APPENDIX OF
A Gödel modal logic over witnessed crisp models

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This Appendix includes supplementary material for the paper:

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Proofs of Section 3

Lemma 1. *Let ρ be an instance of a rule of the calculus $\mathcal{C}_{\mathrm{GW^c}}$, let Γ be the conclusion of ρ and let $\mathfrak{M}$ be a* $\mathrm{GW^c}$*-model. If $\mathfrak{M} \models \Gamma$, then there exists a premise Γ' of ρ such that $\mathfrak{M} \models \Gamma'$.*

Proof. Let $\mathcal{I}$ be an $\mathfrak{M}$-interpretation, let w be a label of $\mathcal{L}_c$ and $w^\star$ a world of $\mathfrak{M}$. By $\mathcal{I}[w \coloneqq w^\star]$ we denote the $\mathfrak{M}$-interpretation $\mathcal{I}'$ such that $\mathcal{I}'(w) = w^\star$ and $\mathcal{I}'(x) = \mathcal{I}(x)$ for every $x \neq w$ (where x is either a label or a constant). The definition of $\mathcal{I}[c \coloneqq r]$, where c is a constant of $\mathcal{L}_c$ and $r \in [0,1]_Q$, is similar.

Let Γ be the conclusion of rule ρ of $\mathcal{C}_{\mathrm{GW^c}}$ and let us assume $\mathfrak{M} \models_{\mathcal{I}} \Gamma$, where $\mathfrak{M} = \langle W, R, e\rangle$; we show that there exists a premise Γ' of ρ and an $\mathfrak{M}$-interpretation $\mathcal{I}'$ such that $\mathfrak{M} \models_{\mathcal{I}'} \Gamma'$.

The case $\rho = \mathrm{Ax}$ has already been discussed in the paper. If ρ is one of the rules $\wedge\triangleleft$, $\wedge\triangleright$, $\vee\triangleleft$, $\vee\triangleright$, the assertion easily follows.

Let us consider the application

$$\dfrac{\overbrace{w : \beta \leq b,\ w : \alpha > b,\ b < t,\ \Gamma_0}^{\Gamma_1}}{\underbrace{w : \alpha \to \beta < t,\ \Gamma_0}_{\Gamma}}\ {\to}{<}$$

By the hypothesis, it holds that $\mathfrak{M} \models_{\mathcal{I}} w : \alpha \to \beta < t$ and $\mathfrak{M} \models_{\mathcal{I}} \Gamma_0$. This implies that $e(\mathcal{I}(w), \alpha \to \beta) < \mathcal{I}(t) \leq 1$, hence $e(\mathcal{I}(w), \alpha) > e(\mathcal{I}(w), \beta)$ and

$e(\mathcal{I}(w), \alpha \to \beta) = e(\mathcal{I}(w), \beta)$, thus $\mathfrak{M} \models_{\mathcal{I}} w : \beta < t$. If $\beta \in \mathcal{V} \cup \{\bot\}$, then b stands for $w : \beta$, hence $\mathfrak{M} \models_{\mathcal{I}} \Gamma_1$. Let us assume $\beta \notin \mathcal{V} \cup \{\bot\}$; in this case b is a fresh constant. Let $r = e(\mathcal{I}(w), \beta)$ ($r \in [0,1]_Q$) and $\mathcal{I}' = \mathcal{I}[b := r]$. We remark that, for every constraint χ not containing b, $\mathfrak{M} \models_{\mathcal{I}} \chi$ iff $\mathfrak{M} \models_{\mathcal{I}'} \chi$. Since $\mathfrak{M} \models_{\mathcal{I}} \Gamma_0$ and b does not occur in Γ_0, we get $\mathfrak{M} \models_{\mathcal{I}'} \Gamma_0$. It is easy to check that $\mathfrak{M} \models_{\mathcal{I}'} w : \beta \leq b$ and $\mathfrak{M} \models_{\mathcal{I}'} w : \alpha > b$ and $\mathfrak{M} \models_{\mathcal{I}'} b < t$, hence $\mathfrak{M} \models_{\mathcal{I}'} \Gamma_1$.

