EE 264: Digital Signal Processing

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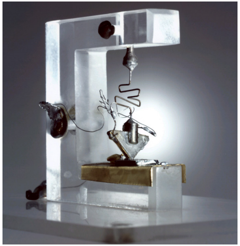
Why **Digital** Signal Processing?

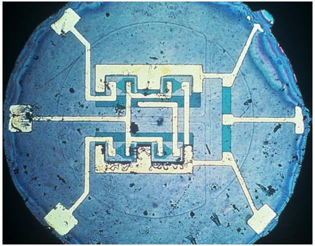
- Flexibility
- Accuracy
- Multi-purpose hardware
- Easy to implement sophisticated operations
- Today we have tremendous computer power

Why **Digital** Signal Processing?

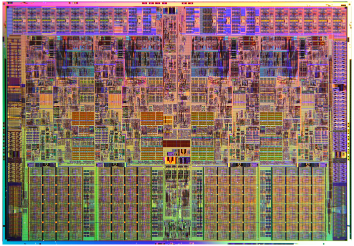
The development of low-cost and high-speed digital electronics paved the way for digital signal processing



First point contact transistor (1947)



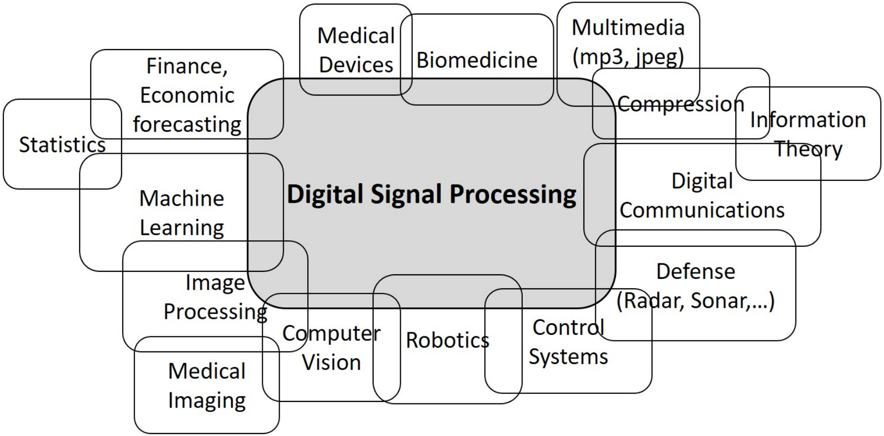
First integrated circuit (1961)



Modern processor (200X)

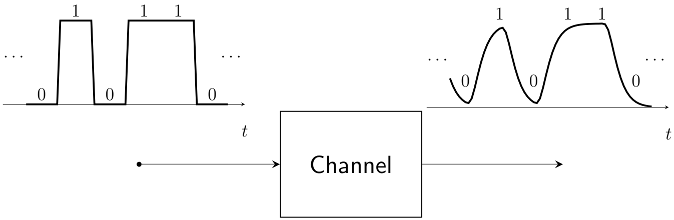
Why learn digital signal processing?

- Present in essentially all fields of modern EE
- Countless applications



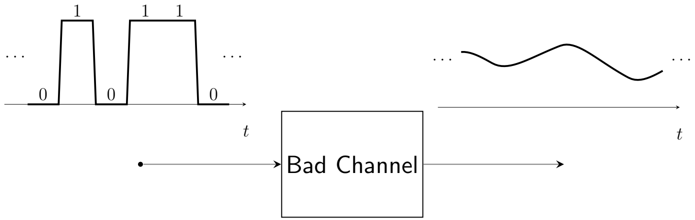
Example: digital communication

Problem:

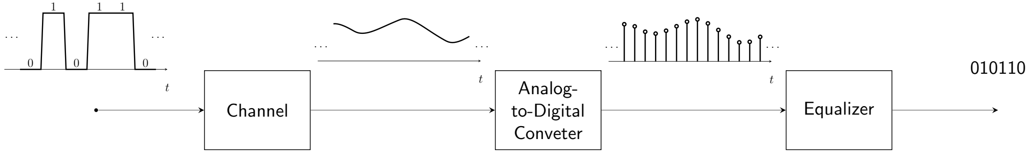


Example: digital communication

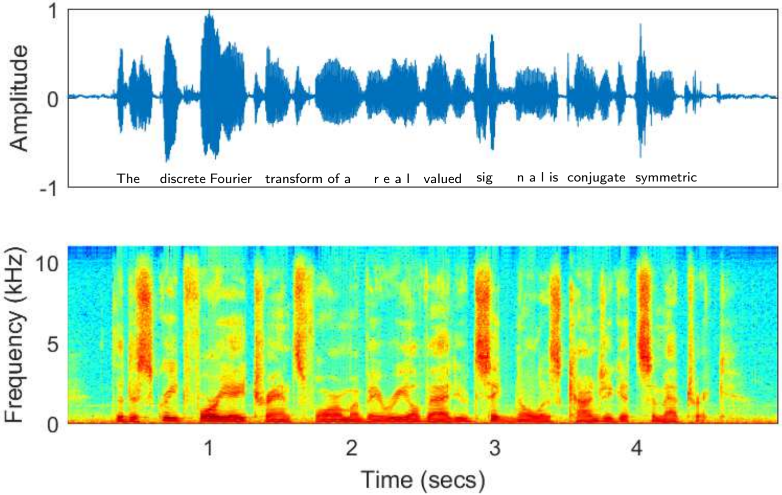
Problem:



