

Chapter 2

Using Python as a Calculator

One of the most important tasks that a computer performs is mathematical computation. In fact, the computation of mathematical functions had been the driving motivation for the invention of ‘computing machines’ by pioneer researchers. As other programming languages do, Python provides a direct interface to this fundamental functionality of modern computers. Naturally, an introduction of Python could start by showing how it can be used as a tool for simple mathematical calculations.

2.1 Using Python as a Calculator

The easiest way to perform mathematics calculation using Python is to use IDLE, the interactive development environment of Python, which can be used as a fancy calculator. To begin with the simplest mathematical functions, including integral addition, subtraction, multiplication and division, can be performed in IDLE using the following mathematical **expressions**¹:

```
>>> 3+5          # addition
8
>>> 3-2          # subtraction
1
>>> 6*7          # multiplication
42
>>> 8/4          # division
2
```

As can be seen from the examples above, a simple Python expression is similar to a mathematical expression. It consists of some numbers, connected by a mathematical **operator**. In programming terminology, number constants (e.g. 3, 5, 2) are called **literals**. An operator (e.g. +, -, *) indicates the mathematical function between

¹The content after the # symbols are comments and can be ignored when typing the examples. Details about comments are given in Chap. 3 or the rest of the book >>> indicates an IDLE command.

its **operands**, and hence the **value** of the expression (e.g. $3 + 5$). The process of deriving the value of an expression is called the **evaluation** of the expression. When a mathematical expression is entered, IDLE automatically evaluates it and displays its value in the next line.

Since expressions themselves represent values, they can be used as operands in longer *composite expressions*.

```
>>> 3+2-5+1
1
```

In the example above, the expression $3+2$ is evaluated first. Its value 5 is combined with the next literal 5 by the $-$ operator, resulting in the value 0 of the composite expression $3+2-5$. This value is in turn combined with the last literal 1 by the $+$ operator, ending up with the value 1 for the whole expression.

In the example, operators are applied from left to right, because $+$ and $-$ have the same priority. In an expression that contains more than one operators, not necessarily all operators have the same priority. For example,

```
>>> 3+2*5-4
9
```

The expression above evaluates to 9, because the multiplication ($*$) operator has a higher priority compared with the $+$ and $-$ operators. The order in which operators are applied is called **operator precedence**, and mathematical operators in Python follow the natural precedence by the mathematical function. For example, multiplication ($*$) and division ($/$) have higher priorities than addition ($+$) and subtraction ($-$). An operator that has higher priority than multiplication is the power operator ($**$).

```
>>> 5 ** 2           # power
25
```

In general, $a ** b$ denotes the value of a to the power of b .

Similar to mathematical equations, Python allows the use of brackets (i.e. ‘(’ and ‘)’) to manually specify the order of evaluation. For example,

```
>>> (3+2)*(5-4)     # brackets
5
```

The value of the expression above is 5 because the bracketed expressions $3+2$ and $5-4$ are evaluated first, before the $*$ operator is applied.

In the examples above, an operator connects two literal operands, and hence they are called *binary operators*. An operator can also be *unary*, taking only a single operand. An example unary operator is $-$, which takes a single operand and negates the number.

```
>>> -(5*1)          # negation
-5
```

There is also a *ternary operator* in Python, which takes three operands. It will be introduced in a later chapter.

Until this point, all the mathematical expressions have been integral, with the values of literal operands and expressions being integers. Take the division operator ($/$) for example,

```
>>> 5/2          # division with integer operands
2
```

The result of the integral division operation is the quotient 2, with the fractional part 1 discarded. To find the remainder of integer division, the *modulo operator* (%) can be used.

```
>>> 5%2          # modulo
1
```

2.1.1 Floating Point Expressions

So far all the expressions in this chapter are integer expressions, of which all the operands and the value are integers. However, for the expressions $5/2$, sometimes the real number 2.5 is a more appropriate value. In computer science, real number are typically called **floating point number**. To perform floating point arithmetics, at least one floating point number must be put in the expression, which results in a **floating point expression**. For example,

```
>>> 5.0/2        # floating point division
2.5
>>> 5/2.0        # floating point division
2.5
>>> 25 ** 0.5    # floating point power
5.0
```

The last example above calculates the positive square root of 25. Regardless of operands, when all the numbers in a Python expression are integers, the expression is an integer expression, and the value of the expression itself is an integer. However, when there is at least one floating point number in an expression, the expression is a floating point expression, and its value is a floating point number. Below are some more examples, which show that +, − and * operators can all be applied to floating point numbers, resulting in floating point numbers.

```
>>> 3.0+5.1      # floating point addition
8.1
>>> 1.0-2.4      # floating point subtraction
-1.4
>>> 5.5*0.3      # floating point multiplication
1.65
```

The observation above leads to an important fact about Python: things have **types**. Literals have types. The literal 3 indicates an integer, and the literal 3.0 indicates a floating point number. Expressions have types, and their types are the types of their values. The type of a literal or expression can be examined by using the following commands:

```
>>> type(3)
<type 'int'>
>>> type(3.0)
```

```

<type 'float'>
>>> type(3+5)
<type 'int'>
>>> type(3+5.0)
<type 'float'>

```

The command *type(x)* returns the type of *x*. This command is a **function call** in Python, which *type* is a built-in function of Python. We call a **function** with specific **arguments** in order to obtain a specific **return value**. In the case above, calling the function *type* with the argument 3 results in the ‘integer type’ return value.

Function calls are also expressions, which are written in a form similar to mathematics function, with a function name followed by a comma-separated list of arguments enclosed in a pair of brackets. The value of a function call expression is the return value of the function. In the above example, *type* is the name of a function, which takes a single argument, and returns the type of the input argument. As a result, the function call *type(3.0)* evaluates to the ‘float type’ value.