Let us consider the application

$$\frac{\overbrace{t \geq 1,\, \Gamma_0}^{\Gamma_1} \qquad \overbrace{w : \beta \leq b,\, w : \alpha > b,\, b \leq t,\, \Gamma_0}^{\Gamma_2}}{\underbrace{w : \alpha \to \beta \leq t,\, \Gamma_0}_{\Gamma}} \to\leq$$

By the hypothesis, $\mathfrak{M} \models_{\mathcal{I}} w : \alpha \to \beta \leq t$ and $\mathfrak{M} \models_{\mathcal{I}} \Gamma_0$, hence $e(\mathcal{I}(w), \alpha \to \beta) \leq \mathcal{I}(t)$. Let us assume $e(\mathcal{I}(w), \alpha) \leq e(\mathcal{I}(w), \beta)$. Then, $e(\mathcal{I}(w), \alpha \to \beta) = 1$, which implies $1 \leq \mathcal{I}(t)$, namely $\mathcal{I}(t) = 1$. It follows that $\mathfrak{M} \models_{\mathcal{I}} t \geq 1$, hence $\mathfrak{M} \models_{\mathcal{I}} \Gamma_1$. Let us assume $e(\mathcal{I}(w), \alpha) > e(\mathcal{I}(w), \beta)$. We have $e(\mathcal{I}(w), \alpha \to \beta) = e(\mathcal{I}(w), \beta)$, hence $\mathfrak{M} \models_{\mathcal{I}} w : \beta \leq t$. If $\beta \in \mathcal{V} \cup \{\bot\}$, then $b = w : \beta$ and we get $\mathfrak{M} \models_{\mathcal{I}} \Gamma_2$. Assume $\beta \notin \mathcal{V} \cup \{\bot\}$ (thus b is a fresh constant), let $r = e(\mathcal{I}(w), \beta)$ ($r \in [0,1]_Q$) and $\mathcal{I}' = \mathcal{I}[b := r]$. Since $\mathfrak{M} \models_{\mathcal{I}} \Gamma_0$ and b does not occur in Γ_0, we get $\mathfrak{M} \models_{\mathcal{I}'} \Gamma_0$. Moreover, one can easily check that $\mathfrak{M} \models_{\mathcal{I}'} w : \beta \leq b$ and $\mathfrak{M} \models_{\mathcal{I}'} w : \alpha > b$ and $\mathfrak{M} \models_{\mathcal{I}'} b \leq t$; we conclude $\mathfrak{M} \models_{\mathcal{I}'} \Gamma_2$.

Let us consider the application

$$\frac{\overbrace{w : \alpha \leq a,\, w : \beta \geq a,\, 1 \rhd t,\, \Gamma_0}^{\Gamma_1} \qquad \overbrace{w : \beta \rhd t,\, \Gamma_0}^{\Gamma_2}}{\underbrace{w : \alpha \to \beta \rhd t,\, \Gamma_0}_{\Gamma}} \to \rhd$$

By the hypothesis, $\mathfrak{M} \models_{\mathcal{I}} w : \alpha \to \beta \rhd t$ and $\mathfrak{M} \models_{\mathcal{I}} \Gamma_0$, hence $e(\mathcal{I}(w), \alpha \to \beta) \rhd \mathcal{I}(t)$. We can reason as in the cases concerning rules $\to<$ and $\to\leq$. If $e(\mathcal{I}(w), \alpha) > e(\mathcal{I}(w), \beta)$, then $e(\mathcal{I}(w), \alpha \to \beta) = e(\mathcal{I}(w), \beta)$, hence $\mathfrak{M} \models_{\mathcal{I}} \Gamma_2$. Let us assume $e(\mathcal{I}(w), \alpha) \leq e(\mathcal{I}(w), \beta)$. In this case $e(\mathcal{I}(w), \alpha \to \beta) = 1$, which implies $1 \rhd \mathcal{I}(t)$. We can prove that $\mathfrak{M} \models_{\mathcal{I}'} \Gamma_1$, where $\mathcal{I}' = \mathcal{I}$ if $\alpha \in \mathcal{V} \cup \{\bot\}$, $\mathcal{I}' = \mathcal{I}[a := e(\mathcal{I}(w), \alpha)]$ otherwise.

