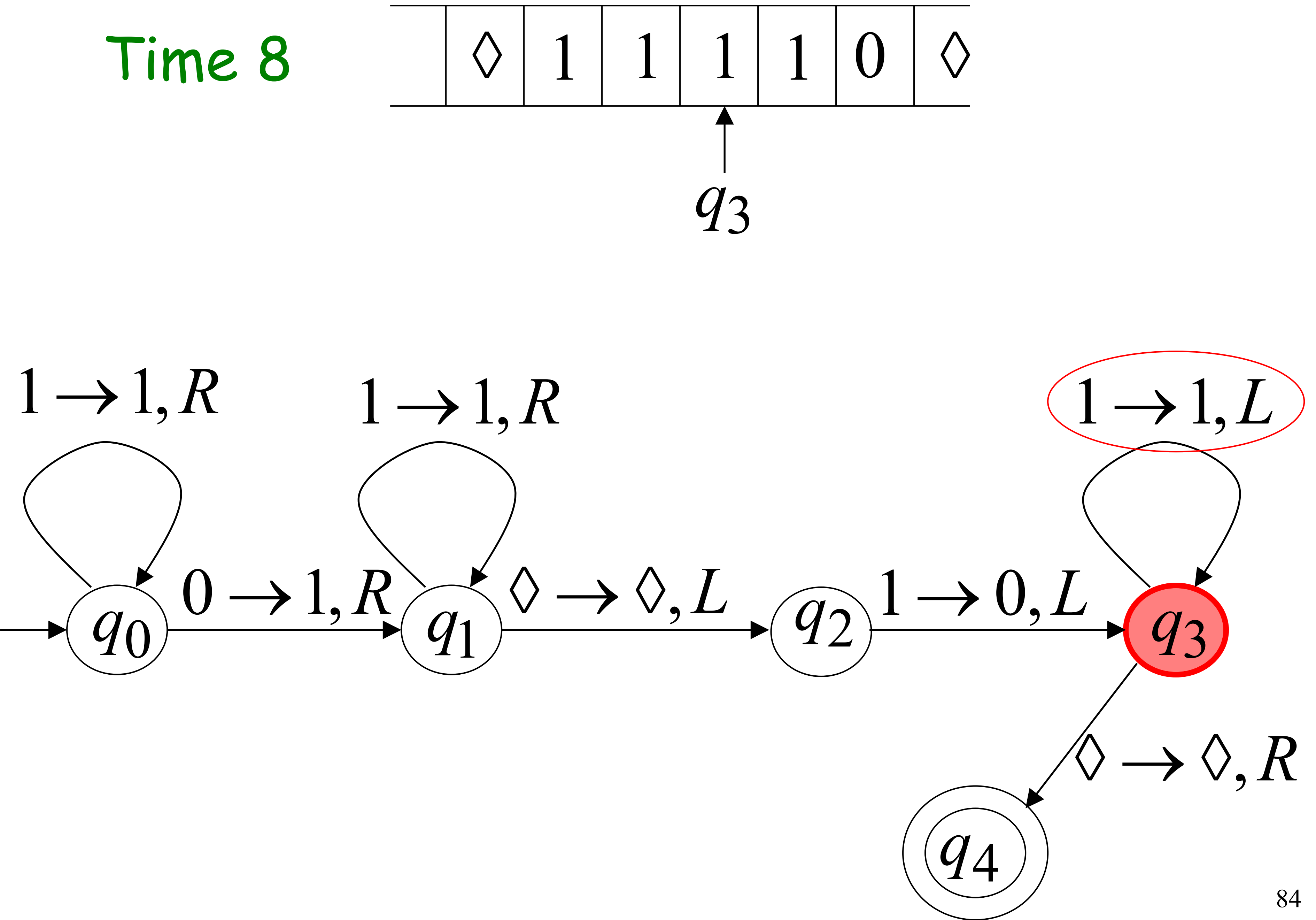


◊
1
1
1
1
0
◊
q_3



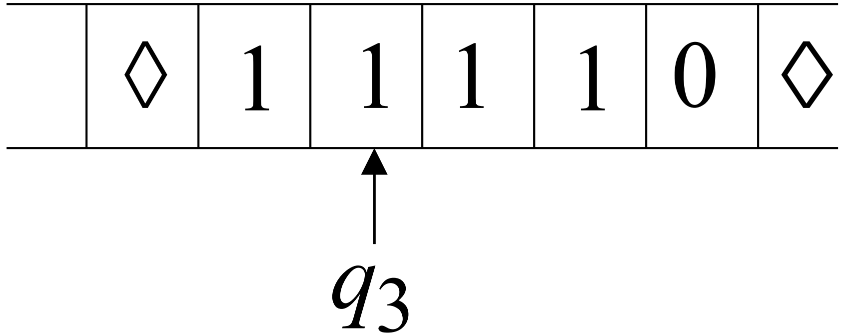
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 8

