Quantitative Macro: Lessons Learnt from Eleven Replications

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Abstract

I replicate all tables and figures from eleven papers in Quantitative Macroeconomics, with an emphasis on incomplete market heterogeneous agent models. I report three main findings: (i) all (non-welfare related) major findings of the papers replicate, (ii) welfare findings based on linear approximation methods —1st-order perturbation, linear and log-linearization around steady-state, and linear-quadratic methods— should be treated as quantitatively incorrect, (iii) decisions around methods for discretizing exogenous shocks have a large and unappreciated influence on results and should be prominently discussed in papers. While some aspects of the papers do not replicate exactly, rather than nitpick in the body of this paper I instead describe some lessons learnt that may be useful for practitioners working with Quantitative Macroeconomic models. The replications use global methods allowing for non-linearities and I argue that these are important and need to be more widely used; many of the original papers use local and/or linear numerical methods and I shed light on the strengths and weaknesses of these. I provide a checklist that researchers can use when trying to check that their work will be reproducible. Matlab codes implementing the replications using the VFI Toolkit are provided, and full results of all replications are given in the online appendix. I conclude with three core points for best practice: (i) codes be made directly available (e.g., on github, not only 'on request', and not just inside a zip file), (ii) report not just baseline parameters but also hyperparameters, equilibrium values, non-baseline parameters and initial conditions, and (iii) replication means rewriting codes from scratch, not just re-running available codes.

Keywords: Quantitative Macroeconomics, Numerical methods, Replication.

JEL Classification: E00; C68; C63; C15

*Thanks to the following who answered emails of questions about replicating their paperss, often providing code or comments: Javier Díaz-Giménez, Mark Huggett, Ayse Imrohoroglu, Guido Lorenzoni, Carlos Urrutia. (This is essentially a full list of those contacted; if people dislike replication of their work such mentality is not present among Macroeconomists!) Further thanks to the following for helpful conversations and feedback, about specific replications or about replication in general: Chris Carroll, Alessandro Di Nola, Chris Edmond, Denise Manfredini, and Nathan Palmer. Special thanks to Shutao Cao for patiently listening to my innumerable issues around the minutiae of papers which I was struggling to replicate; most often due to errors on my part. Kirkby: Victoria University of Wellington. Please address all correspondence about this article to Robert Kirkby at <robertdkirkby@gmail.com>, robertdkirkby.com.

Contents

Imitation is the sincerest [form] of flattery.
— Colton, Charles Caleb (1824)

I quasi-replicate a number of classic papers in Quantitative Macroeconomics. The replications are quasi-replications in two senses: I do not attempt to use the same numerical methods to solve the model as the original authors, and I (only) replicate all figures and tables relating to the model.[1] Our interest is not in nitpicking about where the original papers report a 'wrong number' (whether due to typo, coding error, etc.), and for this reason I relegate all the actual replicated tables and figures to the appendix. The focus of this paper is instead on the lessons to be learned from these replications and on providing some suggestions for best practice based based on the experience of performing the replications.

My main finding is that there is no replication crisis in the Macroeconomics of Quantitative Macroeconomics, but there is a minor crisis in the Quantitative. By this I mean that the major conclusions from the all the papers replicated are unchanged, but most of the papers contain some numbers that are incorrect by a magnitude that is quantitatively important.[2] As well as cases where the numbers are simply quantitatively 'too large' or 'too small' I find that a number of papers contain numerical errors that disappear on replication but formed the basis of a paragraph or two of discussion in the original paper: many papers contain a minor conclusion that does not hold up to replication.

No paper is perfect: Every paper contained at least one 'substantive' error –e.g., an interest rate that was at least 1 percentage point from the replicated value– demonstrating the importance of replication. I further conjecture that the quantitative Ramsey-optimal policy literature likely faces a replication crisis because of it's widespread continuing use of linear-quadratic numerical methods combined with its focus on the social welfare function; the precise grounds for this conjecture are explained in detail below.

Replication is typically thought of as relating to data and statistics. So why replicate computational results from Quantitative Macroeconomics? The main reason is the exact same reason underlying the importance of replication to data and statistics: establishing the reliability of existing results. The need to do so follows directly from thinking of computational models as a form of laboratory in which we run experiments (Bona and Santos, 1997). A secondary use for replications follows as Economists often learn to write code by solving existing models and replication provides

[1]Two main aspects of the papers therefore remain unreplicated: any tables or figures relating purely to the empirical data, and any results reported in the text but without appearing in any table or figure.

[2]This does not mean all papers in Quantitative Macroeconomics replicate. Two examples: Hatchondo, Martinez, and Sapriza (2010) show that some important, but not the main, findings of the sovereign debt papers of Aguiar and Gopinath (2006) and Arellano (2008) fail to replicate; they were numerical error. Takahashi (2014) shows that main finding of Chang and Kim (2007) fails to replicate as it was numerical error (reply of Chang and Kim (2014)).

the needed reliable solutions for this.[3] If anything simple mistakes may be more common when computing Quantitative Macroeconomic models than in other parts of Economics as they depend not only on using data and statistics but also require substantial coding. An additional reason is to understand the influence on Macroeconomics of the choice of which numerical methods are used to solve the model: I document some interesting examples of the importance of this.

The following is a list of papers I replicate, along with a very short description of what motivated their selection. The choices are undoubtedly biased by your authors personal history: working on general equilibrium heterogeneous agent models with incomplete markets, and having the good fortune to do my PhD under the supervision of Javier Díaz-Giménez. In defense of the selection I simply note that having published any one of these papers would improve the Wu-index of any Macroeconomist.

- Hansen (1985): represents a key contribution in developing what might now be called the 'basic Real Business Cycle' model.[4]
- Imrohoroglu (1989): first to numerically solve a heterogenous agent model. It is a pre-general equilibrium, incomplete markets heterogeneous agent model. Early example of how heterogeneity can lead to different implications from representative agent models.
- Díaz-Giménez, Prescott, Alvarez, and Fitzgerald (1992): another of the early, pre-general equilibrium, heterogeneous agent models; early example of alternative ways of modelling money in heterogeneous agent models.
- Hopenhayn and Rogerson (1993): study the impact of a distortion in heterogeneous firm models with entry and exit, with a focus on distortions to aggregate productivity resulting from misallocation of the factors of production.
- Huggett (1993): landmark contribution to general equilibrium in heterogeneous agent models.
- Aiyagari (1994): landmark contribution to general equilibrium in heterogeneous agent models.[5]
- Hubbard, Skinner, and Zeldes (1994): finite-horizon model of role of precautionary savings. (shares model with Hubbard, Skinner, and Zeldes (1995)).
- Huggett (1996): general equilibrium heterogeneous agent OLG model, important contribution to understanding how to model the wealth distribution.

[3]I personally become interested in the issues of numerical error and replication after a 'lost week' spent trying to understand why, when first learning to solve heterogeneous agent models, my codes would not replicate the results of the bottom right corner of Table 2 of Aiyagari (1994), something I now know is because my results were more accurate and the originals contained substantial numberical error.

[4]This replication of Hansen (1985) was originally presented in Kirkby (2017b), however it seems appropriate to repeat it here as part of a larger replication study.

[5]This replication of Aiyagari (1994) was originally presented in Kirkby (2019), however it seems appropriate to repeat it here as part of a larger replication study.

- Castaneda, Díaz-Giménez, and Ríos-Rull (2003): shows one way heterogeneous agent models can generate realistic levels of inequality.

- Restuccia and Urrutia (2004): OLG model that studies intergenerational correlation of earnings. The agents problem differs substantially for the 'young' and 'old' agents.

- Restuccia and Rogerson (2008): study the impact of firm-specific distortions in heterogeneous firm models with entry and exit.

- Guerrieri and Lorenzoni (2017): study the impact of a credit-crisis on the economy by looking at the transition path for a heterogeneous agent endowment economy facing a surprise tightening of the budget constraint.

The codes implementing these replications are all available at github.com/vfitoolkit/vfitoolkit-matlab-replication. Note that this covers a range of 'model-types' including partial and general equilbrium; finite and infinite horizon, including overlapping generations; stationary equilibrium and transition paths; and agent entry and exit. And involves analysing a variety of model 'outputs' including time-series properties, cross-sectional distributions, aggregates, and panel-data.

Replication of these papers was performed using discretized value function iteration with simulations and agents distributions computed on discretized state space; these methods have known reliable convergence properties to the true solution under conditions that are applicable to a broad class of Macroeconomic models (Kirkby, 2017a, 2019) as well as performing well on accuracy in comparisons with other methods (Aruoba et al., 2006; Santos, 2000; Peralta-Alva and Santos, 2014) as long as sufficiently large grids are used. Implementation of the replications makes use of the VFI Toolkit for Matlab (Kirkby, 2017b), which has the advantage that most of the lines of code involved in the replications have been widely tested and hopefully therefore less likely to contain errors.[6]

The only 'substantial' failure to replicate is the welfare results of early papers. This appears to be explained by the use of linear-quadratic methods, while we use non-linear methods to solve the models. For papers such as Imrohoroglu (1989) and Díaz-Giménez, Prescott, Alvarez, and Fitzgerald (1992) the methods used solved the policy function with enough accuracy that their findings on model statistics related to policies and stationary distributions replicate fine. However those same methods led to highly inaccurate welfare evalutations as the value functions were not accurately computed. This finding is not entirely novel, but it's importance is widely underappreciated. Kim and Kim (2007) show that 1st-order approximation methods deliver incorrect welfare results if even when using the correct (to 1st-order) optimal policies (although these can be largely avoided

[6] Coding errors are a genuine concern. A recent replication by Bédécarrats, Guérin, Morvant-Roux, and Roubaud (2019) of Crepon, Devoto, Duflo, and Parienté (2015), an empirical analysis of a field experiment, found numerous coding errors, and that analysis would likely have contained way less lines of code than most quantitative Macroeconomics papers.

by putting the 1st-order solution into the unapproximated welfare function), while Judd, Maliar, and Maliar (2017) show further that 1st-order solution methods are simply incorrect for many Macroeconomic models, deriving *minimum* error bounds that are large enough to be troubling. I conjecture that this problem, innaccurate welfare results, is likely widespread in early Quantitative Macroeconomics papers and recommend that any welfare result from pre-2000 should be treated as quantitatively incorrect until replicated. The continued widespread use of linear-quadratic methods in Ramsey optimal policy where maximizing the welfare function is part of the computational exercise leaves some major open questions about the results of that literature until replication studies are undertaken in that area. Loosely related, first-order (and second-order) perturbation methods have also been shown to give incorrect solution to the Diamond-Moretensen-Pissarides model of search-labor markets (Piccione and Rubinstein, 2007).[7]

One topic that requires much greater discussion in Quantitative Macroeconomics papers is the discretiziation of shocks.[8] Many papers contain a substantial discussion calibration and some robustness exercises to parameter values. The choice of discretiziation method by contrast rarely warrants more than a passing mention, often in a footnote, despite being vastly more important in most models than many parameters. In practice the discretization choices play a key role in determining income risk, the distributions of earnings and wealth, and the degree of market incompleteness. Quantitative Macroeconomic papers would be much improved by treating these choices of shock discretization to the same level of discussion, analysis and sensititivity as any other modelling decision. As an example of their importance Guerrieri and Lorenzoni (2017) use the Tauchen method to discretize an AR(1) shock in a study of the credit crisis that followed the Great Financial Crisis of 2007. Just changing the hyperparameter of the Tauchen method to other reasonable values can cause the zero-lower bound on interest rates to bind for decades, rather than the few years in the baseline model (and seen in reality). A more in-depth discussion about discretizing shocks makes up part of Section 2.

Replication: Controversy about replication has recently raged in Psychology where a project by the Open Science Collaboration to repeat one hundred influential studies was able to successfully replicate the original results in only around 40% of cases (Collaboration, 2015).[9] Closer to home for Economists have been the controversies about the results of Reinhart and Rogoff on the relationship between government debt and economic growth, and Miguel and Kremer (2004) on the effects of de-worming on education in Kenya.[10] Within the field of lab experiments in Economics Camerer

[7]The 'appropriate' level of approximation will always be context dependent. Our point here is that the level of approximation resulting from 1st-order pertubations, log-linear approximations, and linear-quadratic return functions is innappropriate for most dynamic stochastic Economic models. These methods create 'economically significant' numerical approximation error in model outcomes that are of interest.

[8]Numerical quadrature methods are standard for evaluating integrals/expectations, and discretizing shocks is required as part of this. Alternatives exist, like Monte-Carlo integration, but are not used as they are too slow.

[9]An EconTalk podcast on the study may interest readers. 'Around 40%' refers to the passage from the abstract stating that: '39% of effects were subjectively rated to have replicated the original result'. A similar effort now underway in Economics can be found at replicationnetwork.com

[10]Reinhart and Rogoff originally argued that there was a cut-off for Government debt of around 90%-of-GDP, below

et al. (2016) try to replicate 18 studies published in American Economic Review and Quarterly Journal of Economics during 2011-2014, and conclude that replication is successful in 60-80% of the papers (depending on exact metric of 'success'). In a related study Dreber et al. (2015) find that prediction markets in which people can bet on which replications will succeed and fail did well in sense that when they predicted a replication would fail it did (when prediction markets predicted it the replication would succeed this was largely unrelated to outcome of replication); this suggests that informally the profession is aware of certain existing results that are unlikely to replicate.

While replication is important it is not a panacea for all problems. Other loosely related issues include the bias of publication to only publish statistically signicant results (Brodeur et al., 2016). The problems don't just lie with the studies themselves, newspapers rarely report on null-findings and rarely do follow-ups to reporting on results that fail to hold in reproduction studies (Dumas-Mallet et al., 2017).[11] Replications are also often potentitaly difficult, expensive, and time-consuming: a recent effort to replicate 50 papers studying Cancer, with a budget in excess of $1.3 million, ended up replicating just 18. Certainly, the replications in the present paper consumed a lot of time.

The only other replication study in Quantitative Macroeconomics that we are aware of is Chang and Li (2015), although they do not actually look at replication per-se. They take a very different approach and rather than try to replicate the results of the papers, their interest is instead whether the original authors of the papers supply codes and when they do whether these codes can simply be run to recreate the results of the papers. A similar approach is taken by Gertler et al. (2018) who find that in 203 papers from top Economics journals while many provide code only in 37% of cases did it actually run, and in only 14% of cases was there both raw data and the code that generates the papers results (tables and figures) from this data. These approaches are in line with the AEA (American Economic Association) Code and Data Policy,[12] although the interest of the AEA policy is about ensuring that a study is replicable, rather than whether a study has been replicated. While subtle, the distiction is important as replicable can be thought of as true even though the code or data-treatment contains errors; that original code runs and reproduces tables and figures in no way tests for the existence of errors in the code itself although it does make it

which there was little relationship with economic growth and above which there was a strong negative correlation; but the statistical significance of the specific 90%-of-GDP cut-off was shown to be due to Excel error (Herndon, Ash, and Pollin, 2013); the broader negative correlation holds, only the cut-off failed to replicate. Miguel and Kremer (2004) argued, based on a randomized controlled trial, that de-worming of children in Kenya had large positive effects on school attendance and educational outcomes. Two studies, one a replication and another a re-analysis questioned some aspects of the results. At the end of the day the results of the original study appear to stand-up well (more: links,short video).

[11] "This year, a study looked at how newspapers reported on research that associated a risk factor with a disease, both lifestyle risks and biological risks. For initial studies, newspapers didn't report on any null findings, meaning those that had results without expected outcomes. They rarely reported null findings even when they were confirmed in subsequent work. Fewer than half of the "significant" findings reported on by newspapers were later backed by other studies and meta-analyses. Most concerning, while 234 articles reported on initial studies that were later shown to be questionable, only four articles followed up and covered the refutations. "(source)

[12] This AEA Code and Data Policy applies to all journals published by the AEA, including but not limited to: American Economic Review and American Economic Journal: Macroeconomics.

much easier to detect and resolve them.

This current approach to replication in Quantitative Economics obviously misses any issues of whether the original results were themselves correct, which is the main point of replication. While availability of code is important, the conventional wisdom that being able to run the code equates to replication entirely misses the point. Replication necessarily involves writing new code as simply running existing codes includes replicating all the errors made in the original when treating the data and writing the code. Availablility of code is important because code often contains information unintentionally missing from a published paper. For example, papers simply forget to state some initial condition, or the weights used during calibration, or the formula for a certain moment, or parameter values of a counterfactual exercise, etc.

Zimmermann (2015) suggests the need for a Journal of Replication in Economics as a way to overcome the current status quo in which academics typically recieve little to no recognition or reward for performing replications. An online effort by ReplicationWiki, hosted by the University of Göttingen, aims to provide a clearinghouse for replications, on the assumption that people already perform replications and simply need some outlet for them. In computational economics a new Journal of Open Source Economics is being planned to allow greater recognition for contributions of codes/software, and EconARK is developing high-quality guidelines for what counts as replication (with tiers of Bronze, Gold and Platinum envisioned).[13] Even the US Defense Advanced Research Projects Agency (DARPA) is involved in trying to improve the reliability of the social sciences. Nor can citations nor a large following literature be relied on as a substitute for replication: oestrogen receptor cycling in the field of breast cancer research was built on two papers each of which has more than 1000 citations over nearly 20 years, but has now been found to be completely incorrect with neither of the original papers being replicable (Holding, 2019). Christensen, Freese, and Miguel (2019) is a recent book that describes many of these issues, problems, and possible solutions, but with a focus on purely empirical work based on regressions and randomized controlled-trails.[14] It provides a good guide for those interested in improving the reproducibility of their own work.

By making replication easier to perform it is hoped that issues such as robustness of model prediction and sensitivity to parameters and model specification will become easier to perform.

[13] Personal communications with Chris Carroll and Feder Isharkov; see also this link to draft replication standards.

[14] Randomized controlled-trials (RCTs) provide a gold-standard, but not a silver bullet. One issue is whether randomization ends up truely random. An EconTalk podcast with James Heckman describes an RCT for a drug to treat AIDS. Participants randomly received the AIDS drug (treatment) or a placebo (control). Because at that time AIDS was a death-sentence the participants were so terrified that they met up outside the lab, put all their pills into a bowl, and then each took a handful containing a mixture of drug and placebo. The Doctors performing the trail were unaware that their randomization had failed. A second issue for many RCTs is lack of power to find effects due to small sample sizes. An example is documented for Microfinance initiatives by Dahal and Fiala (2019), who find that of all eight peer-reviewed RCT publications not a single one has sufficient sample size to have the enough power to find a statistically significant result of the likely (as indicated by point estimates) size of such a result. Note, the issue is not just the 'raw' sample size but also the compliance or take-up rate (what fraction of those offered microfinance loans actually use them); the problems with the AIDS study could be viewed as their having zero net compliance rate (no actual treatment of the treated, relative to control) and hence no statistical power.

The importance of developing computational packages such as Dynare, EconARK, QuantEcon and VFI Toolkit should viewed as part of contributing to this. The literature on empirical regressions has begun developing tools to address these issues of specification searching with a good overview provided by Chapter 7 of Christensen, Freese, and Miguel (2019).[15] Quantitative Macroeconomics would also benefit from such an approach, and simple replication of existing results is a first step on the road to being able to solve models easily enough to make this possible.

For researchers interested in trying to ensure that their own computational work is reproducible Section 3 presents a checklist, based on my experience with difficulties commonly encountered. This checklist is strictly intended as an aid for researchers, not as a requirement to be imposed. Naturally it will be incomplete but should help researchers who wish to make their work more transparent and reproducible avoid the oversights most common in the literature.

The rest of this paper simply describes some general lessons learnt from the process of replicating these papers. Much of what follows might be misread as picking on certain authors/papers by calling out their minor errors. This is far from my intention, which is to understand where common errors are being made and how the profession might do better. The best defense of my intentions is that any author/paper which appears in this work was one I have chosen to spend a few days of my life in replicating as I thought it was sufficiently important in the development of Quantitative Macroeconomics.[16] After all, [replication] is the sincerest form of flattery!

1 Lessons Learnt from Common Issues

The main pitfalls involve a few that Macroeconomists can learn from and one that they cannot. The common pitfall from which there is nothing to be learned is that coding bugs do occour, this appears to have affected a small fraction of the numbers reported in the papers; as a friend expressed it, if you start with n bugs and squash one you are left with n bugs. The pitfalls from which Macroeconomists can learn are now described as a series of recommendations for practitioners based on issues encountered in the replications.

Recommendation: *Graph agent cumulative density functions, rather than probability density functions.*

Issue: Graphing Probability Distributions. We suggest that researchers plot cumulative

[15] Blinder and Watson (2016) provide the odd case of a paper that sets out to show that out of a few tens of possible specifications only one leads to a statistically significant result. Rather than concluding that this sole statistical significance is likely the result of specification searching across various regressions leading to spurious significance, they instead present it as a robustness exercise. Riffing on the article entitled *Let's Take the Con Out of Econometrics* (Leamer, 1983) one might conclude that they claim to have turned the con into a pro! This is my personal opinion and the reader should obviously treat it as such; both authors have plenty of other good papers and I am a big fan of the other work of Mark Watson in particular, especially on understanding long-run relationships between variables.

[16] For many of the replications reported here I chose to spend a few days replicating, but actually ended up spending a few weeks and in some cases months.

density functions, rather than probability density functions. Probabilitity density functions can mislead for two reasons: first, they obviously depend on the number of grid points used; second, they appear more sensitive to numerical error. Since many solution methods in quantitative economics involve discretizing shock processes this leads to very different looking probability density functions when the number of grid points used to discretize the shock changes; loosely, doubling the number of grid points would halve the probability mass at each point. This issue is minimized but not eliminated when using cumulative density functions. Probability density functions can also be misleading in the sense that they are very sensitive to numerical error. For example Imrohoroglu (1989) graphs the probability density function, finding two spikes, and provides an interpretation of the intuition said to underlie the existence of these spikes. In fact these spikes were numerical approximation error and disappear when the grid is made much finer.

One alternative approach is to parametrize the probability density — say as a basic polynomial, or Chebyshev polynomials, etc. — but this approach is limited if the interest is in, eg., inequality and the shares of Total Income held by the Top 1% as the parametrization will implicitly impose strong assumptions on these shares.[17] Comparing a number of alternatives I concluded that when probability density functions are plotted the best performance comes from graphing kernel-smoothed density functions estimated from the discretized probability mass function.[18]

Recommendation: Record all parameters in single location. *This does not mean one place in the codes, it could be a saved file produced by codes, or in a structure generated within the codes. Projects such as EconARK, QuantEcon, and VFI Toolkit favour data structures that can then be easily exported as JSON files, allowing them to be easily imported by other software.*

Issue: Only baseline case parameters are provided. Papers essentially always provide all the parameter values for their baseline calibrations (a few do not report the final value of things such as general equilibrium prices that would be of much use for replications when trying to understand where differences may be arising). However a number of papers do not report all the parameter values for alternative calibrations, such as those used for 'policy experiments' or difference 'cases' (e.g., Castaneda, Díaz-Giménez, and Ríos-Rull (2003) and Hubbard, Skinner, and Zeldes (1994)). Such parameter values would be appropriate for inclusion in a technical computational appendix.

Issue: Naming variables. Many papers use different names for variables in their papers and code, complicating reading the code for anyone else. Ideally this would not occour, but a more reasonable solution might be the provision of dictionaries anywhere this does occour.

Issue: Reporting parameter values Three main problems occour: First, the reported parameter is for a different time-period to the model (e.g., report the annual value, when model period is two months). Second, reported standard deviation is for the stochastic process, but equations describe it as being for the innovations to that process. Third, parameters that vary over life-cycle are only

[17] In theory they needn't impose any strong assumptions on the shares as the order of the polynomials approaches infinity. But in practice the polynomials are typically low-order, as otherwise most of the computational advantages to using them are lost.

[18] The codes replicating Imrohoroglu (1989) contain a commented out section comparing a few alternatives.

reported as a Figure (so exact values are unavailable).[19] To be more precise about the second of these, many papers will, e.g., have an AR(1) process and describe σ to be the standard deviation of the innovations, but then when reporting the calibrated variables instead report σ as the standard deviation of the AR(1) process itself. My own suggestion is to use a notation that always specifically emphasises when, e.g, a standard deviation is that for innovations ϵ to the AR(1) process z call it σ_ϵ, and when for the AR(1) itself call it σ_z. This simply helps to differentiate between the two standard deviations which are otherwise often and easily mixed up by accident during writing.
Issue: Calibration Details. Many papers will describe which moments were targeted by the calibration. But they will not provide details on how the calibration itself was implemented. While in earlier papers this was fine as most moments are targeted independently more recent papers often jointly target a number of moments. This typically will mean they have implemented a single-objective optimization that assigns each target moment a weight (multi-objective optimization is also a possibility but based on informal conversations seems rarely used by Economists). These weights are not typically reported (eg., Castaneda, Díaz-Giménez, and Ríos-Rull (2003) do not provide such detail). I suggest that papers should more often include a technical computational appendix which provides this kind of detail. Along the same lines the initial values from which such optimization takes place are almost never given. The availability of codes turned out to be important factor in mitigating this. For example Auerbach and Kotlikoff (1987) describe the calibrated values of their age-dependent parameter e, but do not explain that these are in fact the log values, and that one must take their exponential and then normalize them so that the age one value of e_1 is set to 1; Figure 5.2 made it clear that something was missing in the original description of the calibrated values of e and as their codes are available it was easy enough to find out what.

Recommendation: Upload codes to an online and publicly available repository.
Issue: Availability of Codes. In a few cases the original codes are available from the authors website. In most cases however one had to contact the author directly, and even then some authors no longer had codes (to be fair some of these papers are from early 1990s). As an extreme example the codes for Aiyagari (1994) are unavailable online and the author is deceased. While there is an increasing requirement from journals to provide codes[20] the most obvious improvement would be an increased use of github to make codes publicly available; journals that already provide their own online code repositories are a perfectly satisfactory substitute/complement. This issue appears to already be well recognized in Economics and is therefore likely 'already solved' as it were. Current approaches typically have journals provide codes in downloadable zip files making the process much more onerous that if each Journal simply uploaded all codes to its own github repository or similar; this would make them all instantly searchable and easily accessed and read. The importance of making codes available is the clearest lesson from the replications reported in this paper. Where

[19] As concrete examples, these issues occour in Díaz-Giménez, Prescott, Alvarez, and Fitzgerald (1992), Restuccia and Urrutia (2004), and Huggett (1996) respectively.
[20] For example, providing codes is now required by all the 'Top 5' Economics journals.

authors provided codes (often on email request) these were able to resolve many other problems that arose during replication for many of the reasons described elsewhere in this paper.

Further issues, without recommended solutions:

Issue: Parameter Robustness and Numerical Approximation Errors. Many papers have a 'default' parametrization and have performed some kinds of tests to make check that their numerical methods are performing well at minimizing numerical error. They then look at how changing parameters would change certain model outputs. Often these tests will, eg., induce further curvature into certain parts of the solution and this interacts with the numerical methods to worsen their performance. For example Aiyagari (1994) reports the degree of precautionary savings (eg., as the resultant interest rate) for various parametrizations. While the results relating to low-risk and low-risk-aversion are numerically accurate, those relating to high-risk and high-risk-aversion contain substantial numerical error.
The results of tests for the magnitude of numerical errors, such as Euler Equation residuals (Santos, 2000), are sensitive to the parameter values. This fact is known to be the case from the theory underlying such tests but the issue is often ignored in practice. One possibility would be that when measures of numerical accuracy are presented they should be reported across the range of parameter values that are made use of in the model. An alternative might be for the profession to move more towards the use of *adaptive* numerical methods, such as those in Brumm and Scheidegger (2017), which assess approximation errors and then update based on them as part of the solution method itself. Both of these suggestions are rather onerous so for the present simply having researchers more aware of this issue might be the best approach.

Issue: Welfare Evaluations. Some of the replicated papers used linear-quadratic methods (Díaz-Gímenez, 2001) to solve the value function problem. Replication of these papers often showed high accuracy in variables that depend on the stationary distribution and policy function. However the welfare calculations appear to contain substantial numerical error. It is suspected, but not known, that this relects that linear-quadratic methods perform fine for computing policy functions but provide a poor approximation of the actual value function itself. Since welfare calculations are based on the value function itself they were therefore erroneous. This illustrates how numerical errors in different aspects of the model can be very different. It is common practice to report the results of tests for the magnitude of numerical errors, such as Euler Equation residuals which look at the policy function. It is important to understand the conditions under which these also imply limited numerical errors elsewhere in the model (Santos, 2000; Fernandez-Villaverde, Rubio-Ramirez, and Santos, 2006; Kirkby, 2019). In the current instance of the errors in the value function and linear-quadratic methods the theory relating the value function and Euler equation residuals (Santos, 2000) does not apply.[21]

[21] It does not apply for two reasons: first the linear-quadratic methods themselves, secondly as there are periodically-binding constraints.

Issue: Formulae for model statistics. Typically, when reporting model statistics papers provide a verbal description of how they are calculated, but rarely include an explicit equation. This lead to some difficulties in replication. For example, in Díaz-Giménez, Prescott, Alvarez, and Fitzgerald (1992) most statistics could be replicated exactly, but a few table entries could not, it seems likely this is simply because I was unable to turn the verbal descriptions into the precise formula. Another example: Restuccia and Urrutia (2004) calculate 'cross-sectional disparity' as the standard deviation of log earnings, but what is unclear from the written description is that in this two-period OLG model the 'cross-section' is computed conditional on age being 2, not across the whole model economy. One solution would be to put more formulas in Technical appendices, however this seems overly onerous given that the same issue can largely be solved by improved availability of codes.

2 Influence of Numerical Methods on Economics

The need for greater discussion in Quantitative Macroeconomics papers of the discretiziation of shocks —on par with the usual discussion of parameter choices and the sensititivity of results— stems from the large influence these have in many models on driving both modelling choices and quantitative results.

The main discretization methods used all relate to AR(1) shock processes with normally distributed innovations, namely the Tauchen and Rouwenhorst methods (Tauchen, 1986; Rouwenhorst, 1995). Both perform acceptably in most situations as long as sufficient grid-points are used although the later is to be preferred when shocks are highly persistent. When these are used the most important thing is that both grid-size and hyperparameters need to be reported, and some sensititivity/robustness analysis to these choices should be performed. The most common 'error' in the literature is simply to choose 'too few' grid-points and ignore the large quantitative impact of this in driving results. Variations of these exist (Tauchen and Hussey, 1991; Adda and Cooper, 2003; Floden, 2008) but I recommend against their use[22] as they typically perform worse than the Rouwenhorst method and lack the transparency of the Tauchen method.[23,24] The same is true for finite-horizon models with AR(1) shock processes with normally distributed innovations where the parameters are age-dependent Fella, Gallipoli, and Pan (2019); the natural extension of the

[22]The Tauchen-Hussey method (Tauchen and Hussey, 1991) in particular should no longer be used. It's poor performance is well documented and the existing alternatives are just as easy to implement (Toda, 2020). An indication of how widespread this method is comes from it's inclusion as a central part of the textbook and toolkit of Miranda and Fackler (2002), and it's inclusion in QuantEcon as a standard numerical quadrature method (the algorithm is often coded as the use of a function called *qnwnorm*).

[23]The Tauchen (1986) is transparent in the sense that it forces the researcher to specifically choose the hyperparameter value, henceforth Tauchen's q, which determines the the maximum and minimum grid points as being plus/minus $q/2$ standard deviations. This being forced to explicitly choose the hyperparameter means the researcher is aware of the choice, and likely aware of the role it plays in determining model results.

[24]The superior performance of the Rouwenhorst method is documented for stochastic Real Business Cycle by Kopecky and Suen (2010) and the Diamond-Mortensen-Pissarides search model by Piccione and Rubinstein (2007).

Rouwenhorst method performs best, and the natural extension of the Tauchen method is transparent. The main point here though is not so much which method is used, but that these choices need to be discussed in the papers at least as much as any other calibration choice; they only become irrelevant with grid-sizes of a magnitude almost never seen in practice.

The focus of all of these common discretization processes on normally-distributed shocks also seems misguided. Given that discrete Markov processes will be used to compute the models, why run the data through the straight-jacket of an AR(1) process before it reaches the model? Why not go more directly from data to discrete Markov process? This approach allows much more general and realistic shock processes to be used, and is likely to be especially important in any attempts to model income risks, rare disasters (and more broadely the impacts of climate change), and asset prices. Several methods to do this already exist and the literature would be improved by their more widespread adoption; again, alongside more discussion in papers of these discretization choices and their impact on results. Some existing approaches include the quadrature method of Farmer and Toda (2017) which allows more non-parametric approximations, the approach of Castaneda, Díaz-Giménez, and Ríos-Rull (2003) who simply calibrate a four-state Markov directly, and the use of histograms to create 'bins' and then simply 'count-and-normalize' transitions to implement the maximum-likelihood estimator of a finite-state markov (Kirkby (2017b) explains this in detail for model of Hansen (1985)).

Beyond just the choice of discretizing shock processes, the reporting of various choices of numerical methods and hyperparameters would ideally also be more widely discussed in papers. But given the onerous nature of trying to test for sensititivity/robustness of these choices this is probably best left to replication studies using different methods.

One article I would have liked to replicate but did not is Kydland and Prescott (1982). The reason is itself an interesting example of the important role played by the choice of numerical methods, especially those that involve large amounts of approximation. The model of Kydland and Prescott (1982) contains a six-dimensional state variable, making it prohibitively complicated for the discretized value function iteration methods I use in our replications. The model can however be easily solved using the linear-quadratic value function iteration methods used by Kydland and Prescott (1982), which involves solving for six co-efficients, rather than a full six-dimensional object. This is because using linear-quadratic value function iteration methods means that the full distribution of the shocks does not matter for evaluating expectations of next periods value function, only their conditional mean.

The issue of the use of linear and log-linear, and first- and second-order perturbation in welfare evalutations has already been described in the Introduction. The results of Judd, Maliar, and Maliar (2017) showing that the minimum error bounds on linear, log-linear, and first-order approximations are large enough to be problematic for most Economic models should dissuade Economists from using them in any application. This is especially true thanks to the implementation of second-order and higher methods in many available codebases (including Dynare). Users should also be aware that first-order methods imply only the conditional mean matters for expectations, and that with second-order only the conditional mean and conditional second moment matter; this means they are, e.g., simply unusable for any study of the impact of rare events/disasters or conditional changes in volatility. Wherever possible Economists should be making greater use of global non-linear solution methods.[25]

3 A Checklist for Reproducibility

Table 1 is provided to act as a simple checklist that researchers interested in ensuring reproducibility of their work can use to avoid common omissions. The table is not intended to be comprehensive, but is intended to make it easier to avoid omissions that are common in the existing literature.

[25] Linear methods are sometimes the only way of solving large models, and I would not advocate abandoning them for doing so. But wherever a choice is feasible much greater use of global non-linear methods should occour. For example, there is no excuse for the use of linear-methods to solve mid-size representative agent DSGE models in Dynare.