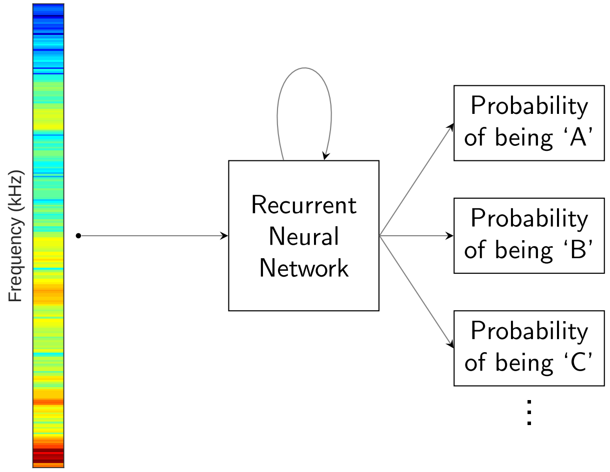
Solution:



Example: speech recognition



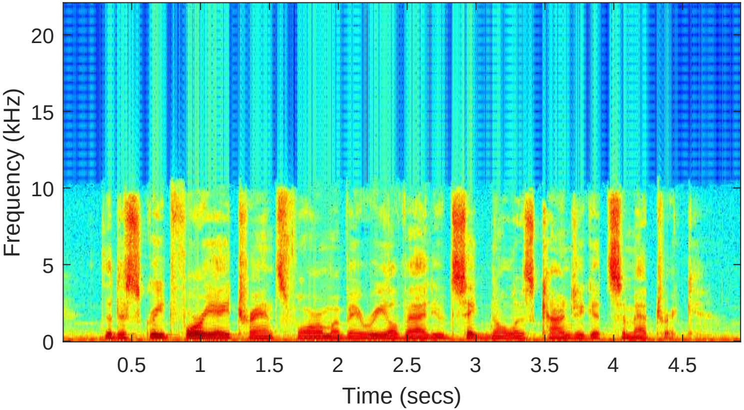
Example: speech recognition



Slice of the spectrogram

Example: speech recognition

Spectrogram of the same speech signal, now recorded with sampling rate of 44.1 kHz



More on spectrograms and short-time Fourier transform on lecture 11.

Digital processing of analog signals

Analog-to-digital converter (ADC)

- Performs filtering, sampling, and quantization
- Sampling rate may be of tens of kHz (audio processing), or it may be of tens of GHz (optical communications)

Digital signal processor

- Performs some operation e.g., filtering, FFT, etc
- May be implemented on PCs with 64-bit floating-point precision, or on ASICs with limited arithmetic precision (e.g., 6 bits).

Digital-to-analog converter (DAC)

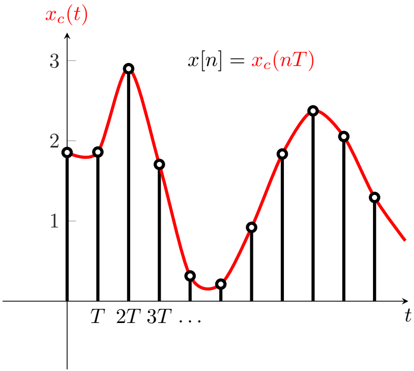
- Performs quantization and reconstruction (filtering)
- Sampling rate could be similar to ADC

Review

- Discrete-time signals and systems
- Discrete-time Fourier transform (DTFT)
- The z-transform
- Difference equations

Discrete-time signals

Discrete-time signals (or simply *sequences*) may be inherently discrete or they may be obtained by sampling a continuous-time signal. More about sampling on lectures 4 and 5.

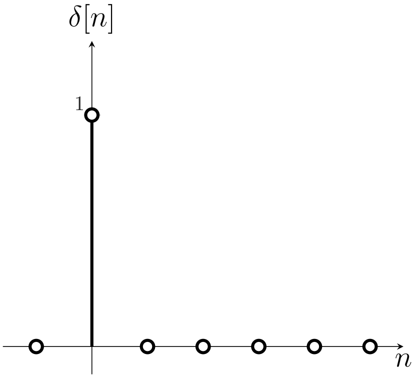


$x[n]$ is only defined for $n \in \mathbb{Z}$

Basic sequences

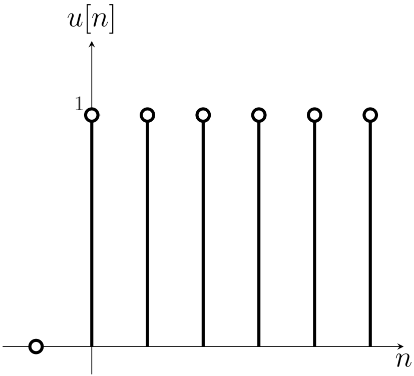
Unit impulse:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$