Intuitively the return value of a function call is decided by both the function itself and the arguments of the call. To illustrate this, consider two more functions. The *int* function takes one argument and converts it into an integer, while the *float* function takes one argument and converts it into a floating point number.

```

>>> float(3)
3.0
>>> int(3.0)
3
>>> float(3)/2
1.5
>>> 3*float(3-2*5+4)**2
27.0

```

As can be seen from the examples above, when the function is *type*, the return values are different when the input argument is 3 and when the input argument is 3.0. On the other hand, when the input argument is 3, the return value of the function *type* differs from that of the function *float*. This shows that both the functions and the arguments determine the return value.

The last two examples above is a composite expression, in which the function call *float(3)* is evaluated first, before the resulting value 3.0 is combined with the literal 2 by the operator */*. Function calls have higher priorities than mathematical operators in operator precedence.

The *int* function converts a floating point number into an integer by discarding all the digits after the floating point. For example,

```

>>> int(3.0)
3
>>> int(3.1)
3
>>> int(3.9)
3

```

In the last example, the return value of *int(3.9)* is 3, even though 3.9 is numerically closer to the integer 4. For floating-point conversion by rounding up an integer, the *round* function can be used.

```
>>> round(3.3)
3
>>> round(3.9)
4
```

The *round* function can round up a number not only to the decimal point, but also to a specific number of digits after the decimal point. In the latter case, two arguments must be given to the function all, with the second input argument indicating the number of digits to keep after the decimal point. The following examples illustrate this use of the *round* function with more than one input arguments. Take note of the comma that separates two input arguments.

```
>>> round(3.333, 1)
3.3
>>> round(3.333, 2)
3.33
```

A floating point operator that results in an integer value is the integer division operator (`//`), which discards any fractional part in the division.

```
>>> 3.0//2
1
>>> 3.5//2
1
```

Correspondingly, the modulo operator can also be applied to floating point division.

```
>>> 3.5%2
1.5
```

Another useful function is *abs*, which takes one numerical argument and returns its absolute value.

```
>>> abs(1)
1
>>> abs(1.0)
1.0
>>> abs(-5)
5
```

One final note on floating point numbers is that their literals can be expressed by a **scientific notation**. For example,

```
>>> 3e1
30.0
>>> 3e-1
0.3
>>> 3E2
300.0
```

The notations xey and xEy have the same meaning. They indicate the value of $x \times 10^y$.

2.1.2 Identifiers, Variables and Assignment

The set of arithmetic expressions introduced above allows simple calculations using IDLE. For example, suppose that the annual interest rate of a savings account is 4 %. To calculate the amount of money in the account after three years, with an initial sum of 3,000 dollars is put into the account, the following expression can be used.

```
>>> 3000*1.04**3
3374.592
```

One side note is that brackets can be used to explicitly mark the intended operator precedence, even if they are redundant. In the case above, $3000 * 1.04 * *3$ can be written as $3000 * (1.04 * *3)$ to make the operator precedence more obvious. In general, being more explicit can often make the code easier to understand and less likely to contain errors, especially when there are potential ambiguities (e.g. non-intuitive or infrequently used operator precedence).

For a second example, suppose that the area of a square is 10 m^2 . The length of each edge can be calculated by:

```
>>> 10**0.5
3.1622776601683795
```

The result can be rounded up to the second decimal place.

```
>>> round(10**0.5, 2)
3.16
```

For notational convenience and to make programs easier to maintain, Python allows names to be given to mathematical values. An equivalent way of calculating the edge length is:

```
>>> a=10
>>> w=a**0.5
>>> round(w, 2)
3.16
```

In the example above, a denotes the area of the square, and w denotes its width. The use of a and w makes it easier to understand the underlying physical meanings of the values. a and w are called **identifiers** in Python. Each Python identifier is bound to a specific value. In the example, a is bound to 10 and w is bound to $\sqrt{10}$. Identifiers can be bound to new value:

```
>>> x=1
>>> x
1
>>> x=2
>>> x
2
```

In the example, the value of x is first 1, and then 2. Because identifiers can change their values, they are also called **variables**.

The $=$ sign in the above example is not an operator, and hence the commands $a = 10$ and $w = \text{round}(a ** 0.5)$ are not expressions. They bare no values. Instead,

Table 2.1 List of keywords in Python

| | | | | |
|----------|--------|---------|-------|-------|
| and | as | assert | break | class |
| continue | def | del | elif | else |
| except | exec | finally | for | from |
| global | if | import | in | is |
| lambda | not | or | pass | print |
| raise | return | try | while | with |
| yield | | | | |

the `=` sign denotes an *assignment statement*, which binds an identifier to a value. Here a **statement** is a command to be executed by Python, and statements are the basic execution units in Python. There are different types of statements, as will be introduced in this book. In an assignment statement, the identifier to which a value is assigned must be on the left hand side of `=`, and the value to assign to the identifier, which can be any expression, should be on the right hand side of `=`. Python gives a name to a value by binding the value to an identifier.

An intuitive difference between identifiers and literals is that the former are names while the latter are values. Formally, an identifier must start with a letter or underscore (`_`), and contain a sequence of letters, numbers and underscores. For example, *area*, *a*, *a0*, *area_of_square* and *_a* are all valid identifiers, but *0a*, *area of square* or *a!* are not valid identifiers. An additional rule is that identifiers must not be **keywords** in Python, which are a list of reserved words. There are 31 keywords in total, which are listed in Table 2.1. Each keyword can be associated with one or more statements, which will be introduced in the subsequent chapters.

