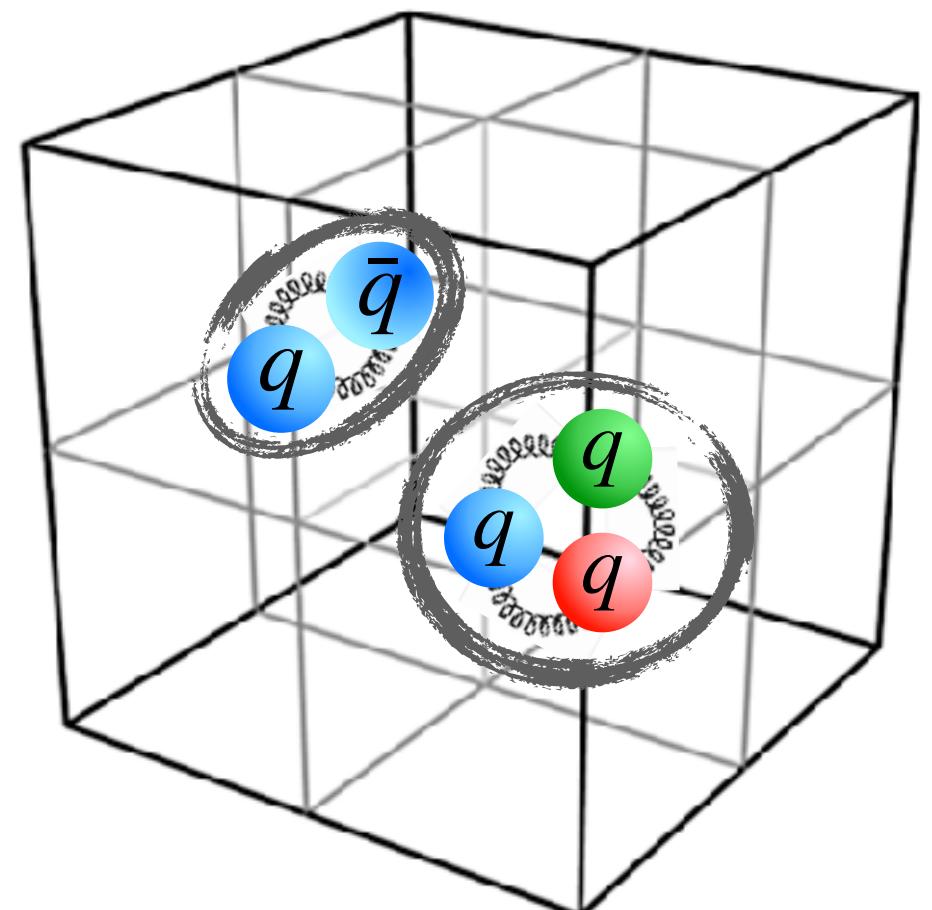
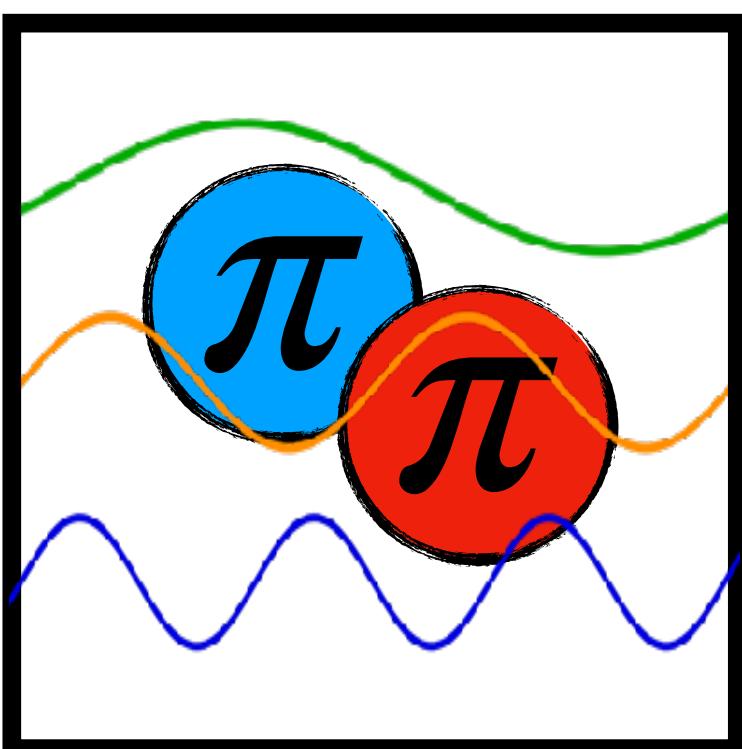


An Introduction to Hadron Spectroscopy from Lattice QCD

Fernando Romero-López

Uni Bern

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Important Links:

Slides + Notebooks: <https://github.com/ferolo2/LQCDclass>

Please, rate this course:



Goals:

- How does the QCD spectrum look like?
- How can we investigate the QCD spectrum with lattice QCD?
- What are the finite-volume effects in QCD stable particles?
- How to obtain resonance properties from lattice QCD?
- What are the finite-volume effects in scattering states?

Outline

1. The Hadron Spectrum
2. Lattice QCD spectroscopy
3. Finite-volume effects: stable particles
4. Scattering processes and resonances
5. Finite-volume effects: multi-hadron states
6. Selection of recent results

The Hadron Spectrum

Quantum Chromodynamics

Quantum chromodynamics is conceptually simple. Its realization in nature, however, is usually very complex.

Frank Wilczek

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$$\mathcal{L}_{QCD} = \sum_i^{N_f} \bar{q}_i \left(D_\mu \gamma^\mu + \textcolor{red}{m}_i \right) q_i + \frac{1}{4g_s^2} G_{\mu\nu}^a G_a^{\mu\nu}$$

Quantum Chromodynamics

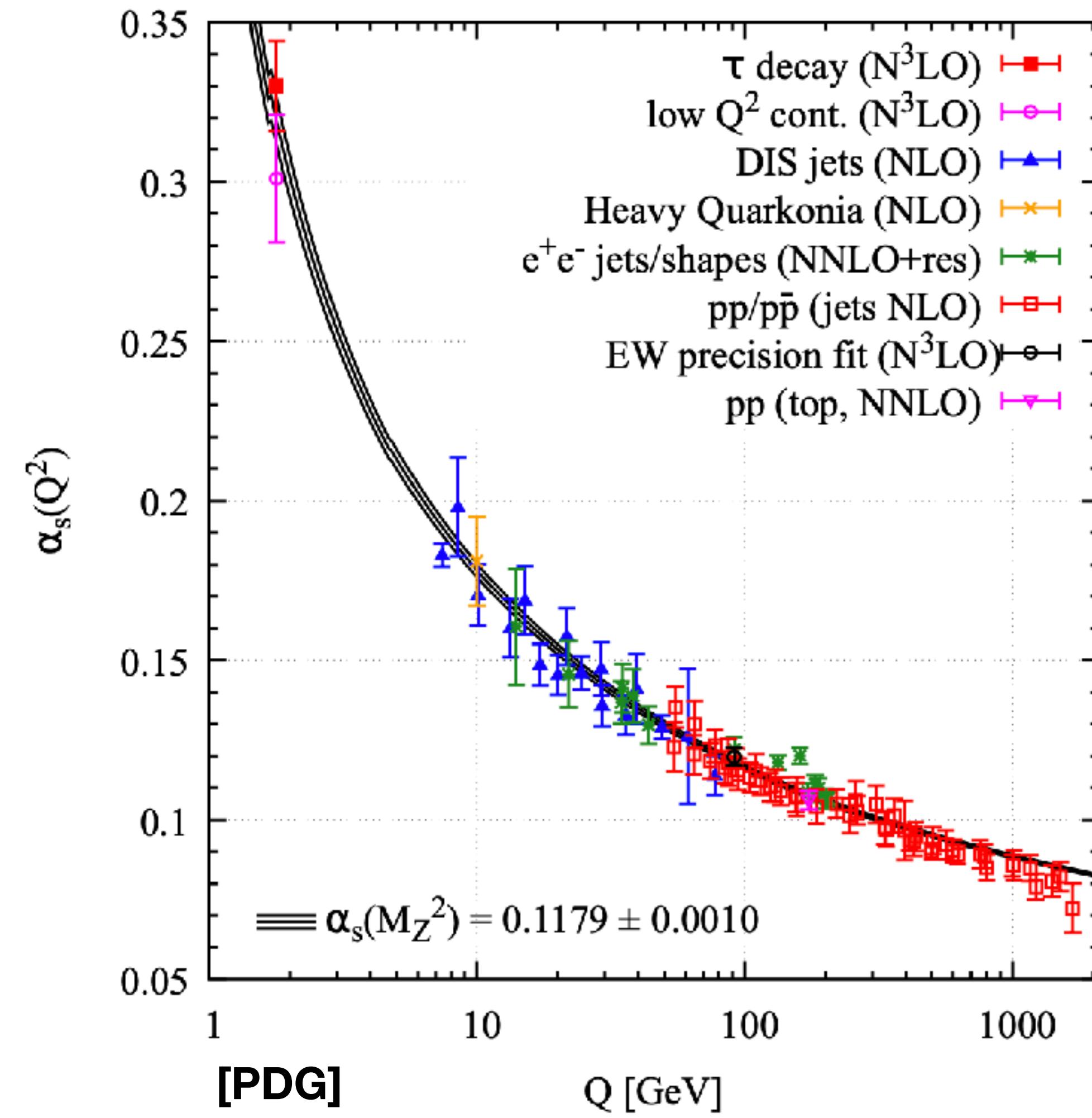
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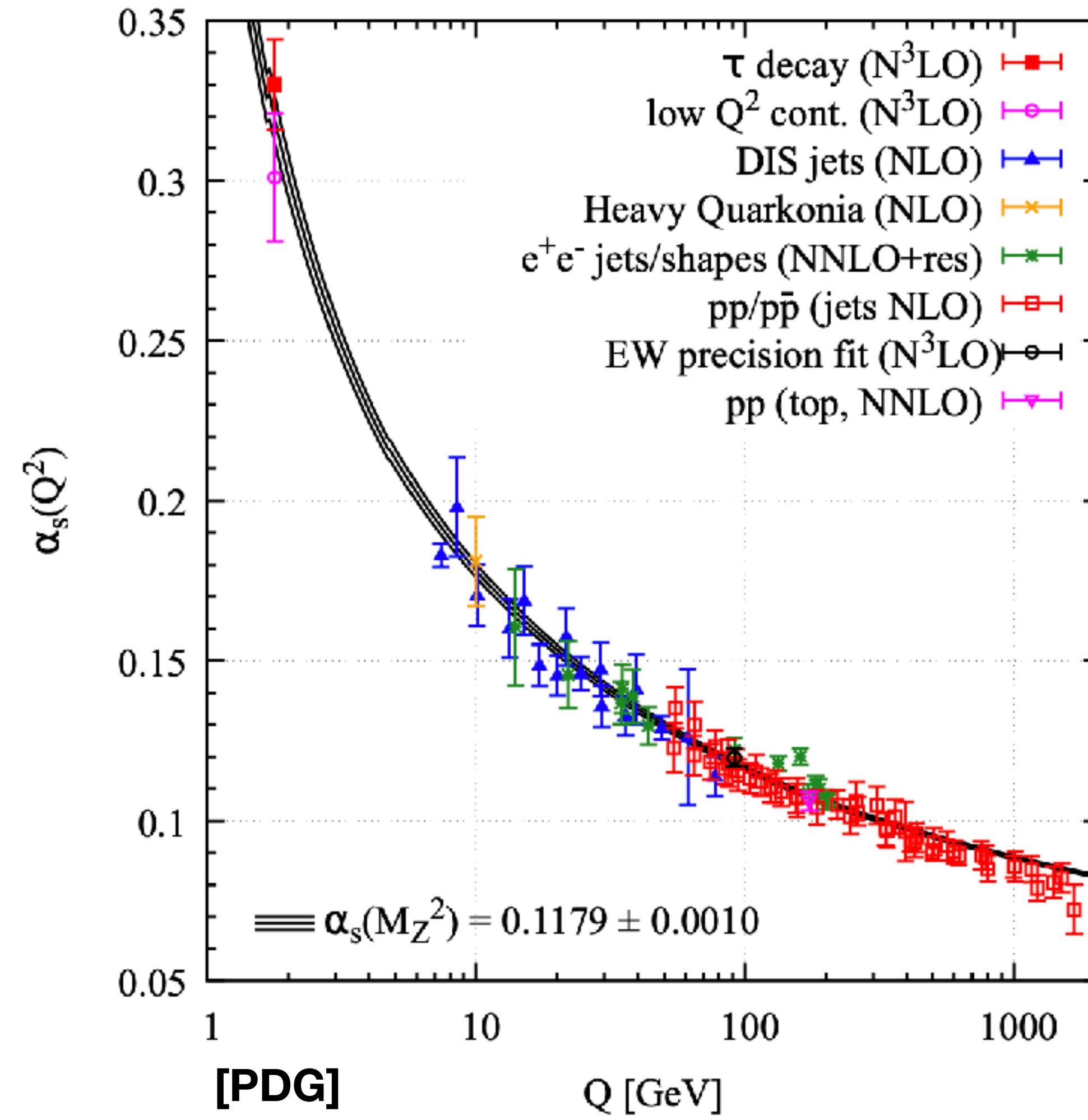
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The QCD coupling

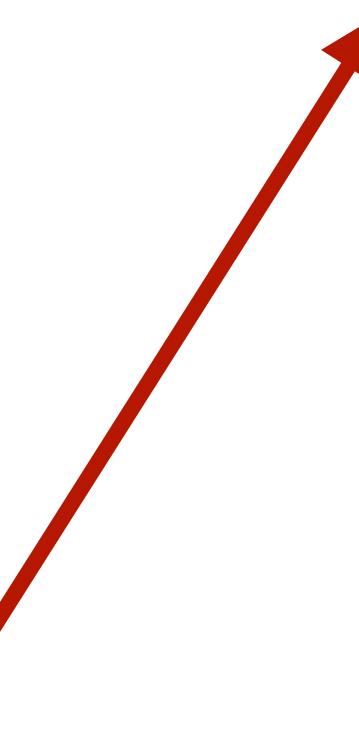


The QCD coupling



Perturbation theory:

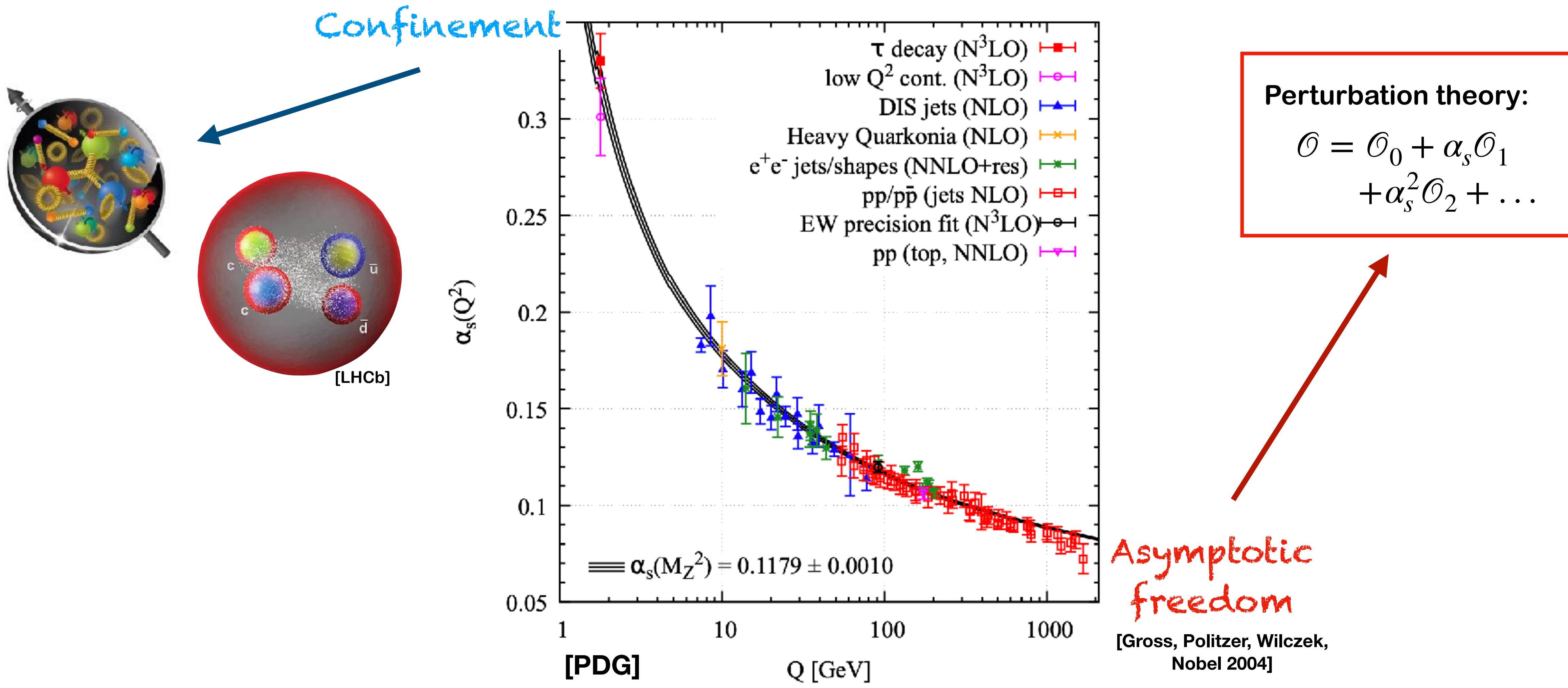
$$\mathcal{O} = \mathcal{O}_0 + \alpha_s \mathcal{O}_1 + \alpha_s^2 \mathcal{O}_2 + \dots$$



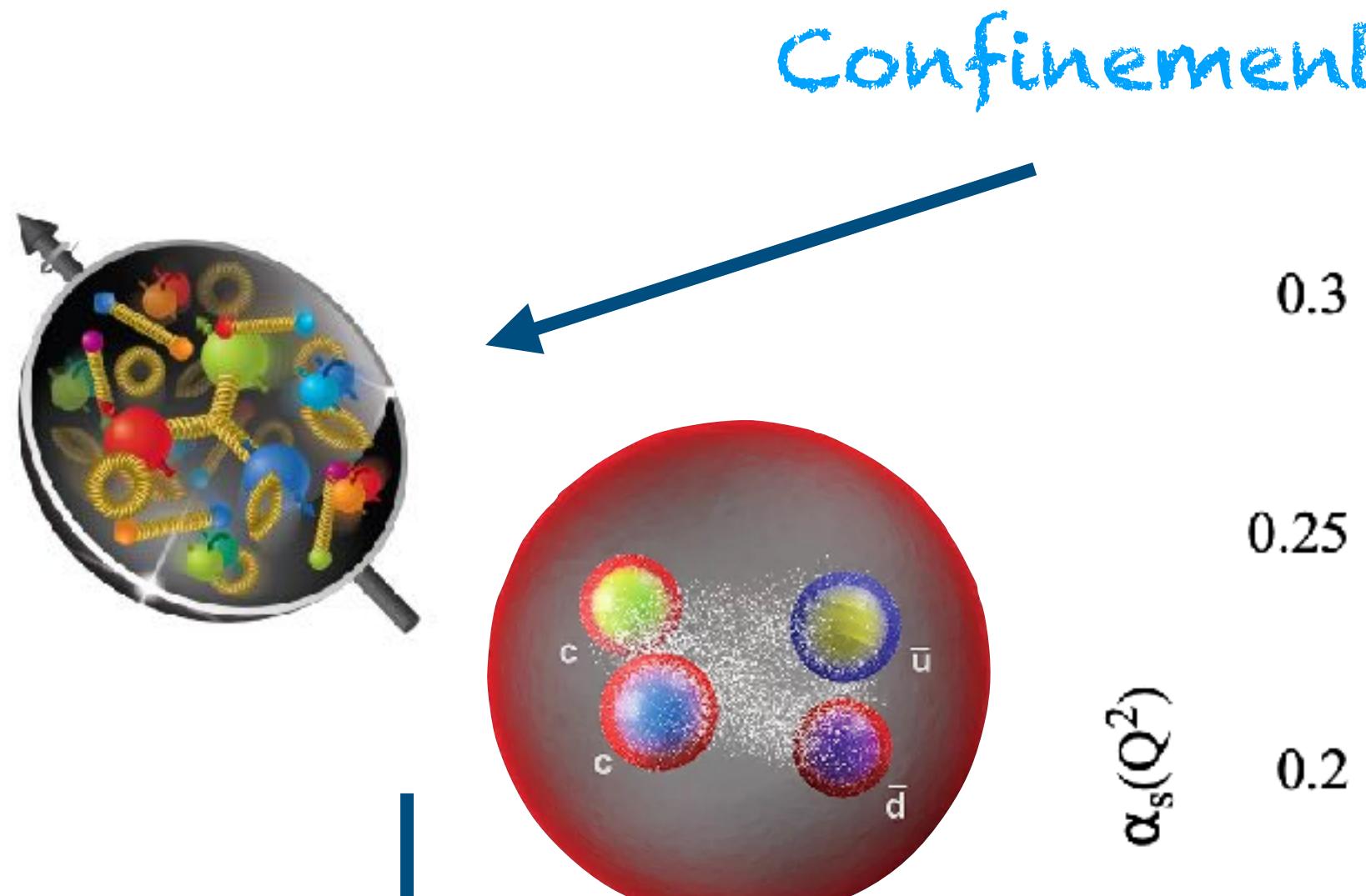
Asymptotic
freedom

[Gross, Politzer, Wilczek,
Nobel 2004]

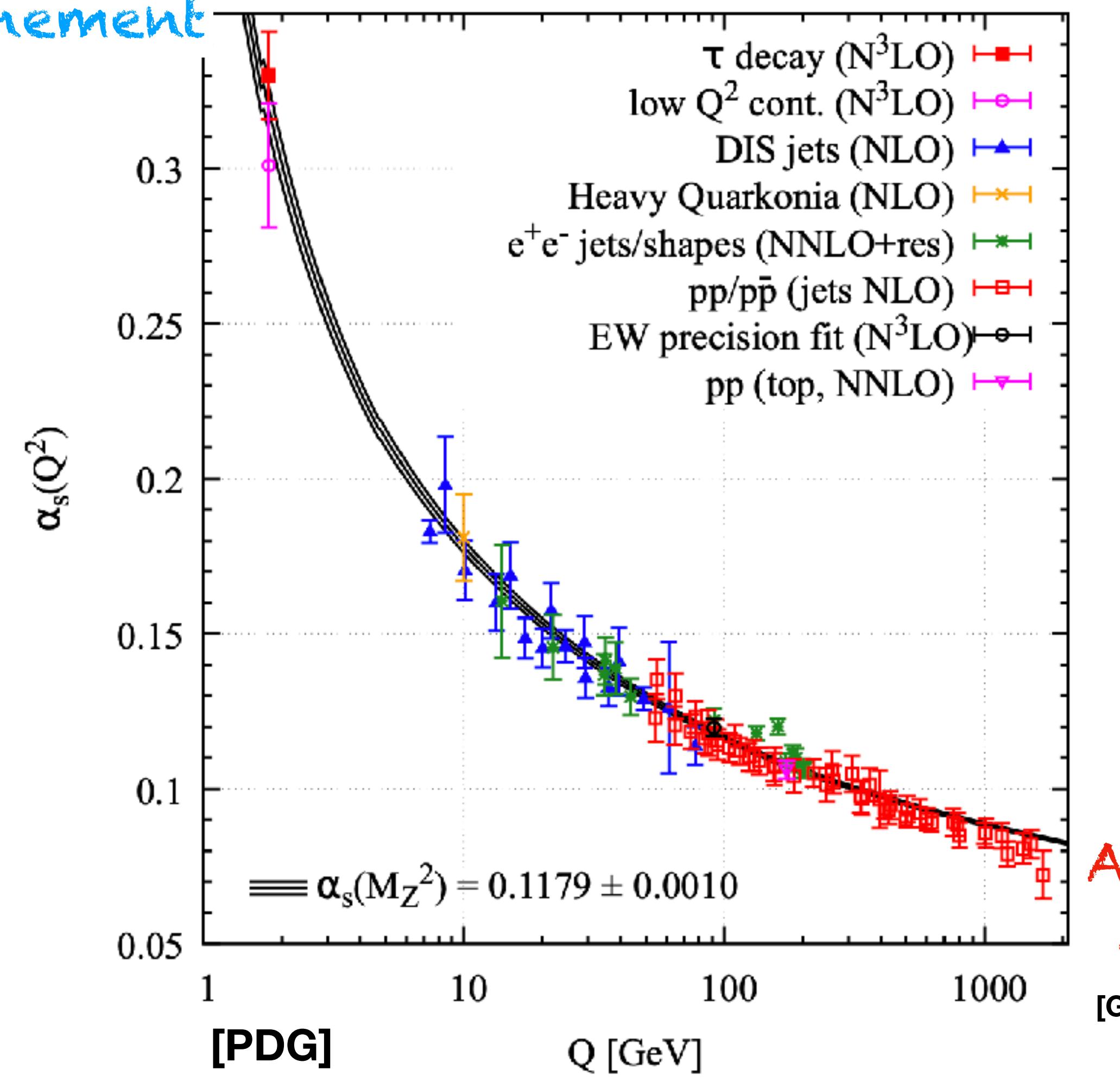
The QCD coupling



The QCD coupling



Non-perturbative
at low energies
Lattice QCD



Perturbation theory:

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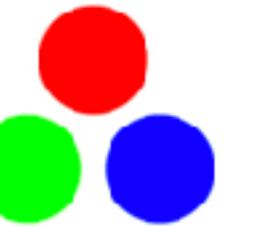
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Hadrons ≠ Confinement

Our understanding of the SM is limited by the complexity of the strong force

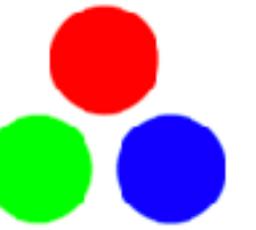
- Quarks and gluons carry the strong charge: the so-called “color”



Hadrons ≠ Confinement

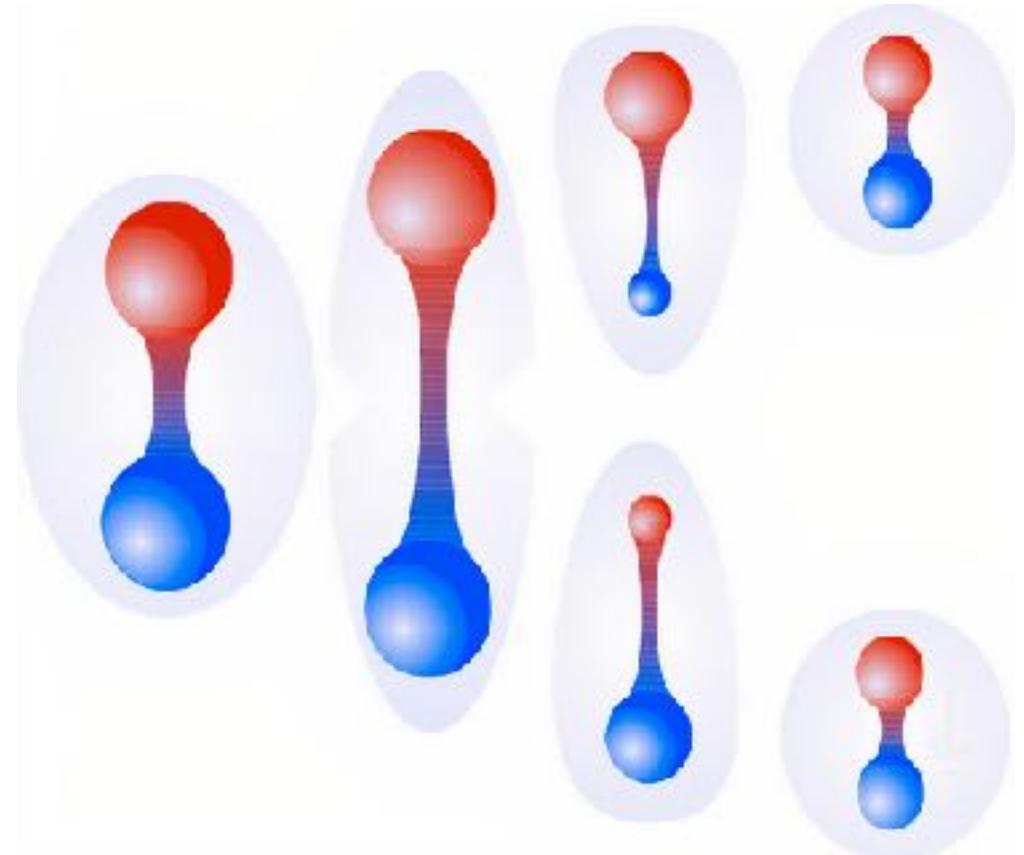
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Confinement

Quarks and gluons can only be found within colorless composite states. These are called “hadrons”.

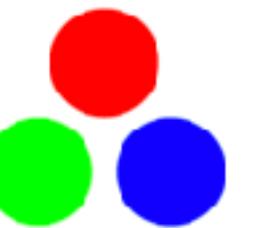


source: IFIC

Hadrons ≠ Confinement

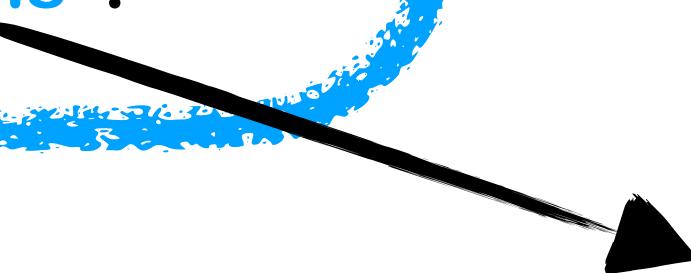
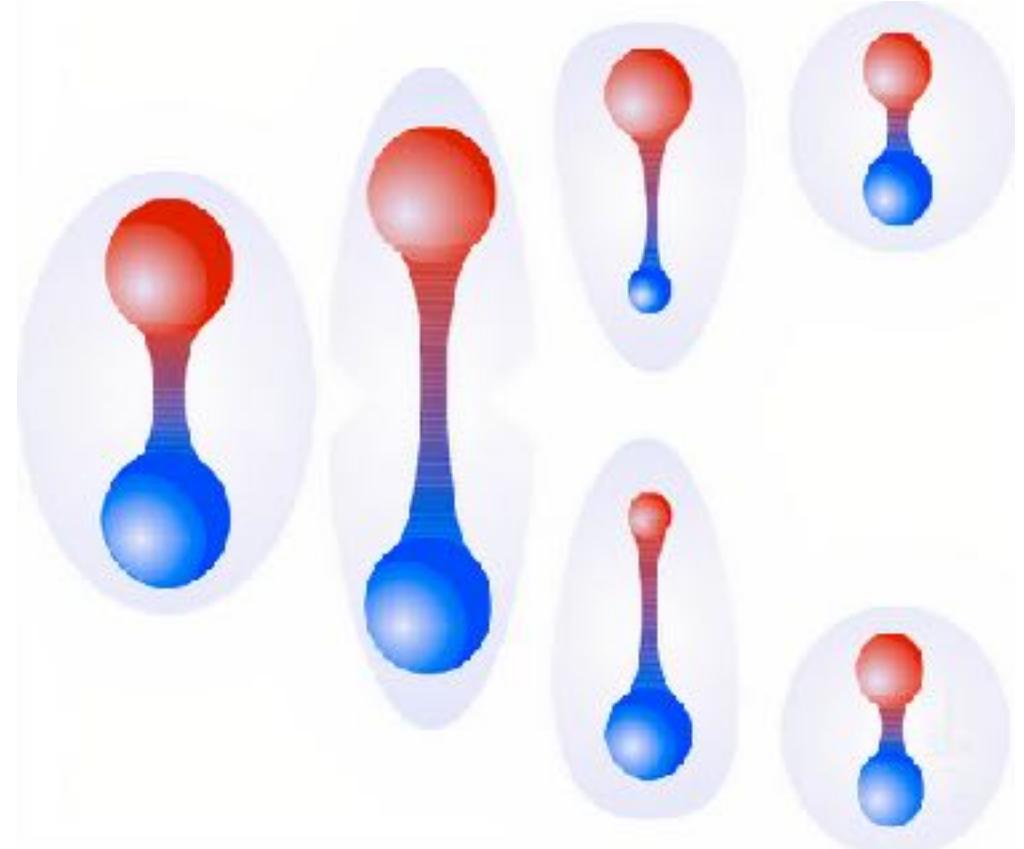
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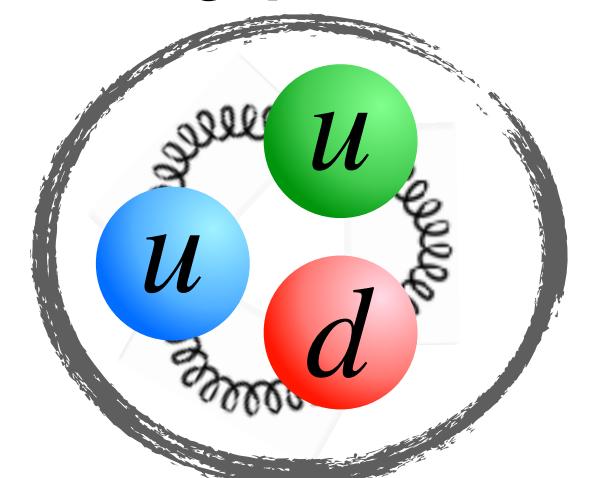