Table 1: Checklist for Reproducibility

Item	Tick
Copy of codes uploaded, preferably to a third-party repository (github, OSF, dataverse)	
If not obvious from filenames, uploaded codes includes a Readme file explaining what to run.	
Readme file may also describe what software (and versions) were used. What hardware was used.	
Readme file may also give rough guidance on runtimes (a few minutes/hours/weeks).	
Parameters: In codes, parameters are stored in a data structure that can be exported as JSON.	
Parameters: Where parameter names differ between paper and codes a ’dictionary’ is provided.	
Parameters: Include general equilibrium values, initial conditions, alternative calibrations.	
Parameters Bonus: Include hyperparamers for numerical methods used.	
Explicit formulae provided for all model statistics reported in paper.	
When codes/functions are taken from previous projects mention their source.	
Bonus: Codes contain easy to understand comments and variable names.	
Bonus: Codes make it clear which parts of code are generating which results in paper.	
Bonus: In paper describe the numerical methods used, even just ’same as paper X’.	

4 Conclusions

We end simply with an inculcation to the importance of reproducibility of results in Economic Science, and in Science more generally: "Non-reproducible single occurrences are of no significance to science." — Popper, Karl (1934, The Logic of Scientific Discovery)

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A The Replicated Figures and Tables

This Appendix contains all of the replicated Figures and Tables from each of the papers replicated. It is organised as a subsection for each paper. We comment on the output only when it differs notably from the results of the original paper. A brief mathematical description of the baseline model being solved is given for each paper. For full descriptions of the models being solved, including their economic use and interpretation, please consult the papers themselves.

Codes which perform these replications, creating all the Tables and Figures from scratch, can be found at: https://github.com/vfitoolkit/vfitoolkit-matlab-replication

These codes were all implemented in Matlab, and for purposes of this paper were run in Matlab (versions between 2018a and 2019b) using the VFI Toolkit (vfitoolkit.com). They were run on a variety of computers all running Linux (Kubuntu is the best distro ;), with NVIDIA gpus (with 2gb to 6gb GDDR ram) and from two to twenty CPU cores and with memory of 16gb to 32gb.

The replication codes were written with robustness, transparency and ease to follow what is being done in mind, and with little to no concern for run-time (many unnecessary objects are computed). Some therefore take a few days to run.

A.1 Hansen (1985) - Indivisible Labor and the Business Cycle

This replication is a copy of that in Kirkby (2017b).

The VFI Toolkit is now used to replicate the results of Hansen (1985). This turns out to be an interesting illustration of the influence of numerical methods on Macroeconomics. Hansen models the process on the productivity shock as an AR(1) with log-normal innovations; this works because linear-quadratic dynamic programming is used as the solution method, so the distribution of innovations is irrelevant. Many solution methods are simply unable to deal with the nonlinearities and asymmetries involved in modeling log-normal innovations. Since the VFI Toolkit is based on using robust global solution methods it is able to handle log-normal innovations, and as seen in the replication results the assumption of log-normal innovations, when modelled in full, has important implications.

The choice of productivity shock process also turns out to be an interesting illustration of the influence of numerical methods on Macroeconomics. Hansen (1985) models the process on the productivity shocks as an AR(1) with log-normal innovations, chosen as this way the productivity process is never negative. In contrast most models chose to model productivity as the having the log of productivity being AR(1) with normally distributed innovations.[26] The latter approach is

[26] Ie. $z_t = \rho z_{t+1} + \epsilon_t$, where $\epsilon \sim logN(0, \sigma_\epsilon)$, versus the more common $log(z_t) = \rho log(z_{t-1}) + \epsilon_t$, where $\epsilon \sim N(0, \sigma_\epsilon)$. Both satisfy that the shock process is strictly non-negative, which can be important for theoretical reasons.

more common not because it is considered in any way more realistic but because it can be more easily handled using common numerical methods, such as perturbation, which don't deal well with substantial asymmetries or nonlinearities. This shows how the widespread use of certain numerical methods, especially those only able to solve certain kinds of models, has had a strong influence on Macroeconomic modeling; including on quantitative Macroeconomic models which are commonly thought to be less susceptible to strict functional form restrictions than purely analytical models.

Before proceeding to the replication itself I give a quick explanation of why Hansen (1985)'s choice of using linear-quadratic dynamic programming to solve the model allowed modelling the process on the productivity shocks as an AR(1) with log-normal innovations: With linear-quadratic dynamic programming the assumption of log-normal innovations is irrelevant as only the conditional mean of the productivity shocks matters.[27] That only the conditional mean of variables matters is also true of first-order perturbation methods (with second-order perturbations the conditional second-moments (and hence variance) also matter, but not higher moments, etc.). While linear-quadratic dynamic programming can solve the model with log-normal innovations it implicitly involves assuming that the log-normal distribution of those shocks is entirely irrelevant. As discussed below the replication results show this is in fact untrue.

A.1.1 Model

I directly present the model as the value function problem to be solved. Readers interested in knowing the motivation behind looking at these models are referred to Hansen (1985).[28] There are two models, one with 'divisible labor', the other 'indivisible labor'. The only difference mathematically is in the utility function.[29] The value function problem to be solved, with a general utility function, is given by

$$
\begin{aligned}
V(k,z) = \max_{c,i,k',y,h} & \; u(c,h) + \beta E[V(k',z')|z] \\
\text{s.t.} \quad & y = z k^{\alpha} h^{1-\alpha} \\
& c + i = y \\
& k' = i + (1-\delta)k \\
& z' = \rho z + \epsilon'
\end{aligned}
$$

where $\epsilon \sim log - normal$ with mean $1-\rho$ and standard deviation σ_ϵ.[30]. k is capital stock, z is productivity shock, c is consumption, y is output, and h is hours worked. The divisible labor

[27] For more on linear-quadratic dynamic programming, see Díaz-Gímenez (2001). Since the log-normal innovations are independent and identically distributed, the conditional mean of the innovations is just equal to their unconditional mean; ie. because of the choice of numerical method only the mean value of the innovations is relevant. Hence the actual log-normality of the innovations is entirely ignored by the solution method used.

[28] Hansen (1985) uses slightly different notation, α he calls θ, z_t he calls λ_t, and ρ he calls γ.

[29] The 'indivisible labor' economy uses lotteries to ensure that the aggregate labor supply looks divisible.

[30] So $exp(\epsilon)$ is distributed as $N(\mu_{ln}, \sigma^2_{ln})$, where $\mu_{ln} = log(1-\rho) - \sigma^2_{ln}/2$, $\sigma_{ln} = \sqrt{log(1 + \frac{\sigma^2_\epsilon}{(1-\rho)^2}}$

economy has utility function $u(c, h) = log(c) + Alog(1 - h)$, while the indivisible labor economy has utility function $u(c, h) = log(c) + B(1 - h)$.

Hansen (1985) calibrates the model to quarterly US data for the period 1955:Q3-1984:Q1. This leads to the parameter values $\alpha = 0.36$, $\delta = 0.025$, $\beta = 0.99$, $A = 2$, $h_0 = 0.53$, $\rho = 0.95$, and $\sigma_\epsilon = 0.00712$; with $B = -Alog(1 - h_0)/h_0$.

I also present results for an 'alternative productivity process' using the more standard approach of modelling the log of productivity as being an AR(1) process with normally distributed innovations. This method is popular as it both maintains the required assumption that productivity is always positive, and is easier to implement by using the Tauchen method to approximate the log of productivity. The parameters for this alternative productivity process are chosen to ensure that it has the same mean and variance as the original productivity process.

A.1.2 Implementation

The only difficulty arose from the productivity shocks being AR(1) with a log-normal innovation. Standard quadrature methods, such as Tauchen, cannot be used for such a shock process.[31] A large fraction of the lines of code involved in replication thus involve dealing with this. It was implemented by first simulating a lengthy time series for the productivity shocks, then using Matlab's built-in histogram routines to divide this into a grid, and then creating the transition matrix by simply counting transitions and the normalising the resulting matrix.[32]

The codes implementing the model can be found at github.com/vfitoolkit/vfitoolkit-matlab-replication. The grids used are 51 points on the hours worked choice, evenly spaced from zero to one. 501 points on the next periods capital choice, half evenly spaced from zero to steady state capital level, half evenly spaced from zero to twice the steady state capital level. 31 points on the productivity shock, either chosen by the quadrature method described above for the case of log-normal innovations, or using Tauchen method with q=3 for the alternative case of log productivity being AR(1) with normally distributed innovations. These grid sizes seem adequate for convergence, as assessed by comparing the results with a reduction of the grids to 31-351-21 points, respectively (viewed another way the limitation to 100 simulation samples, done following Hansen (1985), provides at least as much noise as the grid size).

[31] One could use the Tauchen method to discretize the (exponential of) the lognormal innovation itself and then take the log. This would involve an increase in the state space; to z and ϵ instead of just z. In combination with the use of pure discretization it also necessitates then creating another grid for the AR(1) productivity shocks themselves, and finding the transition matrix for this shock.

[32] Counting transitions and then normalizing is a standard and well behaved estimator for Markov transition matrices. The choice of using histogram routines to choose the grid was an arbitrary assumption.

Table 2: Replication of Table 1 from Hansen (1985)

Standard deviations in percent (a) and correlations with output (b) for US and model economies.

	Quarterly U.S. Time Series[a] 1955:Q3-1984:Q1		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.54 (0.28)	1.00 (0.00)	2.18 (0.28)	1.00 (0.00)
Consumption	1.29	0.85	1.52 (0.18)	0.18 (0.22)	1.56 (0.19)	0.24 (0.17)
Investment	8.60	0.92	7.12 (1.88)	0.75 (0.09)	8.95 (1.69)	0.85 (0.03)
Capital Stock	0.63	0.04	0.37 (0.15)	-0.02 (0.11)	0.49 (0.13)	-0.02 (0.09)
Hours	1.66	0.76	1.41 (0.38)	0.77 (0.10)	2.34 (0.35)	0.91 (0.03)
Productivity	1.18	0.42	0.97 (0.12)	0.43 (0.20)	0.95 (0.09)	0.04 (0.17)

Table 3: Original Table 1 from Hansen (1985)

Standard deviations in percent (a) and correlations with output (b) for US and model economies.

	Quarterly U.S. Time Series[a] 1955:Q3-1984:Q1		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)	1.76 (0.21)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)	0.51 (0.08)	0.87 (0.04)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)	5.71 (0.70)	0.99 (0.00)
Capital Stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)	0.47 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)	1.35 (0.16)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)	0.50 (0.07)	0.87 (0.03)

[a] The US time series used are real GNP, total consumption expenditures, and gross private domestic investment (all in 1972 dollars). The capital stock series includes non-residential equipment and structures. The hours series includes total hours for persons at work in non-agricultural industries as derived from the *Current Population Survey*. Productivity is output divided by hours. All series are seasonally adjusted, logged and detrended.

[b] The standard deviations and correlations with output are sample means of statistics computed for each of 100 simulations. Each simulation consists of 115 periods, which is the same number of periods as the US sample. The numbers in parentheses are sample standard deviations of these statistics. Before computing any statistics each simulated time series was logged and detrended using the same procedure used for the US time series.

[c] Hansen (1985) models productivity as an AR(1) with log-normal innovations. The 'Alternative Productivity process' models the log of productivity as an AR(1) with normal innovations.

A.2 Imrohoroglu (1989) - Cost of Business Cycles with Indivisibilites and Liquidity Constraints

The model is the first ever to numerically solve a heterogenous agent model. It is a partial equilibrium heterogenous agent model with idiosyncratic and aggregate risk. The household problem has one endogenous state (assets, a), and two exogenous shocks (one idiosyncratic s, one aggregate z; s is employed (or unemployed), z is good times (or bad times)). The value function problem to be solved is

$$\begin{aligned} V(a,s,z) = \max_{c,a'} &\frac{c^{1-\mu}}{1-\mu} + \beta E[V(a',s',z')|s,z] \\ \text{s.t.} \quad & c + a'/(1+r) = a + e \\ & a' \geq \underline{\text{a}} \end{aligned}$$

where (s, z) follows a markov process with two states for s and two states for z. The transition matrix

$$\begin{aligned} \pi(s',z'|s,z) =& [0.9141, 0.0234, 0.0587, 0.0038; 0.5625, 0.3750, 0.0269, 0.0356; \\ & 0.0608, 0.0016, 0.8813, 0.0563; 0.0375, 0.0250, 0.4031, 0.5344] \end{aligned}$$

allows for transition of s to depend on z. $r = r_l$ if $a' > 0$, $r = r_b$ if $a' < 0$. $e = y$ if s is employed, $e = \theta y$ if s is unemployed.

Calibration is $\beta = 0.995$, $r_l = 0$, $r_b = 0.08$, $\theta = 0.25$, $y = 1$, $\sigma = 1.5$. y is a normalization, the value to which y is normalized is not explicitly given in Imrohoroglu (1989) and I am guessing that it is to 1. Paper considers alternative asset/insurance environments and in one of these $\underline{\text{a}} = 0$; in the others there is no $\underline{\text{a}}$ (viewed another way it is negative infinity). In one alternative calibration $\sigma = 6.2$. Paper also considers an alternative in which there is only one z state, in this case $\pi(s', z'|s, z) = \pi(s'|s) = [0.9565, 0.0435; 0.5, 0.5]$.

Solving the model involves solving this value function and then finding the stationary agent distribution, all in standard manner. All model output is just functions of these.

Find that Imrohoroglu (1989) gets the correct policy functions, and largely accurate agents distribution, except for identifying two spikes which are an artifact of numerical error (this is clear as they disappear as the grid size is increased; ie., grid spacing reduced). The numbers for Table 2 appear accurate for storage only economy, but those for intermediation economy appear incorrect; changing the parameter values suggests it is likely related to an error relating to the interest rate on borrowing, but without original codes it is unclear. Numbers from Table 1 relating to welfare evaluations do not appear accurate. As discussed in body of this paper it seems likely that this is due to original paper getting the policy functions sufficiently accurate, but not the value function.

Table 4: Table 1 of Imrohoroglu (1989)
Cost of Business Cycles as a Percentage of Consumption

Risk Aversion Parameter	For Economies with Perfect Insurance	For Economies with Only a Storage Technology
$\sigma = 1.5$	0.051	0.103
$\sigma = 6.2$	0.065	0.269

Replication of Table 1 of Imrohoroglu (1989) using grid size $n_a = 701$, $n_s = 2$, $n_z = 2$
Note that these are simply evaluated at the mean of the utility, not as an expectation across agent distribution of their individual compensating variation based on value function. The later would better capture the heterogeneity of the costs

Table 5: Original Table 1 of Imrohoroglu (1989)

COST OF BUSINESS CYCLES AS A PERCENTAGE OF CONSUMPTION

Risk Aversion Parameter	For Economies with Perfect Insurance	For Economies with Only a Storage Technology
$\sigma = 1.5$	.080	.300
$\sigma = 6.2$	.300	1.500

The formula for the welfare calculations is (as confirmed/provided by Ayse Imrohoroglu by email):
For the welfare results here is the way to calculate them: Cost of businsess cycles is found by the consumption equivalence (change in welfare in terms of %-change in consumption across all possible states)
$CE = (V1/V0)^{(}1/(1-\sigma)) - 1.$
where V0 is average value of no-BC and V1 is that of BC
In the storage only case (risk aversion 1.5)
V1= -414.29
V0=-414.06
CE=0.1%

Ayse Imrohoroglu and Kanika Aggarwal have performed another independent replication of this paper and get the same results as this replication (communication by email from Ayse Imrohoroglu). We conclude that the results of this replication can be considered correct.

Table 6: Table 2 of Imrohoroglu (1989)
Properties of the Equilibrium

Time Average of	Economies with an Intermediation Technology	Economies with Only a Storage Technology
Assets Borrowed	0.613	-0.000
Assets Stored	-0.436	2.346
Assets Saved	0.177	2.346
Income	0.934	0.940
Consumption	0.934	0.940

Replication of Table 2 of Imrohoroglu (1989) using grid sizes $n_a = 701$, $n_s = 2$, $n_z = 2$.

Table 7: Original Table 2 of Imrohoroglu (1989)

Properties of the Equilibrium

Time Average of	Economies with an Intermediation Technology	Economies with Only a Storage Technology
Assets borrowed	.480	.000
Assets stored	.220	2.400
Assets saved	.700	2.400
Income	.940	.940
Consumption	.935	.940

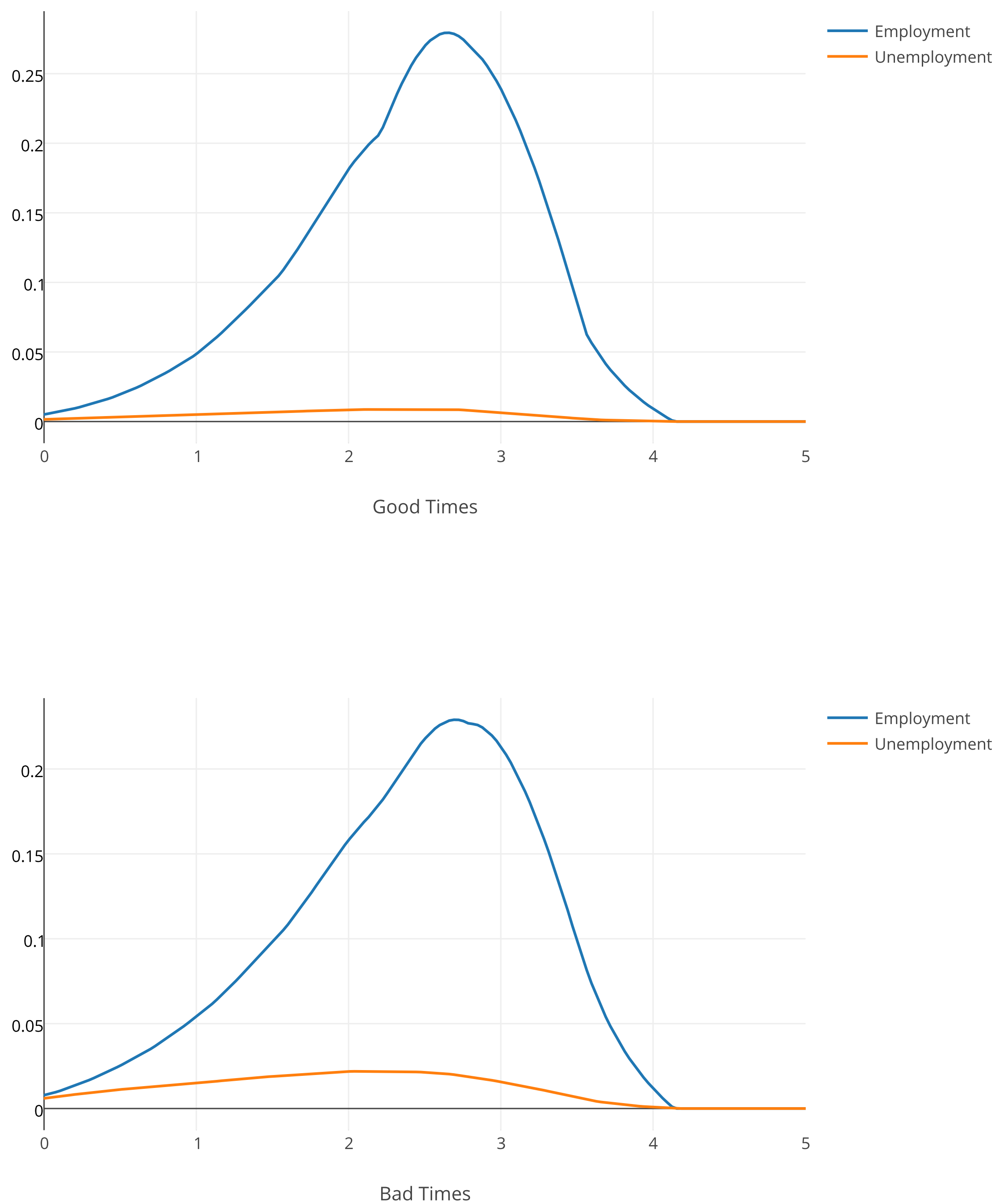


Figure 1: Figure 1 of Imrohoroglu (1989)

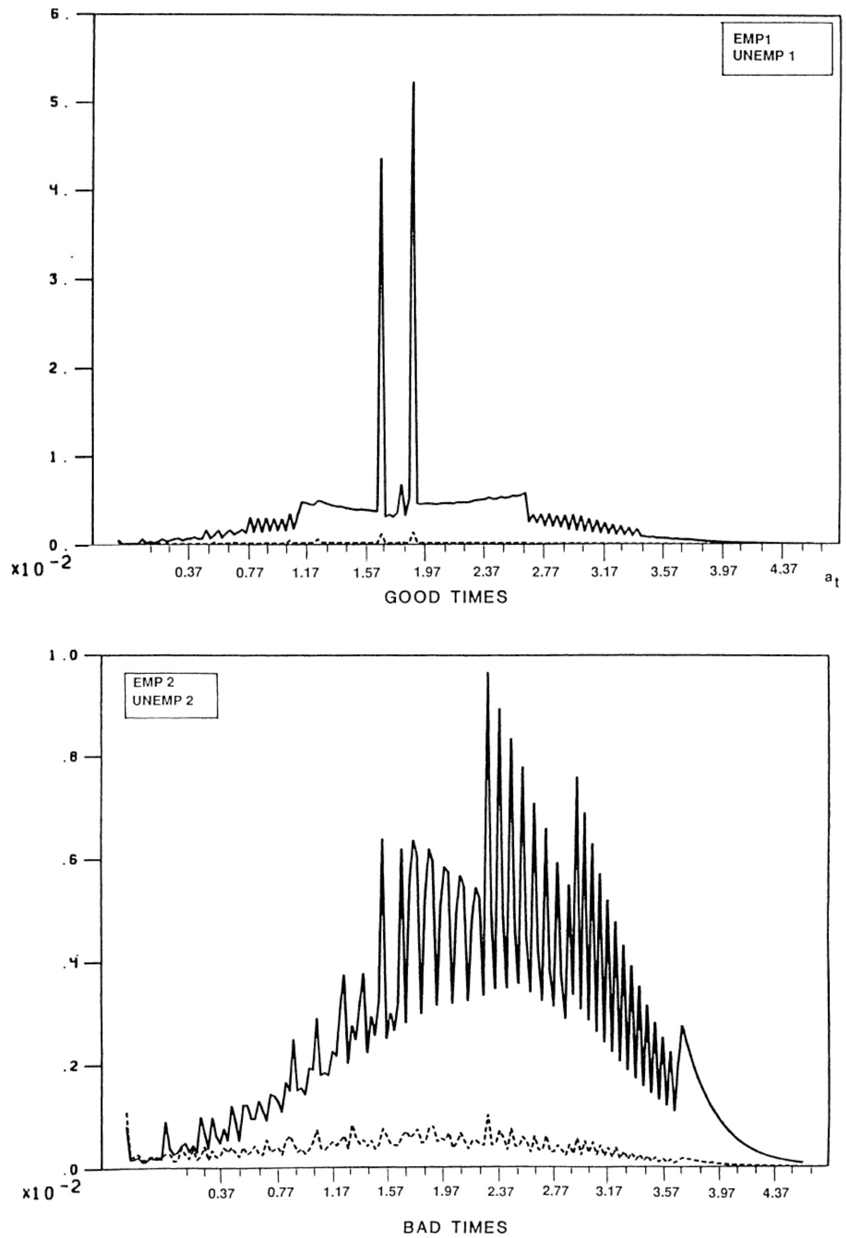


Figure 2: Original Figure 1 of Imrohoroglu (1989)

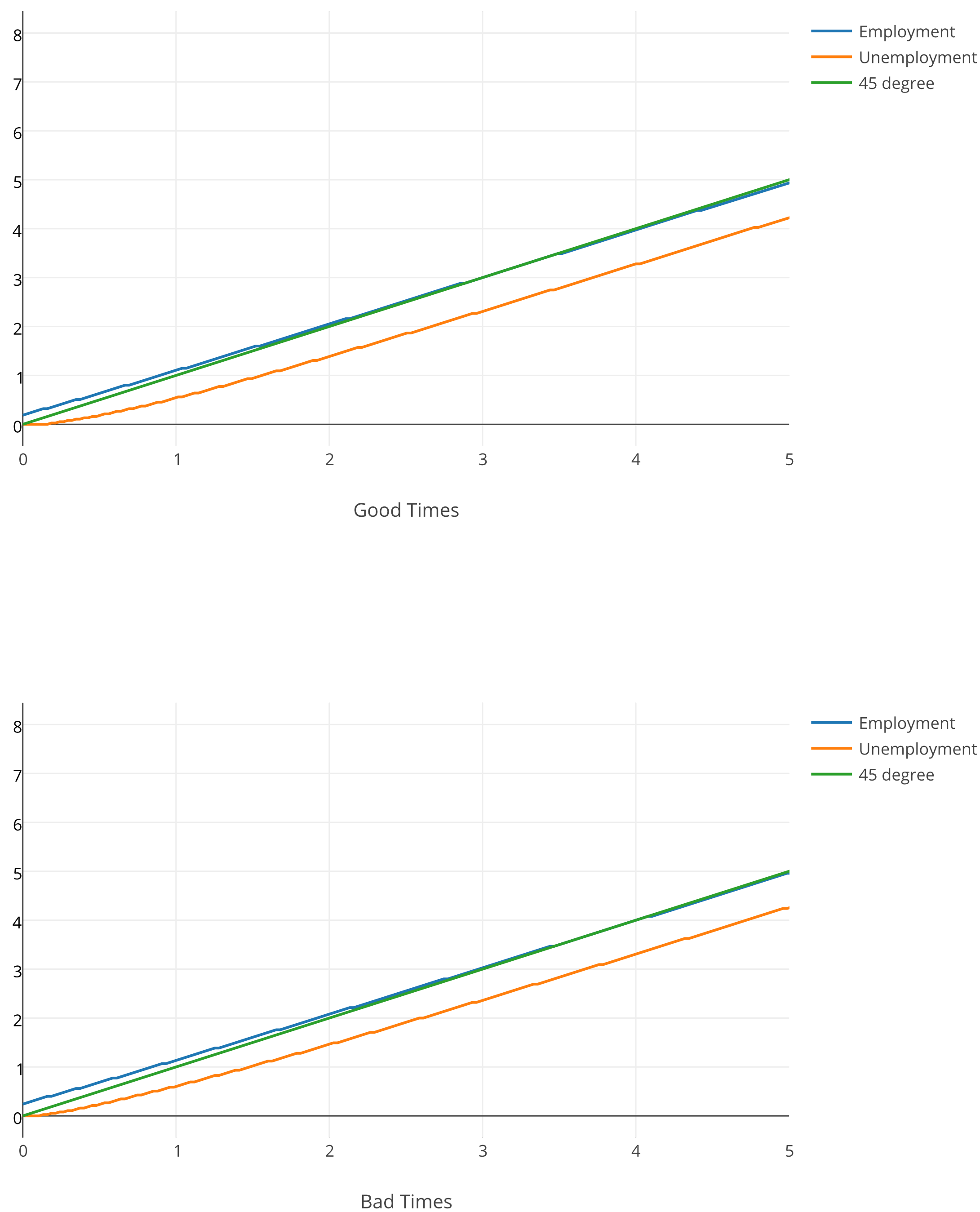


Figure 3: Figure A1 of Imrohoroglu (1989)

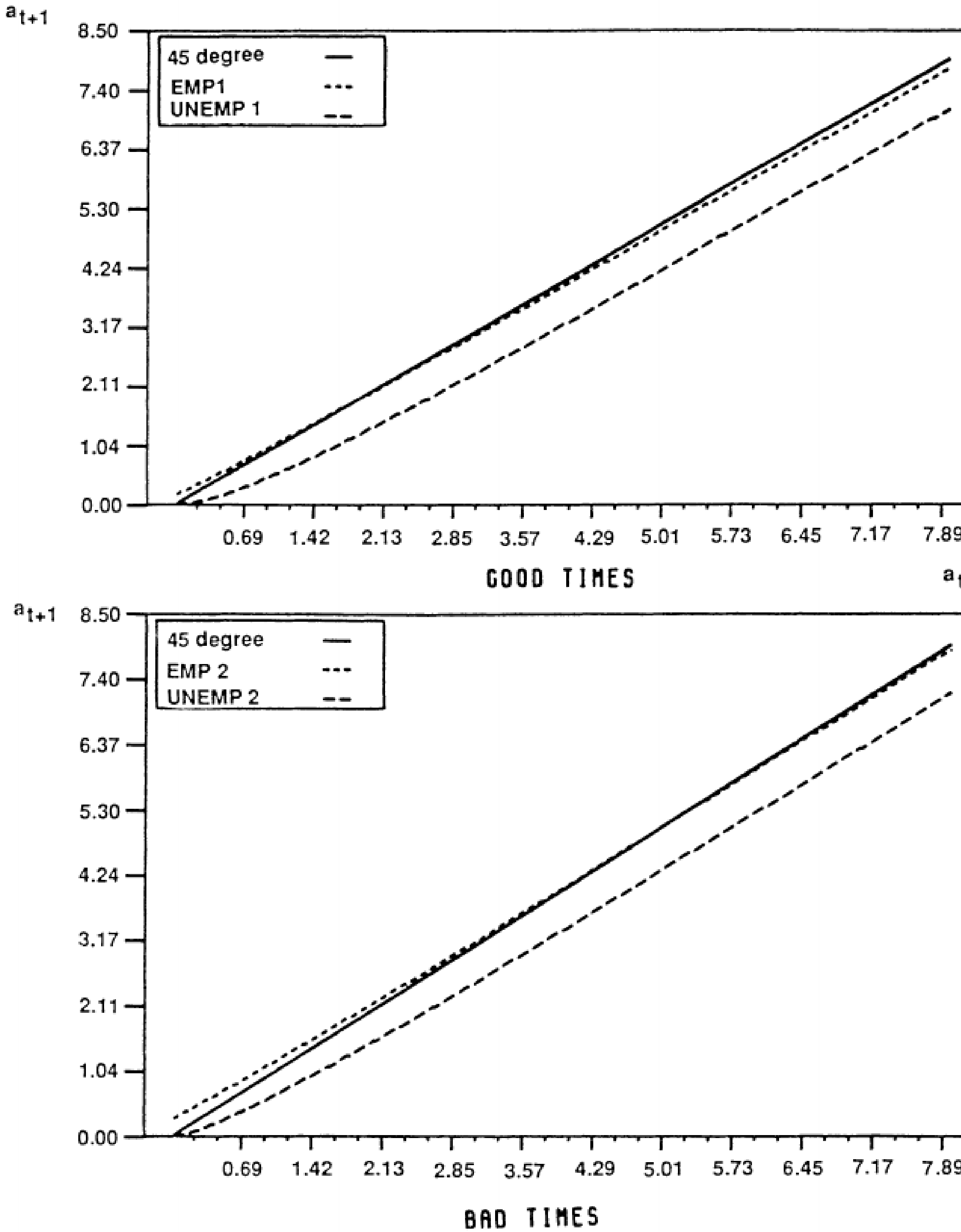


Figure 4: Original Figure A1 of Imrohoroglu (1989)

A.3 Diaz-Gimenez, Prescott, Alvarez, and Fitzgerald (1992) - Banking in Computable General Equilibrium Economies

First we simply note that the parameter value for μ reported in Table 6 of original paper is incorrect due to a typo; given as 0.05, the correct value is 0.00625 (the codes implementing the model presumably use this correct value as we successfully replicate the Tables 7a and 7b of model outputs). The value given of 0.05 is the annual value associated with this parameter, but μ itself should be the value corresponding to the model period of one-eigth of a year.

The model is a partial equilibrium heterogenous agent model with idiosyncratic but no aggregate risk. The household problem has two endogenous states (bank deposits (loan if negative) and housing), one further decision variable (work or not), and one exogenous shock (the model is described as having two, but the second was not used in baseline calibration nor any part of the replication). The value function problem to be solved has a return function which depends on the state of the exogenous shock (which captures, working, retirement, and death) and on whether agent owns a house (K). In the first two states without a house it is

$$\frac{(C^{\alpha}(1-\alpha_k/\alpha)^{\alpha-\alpha_k}(\frac{\alpha_k}{\gamma\alpha})^{\alpha_k}(\tau-N)^{1-\alpha})^{1-\Psi}}{1-\Psi)} \tag{1}$$

In the first two states with a house it is

$$\frac{(C^{\alpha-\alpha_k}K^{\alpha_k}(\tau-N)^{1-\alpha})^{1-\Psi}}{1-\Psi} \tag{2}$$

in the third state (retirement) it is

$$\delta_r\frac{C^{\alpha*(1-\Psi)}}{1-\Psi} \tag{3}$$

in the fourth state (death) is it zero. Consumption is given by $C = A + (1-\theta) * (w(s,z) * N + i_d d - i_l l) + x_s - x_d + l - d + \omega(s,z) - \mu * K'$. l and d are (one plus inflation times) the absolute value of bank deposits when they are negative and positive respectively. If K'¡K then $x_s = \phi * (K - K')$ and $x_d = 0$, else $x_s = 0$ and $x_d = K' - K$. The exogenous states s and z evolve according to independent exogenous finite-valued markov processes. The model contains a substantial number of parameters so the reader is referred to either the original paper or the codes implementing the replication for their value. The value function problem is otherwise standard.

Simulation of the agents distribution is complicated by the presence of stochastic death together with no altruism towards future generations. A customized code therefore had to be written to compute the stationary distribution of agents as the standard commands of the VFI Toolkit would not have worked. Interestingly this combination of stochastic death and no altruism does not appear to have been used elsewhere in the literature suggesting that economic modelling choices do evolve based on ease of computation.

Paper does not provide explicit formulae for many of the model statistics reported in the Tables 7a and 7b (the 'model economy national accounts'); you can figure them out based on knowledge of national accounting, but this is not a trivial exercise and took some time.

Tables 7a and 7b reporting various statistics of the stationary distribution of agents replicates to high accuracy, although lack of explicit formula meant that doing this replication was very involved, and some numbers have not been replicated. In contrast the welfare results reported in Tables 8 and 9 suggest substantial error in the original paper, as explicit formula was provided by original paper confidence in the replication numbers is high. It is suspected, but not known, that this reflects that while the linear-quadratic methods used to solve the model in the original paper did fine at solving for the policy functions they did a poor job of approximating the actual value function and so led to misleading welfare calculations as these depend on changes in the value function.

The paper performs two experiments, results of the first form Table 9. Results of the second experiment do not appear in any Table or Figure and so it was not done as part of this replication.

Table 8: Tables 7a and 7b of Diaz-Gimenez, Prescott, Alvarez, & Fitzgerald (1992)

Calibrated economys steady-state balance sheet data

	Stock/GNP
Household sector	
Tangible Capital	2.71
Deposits	1.00
Loans	0.46
Net Worth	3.25
Government Sector	
Reserves	0.01
Debt	0.55

Calibrated economys steady-state NIPA data

	Percent of GNP
Value added by Sectors	
Housing	
Banking	3.02
Goods producing	
Production	
Consumption	
Goods	50.26
Housing	
Maintainence	13.54
Banking Services	
Government Purchases	16.56
Investment	0.86

Replication of Tables 7a & 7b of Diaz-Gimenez, Prescott, Alvarez & Fitzgerald (1992) using grid sizes $n_a = 800$, $n_s = 4$, $n_z = 1$ (Capital and Labour supply are discrete variables with two possible values each)

Missing values for Housing value added, Goods producing value added, Consumption, Housing, and Banking Services reflect numbers that could not be replicated. Since original paper does not report explicit formulae for their calculation it is unclear if this was simply an inability to find correct formula or actual error in original paper.

Table 9: Original Tables 7a and 7b of Diaz-Gimenez, Prescott, Alvarez, & Fitzgerald (1992)

Table 7a

Calibrated economy's steady-state balance sheet data.