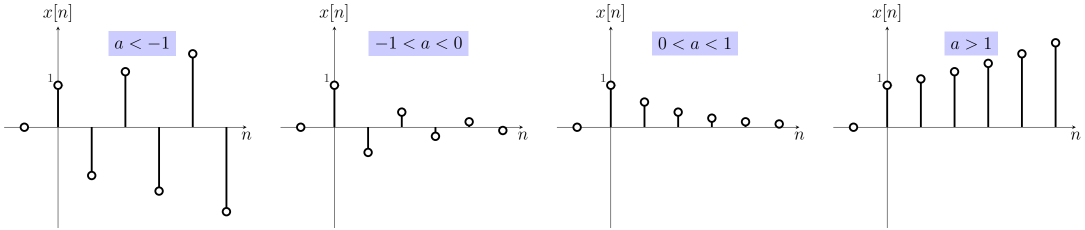
Unit step:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



Basic sequences

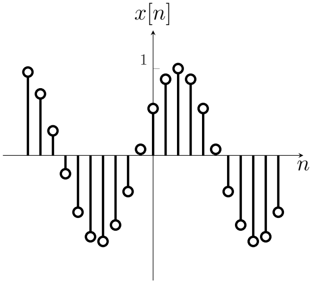
Right-sided exponential: $x[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$



- Exponential exhibits oscillatory behavior for $a < 0$.
- Exponential is **unbounded** if $|a| > 1$.

Basic sequences

Sinusoids: $x[n] = \cos(\omega_0 n + \phi)$



Differently from continuous-time sinusoids, discrete-time sinusoids...

- are periodic only if ω_0/π is rational.
- have maximum frequency $\omega = \pi$.

Complex exponentials:

$$e^{j(\omega_0 n + \phi)} = \cos(\omega_0 n + \phi) + j \sin(\omega_0 n + \phi) \qquad \text{(from Euler's equation)}$$

Classification of discrete-time signals

Energy and power signals

Energy signals have finite energy

$$E \equiv \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Power signals have infinite energy (e.g., periodic signals), but they have finite average power

$$P \equiv \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty$$

Symmetry

$$x[n] = x[-n] \qquad \text{(even symmetry)}$$

$$x[n] = -x[-n] \qquad \text{(odd symmetry)}$$

Any signal $x[n]$ can be decomposed as a sum of an even ($x_e[n]$) and an odd ($x_o[n]$) component:

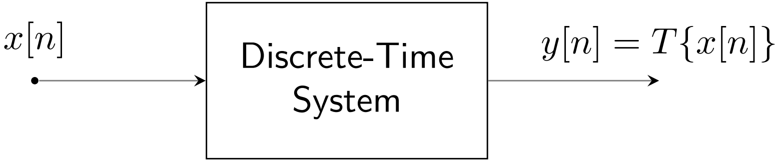
$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e[n] = \frac{1}{2}(x[n] + x[-n]) \qquad \text{(even component)}$$

$$x_o[n] = \frac{1}{2}(x[n] - x[-n]) \qquad \text{(odd component)}$$

Discrete-time systems



Some important properties

Linearity or superposition

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

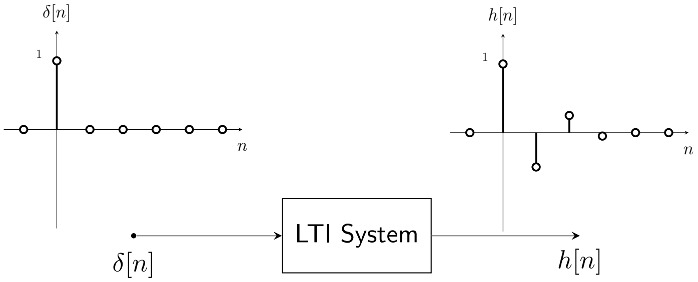
Time invariance or shift invariance

$$\text{If } d[n] = x[n - n_d], \text{then } T\{d[n]\} = y[n - n_d]$$

A time shift of the input causes an equal time shift of the output

Linear time-invariant (LTI) systems

LTI systems are **completely characterized** by their impulse response.



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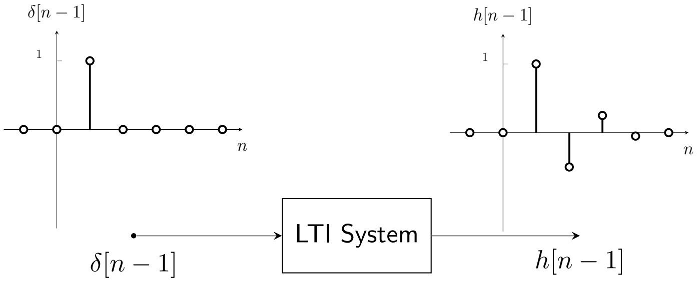