Confinement

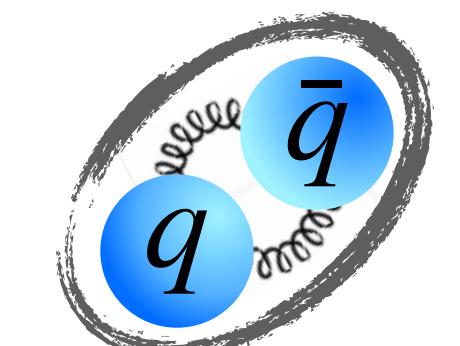
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baryons
e.g. proton



mesons
e.g. π, K

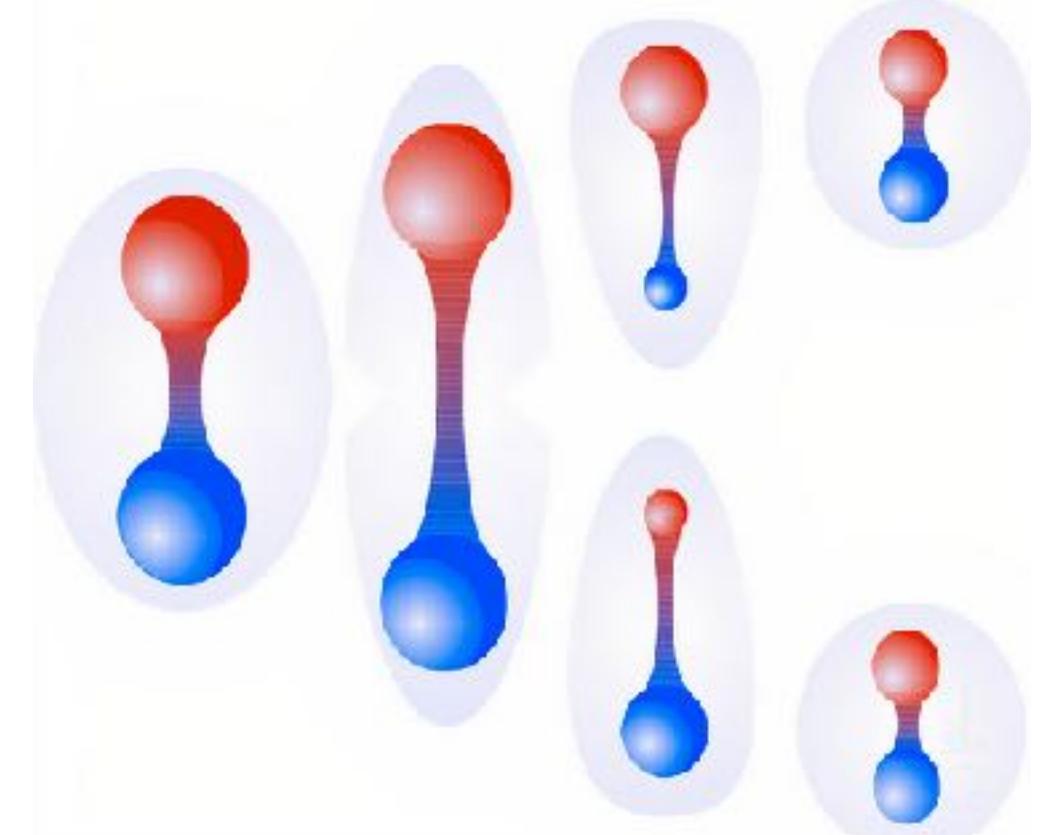
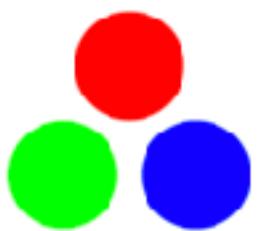


standard
hadrons

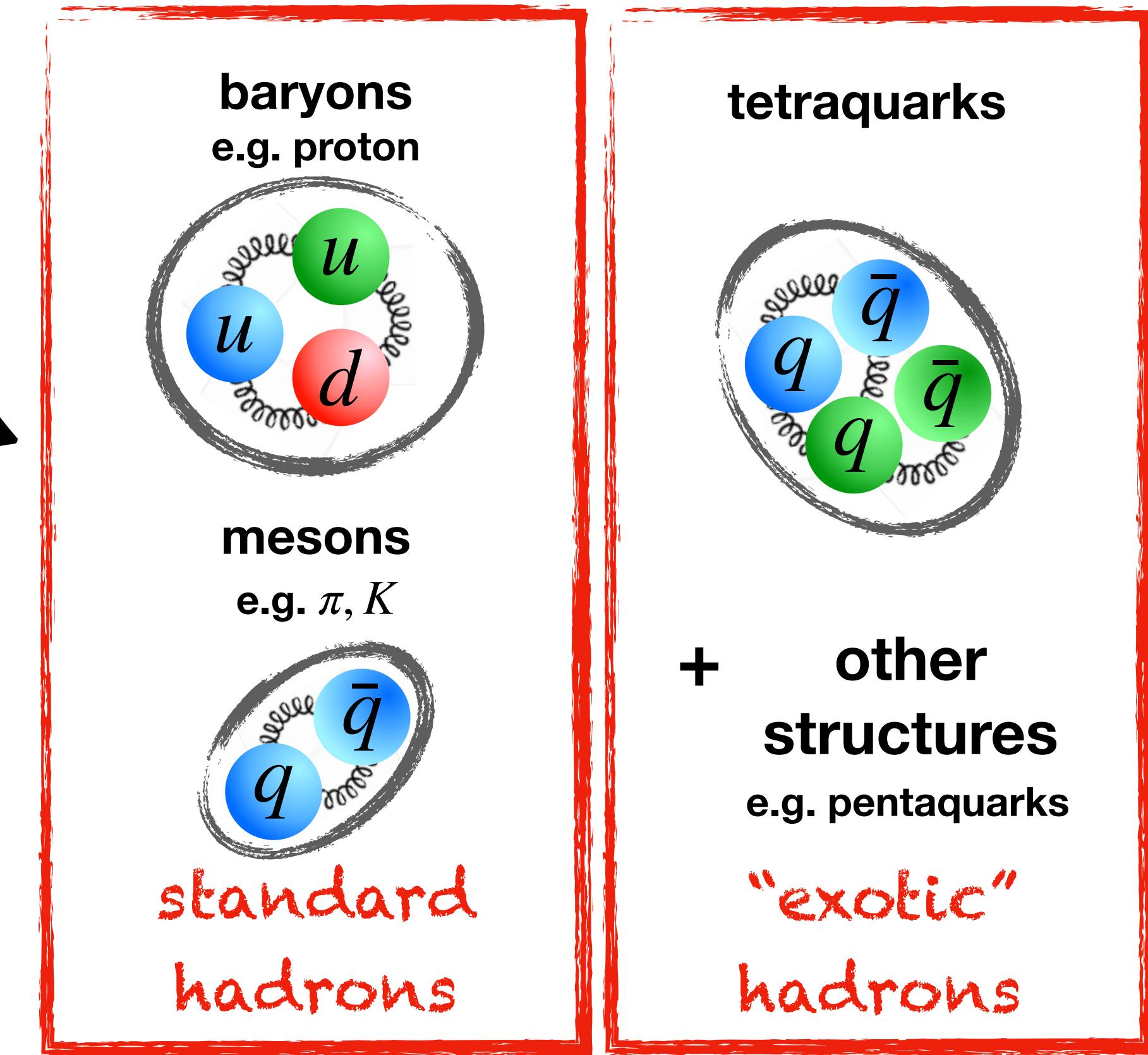
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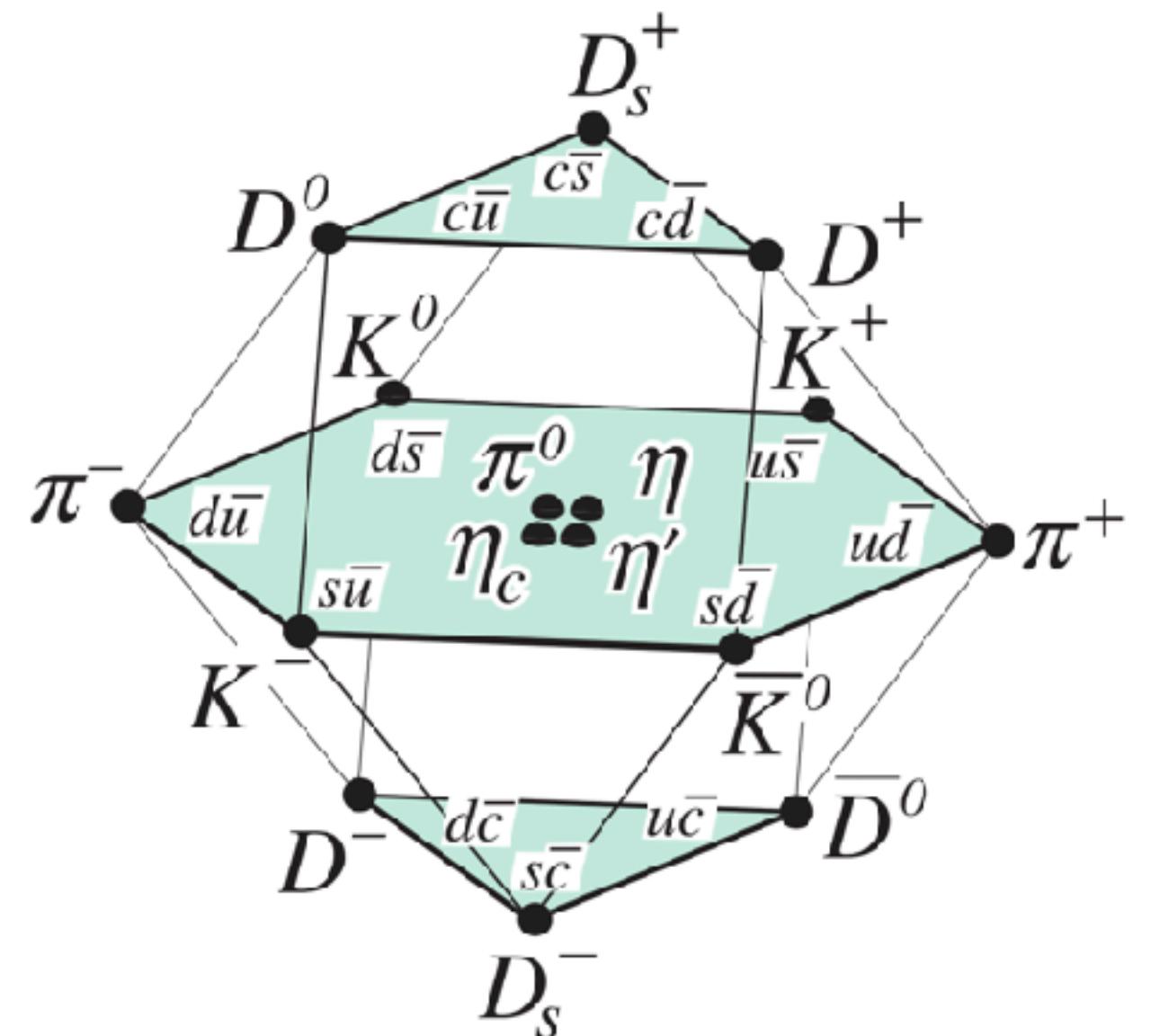


The QCD Fock space

- The only stable hadron in Nature is the **proton**
- In Lattice QCD, we (typically) treat QCD in isolation. Several hadrons become stable: π, K, D, p, n, Σ

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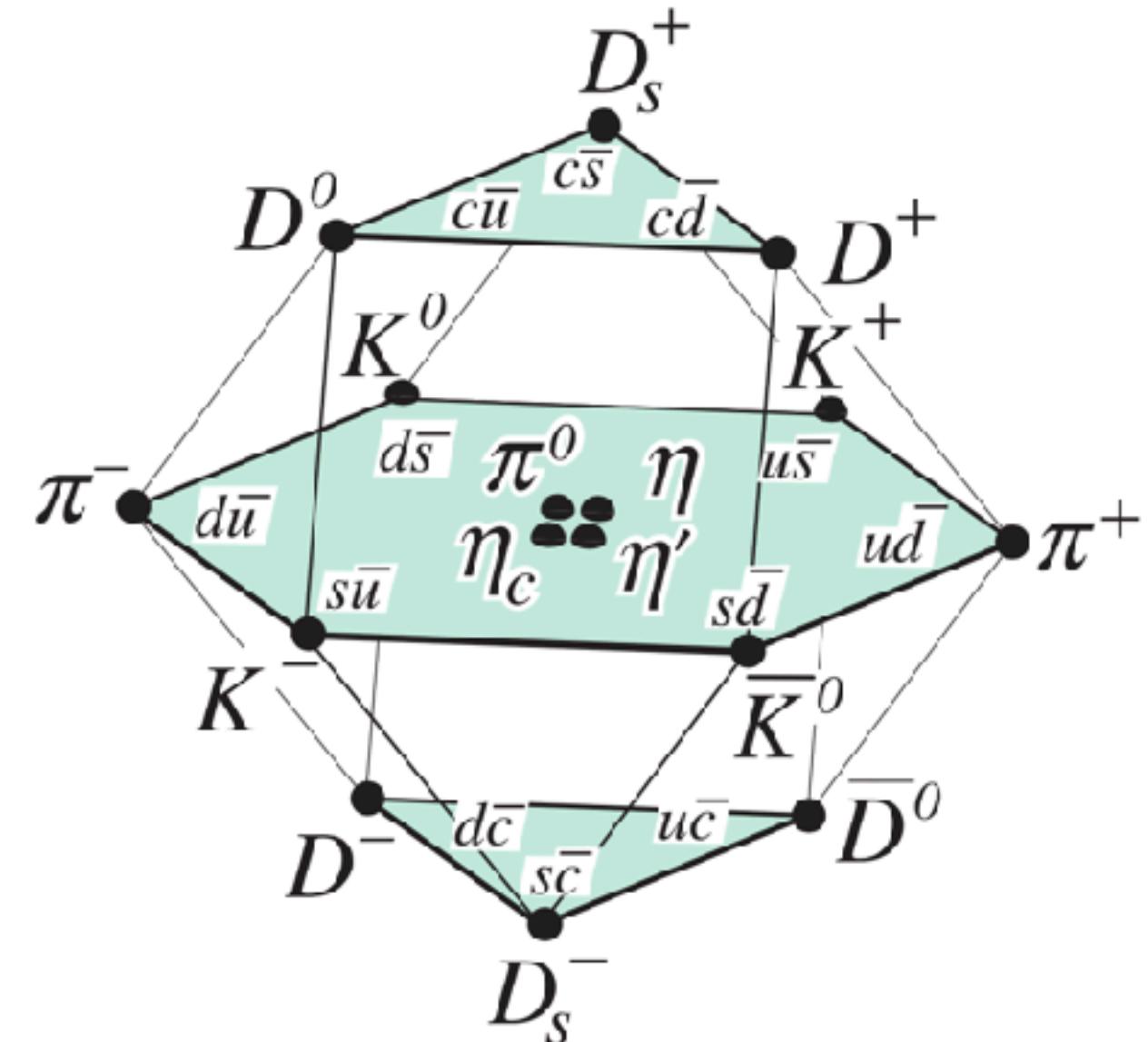


$|\pi\rangle, |K\rangle \in \text{QCD Fock}$

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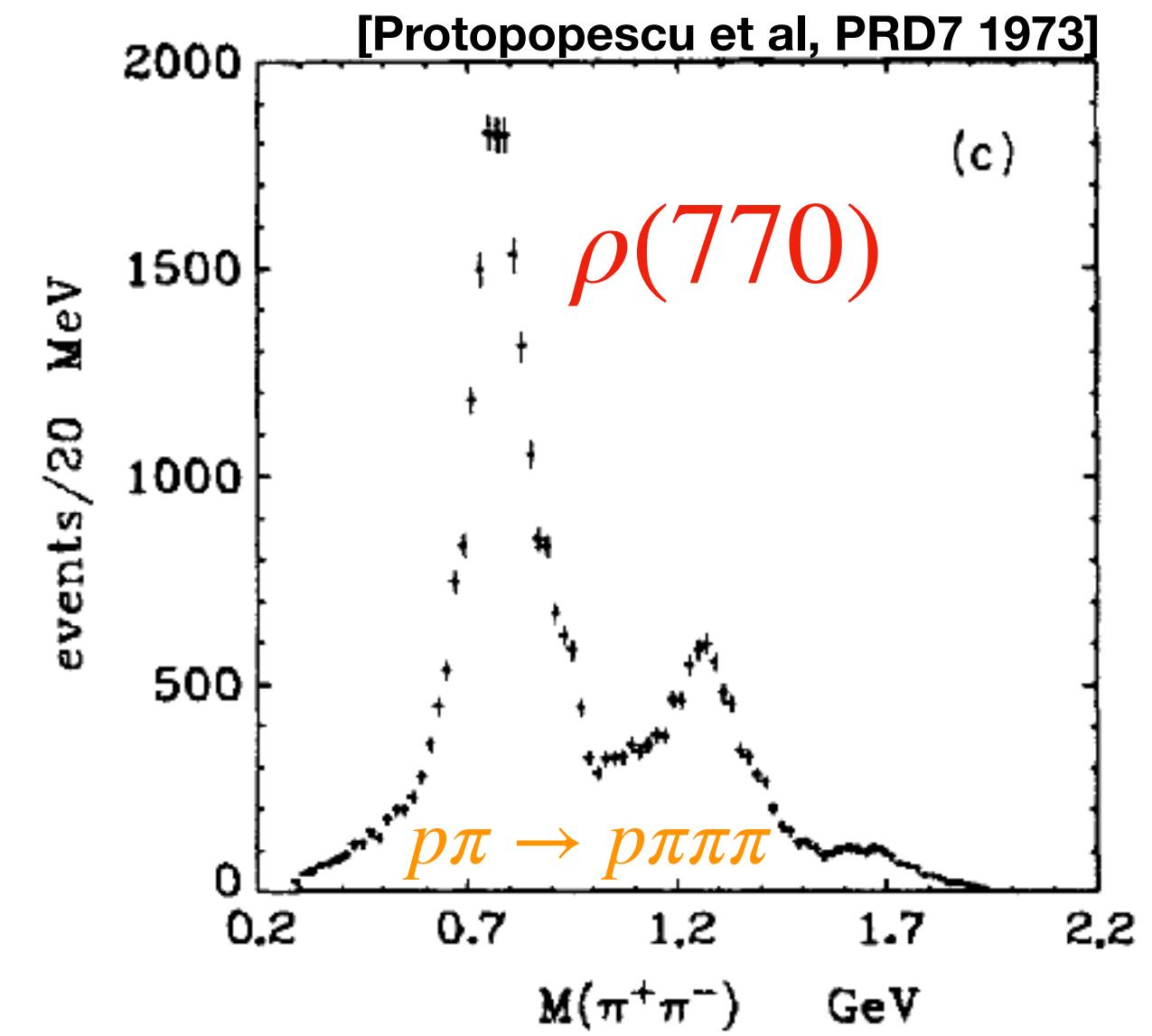
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$|\pi\rangle, |K\rangle \in \text{QCD Fock}$

► Resonances show up in scattering processes



$|\rho\rangle \notin \text{QCD Fock}$

The quark model picture

- Crude classification of the hadron spectrum but still useful for intuition
[Gell-Mann '64 & Zweig '64]
- Construct color singlets combining quarks and antiquarks

Meson:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \boxed{\mathbf{1}} \oplus \mathbf{8}$$

$$q \rightarrow \mathbf{3} \text{ irrep of } \mathrm{SU}(3)_c$$

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Baryon:

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color singlet

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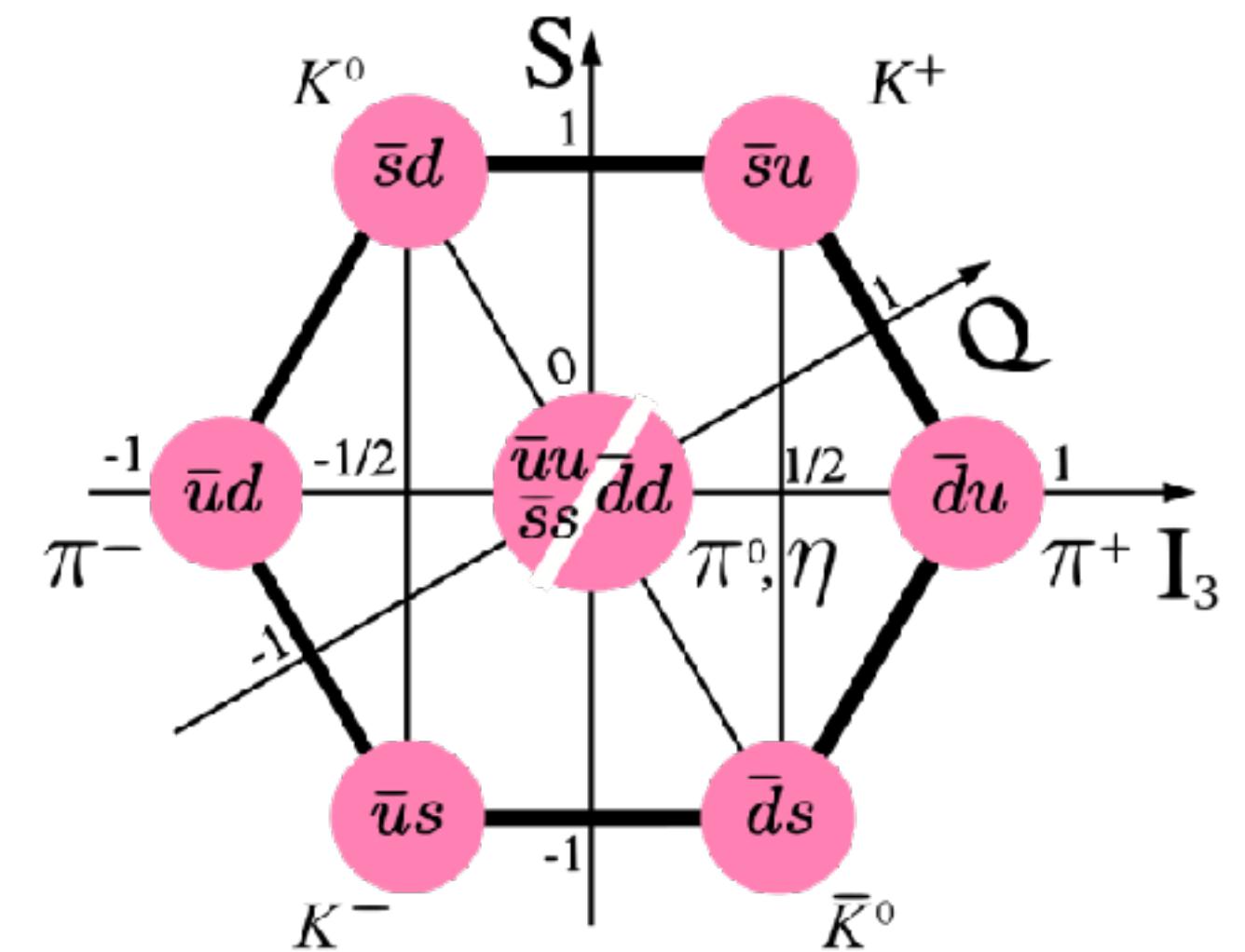
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singlet + octet

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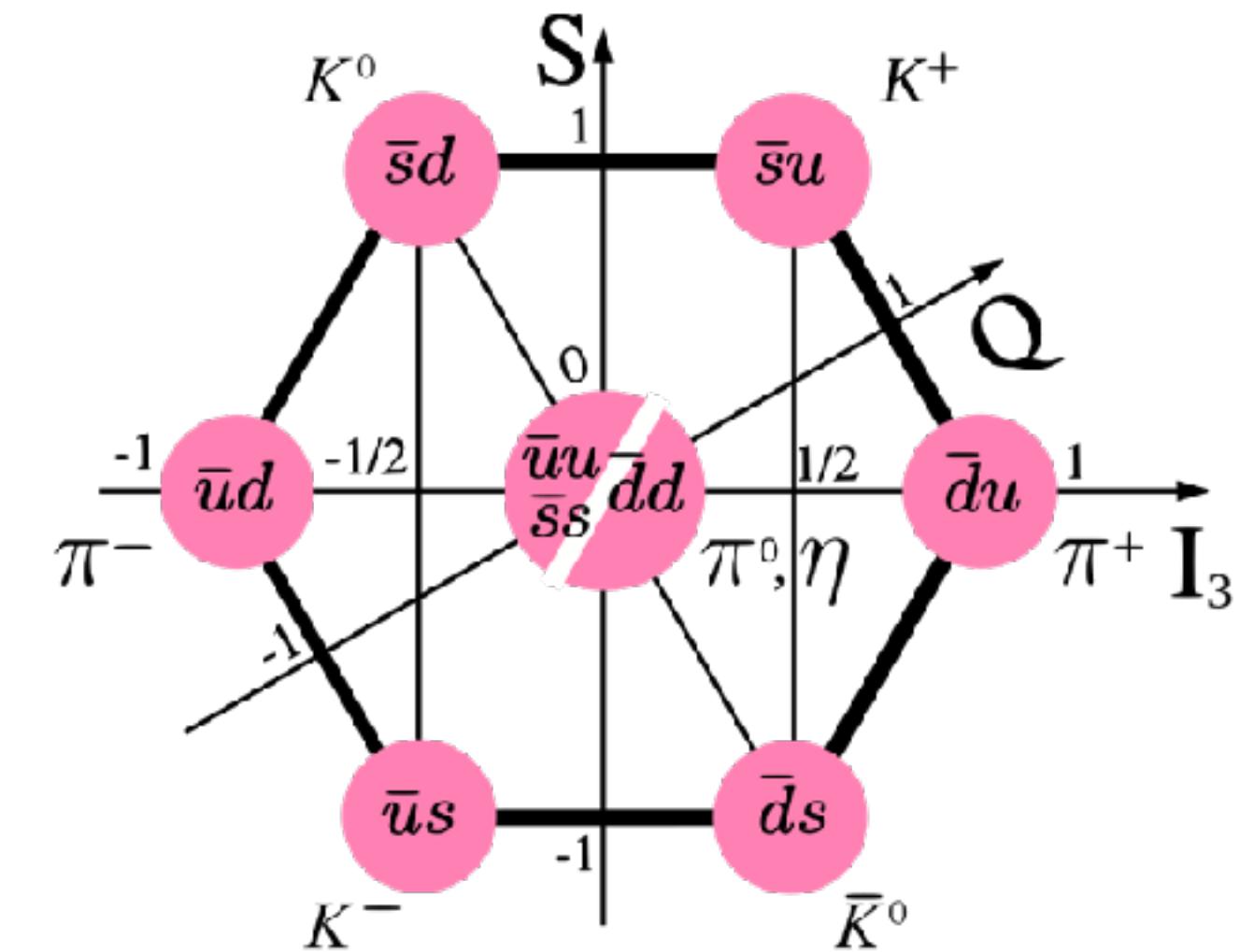
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singlet + octet

- Many hadrons do not fit in this picture: “exotics”
- ▶ They are not mesons/baryons (e.g. tetraquarks) or cannot be explained through multi-quark objects

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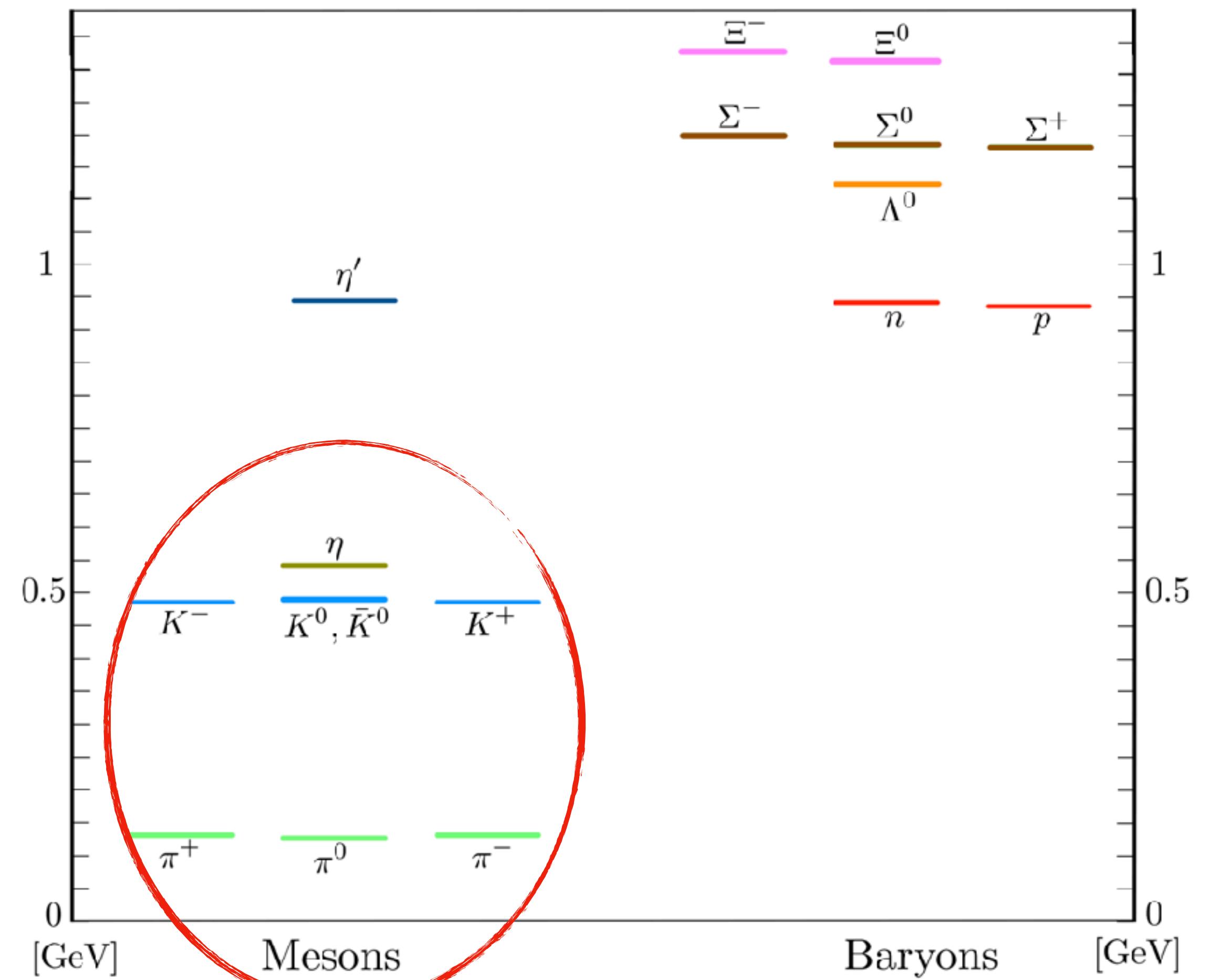