	Stock/GNP
Household sector	
Tangible capital	2.71
Deposits	1.01
Loans	0.46
Net worth	3.26
Government sector	
Reserves	0.01
Debt	0.54

Table 7b

Calibrated economy's steady-state NIPA data.

	Percent of GNP
Value added by sectors	
Housing	15.74
Banking	3.01
Goods producing	81.25
Products	
Consumption	82.61
Goods	50.29
Housing	29.30
Maintenance	13.56
Banking services	3.10
Government purchases	16.55
Investment	0.84

Table 10: Table 8 of Diaz-Gimenez, Prescott, Alvarez, & Fitzgerald (1992)
Benefits of Switching to a policy of a zero after-tax real return on deposits

Current after-tax	Benefits as percent of wealth					
real return	Measured at Mean			Measured across Stationary Dist		
on deposits	S	S_1	S_2	M	M_1	M_2
-2.00	3.55	4.03	-0.48	3.31	3.78	-0.48
-4.00	5.37	5.92	-0.56	5.07	5.63	-0.55
-6.00	6.47	7.13	-0.66	6.16	6.82	-0.65

Replication of Table 8 of Diaz-Gimenez, Prescott, Alvarez & Fitzgerald (1992) using grid sizes $n_a = 800$, $n_s = 2$ Original paper incorrectly describes the first measure as evaluated at the steady-state, it is in fact being evaluated at the mean of the stationary distribution. Note that Table 8 produces results relating not directly to the model of the paper itself, but to the model of Imrohoroglu & Prescott (1991), which is a subcase of the model, hence the different grid sizes used for this Table compared to others from this paper.

Table 11: Original Table 8 of Diaz-Gimenez, Prescott, Alvarez, & Fitzgerald (1992)
Benefits of switching to a policy of a zero after-tax real return on deposits.

Current after-tax real return on deposits	Benefits as percent of wealth					
	Steady-state measure			Our measure		
	S	S_1	S_2	M	M_1	M_2
− 2.0%	0.20	2.62	−2.42	0.18	0.57	−0.39
−4.0	0.41	2.99	−2.58	0.36	1.08	−0.71
−6.0	0.63	3.42	−2.79	0.56	1.55	−0.99

Table 12: Table 9 of Diaz-Gimenez, Prescott, Alvarez, & Fitzgerald (1992)
Welfare benefits of switching to a policy of less-negative after-tax real return on deposits

Current policy		New policy		Benefits (percent of wealth)		
Inflation on deposits	After-tax real return	Inflation rate	After-tax real return	Total	Private	Public
4%	-0.80%	4%	-0.80%			
5	-1.80	4	-0.80	1.24	1.15	0.09
6	-2.80	4	-0.80	1.87	1.63	0.23

Replication of Table 9 of Diaz-Gimenez, Prescott, Alvarez & Fitzgerald (1992) using grid sizes $n_a = 800$, $n_s = 4$.

Table 13: Original Table 9 of Diaz-Gimenez, Prescott, Alvarez, & Fitzgerald (1992)
Welfare benefits of switching to a policy of less-negative after-tax real return on deposits.

Current policy		New policy		Benefits (percent of wealth)		
Inflation rate	After-tax real return	Inflation rate	After-tax real return	Total	Private	Public
4%	−0.7%	4%	−0.7%	—	—	—
5	−1.6	4	−0.7	2.00	0.90	1.10
6	−2.5	4	−0.7	3.58	1.29	2.29

A.4 Hopenhayn and Rogerson (1993) - Job Turnover and Policy Evaluation: A General Equilibrium Analysis

Hopenhayn and Rogerson (1993) study the impact of factor misallocation, specifically the misallocation of labor due to firing costs, in a model of heterogeneous firms. The model has both endogenous entry and endogenous exit of firms.

The problem of an existing firm is to choose employment to maximize (present discounted value of expected) profits, subject to firing costs which must be payed if employment is lower this period than last period. Hiring is costless. Firms have a decreasing-returns-to-scale production function, with labor being the only factor of production, and face a fixed cost of production. Alternatively, a firm can decide to exit in which case it incurs the firing costs of all remaining employees and ceases to exist. The technology level in the production function, z, follows an AR(1) process in logs.

The (existing) firm's problem is thus,

$$V(n,z) = \max_{n',x} \mathbb{1}_{\{x==0\}}(pz(n')^{\alpha} - wn' - pc_f - \tau(\mathbb{1}_{n'<n}(n-n'))) - \mathbb{1}_{\{x==1\}}\tau n \\ +\beta \mathbb{1}_{\{x==0\}} E[V(n',z')|z] \quad (4)$$

where n is 'lag' of employment (last period employment; so n' is this employment this period); z is (idiosyncratic) technology level; $x \in \{0,1\}$ is the exit decision; p is the price level; w is the wage;

c_f is the fixed-cost of production.

The problem facing potential entrant firms is to decide whether or not to enter. A firm that decides to enter must pay a fixed cost of entry, c_e, and will enter as with technology level z drawn from a distribution of entrants η calibrated as a uniform distribution over the lower two-thirds of the range of possible z values, and a 'lagged employment' value of zero.[33] "Once this [entry] cost has been paid, the entrant is in the same position as an incumbent that has chosen to remain in the productive sector and had zero employees in the previous period", according to Hopenhayn and Rogerson (1993) on pg 919; however, in direct conflict with this their Footnote 5 on page 922 states that "Note that we are assuming that a new entrant bears only the fixed cost of entry and does not pay the cost cf"; meaning need to make a slight modification to the existing firm's problem for new entrants (I will ignore this dependence notationally elsewhere). This replication reports the results with Footnote 5 imposed; the codes allow for it to be turned on and off. The mass of new entrants is given by parameter N_e. The problem faced by potential entrants is thus to choose to enter or not based on whether or not $E_\eta[V] > c_e$.

The model also includes a representative household with utility function $\sum_{t=1}^{\infty} \beta^t [log(c) - aN]$; note that there is no aggregate uncertainty in this model. This model gives us the 'demand function' (or equally, the condition for goods market clearance), which enters the model as a general equilibrium condition. It provides household-side assumptions that deliver the infinitely elastic labor-supply (guaranteeing that the labour market clears, regardless of labour demand), and allows for welfare interpretations. It does necessitate additional calibrations, namely A and β to deliver the model values of r and fraction-of-time-worked.[34]

A stationary competitive equilibrium of this model is given by,

Definition 1. *A Stationary Competitive Equilibrium is an agents value function V; agents policy function g; agents exit decision g^x; price of goods p; mass of new entrants N_e; and measure of agents μ; such that*

1. *Given interest rate r, the agents value function V, policy function g, and exit decision g^x, solve the agents problem, as given by equation (4).*
2. *Equilibrium in goods market: $p = \frac{A}{\int z(g^n)^\alpha d\mu}$.*

[33] "We found that a uniform distribution on the lower part of the interval in which realizations of z lie produced a reasonable fit.", pg 930 of Hopenhayn & Rogerson (1993), but no mention of what constitutes 'lower part'. Martin Flodén concludes that roughly the bottom 2/3 (0.65 to be precise) is a good definiting of 'lower' (pg 5): http://martinfloden.net/files/macrolab.pdf Also, "Once this cost has been paid the entrant is in the same position as an incumbent that has chosen to remain in the productive sector and had zero employees in the previous period.", pg 919 of Hopenhayn & Rogerson (1993) tells us that all entrants have zero 'lagged employees'.

[34] To derive the goods-market clearance condition you can first derive $A/C = p$ from the household problem and then combine this with C = real output, and definition of real output as $\int zn^\alpha d\mu$ (note: Y is nominal output in notation of Hopenhayn and Rogerson (1993), so real output can also be found from Y/p. For household side you can get $A/C = p$ from solving $\max_{c,N}[log(c) - AN]$ $s.t.$ $pC = wN + T$, where T is just all lump-sum transfers and other wealth (is not part of standard model notation); note that it is just the standard intratemporal labour/leisure tradeoff condition $-\frac{u_c}{u_N} = \frac{p}{w}$, together with normalization of $w = 1$.

3. *Free-entry condition:* $\int V\eta(n,z)dndz - c_e = 0$

4. *The measure of agents is invariant:*

$$\mu_(n,z) = \int\int\left[\int 1_{n=g(\hat{n},z)}1_{g^x(\hat{n},z)=0}\mu(\hat{n},z)Q(z,dz')\right]d\hat{n}dz + N_e\eta(n,z) \tag{5}$$

Note that the measure of agents μ is not a probability density function, as it includes the mass, N, of agents. Also note that this stationary competitive equilibrium condition can also be defined in terms of finding c_e and treating p as exogenous; Hopenhayn and Rogerson (1993) follow this later in baseline case, and the alternative (p) in other two calibrations with positive values of τ that they solve.

Parameter $a = 0.078$, Hopenhayn & Rogerson (1993) do not report this, but Martin Flodén figures out the following (pg 5): martinfloden.net/files/macrolab.pdf. Hopenhayn & Rogerson (1993) state that they use Tauchen method with 20 grid points to discretize z (which they call s), but do not report the value of the hyperparameter used; $q = 4$ seems to be appropriate based on Table 4 otherwise don't get values of z anywhere near as high as 27.3.

Table 1 relates to data, and so is not part of this replication.

Table 2, I do not replicate 'Co-worker Mean' as I do not know what this means. Hopenhayn and Rogerson (1993) do not provide formulae for any of the statistics reported in the Tables and so I report my interpretation of what their verbal descriptions and the names of the statistics (in some cases they are obvious, the codes implementing the replication provide numerous comments wherever I have made assumptions about how to do this, mostly relating to how to treat entrants and exits). The 0 value for 'Hazard rates by cohort - 1 period' follows directly from imposing footnote 5 of Hopenhayn and Rogerson (1993) (see discussion earlier in this appendix).

The first row of Table 4 contains 'nan' values in replication. This is because according to Hopenhayn and Rogerson (1993) results this was presumably just above the 'exit cutoff', while for me this falls below the 'exit cutoff', so since all firms with this value of z will choose to exit in my replication and the concepts of n_l and n_u are thus undefined. If the grid on n used by Hopenhayn and Rogerson (1993) was truely log-spaced as they indicate, then the numbers they report in Table 4 are rounded to whole numbers in the lower three rows (but not top two rows); I choose not to round these from my actual grid values and instead report all to two decimal places (no good reason for this, the rounding makes just as much sense).

The results of the paper largely replicate, with two main issues. I find a much smaller role of firms with 500+ employees, and I find that firms with 1-19 employees play a smaller role relative to 20-99; smaller is to be understood as relative to original findings. In both cases I beleive this to be because the original paper had 250 points log-spaced from 0 to 5000 employees for a grid on n. I use more than double this. The role of this in creating many more points near 500 employees is clear. As to why it changes relative importance of 1-19 vs 20-99, I suspect that original grid had a point just below 19 and none just above 20, leading firms who might normally choose just above 20 to be forced to choose just below (my grid includes every one of the 101 points from 0 to 100 employees, and then uses log-spaced points from 101 to 5000). Note that with 25+ years of extra computational power it was easy for this replication to use many more points than the original, so I caution against reading too much into this.

Relatedly, Hopenhayn and Rogerson (1993) used 201 points log-spaced from 0 to 5000 for the number of employees of a firm. Together with their use of pure discretization as the solution method this (implicitly) imposes a minimum adjustment size and so their implemented model behaves like one with a (proportional) fixed cost of adjustment plus a linear cost of adjustment, instead of just a linear cost of adjustment in the model itself. The replication uses many more points and so behaves like the linear cost of adjustment model. This intuition explains some of how and where the replication results differ quantitatively from the original. Another explanation is that the limited number of grid points for the number of employees of a firm interacts with the finite nature of productivity z (which takes 20 values) causing productivity distortions; this is largely absent from the replication so the misallocation is smaller.

The utility-adjusted consumption is calculated as $100 * exp(U_i - U_0)$, this formula can be derived definition of consumption-equivalent utility variation as $\bar{c}$ that solves $log(\bar{c}c_0) - aN_0 = log(c_i) - aN_i$, where i is intended to represent the relevate case/tax-rate; note that this ignores the possibility of compensating part of the utility variation as leisure, the present day consensus is to allow for this adjustment of leisure as part of the calculation.[35] This my interpretation of 'The figure for utility-adjusted compensation shows the amount by which consumption would have to be increased in order for utility to reach the same level attained when $\tau = 0$. This measure takes into account the fact that leisure is higher when τ is positive.' (HR1993, pg 934-5).

[35]The 100$*$ comes from the reporting of the numbers as relative to the $\tau = 0$ case; see footnote to replicated Table 3, here Table 16.

Table 14: Table 2 of Hopenhayn & Rogerson (1993)

A: Summary Statistics for Benchmark Model

Average firm size	25.18
Co-worker mean	-
Variance of growth rates (survivors)	0.46
Serial correlation in log(n) (survivors)	0.79
Exit rate of firms	0.30
Turnover rate of jobs	0.24
Fraction of hiring by new firms	0.06
Average size of new firms	5.02
Average size of exiting firms	4.64

B: Size Distribution

	1-19	20-99	100-499	500+
Firms	0.39	0.23	0.07	0.00
Employment	0.11	0.35	0.43	0.11
Hiring	0.14	0.32	0.41	0.13
Firing	0.18	0.40	0.37	0.06
By cohort:				
1 period	0.92	0.08	0.00	0.00
2 periods	0.53	0.45	0.02	0.00
5 periods	0.32	0.54	0.14	0.00
10 periods	0.25	0.52	0.22	0.01
Hazard rates by cohort:				
1 period	0.00			
2 periods	0.85			
5 periods	0.13			
10 periods	0.11			

Note: Do not attempt to replicate Co-worker mean as I do not know definition. Avg size of entering firms is defined in terms of nprime, while avg size of exiting firms is defined in terms of n.

Table 15: Original Table 2 of Hopenhayn & Rogerson (1993)

A. Summary Statistics for Benchmark Model

Average firm size	61.2
Co-worker mean	747
Variance of growth rates (survivors)	.55
Serial correlation in log n (survivors)	.92
Exit rate of firms	.39
Turnover rate of jobs	.30
Fraction of hiring by new firms	.15
Average size of new firm	7.5
Average size of existing firm	4.9

B. Size Distribution

	1–19	20–99	100–499	500+
Firms	.52	.37	.10	.01
Employment	.06	.24	.37	.33
Hiring	.05	.35	.41	.19
Firing	.12	.19	.34	.35
By cohort:				
1 period	.88	.12	.00	.00
2 periods	.54	.45	.01	.00
5 periods	.29	.58	.12	.01
10 periods	.20	.54	.20	.05
Hazard rates by cohort:				
1 period	.75			
2 periods	.32			
5 periods	.15			
10 periods	.10			

Table 16: Table 3 of Hopenhayn & Rogerson (1993)
Effect of Changes in τ (Benchmark Model)

	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$
Price	1.000	1.021	1.040
Consumption (output)	100.0	98.2	96.7
Average Productivity	100.0	99.4	98.3
Total Employment	100.0	98.8	98.4
Utility-adjusted consumption	100.0	98.2	96.8
Average firm size	25.2	25.7	26.6
Layoff costs/wage bill	0.0	0.009	0.013
Job turnover rate	0.24	0.21	0.18
Serial correlation in log(n)	0.79	0.81	0.86
Variance in growth rate	0.45	0.37	0.31

Consumption (output), Average Productivity, Total Employment and Utility-adjusted consumption are all reported relative to $\tau = 0$ (which is set equal to 100). Effectively also true of price which is normalized to one in the $\tau = 0$ case as part of calibration.

Table 17: Original Table 3 of Hopenhayn & Rogerson (1993)

Effect of Changes in τ (Benchmark Model)

	$\tau = 0$	$\tau = .1$	$\tau = .2$
Price	1.00	1.026	1.048
Consumption (output)	100	97.5	95.4
Average productivity	100	99.2	97.9
Total employment	100	98.3	97.5
Utility-adjusted consumption	100	98.7	97.2
Average firm size	61.2	61.8	65.1
Layoff costs/wage bill	0	.026	.044
Job turnover rate	.30	.26	.22
Serial correlation in log(n)	.92	.94	.94
Variance in growth rates	.55	.45	.39

Table 18: Table 4 of Hopenhayn & Rogerson (1993)
Effect of τ on Decision Rules

	$\tau = 0.1$		$\tau = 0.2$	
z	n_l	n_u	n_l	n_u
1.76	NaN	NaN	NaN	NaN
4.24	15.71	18.85	15.71	22.56
10.26	177.75	218.66	171.72	251.00
19.88	1102.50	1355.02	1029.25	1554.71
24.79	2046.63	2430.25	1910.71	2694.11

Note: Hopenhayn & Rogerson (1993) call the first column log(s), but it is clear from the values it should be s, which I call z (the idiosyncratic productivity). Difference in the first column, z, between original and replication represent differences in the grids.

Table 19: Original Table 4 of Hopenhayn & Rogerson (1993)

EFFECT OF τ ON DECISION RULES

	$\tau = .1$		$\tau = .2$	
log s	n_l	n_u	n_l	n_u
1.83	1.36	1.78	1.18	1.98
4.75	21.7	26.7	21.0	32.8
10.5	194	238	181	282
19.9	1,110	1,410	1,036	1,617
27.3	2,610	3,316	2,522	3,935

Table 20: Table 5 of Hopenhayn & Rogerson (1993)
Absolute Deviations from MPL=1/p

	Fraction of Firms within Interval	
Size of Deviation (%)	$\tau = 0.1$	$\tau = 0.2$
0-3	0.32	0.00
3-5	0.38	0.11
5-10	0.27	0.56
10-15	0.00	0.07
>15	0.03	0.27

Table 21: Original Table 5 of Hopenhayn & Rogerson (1993)
ABSOLUTE DEVIATIONS FROM MPL = $1/p$

	FRACTION OF FIRMS WITHIN INTERVAL	
SIZE OF DEVIATION (%)	$\tau = .1$	$\tau = .2$
0–3	.30	.00
3–5	.45	.12
5–10	.15	.78
10–15	.00	.05
>15	.00	.05

A.5 Huggett (1993) - The Risk-Free Rate in Heterogeneous Agent Incomplete Insurance Economies

The model is a heterogeneous agent model with idiosyncratic but no aggregate risk. The household problem has one exogenous state, a, and one exogenous state e, and is given the value function problem,

$$
\begin{aligned}
V(a,e) = \max_{c,a'} & \frac{c^{1-\mu}}{1-\mu} + \beta E[V(a',e')|e] \\
\text{s.t.} \quad & c + qa' = a + e \\
& a' \geq \underline{\text{a}}
\end{aligned}
$$

where e follows a markov process with two states, $e \in \{e_l, e_h\}$, and transition matrix $\pi(e) = [\pi_{e_l,e_l}, \pi_{e_h,e_l}; \pi_{e_l,e_h}, \pi_{e_h,e_h}]$.

The households choices of a' together with the exogenous markov process e imply a transition function P on the state (a,e). Let $\Psi(a,e)$ denote a distribution of agents over (a,e) in the state space $A \times E$.

Definition 2. *A stationary equilibrium for this economy is $c(a,e)$, $a'(a,e)$, q, Φ satisfying*

- *$c(a,e)$ & $a'(a,e)$ are optimal decision rules given q.*
- *Market clear: (i) $\int_{A\times E} c(a,e)d\Psi = \int_{A\times E} ed\Psi$, and (ii) $\int_{A\times E} a'(a,e)d\Psi = 0$.*
- *Agent distribution is stationary: $\Psi(a',e') = \int_{A\times E} P((a,e),(a',e'))d\Psi(a,e)$.*

Note: this definition is lazy on notation; both Φ and P should be defined on the σ-algebra, not on specific points.

So the market clearance (general equilibrium) requirement is for price q to balance borrowing and lending (ie. that integral of a over the stationary agent distribution is zero).[36] Baseline parameter values are given by: $\beta = 0.99322$, $\mu = 1.5$, $\underline{\text{a}} = -2$, $e_h = 1$, $e_l = 0.1$, $\pi_{e_h,e_h} = 0.925$, $\pi_{e_h,e_l} = 0.5$.

Replication involves two Figures, 5 & 6, and two Tables, 22 & 23. Everything replicates pretty accurately (minor numerical differences). The general equilibrium is almost certainly unique (graphs of market clearance condition not shown).

[36]The second clearance condition in the definition of stationary equilibrium will follow by Walras' law).

Table 22: Table 1 of Huggett (1993)
Coefficient of Relative Risk Aversion μ=1.5

Credit Limit (-$\underline{\text{a}}$)	Interest Rate (r)	Price (q)
-2	-7.3 %	1.0127
-4	1.1 %	0.9982
-6	3.1 %	0.9949
-8	3.8 %	0.9938

Replication of Table 1 of Huggett (1993) using grid sizes $n_a = 512$, $n_e = 2$, $n_q = 551$

Table 23: Table 2 of Huggett (1993)
Coefficient of Relative Risk Aversion μ=3.0

Credit Limit (-$\underline{\text{a}}$)	Interest Rate (r)	Price (q)
-2	-23.6 %	1.0458
-4	-4.3 %	1.0073
-6	0.9 %	0.9985
-8	2.7 %	0.9956

Replication of Table 2 of Huggett (1993) using grid sizes $n_a = 512$, $n_e = 2$, $n_q = 551$

Table 1
Coefficient of relative risk aversion $\sigma = 1.5$.

Credit limit ($\underline{a}$)	Interest rate (r)	Price (q)
− 2	− 7.1%	1.0124
− 4	2.3%	0.9962
− 6	3.4%	0.9944
− 8	4.0%	0.9935

Table 2
Coefficient of relative risk aversion $\sigma = 3.0$.

Credit limit ($\underline{a}$)	Interest rate (r)	Price (q)
− 2	− 23 %	1.0448
− 4	− 2.6%	1.0045
− 6	1.8%	0.9970
− 8	3.7%	0.9940

Table 24: Original Tables of Huggett (1993)

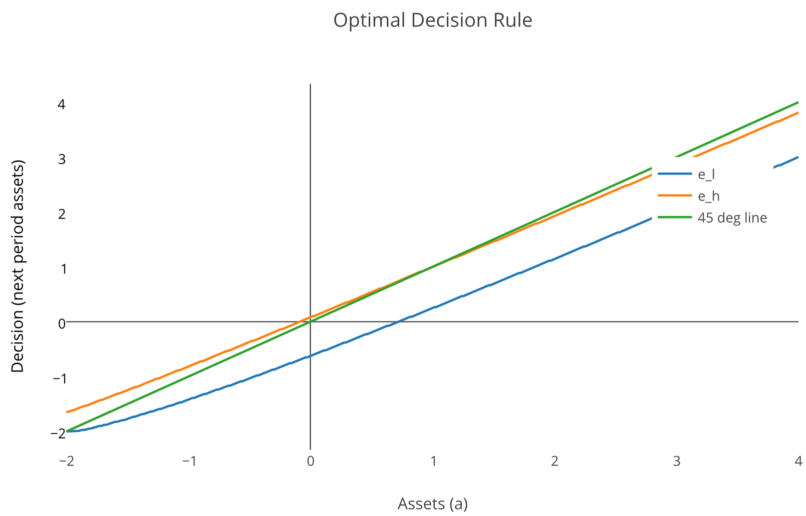


Figure 5: Figure 1 of Huggett (1993)

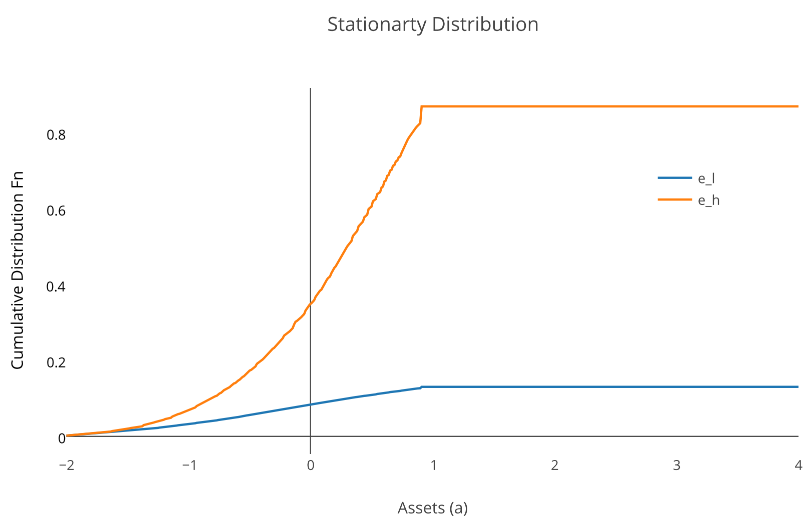


Figure 6: Figure 2 of Huggett (1993)

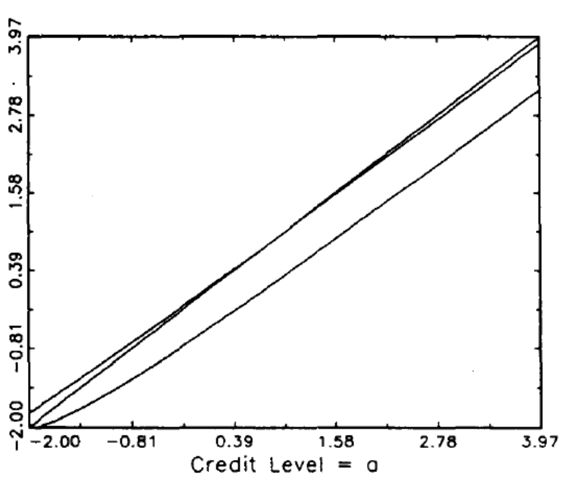


Fig. 1. Optimal decision rule.

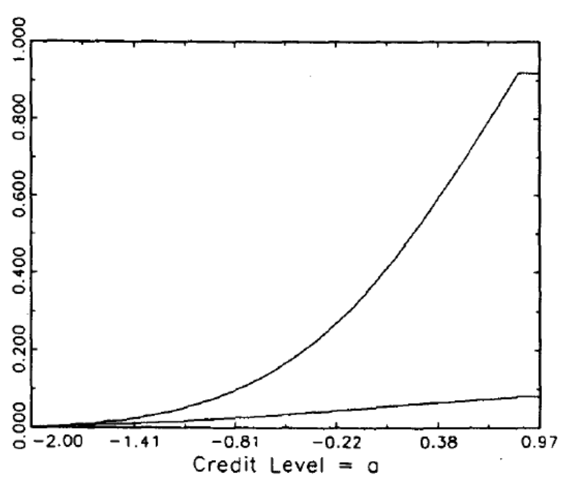


Fig. 2. Stationary distribution.

Figure 7: Original Figures of Huggett (1993)

A.6 Aiyagari (1994) - Uninsured Idiosyncratic Risk and Aggregate Saving

This replication orginally appeared as part of Kirkby (2019).

In the model of Aiyagari (1994), infinitely lived households face a stochastic income—due to exogenous stochastic labor supply—and make consumption-savings decisions; given an interest rate. Therefore, the exogenous shock (z) is the labor supply h, the endogenous state (x) is the capital holdings k, and the decision variable (y) is the next periods capital k'. The state of a household is their current capital holding and their exogenous labor supply shock, (k, h). Individual household capital holdings, given an interest rate, aggregate to give aggregate capital holdings. The market clearance condition is that the interest rate will be determined by perfect competition in the goods market together with a representative firm with Cobb-Douglas production function. A general equilibrium is the condition when an interest rate determines household capital holdings, which in turn determine an interest rate by the market clearance condition, and when this latter interest rate is the same as the first. In short,

Definition 3. *A Competitive Equilibrium is an agents value function $V(k, h)$; agents policy function $k' = g(k, h)$; an interest rate r and wage w; aggregate capital K and labor H; and a measure of agents $\mu(k, h)$; such that*

1. *Given prices r & w, the agents value function $V(k, h)$ and policy function $k' = g(k, h)$ solve the agents problem:*

$$\begin{aligned} V(k,h) = &\max_{k'} \left\{ u(c) + \beta \int V(k',h')Q(h,dh') \right\} \\ &s.t. \quad c + k' = wh + (1+r)k \\ &\qquad\quad c \geq 0,\ k' \geq \underline{k} \end{aligned}$$

2. *The aggregates are determined by individual actions: $K = \int k d\mu(k, h)$, and $H = \int h d\mu(k, h)$*
3. *Markets clear (in terms of prices): $r - (\alpha K^{\alpha-1}H^{1-\alpha} - \delta) = 0$.*
4. *The measure of agents is invariant:*

$$\mu(k,h) = \int\int \left[\int 1_{k=g(\hat{k},h)} \mu(\hat{k},h) Q(h,dh') \right] d\hat{k}dh \tag{6}$$

where h is the labor supply shock, which takes values in $Z = \{h_1, ..., h_{n_h}\}$ and evolves according to the Markov transition function $Q(h, h')$. Note that the wage is residually determined by r.[37] The market clearance condition is more commonly expressed as $r = \alpha K^{\alpha-1}H^{1-\alpha} - \delta$, that the interest equals the marginal product of capital (minus the depreciation rate). Since $H = E(h) = 1$, the Cobb-Douglas production function is really only based on the aggregate capital (in the sense that H is a fixed constant).

[37] The wage, which is given by the derivative of the Cobb-Douglas production with respect to labor, can be rewritten as a function of the interest rate and the parameters of the production function.

Table 25: Accuracy of the Tauchen Method in Aiyagari (1994)
Markov Chain Approximation to the Labour Endowment Shock
Markov Chain σ/Markov Chain ρ

σ/ρ	0.00	0.30	0.60	0.90
0.2	0.20/0.01	0.20/0.29	0.20/0.58	0.20/0.90
0.4	0.40/0.00	0.40/0.28	0.40/0.60	0.40/0.90

Replication of Table 1 of Aiyagari (1994) using grid sizes $n_k = 512$, $n_z = 21$, $n_p = 551$ The mesh size, the maximum space between two consecutive grid points, is $\epsilon_k = 0.3822$, $\epsilon_z = 0.3465$, $\epsilon_p = 0.0004$

Table 26: General Equilibrium Interest Rates in Aiyagari (1994)

A. Net Return to Capital in %/Aggregate savings rate in % ($\sigma = 0.2$)			
ρ/μ	1	3	5
0.0	4.1326/17.76	4.0645/22.85	3.9850/26.02
0.3	4.1326/28.15	4.0304/22.59	3.9169/23.78
0.6	4.1099/24.10	3.9510/24.44	3.7352/24.40
0.9	4.0077/24.38	3.5763/25.00	3.0427/26.18
B. Net Return to Capital in %/Aggregate savings rate in % ($\sigma = 0.4$)			
ρ/μ	1	3	5
0.0	4.0872/23.44	3.8601/24.23	3.5422/24.86
0.3	3.9964/24.01	3.5763/24.88	3.0654/26.00
0.6	3.8488/24.30	3.0654/25.99	2.2252/28.10
0.9	3.5876/24.74	2.0890/28.60	0.6585/33.21

Replication of Table 2 of Aiyagari (1994) using grid sizes $n_k = 512$, $n_z = 21$, $n_p = 551$ The mesh size, the maximum space between two consecutive grid points, is $\epsilon_k = 0.3822$, $\epsilon_z = 0.3465$, $\epsilon_p = 0.0004$

Table 27: Original Version of General Equilibrium Interest Rates in Aiyagari (1994)

A. Net Return to Capital in %/Aggregate savings rate in % ($\sigma = 0.2$)			
ρ/μ	1	3	5
0.0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.12	3.2903/25.51	2.5260/27.32
B. Net Return to Capital in %/Aggregate savings rate in % ($\sigma = 0.4$)			
ρ/μ	1	3	5
0.0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/28.64	-0.3456/37.63

Original Table 2 of Aiyagari (1994).

We start with the results for the model of Aiyagari (1994). The functional forms and calibrated parameter values are as follows. The utility function is parameterized as $u(c) = \frac{c^{1-\mu}}{1-\mu}$. The shock process is $z' = \rho z + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$. The discount rate is $\beta = 0.96$, capital share of production is $\alpha = 0.36$, and the depreciation rate is $\delta = 0.08$. Following Aiyagari (1994), we consider varying the parameters $\mu \in \{1, 3, 5\}$, $\rho \in \{0, 0.3, 0.6, 0.9\}$ and $\sigma \in \{0.2, 0.4\}$. The grid on the exogenous shocks is given by the Tauchen method with $n_z = 21$. The grid on the assets is $n_k = 512$ points, one-third evenly spaced on the interval $[0,K_{ss}]$, one-third evenly spaced on the interval $(K_{ss}, 3K_{ss}]$, and the final third evenly spaced on the interval $(3K_{ss},15K_{ss}]$; where $K_{ss} = (\frac{r_{ss}+\delta}{\alpha})^{\frac{1}{\alpha-1}}$ and $r_{ss} = 1/\beta - 1$ are the steady-state capital stock and interest rate of the corresponding complete markets representative agent economy. The grid on the prices is $n_p = 251$ points, one-third evenly spaced on the interval $[-\delta,0)$, and the remaining two-thirds evenly spaced on the interval $[0,r_{ss}]$.

Table 25 shows that at least as measured by the first-order autocorrelation and variance of the process, the Tauchen method was accurate in discretizing the exogenous process. Although in my experience, the weakness of the Tauchen method tends to be related to the choice of the parameter q, an issue not addressed by this Table.Thus, these results confirm those of Aiyagari (1994).

A comparison of Table 26 with the corresponding Table 2 of Aiyagari (1994) (for readers convenience, the original version of Table 2 from Aiyagari (1994) is reproduced here as Table 27) shows that the original quantitative results that he gives for the equilibrium interest rates display the correct qualitative behavior; decreasing both in risk-aversion (μ) and in the riskiness of earnings (σ & ρ). However, quantitatively, the results of Aiyagari (1994) are quite inaccurate due to the roughness of the numerical approximations used. In particular, the degree of precautionary savings in the high-risk (high σ and/or ρ), high-risk-aversion (high μ) cases were substantially overestimated by Aiyagari (1994); for instance, compare $\mu = 5$, $\sigma = 0.2$, $\rho = 0.9$, or $\mu = 3, \sigma = 0.4, \rho = 0.9$.

A.7 Hubbard, Skinner and Zeldes (1994) - The Importance of Precautionary Motives in Explaining Individual and Aggregate Savings

Hubbard, Skinner, and Zeldes (1994) present a finite-horizon model of the life-cycle to understand individual and aggregate savings (is partial equilibrium). Model has three permanent/fixed types, two exogenous shocks (income and health expeditures), and one endogenous state (assets). Implementing it involves solving the value function problem, and then simulating panel data and life-cycle profiles. The same model is also used by Hubbard, Skinner, and Zeldes (1995). From the perspective of designing algorithms that are capable of handling a wide variety of situations the models key challenges are a consumption floor (which introduces a non-concavity in the return function) and the need to handle permanent/fixed types and age-dependence (both of parameters, and more difficultly of shocks and shock transition probabilities). I note that the original paper had three authors, three programmers, and six research assistants versus the one person that it took to replicate all of the computational half of the paper showing the advance in computing power and the usefulness of efforts like VFI Toolkit and QuantEcon.