from time invariance

Linear time-invariant (LTI) systems

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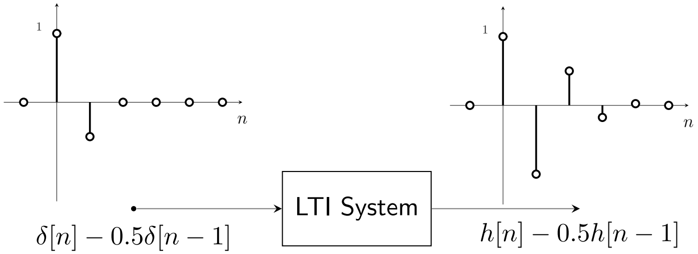

from linearity

Linear time-invariant (LTI) systems

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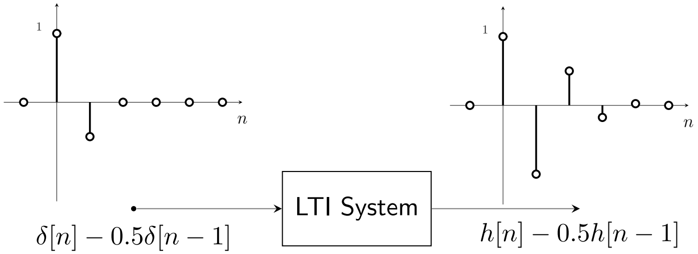


The convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Shorthand notation: $y[n] = x[n] * h[n]$ or $y[n] = (x * h)[n]$

Other properties of discrete-time systems

Memorylessness

A system is **memoryless** if its output at time n, $y[n]$, depends only on the present input $x[n]$.

Causality

A system is **causal** if its output at time n, $y[n]$, depends only on the present and past samples of the input $\{x[n], x[n-1], x[n-2], \ldots\}$. All physical systems are causal.

For **LTI systems**, causality implies $h[n] = 0, n < 0$

$$y[n] = \sum_{k=0}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{n} x[k]h[n-k] \qquad \text{(Convolution sum for causal systems)}$$

Stability

A system is **BIBO stable** if every bounded input ($|x[n]| < B_x < \infty$) produces a bounded output ($|y[n]| < B_y < \infty$)
For **LTI systems**,

$$
\begin{aligned}
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| && \text{(Convolution sum)} \\
&\leq \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| && \text{(Triangle inequality)} \\
&\leq B_x \sum_{k=-\infty}^{\infty} |h[k]| && (\text{Bounded input } |x[n]| < B_x < \infty) \\
&< B_y < \infty && \left(\text{only if } \sum_{k=-\infty}^{\infty} |h[k]| < \infty\right)
\end{aligned}
$$

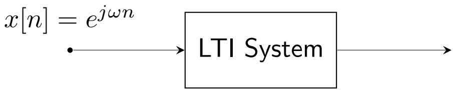
Therefore, an LTI system is BIBO stable only if its impulse response $h[k]$ is **absolute summable**.

Classify the following discrete-time systems

System	Linear	Time invariant	Memoryless	Causal	BIBO stable
Constant offset $y[n] = x[n] + C, C \neq 0$					
Time shift $y[n] = x[n - n_d]$					
Squaring $y[n] = x^2[n]$					
Accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$					
Compressor $y[n] = x[Mn], M > 1$					
Differentiator $y[n] = x[n] - x[n-1]$					
A difference equation $y[n] = x[n] + y[n-1]$					

Frequency-domain representation of LTI systems

Let's use the convolution sum to compute the output of an LTI system to a complex exponential:



$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right)}_{H(e^{j\omega})} e^{j\omega n} = H(e^{j\omega}) e^{j\omega n}$$

- $H(e^{j\omega})$ is a complex number that depends only on the impulse response of the LTI system
- The output of an LTI system to a complex exponential is also a complex exponential of same frequency, but with possibly different amplitude and phase

Conclusions

- Complex exponentials are **eigenfunctions** of LTI systems, and $H(e^{j\omega})$ are the corresponding eigenvalues.
- By representing signals as a sum of complex exponentials, we can readily calculate their output to an LTI system: $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

Discrete-time Fourier transform (DTFT)

Definition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{(Direct transform)}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{(Inverse transform)}$$

Important:

- The DTFT is periodic with period 2π: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$
- If $x[n]$ is **absolute summable** i.e., $\sum_{k=-\infty}^{\infty} |x[k]| < \infty$, then the DTFT exists. This is a *sufficient* condition, but it is *not a necessary* condition.
- Absolute summability also implies that the DTFT **converges uniformly** to a continuous function of ω.