Low-lying hadron spectrum

Light meson octet

Pseudo-Nambu-Goldstone
bosons from spontaneous chiral
symmetry breaking

$$M_m^2 \propto m_q$$



Low-lying hadron spectrum

heavy singlet meson

Mass generated through the chiral anomaly

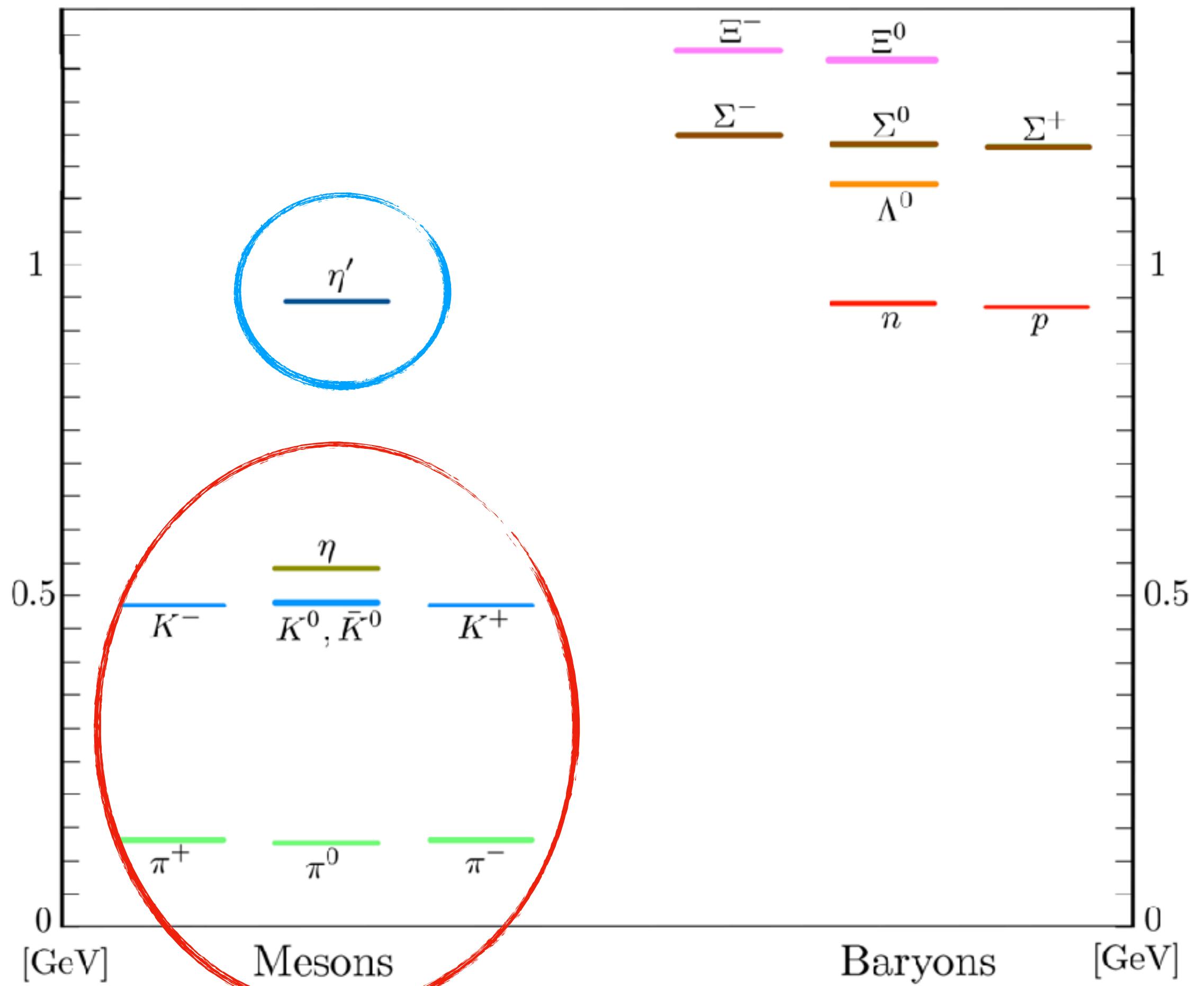
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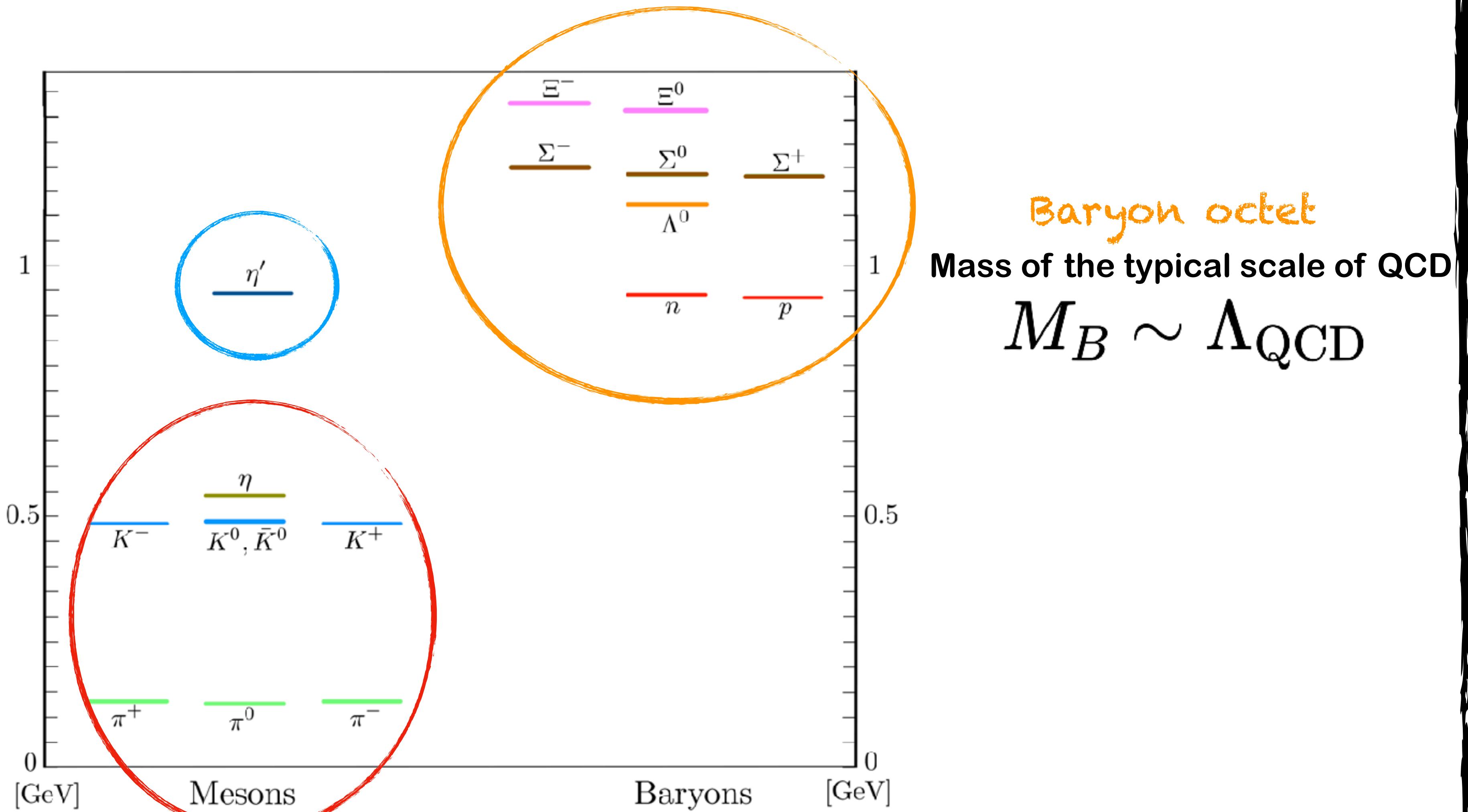
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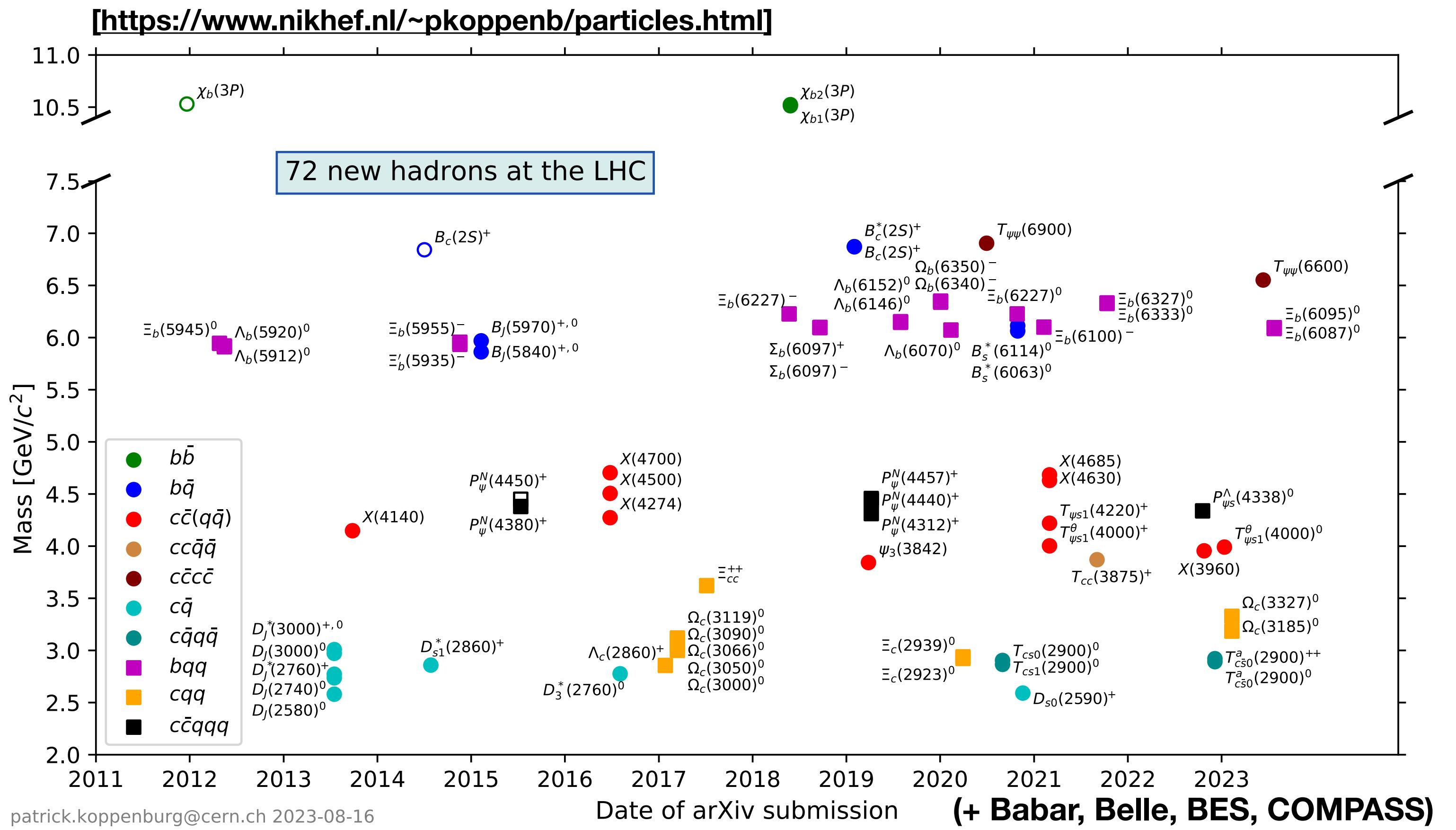
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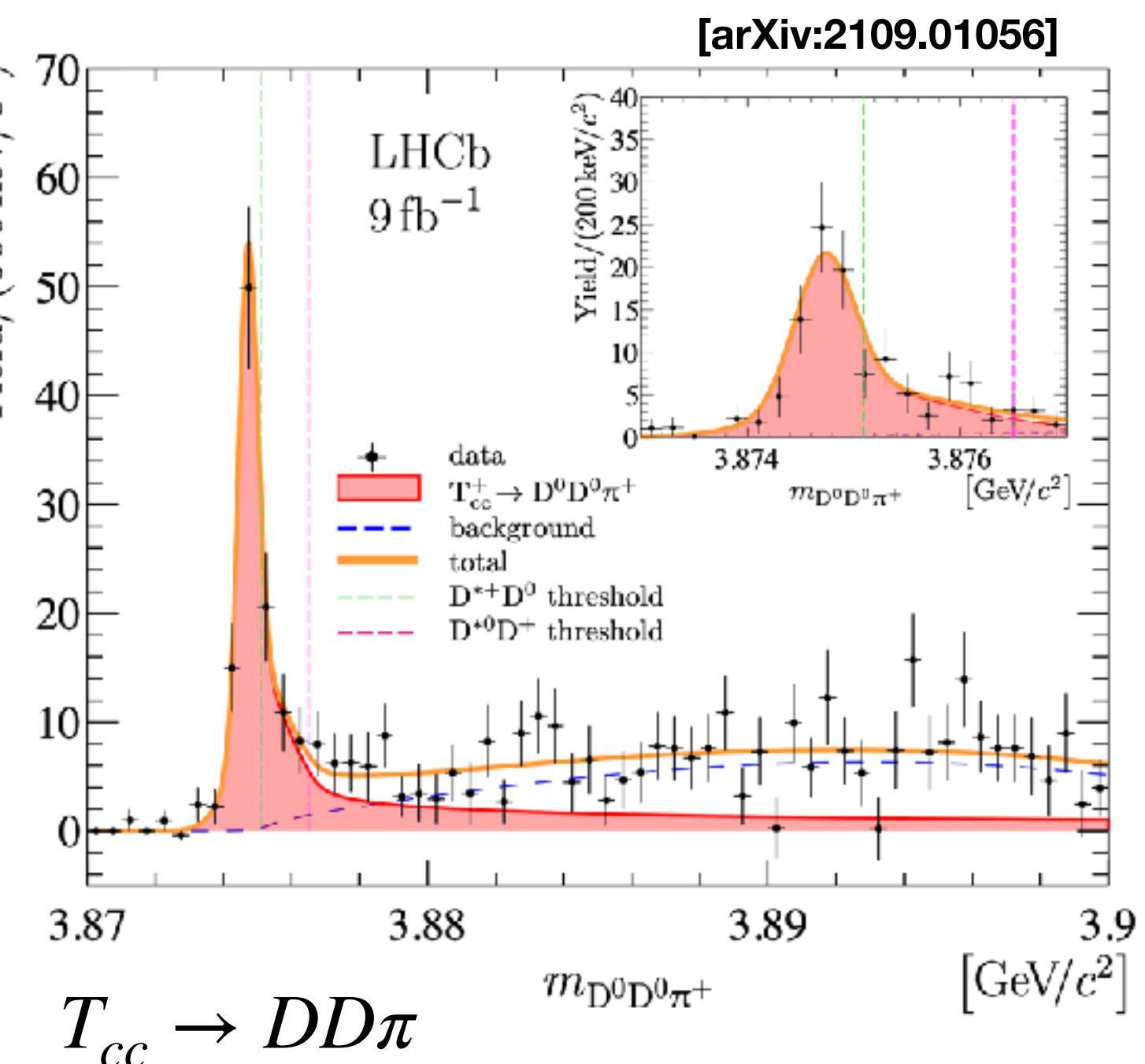
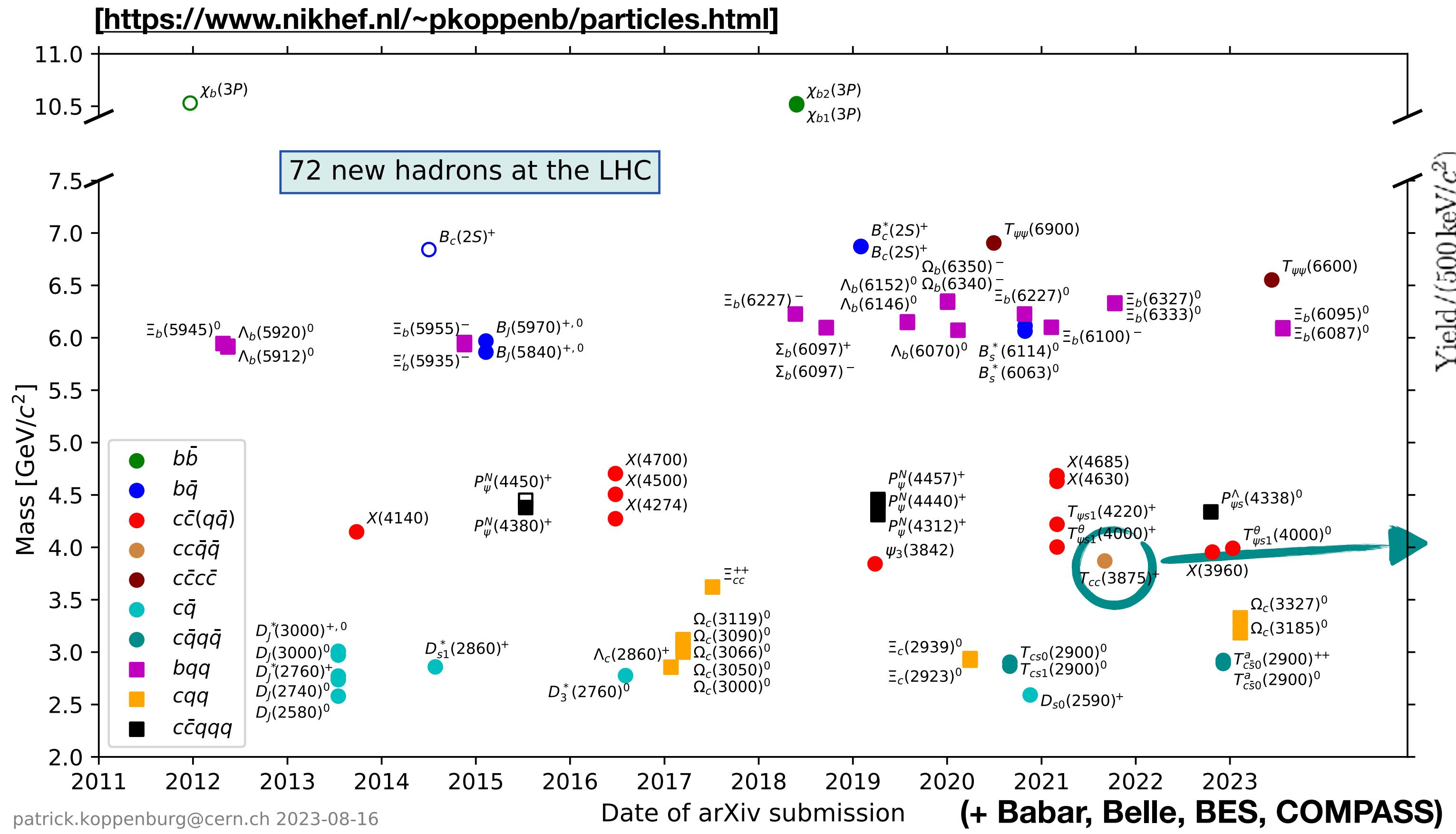
An experimental conundrum

A growing hadron spectrum still requires first principles understanding



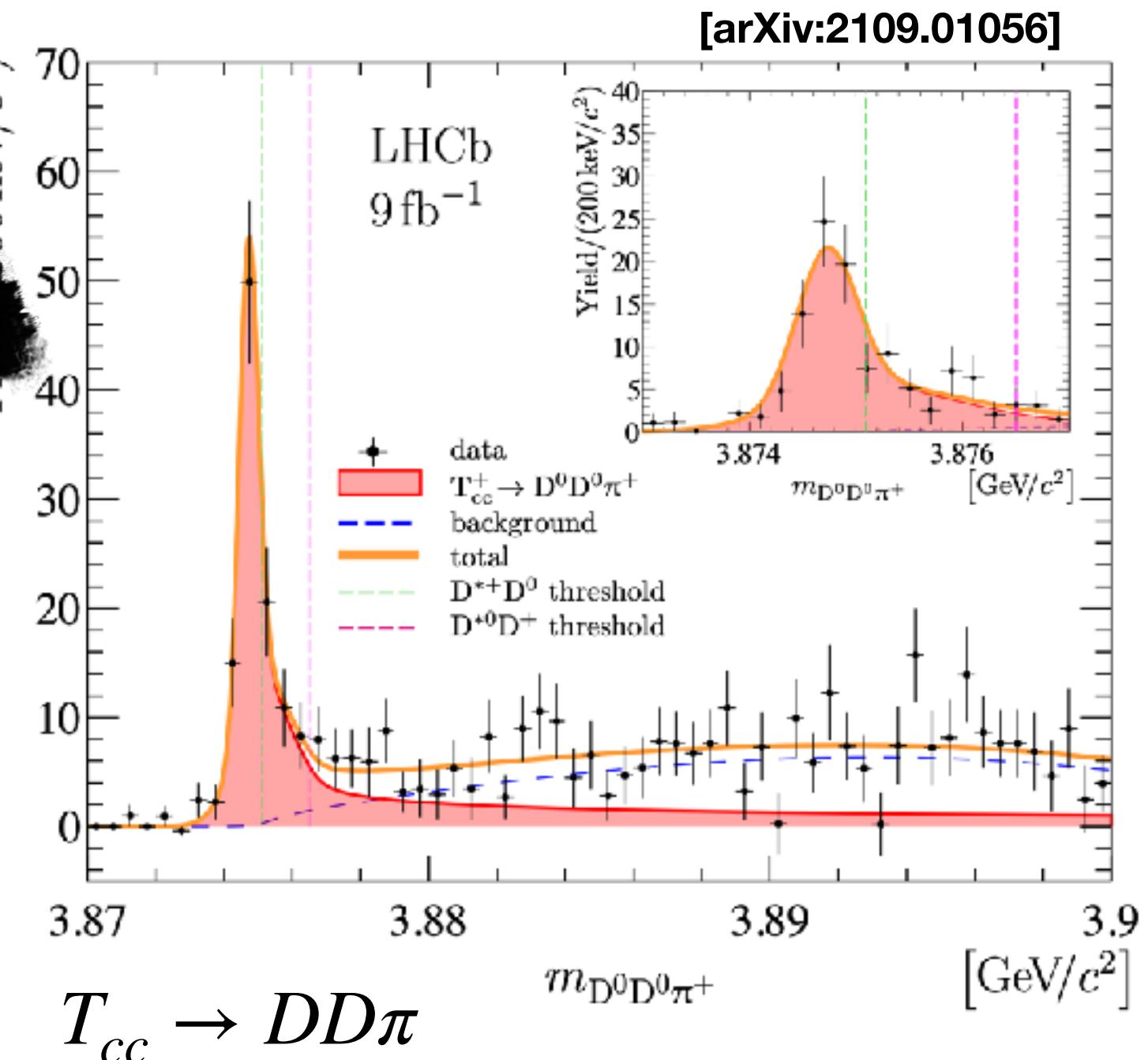
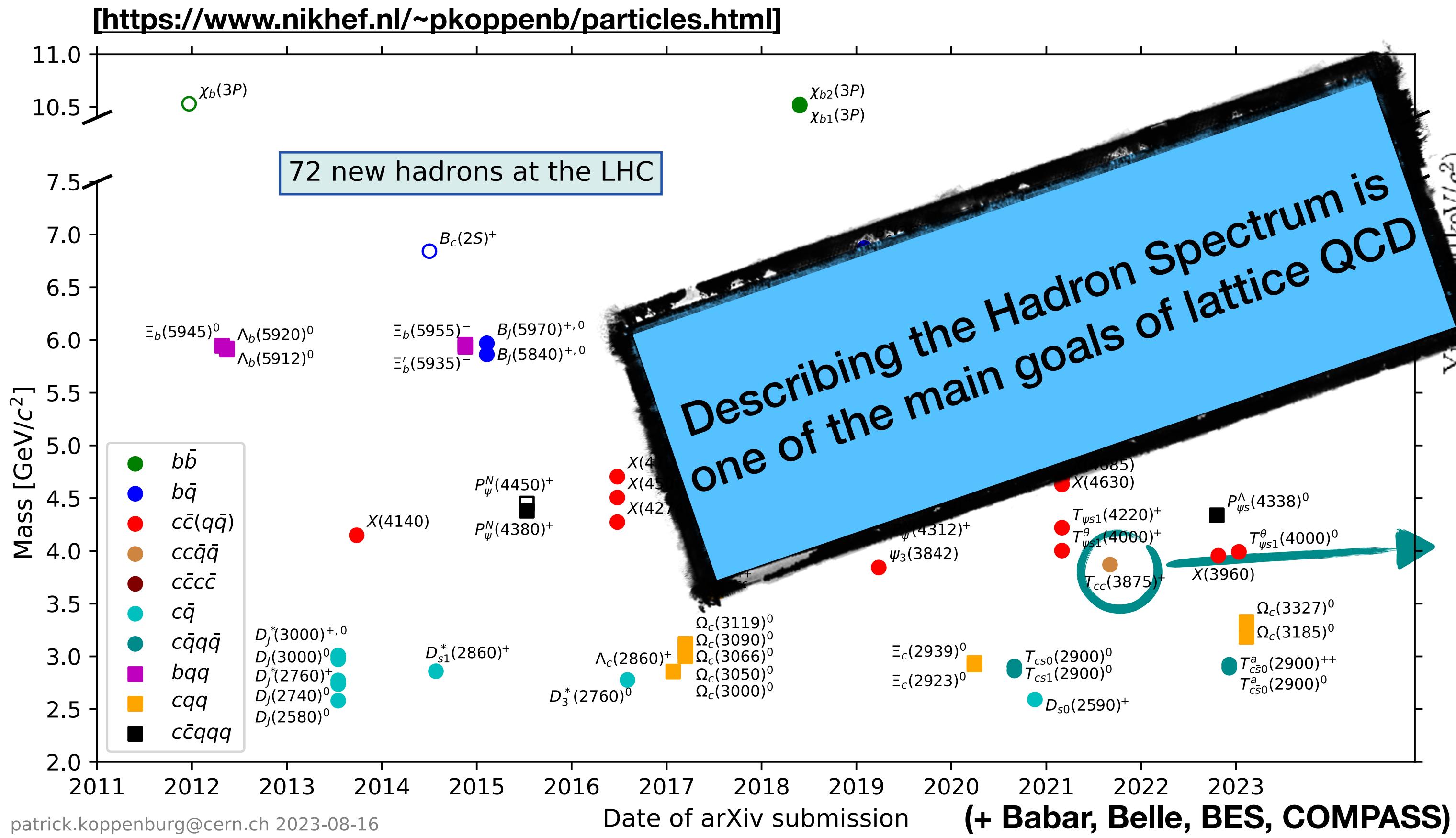
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Lattice QCD spectroscopy

Energies from correlations

- We measure **energy levels** and **matrix elements**: "Spectral decomposition"

$$\begin{aligned} C(t) &= \langle \mathcal{O}^\dagger(t) \mathcal{O}(0) \rangle = \sum_n \langle 0 | \mathcal{O}^\dagger(t) | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle \\ &= \sum_n \langle 0 | e^{Ht} \mathcal{O}^\dagger(0) e^{-Ht} | n \rangle \langle n | \mathcal{O}(0) | 0 \rangle \\ &= \sum_n \left| \langle 0 | \mathcal{O}(0) | n \rangle \right|^2 e^{-E_n t} \end{aligned}$$

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Ground state

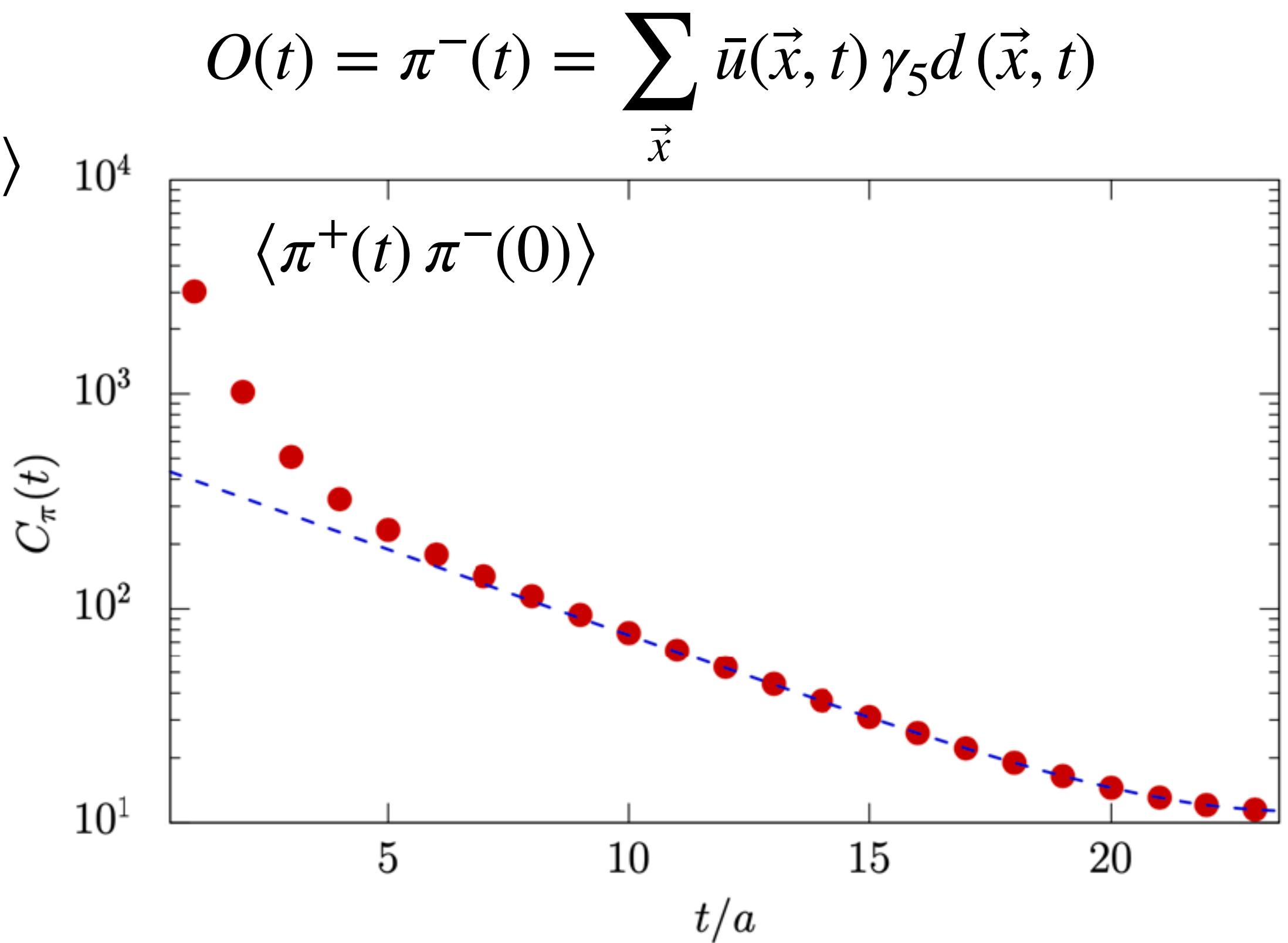
$$\lim_{t \rightarrow \infty} C(t) = A_0 e^{-E_0 t}$$

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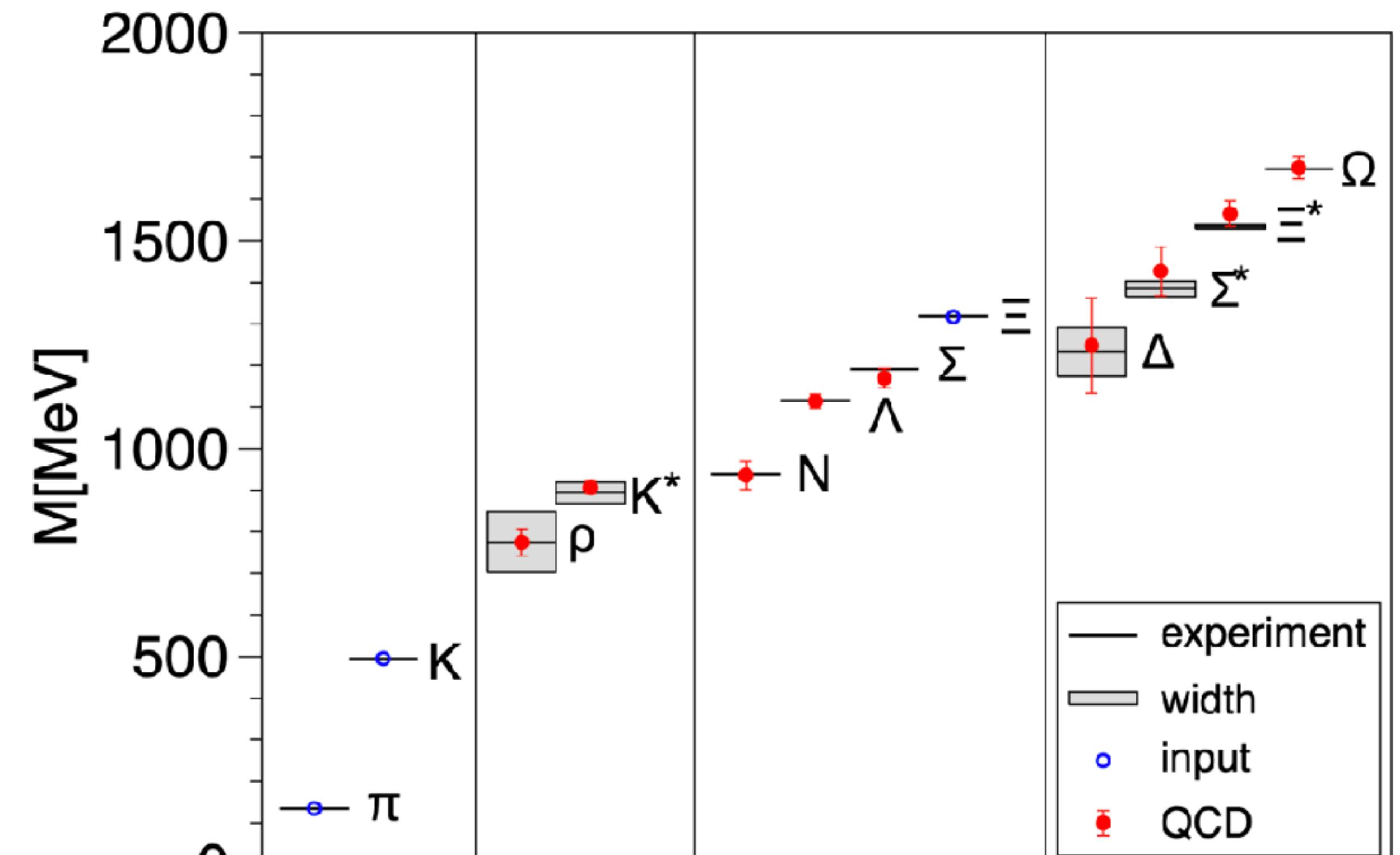
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Ground state

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QCD spectrum from LQCD

- Consider isospin-symmetric QCD ($m_u = m_d$)
- Neglect QED effects in hadrons
- Need a 3 inputs to fix quark masses
 - ▶ Fix light and strange quark mass
 - ▶ Fix lattice spacing
- Reproduce the lowest-lying hadrons!

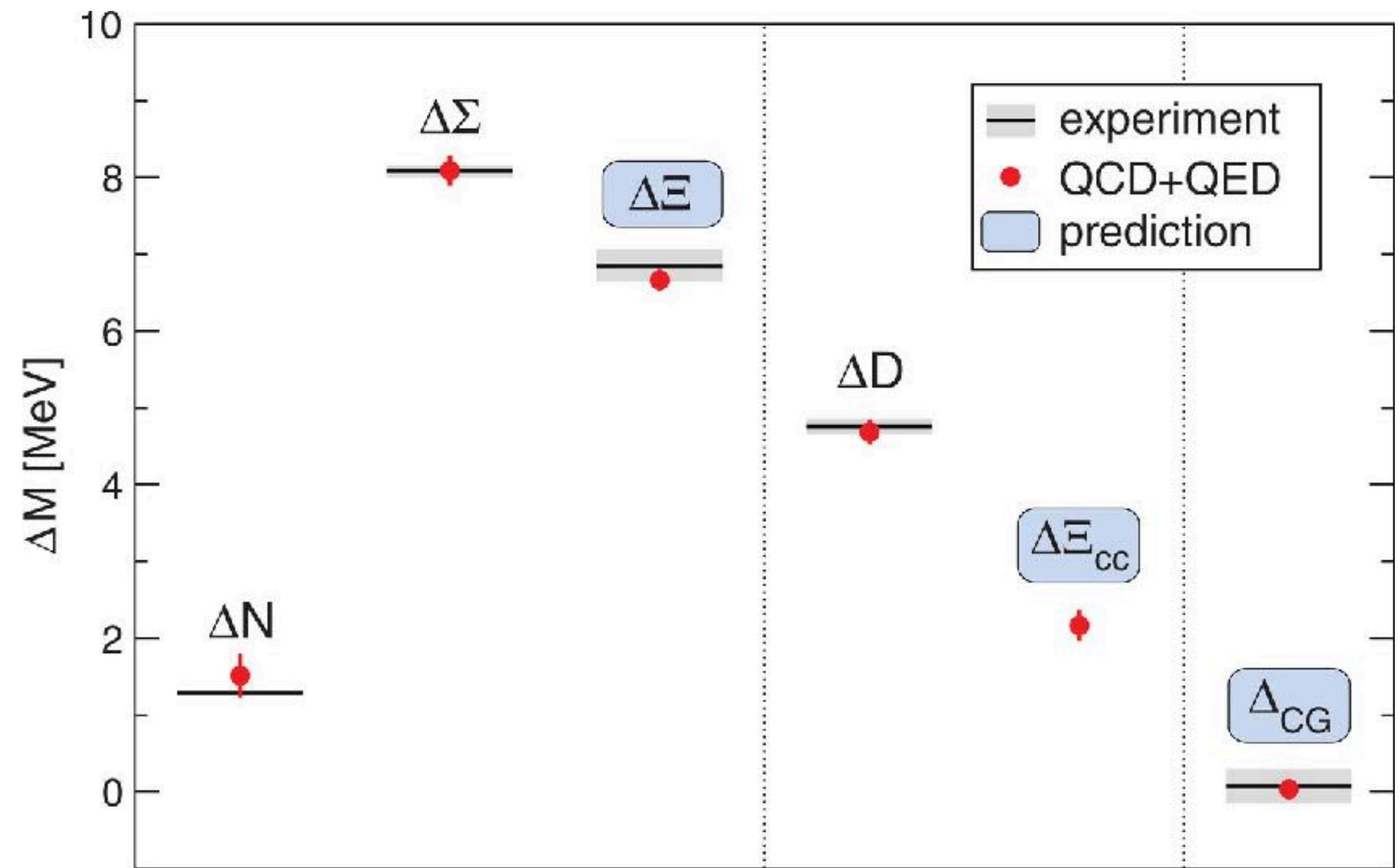


[<https://www.science.org/doi/full/10.1126/science.1163233>]

[BMW collaboration, 2008]

QCD spectrum from LQCD

- More recently: add QED and $m_u \neq m_d$
- Can reproduce neutron/proton mass difference
- More precise than some experimental results

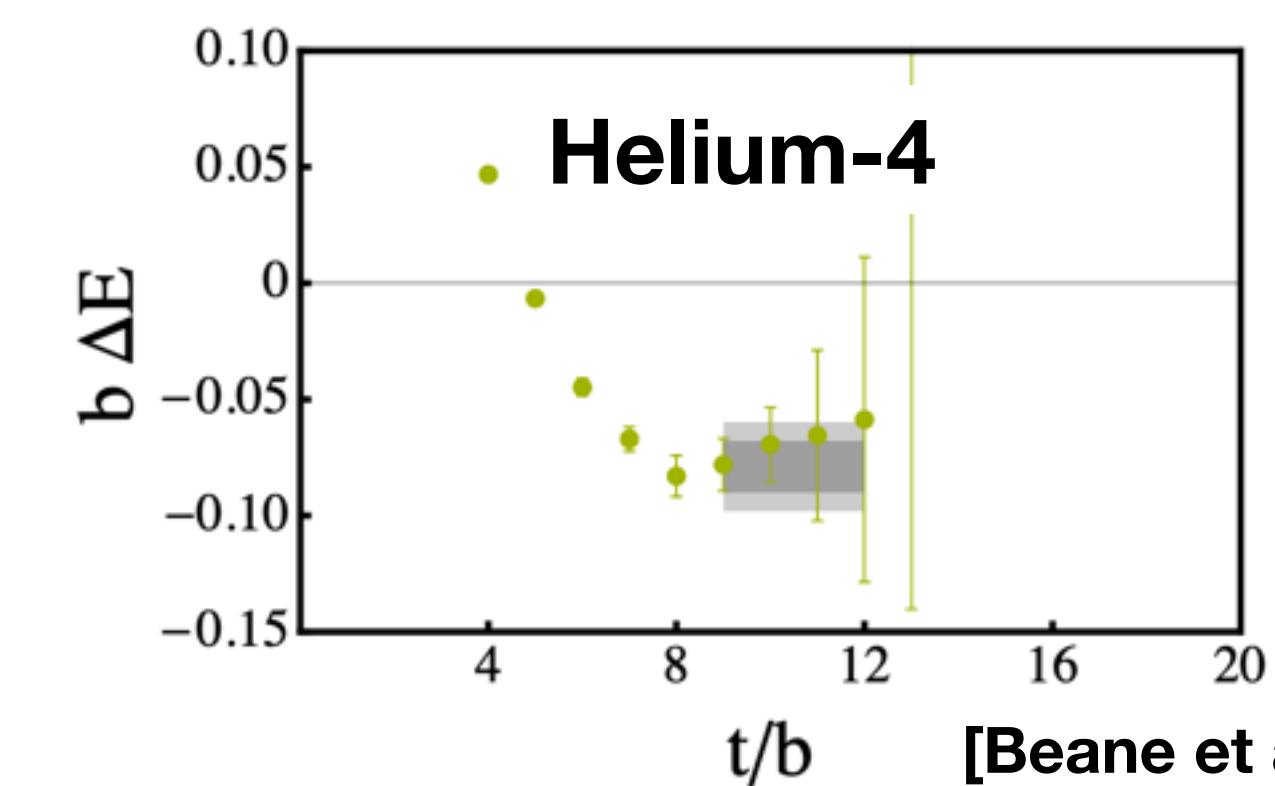
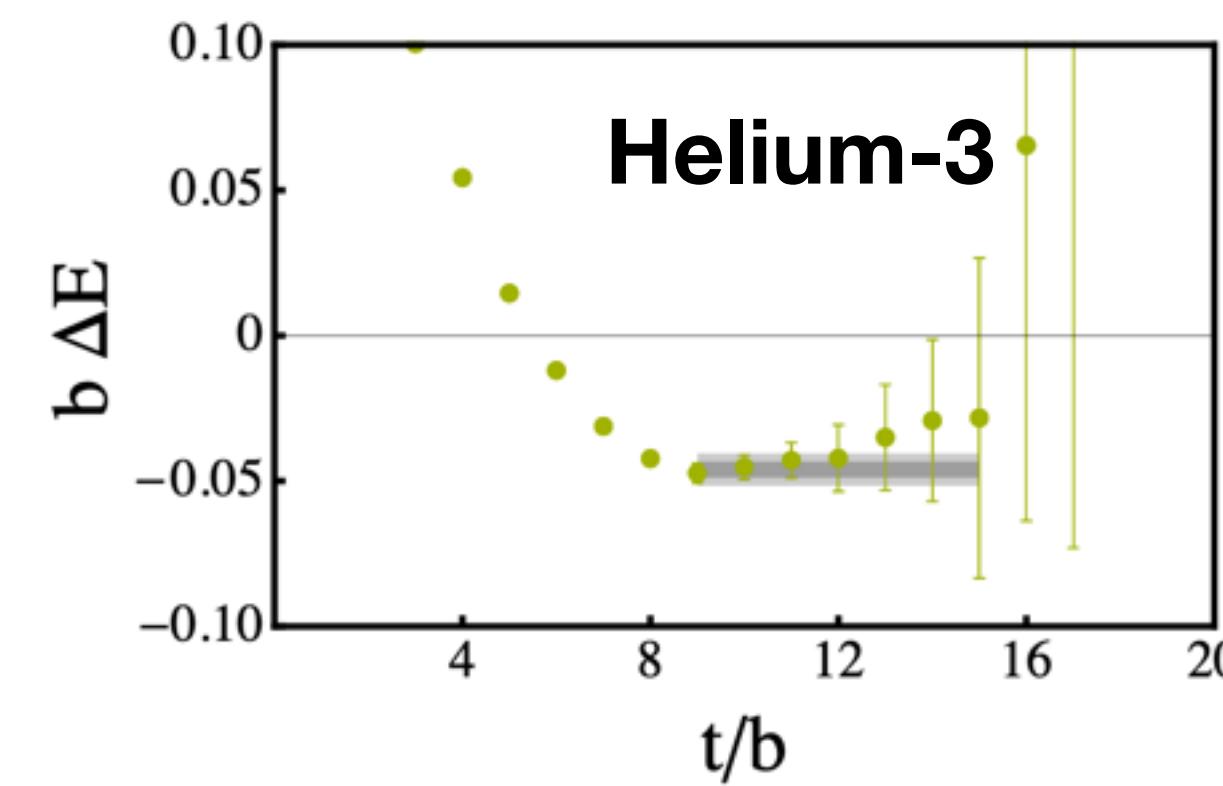
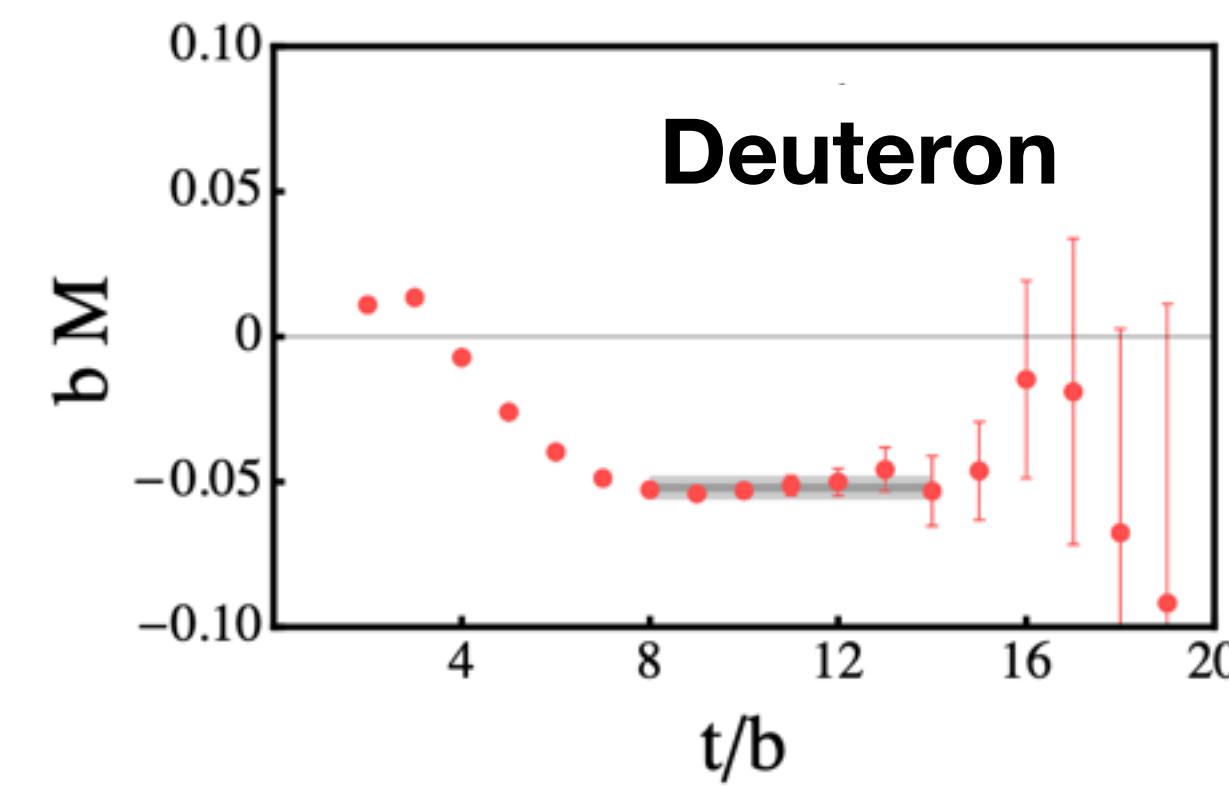


[BMW collaboration (2015)]

[<https://www.science.org/doi/full/10.1126/science.1257050>]

Towards Nuclear Physics

- Lattice QCD can in principle compute properties of nuclei. In practice, it is still very preliminary.
- Signal-to-noise problem grows rapidly with the baryon number: hard to control excited states.