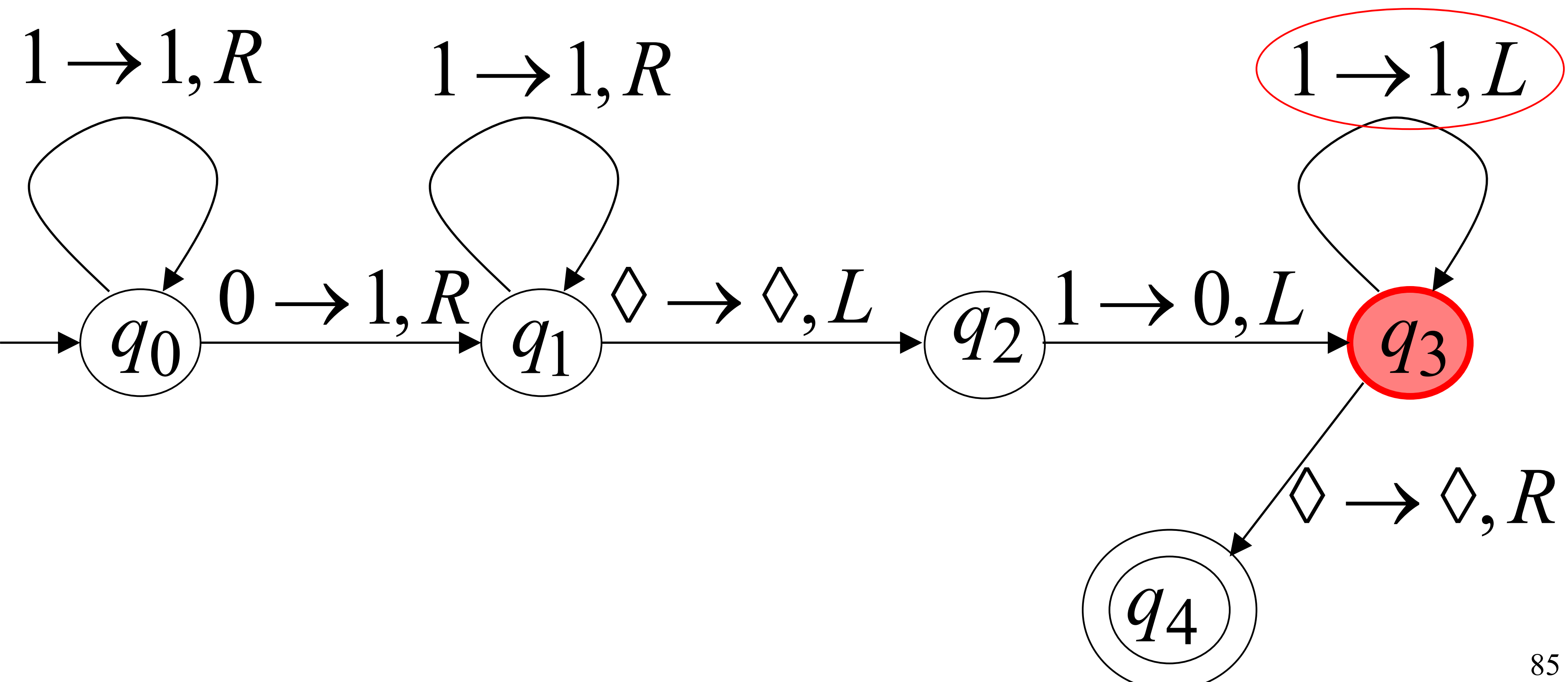


◊
1
1
1
1
0
◊
q_3
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 9

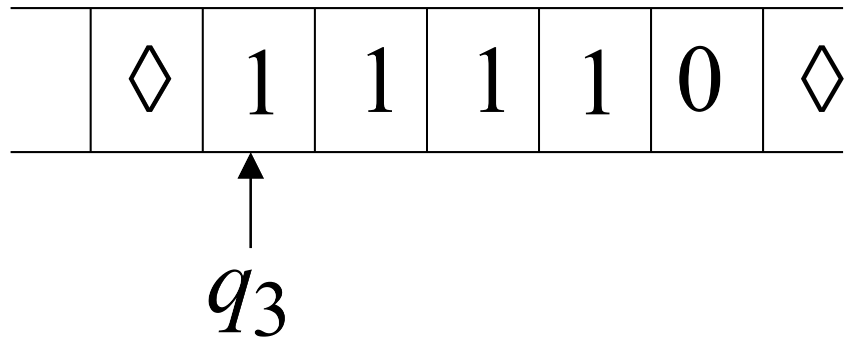


◊ 1 1 1 1 0 ◊
q_3