For another example problem, suppose that a ball is tossed up on the edge of a cliff with an initial velocity v_0 , and that the initial altitude of the ball is 0m. The question is to find the vertical position of the ball at a certain number of seconds t after the toss. If the initial velocity of 5 m/s and the time is 0.1 s, the altitude can be calculated by:

```
>>> v0=5
>>> g=9.81
>>> t=0.1
>>> h = v0*t-0.5*g*t**2
>>> round(h, 2)
0.45
```

To further obtain the vertical location of the ball at 1 s, only t and h need to be modified.

```
>>> t=1
>>> h=v0*t-0.5*g*t**2
>>> round(h, 2)
0.09
```

Note that the value of h must be calculated again after the value of t changes. This is because an assignment statement binds an identifier to a value, rather than

establishing a mathematical correlation between a set of variables. When $h = v_0 * t - 0.5 * g * g * t ** 2$ is executed, the right hand side of $=$ is first evaluated according to the current values of v_0 , g and t , and then the resulting value is bound to the identifier h . This is different from a mathematical equation, which establishes factual relations between values. When the value of t changes, the value of h must be recalculated using $h = v_0 * t - 0.5 * g * g * t ** 2$. For another example,

```
>>> x=1
>>> x=x+1
>>> x
2
```

There are three lines of code in this example. The first is an assignment statement, binding the value 1 to the identifier x . The second is another assignment statement, which binds the value of $x + 1$ to the identifier x . When this line is executed, the right hand side of $=$ is first evaluated, according to the current value of x . The result is 2. This value is in turn bound to the identifier x , resulting in the new value 2 for this identifier. The third line is a single expression, of which the value is displayed by IDLE. Think how absurd it would be if the second line of code is treated as a mathematical equation rather than an assignment statement!

An equivalent but perhaps less ‘counter-intuitive’ way of doing $x = x + 1$ is $x += 1$.

```
>>> x=1
>>> x+=1
>>> x
2
```

The same applies to $x = x - 3$, $x = x * 6$, and other arithmetic operators.

```
>>> x-=3
>>> x
-1
>>> x*=6
>>> x
-6
```

In general, $x <op> = y$ is equivalent to $x = x <op> y$, where $<op>$ can be $+$, $-$, $*$, $/$, $\%$ etc. The special assignment statements $+=$, $-=$, $*=$, $/=$ and $\%=$ can be used as a concise alternative to a $=$ assignment statement when it incrementally changes the value of one variable.

Given the fact above, it is not difficult to understand the outputs, if the following commands are entered into IDLE to find the position of the ball after 3 s in the previous problem.

(continued from above)

```
>>> t=3
>>> round(h, 2)
0.09
>>> h=v0*t-0.5*g*t**2
>>> round(h, 2)
-29.15
```


The commands above are executed sequentially and individually. When t is bound to the new value 3, h is not affected, and remains 0.09. After the new h assignment is executed, its value changes to -29.15 according to the new t . *Cascaded assignments*. Several assignment statements to the same value can be cascaded into a single line. For example, $a = 1$ and $b = 1$ can be cascaded into $a = b = 1$.

```
>>>a=b=1
>>>a
1
>>>b
1
```

2.2 The Underlying Mechanism

Floating point arithmetic can be inaccurate in calculators; the same happens in Python.

```
>>> x=1.0/7
>>> x+x+x+x+x+x+x
0.9999999999999998
>>> 3.3%2
1.2999999999999998
```

In both cases, the expression is evaluated to an imprecise number. The main reason is limitation of memory space. A floating point number can contain an infinite amount of information, if there is an infinite number of digits after the decimal point. However, the amount of information that can be processed or stored by a calculator or a computer is finite. As a result, floating point numbers cannot be stored to an arbitrary precision, and floating point operators cannot be infinitely precise. The error that results from the imprecise operations is called **rounding off error**.

At this point it is useful to know a little about the underlying mechanism of Python, so that deeper understanding can be gained on facts such as rounding off errors, which enables more solid programs to be developed. This section discusses the representation and storage of numbers, the way in which arithmetic operations are carried out by Python, and the underlying mechanisms of identifiers and assignment statements.

The basic architecture of a computer is shown in Fig. 2.2 in the previous chapter. On the bottom of the figure, the main hardware components are shown, which include the CPU, memory and devices. Among the three main components, the CPU is the most important; it carries out computer instructions, including arithmetic operations. The typical way in which arithmetic operations are executed is: the CPU takes the operands from the memory, evaluates the result by applying the operator on them, and then stores the result back into the memory. The memory can be regarded as a long array of information storage units. Each unit is capable of storing a certain amount of information, and the index of its location in the long array is also referred to as its **memory address**. Devices are the channel through which computers are connected

to the physical world. Keyboards, mouses, displays, speakers, microphones are a few commonly-used devices. An important device is the hard disk, on which the OS defines a file system for external storage of information.

2.2.1 Information

Computers are information-processing machines. The rounding-off error examples show that the basic unit of storage and computation can only accommodate a limited amount of information. But what is information, and how can one store information? A short answer to the first question is that, information is represented in computers as *discrete numbers*, or integers. It is an abstraction of real physical quantities, such as the pitch of sound, the colour, and the alphabet. Signals from input devices are transformed into discrete numbers before being processed by a computer, and output devices turn discrete values back into physical signals.

For example, letters typed on a keyboard are mapped into discrete numbers (e.g. 'A' \rightarrow 64) before being stored into the memory. Sound waves received by a microphone are sampled at a certain rate (e.g. 256,000 times a second), and then turned into an array of discrete values. Such type of sound information can be processed (e.g. denoised) or transformed (e.g. enlarged), and then passed to a sound output device and transformed back into sound waves. A black and white display transforms a grid of numbers into a grid of pixels, each number depicting the brightness of a pixel on the display. A robot can move according to input numbers that indicate desired velocity and direction. In all these cases, devices act as a channel between computers and the physical world.