From the perspective of replication the article fails to mention a number of important points. First, it does not give the masses of each of the three different fixed (education) types (e.g., what fraction of the total population is of type 'college'). This is not a problem for all the figures but is for most entries in the tables.[38] Second, the article does not mention a number of assumptions for the non-baseline models; for example it is not mentioned that for the 'certainty' model lifespans are limited to age 80 rather than 100 (period 80), and while it is mentioned that a correction is made for the mean value of stochastic income, no equivalent correction for mean value of medical expenses is mentioned and appears not to have been made. Third, it does not describe the initial conditions for how agents first appear in the panel data simulations. It is also worth observing that the regressions in Table 3 are presumably based on the same panel data simulations as where used for Tables 1 & 2, in which case they do not account for the stochastic probability of death (the formulae in Appendix for how to calculate various elements of Tables 1 and 2 implicitly state that the underlying panel data must not include the stochastic probability of death as they include corrections for it).

The Figures of the paper all reproduce. The exact numbers of Tables 1, 2, and 3 do not, but the general outline of the results and findings for Tables 1, 2 & 3 is unchanged and support the conclusions of Hubbard, Skinner, and Zeldes (1994). While the results of Table 2 point in broadly the same direction they are much weaker in the replication.

[38] I have 'ballparked' them based on alternative sources for 1984 as fractions of US population (pg 1 of Russell Sage foundation Educational Attainment and Achievement chartbook.)

Table 28: Table 1 of Hubbard, Skinner and Zeldes (1994)

	Parameter Assumptions		—-Asset-Income Ratio—-				—-Savings Rate—-			
			No High School	High School	College	Aggre-gate	No High School	High School	College	Aggre-gate
Certain Lifespan,	$\delta = 0.03$,	$\gamma = 3$	7.03	5.78	4.13	5.51	0.08	0.07	0.05	0.06
earnings, and	$\delta = 0.03$,	$\gamma = 1$	4.63	3.93	3.05	3.79	0.06	0.05	0.04	0.05
out of pocket	$\delta = 0.03$,	$\gamma = 5$	3.86	2.33	0.78	2.14	0.04	0.02	0.01	0.02
medical	$\delta = 0.015$,	$\gamma = 3$	7.53	6.25	4.50	5.96	0.09	0.07	0.05	0.07
expenses	$\delta = 0.10$,	$\gamma = 3$	5.35	3.84	2.29	3.63	0.06	0.04	0.03	0.04
Uncertain Lifespan,	$\delta = 0.03$,	$\gamma = 3$	7.66	6.11	4.35	5.86	0.17	0.13	0.10	0.13
earnings, and										
medical										
expenses										
$\bar{C} = \$1$										
Uncertain Lifespan,	$\delta = 0.03$,	$\gamma = 3$	4.82	4.72	3.75	4.45	0.11	0.11	0.10	0.11
earnings, and	$\delta = 0.03$,	$\gamma = 1$	3.22	3.01	2.21	2.81	0.06	0.06	0.05	0.06
medical	$\delta = 0.03$,	$\gamma = 5$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
expenses	$\delta = 0.015$,	$\gamma = 3$	5.67	5.32	4.33	5.09	0.14	0.13	0.11	0.12
$\bar{C} = \$7000$	$\delta = 0.10$,	$\gamma = 3$	2.71	2.02	1.62	2.02	0.06	0.04	0.04	0.04

Note: Based on baseline grid size for assets of 751, and shocks of 15 and 15.

Table 1
Simulated Asset-Income Ratios and Saving Rates

	Parameter Assumptions	Asset-Income Ratio: No High School	Asset-Income Ratio: High School	Asset-Income Ratio: College	Asset-Income Ratio: Aggregate	Saving Rate: No High School	Saving Rate: High School	Saving Rate: College	Saving Rate: Aggregate
Certain lifespan, earnings, and out of pocket medical expenses	δ=.03, γ=3	2.13	2.53	2.27	2.37	.021	.025	.023	.024
	δ=.03, γ=1	2.13	2.53	2.27	2.37	.021	.025	.023	.024
	δ=.03, γ=5	2.13	2.53	2.27	2.37	.021	.025	.023	.024
	δ=.015, γ=3	2.79	3.27	2.78	3.03	.028	.032	.028	.030
	δ=.10, γ=3	0.68	0.62	0.86	0.70	.007	.006	.009	.007
Uncertain lifespan, earnings, and medical expenses $\bar{C}$ = \$1	δ=.03, γ=3	7.40	6.06	4.71	5.99	.167	.133	.119	.136
Uncertain lifespan, earnings, and medical expenses $\bar{C}$ = \$7000	δ=.03, γ=3	4.72	4.74	4.20	4.59	.108	.108	.108	.108
	δ=.03, γ=1	2.72	2.81	2.31	2.65	.053	.054	.051	.053
	δ=.03, γ=5	5.75	5.74	5.23	5.60	.140	.137	.139	.138
	δ=.015, γ=3	5.58	5.66	5.09	5.48	.131	.131	.132	.131
	δ=.10, γ=3	2.11	1.92	1.59	1.87	.044	.039	.040	.040

Source: Authors' calculations.
Note: The actual ratios (using data from the 1984 PSID) of total net worth to total family income are 3.69, 3.80, and 4.80 for the three groups, respectively. The ratio of private net worth to aggregate disposable income, using 1984 data from the Federal Reserve's *Flow of Funds Accounts*, is 4.64.

Table 29: Original Table 1 of Hubbard, Skinner and Zeldes (1994)

Table 30: Table 2 of Hubbard, Skinner and Zeldes (1994)

Percentage of Households with Consumption Approximately Equal to Income
(Absolute Average Savings Rate < 0.5 Percent of Income

	PSID			Simulated $\delta = 0.03$, Floor=$7000			Simulated $\delta = 0.10$, Floor=$1		
Age	NHS	HS	Col.	NHS	HS	Col.	NHS	HS	Col.
<29	0.362	0.059	0.060	0.092	0.148	0.229	0.002	0.005	0.088
30-39	0.157	0.064	0.040	0.091	0.075	0.177	0.075	0.080	0.237
40-49	0.067	0.017	0.025	0.090	0.066	0.085	0.164	0.173	0.186
50-59	0.103	0.032	0.011	0.137	0.138	0.124	0.137	0.158	0.146
60-69	0.095	0.020	0.038	0.097	0.133	0.161	0.084	0.095	0.127
Total	0.116	0.038	0.031	0.073	0.092	0.118	0.081	0.121	0.236
For Households with Initial Assets < 0.5 x Average Income									
<29	0.389	0.080	0.069	0.103	0.155	0.234	0.000	0.000	0.086
30-39	0.205	0.101	0.052	0.110	0.093	0.208	NaN	0.000	0.292
40-49	0.133	0.045	0.031	0.066	0.060	0.111	NaN	0.000	0.265
50-59	0.272	0.114	0.135	0.053	0.034	0.067	NaN	NaN	0.155
60-69	0.302	0.083	0.381	0.095	0.068	0.071	NaN	0.331	0.453
Total	0.252	0.087	0.060	0.086	0.147	0.218	0.040	0.238	0.464

NHS=No high-school, HS=High-school, Col.=College. Numbers for PSID are those of original study, not part of replication.

Note: Based on baseline grid size for assets of 751, and shocks of 15 and 15.

Table 31: Original Table 2 of Hubbard, Skinner and Zeldes (1994)

Table 2:
Percent of Actual (PSID) and Simulated Households With Consumption Approximately Equal to Income (Absolute Average Saving Rate < 0.5 Percent of Income)

	PSID			Simulated $\delta = .03$, Floor = \$7000			Simulated $\delta = .10$, Floor = \$1		
Age	NHS	HS	Col.	NHS	HS	Col.	NHS	HS	Col.
<29	.362	.059	.060	.332	.163	.263	.000	.000	.019
30-39	.157	.064	.040	.231	.093	.167	.000	.000	.073
40-49	.067	.017	.025	.212	.045	.035	.023	.005	.018
50-59	.103	.032	.011	.148	.027	.002	.023	.026	.007
60-69	.095	.020	.038	.131	.038	.015	.018	.005	.031
Total	.116	.038	.031	.204	.067	.086	.013	.007	.033
	For Households with Initial Assets < 0.5 × Average Income								
<29	.389	.080	.069	.494	.226	.377	.000	.000	.040
30-39	.205	.101	.052	.484	.231	.314	.000	n/a	.070
40-49	.133	.045	.031	.613	.310	.189	.000	n/a	.000
50-59	.272	.114	.135	.565	.331	.000	n/a	n/a	.000
60-69	.302	.083	.381	.735	.362	.360	n/a	n/a	n/a
Total	.252	.087	.060	.545	.250	.303	.000	.000	.043

Source: PSID and authors' calculations.

Notes: Average saving rates equal average annual saving during a five-year period (1984–89 for data from the PSID) divided by the average annual real income over the period. The tabulations reported in the second table are based on households whose initial assets (in 1984 in the PSID data) are less than half their average income. The symbol n/a denotes no simulated household in this cell.

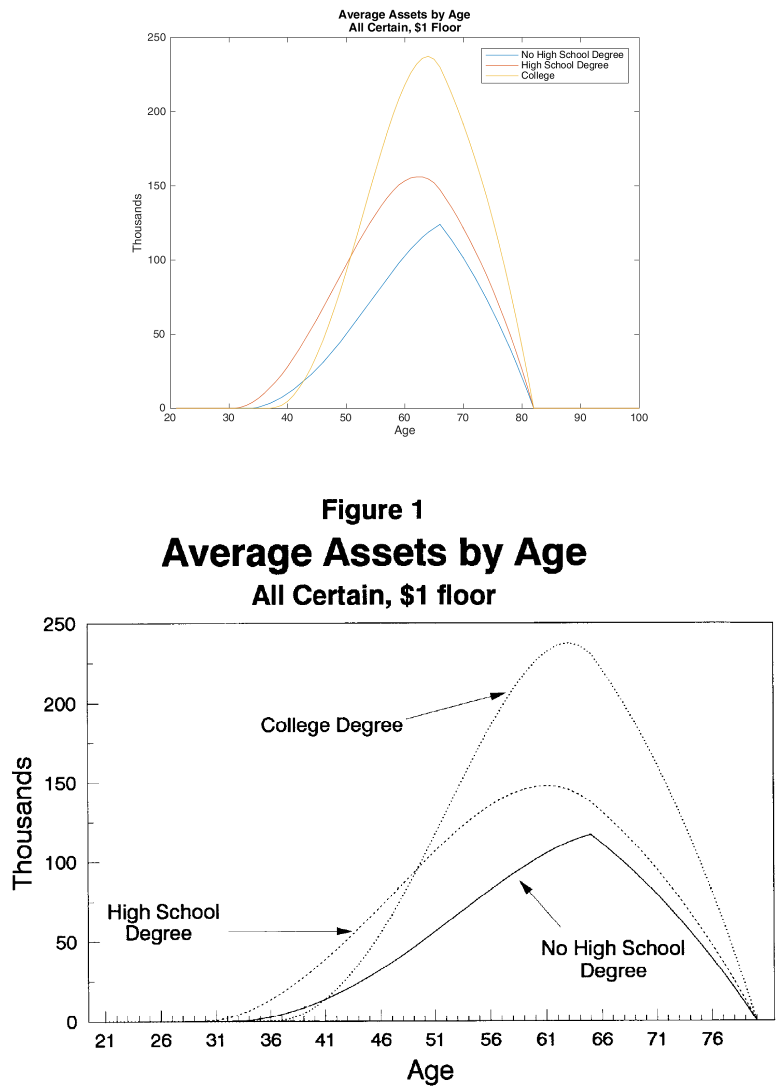


Figure 8: Figure 1 of Hubbard, Skinner and Zeldes (1994)

Table 32: Table 3 of Hubbard, Skinner and Zeldes (1994)

Dependent Varialbe	$\Delta C*$	$\Delta ln(C)*$	$\Delta ln(C)**$	$\Delta ln(C)**$
		$\delta = 0.03$, Floor=$7000		
ΔY	0.397			
	(26.94)			
$\Delta ln(Y)$		0.648	0.640	0.431
		(43.78)	(45.58)	(14.27)
Age				0.002
				(8.50)
Age^2				-0.000
				(-13.46)

Source: Simulated data from model under the benchmark case ($\delta = 0.03$, $\gamma = 3$) with a consumption floor of $7000. Absolute values of t-statistics are in parentheses. The Campbell and Mankiw (1989) coefficient (in levels, corresponding to the first column) is 0.469, and the Lusardi (1993) coefficient (in logs, corresponding to columns 2 through 4) is 0.409.

*: Instruments are two and three year lags of consumption and income, as well as age and age-squared.

**: Instruments are one, two and three year lags of consumption and income, as well as age and age-squared.

Note: Original regressions by Cambell-Mankiw were on aggregate data. Here are on microdata.

Note: Based on baseline grid size for assets of 751, and shocks of 15 and 15.

Table 33: Original Table 3 of Hubbard, Skinner and Zeldes (1994)

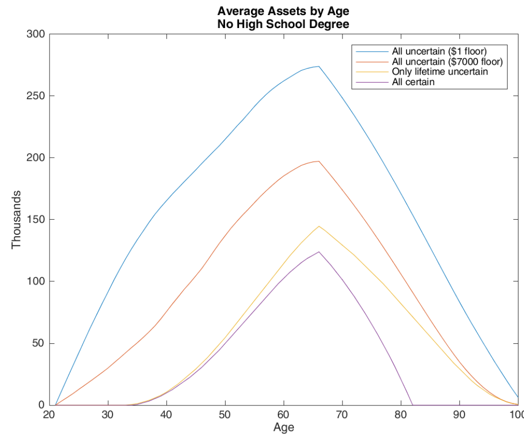
Table 3:
Campbell-Mankiw-Lusardi Euler Equations
Using Simulated Data from the Dynamic Programming Model

Dependent Variable	ΔC^*	$\Delta ln(C)^*$	$\Delta ln(C)^{**}$	$\Delta ln(C)^{**}$
		$\delta = .03$, Floor = $7000		
ΔY	0.411			
	(38.74)			
$\Delta ln(Y)$		0.512	0.498	0.489
		(40.11)	(41.29)	(14.63)
Age				0.109
				(0.61)
Age^2				-0.163
				(1.10)

Source: Simulated data (38520 observations) from the dynamic programming model under the benchmark case ($\delta = 0.03, \gamma = 3$) with a consumption floor of $7000. Absolute values of t-statistics are in parentheses. The Campbell and Mankiw (1989) coefficient (in levels, corresponding to the first column) is 0.469, and the Lusardi (1993) coefficient (in logs, corresponding to columns 2 through 4) is 0.409.

* Instruments are ΔC and ΔY, each lagged two and three years, age, and age^2.

** Instruments are ΔC and ΔY, each lagged one, two, and three years, age, and age^2.



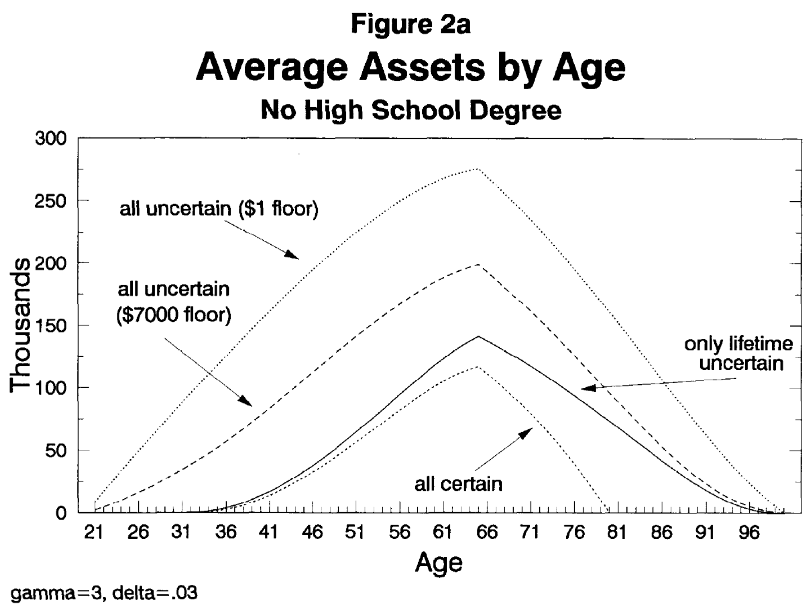
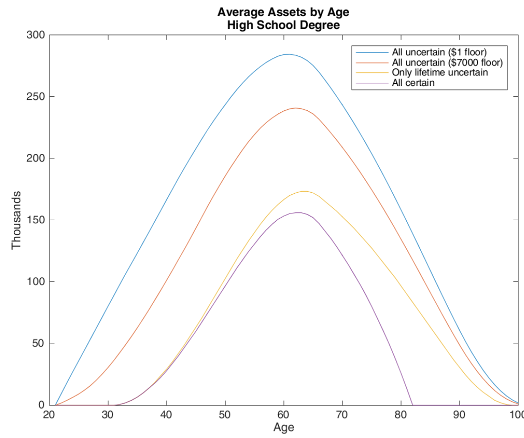


Figure 9: Figure 2a of Hubbard, Skinner and Zeldes (1994)



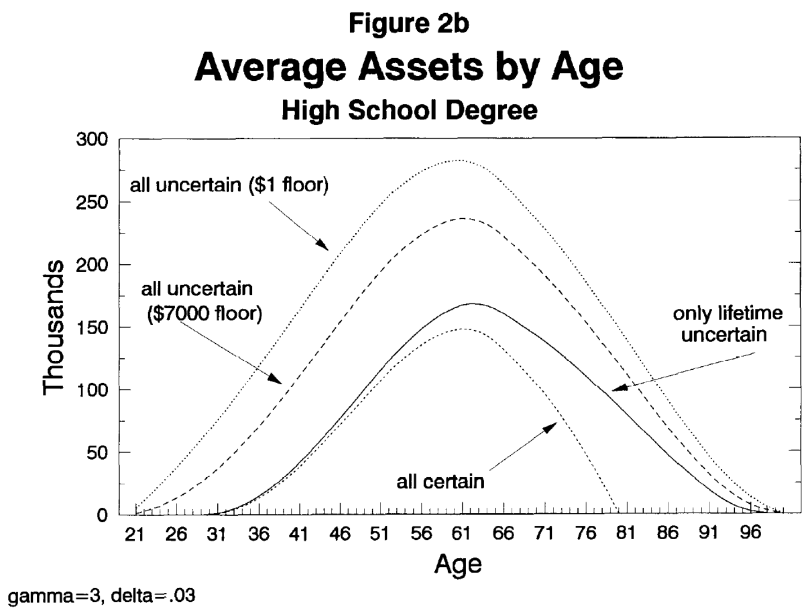
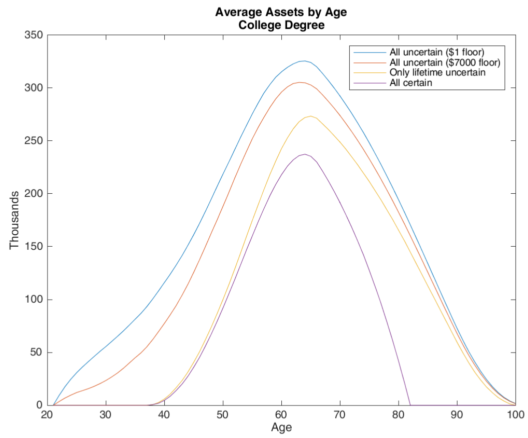


Figure 10: Figure 2b of Hubbard, Skinner and Zeldes (1994)



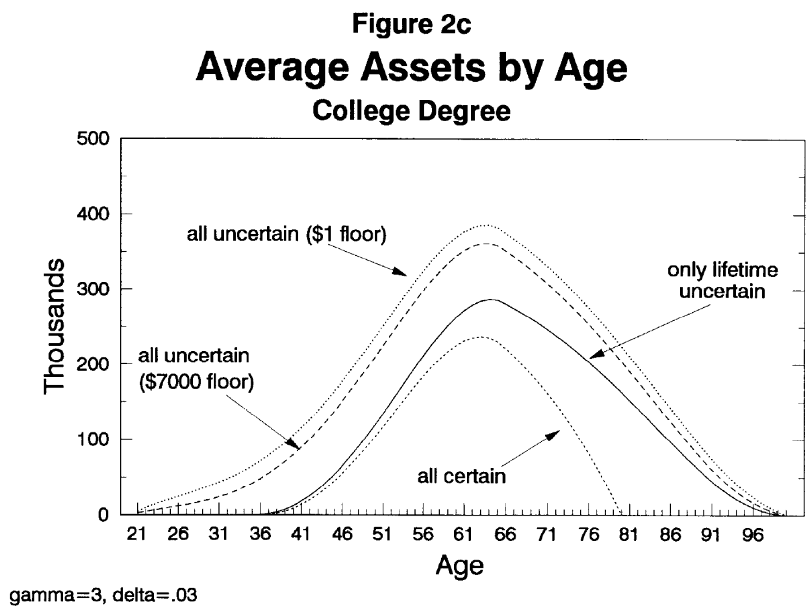
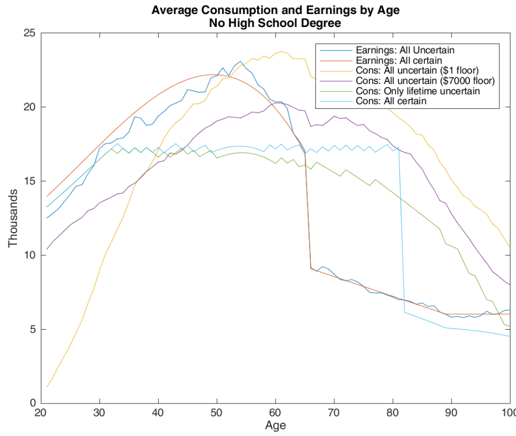


Figure 11: Figure 2c of Hubbard, Skinner and Zeldes (1994)



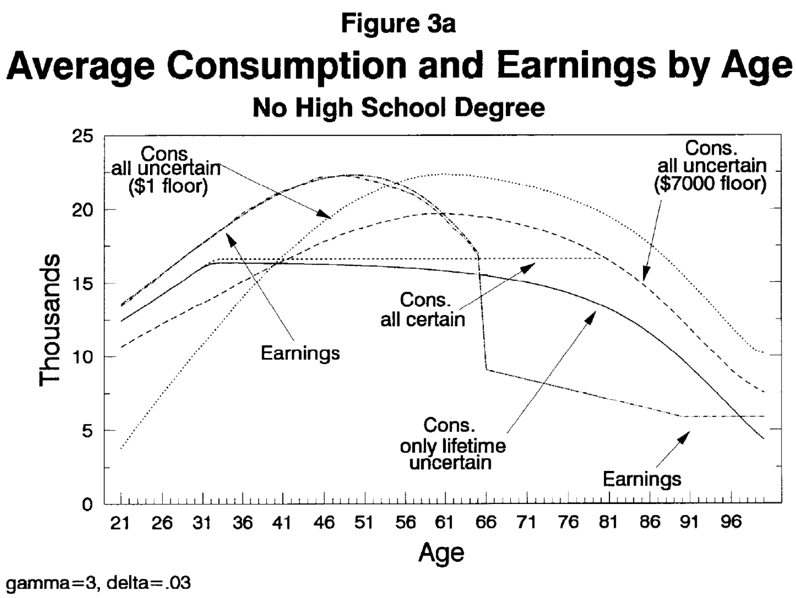
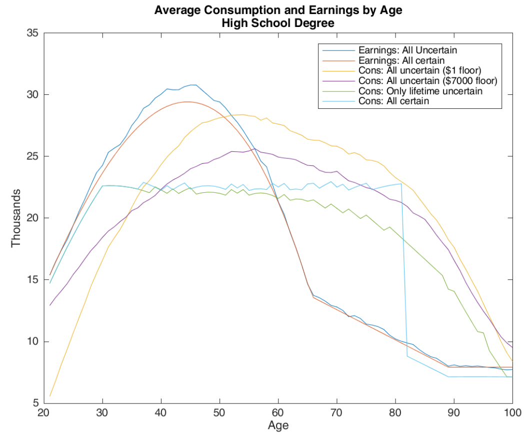


Figure 12: Figure 3a of Hubbard, Skinner and Zeldes (1994)



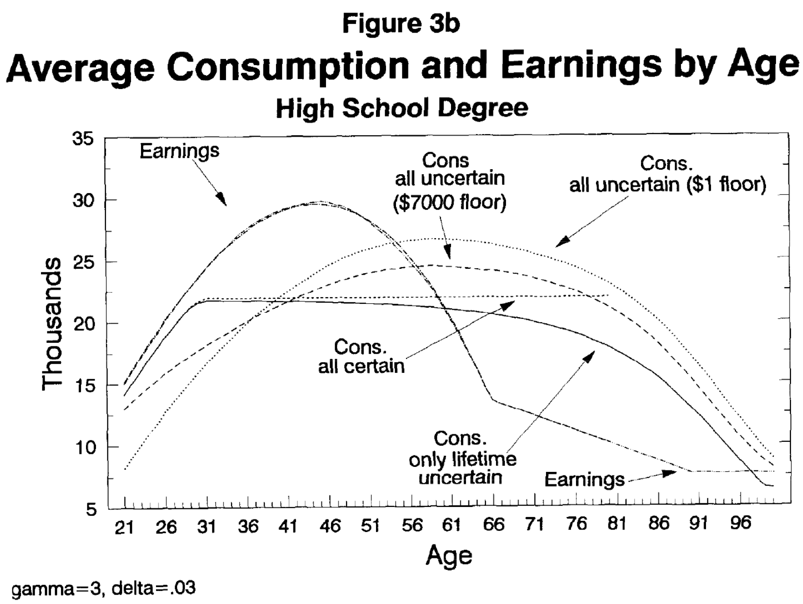
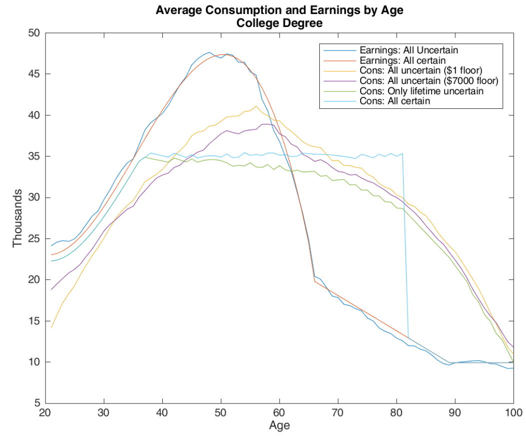


Figure 13: Figure 3b of Hubbard, Skinner and Zeldes (1994)



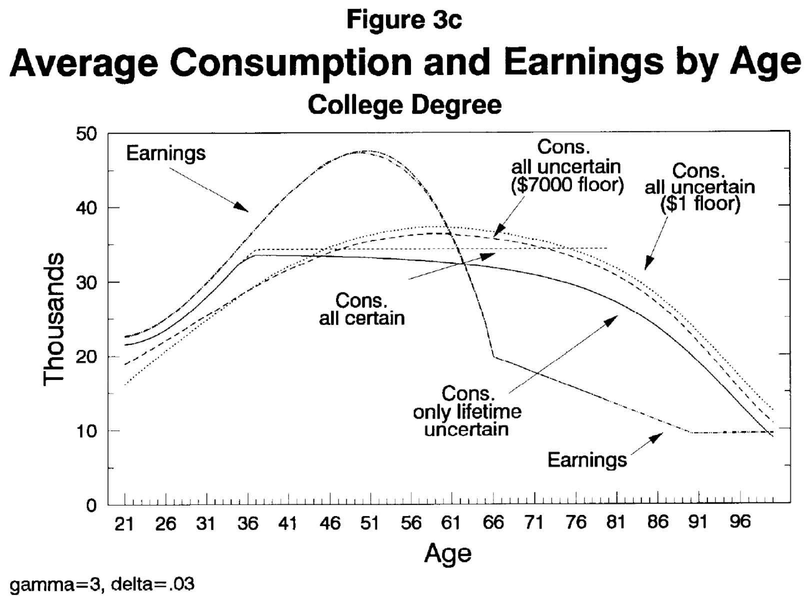


Figure 14: Figure 3c of Hubbard, Skinner and Zeldes (1994)

A.8 Huggett (1996) - Wealth Distribution in Life-cycle Economies

Huggett (1996) explores the ability of a consumption-savings life-cycle model to generate a realistic wealth distribution. He shows that uncertain lifespan and incomes shocks are key to getting the right general form of the wealth distribution, but are unable to get the large shares of the very top percentiles of the wealth distribution. He notes that the model seems capable of generating the low wealth levels of the bottom of the distribution (due to pensions, which are not too tightly linked to income). Transfers (inheritance) seem to play a minor role, but that may be as they are not as narrowly focused on just one part of the population (they are spread lump-sum across the whole population).

In terms of replication the only subtlety was that the definition of the exogenous shock transitions was non-standard and so had to be implemented specifically. I had originally misread the exogenous shock as being 19 states, 18 with an 'extra', but in fact the 'extra' is intended to be understood as part of the 18. One thing not clear from paper is what is the 'unnormalized' deterministic earnings profile, since Figure 1 is the normalized earnings, and these need to be multiplied by 0.5289; without this I was unable to replicate Figure 2, with this Figure 2 replicates exactly.[39] Figure 1 of Huggett (1996) is technically computational, but since it just shows some of the calibrated parameter values I omit it as it trivially replicates (it is generated by the replication codes for any interested reader).

The model is a general equilibrium OLG model. The finite-horizon value function problem has one exogenous state (an earnings shock), one endogenous state (assets), and 79 periods. The household value function problem is given by

$$\begin{aligned} V(a,z,j) = &\max_{c,a'} \frac{c^{1-\sigma}}{1-\sigma} + \beta s_{j+1} E_j[V(a',z',j)|z] \\ &\text{subject to } c + a' \leq a(1+r(1-\tau)) + (1-\tau-\theta)e(z,j)w + T + b_j \\ &\qquad\qquad a' \geq \underline{a} \end{aligned}$$

There are $J = 79$ periods and $V(a,z,J+1) = 0$ for all a & z. So household faces income shocks ($e(z,j)$) and solve a consumption-savings problem of choosing consumption c and next period assets a'. There are some basic taxes τ and θ, the later of which is used to fund pensions b_j that are received once retirement age is reached. When people die their assets are redistributed lump-sum accoss the living as T.[40] The lower bound on assets $\underline{a}$ is either $-w$ or 0 depending on the calibration.

The earnings process $e(z,j)$ consists of a (log) deterministic earnings profile $\bar{y}_j$ which is multiplied by a (log) AR(1) transitory shock, (log of) z_j: so, $e(z,j) = exp(\bar{y}_j + z_j)$, and $z_j = \gamma z_{j-1} + \epsilon$. The

[39] A friend passed me an old Homework handout by Dean Corbae from 2014 which involved a slightly simplified version of this model, and which describe this 0.5289 and included files containing the deterministic earnings profile. The 0.5289 is related to the share of the population of working age and is used to normalize the model so that average earnings across the whole population are equal to 1. It is not essential in any way to solving the model but means that certain model moments, namely aggregate earnings, are known without computing them.

[40] Paper states the additional constraint that $a' \geq 0$ for $j = J$, but is clear from Figure 2 that this was not actually imposed in code.

ϵ shock has 17 states equally spaced from -4 to +4 standard deviations, with one extra (18th) state at +6 standard deviations; it is normally distributed so transition probabilities can be easily computed by quadrature (same principle as used by Tauchen method).

The initial distribution of the shocks on the z-grid is given by quadrature using σ_{z_1} as the standard deviation of z_1. All households start with zero assets. The simulation of the agents distribution is standard, based on optimal policy function and idiosyncratic shocks.

The model has three general equilibrium constraints, the first is that the interest rate r equals the marginal product of capital minus the depreciation rate δ (Huggett (1996) considers this in terms of a general equilibrium condition for K/Y, rather than interest rates, but the two are mathematically equivalent). The second is on the benefit rate for pensions b, which must equal the revenue raised by the payroll tax θ. The third is that the the (total across the population of the) lump-sum transfer of accidental bequests T much equal the assets left behind by people on dying.

For more details on the model see Huggett (1996), or the example code at github.com/vfitoolkit/VFItoolkit-matlab-examples (this example is provided in addition to the replication codes, and is intended to show how to use the VFI Toolkit to solve this kind of model).