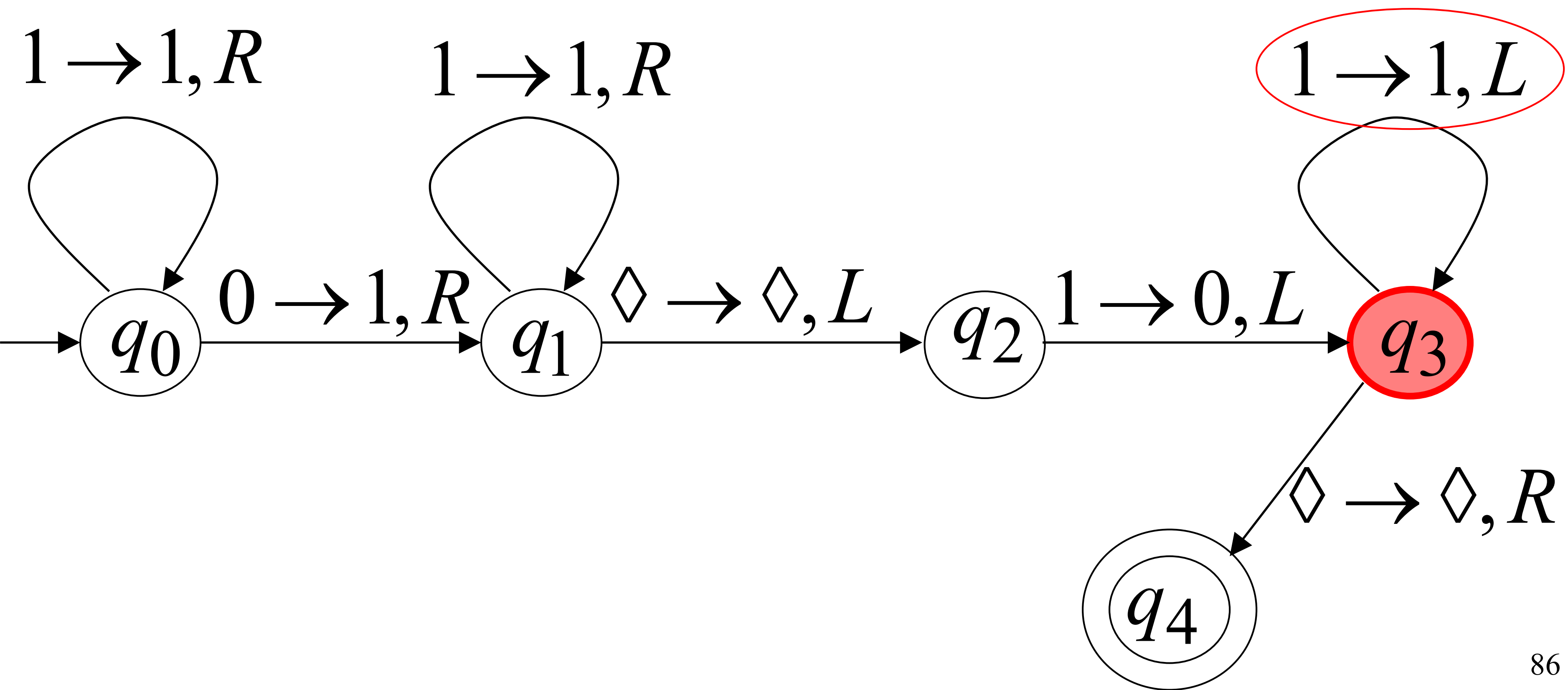


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 10

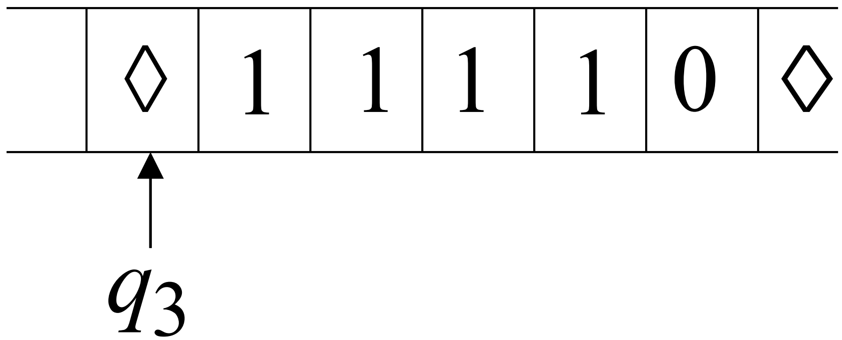


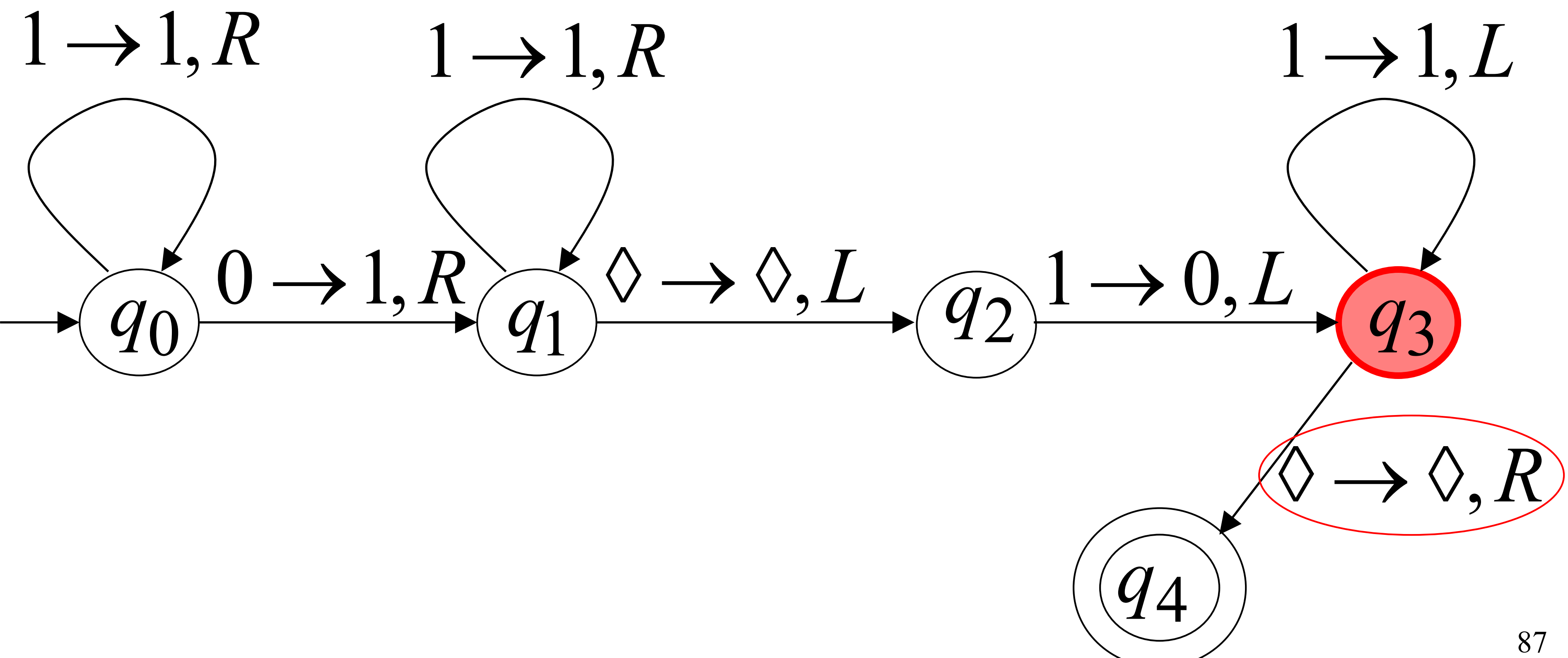
◊ 1 1 1 1 0 ◊
q_3



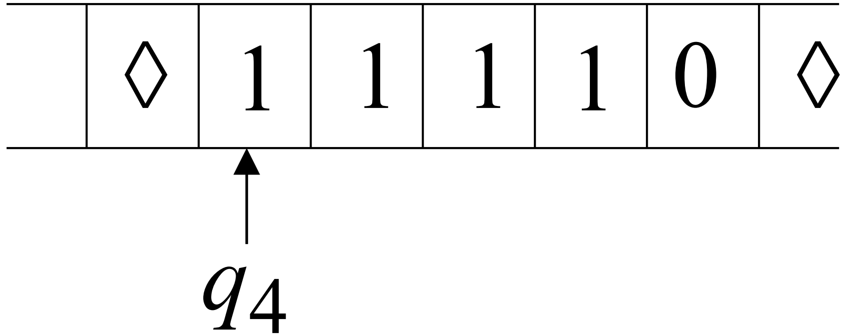
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
q_0
$0 \to 1, R$
q_1
$\Diamond \to \Diamond, L$
q_2
$1 \to 0, L$
q_3
$\Diamond \to \Diamond, R$
q_4