To answer the second question above, the easiest data storage medium that can be found is probably some material that can have two states (e.g. high-voltage vs. low-voltage, solid vs. liquid, hot vs. cold). A basic storage unit made of such material can store a **binary** value, denoted as 0 or 1, each representing a distinct state of the material. Larger integers can be stored by using a combination of multiple basic storage units. For example, the combination of *two* basic storage units can store *four* distinct values: 00, 01, 10 and 11. An illustration is shown in Fig. 2.1. In general, the total number of *possible* states by combining n basic storage units is 2^n . On the other hand, at any time, the combined units can only have *one actual* combined state, which can be represented by a unique array of 0s and 1s.

It is natural to associate an array of 0s and 1s with a discrete number (integer). One way to number distinct states of N basic storage units $s_0, s_1, \dots, s_N, s_i \in \{0, 1\}$ is to interpret each distinct state as a non-negative **binary number**, treating the value of $s_N s_{N-1} \dots s_0$ as $2^N * s_N + 2^{N-1} * s_{N-1} + \dots + 2^0 * s_0 = \sum_{i=0}^N s_i \cdot 2^i$. For example, the state 101 corresponds to the integer $1 * 2^2 + 0 * 2^1 + 1 * 2^0 = 1 * 4 + 0 * 2 + 1 * 1 = 5$, and 1101 corresponds to the integer $1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0 = 1 * 8 + 1 * 4 + 0 * 2 + 1 * 1 = 13$. In this way, a natural connection is established between the states of data storage materials and integers, which represent information. In computer science, each binary-valued digit is called a *bit*, and a combination of

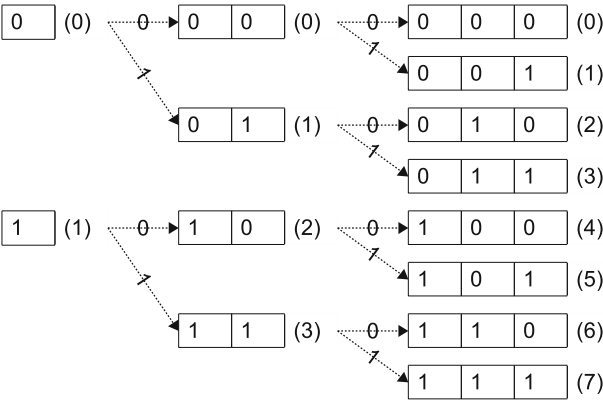


Fig. 2.1 Distinct states that can be represented by a combination of binary storage units

8 bits is a **byte**. A byte storage unit can store $2^8 = 256$ distinct numbers, which can be indexed by the integers 0–255. Most modern computers treat 8 bytes, or 64 bits, as a **word**, which serves as the basic units for storage.

As mentioned earlier, a digital number represents abstract information. Numbers, alphabets, sounds and all other types of information are abstracted and represented by binary numbers in computers. In **information theory**, the amount of information is measured by bits. A 64-bit word can represent 64 bits of information, which translates to 2^{64} distinct integers. The abstract information, however, can be **interpreted** in different ways. The aforementioned interpretation of information as non-negative integer values is just one example, which can be used to represent the brightness of a pixel on a black and white monitor.

Another example is the *2's complement* interpretation of **signed integers**, which uses 64 bits to represent an integer that ranges from -2^{63} to $2^{63} - 1$. In this representation, the first bit always indicates the sign: when it is 0, the number is positive; when it is 1, the number is negative. When the number is positive, the remaining 63 bits indicate the absolute value of the number. Negative numbers are represented in a slightly more complicated way. Rather than concatenating the sign bit (i.e. 1) with a 63-bit representation of the absolute value, a negative number is represented by first inverting its absolute value bit by bit, and then adding 1 to the result. For the convenience of illustration, take a byte-sized number for example. To find the representation of -13 , two steps are necessary. First, the absolute value, 13, or 00001101 in binary form, is inverted bit by bit into 11110010. Then 1 is added to the back of the number:

$$\begin{array}{r} 11110010 \\ +) 00000001 \\ \hline 11110011 \end{array}$$

The resulting number, 1110011, is the binary representation of -13 in 2's complement form. Note that the first bit is 1, indicating that it is a negative number. For

another example, to represent -16 , its absolute value 00010000 is first inverted bit by bit into 11101111 , and then 1 is added to the number,

```

  1 1 1 0 1 1 1 1
+) 0 0 0 0 0 0 0 1
-----
  1 1 1 1 0 0 0 0

```

As a result, -13 and -16 are 11110011 and 11110000 , respectively, according to 2's complement representation. The same process of negative number interpretation applies to 64-bit words. The advantage of 2's complement representations is that the addition operation between numbers can be performed in the same way regardless of whether negative numbers are involved or not. For example, $-13 + 13$ can be performed by

```

  1 1 1 1 0 0 1 1
+) 0 0 0 0 1 1 0 1
-----
  1 0 0 0 0 0 0 0

```

With the first bit being discarded (it runs out of the 8-bit boundary, and hence cannot be recorded by 8-bit physical media), the result is 0, the correct answer.