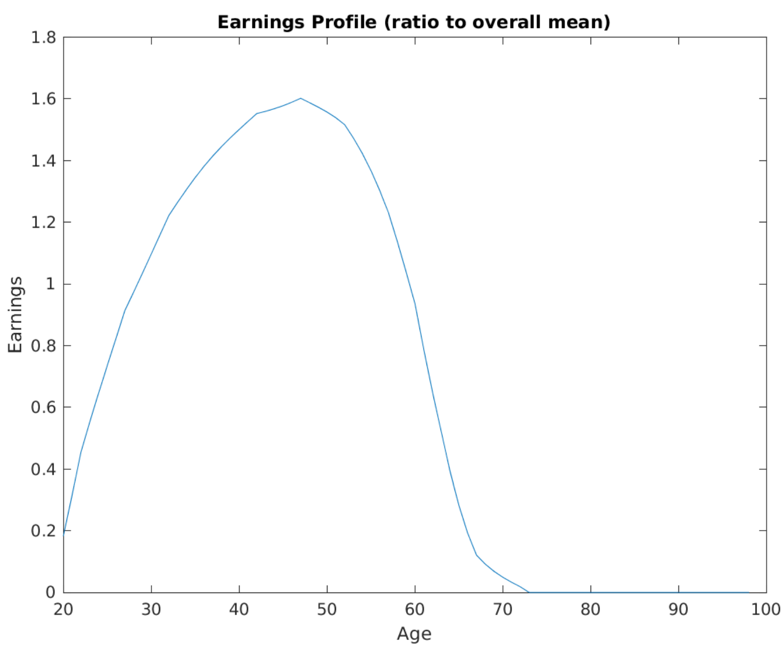


Figure 15: Figure 1 of Huggett (1996)

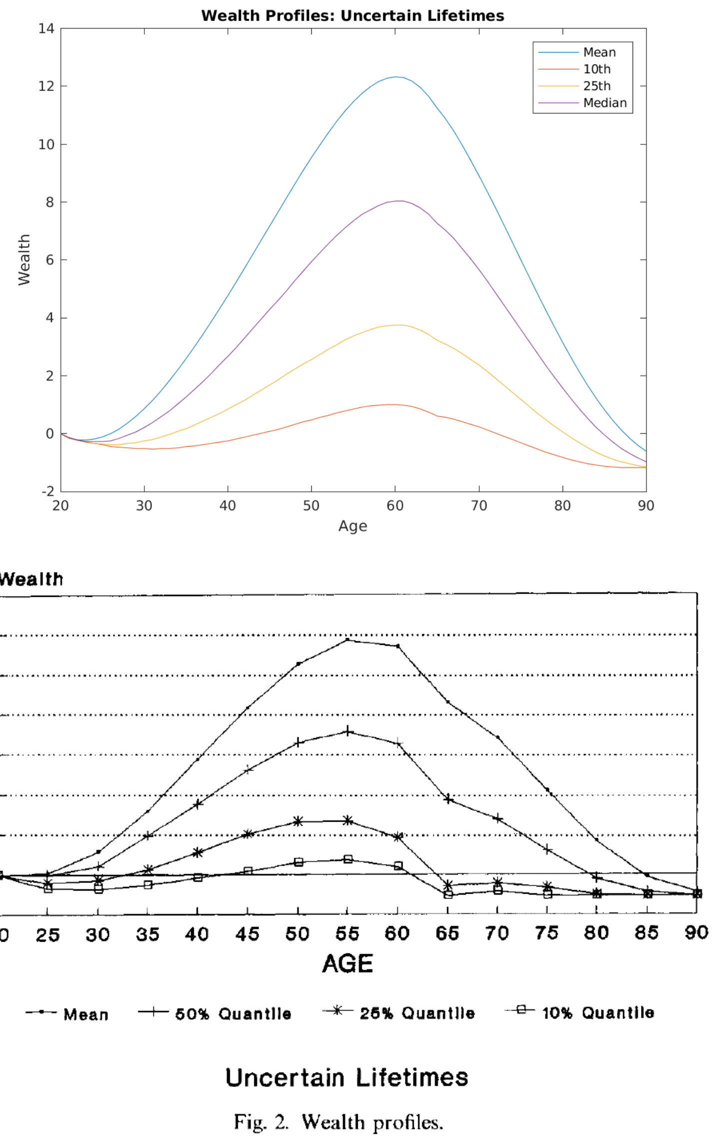


Figure 16: Figure 2 of Huggett (1996)

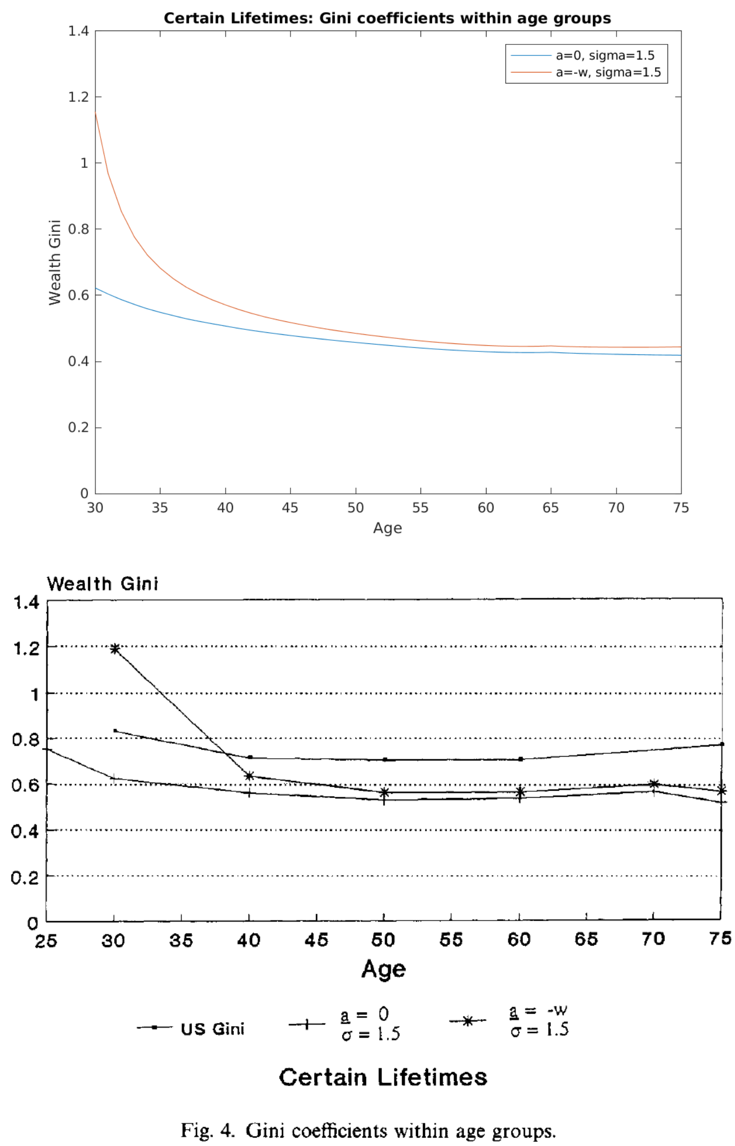


Figure 17: Figure 4 of Huggett (1996)

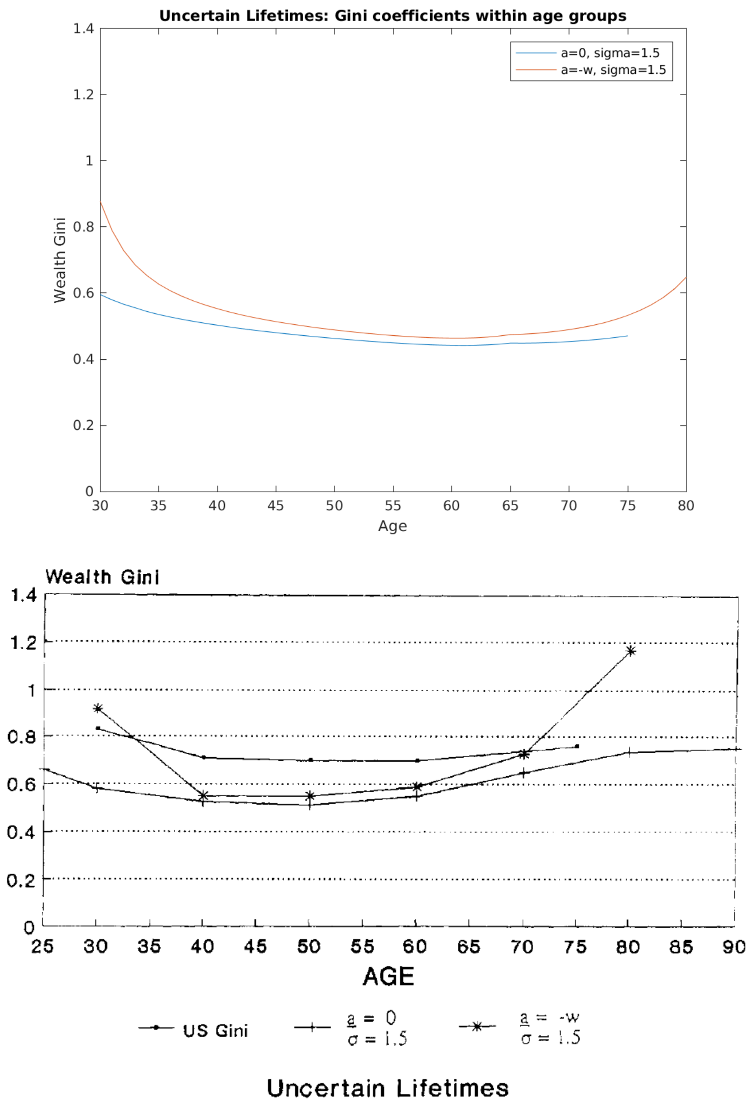


Fig. 5. Gini coefficients within age groups.

Figure 18: Figure 5 of Huggett (1996)

Table 34: Table 3 of Huggett (1996)

Wealth Distribution (risk aversion coefficient $\sigma = 1.5$)

Credit limit $\underline{a}$	Earnings shock σ_ϵ^2	K/Y	Transfer wealth ration	Wealth Gini	Percentage wealth in the top 1%	negative 5%	20 %	Zero or wealth (%)
US Economy		3.0	0.78–1.32	0.72	28	49	75	5.8–15.0
Certain Lifetimes								
0	0.000	4.1	0.00	0.64	8.4	27.0	63.4	17.2
-w	0.000	4.0	-0.00	0.70	9.2	29.1	68.0	29.8
0	0.045	4.2	-0.00	0.66	9.1	28.8	66.0	17.0
-w	0.045	4.1	0.00	0.70	9.7	30.6	69.3	28.2
Uncertain Lifetimes								
0	0.000	4.0	0.43	0.62	8.0	25.9	61.7	15.4
-w	0.000	3.9	0.39	0.68	9.0	28.3	66.2	26.7
0	0.045	4.1	0.40	0.65	9.2	29.0	65.6	15.1
-w	0.045	4.0	0.36	0.70	9.9	30.8	69.2	26.7

Note: Based on baseline grid size for assets of 1501, and shocks of 18.

Figure 19: Original Table 3 of Huggett (1996)

Table 3
Wealth distribution (risk aversion coefficient $\sigma = 1.5$)

Credit limit $\underline{a}$	Earnings shock σ_ε^2	K/Y	Transfer wealth ratio	Wealth Gini	Percentage wealth in the top 1%	5%	20%	Zero or negative wealth (%)
US economy		3.0	0.78–1.32	0.72	28	49	75	5.8–15.0
Certain lifetimes								
0.0	0.00	2.9	0.0	0.47	2.4	11.6	42.8	14.0
$-w$	0.00	2.8	0.0	0.54	2.7	12.7	46.6	25.0
0.0	0.045	3.2	0.0	0.70	10.8	32.4	68.9	19.0
$-w$	0.045	3.1	0.0	0.74	11.1	33.8	72.3	24.0
Uncertain lifetimes								
0.0	0.00	3.1	1.03	0.46	2.5	11.7	42.8	11.0
$-w$	0.00	3.0	1.07	0.49	2.6	12.1	44.3	12.0
0.0	0.045	3.4	0.84	0.69	10.9	32.9	70.0	17.0
$-w$	0.045	3.2	0.89	0.76	11.8	35.6	75.5	24.0

Table 35: Table 4 of Huggett (1996)

Wealth Distribution (risk aversion coefficient $\sigma = 3.0$)

Credit limit $\underline{a}$	Earnings shock σ_ϵ^2	K/Y	Transfer wealth ration	Wealth Gini	Percentage wealth in the top 1%	negative 5%	20 %	Zero or wealth (%)
US Economy		3.0	0.78–1.32	0.72	28	49	75	5.8–15.0
Certain Lifetimes								
0	0.000	3.3	0.00	0.66	8.7	27.9	66.0	21.2
-w	0.000	3.2	0.00	0.79	10.6	32.8	74.9	38.8
0	0.045	3.6	0.00	0.65	9.1	28.8	65.7	14.7
-w	0.045	3.5	0.00	0.69	9.6	30.1	68.2	25.3
Uncertain Lifetimes								
0	0.000	3.2	0.72	0.65	8.6	27.2	64.4	20.3
-w	0.000	3.1	0.77	0.77	10.2	31.6	72.6	35.9
0	0.045	3.4	0.54	0.65	9.2	28.8	65.6	14.9
-w	0.045	3.3	0.51	0.69	9.6	30.2	68.2	25.3

Note: Based on baseline grid size for assets of 1501, and shocks of 18.

Figure 20: Original Table 4 of Huggett (1996)

Table 4
Wealth distribution (risk aversion coefficient $\sigma = 3.0$)

Credit limit $\underline{a}$	Earnings shock σ_ε^2	K/Y	Transfer wealth ratio	Wealth Gini	Percentage wealth in the top 1%	5%	20%	Zero or negative wealth (%)
US economy		3.0	0.78–1.32	0.72	28	49	75	5.8–15.0
Certain lifetimes								
0.0	0.00	2.3	0.0	0.51	2.7	13.0	46.1	21.0
$-w$	0.00	2.0	0.0	0.62	3.3	14.7	52.5	29.0
0.0	0.045	2.9	0.0	0.66	10.5	32.0	66.6	3.0
$-w$	0.045	2.8	0.0	0.73	11.4	34.0	73.1	23.0
Uncertain lifetimes								
0.0	0.00	2.5	2.54	0.50	2.6	12.6	45.3	21.0
$-w$	0.00	2.3	4.30	0.61	3.1	14.3	51.1	29.0
0.0	0.045	3.0	1.28	0.72	12.1	35.7	71.7	19.0
$-w$	0.045	2.8	1.75	0.84	13.8	40.4	80.2	40.0

A.9 Castaneda, Diaz-Gimenez, and Rios-Rull (2003) - Accounting for the U.S. Earnings and Wealth Inequality

Presents a model capable of capturing the inequality in the US earnings and wealth distributions, the later being a breakthrough at that time. Key to the model is the process of labour efficiency units combined with use of dynasties (infinite horizon value functions). Very high income dynasties expect their descendents to be much lower income and so save large fractions of their income, this leads to high levels of wealth inequality.

Only the first 9 Tables are reproduced here. Tables 10 and up are based on alternative calibrations and the parameter values underlying those alternatives have not survived and nor have the weights used as part of those alternative calibrations. Hence they could not be replicated. The replication codes do include code that implements alternative calibrations, and which work for a given combination of weights, but without the 'correct' weights they are not actually used for anything.

The model to be solved is now given, starting with the value function problem of the household.

$$V(a,s) = \max_{\substack{c \geq 0 \\ \tilde{a} \in \mathcal{A} \\ 0 \leq h \leq \ell}} \frac{c^{1-\sigma_1}}{1-\sigma_1} + \frac{(\ell - h)^{1-\sigma_2}}{1-\sigma_2} + \beta \sum_{s' \in S} \Gamma_{ss'} \, V[a'(\tilde{a}), s'], \tag{7}$$

$$s.t. \quad c + \tilde{a} = y - \tau(y) + a, \tag{8}$$

$$y = a\,r + e(s)\,h\,w + \omega(s), \tag{9}$$

$$\tau(y) = a_0[y - (y^{-a_1} + a_2)^{-1/a_1}] + a_3 y \tag{10}$$

$$a'(\tilde{a}) = \begin{cases} \tilde{a} - \tau_e(\tilde{a}) \text{ if } s \in \mathcal{R} \text{ and } s' \in \mathcal{E}, \\ \tilde{a} \text{ otherwise.} \end{cases} \tag{11}$$

$$\Gamma_{ss'} \tag{12}$$

Note: for VFI Toolkit this is a Case 2 value function problem (in nomenclature of SLP1989), as next period assets cannot be chosen directly due to existence of estate taxes. Notice that household income, which we denote by y, includes three terms: capital income, $y_k = a\,r$, labor income, $y_l = e(s)\,h\,w$, and retirement pensions, ω. Every household can earn capital income. Only workers can earn labor income. And only retirees receive retirement pensions. We denote labour supply by h, while CDGRR2003 use l. The estate taxes are

$$\tau_e(\tilde{a}) = \begin{cases} 0 \text{ for } \tilde{a} \leq \underline{a} \\ (a - \tilde{a})\tau_e \text{ for } \tilde{a} > \underline{a} \end{cases}$$

The retirement benefit $\omega(s)$ is zero for working age (first four states of s) and a constant for retirees (fifth to eigth states of s).

There is a representative firm with Cobb-Douglas production function (and perfect competition): $Y = K^{\theta}L^{1-\theta}$, where L is labour supply $L = \int he(s)d\mu$ (μ is agent distribution), as distinct from hours worked $H = \int hd\mu$. (Note: I depart from CDGRR2003 in that I use the reverse the notation of l and H relative to them.)

Since the actual definition for a stationary competitive equilibrium in this economy is essentially the same as that of Aiyagari (1994) I largely omit it here (I skip the definitions of some aggregates, and the law-of-motion for the agent distribution). I do provide the general equilibrium conditions. There are two. The first general equilibrium condition is the 'same' as that of Aiyagari (1994), namely that the interest rate (the rate of return on capital) is equal to the marginal product of capital (minus the depreciation rate, as it is the net return); $r = \theta K^{\theta-1}L^{1-\theta} - \delta$.[41] The second general equilibrium condition is the Government budget constraint, $Tr + G = T$, where T is tax revenue (from income and estate taxes), and Tr is transfers (the retiree benefits ω). Finding the general equilibrium amounts to finding r and G that satisfy these two general equilibrium conditions.

An important part of this 'infinite-horizon dynasties' setup is the transition process on s which determines 'stochastic aging' and also 'productivity level'. s has eight states, four producitivity levels crossed with two 'ages' (worker and retiree). The eight-by-eight transition matrix $\Gamma_{ss'}$ plays an important role in this model and is described in full below. This description also contains a correction to a typo in the original paper (in the equations relating to ϕ_1 and ϕ_2.

We use the one-dimensional shock, s, to denote the household's random age and random endowment of efficiency labor units jointly.[42]

The process on s is independent and identical across households, and follows a finite state Markov chainwith conditional transition probabilities given by $\Gamma = \Gamma(s' \mid s) = Pr\{s_{t+1} = s' \mid s_t = s\}$, where s and $s' \in S$. We assume that s takes values in one of two possible J–dimensional sets, $\mathcal{E}$ and $\mathcal{R}$. Therefore the formal description of set S is $S = \mathcal{E} \cup \mathcal{R} = \{1, 2, \dots, J\} \cup \{J+1, J+2, \dots, 2J\}$. When a household draws shock $s \in \mathcal{E}$, it is a worker and its endowment ofefficiency labor units is $e(s) > 0$. When a household draws shock $s \in \mathcal{R}$ it is a retiree. When a household's shock changes from $s \in \mathcal{E}$ to $s' \in \mathcal{R}$, we say that it has retired and when it changes from $s \in \mathcal{R}$ to $s' \in \mathcal{E}$, we say that it has died and has been replaced by a working-age descendant. When a household dies, its estate is liquidated, and its descendant inherits a fraction $1 - \tau_e(\tilde{a})$ of the estate, where $\tilde{a}$ denotes the value of the household's stock of wealth at the end of the period, and $\tau_e(\tilde{a})$ represents estate taxes.

[41] Note that unlike Aiyagari (1994), labour supply is endogenous and so actually has to be calculated as part of evaluating this.

[42] To ease interpretation, we use $e(s)$ to denote the endowment of efficiency labour units, and simply have s take integer values. In principle, s and $e(s)$ could be combined to be one object. By seperating the efficiency labor units, $e(s)$, from the actual values taken by s, it is easier to see how earnings ability is transferred from one generation to the next, the transitions of s, without being confused by the fact that $e(s) = 0$ in all of the retirement states.

This specification of the joint age and endowment process implies that the transition probability matrix, Γ, controls the demographics of the model economy, the life-cycle profile of earnings, and their intergenerational persistence (in combination with hours worked choices). When we come to calibrating this markov process it is done based on these issues; demographics, life-cycle profile of earnings, and intergenerational persistence of earnings.

To specify the process on s (and the values for $e(s)$) we must choose the values of $(2J)^2 + J$ parameters, of which $(2J)^2$ are the conditional transition probabilities and the remaining J are the values of the endowment of efficiency labor units. To reduce this large number of parameters, we impose some additional restrictions on matrix Γ. To understand these restrictions better, it helps to consider the following partition of matrix Γ:

$$\Gamma = \begin{bmatrix} \Gamma_{\mathcal{E}\mathcal{E}} & \Gamma_{\mathcal{E}\mathcal{R}} \\ \Gamma_{\mathcal{R}\mathcal{E}} & \Gamma_{\mathcal{R}\mathcal{R}} \end{bmatrix}$$

Submatrix $\Gamma_{\mathcal{E}\mathcal{E}}$ contains the transition probabilities of working-age households that are still of working-age one period later. Since we impose no restrictions on these transitions, to characterize $\Gamma_{\mathcal{E}\mathcal{E}}$ we must choose the values of J^2 parameters.

Submatrix $\Gamma_{\mathcal{E}\mathcal{R}}$ describes the transitions from the working-age states into the retirement states. The value of this submatrix is $\Gamma_{\mathcal{E}\mathcal{R}} = p_{e\varrho}I$, where $p_{e\varrho}$ is the probability of retiring and I is the identity matrix. This is because we assume that every working-age household faces the same probability of retiring, and because we use only the last realization of the working-age shock to keep track of the earnings ability of retirees. Consequently, to characterize $\Gamma_{\mathcal{E}\mathcal{R}}$ we must choose the value of only one parameter.

Submatrix $\Gamma_{\mathcal{R}\mathcal{E}}$ describes the transitions from the retirement states into the working-age states that take place when a retiree exits the economy and is replaced by a working-age descendant. The rows of this submatrix contain a two parameter transformation of the stationary distribution of $s \in \mathcal{E}$, which we denote by $\gamma^*_{\mathcal{E}}$. This transformation allows us to control both the life-cycle profile of earnings and its intergenerational correlation. Intuitively, the transformation amounts to shifting the probability mass from $\gamma^*_{\mathcal{E}}$ towards both the first row of $\Gamma_{\mathcal{R}\mathcal{E}}$ and towards its diagonal. The exact description of this is given below. Consequently, to characterize $\Gamma_{\mathcal{R}\mathcal{E}}$ we must choose the value of the two shift parameters.

Finally, submatrix $\Gamma_{\mathcal{R}\mathcal{R}}$ contains the transition probabilities of retired households that are still retired one period later. The value of this submatrix is $\Gamma_{\mathcal{R}\mathcal{R}} = p_{\varrho\varrho}I$, where $(1 - p_{\varrho\varrho})$ is the probability of exiting the economy. This is because the type of retired households never changes, and because we assume that every retired household faces the same probability of exit. Therefore, to identify this submatrix we must choose the value of only one parameter.

To keep the dimension of the process on s as small as possible while still being able to achieve our calibration targets, we choose $J = 4$. Therefore, to characterize the process on s (and the values of $e(s)$), we must choose the values of $(J^2+4)+J=24$ parameters. Notice that we have not yet imposed that Γ must be a Markov matrix. When we do this, the number of free parameters is reduced to 20.

In practice the easiest is actually to normalize $\Gamma_{\mathcal{E}\mathcal{E}}$, with normalization of Γ following trivially from this. The choice of the normalization is, for computational reasons best done to the diagonals of $\Gamma_{\mathcal{E}\mathcal{E}}$; e.g., set $p^{ee}_{22} = 1 - p^{ee}_{21} - p^{ee}_{23} - p^{ee}_{24}$; where $\Gamma_{\mathcal{E}\mathcal{E}} = (1-p_{e\varrho}) * [p^{ee}_{ij}]$. Since the non-diagonals are smaller by having the diagonals given by whatever was leftover to make the row sum up to one we avoided the problem that they may end up being negative — something that occoured in an earlier version of the codes where we used the last element of each row for the normalization. This saves throwing out many 'evaluations' of parametrizations (during calibration) due to that specific vector of parameters containing negative elements in the transition matrix. (Note: this normalization of diagonals is part of the replication, and is different (improved) from that of the original paper.)

The following explains the definition of parameters ϕ_1 and ϕ_2, and how they affect the transition matrix. Let p_{ij} denote the transition probability from $i \in \mathcal{R}$ to $j \in \mathcal{E}$, let γ^*_i be the invariant measure of households that receive shock $i \in \mathcal{E}$, and let ϕ_1 and ϕ_2 be the two parameters that shift the probability mass towards the diagonal and towards the first column of submatrix $\Gamma_{\mathcal{E}\mathcal{E}}$, then the recursive procedure that we use to compute the p_{ij} is the following:

- Step 1: First, we use parameter ϕ_1 to shift the probability mass from a matrix with vector $\gamma^*_{\mathcal{E}} = (\gamma^*_1, \gamma^*_2, \gamma^*_3, \gamma^*_4)$ in every row towards its diagonal, as follows:

$$
\begin{aligned}
p_{51} &= \gamma_1^* + \phi_1\gamma_2^* + \phi_1^2\gamma_3^* + \phi_1^3\gamma_4^* \\
p_{52} &= (1-\phi_1)[\gamma_2^* + \phi_1\gamma_3^* + \phi_1^2\gamma_4^*] \\
p_{53} &= (1-\phi_1)[\gamma_3^* + \phi_1\gamma_4^*] \\
p_{54} &= (1-\phi_1)\gamma_4^* \\
p_{61} &= (1-\phi_1)\gamma_1^* \\
p_{62} &= \phi_1\gamma_1^* + \gamma_2^* + \phi_1\gamma_3^* + \phi_1^2\gamma_4^* \\
p_{63} &= (1-\phi_1)[\gamma_3^* + \phi_1\gamma_4^*] \\
p_{64} &= (1-\phi_1)\gamma_4^* \\
p_{71} &= (1-\phi_1)\gamma_1^* \\
p_{72} &= (1-\phi_1)[\phi_1\gamma_1^* + \gamma_2^*] \\
p_{73} &= \phi_1^2\gamma_1^* + \phi_1\gamma_2^* + \gamma_3^* + \phi_1\gamma_4^* \\
p_{74} &= (1-\phi_1)\gamma_4^* \\
p_{81} &= (1-\phi_1)\gamma_1^* \\
p_{82} &= (1-\phi_1)[\phi_1\gamma_1^* + \gamma_2^*] \\
p_{83} &= (1-\phi_1)[\phi_1^2\gamma_1^* + \phi_1\gamma_2^* + \gamma_3^*] \\
p_{84} &= \phi_1^3\gamma_1^* + \phi_1^2\gamma_2^* + \phi_1\gamma_3^* + \gamma_4^*
\end{aligned}
$$

• Step 2: Then for $i = 5, 6, 7, 8$ we use parameter ϕ_2 to shift the resulting probability mass towards the first column as follows:

$$
\begin{aligned}
p_{i1} &= p_{i1} + \phi_2 p_{i2} + \phi_2^2 p_{i3} + \phi_2^3 p_{i4} \\
p_{i2} &= (1-\phi_2)[p_{i2} + \phi_2 p_{i3} + \phi_2^2 p_{i4}] \\
p_{i3} &= (1-\phi_2)[p_{i3} + \phi_2 p_{i4}] \\
p_{i4} &= (1-\phi_2)p_{i4}
\end{aligned}
$$

Table 36: Table 3 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)
Parameter Values for the Benchmark Model Economy

Preferences		
Time discount factor	β	0.942
Curvature of consumption	σ_1	1.500
Curvature of leisure	σ_2	1.016
Relative share of consumption and leisure	χ	1.138
Endowment of productive time	ℓ	3.200
Age and endowment process		
Common probability of retiring	$p_{e\varrho}$	0.022
Common probability of dying	$1 - p_{\varrho\varrho}$	0.066
Life cycle earnings profile	ϕ_1	0.969
Intergenerational persistence of earnings	ϕ_2	0.525
Technology		
Capital share of income	θ	0.376
Capital depreciation rate	δ	0.059
Fiscal policy		
Government consumption	G	0.296
Normalized Retirement pensions	ω	0.696
Income tax function parameters	a_0	0.258
	a_1	0.768
	a_2	0.491
	a_3	0.144
Estate tax function parameters:		
Tax-exempt level	$\underline{z}$	14.101
Marginal tax rate	τ_E	0.160

Some minor changes to the precise description of the parameters are made from the original.

Table 37: Original Table 3 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

TABLE 3
PARAMETER VALUES FOR THE BENCHMARK MODEL ECONOMY

	Parameter	Value
Preferences:		
Time discount factor	β	.924
Curvature of consumption	σ_1	1.500
Curvature of leisure	σ_2	1.016
Relative share of consumption and leisure	χ	1.138
Productive time	ℓ	3.200
Age and employment process:		
Common probability of retiring	$p_{e\varrho}$	.022
Common probability of dying	$1 - p_{\varrho\varrho}$	.066
Earnings life cycle controller	ϕ_1	.969
Intergenerational earnings persistence controller	ϕ_2	.525
Technology:		
Capital share	θ	.376
Capital depreciation rate	δ	.059
Government policy:		
Government expenditures	G	.296
Normalized transfers to retirees	ω	.696
Income tax function parameters	a_0	.258
	a_1	.768
	a_2	.491
	a_3	.144
Estate tax function parameters:		
Tax-exempt level	$\underline{z}$	14.101
Marginal tax rate	τ_E	.160

Table 38: Tables 4 and 5 of Castaneda, Diaz-Gimenez and Rios-Rull (2003) Transition Probabilities of the Process on the Endowment of Efficiency Labor Units for Working-Age Households That Remain at Working Age One Period Later $\Gamma_{\mathcal{E}\mathcal{E}}$ (%)

			$\Gamma_{\mathcal{E}\mathcal{E}}$ (%) From s To s'			
	$e(s)$	γ_s^* (%)	$s'=1$	$s'=2$	$s'=3$	$s'=4$
$s=1$	1.00	69.15	98.43	1.17	0.40	0.01
$s=2$	3.15	19.67	3.14	96.48	0.38	0.00
$s=3$	9.78	11.13	1.53	0.44	98.01	0.02
$s=4$	1061.00	0.05	10.90	0.50	6.25	82.35

Table 39: Original Tables 4 and 5 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

TABLE 4

TRANSITION PROBABILITIES OF THE PROCESS ON THE ENDOWMENT OF EFFICIENCY LABOR UNITS FOR WORKING-AGE HOUSEHOLDS THAT REMAIN AT WORKING AGE ONE PERIOD LATER, $\boldsymbol{\Gamma}_{\varepsilon\varepsilon}$ (%)

	To s'			
FROM s	$s'=1$	$s'=2$	$s'=3$	$s'=4$
$s=1$	96.24	1.14	.39	.006
$s=2$	3.07	94.33	.37	.000
$s=3$	1.50	.43	95.82	.020
$s=4$	10.66	.49	6.11	80.51

TABLE 5

RELATIVE ENDOWMENTS OF EFFICIENCY LABOR UNITS, $e(s)$, AND THE STATIONARY DISTRIBUTION OF WORKING-AGE HOUSEHOLDS, $\boldsymbol{\gamma}_{\varepsilon}^*$

	$s=1$	$s=2$	$s=3$	$s=4$
$e(s)$	1.00	3.15	9.78	1,061.00
$\boldsymbol{\gamma}_{\varepsilon}^*$ (%)	61.11	22.35	16.50	.0389

Table 40: Table 6 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

Values of the Targeted Ratios and Aggregates in the United States and in the Benchmark Model Economies

	K/Y	I/Y	G/Y	Tr/Y	T_E/Y	$mean(h)$	CV_C/CV_H	$e_{40/20}$	$ho(f,s)$
Target (USA)	3.13	18.6%	20.2%	4.9%	0.20%	30.0%	3.00	1.30	0.40
Benchmark	3.62	0.2%	0.2%	0.0%	0.00%	0.3%	0.34	1.02	0.71

Note: Variable $mean(h)$ (column 6) denotes the average share of disposable time allocated to the market. The statistic CV_c/CV_h (column 7) is the ratio of the coefficients of variation of consumption and of hours worked. $e_{40/20}$ is the ratio of average earnings of 40 year old to 20 year old. $ho(f,s)$ the intergenerational correlation coefficient between lifetime earnings of father and so. Note that model actually has households, while data is individuals.

Table 41: Original Table 6 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

TABLE 6

Values of the Targeted Ratios and Aggregates in the United States and in the Benchmark Model Economies

	K/Y (1)	I/Y (2)	G/Y (3)	Tr/Y (4)	T_E/Y (5)	h (6)	$\mathrm{CV}_c/\mathrm{CV}_l$ (7)	$e_{40/20}$ (8)	$\rho(f, s)$ (9)
Target (United States)	3.13	18.6%	20.2%	4.9%	.20%	30.0%	3.00	1.30	.40
Benchmark	3.06	18.1%	20.8%	4.4%	.20%	31.2%	3.25	1.09	.25

Note.—Variable h (col. 6) denotes the average share of disposable time allocated to the market. The statistic $\mathrm{CV}_c/\mathrm{CV}_l$ (col. 7) is the ratio of the coefficients of variation of consumption and of hours worked.

A.10 Restuccia and Urrutia (2004) - Intergenerational Persistence of Earnings: The Role of Early and College Education

Restuccia and Urrutia (2004) present a general equilibrium 2-period OLG model to study how much of cross-country differences in the intergenerational persistence of earnings can potentially be explained by the interactions between innate ability, income, and private eduction (high income parents being able to spend more on the private education of their child). They also consider how private education can be crowded out by public education, and how this might effect inetergenerational persistence of earnings and economic efficiency. From perspective of computation the complication is that they have dynasties (so care directly about value function of child) and that the problems to be solved by young (1st period) and old (2nd period) households look very different. Using the VFI toolkit to solve this model requires rewriting in an equivalent, but slightly different looking, mathematical form, as described here.

Compared to the original paper there are two main surface/notational differences: variables h, $\hat{b}$ and b are called b, $\hat{\pi}$, and π by Restuccia and Urrutia (2004). And the way the decision to attend college and whether student completes or drops out is determined: here $s \in \{0,1\}$ is the decision to attend, and completion/dropout is determined by exogenous state (shock) $\hat{\theta} \sim U[0,1]$, while in Restuccia and Urrutia (2004) $\theta \in \{0,1\}$ is the decision to attend, and the actual complete/dropout shock value is never denoted as it does not need to be kept.

Table 1 reports σ_b as 0.48, and first eqn on pg 1359 describes σ_b as the standard deviation of the innovations to the AR(1) in logs process on innate ability b. This is incorrect, and the reported value of σ_b in Table 1 corresponds to the standard deviation of log(b); the value for the innovations is just $(1-\rho)^2\sigma_b$. (In their notation, σ_b is called σ_π.) In this replication I define σ_b as the standard deviation of $\log(b)$, thus meaning that the same values are reported for Tables 1 and 2, but it's

Table 42: Table 7 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

Distributions of Earnings and of Wealth in the United States and in the Benchmark Model Economies (%)

		QUINTILE					(TOP GROUPS (Percentile)		
ECONOMY	GINI	First	Second	Third	Fourth	Fifth	90th-95th	95th-99th	99th-100th
				A. Distribution of Earnings					
United States	0.63	-0.40	3.19	12.49	23.33	61.39	12.38	16.37	14.76
Benchmark	0.64	0.00	4.27	12.47	14.87	65.53	14.65	16.91	18.07
				B. Distribution of Wealth					
United States	0.78	-0.39	1.74	5.72	13.43	79.49	12.62	23.95	29.55
Benchmark	0.83	0.06	0.26	1.03	13.22	83.90	16.39	17.28	35.69

Table 43: Original Table 7 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

TABLE 7

Distributions of Earnings and of Wealth in the United States and in the Benchmark Model Economies (%)

		Quintile					Top Groups (Percentile)		
Economy	Gini	First	Second	Third	Fourth	Fifth	90th–95th	95th–99th	99th–100th
				A. Distributions of Earnings					
United States	.63	−.40	3.19	12.49	23.33	61.39	12.38	16.37	14.76
Benchmark	.63	.00	3.74	14.59	15.99	65.68	15.15	17.65	14.93
				B. Distributions of Wealth					
United States	.78	−.39	1.74	5.72	13.43	79.49	12.62	23.95	29.55
Benchmark	.79	.21	1.21	1.93	14.68	81.97	16.97	18.21	29.85

Table 44: Table 8 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

Distribution of Consumption in the United States and in the Benchmark Model Economies (%)

		QUINTILE					(TOP GROUPS (Percentile)		
ECONOMY	GINI	First	Second	Third	Fourth	Fifth	90th-95th	95th-99th	99th-100th
United States:									
Nondurables	0.32	6.87	12.27	17.27	23.33	40.27	9.71	10.30	4.83
Nondurables+*	0.30	7.19	12.96	17.80	23.77	38.28	9.43	9.69	3.77
Benchmark:									
Wealthiest									
1	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	
Entire Sample	0.45	5.62	10.74	10.94	17.92	51.39	12.69	12.36	13.74

*: Includes imputed services of consumer durables.

Table 45: Original Table 8 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

TABLE 8

DISTRIBUTIONS OF CONSUMPTION IN THE UNITED STATES AND IN THE BENCHMARK MODEL ECONOMIES (%)

		QUINTILE					TOP GROUPS (Percentile)		
ECONOMY	GINI	First	Second	Third	Fourth	Fifth	90th–95th	95th–99th	99th–100th
United States:									
Nondurables	.32	6.87	12.27	17.27	23.33	40.27	9.71	10.30	4.83
Nondurables+*	.30	7.19	12.96	17.80	23.77	38.28	9.43	9.69	3.77
Benchmark:									
Wealthiest 1% excluded	.40	5.23	12.96	13.55	20.41	47.85	12.77	14.89	3.83
Entire sample	.46	4.68	11.58	12.07	18.68	52.99	12.82	13.45	11.94

* Includes imputed services of consumer durables.