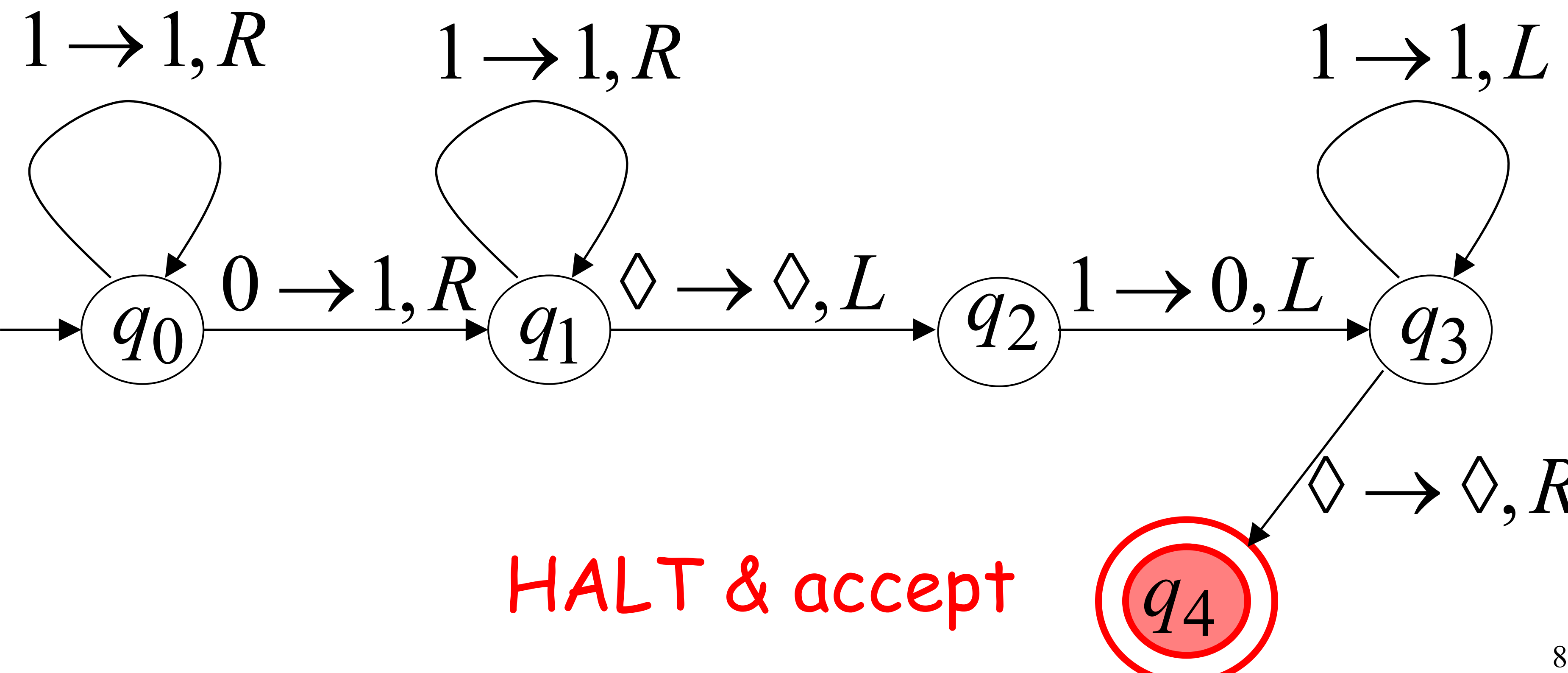
Time 11


◊
1
1
1
1
0
◊
q_3


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 12





HALT & accept

Another Example

The function $f(x) = 2x$ is computable

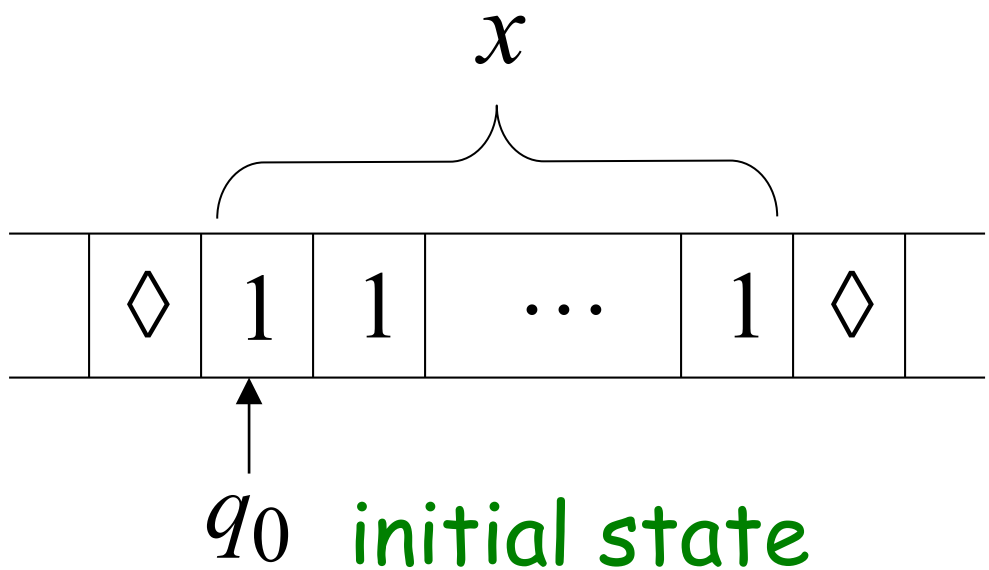
x is integer

Turing Machine: เป็น algolithm copy