A third useful interpretation of information is **floating point numbers**. When sound is concerned, floating point numbers give a more convenient model of the pitches in sound wave samples. As discussed earlier, all floating point numbers cannot be represented using a finite amount of information, and therefore some have to be truncated when represented using a computer word. A standard approach of representing floating point numbers in a finite number of bits is to split the total number of bits into two parts, one representing a base number (also referred to as the *significant*) and the other representing the exponent. For example, from a 64-bit word, 11 bits can be used to denote the exponent (e) and 53 bit the base (b). This type of representation naturally corresponds to the scientific notation of float literals in Python, where $bEe = b \times 10^e$. Of course, abstract information can also be interpreted as letters and other quantities in the physical world, which are out of the scope of this book.

Words are used not only as the basic units of **data storage**, but also as the basic units of **computation**. In digital circuits, of which all modern computers are made, electric signals are represented by binary values, with a high voltage in a wire denoting the value 1, and a low voltage denoting 0. Digital chips, such as CPUs, takes a fixed number of binary signals as input, and have a fixed number of output. 64-bit CPUs perform arithmetic operations on 64-bit operands, yielding 64-bit results by hardware computation. As a consequence, floating point numbers can contain only 64 bits of information, and floating point arithmetics have rounding off errors. In fact, integers are also represented by words on computer hardware, typically in 2's complement form. However, Python provides a new type, *long*, which represents numbers that exceeds the range of 64 bits. Python converts large integers into the *long* type automatically, and performs arithmetic operations between *long* type numbers implicitly by using a sequence of 64-bit integer arithmetic operations, so that programmers can use a large integer in Python without noticing the difference

between *int* and *long*. *Long* type numbers can be identified by examining their types explicitly.

```
>>> type(111)
<type 'int'>
>>> type(2**64)
<type 'long'>
```

A *long* type number can also be specified explicitly using *long* literals, which are integer literals with a 'l' or 'L' added to the end

```
>>> type(111L)
<type 'long'>
```

In summary, information that a computer stores and processes is ultimately represented by a finite number of 0s and 1s (i.e. bits), organized in basic unites (e.g. words). They are interpreted in different ways when turned into specific types.

2.2.2 Python Memory Management

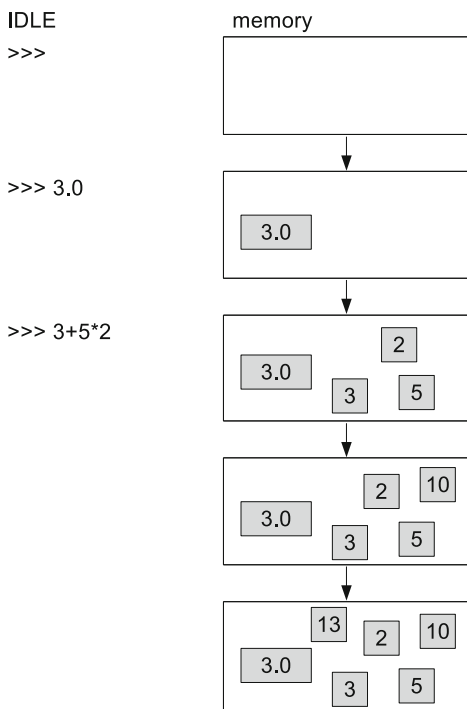
When evaluating an arithmetic expression, Python first constructs an **object** in the memory for each literal in the expression, and then applies the operators one by one to obtain intermediate and final values. All the intermediate and final values are stored in the memory as objects. Python objects are one of the most important concepts in understanding the underlying mechanism of Python. Integers, floating point numbers and instances of many more types to be introduced in this book, are maintained in the memory as Python objects.

Fig. 2.2 illustrates how the memory changes when some arithmetic expressions are evaluated. After IDLE starts, the memory contains some default objects, which are not under concern at this stage, and therefore not shown in the figure. When the expression 3.0 is evaluated, a new *float* object is constructed in the memory. Python always creates a new object when it evaluates a literal. When the expression $3 + 5 * 2$ is evaluated, the integer constants 3, 5 and 2 are constructed in the memory, before the operators $*$ and $+$ are executed in their precedence. When $*$ is executed, Python passes the values of the objects 5 and 2 to the CPU, together with the $*$ operator, and stores the result 10 as a new integer object in the memory. When $+$ is executed, Python invokes the CPU addition operation with the values of the objects 3 and 10, storing the result 13 as a new object.

Identifiers are names of objects in memory, used by Python to access the corresponding objects. Python associates identifiers to their corresponding values, or objects, by using a **binding table**, which is a lookup table.

Figure 2.3 shows an example of the binding table, and how it changes as Python executes assignment statements. After IDLE starts, some default entries are put into the binding table, which are ignored in this figure. When $x = 6$ is executed, the expression 6 is first evaluated, resulting in the integer object 6 in the memory. Then Python adds an entry in the binding table, associating the name x with the object 6.

Fig. 2.2 Example memory structure for expressions



When $y = x * 2$ is executed, the value of the expression $x * 2$ is first evaluated by evaluating x , and 2, and then calculating $x * 2$. When Python evaluates the literal 2, it creates a new object in the memory; then when it evaluates the identifier x , it looks up the binding table for an entry named x , which is bound to the object 6. The final value of the expression $x * 2$ is saved to a memory object 36, and bound to the identifier y .

When the statement $x = 3$ is executed next, the expression 3 is first evaluated, resulting in a new object 3 in the memory, which is bound to the name x in the binding table. The old association between the name x and the object 6 is deleted, since one name can be bound to only one object. Note that the assignment statement *always* binds a name in the binding table with an object in the memory. Like all other Python statements, the execution is rather mechanic. Given this fact, it is easy to understand the reason why the value of y does not change to 9 automatically when $x = 3$ is executed.