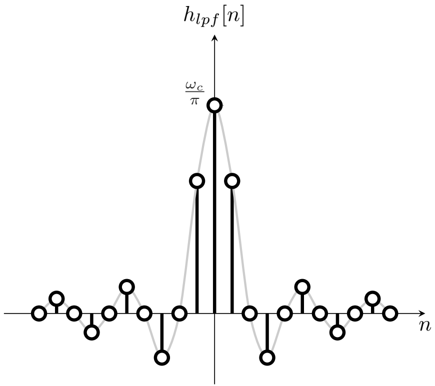
Example: the ideal lowpass filter

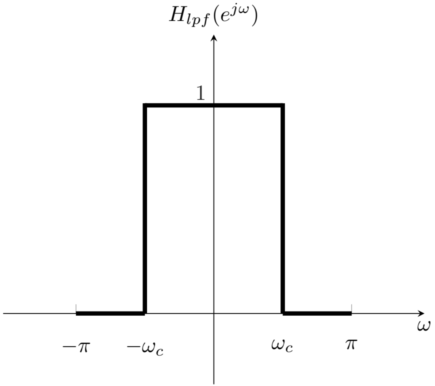
Time domain

$$h_{lpf}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c}{\pi} n\right)$$

Frequency domain

$$H_{lpf}(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$





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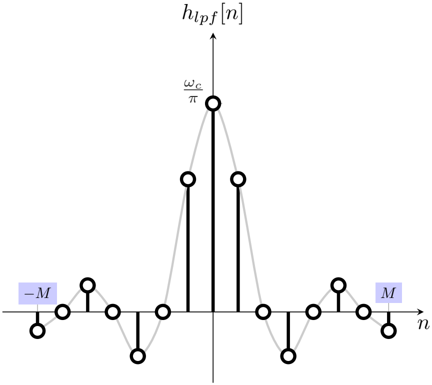
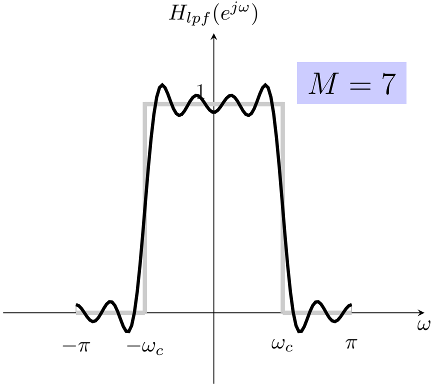


Frequency domain

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$



Example: the ideal lowpass filter

Time domain

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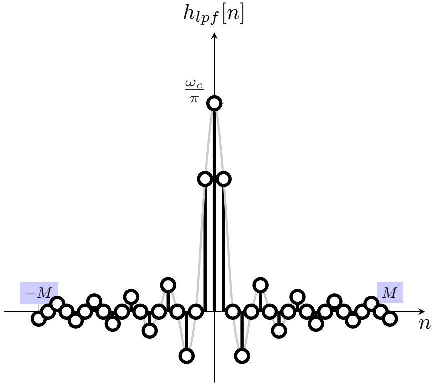
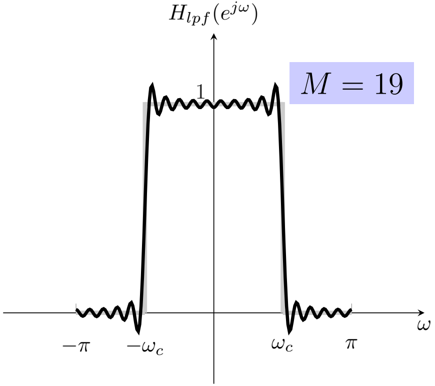


Frequency domain

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$



Convergence of the DTFT

- The sinc function is not absolute summable, but it is **square summable**

$$\sum_{n=-\infty}^{\infty} |\text{sinc}(n)|^2 < \infty$$

- In this case, the DTFT does not converge uniformly; it converges in the mean-square sense
- The oscillations are known as **Gibbs phenomenon**, and they occur whenever there is a discontinuity in the frequency domain
- Interestingly, as $M \to \infty$ the oscillations become more rapid, but the size of the ripples does not decrease
- The DTFT may exist even when sequences are neither absolute summable nor square summable. Examples: a constant, unit step, complex exponentials
- Although all this may seem like a mathematical curiosity, it has important implications in filter design (lecture 9) and in spectrum analysis (lecture 11)

DTFT properties

- Linearity

$$ax_1[n] + bx_2[n] \Longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Time shift, delay for $n_d > 0$, advance for $n_d < 0$

$$x[n - n_d] \Longleftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

- Frequency shift (modulation)

$$e^{j\omega_0 n} x[n] \Longleftrightarrow X(e^{j(\omega - \omega_0)})$$

- Convolution

$$y[n] = x[n] * h[n] \Longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Multiplication of sequences (windowing)

$$x[n]w[n] \Longleftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega - \theta)}) d\theta$$

DTFT properties

- Linear weighting

$$nx[n] \Longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- Time reversal

$$x[-n] \Longleftrightarrow X(e^{-j\omega})$$

- Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})\mathrm{d}\omega$$

if $x[n] = y[n]$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\mathrm{d}\omega \qquad \text{(signal energy)}$$

- Deterministic autocorrelation function

$$c_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m]x[n+m] \Longleftrightarrow X(e^{j\omega})X^*(e^{j\omega}) = |X(e^{j\omega})|^2$$

The z-Transform

The z-transform is a generalization of the DTFT and it is applicable to a broader class of signals and systems.

Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad \text{(Direct transform)}$$

$$x[n] = \frac{1}{2j\pi} \oint_{\mathcal{C}} X(z)z^{n-1}\mathrm{d}z \qquad \text{(Inverse transform)}$$

- The z-transform has a **region of convergence (ROC)**, which is the values of z for which the infinite sum in the direct transform is finite. Without the ROC, the z-transform is an **ambiguous representation** of a signal.
- The inverse transform is given by the contour integral over some complex path $\mathcal{C}$. This is generally laborious, so we'll obtain the inverse z-transform through indirect methods such as **partial fraction expansion**.