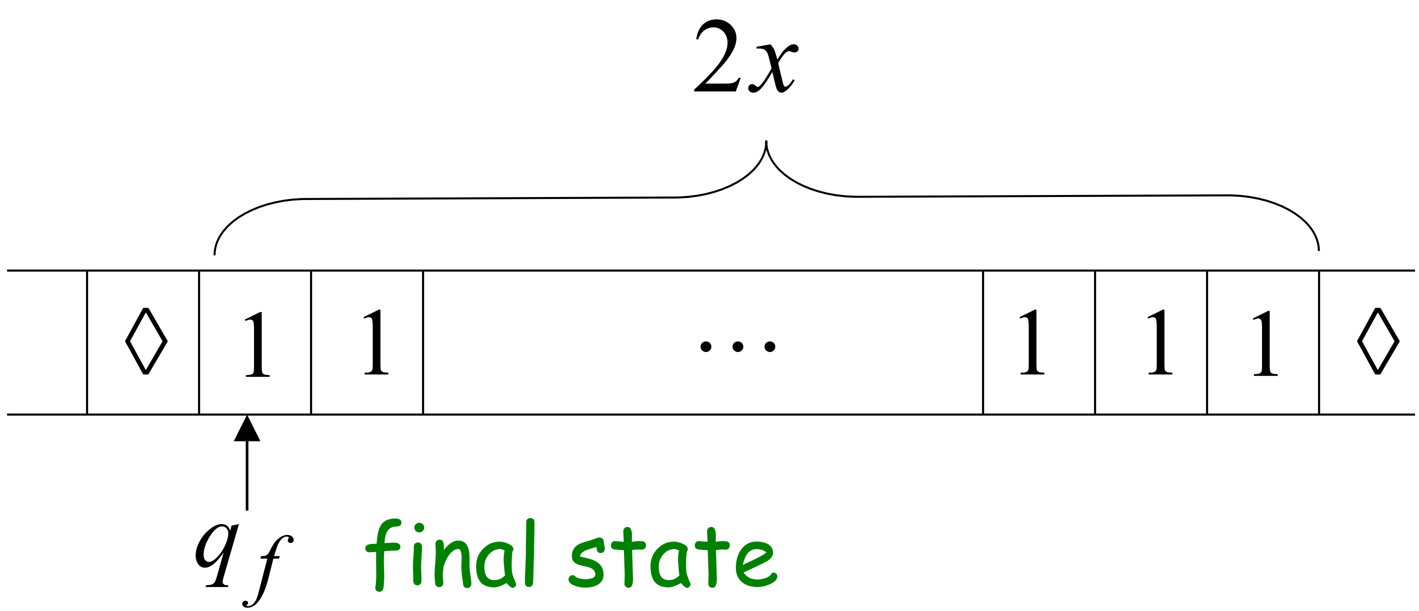
Input string: x unary

Output string: xx unary

Start



Finish



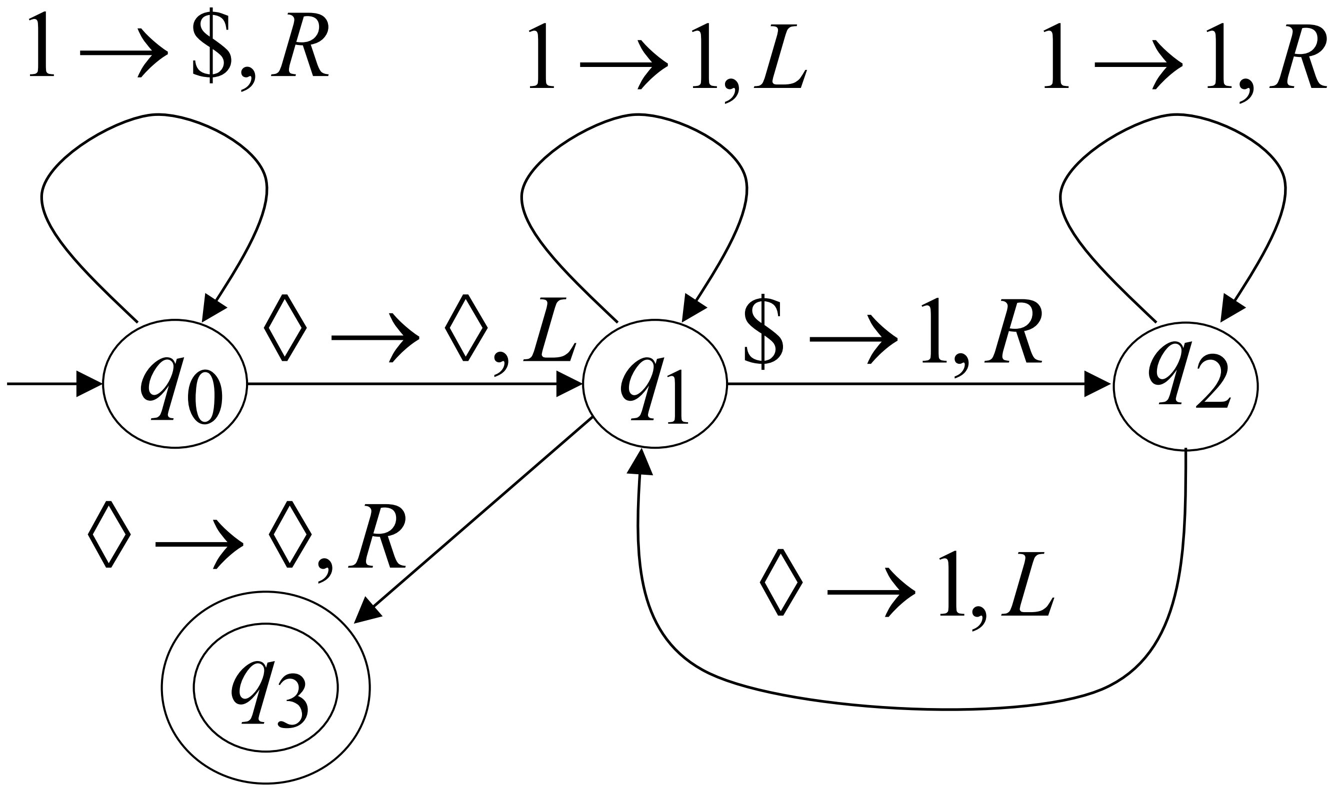
Turing Machine Pseudocode for $f(x) = 2x$

เจอ 1 เปลี่ยนเป็น $

- Replace every 1 with $
- Repeat:
  - Find rightmost $, replace it with 1
  - Go to right end, insert 1