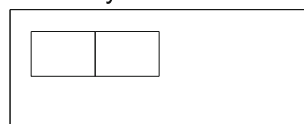
At this stage, the objects 2 and 6 are still in the memory, although they are not bound to any identifiers. As the number of statements increases, the memory can be filled with many such objects. They are no longer used, but still occupy memory space. Python has a **garbage collector** that periodically removes unused objects from the memory, so that more memory space can be available. Note also that Fig. 2.2 does

Fig. 2.3 Example memory structure for assignments

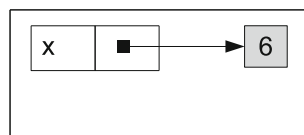
IDLE

>>>

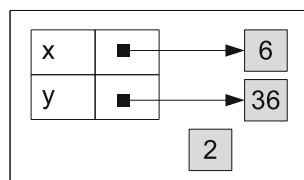
memory



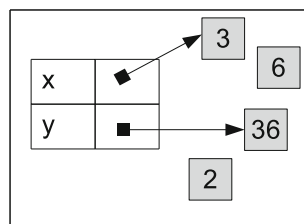
>>> x=6



>>> y=x**2



>>> x=3



not show the binding table, although it exists in the memory, containing identifiers that are irrelevant to the example.

For readers who know C++ and Java variables, it is worth noting that Python variables are not exactly the same as their C++ and Java counterparts. Specifically, the assignment statement in Python changes the value of a variable by changing the binding (i.e. associating the identifier to a different object in the binding table), while the assignment statement of C++ and Java directly changes the value of the object that the identifier is associated with, without changing the binding between identifiers and memory objects. Although in many cases, Python variable can be used in the same way as C++ and Java variables from the programming perspective, an understanding of this difference could be useful in avoiding subtle errors, especially when mutability (Chap. 7) is involved.

Python provides a special statement, the **del statement**, for deleting an identifier from the binding table.

```
>>> x=1
>>> y=2
>>> x
```

```

1
>>> y
2
>>> del x
>>> x
Traceback (most recent call last):
  File "<pyshell#5>", line 1, in <module>
    x
NameError: name 'x' is not defined
>>> y
2

```

As can be seen from the example above, the *del* statement begins with the *del* key word, followed by an identifier. It deletes the identifier from the binding table. As the second last command shows, Python reports a *name error* when the value of *x* is requested after the identifier *x* has been deleted from the binding table. After an identifier is deleted, the object that it is bounded to is not deleted immediately. Instead, the garbage collector will remove it later when no other identifiers are bound to it.

Each Python object has a unique **identify**, which is typically its memory address. Python provides a function, *id*, which takes a Python object as its input argument, and returns the identify of the object. For example,²

```

>>> x=12345
>>> id(x)
4535245720
>>> y=x
>>> id(y)
4535245720
>>> y=23456
>>> id(x)
4535245720
>>> id(y)
4535245672
>>> id(12345)
4535245984

```

When *x* is assigned to the value 12345, it is bound to a new object 12345 in the memory. When *y* is assigned to the value of *x*, it is bound to the same object 12345. Hence the identifies of *x* and *y* are the same. When *y* is reassigned to the value 23456, a new object 23456 is constructed in the memory, occupying a new memory address, and *y* is bound to his object. Hence the identify of *y* changes, while the identify of *x* remains the same. The last command shows the identify of a new object, constructed by the evaluation of the expression 12345. It is different from that of *x*, because every time a literal is evaluated, a new object is constructed in the memory. Here is another example.

```

>>> x=12345
>>> y=x
>>> id(x)

```

²The memory addresses in the examples below are unlikely to be reproduced when the examples are tried again, since the allocation of memory addresses is dynamic and runtime dependent.


```

4462893976
>>> id(y)
4462893976
>>> x=12345
>>> id(x)
4462894072
>>> id(y)
4462893976

```

When x is assigned to the value 12345 the second time, the right hand side of the assignment statement is evaluated first, which leads to a new object having the value 12345 in the memory. x is bound to this new object, while y remains the same. Although the values of x and y are the same, their memory addresses, or identifies are different, because they are bound to two different objects.

Note that the observations above may not hold for small numbers (e.g. 3 instead of 12345). This is because to avoid frequent construction of new objects, Python constructs at initialization a set of frequently used objects, including small integers, so that they are reused rather than constructed afresh when their literals are evaluated.

```

>>> x=10
>>> y=10
>>> id(x)
140621696282752
>>> id(y)
140621696282752

```

In the example above, y is assigned the value 10 after x is assigned the value 10. In each case, no new object is created when the literal 10 is evaluated, because an object with the value 10 has been created in the memory location 140621696282752 to represent all objects with this value.³

In summary, in addition to a value, a Python object also has an identify and a type. Once constructed, the identify and type of an object cannot be changed. For number objects, the value also cannot be changed after the object is constructed.

2.3 More Mathematical Functions Using the *math* and *cmath* Modules

Several mathematical functions have been introduced so far, which include addition, subtraction, multiplication, division, modulo and power. There are more mathematical functions that a typical calculator can do, such as factorial, logarithm and trigonometric functions. These functions are supported by Python through a special **module** called *math*.

A Python module is a set of Python code that typically includes the definition of specific variables and functions. The next chapter will show that a Python module can

³Exactly which small values are represented as frequently accessed objects depends on the Python distribution.

be nothing but a normal Python program. In order to use the variables and functions defined in a Python module, the module must be *imported*. For example,

```
>>> import math
>>> math.pi
3.141592653589793
>>> math.e
2.718281828459045
```

In the example above, the *math* module is imported by the statement *import math*. The *import* statement is the third type of statement introduced in this chapter, with the previous two being the assignment statement and the *del* statement. The *import* statement loads the content of a specific module, and adds the name of the module in the binding table, so that content of the module can be accessed by using the name, followed by a dot (.).