[Beane et al (NPLQCD), 2012]

- Computational cost of Wick contractions grows (naively) factorially
contractions $\sim (3A)!$

Beyond the ground state

- ▶ Compute matrix of Euclidean correlation functions using operators with the same quantum numbers

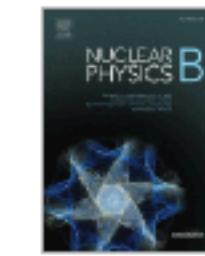
$$C_{ij}(t) = \langle \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0) \rangle$$

$$C_{ij}(t) = \langle 0 | \mathcal{O}_i | n \rangle \langle n | \mathcal{O}_j | 0 \rangle^* e^{-E_n t}$$

- ▶ Variational techniques
(Generalized EigenValue Problem, GEVP)

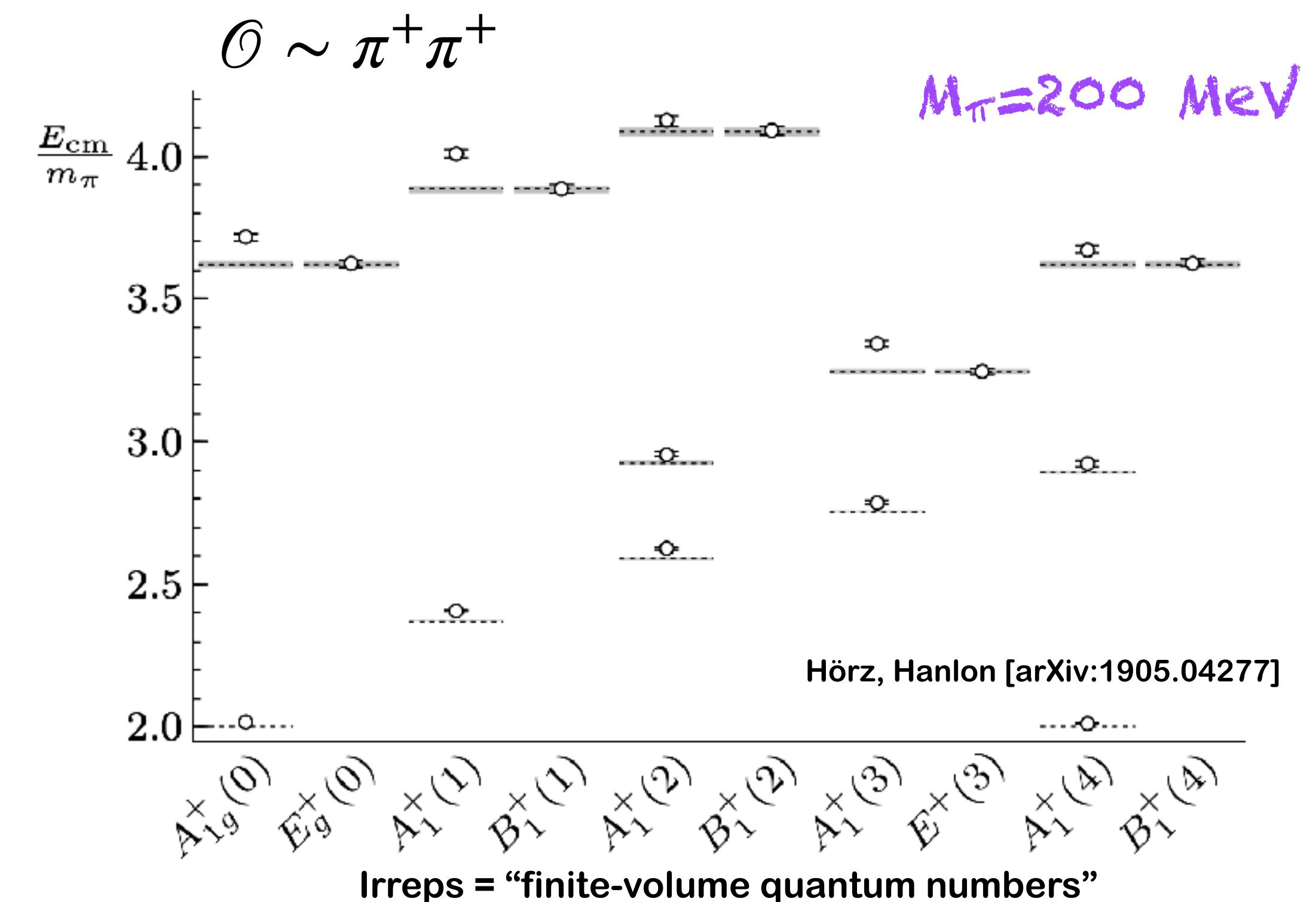


Nuclear Physics B
Volume 339, Issue 1, 23 July 1990, Pages 222-252

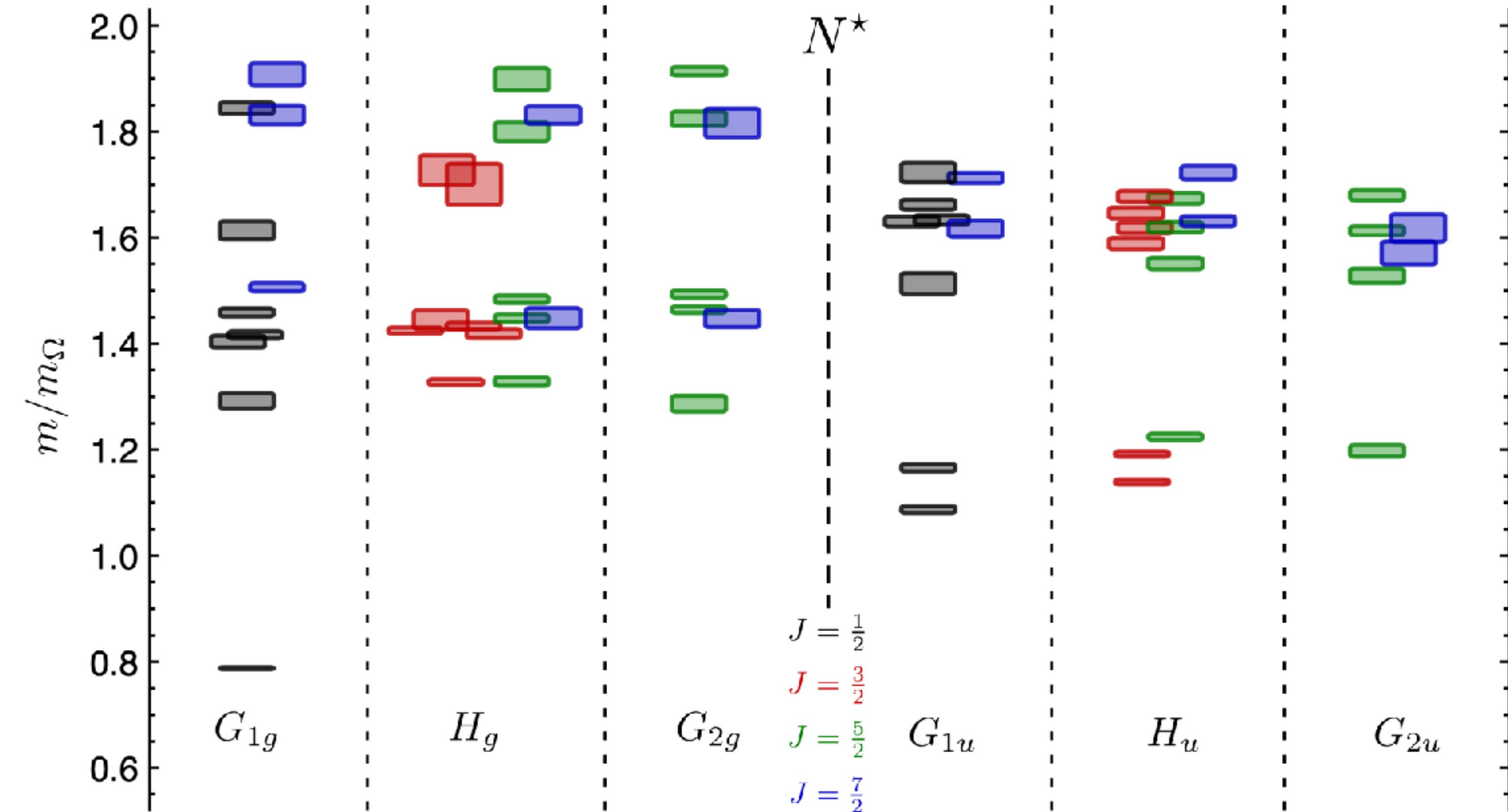


How to calculate the elastic scattering matrix in two-dimensional quantum field theories by numerical simulation

Martin Lüscher, Ulli Wolff^{a,b}

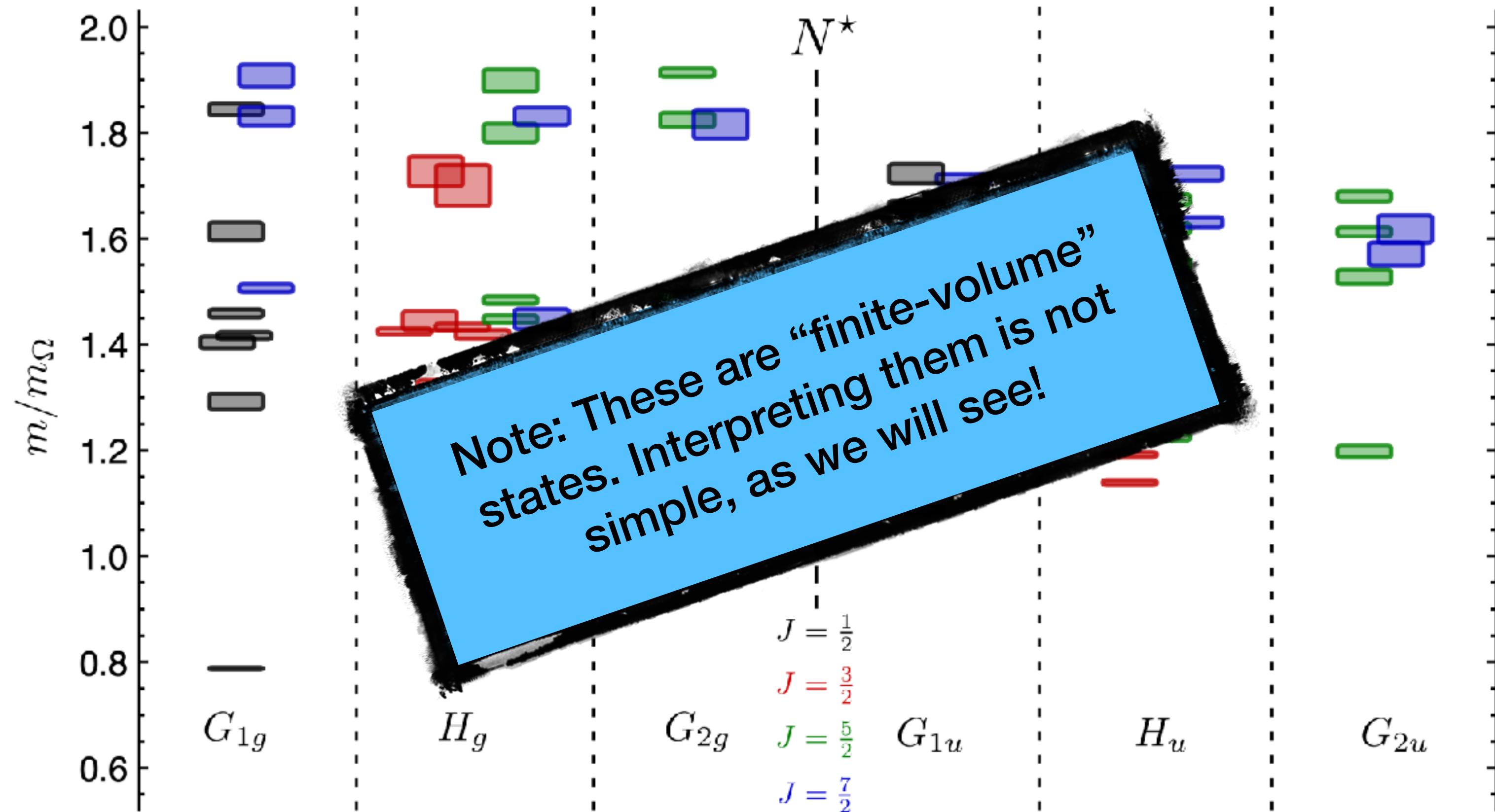


Hadron Spectrum



[HadSpec Collaboration, arXiv:1104.5152]

Hadron Spectrum



[HadSpec Collaboration, arXiv:1104.5152]

Finite-volume effects: stable particles

Stable particle in a box

- We want to compute the mass of a particle in lattice QCD.

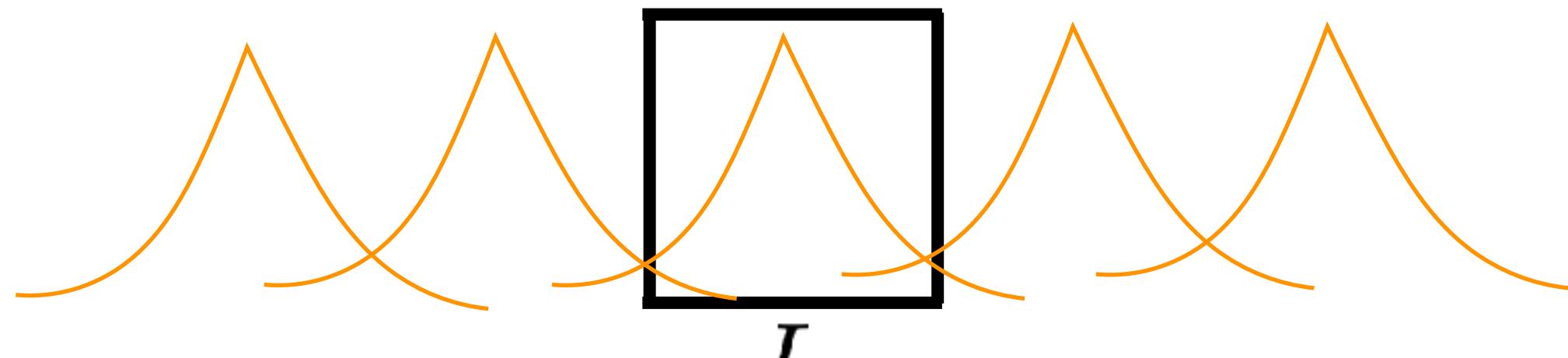
- ▶ Lattice QCD calculations are performed in a finite volume.
- ▶ What are the finite-volume effects?

- Qualitative picture in Quantum Mechanics:

$$\psi(x) = \sqrt{m} \exp(-m|x|) \quad m = \langle \psi | H | \psi \rangle$$

$$\psi_L(x) = \sum_{n \in \mathbb{Z}} \psi(x + nL)$$

- ▶ Finite volume is like infinite periodic copies:



$$\psi_L(x) = \psi_L(x + L) \quad \text{Periodic BCs}$$

$$m(L) = \langle \psi_L | H | \psi_L \rangle \sim m \times \int_{-L/2}^{L/2} dx \sum_{n_1, n_2} \psi(x + n_1 L) \psi(x + n_2 L) = m + O(\exp(-mL))$$

Stable massive particle only “sees” boundary exponentially

Finite volume QFT

- In a QFT is formulated in finite-volume, loop integrals become sums:

$$\int \frac{d^4 k}{(2\pi)^4} \rightarrow \int \frac{dk^0}{2\pi} \frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\vec{n}}$$

- Mass of a particle given by the self-energy:

$$m^2 = m_0^2 + \Sigma(m^2) \xrightarrow{\text{Finite Volume}} m^2(L) = m_0^2 + \Sigma_L(m^2(L))$$

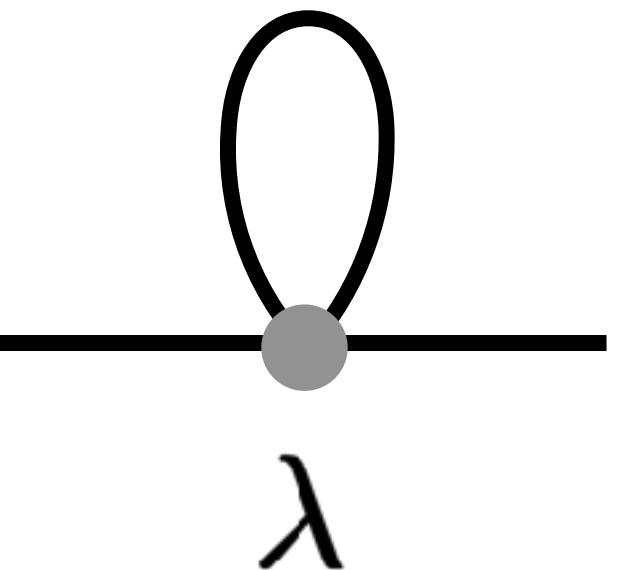
- Mass of a particle given by the self-energy:

$$m(L) - m = \frac{1}{2m} [\Sigma_L(m^2) - \Sigma(m^2)]$$

Example: ϕ^4 theory

- Self-energy given by tad-pole diagram:
- Finite-volume effects given as:

$$\mathcal{L}_I = -\frac{\lambda}{4!} \phi^4$$



$$\Sigma_L - \Sigma = \int \frac{2k^0}{2\pi} \left(\frac{1}{L^3} \sum_{\vec{n}} - \int \frac{d^3 k}{(2\pi)^3} \right) \frac{-\lambda}{k^2 - m^2} \simeq \text{const} \times m \frac{e^{-mL}}{L}$$

EXERCISE Demonstrate that the finite-volume corrections are: $\Delta m(L) \propto \frac{K_1(mL)}{mL} \approx \frac{e^{-mL}}{mL}$

1. Perform the temporal integral in the self energy

2. Use the first term in the Poisson summation formula: $\left(\frac{1}{L^3} \sum_{\vec{n}} - \int \frac{d^3 p}{(2\pi)^3} \right) f(p^2) = \sum_{\vec{n} \neq \vec{0}} \int \frac{d^3 p}{(2\pi)^3} f(p^2) e^{i(n \cdot \vec{p})L}$

Volume dependence

Non-perturbative volume dependence of stable particles

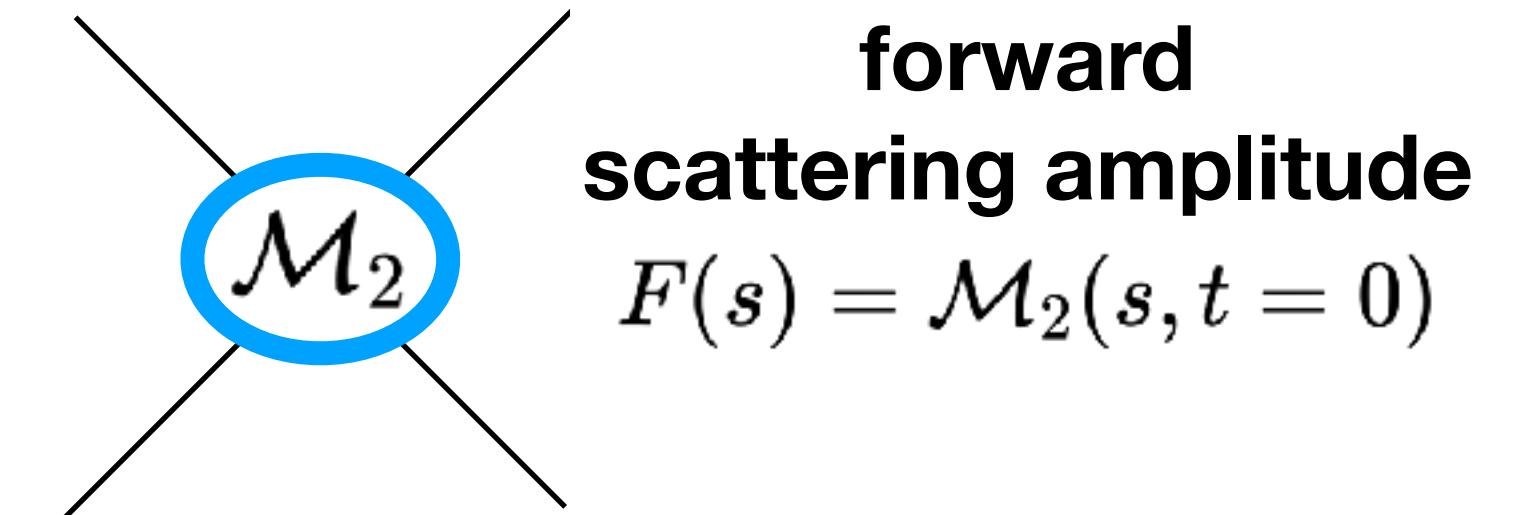
Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories

I. Stable Particle States

M. Lüscher

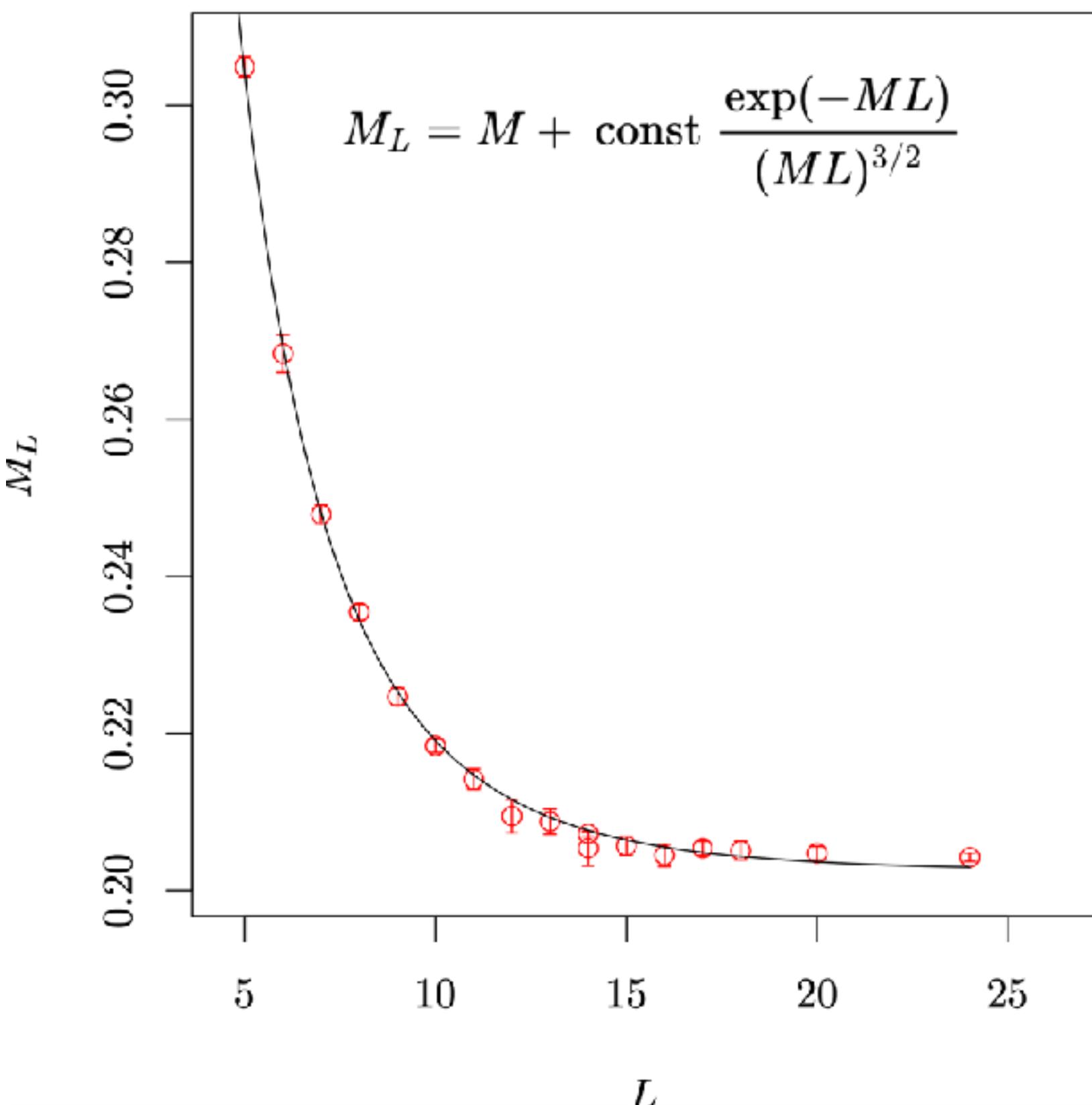
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$$\Delta m = -\frac{3}{16\pi m^2 L} \left\{ \lambda_3^2 e^{-\frac{\sqrt{3}}{2} mL} + \frac{m}{\pi} \int_{-\infty}^{\infty} dy e^{-\sqrt{m^2+y^2}L} F(iy) + O(e^{-\bar{m}L}) \right\}$$



Summary (1)

- Spectroscopy of stable particles leads only to exponentially suppressed effects
- Form of the FV effects is calculable in QFT
- Motivates choice: $M_\pi L > 4$, $\exp(-4) \sim 2\%$ error
- Example: Volume dependence in phi^4 theory
[FRL, Rusetsky, Urbach, 1806.02367]
- Related ideas on the volume dependence of stable particles:
 - ▶ Volume dependence in Chiral Perturbation Theory
[Colangelo et al, 0311023 & 0503014]
 - ▶ Volume dependence of bound states.
[Hansen, Sharpe arXiv:1609.04317]
 - ▶ Finite-volume effects in g-2
[Hansen, Patella arXiv:2004.03935]



Scattering processes and resonances

Scattering basics

- The “Scattering Matrix” is a unitary operator that connects asymptotic free states

$$S_{ab}(E) \equiv \langle \text{out} | \hat{S} | \text{in} \rangle$$

- It is related to the scattering amplitude as:

$$\langle \text{out} | (\hat{S} - 1) | \text{in} \rangle = (2\pi)^4 \delta^{(4)}(P_{\text{in}} - P_{\text{out}}) i\mathcal{M}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{p}_1, \mathbf{p}_2)$$

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- Unitarity imposes important constraints: $\hat{S}\hat{S}^\dagger = 1$

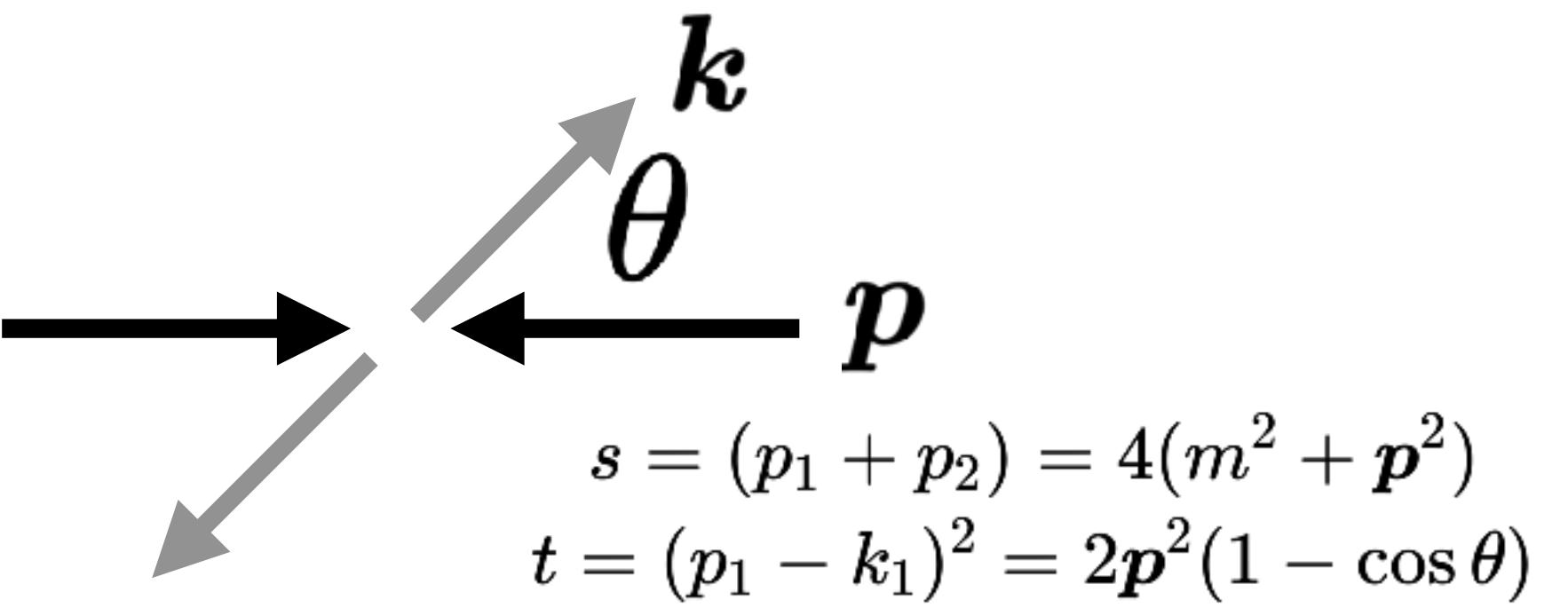
$$\mathcal{M}_2(\mathbf{k}_1, \mathbf{k}_2; \mathbf{p}_1, \mathbf{p}_2) - \mathcal{M}_2^*(\mathbf{p}_1, \mathbf{p}_2; \mathbf{k}_1, \mathbf{k}_2) =$$

$$\frac{i}{2} \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6 4\omega(q_1)\omega(q_2)} \mathcal{M}_2(\mathbf{k}_1, \mathbf{k}_2; \mathbf{q}_1, \mathbf{q}_2) \mathcal{M}_2^*(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_1, \mathbf{q}_2) \times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q_1 - q_2)$$

Two-particle scattering

- Consider two-hadron scattering in their CM frame
 - Amplitude depends on two kinematic variables:

$$\mathcal{M}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{p}_1, \mathbf{p}_2) \equiv \mathcal{M}(s, \cos \theta)$$

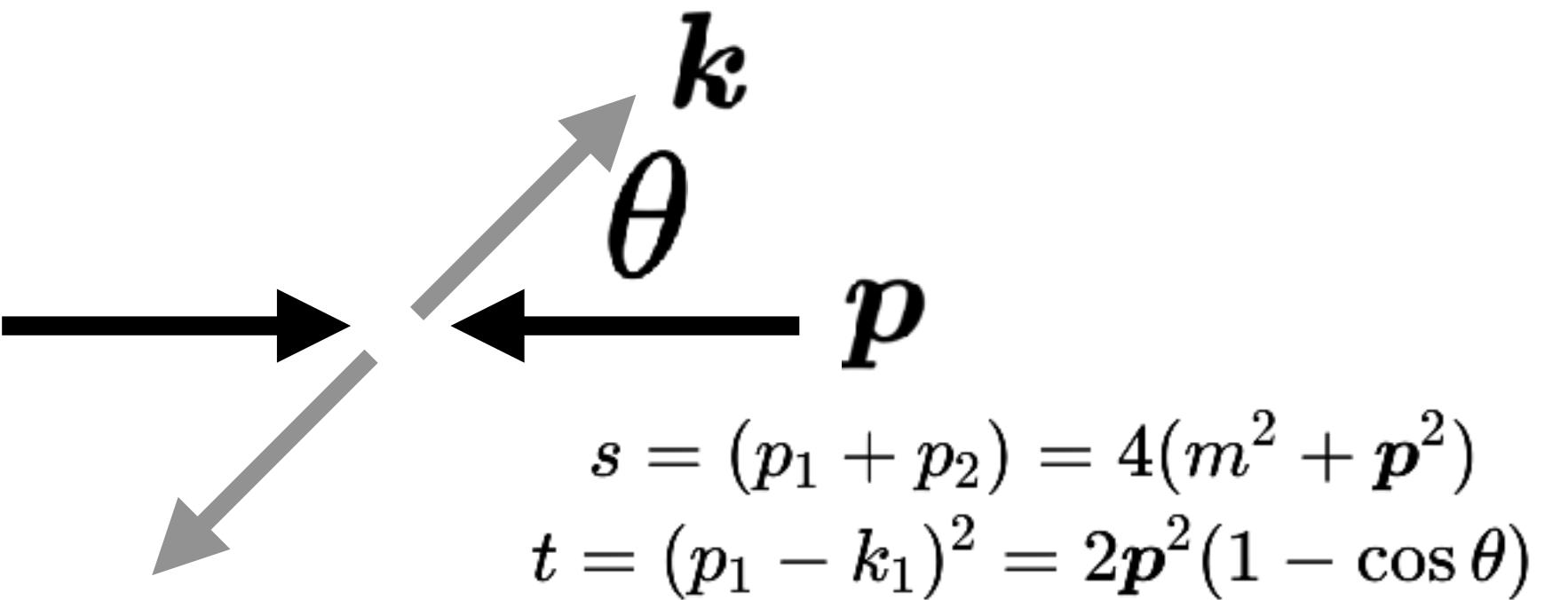


Two-particle scattering

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- We typically work in partial waves:

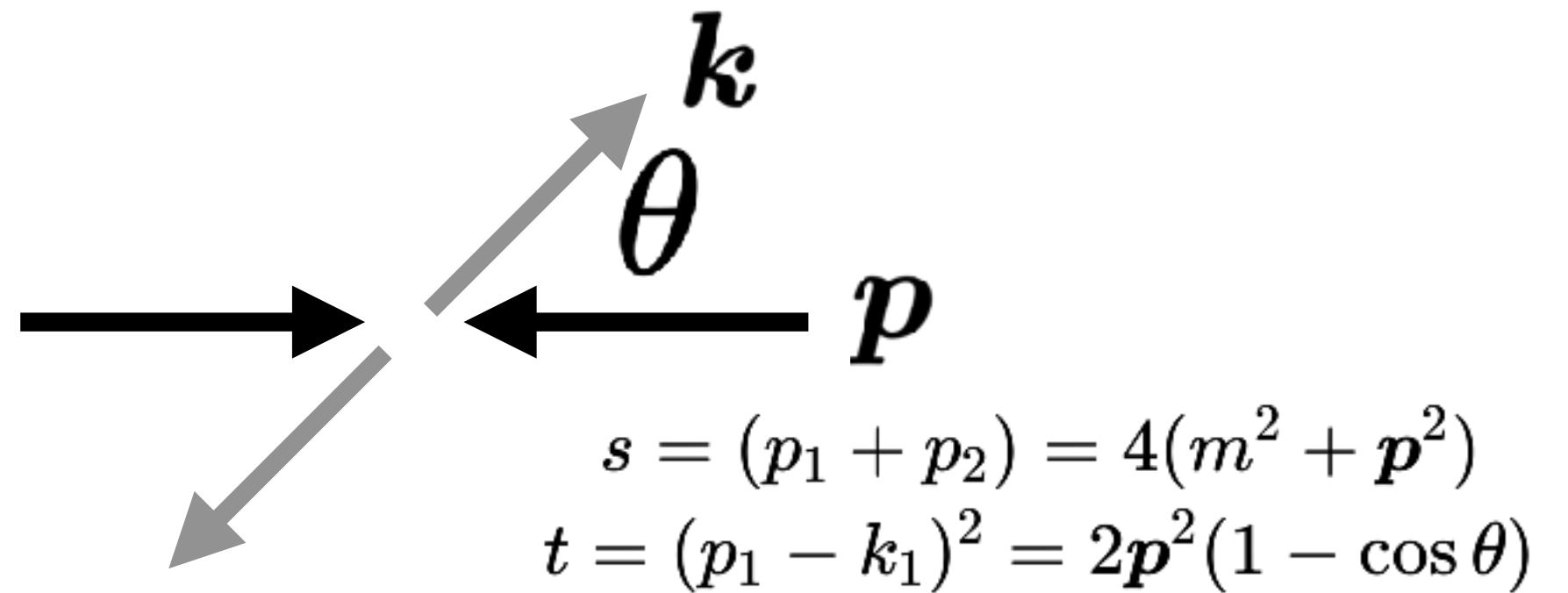
$$\mathcal{M}(s, \cos \theta) = \sum_{\ell} (2\ell + 1) P_{\ell}(\cos \theta) \mathcal{M}_{\ell}(s)$$

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- Unitarity implies that:

$$\mathcal{M}_{\ell} = \frac{16\pi\sqrt{s}}{k \cot \delta_{\ell} - ik}$$

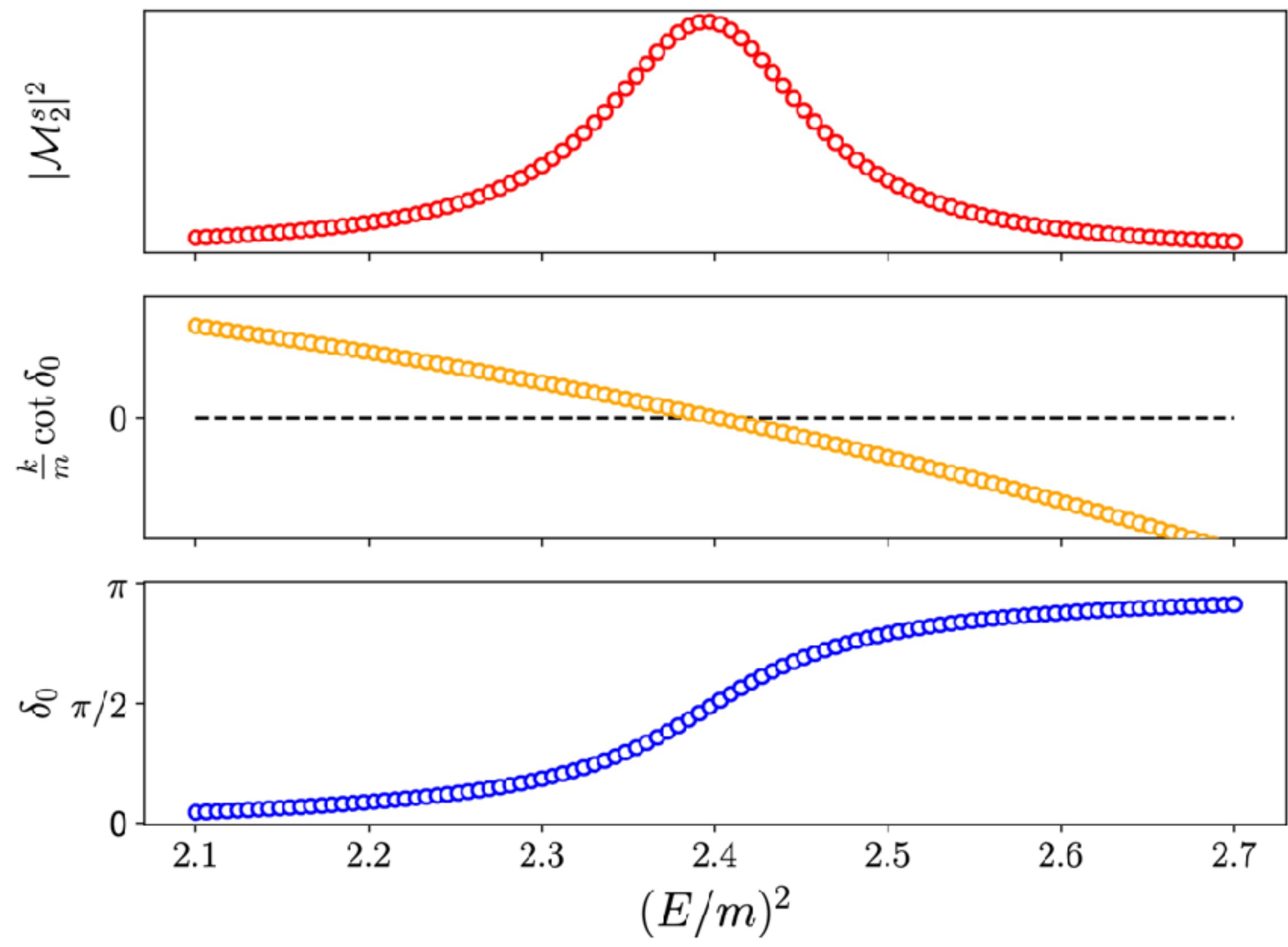
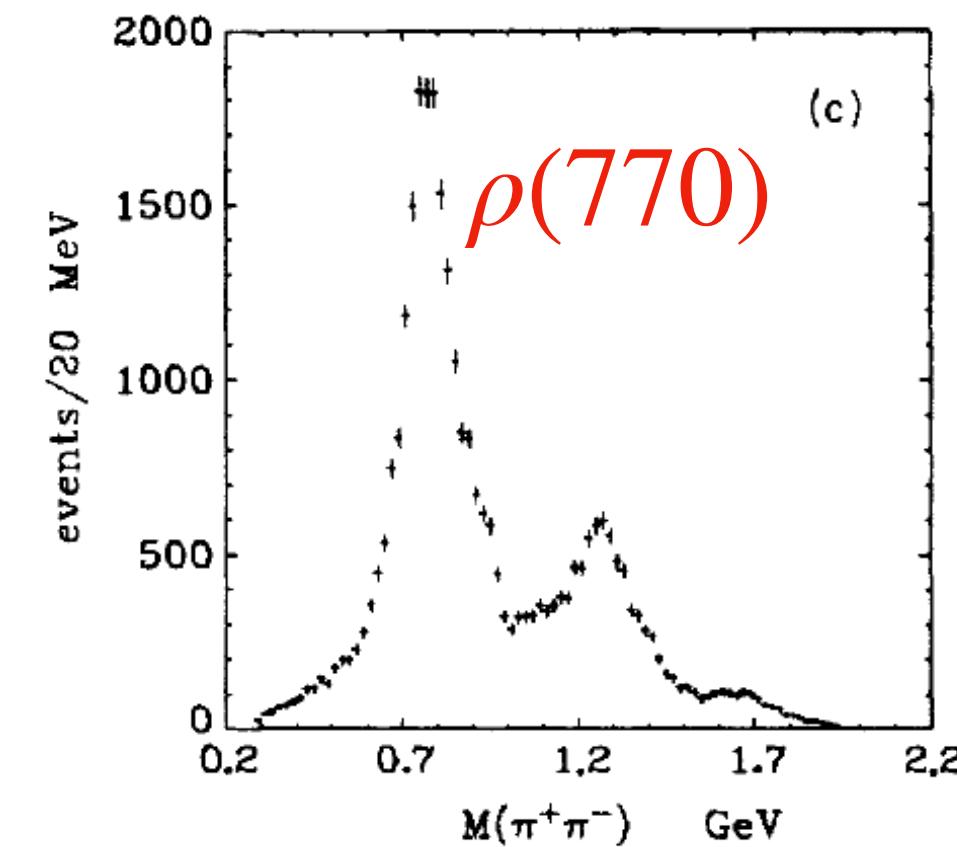
- Where $k^{2\ell+1} \cot \delta_{\ell}$ is a **meromorphic** function, e.g. simple polynomial or rational function
(holomorphic up to isolated points)

Scattering ≠ Resonances

- Resonances typically show up as enhancements in the cross-section

$$\sigma_\ell \propto |\mathcal{M}_\ell|^2 = (16\pi\sqrt{s})^2 \frac{\sin^2 \delta_\ell}{k}$$

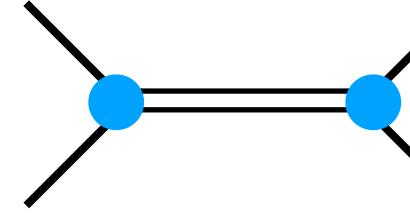
- Maximum when $\delta_\ell = 90^\circ$
- Unitarity bound is $\sin \delta_\ell = 1$



Poles in the complex plane

- The rigorous definition of a hadronic resonance is a pole in the complex plane

pole residue:
a.k.a coupling

$$\mathcal{M}_\ell \sim -\frac{g^2}{s - s_R}$$


$$\sqrt{s_R} = M_R - i \frac{\Gamma}{2}$$

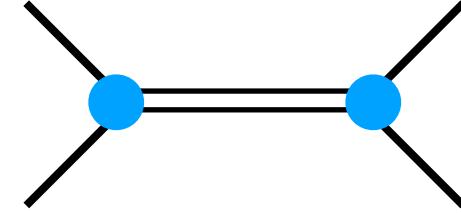
width of
the resonance

mass of
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- Based on the location of the poles, they receive different names

- ▶ Bound states: stable particles, e.g. the deuteron is an NN bound state
- ▶ Resonances: unstable hadrons, e.g. the rho resonance
- ▶ Virtual states: “non-normalizable QM states”, e.g. “dineutron”

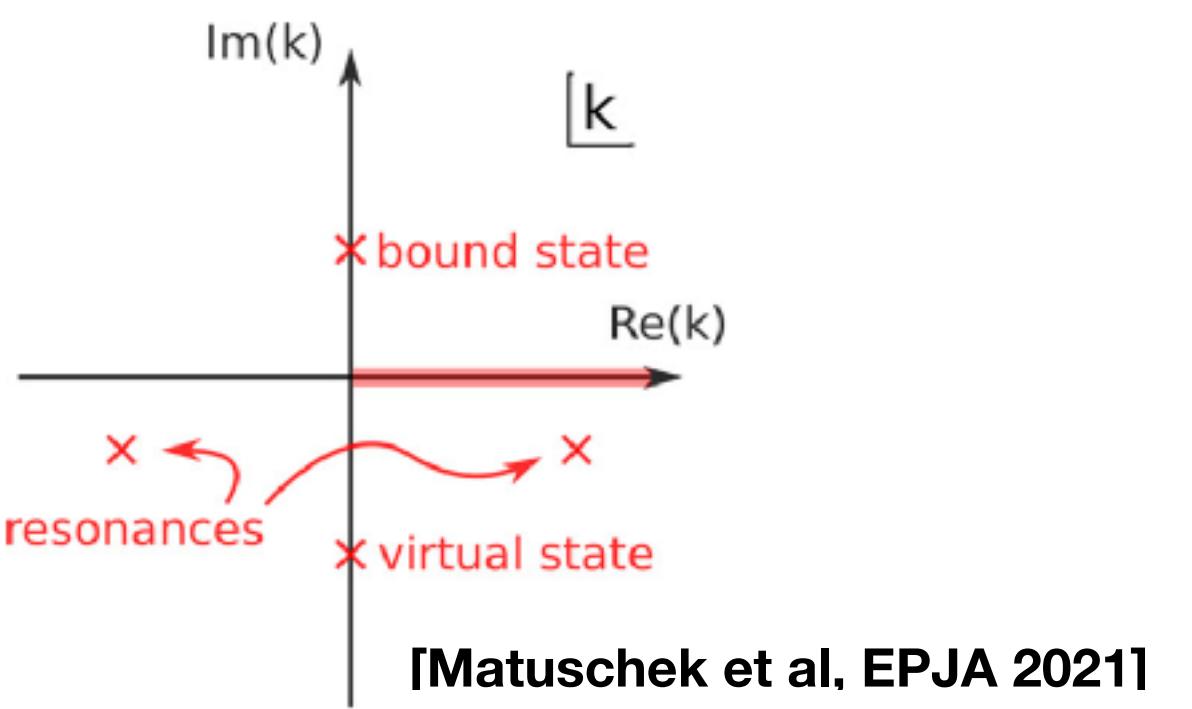


Fig. 1 Naming convention for the poles in the k -plane. The thick red line for positive real valued k marks the physical momenta in the scattering regime

Pole calculation

EXERCISE Compute the pole positions in the complex plane for the following cases

1. $k \cot \delta_0 = -\frac{1}{a_0}$
2. $k \cot \delta_0 = A(k_R^2 - k^2), \quad A \gg 1/k_R$

What kind of poles do these parametrizations lead to?

What is the residue of the pole? What sign does the residue have?

Summary (2)

The S-Matrix contains the physical information of the theory:

$$S_{ab}(E) \equiv \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

Lattice QCD \rightarrow QCD S-matrix

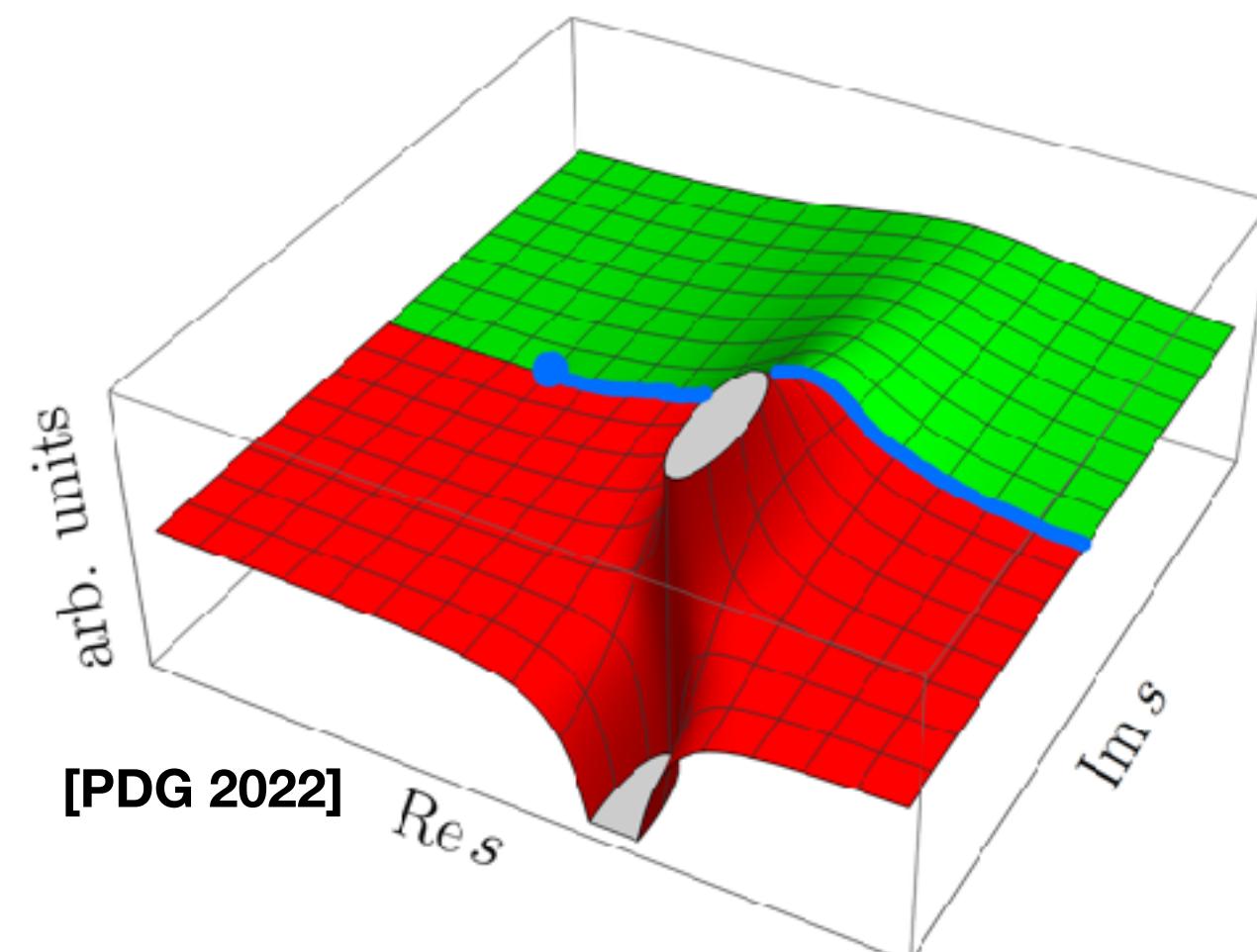
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- Resonances as poles in the S-matrix (or scattering amplitude)



$$\sim \frac{g}{E^2 - E_R^2}$$

$E_R = M_R - i\Gamma/2$

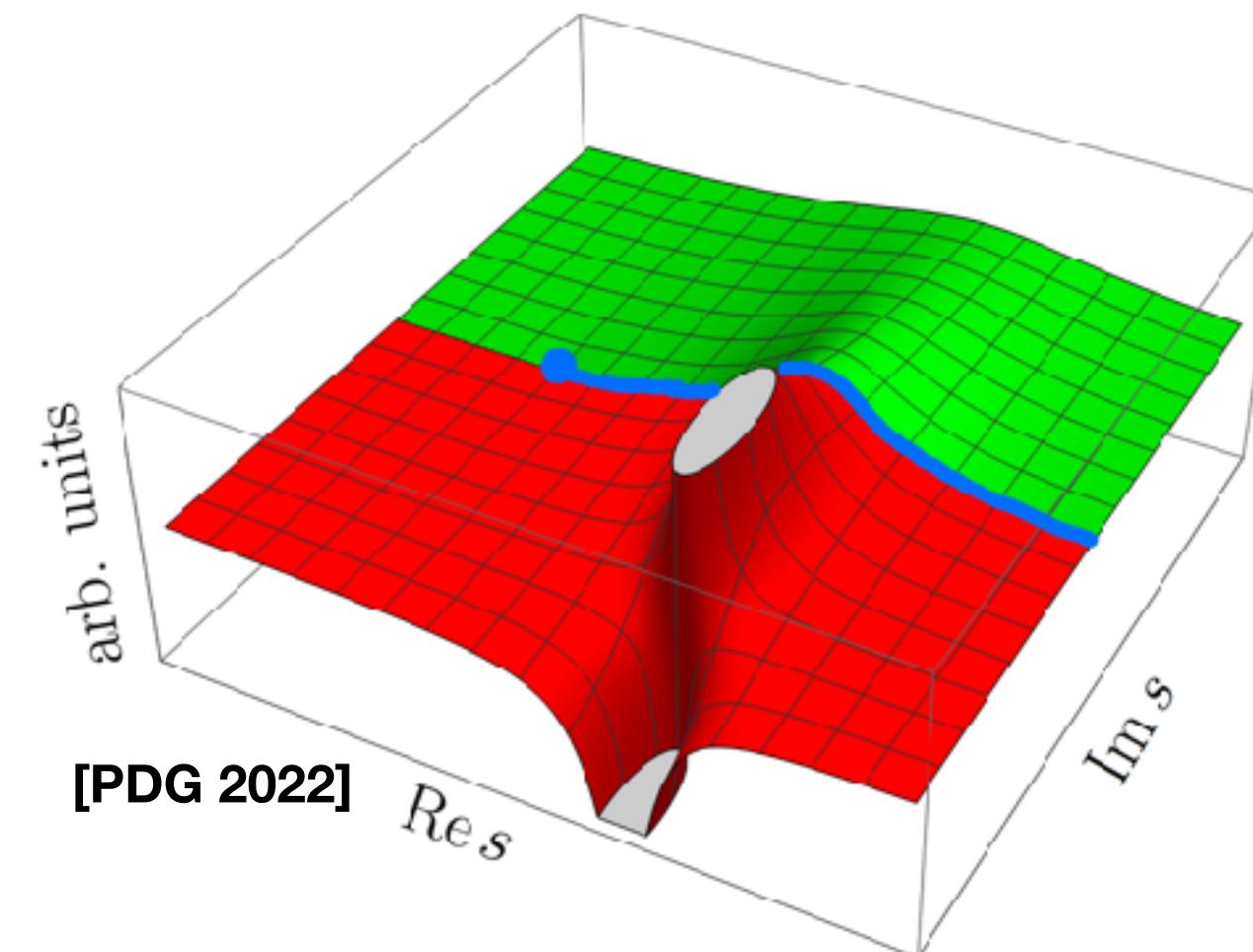
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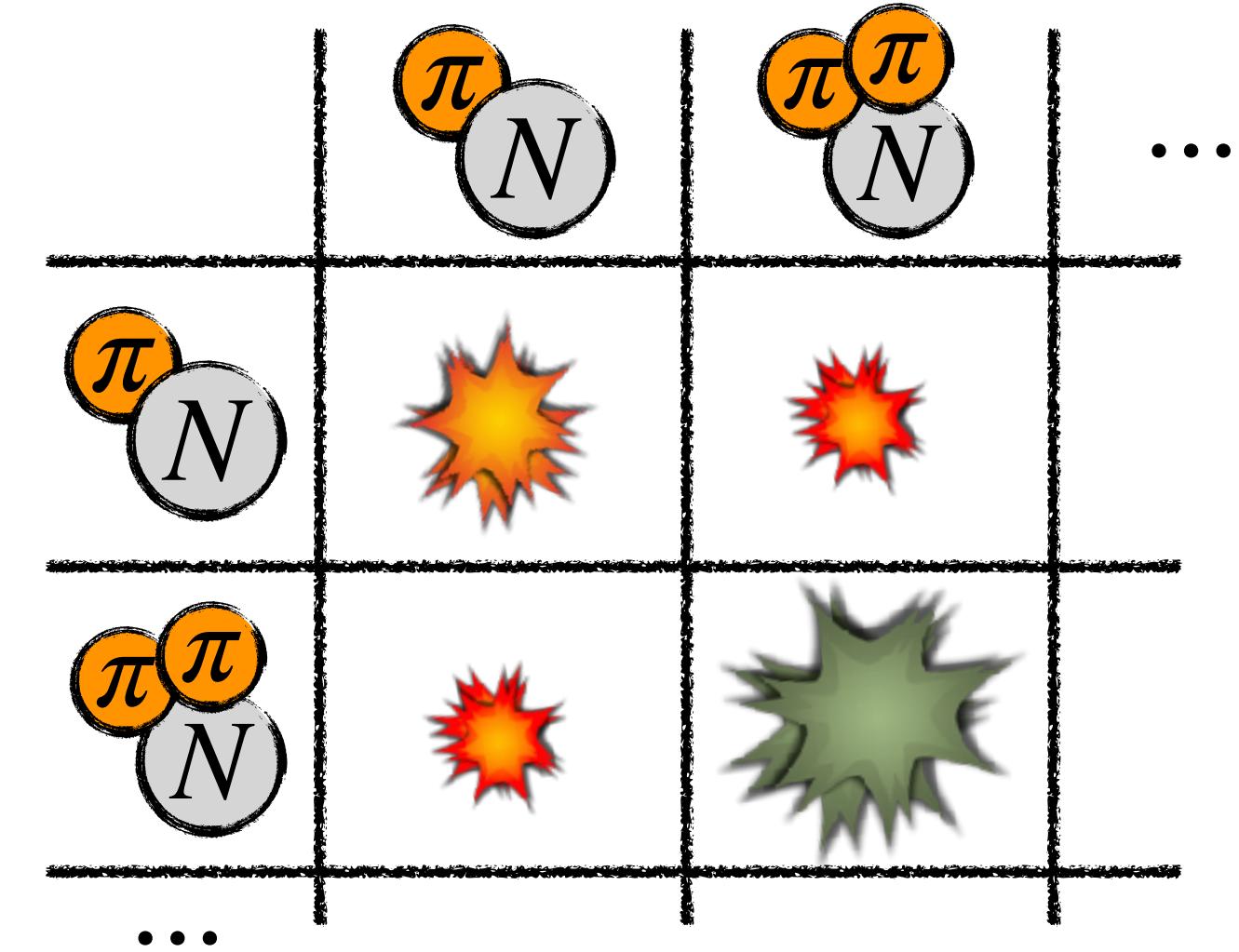
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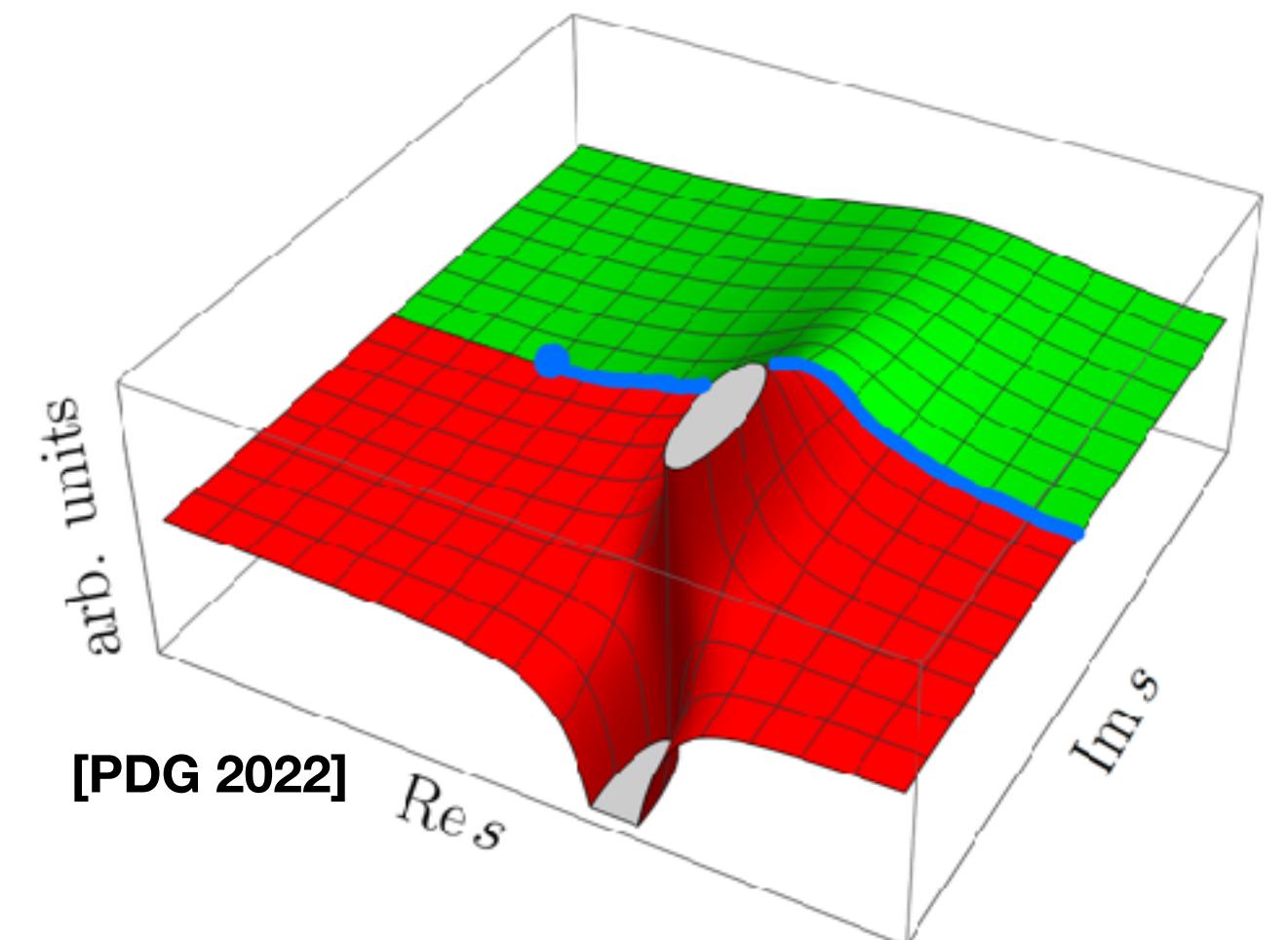
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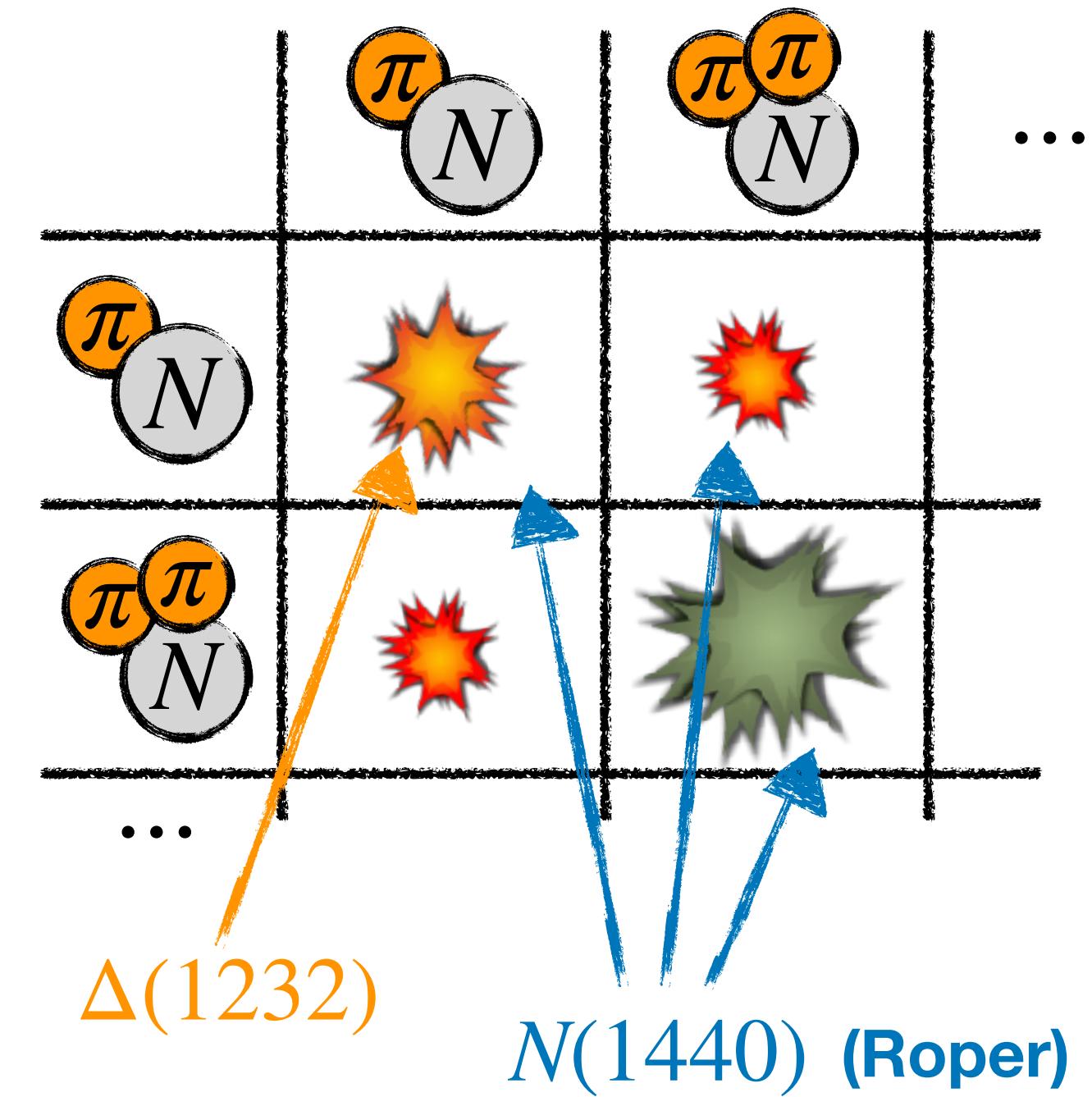
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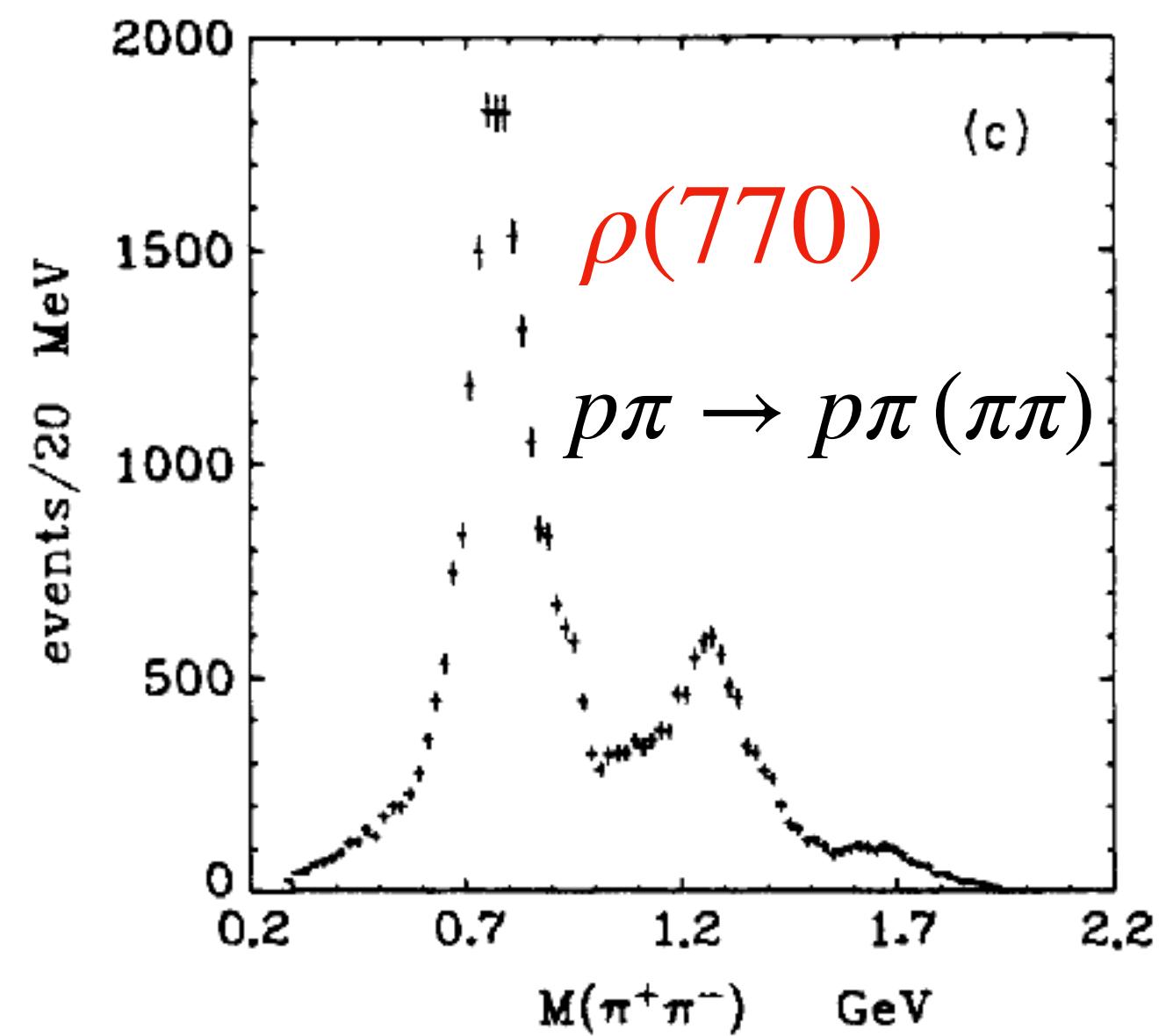