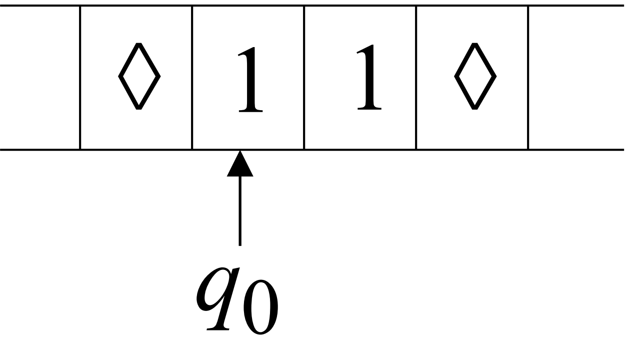
Until no more $ remain

Turing Machine for $f(x) = 2x$

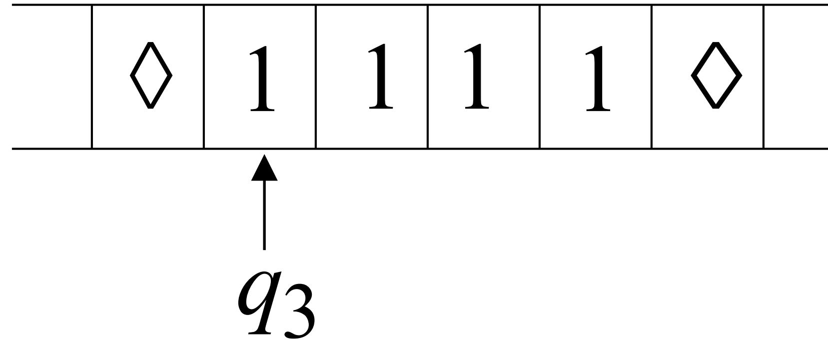


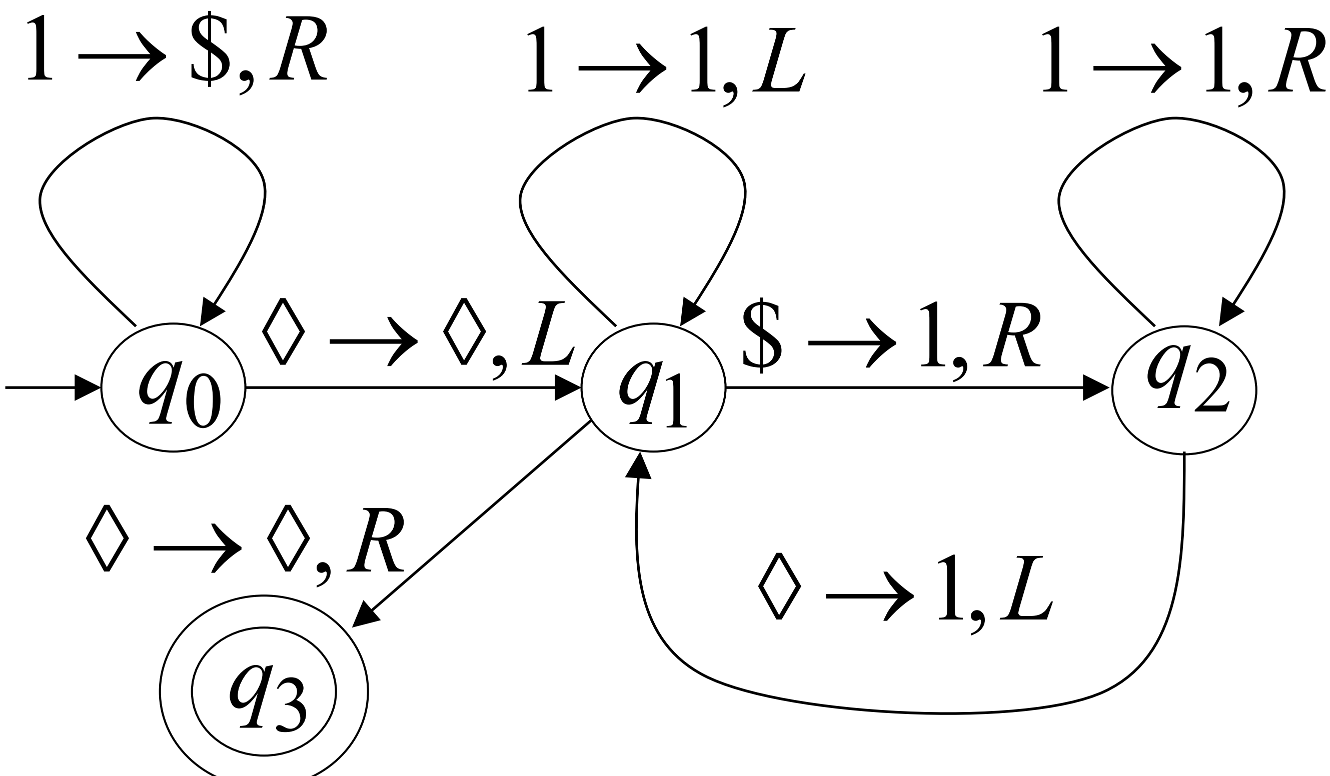
Example

Start



Finish





Another Example

The function

is computable

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \le y \end{cases}$$

Turing Machine for

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \le y \end{cases}$$

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

สามารถทำได้

ทำไม

Turing machine = Algorithm

↳ บางสิ่งที่กริยาเกิด ก → ทำเป็น Algorithm