Two mathematical constants *pi* and *e*, are defined in the *math* module, and accessed by using '*math.*' in the above example. Both *pi* (π) and *e* are defined as floating point numbers, up to the precision supported by a computer word.

Mathematical functions can be accessed in the same way as constants, by using '*math.*'. For example, the *factorial* function returns the factorial of the input argument.

```
>>> import math
>>> math.factorial(3)
6
>>> math.factorial(8)
40320
```

The *math* module provides several classes of functions, including power and logarithmic functions, trigonometric functions and hyperbolic functions. The two basic power and logarithmic functions are *math.pow*(*x*, *y*) and *math.log*(*x*, *y*), which take two floating point arguments *x*, and *y*, and return x^y and $\log_y x$, respectively.

```
>>> import math
>>> math.pow(2, 5)
32.0
>>> math.pow(25, 0.5)
5.0
>>> math.log(1000, 10)
2.9999999999999996
```

Note the rounding off error in the last example. The functions *math.pow* and *math.log* always return floating point numbers. The function *math.log* can also take one argument only, in which case it returns the natural logarithm of the input argument.

```
>>> import math
>>> math.log(1)
0.0
>>> math.log(math.e)
1.0
>>> math.log(10)
2.302585092994046
```

There is also a handy function to calculate the base-10 logarithm of an input floating point number: *math.log10(x)*. The function takes a single floating point argument. As two other special power functions, *math.exp(x)* can be used to calculate the value of e^x , and *math.sqrt(x)* can be used to calculate the square root of x .

The set of trigonometric functions that the *math* module provides include *math.sin(x)*, *math.cos(x)*, *math.tan(x)*, *math.asin(x)*, *math.acos(x)* and *math.atan(x)*, which calculate the sine, the cosine, the tangent, the arc sine, the arc cosine, the arc tangent of x , respectively.

```
>>> import math
>>> math.sin(3)
0.1411200080598672
>>> math.cos(math.pi)
-1.0
>>> math.tan(3*math.pi)
-3.6739403974420594e-16
>>> math.asin(1)
1.5707963267948966
>>> math.acos(1)
0.0
>>> math.atan(100)
1.5607966601082315
```

Note the rounding off errors in some of the examples above. All angles in the functions above are represented by radians. The *math* module provides two functions to convert between radians and degrees: the *math.degrees* function takes a single argument x , and converts x from radians to degrees; the *math.radians* function takes a single argument x , and converts x from degrees to radians.

The set of hyperbolic functions include *math.sinh(x)*, *math.cosh(x)*, *math.tanh(x)*, *math.asinh(x)*, *math.acosh(x)* and *math.atanh(x)*, which calculate the hyperbolic sine, the hyperbolic cosine, the hyperbolic tangent, the inverse hyperbolic sine, the inverse hyperbolic cosine, and the inverse hyperbolic tangent of x , respectively.

There are more functions that the *math* module provides, including *math.ceil(x)*, which returns the smallest integer that is greater than or equal to x , and *math.floor(x)*, which returns the largest integer that is less than or equal to x . It does not make sense to remember all the functions that Python provides for the purpose of programming, although remembering a few commonly-used functions would be useful for the efficiency of programming. A good practice is to keep the **Python documentation** at hand, which is also easily accessible online. For example, searching for the key words ‘Python math’ using a search engine can lead to the Python documentation for the *math* module.

2.3.1 Complex Numbers and the *cmath* Module

Complex literals. In some engineering disciplines, *complex numbers* are commonly useful. A complex number consists of a *real* part and an *imaginary* part. While the real part of a complex number is an arbitrary real number, represented by a floating

point number in Python, the imaginary part is based on j , the imaginary square root of -1 . Python represents the imaginary part of a complex literal by a floating point number, followed by the special character j .

```
>>> c=1j
>>> type(c)
<type 'complex'>
>>> c*c
(-1+0j)
```

In the example above, c is a complex number that has only the imaginary part, $1j$. The square of c is $1j \times 1j = -1$. In Python, if a complex number co-exists in an expression with floating point numbers and integers, the type of the whole expression becomes a complex number. Therefore, the value of the expression $c * c$ is the complex number $(-1 + 0j)$.

$1j$ is a special complex number, with the real part being 0. In general, a complex number is specified by the sum of its real part and imaginary part, as shown by IDLE in the example above.

```
>>> a=1+2j
>>> a
(1+2j)
>>> type(a)
<type 'complex'>
>>> b=3+4j
>>> b
(3+4j)
>>> type(b)
<type 'complex'>
```

Type conversion from integers and floating point number into complex numbers. Similar to the construction of integer and floating number objects using the functions *int* and *float* introduced earlier, a complex number can be constructed from integers and floating point numbers by using the function *complex*.

```
>>> complex(1,2)
(1+2j)
>>> a=complex(-1, 0.5)
>>> a
(-1+0.5j)
```

As shown by the example above, the function *complex* takes two numeric arguments specifying the real and imaginary components, respectively, and returns a complex object.