Finite-volume effects: scattering ≠ resonances

Scattering amplitudes in LQCD

Experiments

- Asymptotic states
- Direct access to cross sections

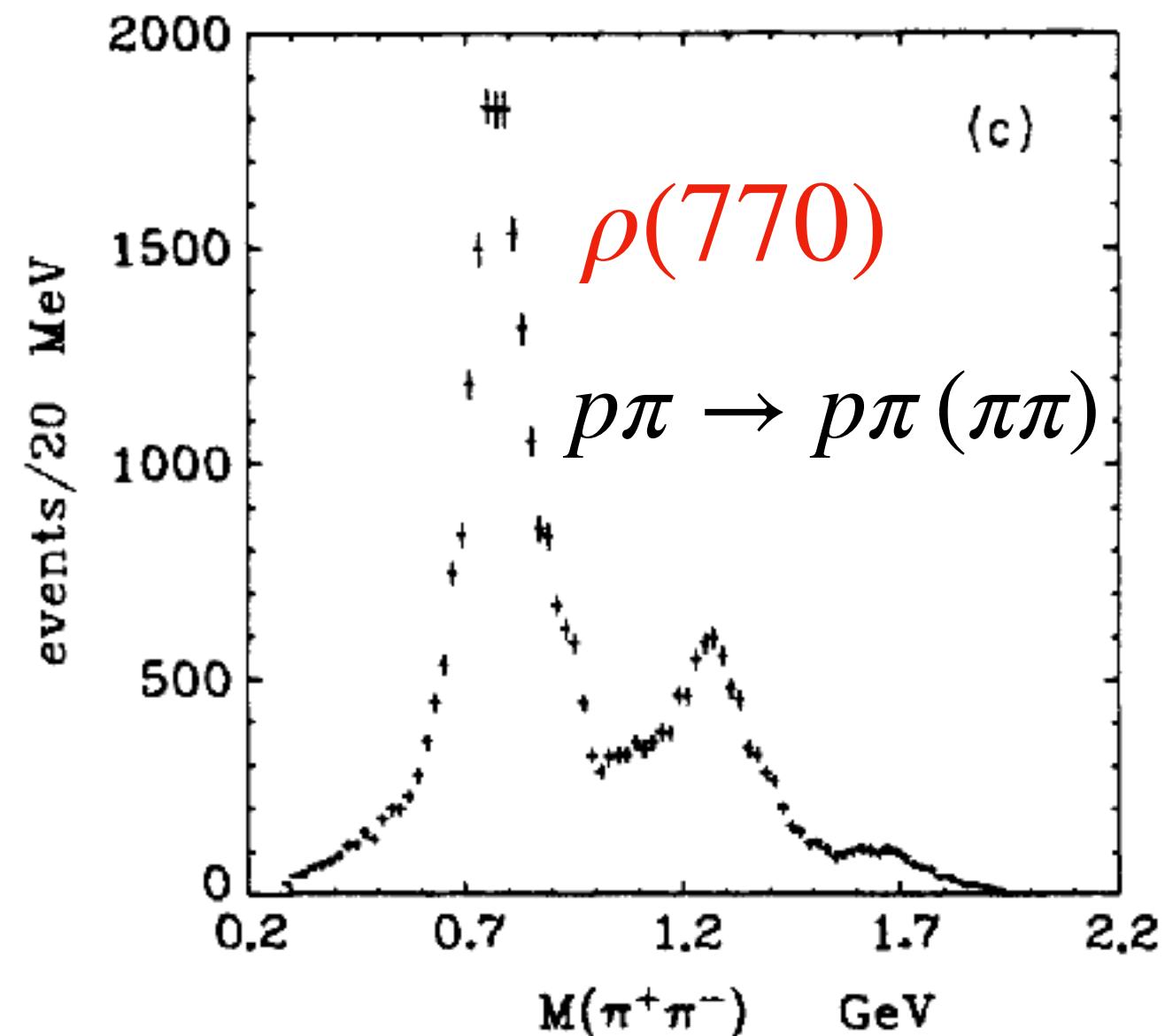


[Protopopescu et al, PRD7 1973]

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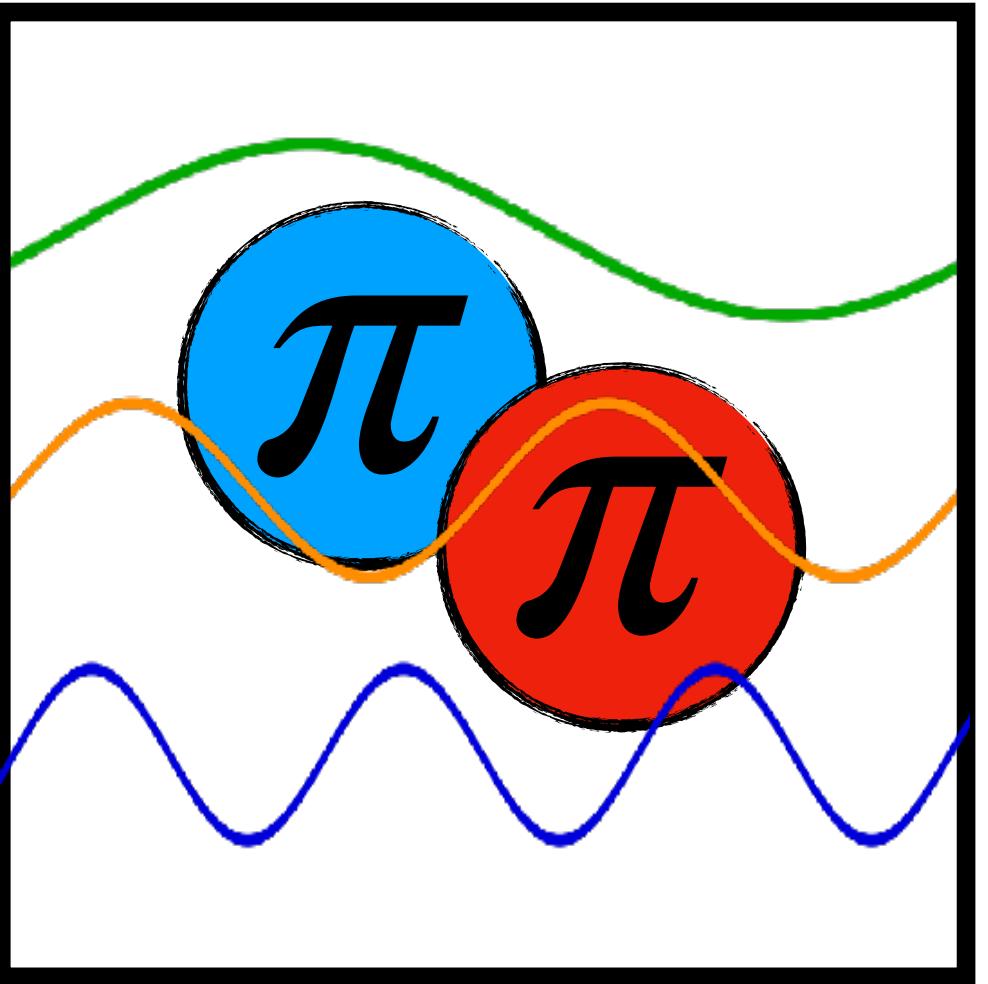


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Lattice QCD

- Euclidean time
- Stationary states in a box

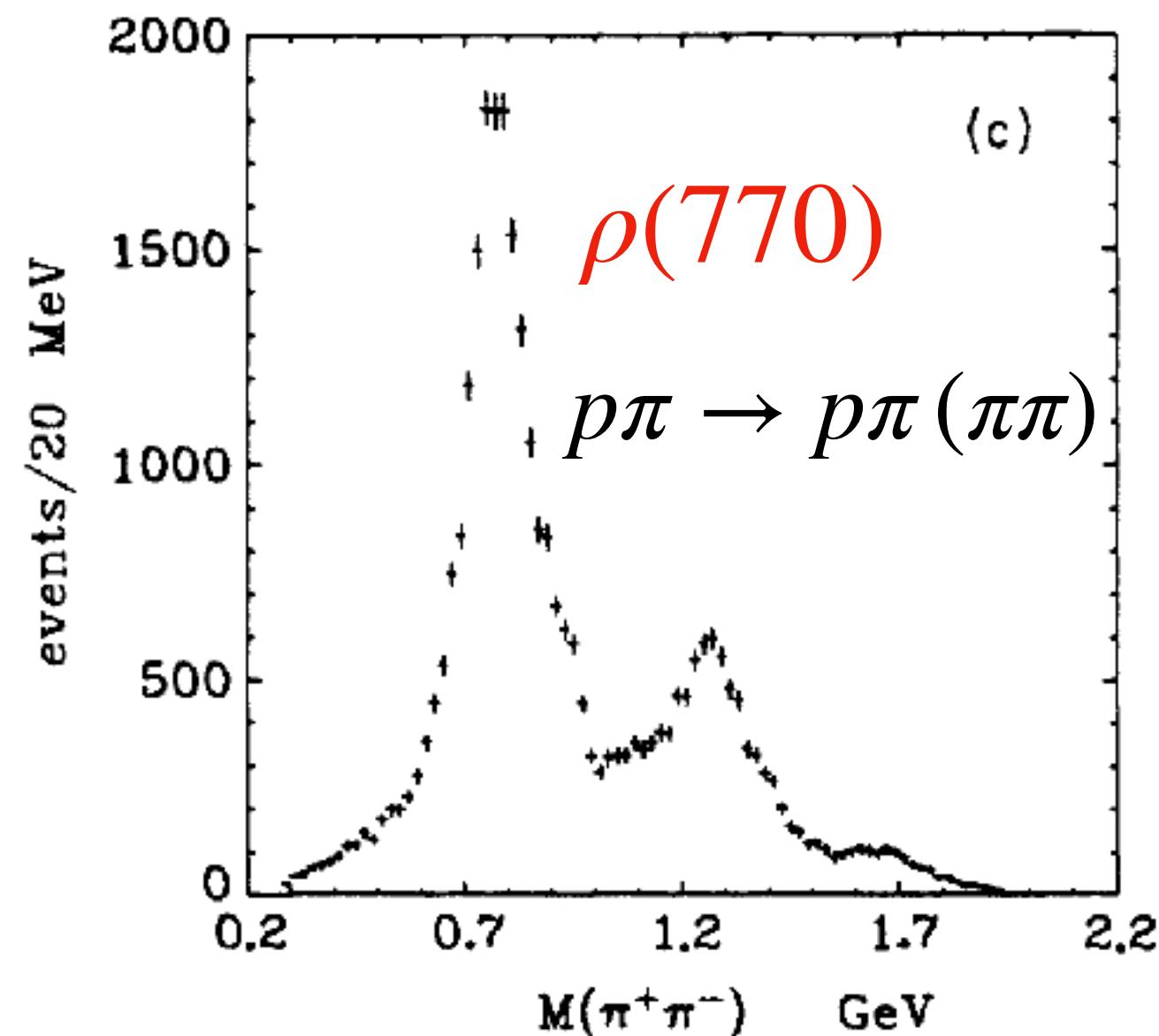
$$C(t) = \langle \mathcal{O}(t)\mathcal{O}^\dagger(0) \rangle = \sum_n \left| \langle 0 | \mathcal{O}(0) | n \rangle \right|^2 e^{-E_n t}$$



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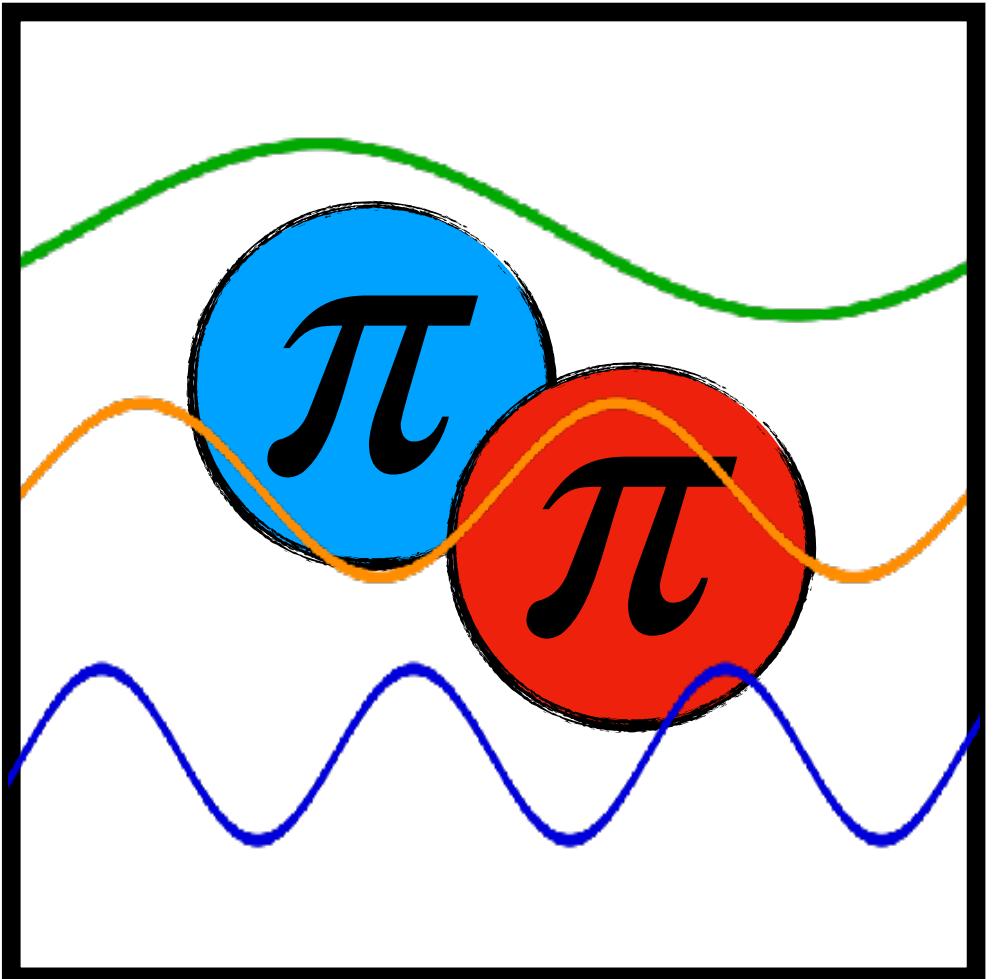


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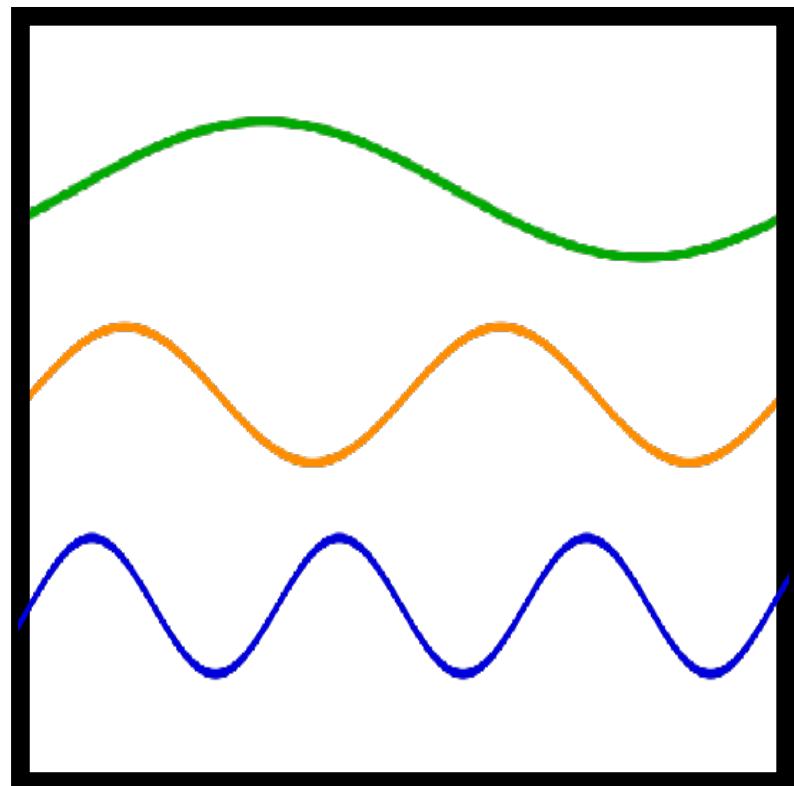


Finite-volume formalism
[Lüscher, 89']

Finite-volume systems

Free scalar particles in finite volume
with periodic boundaries

$$\psi(\vec{x}) = \psi(\vec{x} + \vec{n}L)$$

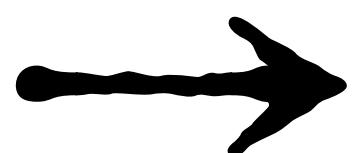


$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$\text{Two particles: } E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^2}$$

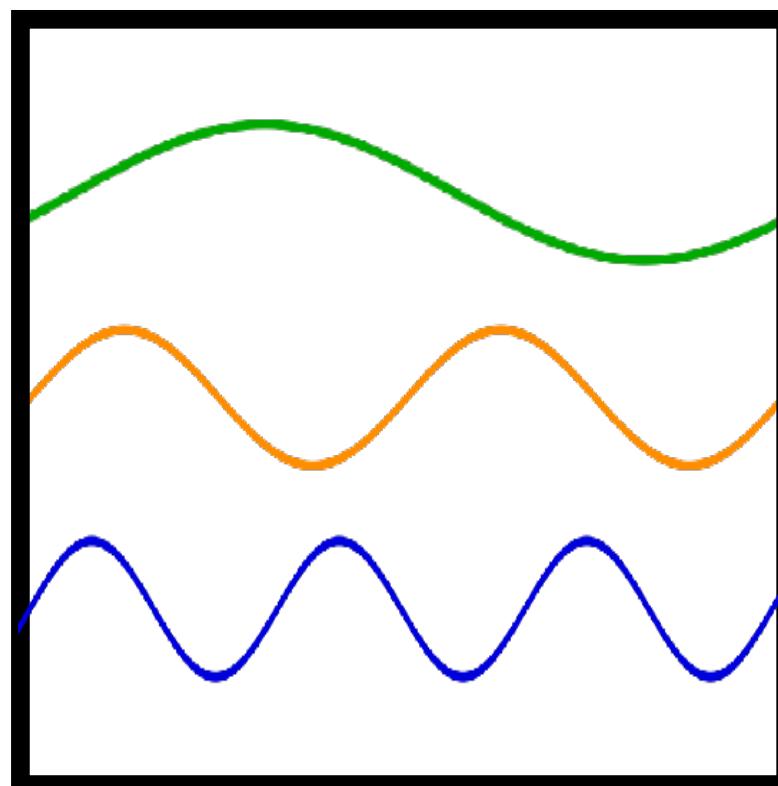
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Interactions change the spectrum:
it can be treated as a perturbation

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Ground state to leading order

$$E_2 - 2m = \langle \phi(\vec{0})\phi(\vec{0}) | \mathbf{H}_I | \phi(\vec{0})\phi(\vec{0}) \rangle$$

$$\Delta E_2 = \frac{\mathcal{M}_2(E=2m)}{8m^2L^3} + O(L^{-4})$$

[Huang, Yang, 1958]

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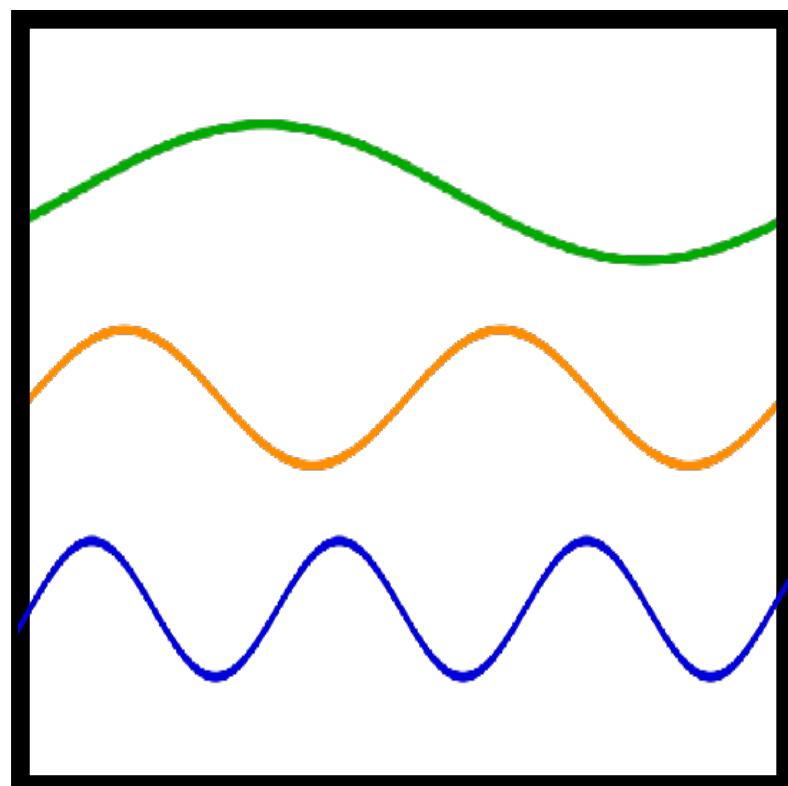
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[Huang, Yang, 1958]

The energy shift of the two-particle ground state
is related to the $2 \rightarrow 2$ scattering amplitude

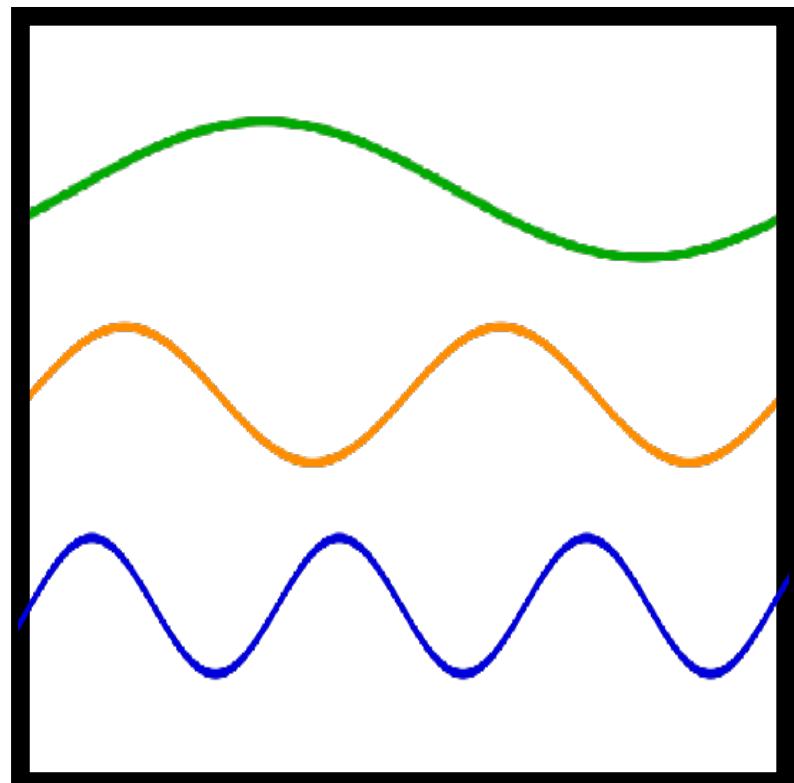
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$$\psi(\vec{x}) = \psi(\vec{x} + \vec{n}L)$$



In general a problem of
Quantum Field Theory
in finite volume

$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Two particles: $E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^2}$

and state to leading order

$$-2m = \langle \phi(\vec{0})\phi(\vec{0}) | H_I | \phi(\vec{0})\phi(\vec{0}) \rangle$$

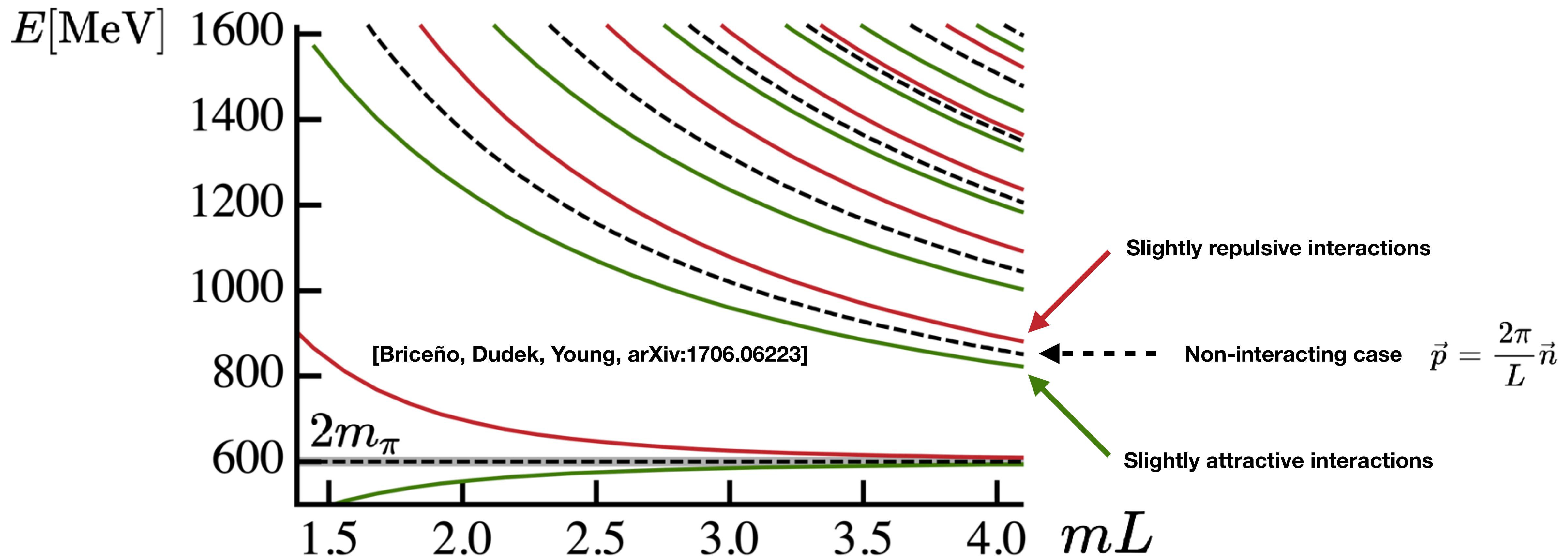
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[Huang, Yang, 1958]

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Key insight

Volume dependence of finite-volume energy states contains scattering information

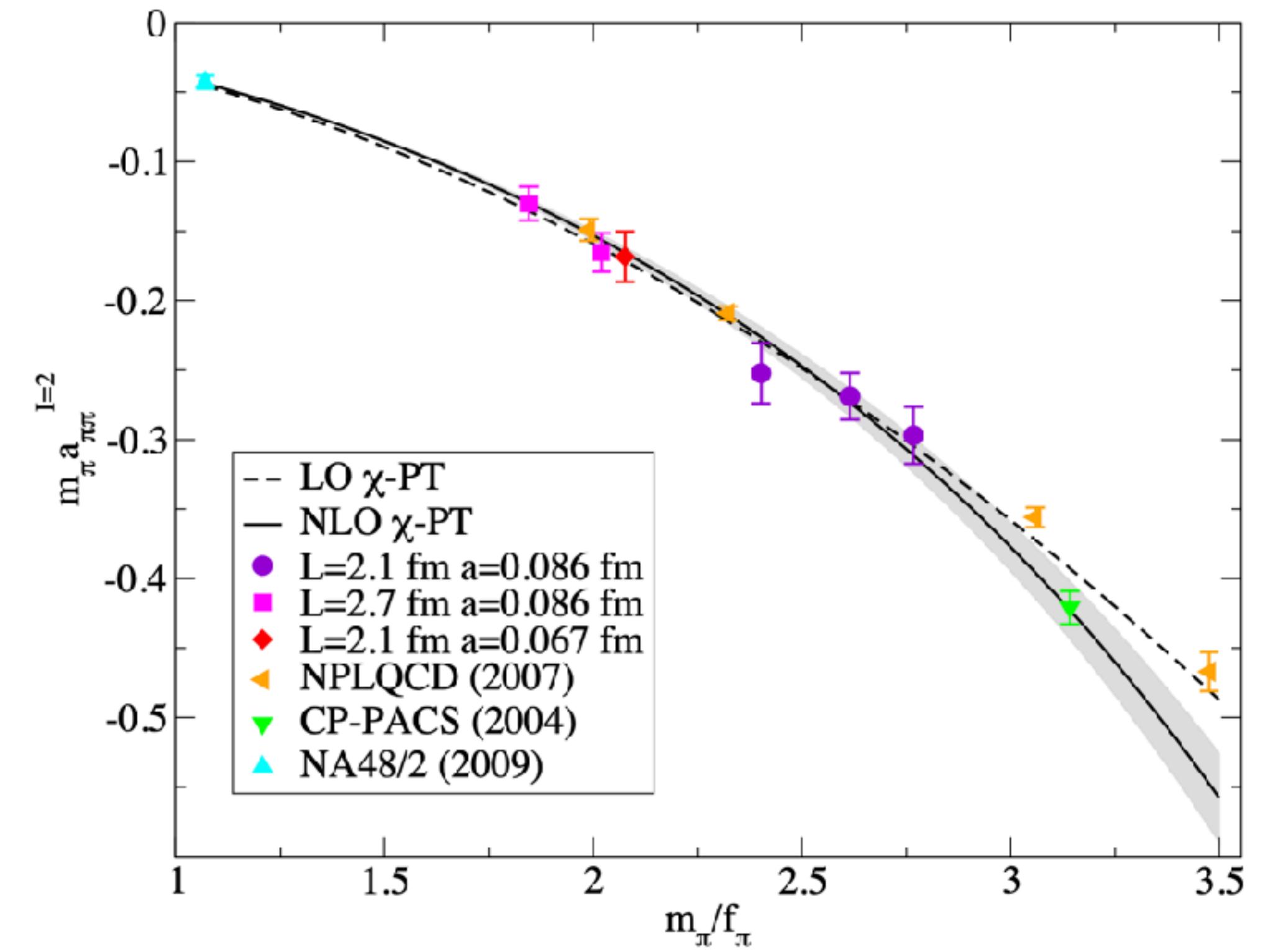


1/L expansions

- Perturbative expansions are useful for
[Lüscher, 89']
 - ▶ Weakly interacting systems (no nearby bound states or resonances)
 - ▶ Just obtaining the scattering length

$$\Delta E_2 = E_2 - 2m = \frac{4\pi a_0}{m L^3} \left\{ 1 + c_1 \left(\frac{a_0}{L} \right) + c_2 \left(\frac{a_0}{L} \right)^2 \right\} + \mathcal{O}(L^{-6})$$

$(c_1 = 2.837, c_2 = 6.375)$



General approach

- In order to derive the full relation, consider the finite-volume correlator:

$$C_L(E, \vec{P}) = \int_L e^{iPx} \langle \mathcal{O}(x) | \mathcal{O}(0) \rangle =$$

[à la Kim, Sachrajda, Sharpe]

General approach

- In order to derive the full relation, consider the finite-volume correlator:

Skeleton expansion

$$C_L(E, \vec{P}) = \int_L e^{iPx} \langle \mathcal{O}(x) | \mathcal{O}(0) \rangle = \text{---} + \text{---} + \text{---} + \dots$$

[à la Kim, Sachrajda, Sharpe]

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$$C_L(E, \vec{P}) = \int_L e^{iPx} \langle \mathcal{O}(x) | \mathcal{O}(0) \rangle = \text{Skeleton expansion} + \sum_{\vec{k}} \text{Finite-volume sums}$$

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$\sum_{\vec{k}}$

Finite-volume sums

$$\mathcal{B}_2 = \text{Feynman diagrams} + \dots$$

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[à la Kim, Sachrajda, Sharpe]

Only exponentially small effects in L

$\sum_{\vec{k}}$

Bethe-Salpeter Kernels

Finite-volume sums

$\mathcal{B}_2 = \dots$

The diagram illustrates the skeleton expansion of the finite-volume correlator. It starts with a horizontal line with two circles containing operators \mathcal{O} . This is followed by a plus sign, then a line with three circles: \mathcal{O} , \mathcal{B}_2 , \mathcal{O} . Another plus sign, then a line with four circles: \mathcal{O} , \mathcal{B}_2 , \mathcal{B}_2 , \mathcal{O} . A blue arrow points down from the \mathcal{B}_2 term to a detailed diagram of \mathcal{B}_2 as a sum of Feynman-like diagrams. The text "Only exponentially small effects in L " is associated with the first term, and "Finite-volume sums" is associated with the final ellipsis.