- Repeat

x - y

แมท 1 - 1

Match a 1 from x with a 1 from y

Until all of x or y is matched

x or y หมดก่อน หรือหมดพร้อมกัน

หลุดลูป

เช็ค

- If a 1 from x is not matched

ลบทั้งหมดที่อยู่บนเทป เขียน

erase tape, write 1 $\quad (x > y)$

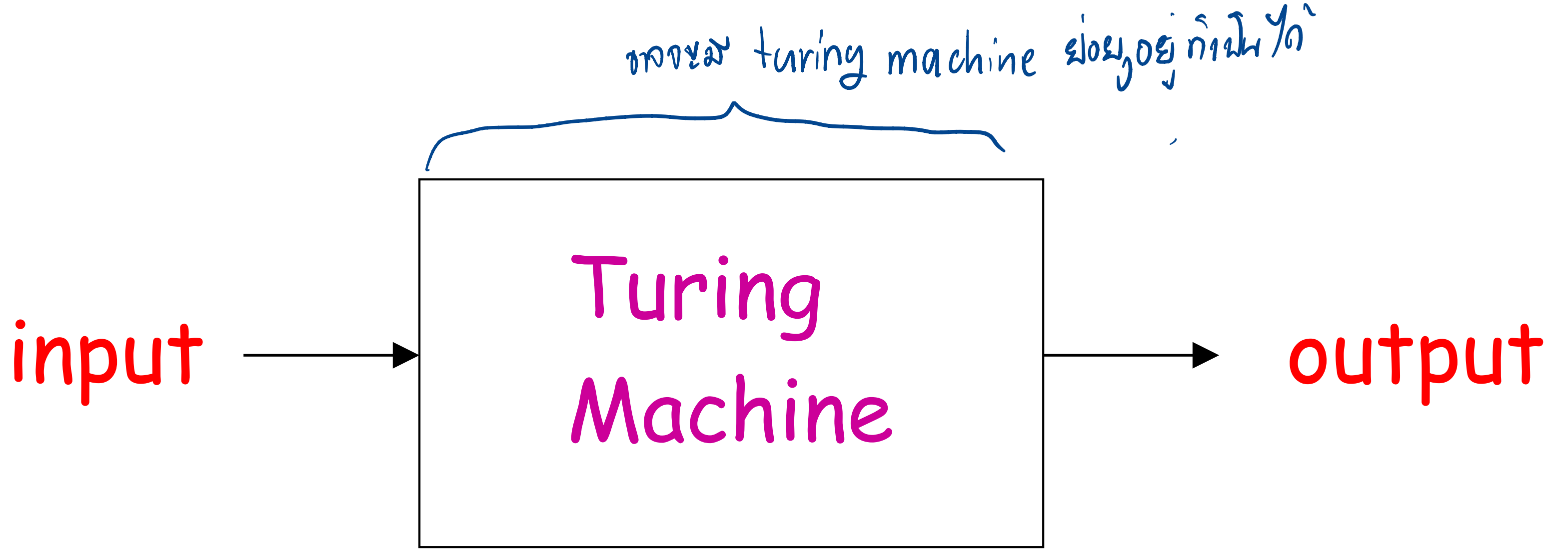
else

เขียน

erase tape, write 0 $\quad (x \le y)$

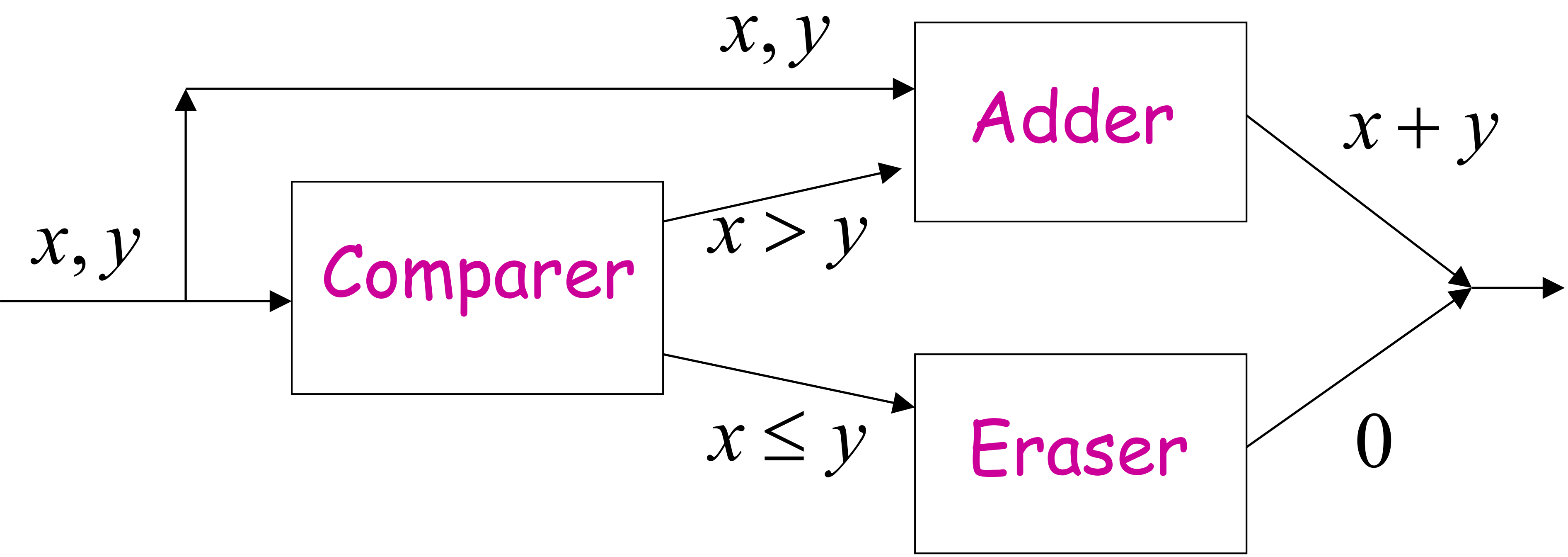
Combining Turing Machines

Block Diagram



Example:

$$f(x, y) = \begin{cases} x + y & \text{if} \quad x > y \\ 0 & \text{if} \quad x \le y \end{cases}$$




That's all Folks!