Table 46: Table 9 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)
Earnings and Wealth Persistence in the United States and in the Benchmark Model Economies: Fraction of Households That Remain In The Same Quintile After Five Years

			(QUINTILE		
ECONOMY	First	Second	Third	Fourth	Fifth
			A. Earnings Persistence		
United States	0.86	0.41	0.47	0.46	0.66
Benchmark	0.77	0.69	0.71	0.61	0.68
			A. Wealth Persistence		
United States	0.67	0.47	0.45	0.50	0.71
Benchmark	0.72	0.68	0.78	0.83	0.90

Table 47: Original Table 9 of Castaneda, Diaz-Gimenez and Rios-Rull (2003)

TABLE 9
Earnings and Wealth Persistence in the United States and in the Benchmark Model Economies: Fractions of Households That Remain in the Same Quintile after Five Years

			Quintile		
Economy	First	Second	Third	Fourth	Fifth
			A. Earnings Persistence		
United States	.86	.41	.47	.46	.66
Benchmark	.76	.55	.65	.80	.80
			B. Wealth Persistence		
United States	.67	.47	.45	.50	.71
Benchmark	.81	.80	.80	.75	.89

definition in the model is corrected to fit these. Relatedly, Restuccia and Urrutia (2004) are not quite explicit on evolution of b, but it appears that b is constant between ages $j = 1$ and $j = 2$, and then follows the aforementioned AR(1)-in-logs between $j = 2$ and $j = 1$.

Note that the 'college dropout premium', $\underline{p}$, is less than one. So model does imply college dropouts receive lower wages than those who do not attend (lower wage-per-unit-of-time, not just lower earnings). This is likely counterfactual.

A.10.1 The Household problem

The household lives for two-periods, with different problems at each age, and with intergenerational linkages.

From the perspective of VFI Toolkit there are two decision (d) variables, three endogenous state (a) variables, and two exogenous state (z) variables. But at each of the two ages, j=1 and j=2, only a certain few of these are actually relevant. (A third exogenous state variable is required later in the paper as part of the exercise underlying Table 7.)

The two decision variables are: $\hat{b}$ acquired ability of the child, and s a schooling choice (decision to send child to college); both decision variables are only relevant at age j=1. The three exogenous state variables are: h human capital, $\hat{b}$ acquired ability of the child, and s whether the child is attending school (college); these are relevant for both ages, only age $j = 2$, and only age $j = 2$ respectively. The two exogenous state variables are: b innate ability of the child, and $\hat{\theta}$ which determines whether a child attending school completes college or drops out; b is relevant at both ages, $\hat{\theta}$ is only relevant at age $j = 2$.

The value function problem for young (age $j = 1$) households is given by

$$
\begin{aligned}
V_1(h, b) = & \max_{c \geq 0,\, e \geq 0, s, \hat{b}} \left\{ \frac{c^{1-\sigma} - 1}{1 - \sigma} + \beta E[V(h', \hat{b}, s, b', \hat{\theta})] \right\} \\
& \text{s.t. } c + e = (1 - \tau) w h \\
& \quad \hat{b} = b(e + g)^{\gamma}, \quad 0 \leq \gamma \leq 1 \\
& \quad s \in \{0, 1\} \\
& \quad h' = \zeta h \\
& \quad b' = b \\
& \quad \hat{\theta} \sim U[0, 1]
\end{aligned}
$$

where w, τ, g, σ, β, ζ and γ are all parameters from the household's perspective. Notice that e is determined by choice of $\hat{b}$ (given b); so VFI Toolkit takes advantage of this and considers that it is $\hat{b}$ that is chosen rather than e (b is multiplying the $(e + g)^{\gamma}$ term).[43]

[43] Consumption c is similarly residually determined by the budget constraint and so not really one of the decision variables.

The value function problem for old (age $j=2$) households is given by

$$
\begin{aligned}
V_2(h,\hat{b},s,b,\hat{\theta}) = \max_{c\geq 0,\, e\geq 0} & \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \beta E[V_1(h',b')] \right\} \\
\text{s.t. } & q(\hat{b}) = min\{\psi_0(1+\hat{b})^{\psi_1}, 1\} \\
& \text{if } s=0:\ c=(1-\tau)(wh+wh') \\
& \text{if } s=1\,\&\, q(\hat{b})<\hat{\theta}:\ c+(1-\kappa(wh))f\underline{\text{n}} = (1-\tau)(wh+wh'(1-\underline{\text{n}})) \\
& \text{if } s=1\,\&\, q(\hat{b})>\hat{\theta}:\ c+(1-\kappa(wh))f\bar{n} = (1-\tau)(wh+wh'(1-\bar{n})) \\
& \text{if } s=0: h'=\bar{b} \\
& \text{if } s=1\,\&\, q(\hat{b})<\hat{\theta}:\ h'=\underline{\text{p}}\bar{b} \\
& \text{if } s=1\,\&\, q(\hat{b})>\hat{\theta}:\ h'=\bar{p}\bar{b} \\
& \kappa(wh) = \min\{\max\{\kappa_1-\kappa_0 wh, 0\}, 1\} \\
& log(b') = \rho_b log(b) + \epsilon_b, \quad \epsilon \sim N(0,\sigma^2_{\epsilon_b})
\end{aligned}
$$

where f, σ, β, ψ_0, ψ_1, τ, w, $\underline{\text{p}}$, $\bar{p}$, $\underline{\text{n}}$, $\bar{n}$ κ_0, κ_1 are all parameters from the household's perspective. Note that s is whether they attend college, and $q(\hat{b}) \lessgtr \hat{\theta}$ determines whether they complete ($>$) or dropout ($<$).[44]

A.10.2 Rest of the Model Economy and General Equilibrium

All sectors of the economy are model as price-taking perfect competition. There is a representative firm with linear production function that solves the static profit maximization problem,

$$\max_{H^f\geq 0}\{Y-wH^f\} \text{ subject to } Y=AH^f$$

The labour market clearance requires

$$H^f = H \equiv \int h d\mu + \int [(1-s)\hat{b} + s(p\hat{b}(1-n))] d\mu$$

Government budget balance, tax revenues equals pulic education expenditure, requires

$$g+\kappa F = \tau Y$$

where $\kappa F \equiv f \int \kappa(wh) snd\mu_{j=2}$. Not needed for computation is that the aggregate resource constraint is given by $Y=AH=C+E+F+g$ where $F\equiv f\int snd\mu_{j=2}$.

Two observations that make this easier to compute, first is that $w=A=1$ (follows from perfect competition so wage equals marginal product of labour). Second, since production function is $Y=AH$ the demand for labour is $H^f \in [0,\infty)$ and so general equilibrium for the labour market is trivially satisfied.

[44] Thus, e.g., $s=1$ and $q(\hat{b})<\hat{\theta}$ corresponds to $\theta=0$ in notation of Restuccia and Urrutia (2004) (they denote different value functions for $s=0,1$, while here were it is simply treated as an endogenous state variable). I use $\hat{\theta}$ notation specifically to emphasize that it is not the same as their θ, although it serves a similar role in the model in determining complete/dropout for university.

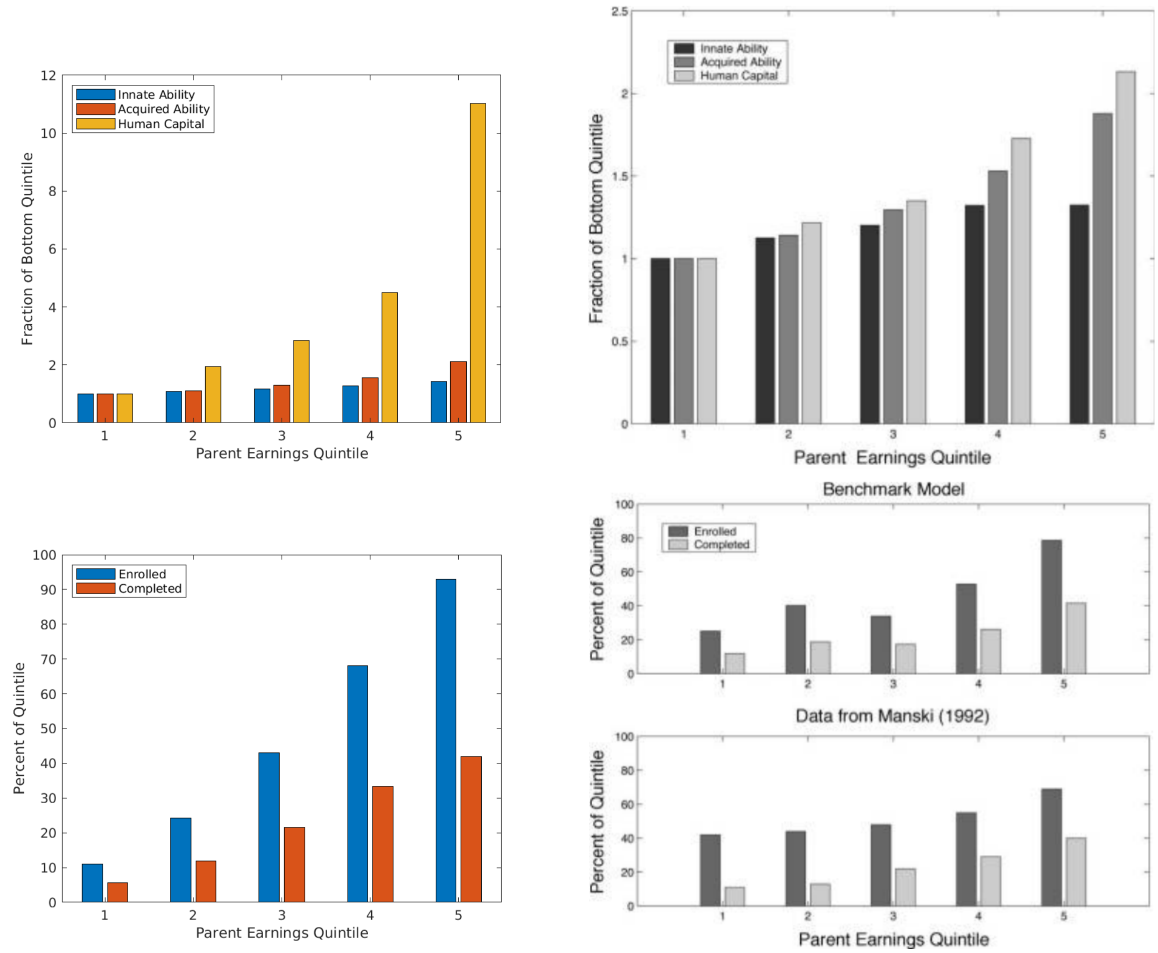


Figure 21: Figures 1 and 2 of Restuccia and Urrutia (2004)

A.10.3 Replication Results

The resulting Tables and Figures of the replication, together with those of the original, are now presented.

Table 48: Table 1 of Restuccia and Urrutia (2004)
Calibration of the Benchmark Economy

Target	Data	Model	Parameter	Value
(i) Fraction of non-college	0.54	0.51	ψ_0	0.27
(ii) Dropout rate	0.50	0.53	ψ_1	1.02
(iii) Early education/GDP	0.044	0.09	γ	0.24
(iv) College/GDP	0.028	0.02	f	0.64
(v) Public/Total college	0.64	0.58	κ_0	0.36
(vi) Average dropout premium	1.41	2.04	$\underline{p}$	0.86
(vii) Average college premium	2.33	2.52	$\bar{p}$	1.48
(viii) std(log earnings)	0.60	0.80	σ_b	0.48
(ix) Intergenerational correlation of earnings	0.40	0.36	ρ_b	0.20

Note: Data column is copy of original from Restuccia & Urrutia (2004), is not part of the replication.

Table 49: Original Table 1 of Restuccia and Urrutia (2004)

TABLE 1—CALIBRATION OF THE BENCHMARK ECONOMY

Target	Data	Model	Parameter	Value
(i) Fraction of non-college	0.54	0.54	ψ_0	0.27
(ii) Dropout rate	0.50	0.50	ψ_1	1.02
(iii) Early education/GDP	0.044	0.043	γ	0.24
(iv) College/GDP	0.028	0.027	f	0.64
(v) Public/total college	0.64	0.64	κ_0	0.36
(vi) Average dropout premium	1.41	1.38	$\underline{p}$	0.86
(vii) Average college premium	2.33	2.37	$\bar{\bar{p}}$	1.48
(viii) std(log earnings)	0.60	0.60	σ_π	0.48
(ix) Intergenerational correlation of earnings	0.40	0.40	ρ	0.20

Table 50: Table 2 of Restuccia and Urrutia (2004)
Disparity and Persistence in the Benchmark Economy

	Innate Ability	Acquired Ability	Earnings
Cross-sectional disparity: std(log x)	0.48	0.54	0.80
Intergenerational correlation:	0.25	0.42	0.36

Note: In model notation these columns are: b, $\hat{b}$, and wh. I follow Restuccia & Urrutia (2004) in reporting as cross-sectional the number conditional on being an elderly household, not cross-sectional over the whole model economy. Restuccia & Urrutia (2004) explain calculation of intergenerational correlation of earnings at bottom of pg 1363. I assume the intergeneration correlations of (log) innate and acquired ability are calculated by the analagous regressions (with modification for acquired ability as is only observed for old).

Table 51: Original Table 2 of Restuccia and Urrutia (2004)

TABLE 2—DISPARITY AND PERSISTENCE IN THE BENCHMARK ECONOMY

	Innate ability	Acquired ability	Earnings
Cross-sectional disparity: std(log x)	0.48	0.51	0.60
Intergenerational correlation	0.20	0.41	0.40

Table 52: Table 3 of Restuccia and Urrutia (2004)

Decision Rules by Parents Earnings and Childs Ability

Panel A: Expenditures in Early Education			
	Young Parent Earnings Tercile		
Child Innate Ability:	I	II	III
Low:	0.0052	0.0281	0.2179
Medium:	0.0045	0.0223	0.1885
High:	0.0042	0.0242	0.1754
Panel B: Childs Acquired Ability			
	Young Parent Earnings Tercile		
Child Innate Ability:	I	II	III
Low:	0.258	0.292	0.389
Medium:	0.422	0.462	0.614
High:	0.742	0.836	1.120
Panel C: College Enrollment Rate (percentage)			
	Old Parent Earnings Tercile		
Child Acquired Ability:	I	II	III
Low:	2.81	0.00	40.45
Medium:	10.37	35.33	82.00
High:	67.69	100.00	93.95

Note: Earnings, Innate ability, and aquired ability are *wh*, *b* and *bhat*. The three panels report *e*, *bhat* and *s* respectively.

Table 53: Original Table 3 of Restuccia and Urrutia (2004)

TABLE 3—DECISION RULES BY PARENTS' EARNINGS AND CHILD'S ABILITY

Panel A: Expenditures in early education			
	Young parent earnings tercile		
Child innate ability	I	II	III
Low	0.0033	0.0188	0.0977
Medium	0.0024	0.0113	0.0952
High	0.0030	0.0142	0.0989
Panel B: Child's acquired ability			
	Young parent earnings tercile		
Child innate ability	I	II	III
Low	0.264	0.289	0.350
Medium	0.432	0.453	0.554
High	0.745	0.814	1.022
Panel C: College enrollment rate (percentage)			
	Old parent earnings tercile		
Child acquired ability	I	II	III
Low	1.15	0.70	8.98
Medium	22.29	26.18	64.86
High	99.74	82.66	82.05

Table 54: Table 4 of Restuccia and Urrutia (2004)
Uniform Transfer Experiment

Transfer to	Young Parents	Old Parents
Increase in education expenditures: (percent of transfer)	197.06	0.00
Parents changing education decisions: (percent of parents in age group)	24.44	NaN

Note: Clockwise from top-left, these are percent change in e, F-kappaF, s, and bhat.

Table 55: Original Table 4 of Restuccia and Urrutia (2004)

TABLE 4—UNIFORM TRANSFER EXPERIMENT

Transfer to:	Young parents	Old parents
Increase in education expenditures (percentage of transfer)	13.96	1.02
Parents changing education decisions (percentage of parents in age group)	29.23	2.54

Note: Uniform lump-sum transfer of 30 percent of the total college tuition cost to either young or old parents, keeping decision rules constant.

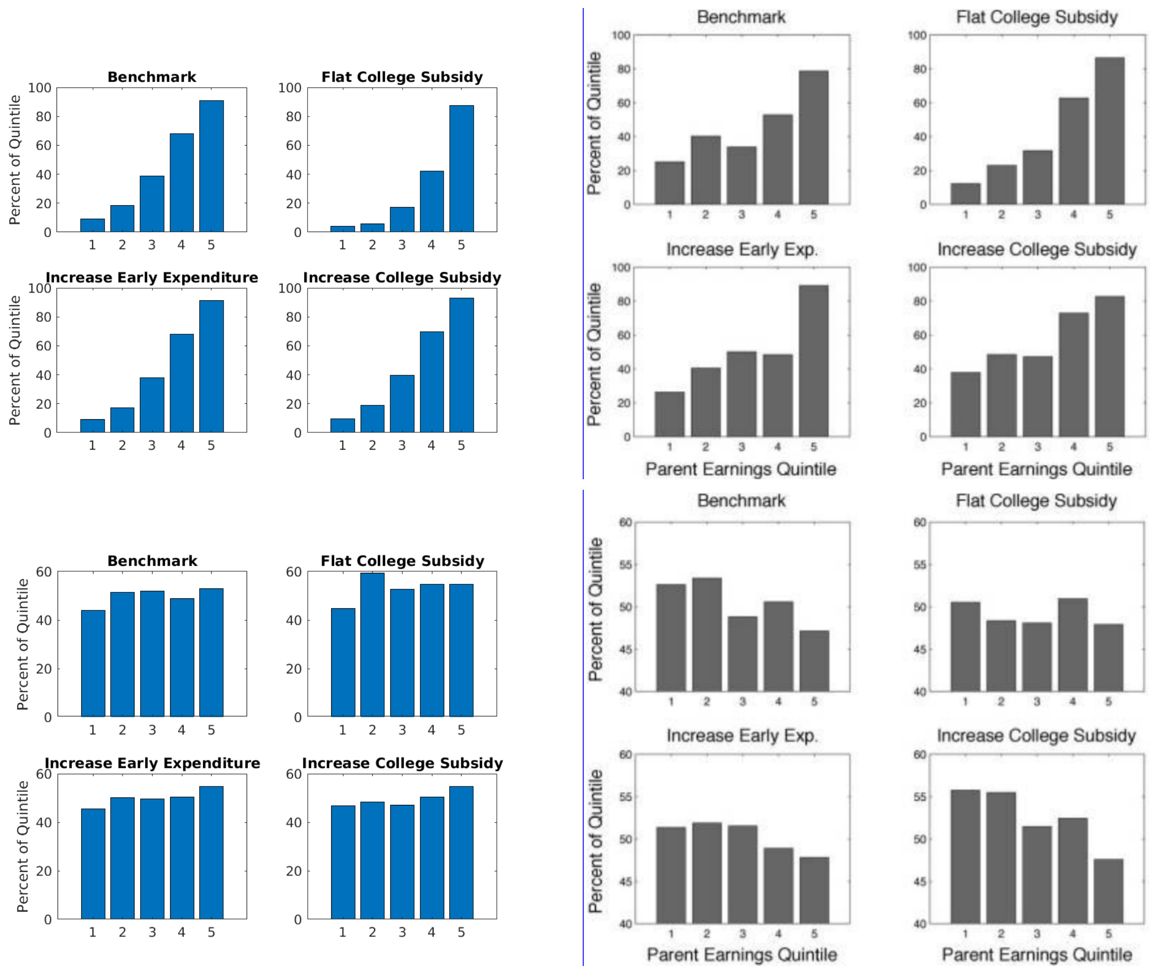


Figure 22: Figures 3 and 4 of Restuccia and Urrutia (2004): Fig 3 shows Enrollment Rates, Fig 4 shows Dropout Rates.

Table 56: Table 5 of Restuccia and Urrutia (2004)
Sensitivity Analysis with Respect to ρ and σ_π

	Benchmark $\rho = 0.2$ $\sigma_\pi = 0.48$	$\rho = 0.1$	$\rho = 0.3$	$\sigma_\pi = 0.4$	$\sigma_\pi = 0.6$
Intergenerational correlation					
Innate ability	0.25	0.24	0.24	0.24	0.22
Acquired ability	0.47	0.45	0.45	0.45	0.44
Earnings	0.43	0.44	0.43	0.42	0.43
Disparity: std(log x)					
Innate ability	0.48	0.48	0.48	0.48	0.48
Acquired ability	0.62	0.62	0.62	0.62	0.62
Earnings	0.80	0.80	0.80	0.80	0.80
Other aggregate statistics					
College enrollment	0.48	0.48	0.48	0.48	0.48
Private early/GDP (as					
hline					

Note: All other parameters are the same as in the benchmark economy; public expenditures in early education adjusted to balance the government budget.

Note: In model notation these columns are: b, $\hat{b}$, and wh. I follow Restuccia & Urrutia (2004) in reporting as cross-sectional the number conditional on being an elderly household, not cross-sectional over the whole model economy. Restuccia & Urrutia (2004) explain calculation of intergenerational correlation of earnings at bottom of pg 1363. I assume the intergeneration correlations of (log) innate and acquired ability are calculated by the analagous regressions (with modification for acquired ability as is only observed for old).

Table 57: Original Table 5 of Restuccia and Urrutia (2004)

TABLE 5—SENSITIVITY ANALYSIS WITH RESPECT TO ρ AND σ_π

	Benchmark $\rho = 0.20$ $\sigma_\pi = 0.48$	$\rho = 0.1$	$\rho = 0.3$	$\sigma_\pi = 0.4$	$\sigma_\pi = 0.6$
Intergenerational correlation					
Innate ability	0.20	0.10	0.30	0.20	0.20
Acquired ability	0.41	0.33	0.47	0.36	0.38
Earnings	0.40	0.33	0.48	0.41	0.38
Disparity: std(log x)					
Innate ability	0.48	0.48	0.48	0.40	0.60
Acquired ability	0.51	0.50	0.54	0.45	0.65
Earnings	0.60	0.58	0.63	0.53	0.76
Other aggregate statistics					
College enrollment	0.46	0.47	0.42	0.32	0.44
Private early/GDP	2.13	2.22	1.99	1.73	2.47

Note: All other parameters are the same as in the benchmark economy; public expenditures in early education adjusted to balance the government's budget.

Table 58: Table 6 of Restuccia and Urrutia (2004)
Sensitivity Analysis with Respect to γ and $\bar{p}$

	Benchmark $\gamma = 0.24$ $\bar{p} = 1.48$	$\gamma = 0.1$	$\gamma = 0.4$	$\bar{p} = 1.4$	$\bar{p} = 1.6$
Intergenerational correlation					
Innate ability	0.25	0.24	0.23	0.24	0.24
Acquired ability	0.47	0.32	0.59	0.44	0.46
Earnings	0.43	0.28	0.52	0.41	0.43
Disparity: std(log x)					
Innate ability	0.48	0.48	0.48	0.48	0.48
Acquired ability	0.62	0.48	0.96	0.61	0.63
Earnings	0.80	0.72	0.88	0.77	0.82
Other aggregate statistics					
College enrollment	0.48	0.86	0.22	0.34	0.66
Private early/GDP (as %)	7.72	2.50	15.88	7.70	7.70

Note: All other parameters are the same as in the benchmark economy; public expenditures in early education adjusted to balance the government budget.

Note: In model notation these columns are: b, $\hat{b}$, and wh. I follow Restuccia & Urrutia (2004) in reporting as cross-sectional the number conditional on being an elderly household, not cross-sectional over the whole model economy. Restuccia & Urrutia (2004) explain calculation of intergenerational correlation of earnings at bottom of pg 1363. I assume the intergeneration correlations of (log) innate and acquired ability are calculated by the analagous regressions (with modification for acquired ability as is only observed for old).

Table 59: Original Table 6 of Restuccia and Urrutia (2004)

TABLE 6—SENSITIVITY ANALYSIS WITH RESPECT TO γ AND $\bar{p}$

	Benchmark $\gamma = 0.24$ $\bar{p} = 1.48$	$\gamma = 0.1$	$\gamma = 0.4$	$\bar{p} = 1.4$	$\bar{p} = 1.6$
Intergenerational correlation					
Innate ability	0.20	0.20	0.20	0.20	0.20
Acquired ability	0.41	0.23	0.54	0.36	0.45
Earnings	0.40	0.22	0.55	0.37	0.43
Disparity: std(log x)					
Innate ability	0.48	0.48	0.48	0.48	0.48
Acquired ability	0.51	0.49	0.56	0.51	0.52
Earnings	0.60	0.60	0.61	0.57	0.64
Other aggregate statistics					
College enrollment	0.46	0.75	0.20	0.25	0.66
Private early/GDP	2.13	0.51	3.57	1.57	2.75

Note: All other parameters are the same as in the benchmark economy; public expenditures in early education adjusted to balance the government's budget.

Table 60: Table 7 of Restuccia and Urrutia (2004)
Adding Post-College Permanent Earnings Shocks

	Benchmark Economy	Small shock $z = \{0.8, 1.2\}$	Large shock $z = \{0.5, 1.5\}$
Intergenerational correlation			
Innate ability	0.24	0.25	0.24
Acquired ability	0.46	0.46	0.45
Earnings	0.45	0.33	0.35
Disparity: std(log x)			
Innate ability	0.71	0.71	0.71
Acquired ability	0.79	0.80	0.81
Earnings	0.81	1.36	1.36
Correlation with log acquired ability			
College enrollment	NaN	0.68	NaN
College enrollment	0.68	NaN	0.68
Educational attainment	NaN	0.33	NaN
Educational attainment	0.33	NaN	0.34
Log earnings	NaN	0.91	NaN
Log earnings	0.34	NaN	0.45
Other aggregate statistics			
College enrollment	0.35	0.35	0.35
Private early/GDP	0.08	0.07	0.07
Average College Premium	2.69	2.26	2.81

Note: Same parameters as in the benchmark economy; public expenditures in early education adjusted to balance the government budget.

Note: In model notation these columns are: b, $\hat{b}$, and wh. I follow Restuccia & Urrutia (2004) in reporting as cross-sectional the number conditional on being an elderly household, not cross-sectional over the whole model economy. Restuccia & Urrutia (2004) explain calculation of intergenerational correlation of earnings at bottom of pg 1363. I assume the intergeneration correlations of (log) innate and acquired ability are calculated by the analagous regressions (with modification for acquired ability as is only observed for old).

Table 61: Original Table 7 of Restuccia and Urrutia (2004)

TABLE 7—ADDING POST-COLLEGE PERMANENT EARNINGS SHOCKS

	Benchmark economy	Small shock $z = \{0.8, 1.2\}$	Large shock $z = \{0.5, 1.5\}$
Intergenerational correlation			
Innate ability	0.20	0.20	0.20
Acquired ability	0.41	0.42	0.41
Earnings	0.40	0.38	0.29
Disparity: std(log x)			
Innate ability	0.47	0.47	0.47
Acquired ability	0.51	0.51	0.52
Earnings	0.60	0.63	0.82
Correlation with log acquired ability			
College enrollment	0.66	0.66	0.66
Educational attainment	0.64	0.64	0.64
Log earnings	0.95	0.91	0.76
Other aggregate statistics			
College enrollment	0.46	0.47	0.48
Private early/GDP	2.14	2.27	2.72
Average college premium	2.38	2.39	2.43

Note: Same parameters as in the benchmark economy; public expenditures in early education adjusted to balance the government's budget.

Table 62: Table 8 of Restuccia and Urrutia (2004)
Policy Experiments

	Benchmark	Increase in early expenditures	Increase in college subsidy	Flat college subsidy
Intergenerational correlation				
Earnings	0.43	0.43	0.43	0.43
Educational attainment	0.00	0.00	0.00	0.00
Consumption	0.39	0.39	0.40	0.38
Expenditures (percent of GDP)				
Private early education	7.72	7.68	7.67	7.89
Public early eduction	0.01	0.01	0.01	0.01
Private college education	0.94	0.94	0.89	1.27
Public college education	1.38	1.38	1.44	0.71
Other aggregate statistics				
College enrollment rate	0.48	0.48	0.48	0.40
College dropout rate	0.13	0.13	0.13	0.11
Aggregate human capital	1.06	1.06	1.06	1.06
Aggregate consumption	1.19	1.18	1.18	1.19

Note: All other parameters are the same as in the benchmark economy; public expenditures in early education adjusted to balance the government budget.

Note: In model notation these columns are: b, $\hat{b}$, and wh. I follow Restuccia & Urrutia (2004) in reporting as cross-sectional the number conditional on being an elderly household for b and wh (young for $\hat{b}$), not cross-sectional over the whole model economy.

Restuccia & Urrutia (2004) explain calculation of intergenerational correlation of earnings at bottom of pg 1363. I assume the intergeneration correlations of (log) innate and acquired ability are calculated by the analagous regressions (with modification for acquired ability as is only observed for old).

Table 63: Original Table 8 of Restuccia and Urrutia (2004)

TABLE 8—POLICY EXPERIMENTS

	Benchmark	Increase in early expenditures	Increase in college subsidy	Flat college subsidy
Intergenerational correlation				
Earnings	0.40	0.36	0.40	0.42
Educational attainment	0.35	0.28	0.26	0.45
Consumption	0.66	0.64	0.67	0.66
Expenditures (percentage of GDP)				
Private early education	2.13	1.77	2.16	2.23
Public early education	2.18	2.94	2.18	2.17
Private college education	0.96	1.03	0.70	0.69
Public college education	1.71	1.71	2.50	1.71
Other aggregate statistics				
College enrollment	0.46	0.50	0.56	0.42
Dropout rate	0.50	0.50	0.52	0.49
Aggregate human capital	2.06	2.15	2.07	2.06
Aggregate consumption	1.92	1.99	1.92	1.91

Note: All other parameters are the same as in the benchmark economy; public expenditures in early education adjusted to balance the government's budget.

A.11 Restuccia and Rogerson (2008) - Policy distortions and aggregate productivity with heterogeneous establishments

The model of Restuccia and Rogerson (2008) studies the role of idiosyncratic distortions to firms on aggregate outcomes. The focus is on comparing outcomes between different stationary competitive equilibria. Because all of the shocks are permanent, while the model is in principle dynamic in practice it is effectively static. Restuccia and Rogerson (2008) take an approach to solving the model that exploits this static nature of the solution, and additionally exploits that it is known to be linear in the mass of entrants. The codes used to replicate the model do not attempt to exploit either of these, and solve *as if* the model was fully dynamic and stochastic with time-varying shocks potentially non-linear in the mass of entrants; while this makes it notably slower than the code provided by RR2008 it illustrates how even code that is written for solving more general problems can often still be used for simple problems as long as speed is not essential.[45]

The code provided by RR2008 (namely RR_model_RED.m) contains an error. It sets variable i as interest rate on line 23. It then overwrites this as part of using i in a for-loop on lines 43-46 so that value of i ends up equal to 'length(datazupper2)'. On line 68 i is used to calculate rho, with the intention that i is still the interest rate, but this is no longer true.[46]

Tables 1-9 essentially replicate perfectly. Table 7 took some time as from paper it is not easy to determine what exactly is being reported. First there is the bottom row of Table 7 which reports Y_s/Y. Page 715 defines Y_s/Y as ' *the output share of establishments that are receiving a subsidy*'. Table 7 relates to Section 6.1 which considers when all but some establishments are taxed and aggregate capital is allowed to change; since the subsidy was set to keep aggregate capital unchanged, this suggests that the subsidy should equal zero, and this interpretation is supported by line 337 of the codes of RR2008 which sets 'sub=0'. This would immediately imply that $Y_s = 0$ by definition (and thus $Y_s/Y = 0$). What is not clear is that in Table 7 you need to reinterpret/redefine Y_s as relating to 'non-taxed' rather than 'subsidised'. The top-left most entry of my Table 6 appears erroneous, but I could not find where in my replication code the problem lies (it is just one number, so at some point diminishing returns kicked in...).

The model has both a standard endogenous entry condition (which introduces a free-entry general equilibrium condition) and a further 'conditional entry' condition. The Conditional entry condition is that after deciding to enter firms draw their initial state, (s, τ), and then *conditional* on this firms can decide whether or not they will actually enter; this might also be thought of as a 'conditional decision to abort entry'. This conditional entry decision is additional to the standard endogenous entry condition and imposes a further general equilibrium condition. In the model of RR2008 we thus have both the free-entry condition and a condition relating to the conditional entry condition on $\bar{e}(s, \tau)$ (which they denoted $\bar{x}(s, \tau)$). There is a third general equilibrium condition, namely choosing the mass of entrants N_e (which they denoted E) to ensure labour market clearance; because the model is linear in N_e this can simply be ignored at first when solving the other two general equilibrium conditions, and then imposed by renormalizing the solutions. This renormalization is the approach taken by RR2008, and which the replication follows for the baseline case. When solving the other cases there is typically an additional requirement that the subsidy rate is determined in general equilibrium to keep aggregate capital equal to it's value in the baseline (see RR2008 for explanation of why they wish to do so), and in this case the renormalization

[45]The full replication codes also solve many 'cases' that are not reported/needed for the actual replication, further adding to run time.

[46]Thanks to Denise Manfredini who brought this to my attention.

relating N_e to the labour market clearance cannot be imposed afterwards (would change aggregate capital stock to no longer equal baseline value) and so this 'renormalization' general eqm condition is treated by replication codes as a standard general equilibrium condition for all the non-baseline cases.

Exit is exogenous, and the rest of the model is standard. The optimal choices of capital and labour (and hence profits) can all be solved in closed form as functions of the exogenous state and the equations are in RR2008, I additionally provide the derivation for these are below (*kbar* and *nbar*).

The problem of an existing firm is to choose capital and labour inputs to maximize expected present value of profits conditional on their idiosyncratic values of productivity, s, and tax/subsidy τ,

$$\begin{aligned} V(s,\tau) = \max_{n,k} \pi(n,k) + \frac{1}{1+r}\lambda E V(s',\tau') \\ \text{s.t} \quad & \pi(n,z) = (1-\tau)y - wn - rk - c_f \\ & y = s k^{\alpha} n^{\gamma} \\ & s' = s, \tau' = \tau \end{aligned}$$

where π is (period) profit; r is the interest rate (in this model equivalent to the rental rate of capital and to the return on capital); n is labour input, k is capital input; w is wage; c_f is fixed-cost of production.[47] since this problem is essentially static (as every period is independent) it can be written as such and solved directly (see RR2008). However to determine entry we still need to calculate the expected present discounted value (as this is what matters for the entry decision) and so the replication codes simply solve the full dynamic programing problem directly, rather than following RR2008 who used based results from summation of sequences to massively simplify this. This simplified decision uses the static case to solve analytically for k and n and give results in the following simplified problem,

$$\begin{aligned} V(s,\tau) = \pi(n,k) + \frac{1}{1+r}\lambda E V(s',\tau') \\ \text{s.t} \quad & \pi(n,z) = (1-\tau)y - wn - rk - c_f \\ & k = \left(\frac{\alpha}{r}\right)^{\frac{1-\gamma}{1-\gamma-\alpha}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma-\alpha}} ((1-\tau)s)^{\frac{1}{1-\gamma-\alpha}} \quad n = \left(\frac{(1-\tau)s\gamma}{w}\right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \quad s' = s, \tau' = \tau \end{aligned}$$

(The replication code uses slight further simplifications of expressions for n, y, and π which are derived in full below.)