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[à la Kim, Sachrajda, Sharpe]

$$\begin{aligned} &= \text{Diagram with two O's} + \text{Diagram with one O and one B}_2 + \text{Diagram with two B}_2's + \dots \\ &\quad \downarrow \\ &\quad \text{Bethe-Salpeter Kernels} \\ &\quad \text{Only exponentially small effects in } L \quad \text{B} = \text{Diagram} + \dots \\ &\quad \sum_{\vec{k}} \rightarrow \int d^3k + \left[\sum_{\vec{k}} - \int d^3k \right] \\ &\quad \text{Finite-volume sums} \end{aligned}$$

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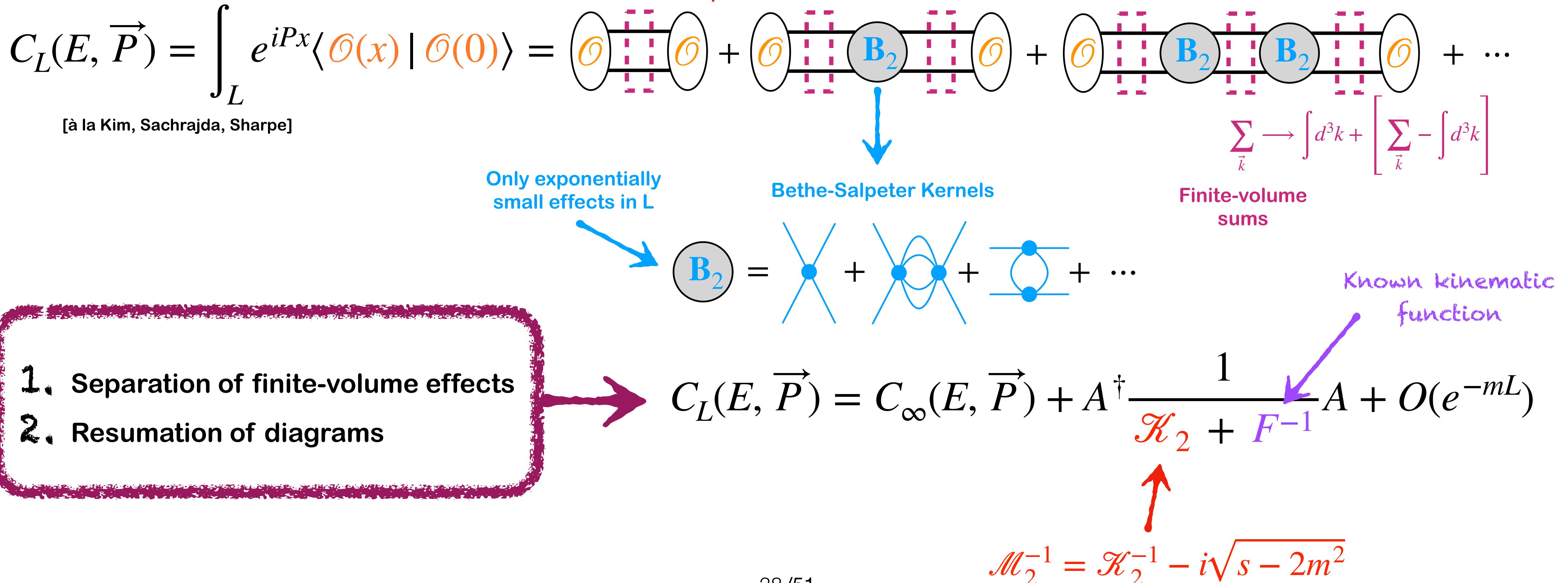
Bethe-Salpeter Kernels

Finite-volume sums

1. Separation of finite-volume effects
2. Resummation of diagrams

General approach

- In order to derive the full relation, consider the finite-volume correlator:



General approach

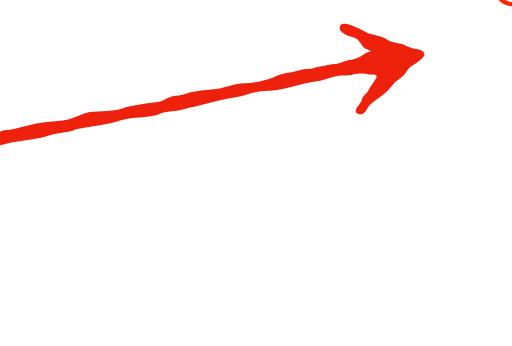
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K-matrix parametrized
in terms of phase shift

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Finite-volume states appear
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! It holds below $E_{cm} < 4m$

Two-particle Quantization Condition

$$\det \left[\mathcal{K}_2(E_n) + F^{-1}(E_n, \vec{P}, L) \right] = 0$$

Scattering
K-Matrix

Known kinematic
function

"QC2"

The quantization condition

- Indirect connection between the spectrum and the two-particle scattering amplitude [Lüscher 89']

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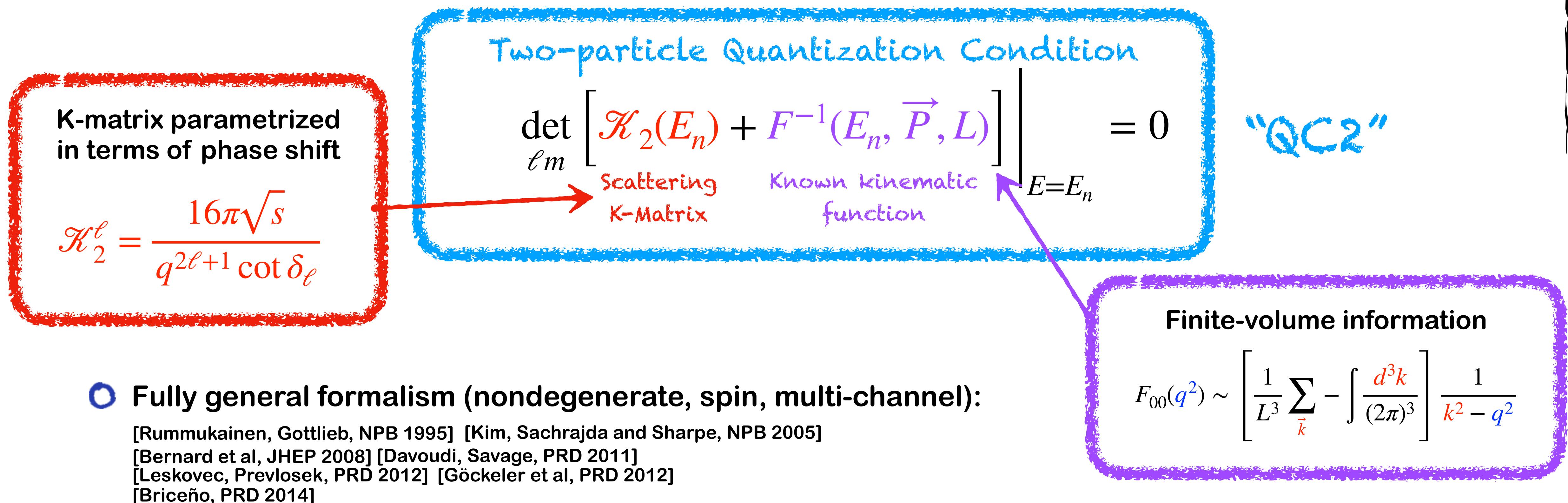
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Example application

Two pions in s-wave

$$\mathcal{K}_2^{s\text{-wave}}(E_n) = \frac{-1}{F_{00}(E_n, \vec{P}, L)}$$

one
energy
level



a phase
shift
point

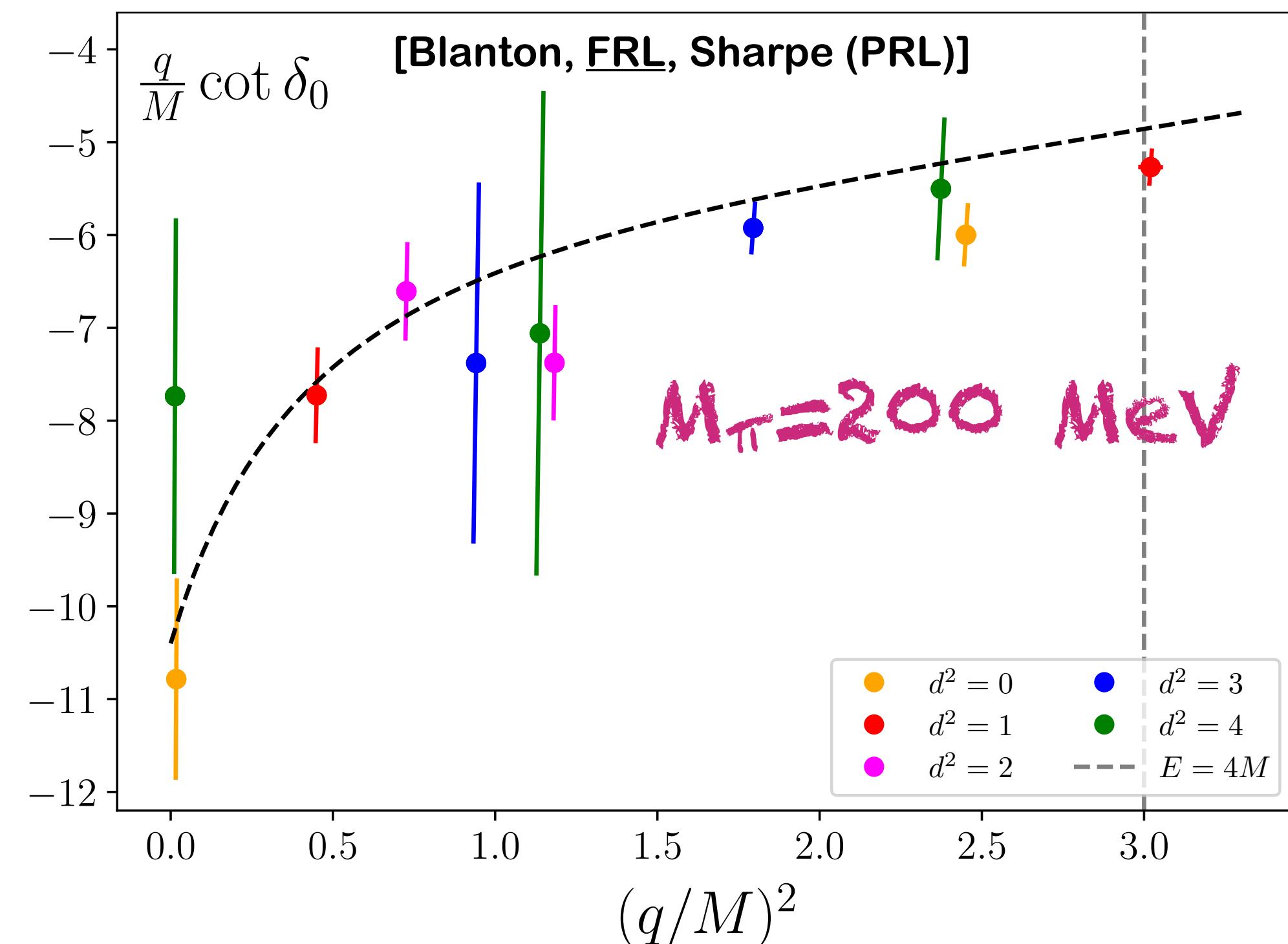
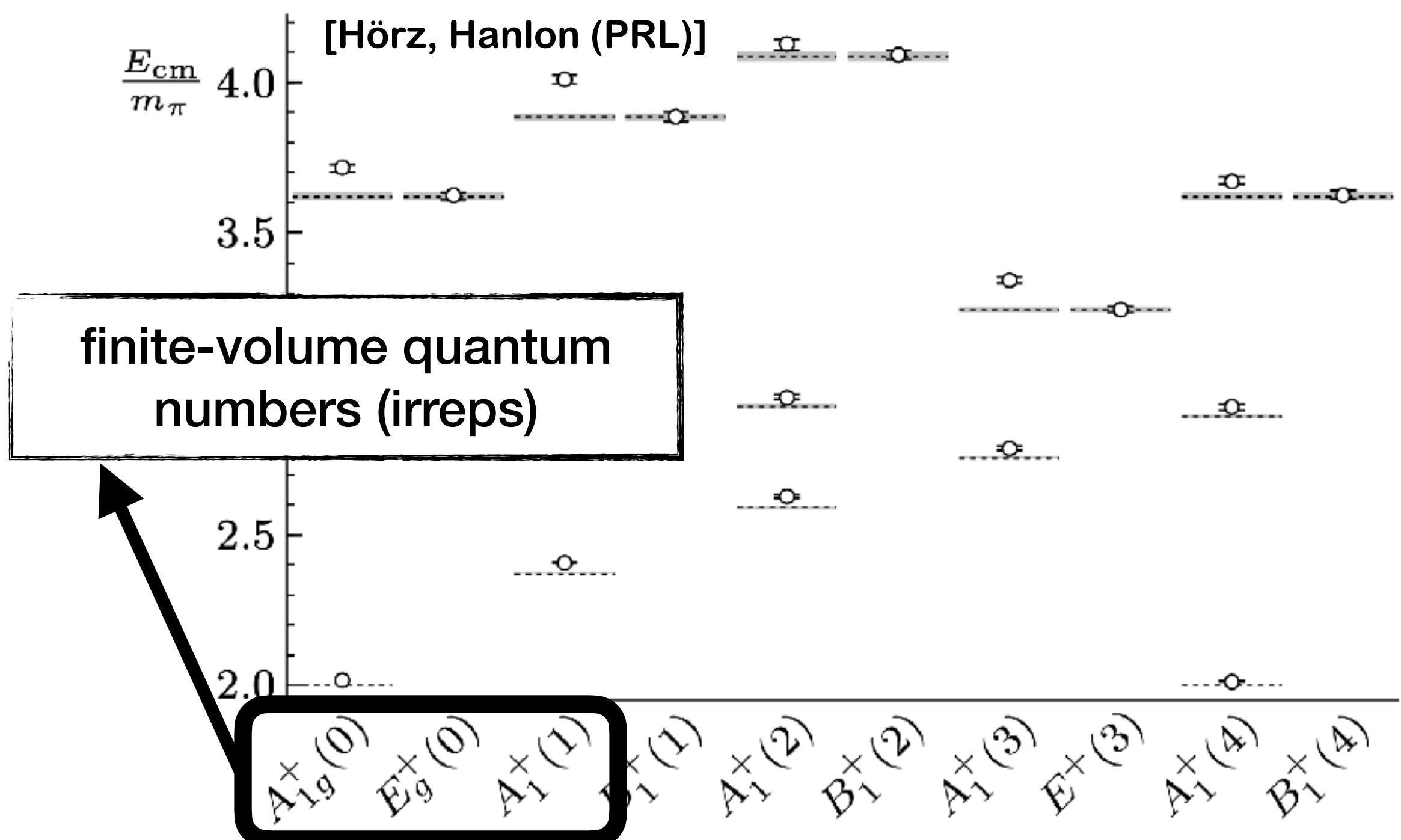
Example application

Two pions in s-wave

$$\mathcal{K}_2^{s-wave}(E_n) = \frac{-1}{F_{00}(E_n, \vec{P}, L)}$$

one
energy
level

a phase shift point



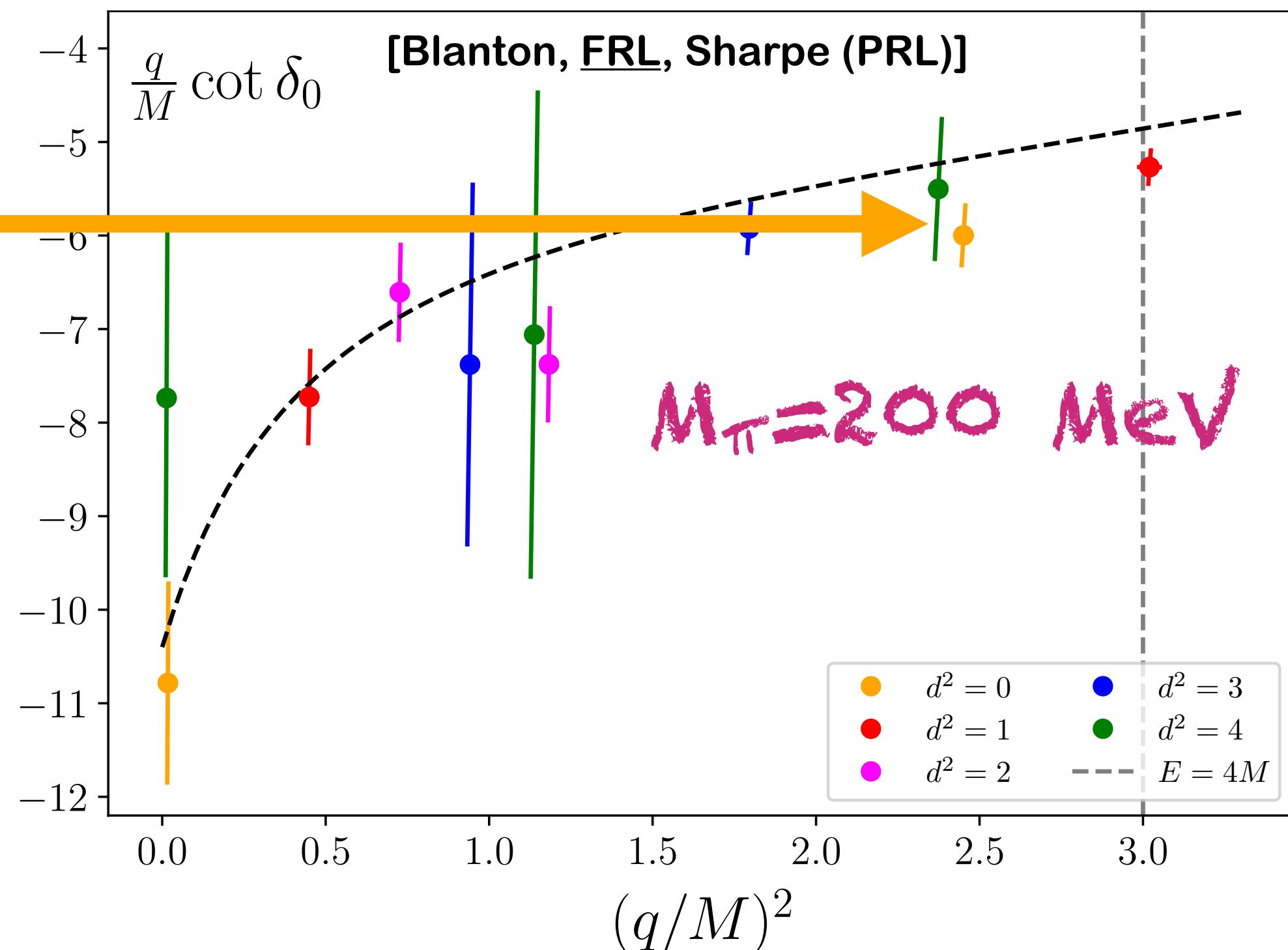
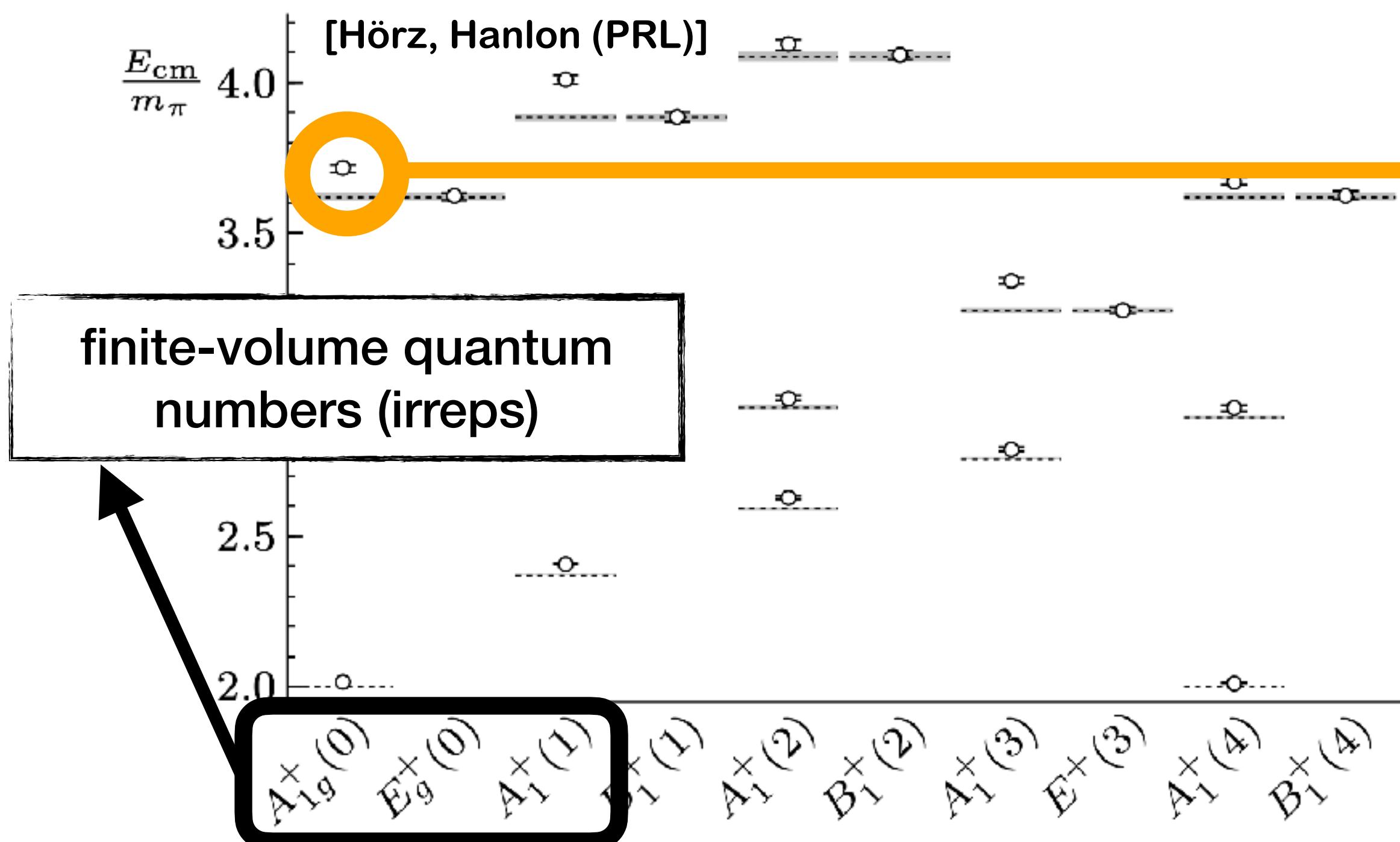
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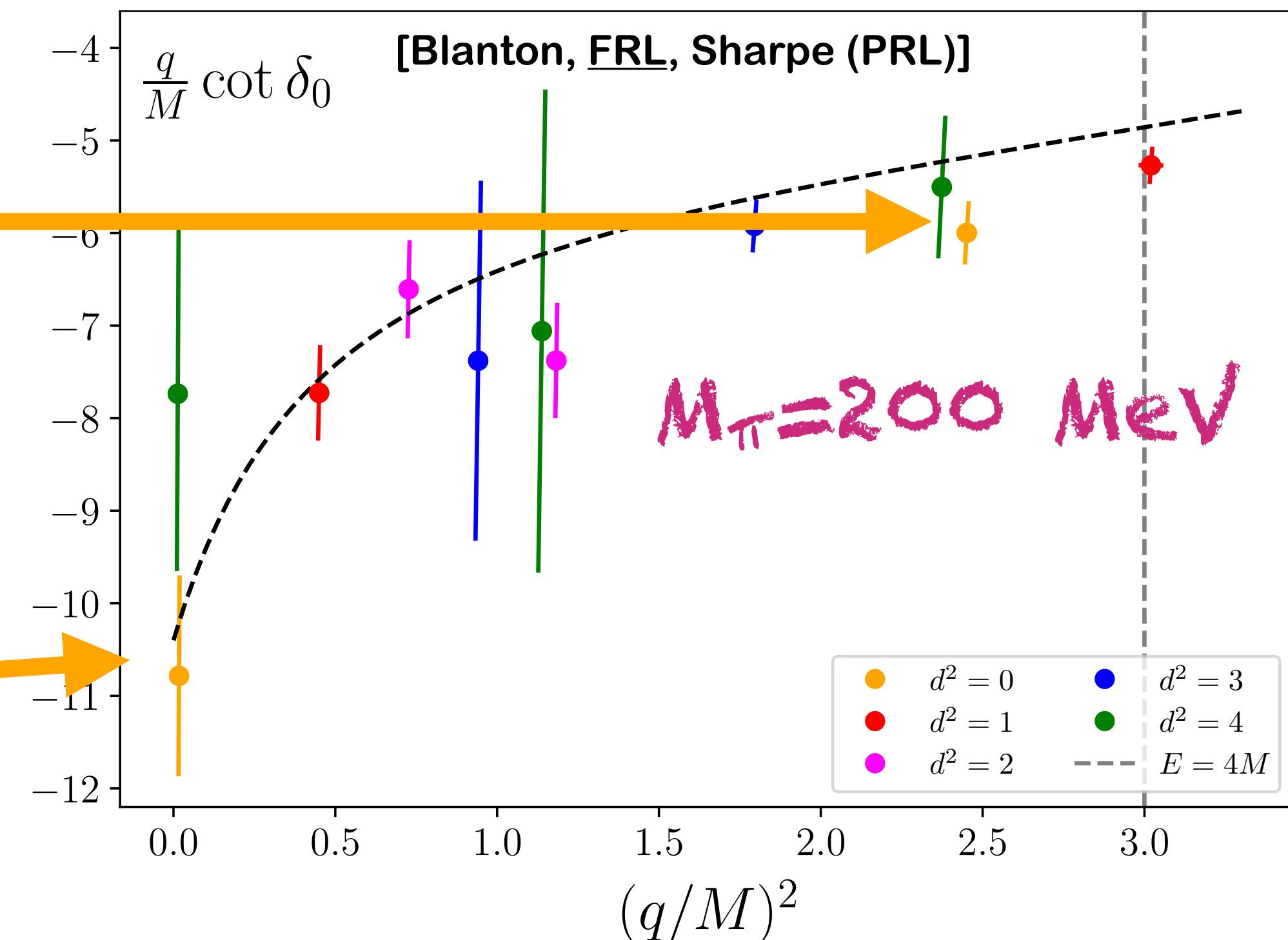
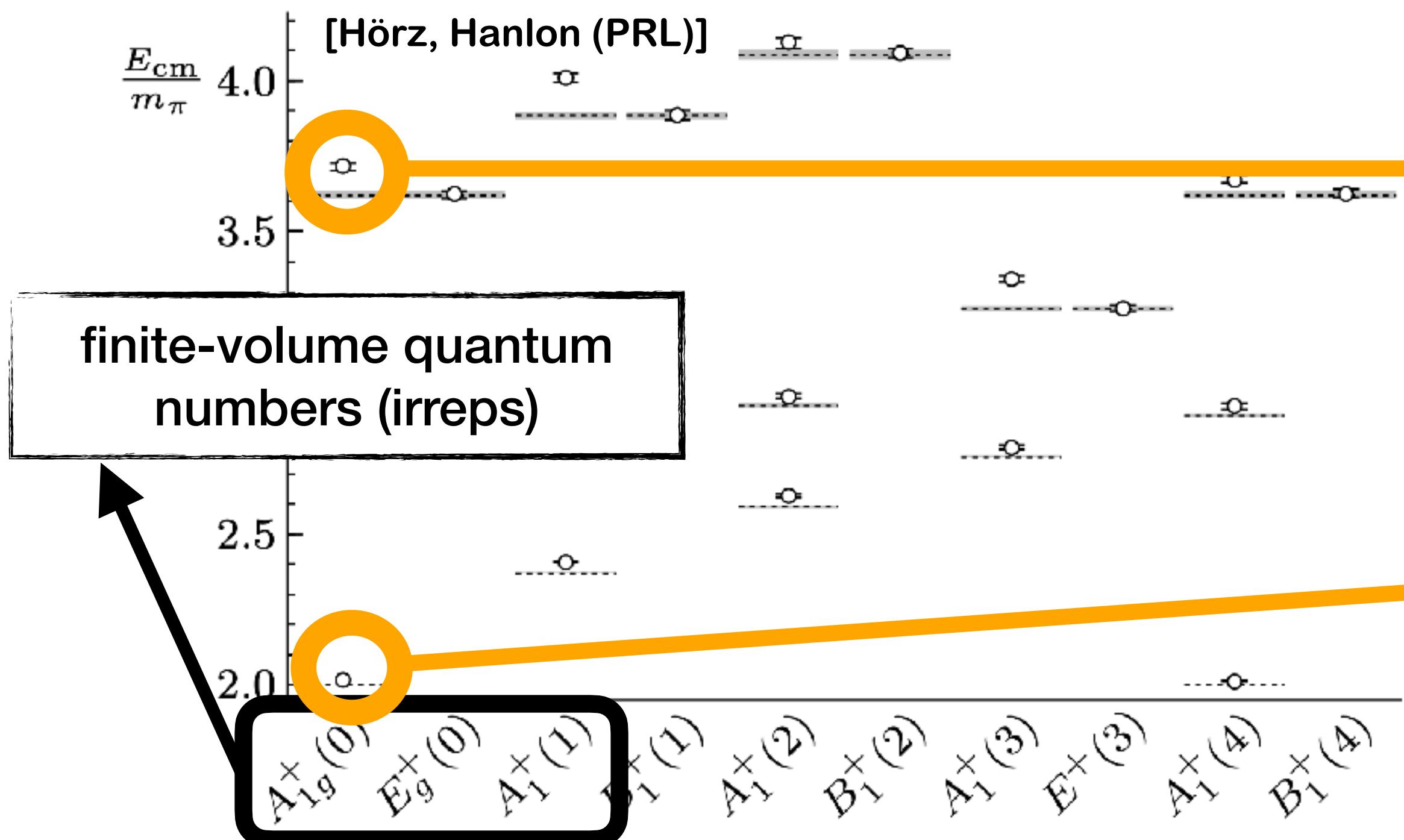
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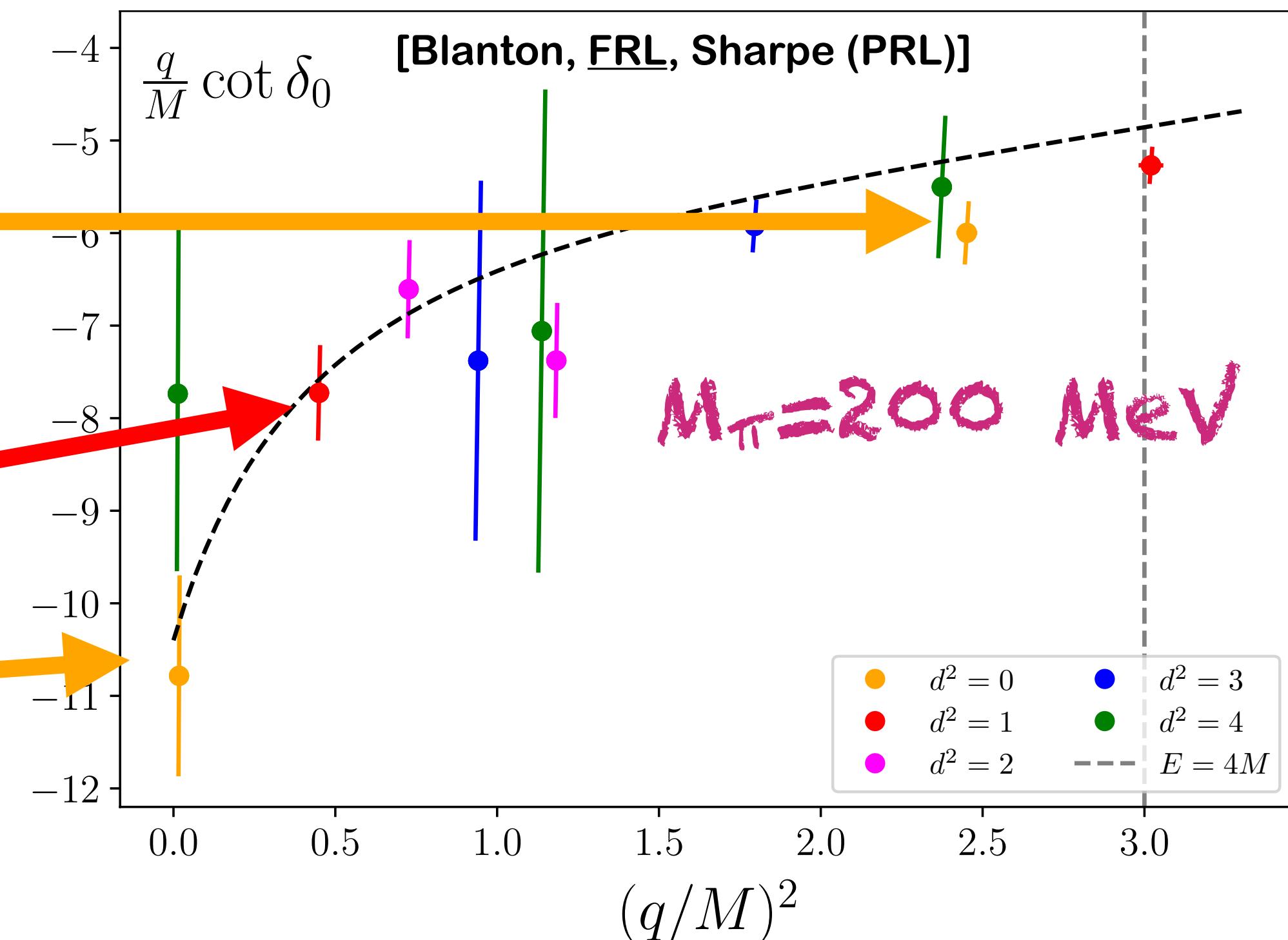
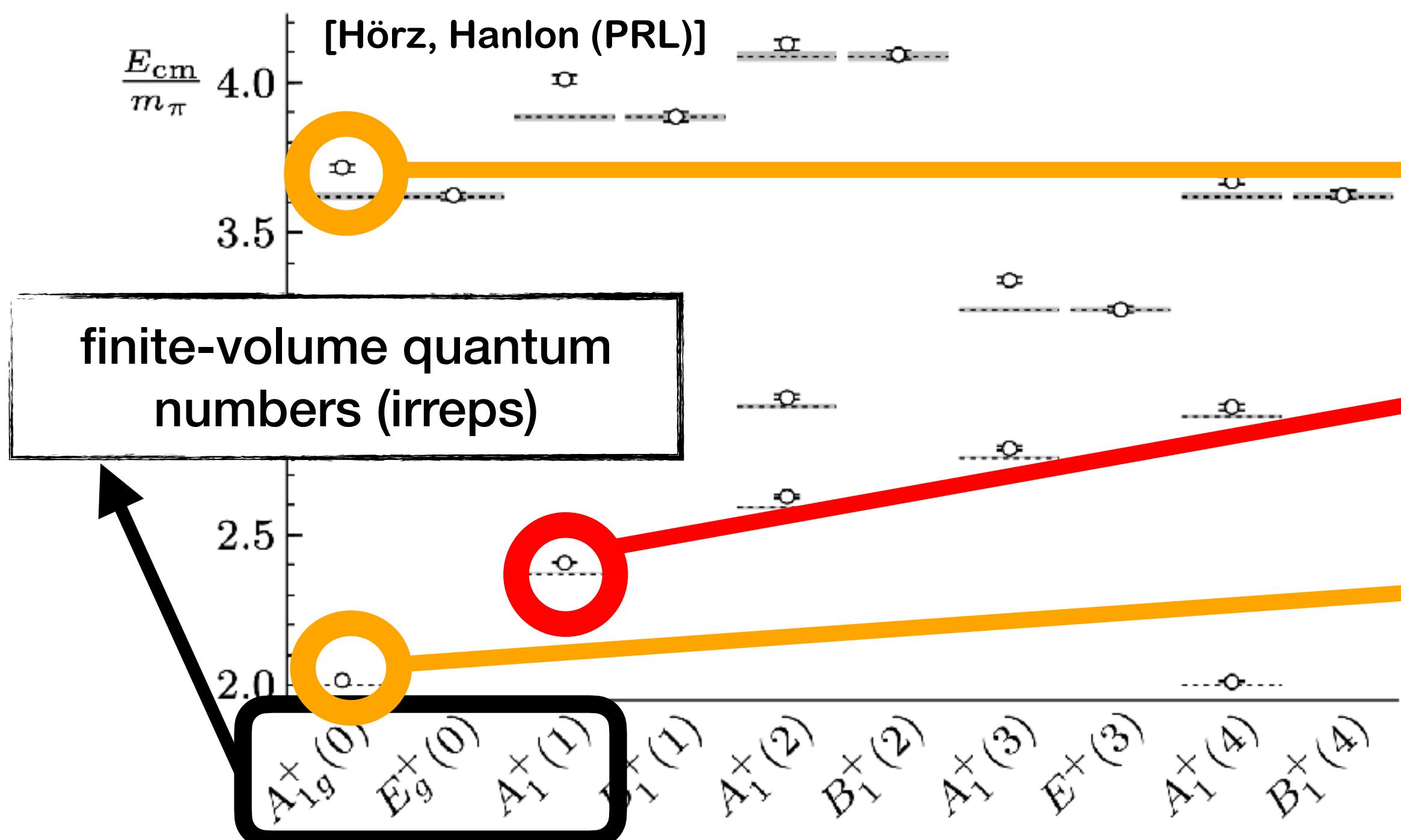
Example application

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$$\mathcal{K}_2^{s\text{-wave}}(E_n) = \frac{-1}{F_{00}(E_n, \vec{P}, L)}$$

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Exercise

EXERCISE Code up the quantization condition and solve it for a system with:

$$mL = 4, \quad \frac{k}{m} \cot \delta_0 = -\frac{1}{ma_0}, \quad ma_0 = 0.1$$

You can use an easy implementation of the zeta function, and this form of the QC2:

$$\mathcal{Z}_0(k^2) = \lim_{\Lambda \rightarrow \infty} \sum_{\vec{n} \in \mathbb{Z}}^{| \vec{n} | < \Lambda} \frac{1}{\vec{n}^2 - (\frac{kL}{2\pi})^2} - 4\pi\Lambda \quad k \cot \delta_0(k^2) = \frac{1}{\pi L} \mathcal{Z}_0(k^2, L)$$

What is the ground-state energy of the system in the frame $\vec{P} = 0$?