The z-Transform

$x[n] = z^n, z \in \mathbb{C}$ is another **eigenfunction** of LTI systems
Proof:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k] z^{(n-k)} && \text{(from convolution sum)} \\
&= \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n && \text{(rearranging)} \\
&= H(z) z^n && \text{(By definition } H(z) \equiv \textstyle\sum_{k=-\infty}^{\infty} h[k] z^{-k} \text{)}
\end{aligned}
$$

- We can always write $z^n = r^n e^{j\omega n}$ (polar coordinates)
- The eigenfunction $e^{j\omega n}$, used in the DTFT, is just a particular case $(r = 1)$ of the eigenfunction used in the z-transform
- The factor r^n helps the z-transform sum converge to a broader class of signals

Relation between DTFT and the z-Transform

DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

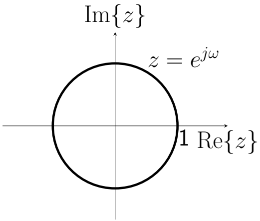
(Direct transform)

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

(Direct transform)

- The DTFT is equal to the z-transform evaluated on the unit circle
- DTFT is periodic with period 2π.
- **Question:** Does the DTFT exist if the ROC of the z-transform does not include the unit circle?



For what will we use the z-transform?

1. Representing LTI systems
2. Determining stability of LTI systems
3. Solving difference equations

Rational z-transforms

Rational z-transforms are a ratio of two polynomials in z^{-1} (or z)

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$
$$= \frac{b_0}{a_0} z^{N-M} \frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

- The **zeros** of a z-transform are the values of z for which $X(z) = 0$
- The **poles** of a z-transform are the values of z for which $X(z) = \infty$. By definition, the ROC cannot contain any pole
- $X(z)$ has M zeros $\{z_1, z_2, \ldots, z_M\}$, which are the roots of the numerator polynomial $B(z)$
- $X(z)$ has N poles $\{p_1, p_2, \ldots, p_M\}$, which are the roots of the denominator polynomial $A(z)$
- If $N - M > 0$, $X(z)$ has more $N - M$ zeros at $z = 0$
- If $N - M < 0$, $X(z)$ has more $M - N$ poles at $z = 0$
- If the coefficients $\{b_0, \ldots, b_M\}, \{a_0, \ldots, a_N\}$ are real, the poles and zeros are either real or they appear in complex conjugate pairs

Examples

1. Right-sided exponential $x[n] = a^n u[n]$

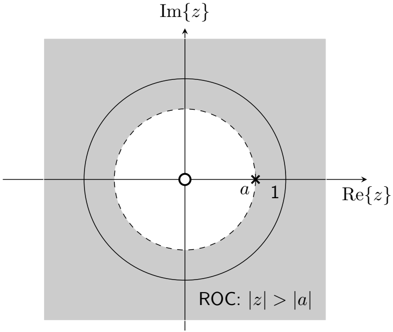
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

This sum converges only if $|az^{-1}| < 1$. Hence, ROC: $|z| > |a|$.

From the sum of an infinite geometric progression:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Pole at $z = a$ and zero at $z = 0$.
The ROC of a causal signal is the **exterior of a circle** whose radius is the magnitude of the outermost pole $|a|$.



Examples

2. Left-sided exponential $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

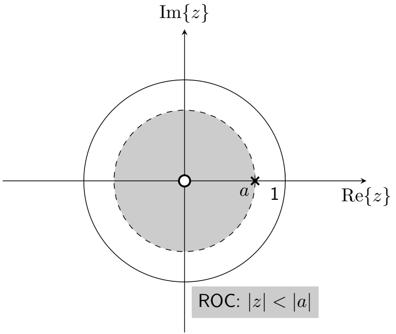
This sum converges only if $|a^{-1}z| < 1$. Hence, ROC: $|z| < |a|$.

Once again from the sum of an infinite geometric progression:

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad \textbf{(Same as before!)}$$

Without the ROC, the z-transform is an **ambiguous representation** of a signal.

The ROC of an anti-causal signal is the **interior of a circle** whose radius is the magnitude of the innermost pole $|a|$.



3. Two-sided exponential $x[n] = -b^n u[-n-1] + a^n u[n]$

We can use the previous results and the linearity property of the z-transform:

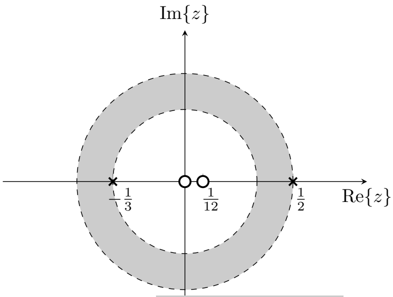
$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-bz^{-1}} + \frac{1}{1-az^{-1}}$$

The ROC is the intersection of the two previous ROCs, i.e., ROC $= |b| < |z| < |a|$.