The calculation of the stationary agents distribution is standard for cases of endogenous entry and exogenous exit, with conditional entry adding the minor change that the distribution of entrants used when iterating the agents distribution is the distribution of potential entrants times the conditional entry decision; again RR2008 avoid this iterating on the agents distribution by taking advantage of the fact that there are no time-varying idiosyncratic shocks, and the exogenous and constant value of the probability of exit to avoid iterating on the agent distribution and instead again just using results from summing infinite sequences (this time on distributions).

The conditional entry decision is $\max_{\bar{e}(s,\tau)}\{\bar{e}(s,\tau)\beta V(s,\tau), 0\}$, for each (s,τ). It is a general equilibrium condition on $\bar{e}$.

[47] Since c_f is a lump-sum and exit is exogenous it does not change any decisions of existing firms. From the perspective of potential entrants it is no different to the fixed-cost of entry c_e, and so plays no (seperate) role there. But it does effect the 'conditional entry' decisions (which c_e does not).

The free-entry condition is $\beta \int V(s,\tau)\bar{e}(s,\tau)dg(s,\tau) - c_e = 0$. Notice that this is the requirement that the (discounted) expected value of being a new entrant is equal to the (fixed) cost of entry; as otherwise there would be more (or less) entry if this did not hold with equality. Notice that the expectation is taken across the distribution of actual entrants (the conditional entry decision times the distribution of potential entrants). This general equilibrium condition is used to determine the wage w (as this in turn determines the value function and so can be chosen to ensure that the free-entry condition holds; other models instead commonly use this condition to instead determine the fixed cost of entry).

There is a (representative) household side of the economy, which essentially provides two general equilibrium conditions. The first can be reduced to just a calibration issue, that $r = 1/\beta - (1-\delta)$, the second is that labour supply equals one (households have endowment of one unit of time which they supply perfectly inelastically), and so leads to the labour market clearance condition that labour demand of the firms must equal labour supply (which equals one). RR2008 solve this by simply renormalizing the mass of potential entrants after solving all the other general equilibrium conditions (which works for this specific model as the total mass is linear in potential entrants, and the decisions are unaffected); the replication codes use this for some but not all cases.

Many of the experiments performed by RR2008 involve setting the subsidy rate so that the aggregate capital remains equal to it's baseline level, which is computationally equivalent to condsidering this (aggregate capital minus baseline aggregate capital equals zero) as an additional 'general eqm' condition. This is the approach taken by the replication codes.

We thus have three general equilibrium conditions: conditional entry, free entry, and labour market clearance. (In the baseline model we can just solve for the first two, the third can be done as a renormalization, and the fourth is not relevant.)

The definition of stationary competitive eqm in this model is standard. It involves finding the parameters w, $\bar{e}$, and N_e ($\bar{e}$ can be thought of as a decision, or just as an equilibrium parameter, denoted $\bar{x}$ by RR2008; N_e is the mass of potential entrants, denoted E by RR2008). These are chosen to satisfy the conditional entry condition, the free-entry condition, labour market clearance.

Replication involves computing many different stationary competitive equilibria and comparing various model outputs from these. However it typically also involves adding a condition that is to choose the subsidy rate τ_s to satisfy the requirement that capital remains equal to it's baseline value. This is done in the codes by simply considering this requirement as if it were a fourth general equilibrium condition.

The results of the replication are Figure 64 and Tables 66-82. Everything appears to replicate just fine.

Derive nbar and kbar: With the output tax, take the FOCs of $(1-\tau)sk^{\alpha}n^{\gamma} - wn - rk - c_f$ w.r.t. k and n to get:

$$\alpha(1-\tau)sk^{\alpha-1}n^{\gamma} - r = 0$$
$$\gamma(1-\tau)sk^{\alpha}n^{\gamma-1} - w = 0$$

rearranging the second of these

$$n^{1-\gamma} = \frac{\gamma(1-\tau)sk^{\alpha}}{w}$$

so

$$n = \left(\frac{(1-\tau)s\gamma}{w} \right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \tag{13}$$

That is nbar done. Now, rearrange first FOC to get

$$k^{1-\alpha} = \frac{\alpha(1-\tau)sn^{\gamma}}{r}$$

substitute our expression for nbar in here for n to get

$$k^{1-\alpha} = ((1-\tau)s)\frac{\alpha}{r}((1-\tau)s)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} k^{\frac{\alpha\gamma}{1-\gamma}}$$

so

$$k = ((1-\tau)s)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\alpha-\gamma}} \left(\frac{\alpha}{r} \right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \tag{14}$$

by returning to sub this into the eqn for *nbar* we can get the alternative formula,

$$n = ((1-\tau)s)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{w} \right)^{\frac{1-\alpha}{1-\alpha-\gamma}} \left(\frac{\alpha}{r} \right)^{\frac{\alpha}{1-\alpha-\gamma}} \tag{15}$$

that completes our derivation of formulae for *kbar* and *nbar* with an output tax. We can also substitute these into the production function, $sk^{\alpha}n^{\gamma}$ to get

$$y = (1-\tau)^{\frac{\alpha+\gamma}{1-\alpha-\gamma}} s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\alpha}{r} \right)^{\frac{\alpha}{1-\alpha-\gamma}} \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\alpha-\gamma}} \tag{16}$$

Derive nbar and kbar with capital tax: With the output tax, take the FOCs of $sk^{\alpha}n^{\gamma} - wn - (1+\tau)rk - c_f$ w.r.t. k and n to get:

$$\alpha sk^{\alpha-1}n^{\gamma} - (1+\tau)r = 0$$
$$\gamma sk^{\alpha}n^{\gamma-1} - w = 0$$

rearranging the second of these

$$n^{1-\gamma} = \frac{\gamma sk^{\alpha}}{w}$$

so

$$n = \left(\frac{\gamma}{w} \right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \tag{17}$$

That is nbar done. Now, rearrange first FOC to get

$$k^{1-\alpha} = \frac{\alpha sn^{\gamma}}{(1+\tau)r}$$

substitute our expression for nbar in here for n to get

$$k^{1-\alpha} = s\frac{\alpha}{(1+\tau)r} s^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} k^{\frac{\alpha\gamma}{1-\gamma}}$$

so

$$k = s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\alpha-\gamma}} \left(\frac{\alpha}{(1+\tau)r}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \tag{18}$$

by returning to sub this into the eqn for *nbar* we can get the alternative formula,

$$n = s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{1-\alpha}{1-\alpha-\gamma}} \left(\frac{\alpha}{(1+\tau)r}\right)^{\frac{\alpha}{1-\alpha-\gamma}} \tag{19}$$

that completes our derivation of formulae for *kbar* and *nbar* with a capital tax. We can also substitute these into the production function, $sk^{\alpha}n^{\gamma}$ to get

$$y = s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\alpha}{(1+\tau)r}\right)^{\frac{\alpha}{1-\alpha-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\alpha-\gamma}} \tag{20}$$

Derive nbar and kbar with labour tax: With the output tax, take the FOCs of $sk^{\alpha}n^{\gamma} - (1+\tau)wn - rk - c_f$ w.r.t. k and n to get:

$$\begin{aligned} \alpha s k^{\alpha-1} n^{\gamma} - r &= 0 \\ \gamma s k^{\alpha} n^{\gamma-1} - (1+\tau)w &= 0 \end{aligned}$$

rearranging the second of these

$$n^{1-\gamma} = \frac{\gamma s k^{\alpha}}{(1+\tau)w}$$

so

$$n = \left(\frac{\gamma}{(1+\tau)w}\right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \tag{21}$$

That is nbar done. Now, rearrange first FOC to get

$$k^{1-\alpha} = \frac{\alpha s n^{\gamma}}{r}$$

substitute our expression for nbar in here for n to get

$$k^{1-\alpha} = s\frac{\alpha}{r} s^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma}{(1+\tau)w}\right)^{\frac{\gamma}{1-\gamma}} k^{\frac{\alpha\gamma}{1-\gamma}}$$

so

$$k = s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{(1+\tau)w}\right)^{\frac{\gamma}{1-\alpha-\gamma}} \left(\frac{\alpha}{r}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \tag{22}$$

by returning to sub this into the eqn for *nbar* we can get the alternative formula,

$$n = s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\gamma}{(1+\tau)w}\right)^{\frac{1-\alpha}{1-\alpha-\gamma}} \left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha-\gamma}} \tag{23}$$

that completes our derivation of formulae for *kbar* and *nbar* with a labour tax. We can also substitute these into the production function, $sk^{\alpha}n^{\gamma}$ to get

$$y = s^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha-\gamma}} \left(\frac{\gamma}{(1+\tau)w}\right)^{\frac{\gamma}{1-\alpha-\gamma}} \tag{24}$$

Table 64: Figure 1 of Restuccia & Rogerson (2008)

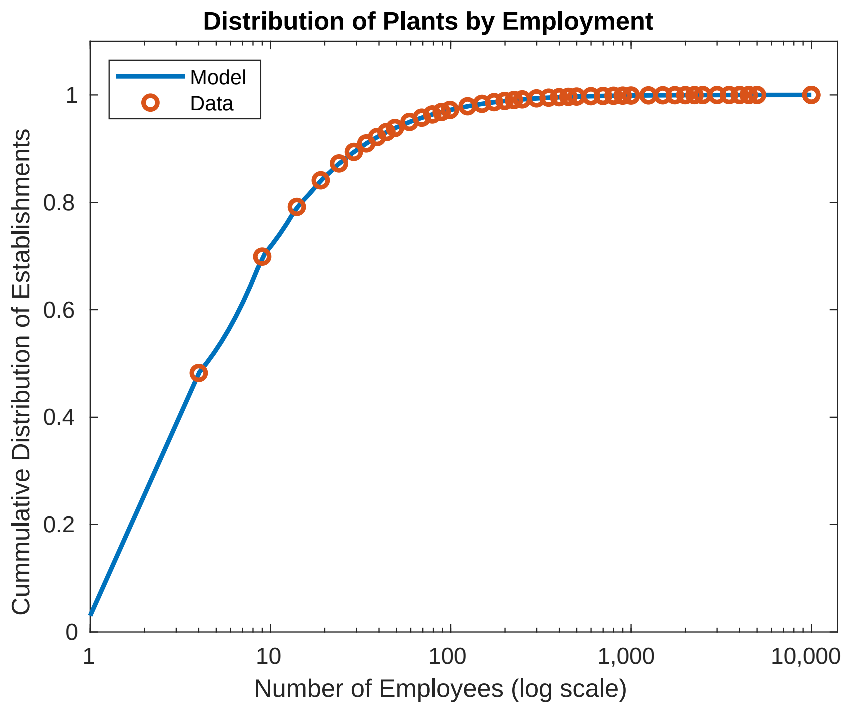


Table 65: Original Figure 1 of Restuccia & Rogerson (2008)

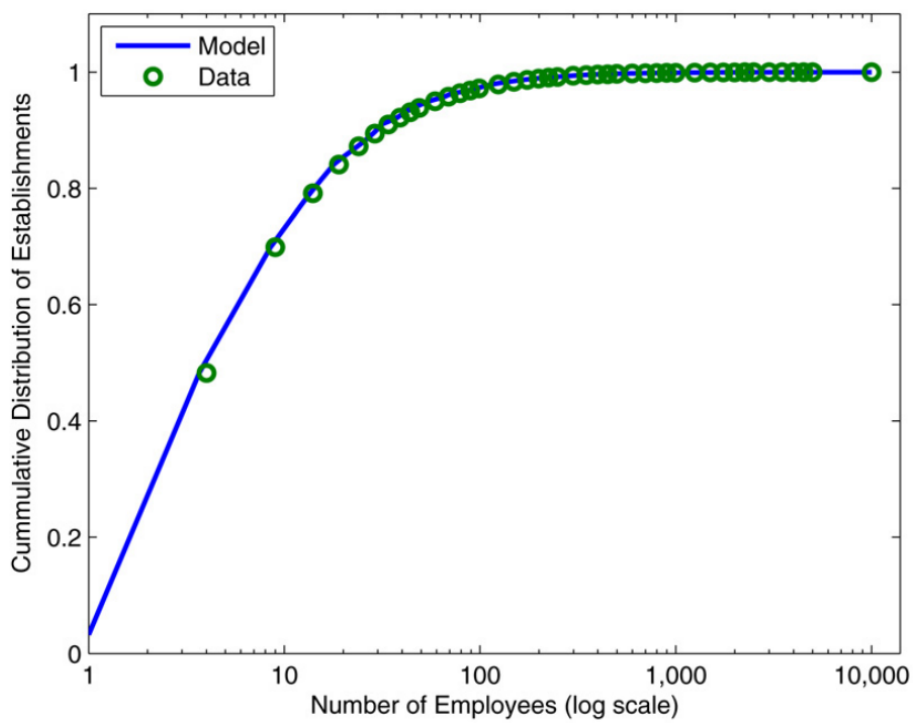


Fig. 1. Distribution of establishments by employment—model vs. data.

Table 66: Table 1 of Restuccia & Rogerson (2008)
Benchmark calibration to US data

Parameters	Value	Target
α	0.283	Capital income share
γ	0.567	Labour income share
β	0.96	Real rate of return
δ	0.08	Investment to output rate
c_e	1.0	Normalization
c_f	0.0	Benchmark rate
λ	0.1	Annual exit rate
s range	[1, 3.98]	Relative establishment sizes
$h(s)$	see Fig. 1	Size distribution of establishments

Table 67: Original Table 1 of Restuccia & Rogerson (2008)

Table 1
Benchmark calibration to US data

Parameter	Value	Target
α	0.283	Capital income share
γ	0.567	Labor income share
β	0.96	Real rate of return
δ	0.08	Investment to output ratio
c_e	1.0	Normalization
c_f	0.0	Benchmark case
λ	0.1	Annual exit rate
s range	$[1, 3.98]$	Relative establishment sizes
$h(s)$	see Fig. 1	Size distribution of establishments

Table 68: Table 2 of Restuccia & Rogerson (2008)
Distribution statistics of benchmark economy

	Establishment size (number of employees)		
	<5	5 to 49	≥50
Share of establishments	0.52	0.42	0.06
Share of output	0.07	0.34	0.60
Share of labour	0.07	0.34	0.60
Share of capital	0.07	0.34	0.60
Share of employment	2.37	15.18	185.19

Table 69: Original Table 2 of Restuccia & Rogerson (2008)

Table 2
Distribution statistics of benchmark economy

	Establishment size (number of employees)		
	< 5	5 to 49	⩾ 50
Share of establishments	0.56	0.39	0.05
Share of output	0.08	0.34	0.58
Share of labor	0.08	0.34	0.58
Share of capital	0.08	0.34	0.58
Average employment	2.4	15.5	183.0

A.12 Guerrieri and Lorenzoni (2017) - Credit Crises, Precautionary Savings, and the Liquidity Trap

Guerrieri and Lorenzoni (2017) study whether a credit-crisis, modelled as a surprise tightening of the budget constaint, can explain the US experience in the recession that followed the Great Financial Crisis of 2007. This is done using, essentially, the model of Huggett (1993) with the addition of endogenous labour, and with a focus on general equilibrium transition paths.

Table 1 contains the parameters. The differences from the original reflect two things. Some are minor differences where I follow the values from the codes provided on Lorenzoni's website. The large differences (B and ϕ) are because I report the actual model parameters, while the GL2017 paper reports the model parameters as a fraction of annual model output (note that the model is quarterly; annual model output is general eqm is roughly 1.64). Obviously it is the actual parameter values that are needed when implementing the model.

A similar issue (actual values, versus value-as-a-fraction-of-annual-income) is present in many of the axes in the Figures from GL2017. This is fine, but often went unmentioned in the paper and so took some time to figure out when replicating. The replication codes produce both the versions that follow the 'figure descriptions' and those that follow the 'what is plotted'. I here only show those that follow the 'figure descriptions', and so the x-axes will sometimes appear different to the originals. Comments in the replication codes describe exactly which graphs this is relevant to.

Brief description of model is as follows. I present it in the form relevant to the replication codes, a fuller description is provided by Guerrieri and Lorenzoni (2017).

Table 70: Table 3 of Restuccia & Rogerson (2008)
Effects of idiosyncratic distortions – uncorrelated case

Variable	τ_t (tax rate on output)			
	0.1	0.2	0.3	0.4
Relative Y	0.98	0.96	0.93	0.92
Relative TFP	0.98	0.96	0.93	0.92
Relative E	1.00	1.00	1.00	1.00
Y_s/Y	0.72	0.85	0.93	0.97
S/Y	0.05	0.08	0.09	0.10
τ_s	0.06	0.09	0.10	0.11

Table 71: Original Table 3 of Restuccia & Rogerson (2008)

Table 3
Effects of idiosyncratic distortions—uncorrelated case

Variable	τ_t			
	0.1	0.2	0.3	0.4
Relative Y	0.98	0.96	0.93	0.92
Relative TFP	0.98	0.96	0.93	0.92
Relative E	1.00	1.00	1.00	1.00
Y_s/Y	0.72	0.85	0.93	0.97
S/Y	0.05	0.08	0.09	0.10
τ_s	0.06	0.09	0.10	0.11

Table 72: Table 4 of Restuccia & Rogerson (2008)
Relative TFP – uncorrelated case

Fraction of estabilishments taxed (%)	τ_t (tax rate on output)			
	0.1	0.2	0.3	0.4
90	0.92	0.84	0.78	0.74
80	0.95	0.89	0.84	0.81
60	0.98	0.94	0.91	0.89
50	0.98	0.96	0.93	0.92
40	0.99	0.97	0.95	0.94
20	1.00	0.99	0.98	0.97
10	1.00	0.99	0.99	0.99

Table 73: Original Table 4 of Restuccia & Rogerson (2008)

Table 4
Relative TFP—uncorrelated distortions

Fraction of establishments taxed (%):	τ_t			
	0.1	0.2	0.3	0.4
90	0.92	0.84	0.78	0.74
80	0.95	0.89	0.84	0.81
60	0.98	0.94	0.91	0.89
50	0.98	0.96	0.93	0.92
40	0.99	0.97	0.95	0.94
20	1.00	0.99	0.98	0.97
10	1.00	0.99	0.99	0.99

Table 74: Table 5 of Restuccia & Rogerson (2008)
Effects of idiosyncratic distortions – correlated case

	τ_t (tax rate on output)			
	0.1	0.2	0.3	0.4
Relative Y	0.90	0.80	0.73	0.68
Relative TFP	0.90	0.80	0.73	0.68
Relative E	1.00	1.00	1.00	1.00
Y_s/Y	0.42	0.67	0.83	0.92
S/Y	0.17	0.32	0.43	0.49
τ_s	0.40	0.48	0.52	0.53

Table 75: Original Table 5 of Restuccia & Rogerson (2008)

Table 5
Effects of idiosyncratic distortions—correlated case

Variable	τ_t			
	0.1	0.2	0.3	0.4
Relative Y	0.90	0.80	0.73	0.69
Relative TFP	0.90	0.80	0.73	0.69
Relative E	1.00	1.00	1.00	1.00
Y_s/Y	0.42	0.67	0.83	0.92
S/Y	0.17	0.32	0.43	0.49
τ_s	0.40	0.48	0.52	0.53

Table 76: Table 6 of Restuccia & Rogerson (2008)
Relative TFP – correlated case

Fraction of estabilishments taxed (%)	τ_t (tax rate on output) 0.1	0.2	0.3	0.4
90	0.99	0.65	0.55	0.50
80	0.84	0.70	0.61	0.56
60	0.88	0.77	0.69	0.65
50	0.90	0.80	0.73	0.68
40	0.92	0.83	0.76	0.73
20	0.95	0.88	0.84	0.80
10	0.97	0.92	0.88	0.86

Table 77: Original Table 6 of Restuccia & Rogerson (2008)

Table 6
Relative TFP—correlated distortions

Fraction of establishments taxed (%):	τ_t 0.1	0.2	0.3	0.4
90	0.81	0.66	0.56	0.51
80	0.84	0.70	0.62	0.57
60	0.88	0.77	0.69	0.65
50	0.90	0.80	0.73	0.69
40	0.92	0.82	0.76	0.72
20	0.95	0.89	0.84	0.81
10	0.97	0.92	0.88	0.86

Table 78: Table 7 of Restuccia & Rogerson (2008)
Taxing all but some exempt establishments ($\tau_t = 0.40$)

Variable	Establishments exempt (%)				
	10	30	50	70	90
Relative Y	0.64	0.57	0.50	0.46	0.43
Relative TFP	0.83	0.73	0.65	0.59	0.55
Relative E	0.35	0.25	0.15	0.09	0.04
Relative w	0.41	0.41	0.41	0.41	0.41
Relative K	0.41	0.41	0.41	0.41	0.41
Y_s/Y	0.09	0.29	0.52	0.72	0.89

Table 79: Original Table 7 of Restuccia & Rogerson (2008)

Table 7
Taxing all but some exempt establishments ($\tau_t = 0.40$)

Variable	Establishments exempt (%)				
	10	30	50	70	90
Relative Y	0.66	0.65	0.65	0.69	0.78
Relative TFP	0.85	0.80	0.78	0.79	0.85
Relative E	0.42	0.47	0.53	0.62	0.75
Relative w	0.42	0.47	0.53	0.62	0.75
Relative K	0.42	0.47	0.53	0.62	0.75
Y_s/Y	0.10	0.31	0.52	0.73	0.89

Table 80: Table 8 of Restuccia & Rogerson (2008)
Idiosyncratic distortions to capital rental rates

Variable	Uncorrelated		Correlated	
	$\tau_t = 0.5$	$\tau_t = 1.0$	$\tau_t = 0.5$	$\tau_t = 1.0$
Relative Y	0.97	0.95	0.89	0.82
Relative TFP	0.97	0.95	0.89	0.82
Relative E	0.97	0.95	0.89	0.82
Y_s/Y	0.74	0.83	0.33	0.46
S/Y	0.03	0.04	0.10	0.14
τ_s	0.14	0.15	0.51	0.51

Table 81: Original Table 8 of Restuccia & Rogerson (2008)

Table 8
Idiosyncratic distortions to capital rental rates

	Uncorrelated		Correlated	
	$\tau_t = 0.50$	$\tau_t = 1.00$	$\tau_t = 0.50$	$\tau_t = 1.00$
Relative Y	0.97	0.95	0.89	0.82
Relative TFP	0.97	0.95	0.89	0.82
Relative E	0.97	0.95	0.89	0.82
Y_s/Y	0.74	0.83	0.33	0.46
S/Y	0.03	0.04	0.10	0.14
τ_s	0.14	0.15	0.51	0.51

Table 82: Table 9 of Restuccia & Rogerson (2008)
Idiosyncratic distortions–outputs vs wages

Variable	Uncorrelated		Correlated	
	Output	Wages	Output	Wages
Relative Y	1.14	0.84	0.65	0.58
Relative TFP	0.98	0.89	0.66	0.67
Relative K	1.70	0.84	0.96	0.58
Relative E	1.70	0.84	0.96	0.58
Relative w	1.70	1.67	0.96	0.99
Y_s/Y	1.00	0.98	0.97	0.79
S/Y	0.50	0.56	0.48	0.45

Table 83: Original Table 9 of Restuccia & Rogerson (2008)

Table 9
Idiosyncratic distortions—output vs. wages

	Uncorrelated		Correlated	
	Output	Wages	Output	Wages
Relative Y	1.14	0.84	0.65	0.58
Relative TFP	0.98	0.89	0.66	0.67
Relative K	1.70	0.84	0.96	0.58
Relative E	1.70	0.84	0.96	0.58
Relative w	1.70	1.67	0.96	1.00
Y_s/Y	1.00	0.98	0.97	0.79
S/Y	0.50	0.56	0.48	0.45

Households face the following value function iteration problem,

$$
\begin{aligned}
V(a,z) = \max_{n,a'} & \frac{c^{1-\gamma}}{1-\gamma} + \frac{1}{1-\omega}\psi - \frac{(1-n)^{1-\eta}}{1-\eta} + \beta E[V(a',z')|z] \\
\text{s.t. } & c + \frac{1}{1+r}a' = a + zn - \tilde{\tau} \\
& a' \geq -\psi
\end{aligned}
$$

where n is labour supply, a is assets, z is exogenous labour productivity and follows a markov process. ω will be used as a 'wedge' to implement the 'New Keynsian sticky wage' transtion, but is otherwise zero. r is the interest rate to be determined in general equilibrium. $\tilde{\tau}$ is lump-sum net tax, which will be a lump-sum tax τ net of unemployment benefit v; $\tilde{\tau} = \tau - v\mathbb{1}_{(z=0)}$.[48] Note that GL2017 refer to z and θ, and set up model using $q = 1/(1+r)$. Household debt (negative values of assets, a) is bounded below by exogenous limit ψ.

There are two other main aspects to the model. The first is a Government which faces the following budget constraint:

$$
\tau + \frac{1}{1+r}B' = uv + B
$$

where B is government debt, and u is the unemployment rate (note that this will be equal to the probability that $z = 0$, which can be calculated directly from the transition matrix for exogenous markov process on z). In the model the government is always considered to take everything but τ as exogenous and then simply set τ to ensure that this budget constraint holds (τ is thus determined in general eqm, but trivially so). The second main aspect is asset market clearance. The net private plus public asset supply must be equal to zero, which gives us that $B + \int a d\mu = 0$ (where $\mu(a,z)$ is agent distribution). General equilibrium will involve finding r to ensure asset market clearance (as in Huggett (1993)).[49]

Much of the replication involves solving stationary general equilibrium problems (finding r), and then general equilibrium transition paths (path of r) for a change/path in borrowing-limit ψ. There are also New-Keynesian general equilibrium transition paths (add restriction $r >= 0$, and use ω to ensure it holds), as well as some Fiscal and Fisher-deflation paths which largely add change/path in B and B' (and initial distribution of a). See GL2017 paper for details. The replication codes contain some clarifying comments about the exact timing of how to model these.

GL2017 state that they use Tauchen method to discretize AR(1) process for (log) z. They provide the parameter values of the AR(1) process (or more accurately the provide the 'annual values', together with formulae for calculating the quarterly value needed for the model). In code, GL2017 take their discretization of z (θ in the notation of paper) from Shimer (2005), and in their codes it is simply imported (i.e., there is no code that generates the actual values based on the properties of the AR(1)). It appears likely (but not known for a fact) that actually the Tauchen-Hussey method, a specific sub-version of the Tauchen method, was used to create the discretization. This choice of the Tauchen-Hussey method plays a very large role in their findings, but from the perspective of replication is perfectly fine. The only issue when replicating was that since Tauchen-Hussey was described as Tauchen, the paper did not report the hyperparameter value that would be required by Tauchen method.

[48]You only receive unemployment benefits in the 'unemployment state' $z = 0$, you do not receive them based on choosing not to work ($n = 0$).

[49]Worth mentioning that there is an equivalence from the perspective of model behaviour when shifting both ψ and B, see GL2017 paper for detailed explanation.

Allowing for the 'different axes' in some of the Figures in this replication, everything replicates just fine.

Table 84: Table 1 of Guerrieri & Lorenzoni (2017)

Parameters Values

Parameter	Explanation	Value	Target/Source
β	Discount Factor	0.9774	Interest rate r=2.5%
γ	Coefficient of relative risk aversion	4	
η	Curvature of utility of leisure	1.5	Average Frisch elasticity=1
ψ	Coefficient on leisure in utility	15.88	Average hours worked 0.4 of endowment
ρ	Persistence of productivity shock	0.967	Persistence of wage process
σ_ϵ	Variance of productivity shock	0.017	Variance of wage process
$\pi_{e,u}$	Transition to unemployment	0.057	Shimer (2005)
$\pi_{u,e}$	Transition to employment	0.882	Shimer (2005)
υ	Unemployment benefit	0.17	40% of average labor income
B	Bond supply	2.7	Liquid assets (flow of funds)
phi	Borrowing limit	1.601	Total gross debt (flow of funds)

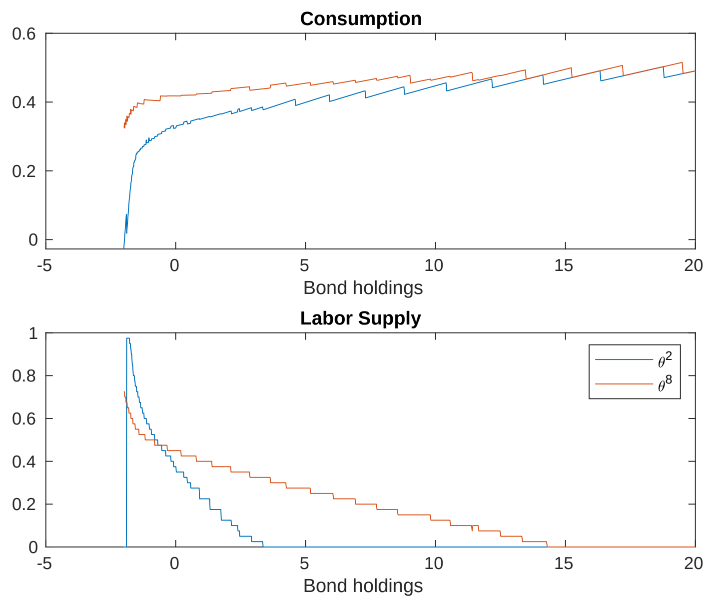
Note: values of B and phi differ from those in Guerrieri & Lorenzoni (2017). The actual model parameters are reported here, while those in paper are the parameter divided by annual output (annual output equals 4 times quarterly output; model is quarterly).

Table 85: Original Table 1 of Guerrieri & Lorenzoni (2017)

PARAMETER VALUES

Parameter	Explanation	Value	Target/source
β	Discount factor	0.9711	Interest rate $r = 2.5\%$
γ	Coefficient of relative risk aversion	4	
η	Curvature of utility from leisure	1.5	Average Frisch elasticity =1
ψ	Coefficient on leisure in utility	12.48	Average hours worked 0.4 of endowment (Nekarda and Ramey 2010)
ρ	Persistence of productivity shock	0.967	Persistence of wage process in Flodén and Lindé (2001)
σ_ε	Variance of productivity shock	0.017	Variance of wage process in Flodén and Lindé (2001)
$\pi_{e,u}$	Transition to unemployment	0.057	Shimer (2005)
$\pi_{u,e}$	Transition to employment	0.882	Shimer (2005)
ν	Unemployment benefit	0.10	40% of average labor income
B	Bond supply	1.6	Liquid assets (flow of funds)
ϕ	Borrowing limit	0.959	Total gross debt (flow of funds)

Note. See the text for details on the targets.



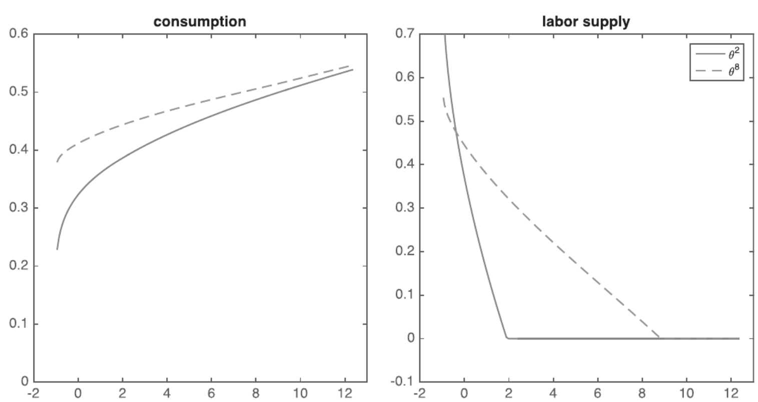
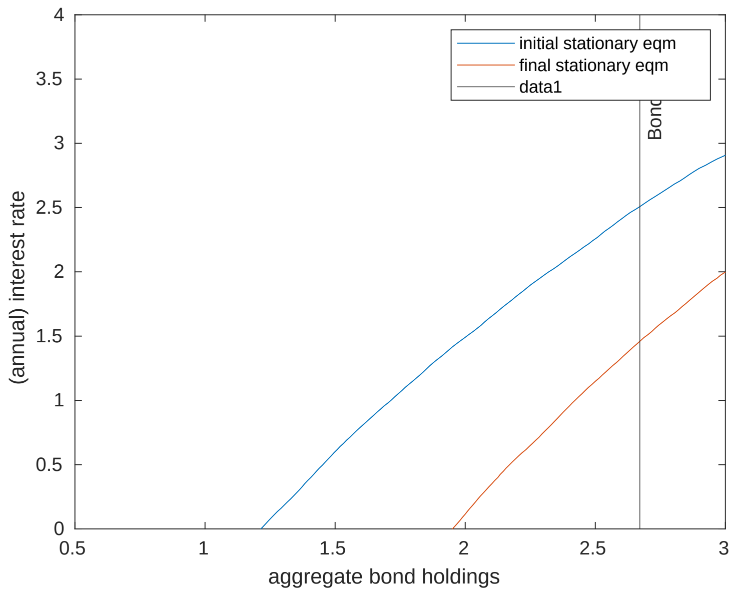


FIGURE I

Optimal Consumption and Labor Supply in Steady State

Figure 23: Figure 1 of Guerrieri & Lorenzoni (2017)



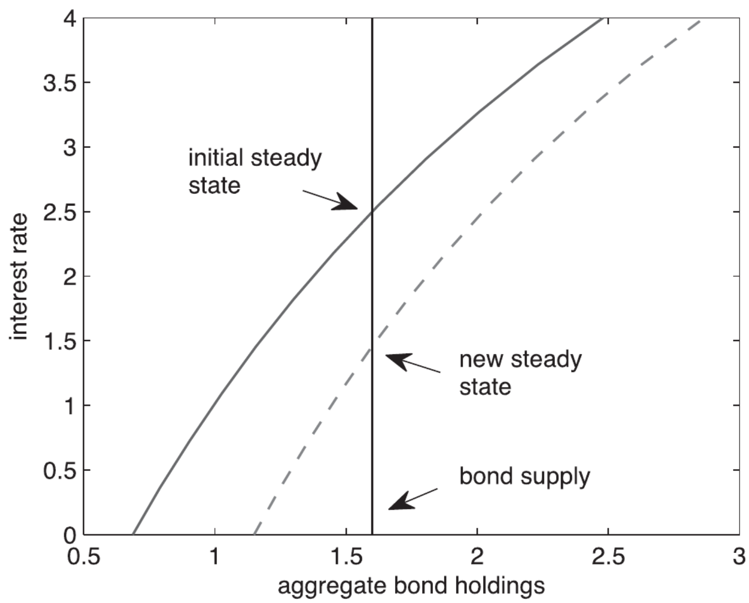
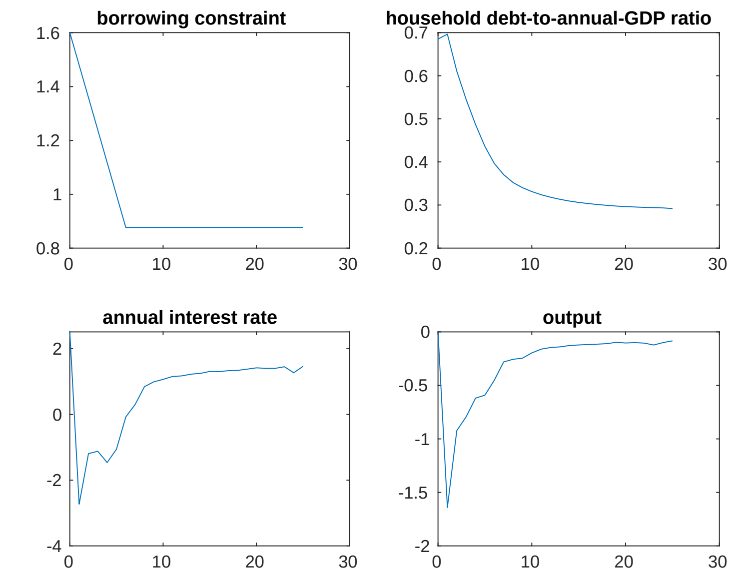


FIGURE II

Bond Market Equilibrium in Steady State

Interest rate is in annual terms.