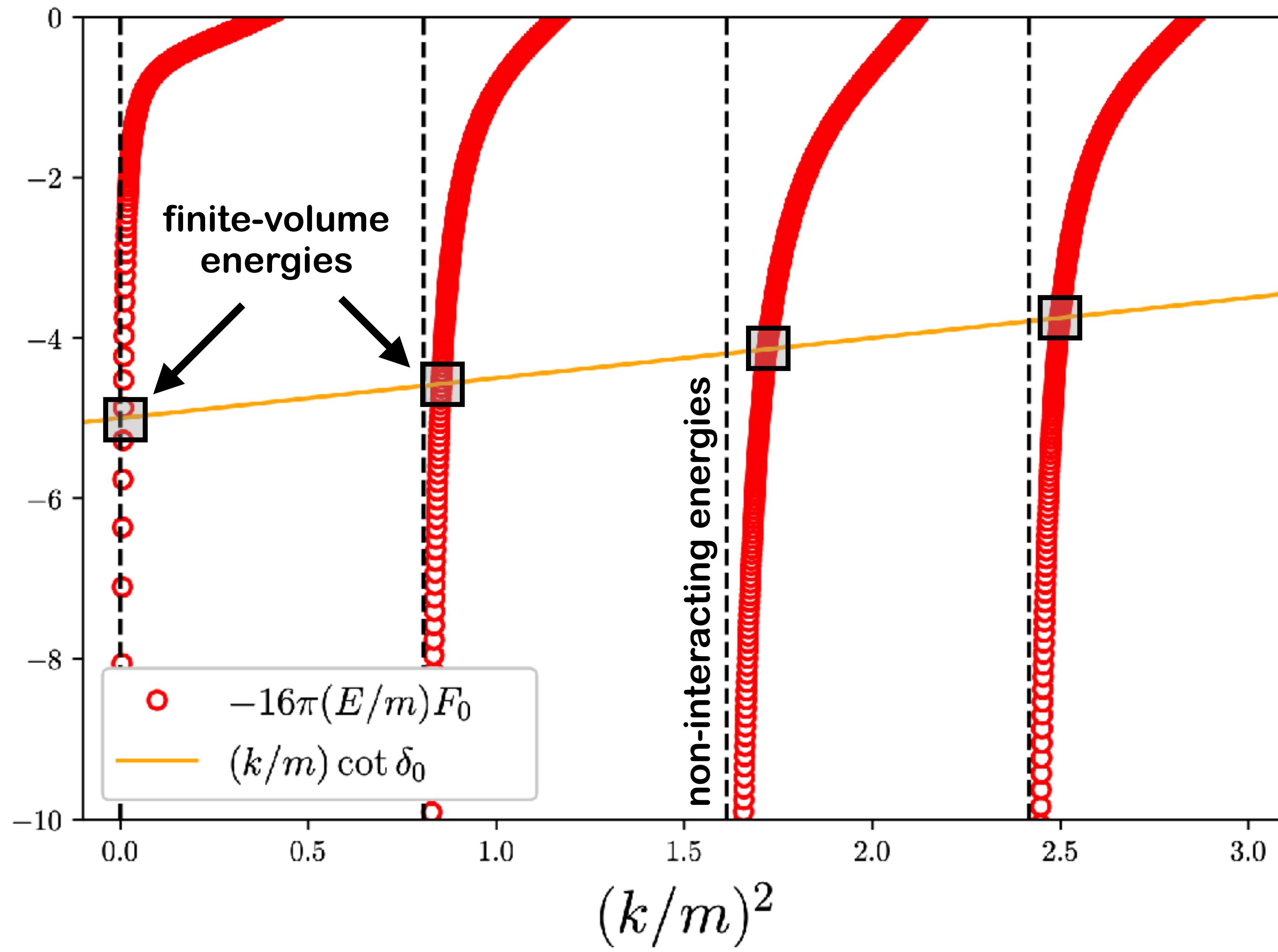
How does it compare to the 1/L expansion?

Solving the QC2

Leading-partial wave approximation:

$$\text{QC2: } \frac{1}{\mathcal{K}_2^{\ell=0}(k^2)} = -F_0(k^2, L)$$

$$\mathcal{K}_2^{\ell=0} = \frac{16\pi\sqrt{s}}{k \cot \delta_0}$$



Summary (3)

- Volume dependence of multi-hadron states contains information about scattering amplitude
- Energy shift to the non-interaction theory can be related to scattering parameters

$$\Delta E_2 = E_2 - 2m = \frac{4\pi a_0}{mL^3} \left\{ 1 + c_1 \left(\frac{a_0}{L} \right) + c_2 \left(\frac{a_0}{L} \right)^2 \right\} + \mathcal{O}(L^{-6})$$

- Two-body quantization condition relates the spectrum to amplitude in generic systems

$$k \cot \delta_0(k^2) = \frac{1}{\pi L} \mathcal{Z}_0(k^2, L)$$

Selection of recent results

The $\Lambda(1405)$

$$\begin{pmatrix} \pi\Sigma \rightarrow \pi\Sigma & \pi\Sigma \rightarrow Kp \\ Kp \rightarrow \pi\Sigma & Kp \rightarrow Kp \end{pmatrix}$$

$$\det_{\ell m} \left[\begin{pmatrix} \tilde{K}_{\pi\Sigma \rightarrow \pi\Sigma} & \tilde{K}_{\pi\Sigma \rightarrow KN} \\ \tilde{K}_{KN \rightarrow \pi\Sigma} & \tilde{K}_{KN \rightarrow KN} \end{pmatrix} + \begin{pmatrix} F_{\pi\Sigma}^{-1}(E_n, \vec{P}, L) & 0 \\ 0 & F_{KN}^{-1}(E_n, \vec{P}, L) \end{pmatrix} \right] = 0$$

Multi-channel K-Matrix

Zeta function

$\Lambda(1405) 1/2^-$

$I(J^P) = 0(\frac{1}{2}^-)$ Status: ****

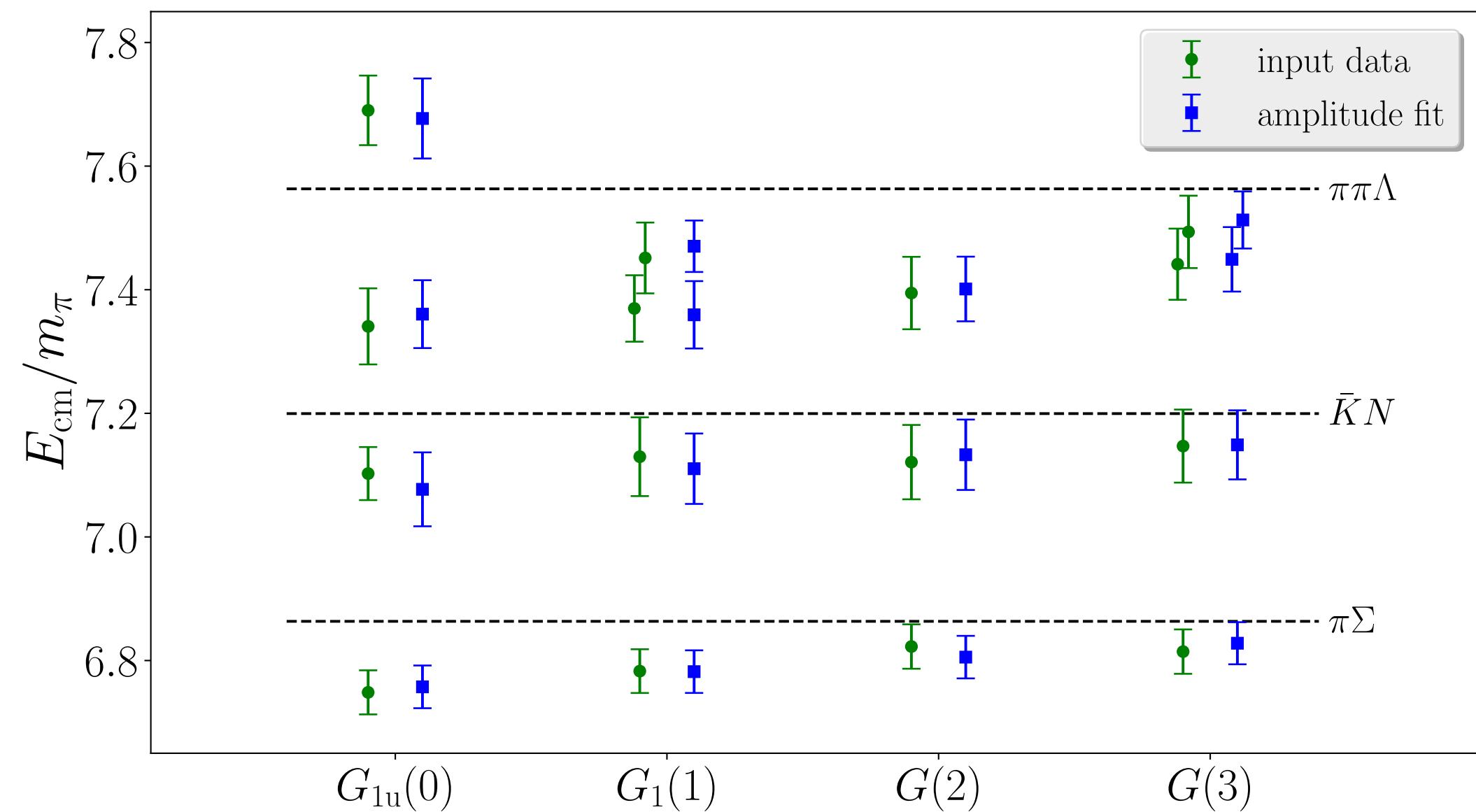


$\Lambda(1380) 1/2^-$

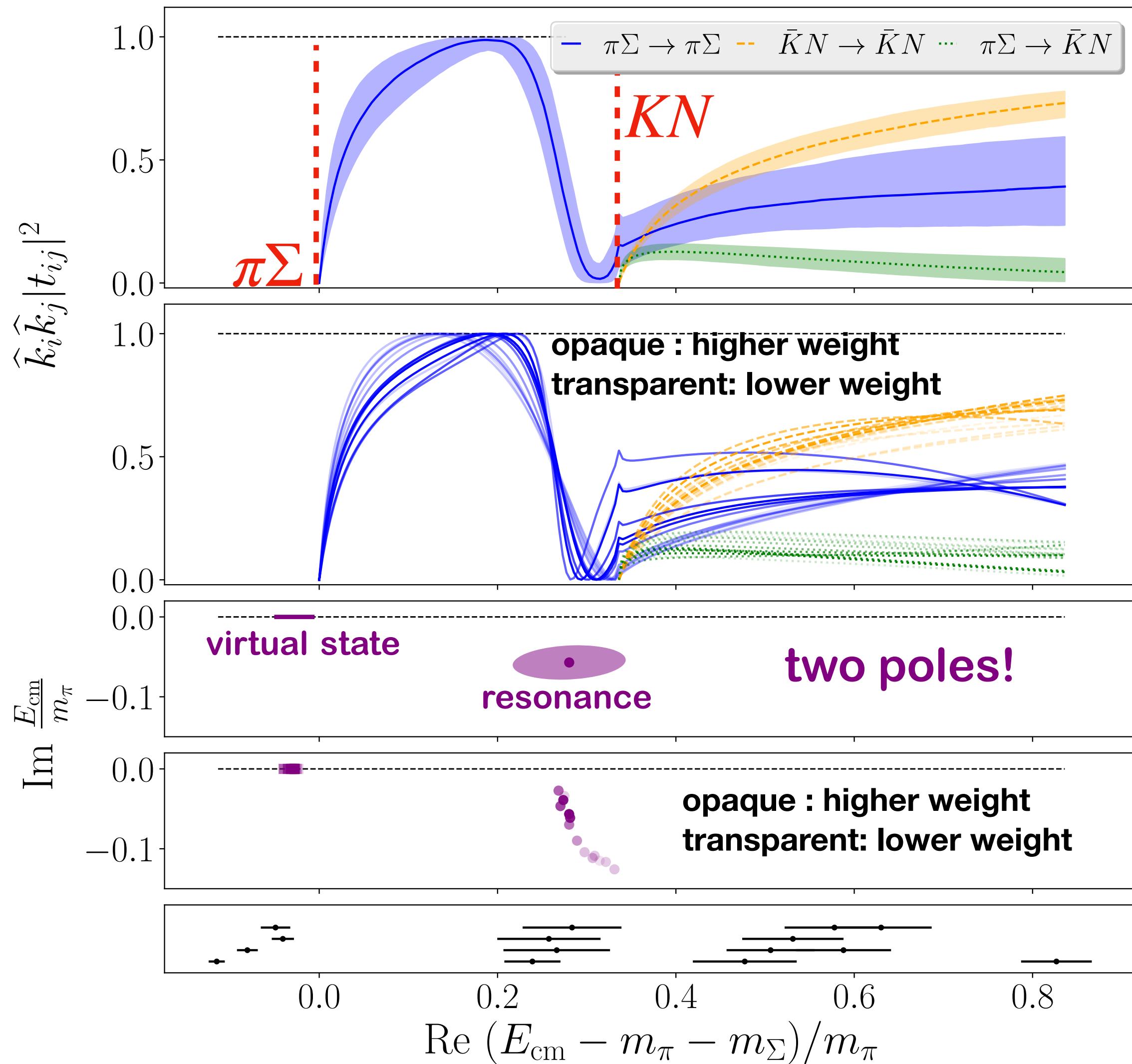
$J^P = \frac{1}{2}^-$ Status: **



- **** Existence is certain.
- *** Existence is very likely.
- ** Evidence of existence is fair.
- * Evidence of existence is poor.



The $\Lambda(1405)$



▶ Scattering amplitudes for “preferred” fit
i.e. with lowest AIC = $\chi^2 - 2 \text{ dof}$

▶ Scattering amplitudes for different parametrizations

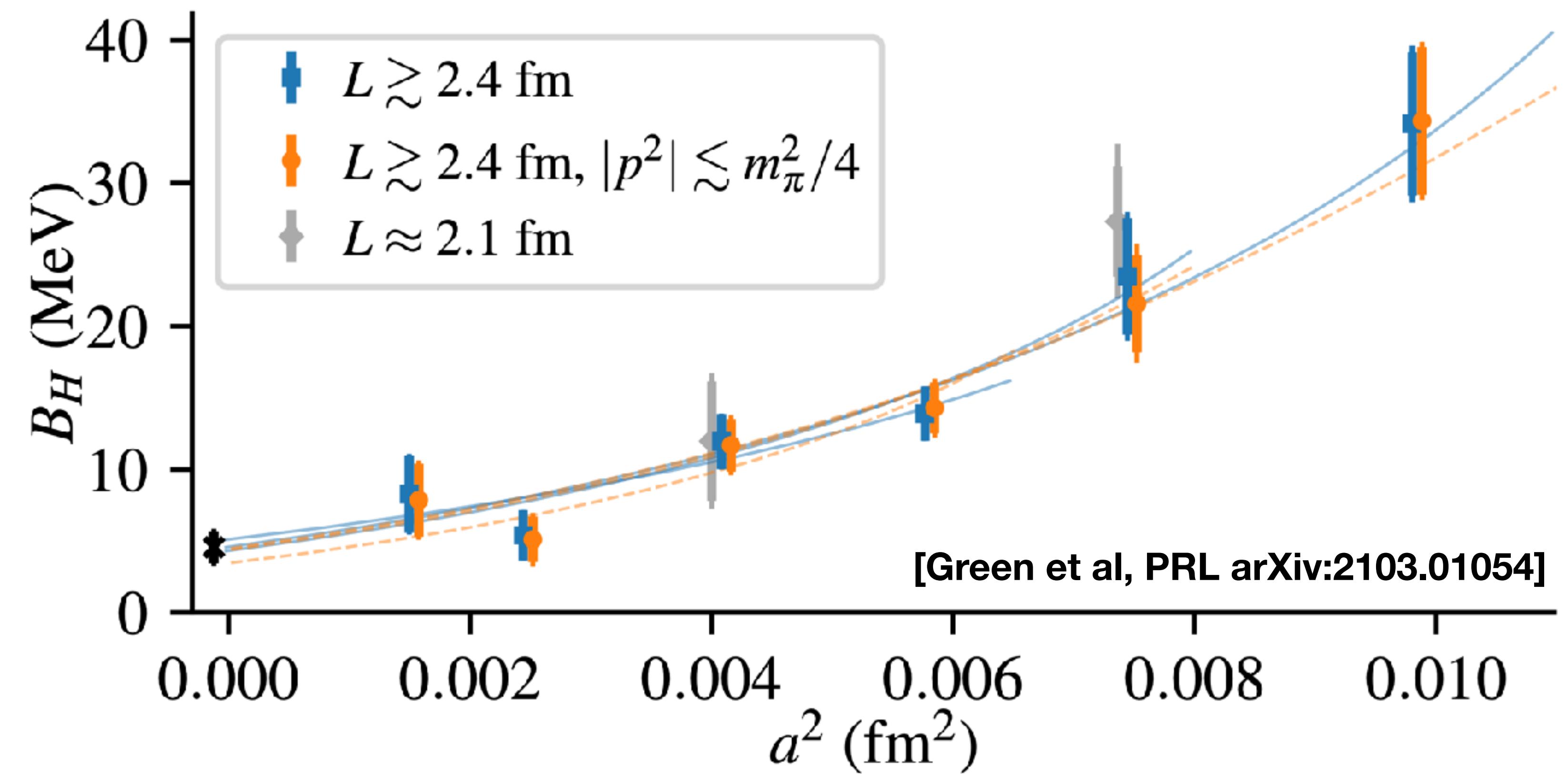
▶ Pole positions for “preferred” fit

▶ Pole positions for different parametrization
All find two poles!

▶ Lattice QCD energies used in fits

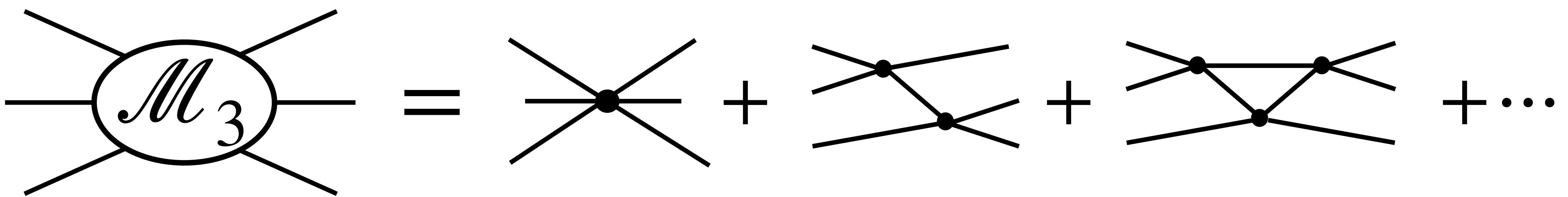
Continuum Limit of H dibaryon

- H dibaryon is a $\Lambda\Lambda$ bound state (uuddss quarks)



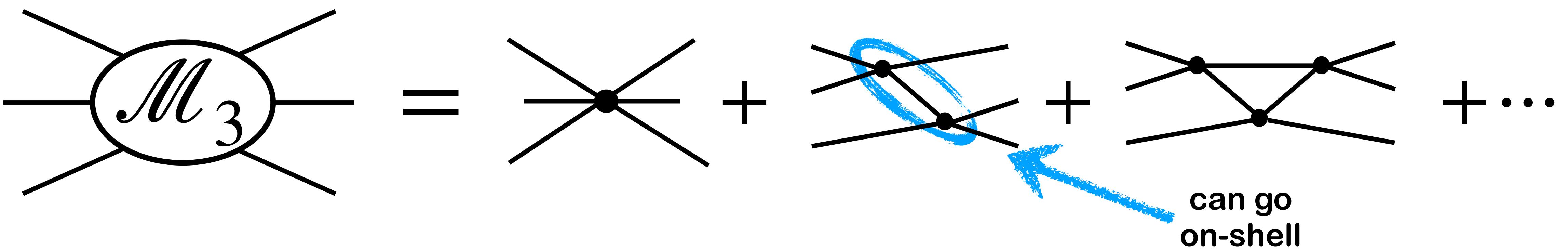
Three-hadron amplitudes

Qualitatively more complicated than the two-particle case!



Three-hadron amplitudes

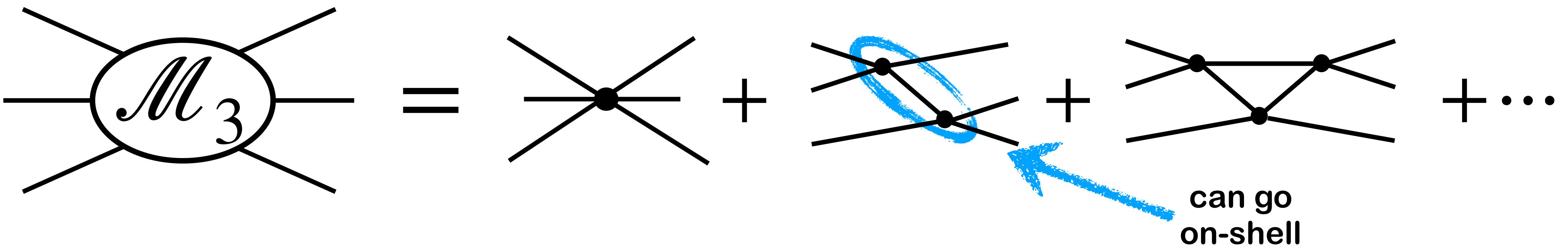
Qualitatively more complicated than the two-particle case!



- Three-particle scattering amplitudes can be divergent for specific kinematics.

Three-hadron amplitudes

Qualitatively more complicated than the two-particle case!



- Three-particle scattering amplitudes can be divergent for specific kinematics.
- They depend also on two-to-two interactions.
 - But any separation between “two-particle” and “three-particle” effects is not well-defined

Quantization Condition

Skeleton expansion

$$C_L = \begin{aligned} & \text{Diagram 1: } B_3 + B_3 B_3 + B_3 B_3 B_3 + \dots \\ & + \text{Diagram 2: } B_2 + B_2 B_2 + \dots \\ & + \text{Diagram 3: } B_2 B_2 + B_2 B_2 B_2 + \dots \\ & + \text{Diagram 4: } B_2 B_2 B_2 + B_2 B_2 B_2 B_2 + \dots \\ & + \dots \\ & + \text{Diagram 5: } B_2 B_3 + B_2 B_2 B_3 + \dots \end{aligned}$$

The equation shows a skeleton expansion for C_L . It consists of a sum of terms, each represented by a diagram. The diagrams are composed of nodes (diamonds and circles) connected by horizontal and vertical lines. Red vertical lines indicate boundaries between different components. Some nodes contain labels like B_2 or B_3 . Ellipses (\dots) are used to indicate repeated patterns.

Quantization Condition

Skeleton expansion

$$C_L = \begin{aligned} & \text{Diagram with } B_3 \text{ in red dashed box} + \text{Diagram with } B_3 \text{ in red dashed box} + \text{Diagram with } B_3 \text{ in red dashed box} + \dots \\ & + \text{Diagram with } B_2 \text{ in red dashed box} + \text{Diagram with } B_2 \text{ in red dashed box} + \dots \\ & + \text{Diagram with } B_2 \text{ in blue dashed box} + \text{Diagram with } B_2 \text{ in blue dashed box} + \dots \\ & + \text{Diagram with } B_2 \text{ in blue dashed box} + \text{Diagram with } B_2 \text{ in blue dashed box} + \dots \\ & + \dots \\ & + \text{Diagram with } B_2 \text{ in red dashed box} + \text{Diagram with } B_3 \text{ in red dashed box} + \dots \end{aligned}$$

Separation of finite and infinite volume terms:

$$= C_\infty(P) + A_3 \frac{1}{\mathcal{K}_{df,3} + F_3^{-1}} A'_3 + O(e^{-mL})$$

Easier derivation: Blanton, Sharpe [2007.16188]

Quantization Condition

Skeleton expansion

$$C_L = \begin{aligned} & \text{Diagram with } B_3 \text{ in the center} \\ & + \text{Diagram with } B_3 \text{ in the center} \\ & + \dots \\ & + \text{Diagram with } B_2 \text{ in the center} \\ & + \text{Diagram with } B_2 \text{ in the center} \\ & + \dots \\ & + \text{Diagram with } B_2 \text{ and } B_2 \text{ on the left} \\ & + \text{Diagram with } B_2 \text{ and } B_2 \text{ on the left} \\ & + \dots \\ & + \text{Diagram with } B_2 \text{ and } B_2 \text{ on the left} \\ & + \text{Diagram with } B_2 \text{ and } B_2 \text{ on the left} \\ & + \dots \\ & + \text{Diagram with } B_2 \text{ and } B_3 \text{ on the left} \\ & + \text{Diagram with } B_2 \text{ and } B_3 \text{ on the left} \\ & + \dots \end{aligned}$$

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Three-particle Quantization Condition
for identical scalars with G-parity

$$\det \left[\mathcal{K}_3(E) + F_3^{-1}(E, \vec{P}, L) \right] = 0$$

"QC3"

Easier derivation: Blanton, Sharpe [2007.16188]

Quantization Condition

Skeleton expansion

$$C_L = \begin{aligned} & \text{Diagram with } B_3 \text{ in a loop} + \text{Diagram with } B_3 \text{ in a loop} + \text{Diagram with } B_3 \text{ in a loop} + \dots \\ & + \text{Diagram with } B_2 \text{ in a loop} + \text{Diagram with } B_2 \text{ in a loop} + \dots \\ & + \text{Diagram with } B_2 \text{ and } B_2 \text{ in a loop} + \text{Diagram with } B_2 \text{ and } B_2 \text{ in a loop} + \dots \\ & + \text{Diagram with } B_2 \text{ and } B_2 \text{ in a loop} + \text{Diagram with } B_2 \text{ and } B_2 \text{ in a loop} + \dots \\ & + \dots \\ & + \text{Diagram with } B_2 \text{ and } B_3 \text{ in a loop} + \text{Diagram with } B_2 \text{ and } B_3 \text{ in a loop} + \dots \end{aligned}$$

Separation of finite and infinite volume terms:

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Three-particle Quantization Condition
for identical scalars with G-parity

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“QC3”

Easier derivation: Blanton, Sharpe [2007.16188]

“Formally” similar to the two-particle case

Quantization Condition

Skeleton expansion

$$C_L = \begin{aligned} & \text{Diagram with } B_3 \text{ in the first box} + \text{Diagram with } B_3 \text{ in the second box} + \dots \\ & + \text{Diagram with } B_2 \text{ in the first box} + \text{Diagram with } B_2 \text{ in the second box} + \dots \\ & + \text{Diagram with } B_2 \text{ in the first box and } B_2 \text{ in the second box} + \dots \\ & + \dots \\ & + \text{Diagram with } B_2 \text{ in the first box} + \dots \end{aligned}$$

Unfortunately, it is
not so simple!

Easier derivation: Blanton, Sharpe [2007.16188]

Separation of finite and infinite volume terms:

$$= C_\infty(P) + A_3 \frac{1}{\mathcal{K}_{df,3} + F_3^{-1}} A'_3 + O(e^{-mL})$$



Three-particle Quantization Condition
for identical scalars with G-parity

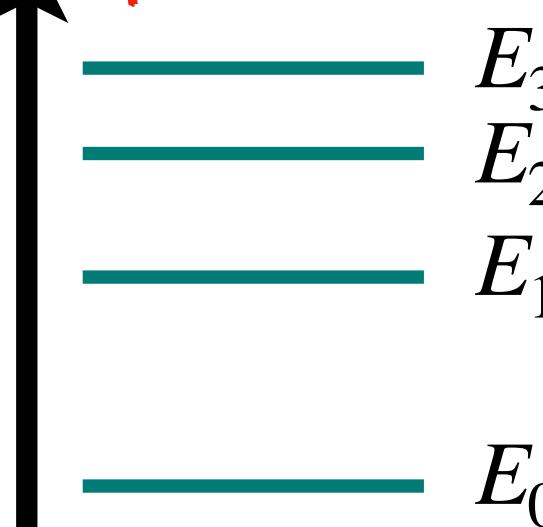
$$\det \left[\mathcal{K}_3(E) + F_3^{-1}(E, \vec{P}, L) \right] = 0$$

“QC3”

“Formally” similar to the two-particle case

Three-hadron formalism

2π and 3π
Spectrum



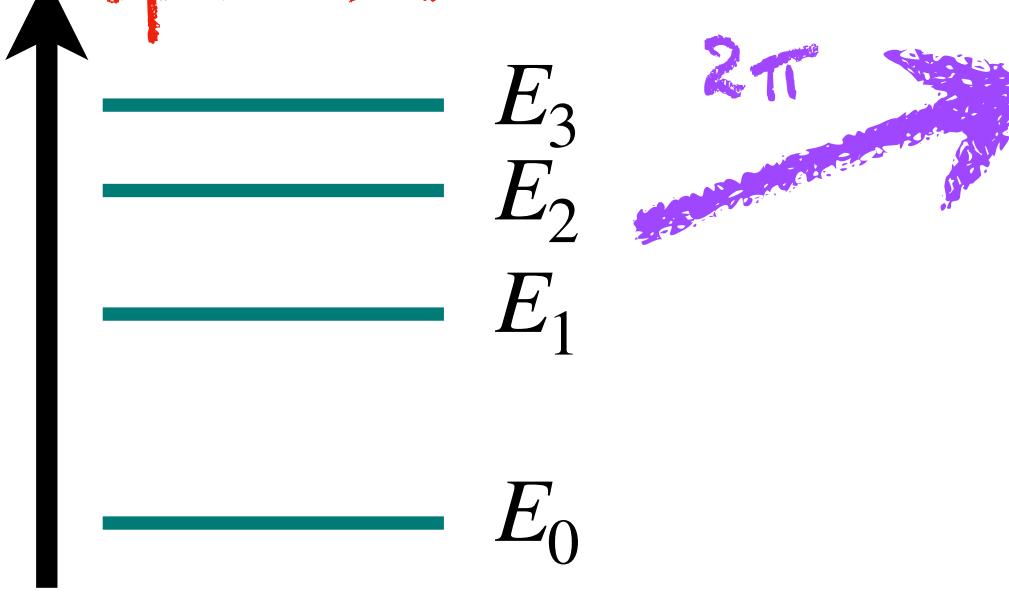
E_3
 E_2
 E_1
 E_0

1. Determine \mathcal{K}_2 and $\mathcal{K}_{df,3}$ from the two and three-pion spectrum

Hansen, Sharpe [arXiv:1408.5933]

Three-hadron formalism

2π and 3π
Spectrum

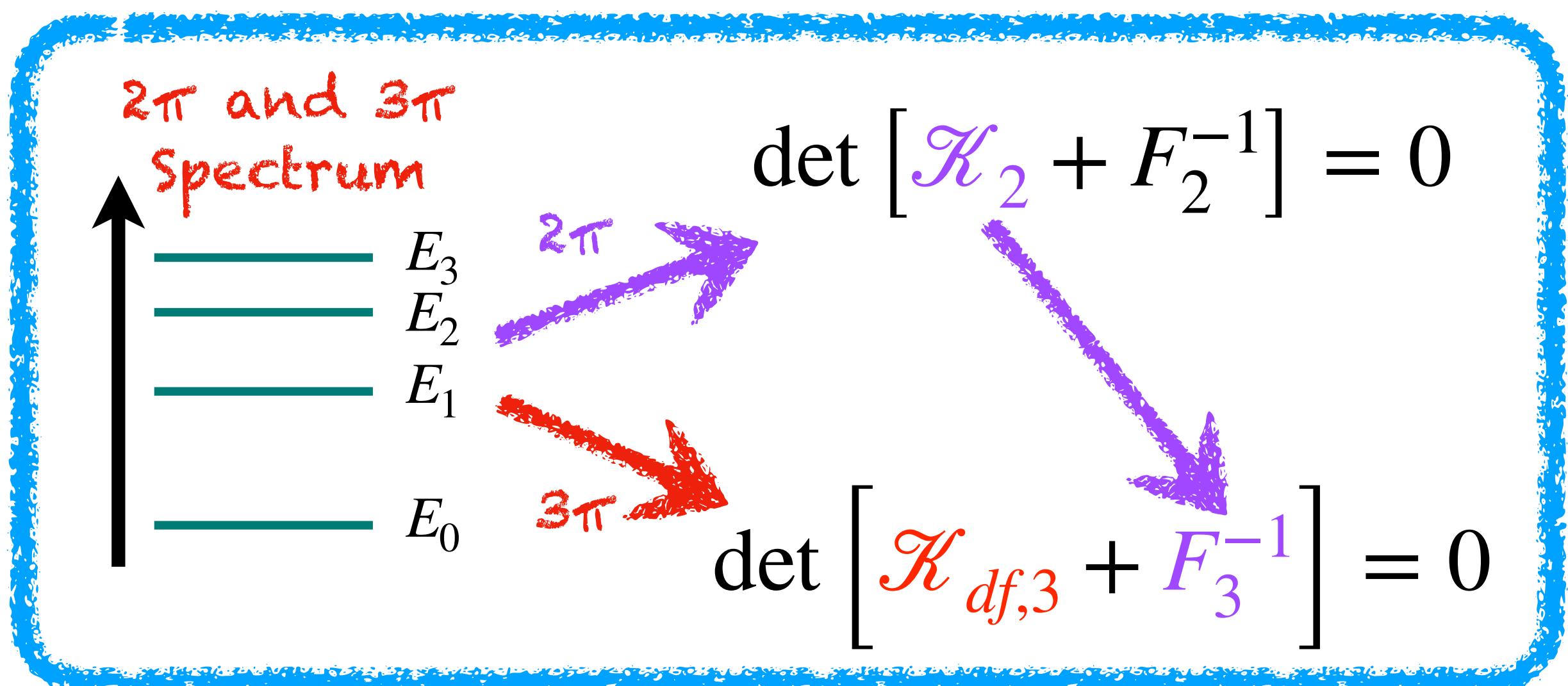


$$\det [\mathcal{K}_2 + F_2^{-1}] = 0$$

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Hansen, Sharpe [arXiv:1408.5933]

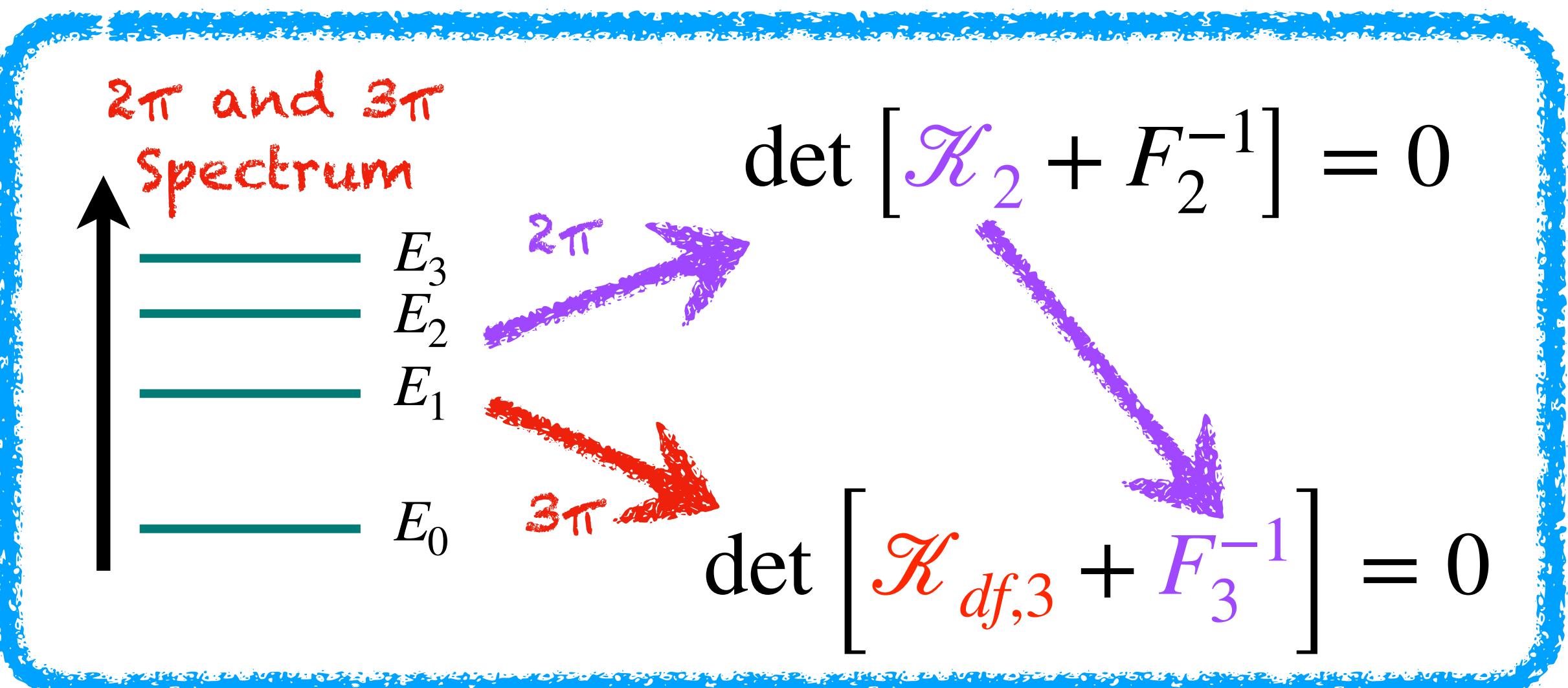
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Three-hadron formalism



1. Determine \mathcal{K}_2 and $\mathcal{K}_{df,3}$ from the two and three-pion spectrum

Hansen, Sharpe [arXiv:1408.5933]

2. Solve integral equations to obtain The physical three-to-three amplitude

Hansen, Sharpe [arXiv:1504.04248]