Assuming $a = -1/3$ and $b = 1/2$

$$X(z) = \frac{2z(z-1/12)}{(z+1/3)(z-1/2)}$$

Poles at $z = -1/3, 1/2$
Zeros at $z = 0, 1/12$
The ROC of a two-sided signal is an **annulus** (ring-shaped region).



ROC: $1/3 < |z| < 1/2$

Properties of the region of convergence

The ROC tells a lot about a signal or system

1. If the ROC is the exterior of a circle (ROC $= \{|z| > |a|\}$), the system/signal is **causal**, where a is the outermost pole. The ROC cannot contain any pole
2. If a **LTI system is BIBO stable** (i.e., $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$), the ROC must contain the unit circle.
   Proof:
   $$\begin{aligned} |H(z)| &= \left| \sum_{n=-\infty}^{\infty} h[n] z^{-n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |h[n]| |z|^{-n} && \text{(From triangle inequality)} \\ &= \sum_{n=-\infty}^{\infty} |h[n]| < \infty && (z = e^{j\omega} \implies |z| = 1) \end{aligned}$$
3. A **causal and stable LTI system** has all poles inside the unit circle. This follows from the first two properties.

Properties of the z-transform

Property	Time-Domain	z-Domain	ROC
Notation	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC $= \{r_2 < \|z\| < r_1\}$ ROC_1 ROC_2
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	at least $\text{ROC}_1 \cap \text{ROC}_2$
Time shifting	$x[n-k]$	$X(z)z^{-k}$	$\text{ROC} - \begin{cases} \{0\}, k > 0 \\ \{\infty\}, k < 0 \end{cases}$
Scaling in the z-domain	$a^n x[n]$	$X(\frac{z}{a})$	$\|ar_1\| < \|z\| < \|ar_2\|$
Time reversal	$x[-n]$	$X(z^{-1})$	$\|\frac{1}{r_1}\| < \|z\| < \|\frac{1}{r_2}\|$
Conjugation	$x[n]^*$	$X(z^*)^*$	ROC
Real part	Re$\{x[n]\}$	$\frac{X(z)+X(z^*)^*}{2}$	includes ROC
Imaginary part	Im$\{x[n]\}$	$\frac{X(z)-X(z^*)^*}{2}$	includes ROC
Differentiation in the z-domain	$nx[n]$	$-z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$	ROC
Convolution	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$	at least $\text{ROC}_1 \cap \text{ROC}_2$
Correlation	$x_1[n] * x_2[-n]$	$X_1(z) \cdot X_2(z^{-1})$	at least $\text{ROC}_1 \cap \text{ROC}_2(z^{-1})$

Linear constant-coefficient difference equation

Many practical problems appear in the form of difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

This difference equation defines an LTI system only if:

1. All coefficients a_k, b_k are constant
2. The initial conditions (or *rest* conditions) are zero $y[-N] = y[-N+1] = \ldots = y[-1] = 0$

Applying the linearity and time-shift properties of the z-transform:

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-jk}}$$

$H(z)$ is a **rational z-transform**.

Example: first-order system

First-order system: $y[n] - ay[n-1] = x[n]$

Calculating the z-transform:

$$\begin{aligned} Y(z) - aY(z)z^{-1} &= X(z) \\ Y(z)(1 - az^{-1}) &= X(z) \\ H(z) = \frac{Y(z)}{X(z)} &= \frac{1}{1 - az^{-1}} \end{aligned}$$

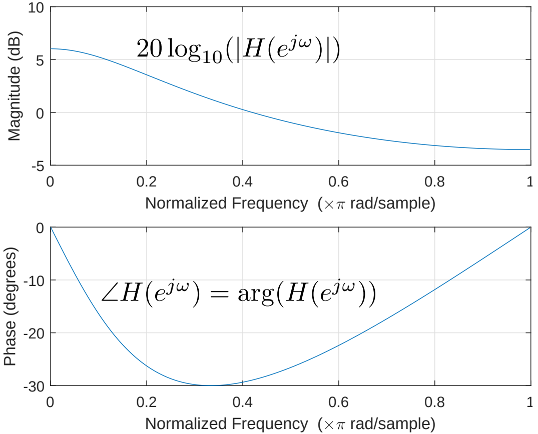
- This corresponds to the exponential: $h[n] = a^n u[n]$.
  Questions: Why the causal exponential? For what values of a is this system stable?
- This system is **autoregressive** i.e., the present output depends on previous outputs
- Autoregressive systems have **infinite impulse response (IIR)**
- Systems with rational z-transforms with non-zero poles are IIR

Let's assume $a = 0.5$, then $y[n] - 0.5y[n-1] = x[n]$

$$H(z) = \frac{1}{1 - 0.5z^{-1}} \xrightarrow{z=e^{j\omega}} H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

In Matlab:

$> \texttt{freqz}(1, [1, -0.5])$



Example: moving average system

Moving average: $y[n] = \frac{1}{M}(x[n] + x[n-1] + x[n-2] + \ldots + x[n-M+1])$

Calculating the z-transform:

$$Y(z) = \frac{1}{M}X(z)(1 + z^{-1} + \ldots + z^{-(n-M+1)})$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M}(1 + z^{-1} + \ldots + z^{-(n-M+1)})$$