Complex operators. Similar to integers and floating point numbers, complex numbers also support arithmetic operations by using operators. The $+$, $-$, $*$, $/$ and $**$ operators for integers and floating point numbers also apply to complex numbers.

```
>>> a=1+2j
>>> b=-1+3j
>>> a+b
5j
>>> a-b
(2-1j)
```

```
>>> a*b
(-7+1j)
>>> a/b
(0.5-0.49999999999999994j)
>>> a**2
(-3+4j)
```

Functions for complex numbers. The *abs* function, when applied to complex numbers, returns the magnitude of the number.

```
>>> a=3+4j
>>> abs(a)
5.0
```

In the example above, the input argument to the function call *abs(a)* is a complex number, and the return value is a floating point number. However, *no* built-in function or operator takes floating point numbers but results in a complex number. In other words, the default domain in which Python handles mathematical expressions is real numbers. For example, trying to obtain the square root of -1 by the **** operator, or using the *math.sqrt* function, will result in an error.

```
>>> (-1)**0.5
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
ValueError: negative number cannot be raised to a
fractional power
>>> import math
>>> math.sqrt(-1)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
ValueError: math domain error
```

This choice of the default mathematical domain for Python is based on the fact that real numbers are the most widely used, while complex numbers are common only in specific fields. Some users of Python might not even know the existence of complex numbers. As a result, the most natural choice is to leave their processing only to specific modules. Python provides a module, *cmath*, for complex numbers.

The *cmath* module. The power and logarithmic functions that the *cmath* module provides include *cmath.exp*, *cmath.log*, *cmath.log10* and *cmath.sqrt*. They bare the same names as their counterparts in the *math* module, with the difference being that they can be applied to complex numbers, and can return complex numbers. For example, to get the square root of -1 in the complex domain, the *cmath.sqrt* function should be used.

```
>>> import cmath
>>> cmath.sqrt(-1)
1j
```

The *cmath* module also provides the trigonometric functions *cmath.sin*, *cmath.cos*, *cmath.tan*, *cmath.asin*, *cmath.acos* and *cmath.atan*, and the hyperbolic functions *cmath.sinh*, *cmath.cosh*, *cmath.tanh*, *cmath.asinh*, *cmath.acosh* and *cmath.atanh*, with exactly the same use as their *math* counterparts except of the domain.

While the $\text{abs}(x)$ function returns the magnitude of a complex number x , the function $\text{cmath.phase}(x)$ returns the phase of x . The $\text{cmath.polar}(x)$ function returns the representation of a complex number x in polar coordinates, while the function $\text{cmath.rect}(r, p)$ returns the complex number x given its polar coordinates (r, p) .

2.3.2 Random Numbers and the random Module

For one last example of modules in this chapter, the *random* module provides functions for generating random numbers. It is useful to a range of mathematical problems, including a branch of numerical simulation methods that will be introduced in this book.

Two important functions provided by the *random* module include *random.random* and *random.randint*. *random.random()* takes no input arguments, and returns a random floating point number in the range $[0.0, 1.0)$. *random.randint(a, b)* takes two integer input arguments a and b , and returns a random number between a and b , inclusive.

```
>>> import random
>>> random.random()
0.1265646141812915
>>> random.random()
0.5701294637390362
>>> random.random()
0.8842027581970576
>>> random.randint(10,20)
12
>>> random.randint(10,20)
17
>>> random.randint(10,20)
17
>>> random.randint(10,20)
19
```

In general, many commonly-used mathematical functions are provided by Python, and it would always be useful to look for a readily-available implementation via the Python documentation and other resources. However, there are also cases where a customized function is needed. The following chapters will introduce step by step how complex functionalities can be achieved by the powerful Python language.

Exercises

1. What are the values of the following expressions?

- (a) $1 + 3 * 2 - 5 + 4$
- (b) $1 + 3 * (2 - 5) + 4$
- (c) $5 ** 2 ** 2 * 3 + 1$
- (d) $5 ** (2 ** 2) * 3 + 1$
- (e) $1 + 3/2$

- (f) $1 + 3.0/2$
- (g) $-2 - 1$
- (h) $-(2 - 1)$
- (i) $3.0 + 3/2$
- (j) $3 + 3/2.0$
- (k) $-1 * 0.5$

2. Use IDLE to calculate the following mathematical values.

- (a) 10^5
- (b) $\sqrt{10}$
- (c) the roots of $x^2 - 7x + 10 = 0$
- (d) $\lg(2 + \sqrt{5})$
- (e) the area of a circle with a radius of 5.5
- (f) $\sin 2.5$
- (g) the complex roots of $x^2 - 2x + 10 = 0$
- (h) $4!$
- (i) $\sum_{k=32}^{128} k$
- (j) $\prod_{k=3}^{17} k$

3. What are the values of the following binary numbers if they are (a) non-negative; or (b) 2's Complements?

- (a) 01001
- (b) 100
- (c) 1100
- (d) 11111
- (e) 11111111

4. Use IDLE to solve the following mathematical problems.

- (a) A car runs at a constant speed of 20 km/h. When it passes another case, the latter starts to accelerate in order to catch up. Assuming that the first case keeps a constant speed, and the second car keeps a constant acceleration of 2 m/s^2 . After how many seconds will the second car catch up with the first one?
- (b) The annual interest rate of a savings account is 4.1 %. John has \$10,000 in his account, and aims at saving \$50,000 within 5 years by depositing a fixed amount of money to his account in the beginning of each year, including this year. How much money does John need to save each year in order to achieve his goal?
- (c) In a shooting exercise, a coach stands 5 m away from a trainee, and throws a target up vertically at 5 m/s. If the trainee must fire her gun exactly 0.5 s after the throwing of the ball, then at which angle should she aim? If she must fire the gun exactly 1 s after the throwing, then at which angle should she aim?

- (d) John deposited an initial sum of \$3000 in his account. After 3 years, the balance reaches \$3335.8 due to composite interest. What is the interest rate per annual?
 - (e) The three sides of a triangle are 3, 4 and 6m, respectively. What is its area?
5. What are the three most important properties of a Python object? Which of them can change after the object is constructed, if the object is a number?
 6. State the main differences between identifiers and literals. Given a token in a program, how does Python know whether it is an identifier or a literal?

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