Figure 24: Figure 2 of Guerrieri & Lorenzoni (2017)



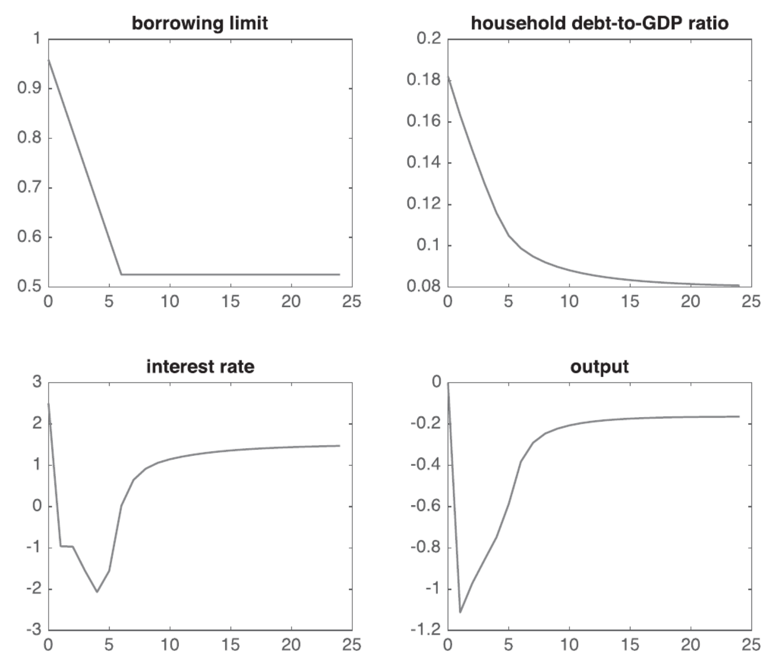
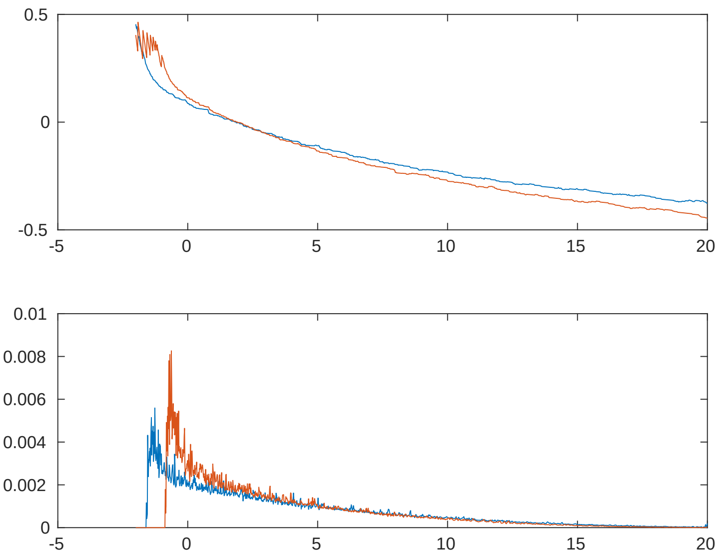


FIGURE III

Interest Rate and Output Responses

Interest rate is in annual terms. Output is in percent deviation from initial steady state.

Figure 25: Figure 3 of Guerrieri & Lorenzoni (2017)



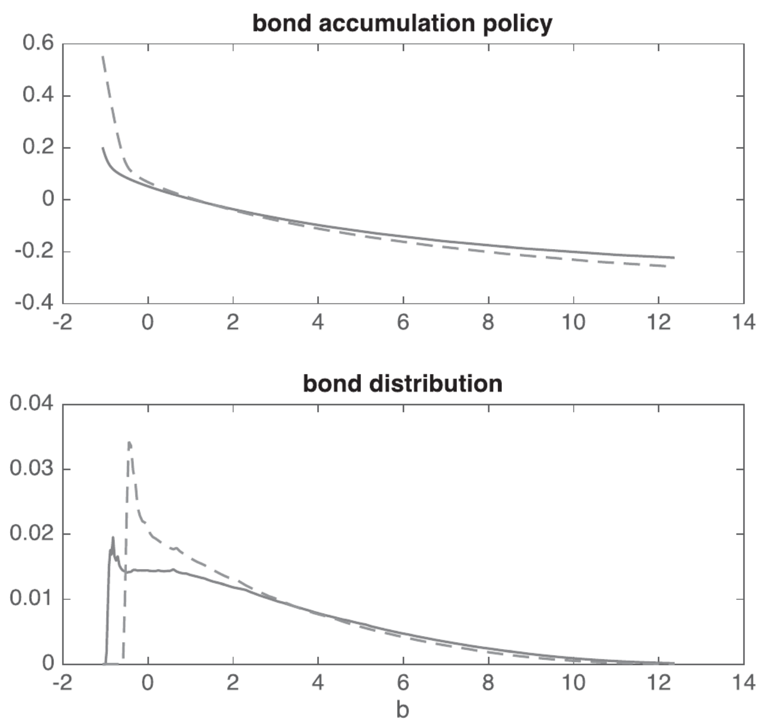
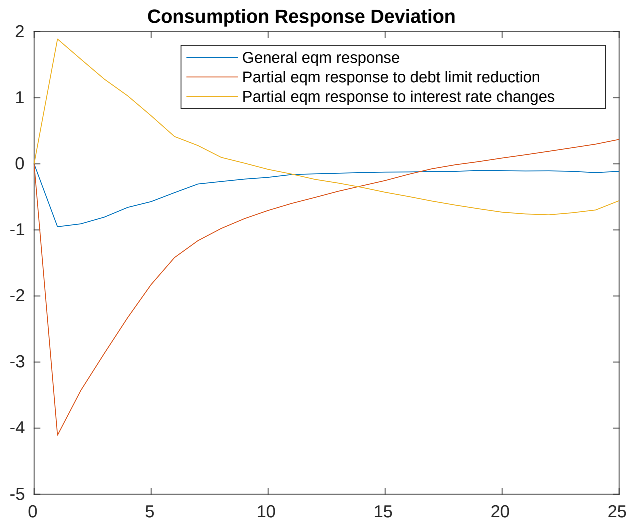


FIGURE IV

Bond Accumulation and Distributions in the Two Steady States

Solid line: initial steady state. Dashed line: new steady state.

Figure 26: Figure 4 of Guerrieri & Lorenzoni (2017)



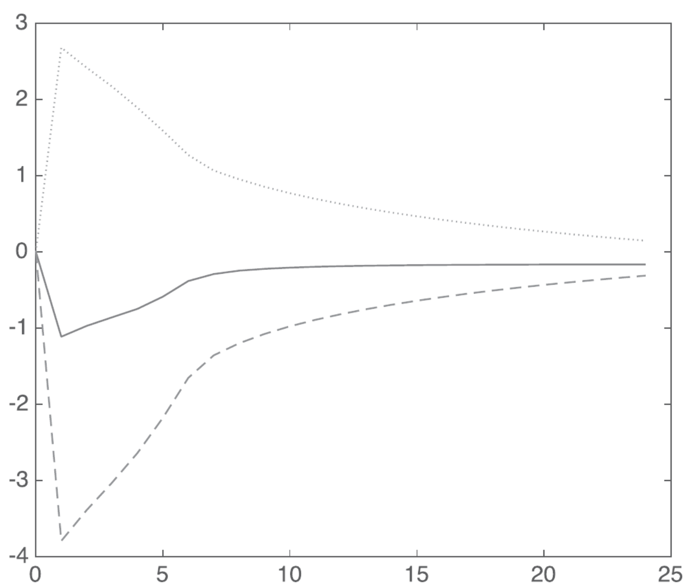
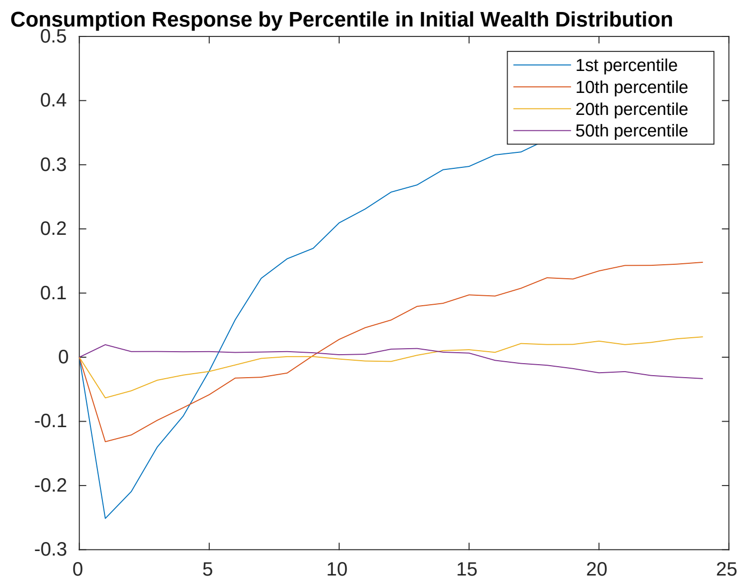


FIGURE V

Consumption Response Decomposition

Percent deviations from initial steady state. Solid line: general equilibrium response. Dashed line: partial equilibrium response to debt limit reduction. Dotted line: response to the equilibrium sequence of interest rate changes.

Figure 27: Figure 5 of Guerrieri & Lorenzoni (2017)



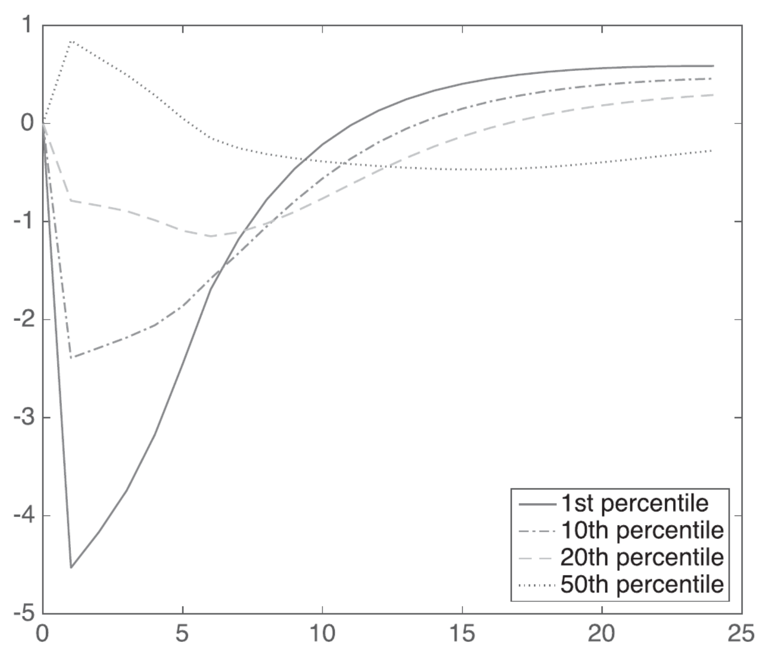
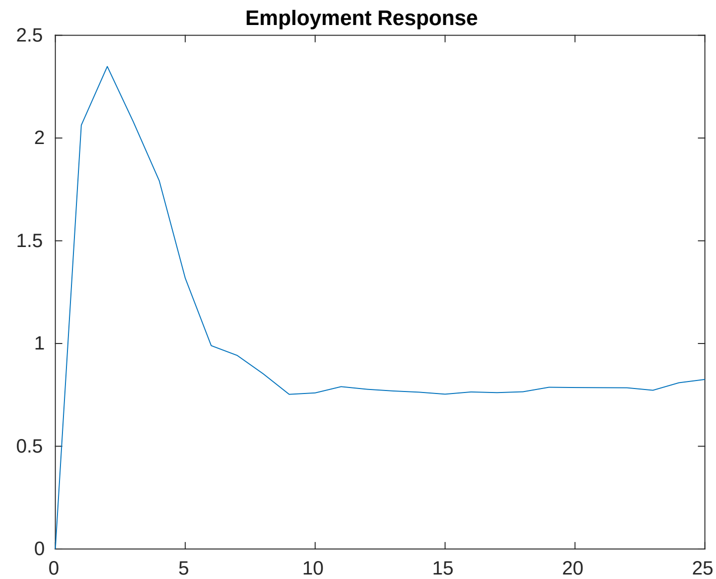


FIGURE VI

Consumption Response by Percentile in Initial Wealth Distribution

Percent deviations from steady-state path conditional on initial wealth being in the reported percentile.

Figure 28: Figure 6 of Guerrieri & Lorenzoni (2017)



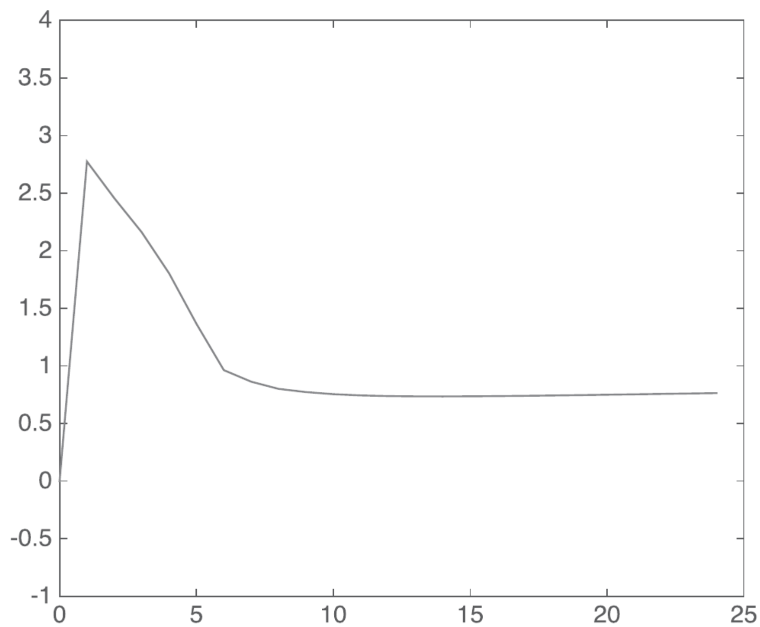
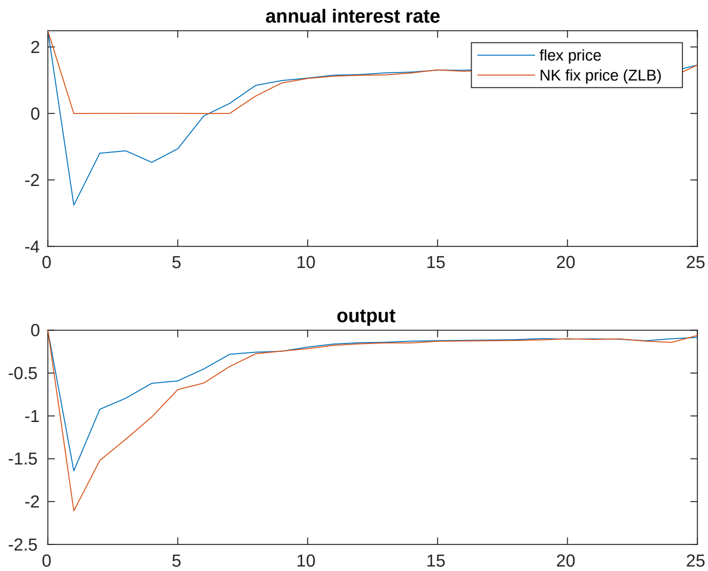


FIGURE VII

Employment Response

Percent deviations from initial steady state.

Figure 29: Figure 7 of Guerrieri & Lorenzoni (2017)



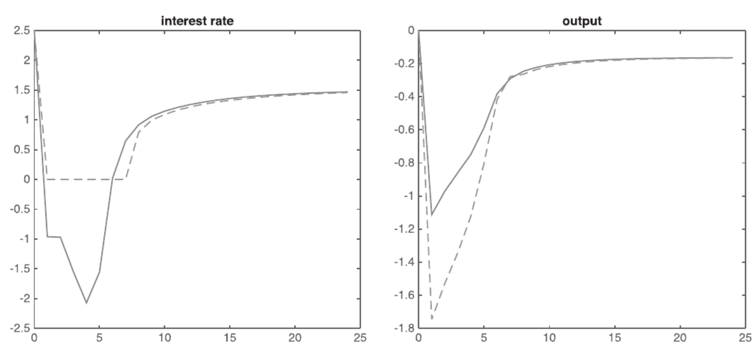
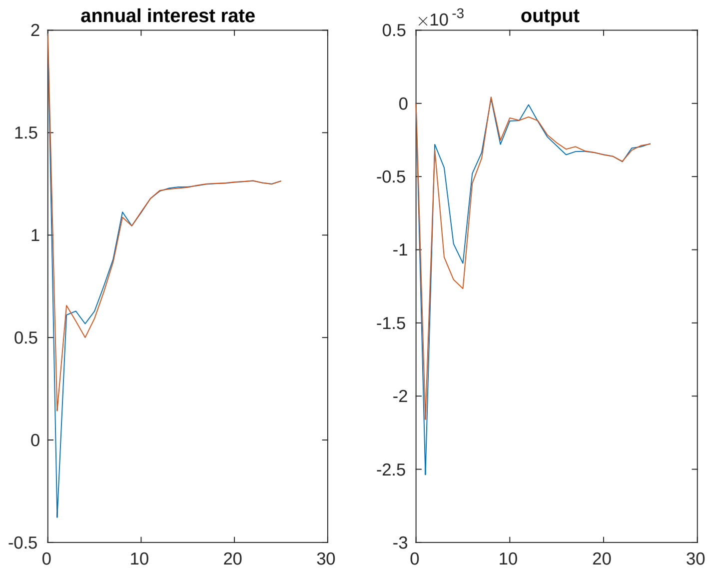


FIGURE VIII

Responses with Fixed Wages

Solid line: flexible price economy. Dashed line: economy with fixed wages. Interest rate in annual terms. Output in percent deviation from initial steady state.

Figure 30: Figure 8 of Guerrieri & Lorenzoni (2017)



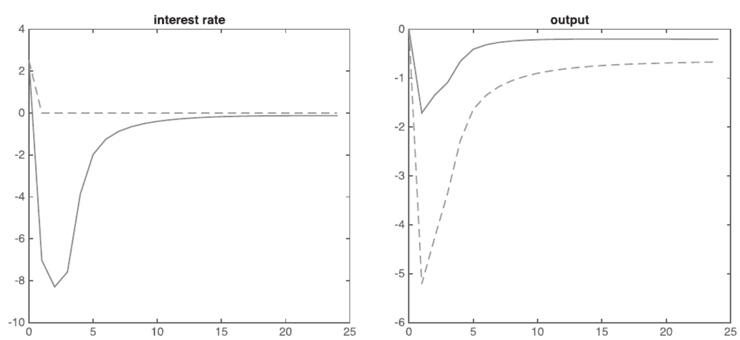
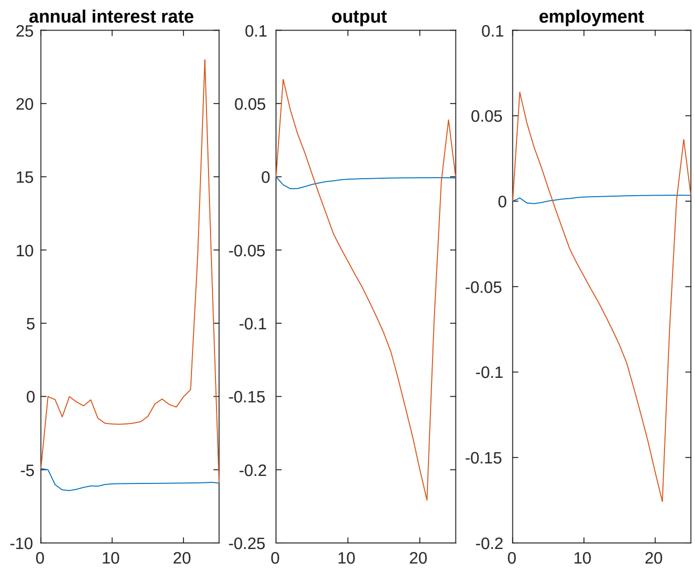


FIGURE IX

Interest Rate and Output: Median Wealth Calibration

Solid line: flexible price economy. Dashed line: economy with fixed wages. Interest rate in annual terms. Output in percent deviation from initial steady state.

Figure 31: Figure 9 of Guerrieri & Lorenzoni (2017)



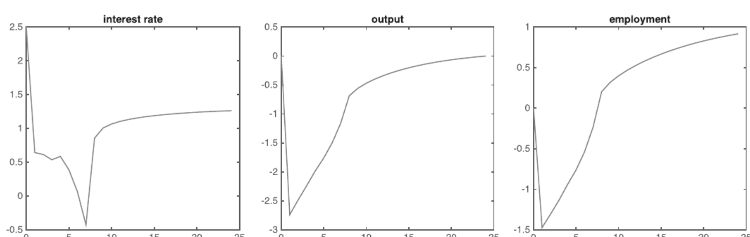
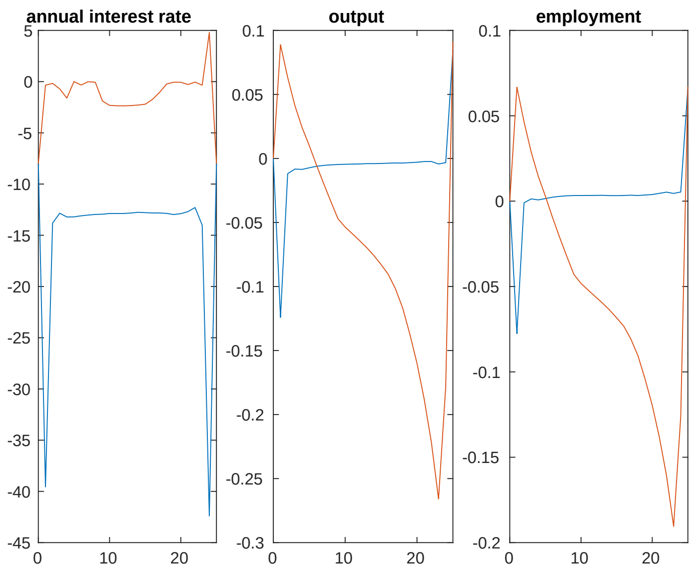


FIGURE X

Interest Rate, Output, Employment: Low ψ Calibration

Interest rate is in annual terms. Output is in percent deviation from initial steady state.

Figure 32: Figure 10 of Guerrieri & Lorenzoni (2017)



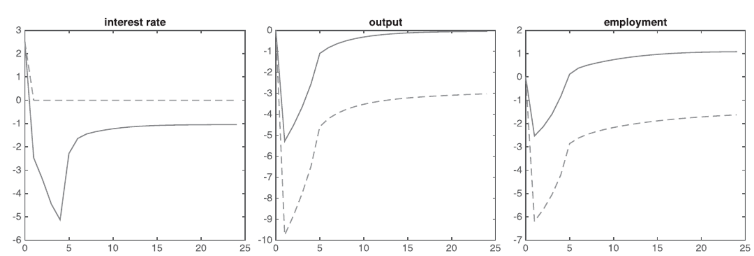
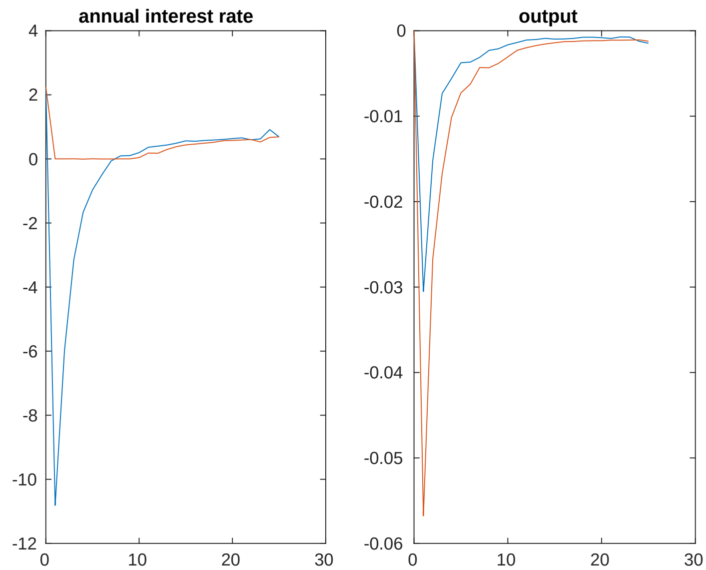


FIGURE XI

Interest Rate, Output, Employment: Median Wealth and Low ψ Calibration

Solid line: flexible price economy. Dashed line: economy with fixed wages. Interest rate is in annual terms. Output is in percent deviation from initial steady state.

Figure 33: Figure 11 of Guerrieri & Lorenzoni (2017)



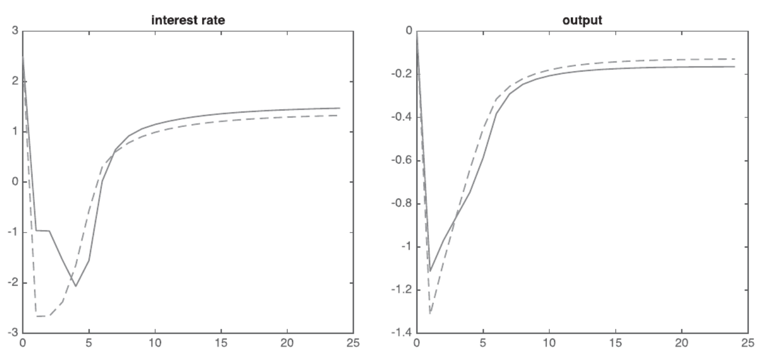
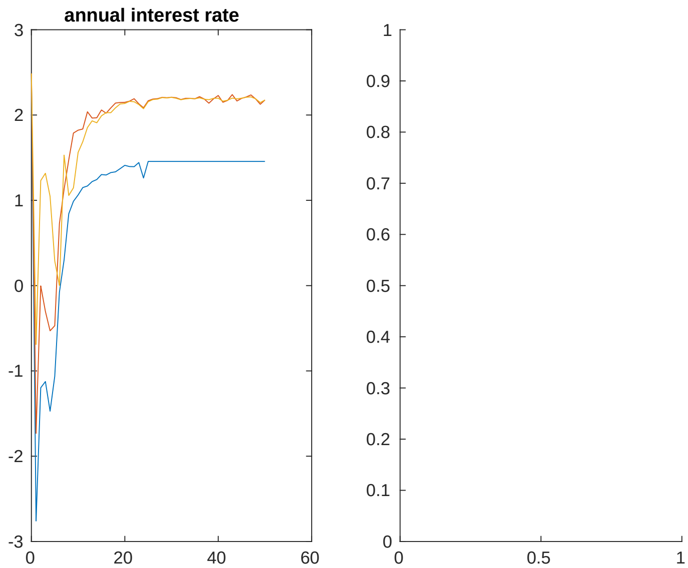


FIGURE XII

Interest Rate and Output Responses: $\gamma = 6$ Calibration

Interest rate is in annual terms. Output is in percent deviation from initial steady state.

Figure 34: Figure 12 of Guerrieri & Lorenzoni (2017)



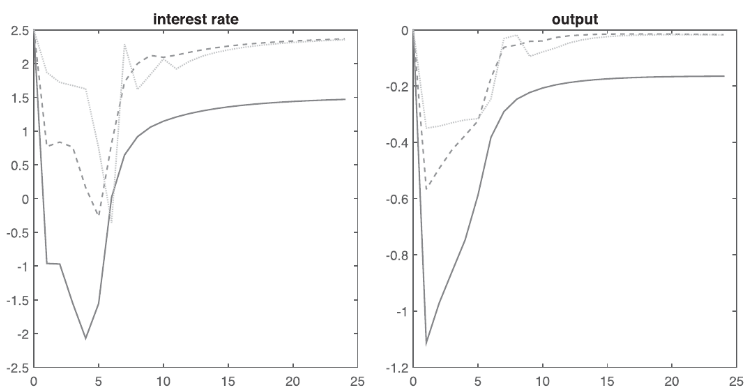
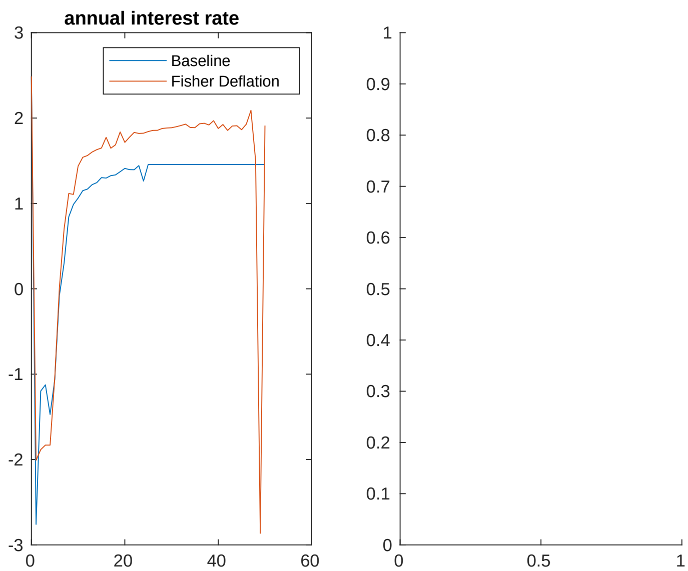


FIGURE XIII

Fiscal Policy

Solid line: baseline. Dashed line: temporary reduction in lump-sum tax. Dotted line: temporary increase in unemployment benefits. Interest rate is in annual terms. Output is in percent deviation from initial steady state.

Figure 35: Figure 13 of Guerrieri & Lorenzoni (2017)



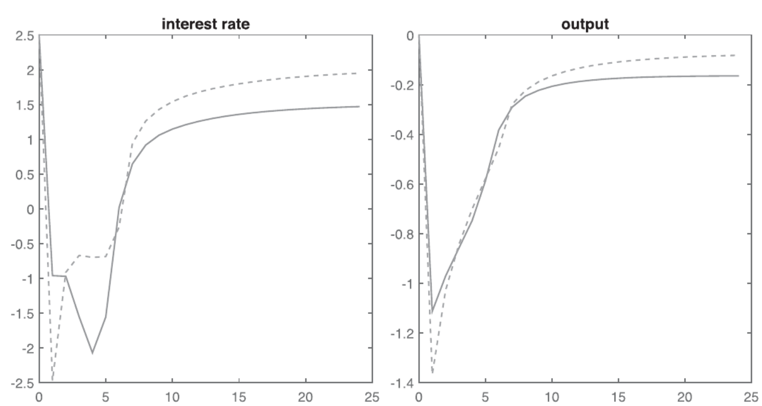


FIGURE XIV

Deflation: the Fisher Effect

Solid line: baseline. Dashed line: 10% deflation at $t = 0$. Interest rate is in annual terms. Output is in percent deviation from initial steady state.

Figure 36: Figure 14 of Guerrieri & Lorenzoni (2017)

./SavedOutput/GuerrieriLorenzoni2017/Graphs/GuerrieriLorenzoni2017_Figure15.pd

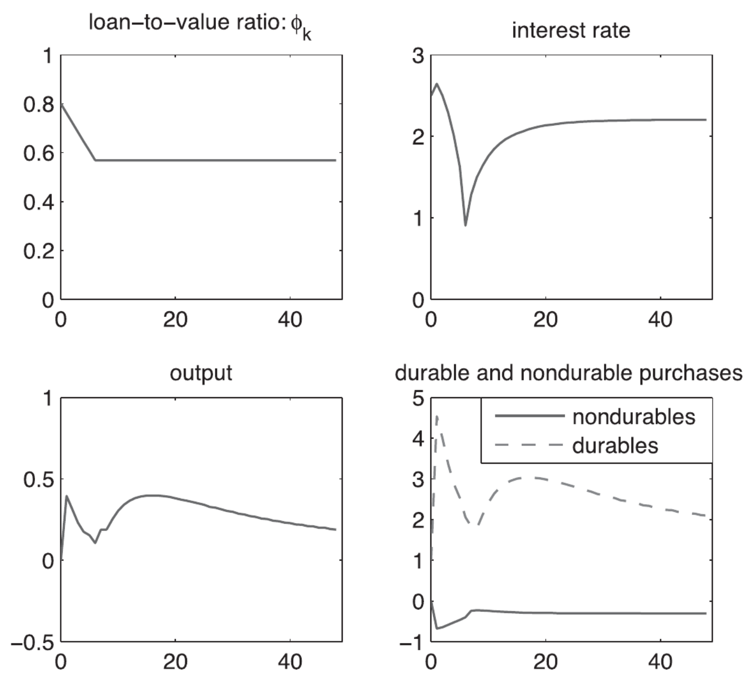


FIGURE XV

Responses to a Shock to the Borrowing Limit ϕ_k

Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

Figure 37: Figure 15 of Guerrieri & Lorenzoni (2017)

./SavedOutput/GuerrieriLorenzoni2017/Graphs/GuerrieriLorenzoni2017_Figure16.pd

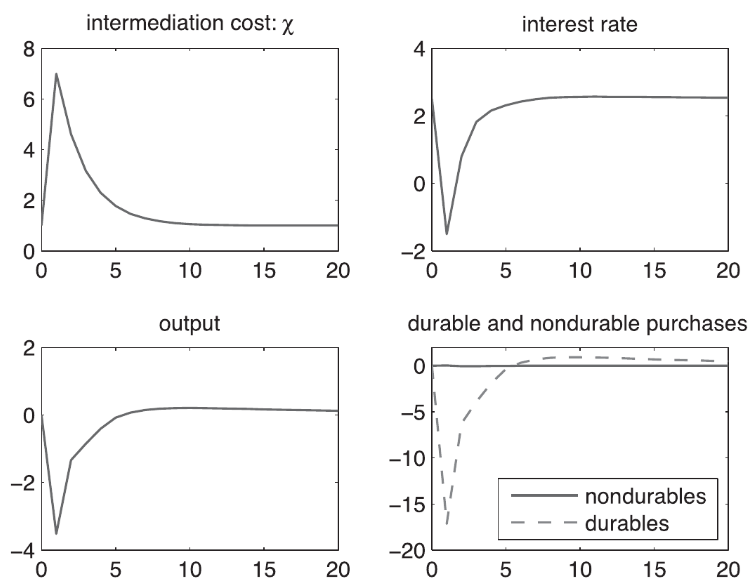


FIGURE XVI

Responses to a Temporary Shock to the Intermediation Cost χ

Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

Figure 38: Figure 16 of Guerrieri & Lorenzoni (2017)