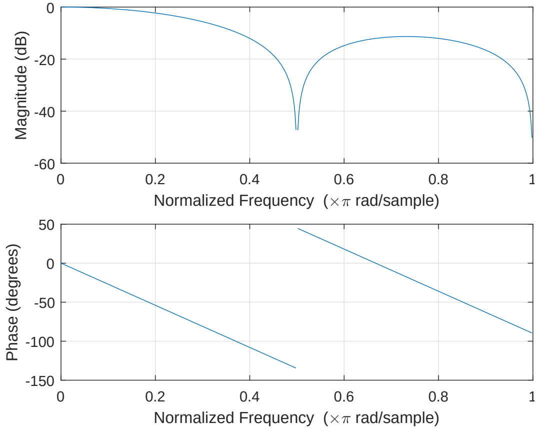
- This system has impulse response $h[n] = 1/M(\delta[n] + \delta[n-1] + \ldots + \delta[n-M+1])$
- The impulse response only depends on a finite number of previous inputs. Hence, this system has a **finite impulse response (FIR)**

Let's assume $M = 4$, then $y[n] = \frac{1}{4}(x[n] + x[n-1] + x[n-2] + x[n-3])$

$$H(z) = \frac{1}{4}(1 + z^{-1} + \ldots + z^{-3}) \xrightarrow{z=e^{j\omega}} H(e^{j\omega}) = \frac{1}{4}(1 + e^{-j\omega} + \ldots + e^{-3j\omega})$$

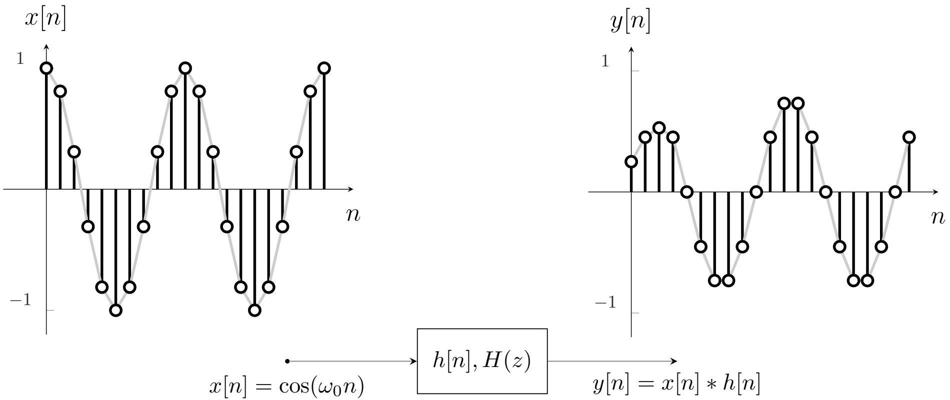
In Matlab:

$>$ `freqz([1,1,1,1]/4,1)`



Example: output of a moving average filter

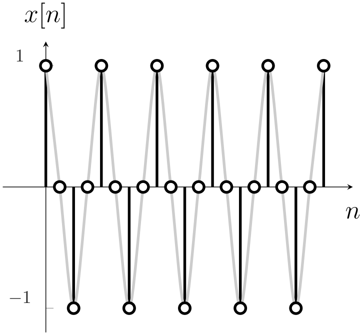
Suppose the input signal frequency is $\omega_0 = 0.2\pi$

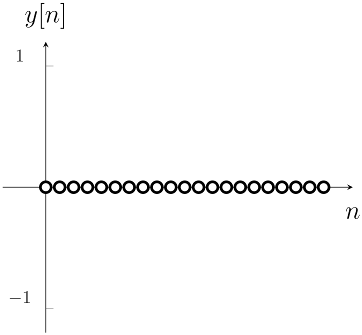


$$h[n] = \frac{1}{4}(\delta[n] + \ldots + \delta[n-3]) \Longleftrightarrow H(z) = \frac{1}{4}(1 + z^{-1} + \ldots + z^{-3})$$

Example: output of a moving average filter

What would happen if $\omega_0 = 0.5\pi$?





$$x[n] = \cos(\omega_0 n) \longrightarrow \boxed{h[n], H(z)} \longrightarrow y[n] = x[n] * h[n]$$

$$h[n] = \frac{1}{4}(\delta[n] + \ldots + \delta[n-3]) \iff H(z) = \frac{1}{4}(1 + z^{-1} + \ldots + z^{-3})$$

Summary

- Systems can be linear, time-invariant, memoryless, causal, and stable
- Linear and time-invariant (LTI) systems are completely characterized by their impulse response
- We can use the convolution sum to calculate the output of an LTI system to any signal
- The complex exponential $e^{j\omega n}$, and more generally z^n, are eigenfunctions of LTI systems
- Frequency-domain representation of signals and systems
  - Discrete-time Fourier transform (DTFT)
  - z-transform and ROC. Without specifying the ROC the z-transform is ambiguous
- The DTFT is equivalent to the z-transform evaluated on the unit circle: $H(e^{j\omega}) = H(z = e^{j\omega})$, provided that the unit circle is in the ROC of the z-transform