Let us consider the application

$$\frac{\overbrace{1 \lhd t,\, \Phi^{0,1}(\Gamma_0)}^{\Gamma_1} \qquad \overbrace{w_1 : \alpha \lhd t,\, \Phi^{\Box,\Diamond}(\Gamma, w, w_1),\, \Gamma_0}^{\Gamma_2}}{\underbrace{w : \Box\alpha \lhd t,\, \Gamma_0}_{\Gamma}} \Box\lhd$$

By the hypothesis, $\mathfrak{M} \models_{\mathcal{I}} w : \Box\alpha \lhd t$ and $\mathfrak{M} \models_{\mathcal{I}} \Gamma_0$, hence $e(\mathcal{I}(w), \Box\alpha) \lhd \mathcal{I}(t)$. Let us assume that the world $\mathcal{I}(w)$ of $\mathfrak{M}$ has no R-successors. For every formula φ, $e(\mathcal{I}(w), \Box\varphi) = 1$ and $e(\mathcal{I}(w), \Diamond\varphi) = 0$, and this implies that $\mathfrak{M} \models_{\mathcal{I}} \Gamma_1$. Let us assume that $\mathcal{I}(w)$ has at least one R-successor. Since $\mathfrak{M}$ is a GW$^{\mathrm{c}}$-model, there exists a world $w^\star$ of $\mathfrak{M}$ such that $\mathcal{I}(w)Rw^\star$ and $e(\mathcal{I}(w), \Box\alpha) = e(w^\star, \alpha)$. Let $\mathcal{I}' = \mathcal{I}[w_1 := w^\star]$. We remark that, for every constraint χ not containing w_1, $\mathfrak{M} \models_{\mathcal{I}} \chi$ iff $\mathfrak{M} \models_{\mathcal{I}'} \chi$. Since $\mathfrak{M} \models_{\mathcal{I}} \Gamma_0$ and w_1 does not occur in Γ_0, we get $\mathfrak{M} \models_{\mathcal{I}'} \Gamma_0$. By the fact that $e(\mathcal{I}(w), \Box\alpha) \lhd \mathcal{I}(t)$ and $e(\mathcal{I}(w), \Box\alpha) = e(w^\star, \alpha)$, we have $e(w^\star, \alpha) \lhd \mathcal{I}(t)$, and this implies $\mathfrak{M} \models_{\mathcal{I}'} w_1 : \alpha \lhd t$. Let $\chi \in \Phi^{\Box,\Diamond}(\Gamma, w, w_1)$ and assume that $\chi = w_1 : \beta \rhd t'$. We have $w : \Box\beta \rhd t' \in \Gamma_0$, hence $e(\mathcal{I}(w), \Box\beta) \rhd \mathcal{I}(t')$. Since $\mathcal{I}(w)Rw^\star$, it follows that $e(w^\star, \beta) \rhd \mathcal{I}(t')$, and this implies $\mathfrak{M} \models_{\mathcal{I}'} \chi$. Similarly, if $\chi = w_1 : \beta \lhd t'$, namely $w : \Diamond\beta \lhd t' \in \Gamma_0$, we get $\mathfrak{M} \models_{\mathcal{I}'} \chi$. We conclude $\mathfrak{M} \models_{\mathcal{I}'} \Gamma_2$.

The case where ρ is rule $\Diamond\rhd$ is similar. ■

Proofs of Section 4

Lemma 11. *Let $\mathcal{K}$ be a* GW^{c}*-bimodel and* Rn *a ranking over $\mathcal{K}$.*

(i) $\mathrm{Rn}(x, \alpha \star \beta) = \mathrm{Rn}(x, \alpha) \star \mathrm{Rn}(x, \beta)$*, for* $\star \in \{\wedge, \vee, \to\}$.
(ii) $\mathrm{Rn}(x, \Box\alpha) = \min(\{\mathrm{Rn}(y, \alpha) \mid xSy\} \cup \{1\})$.
(iii) $\mathrm{Rn}(x, \Diamond\alpha) = \max(\{\mathrm{Rn}(y, \alpha) \mid xSy\} \cup \{0\})$.

Proof. Let $\mathcal{K} = \langle X, \leq, S, V \rangle$ and let $x \in X$. An element $x_1 \in X$ is the immediate $\leq$-predecessor of x if $x_1 < x$ and, for every $y \in X$ such that $x_1 \leq y \leq x$, either $y = x_1$ or $y = x$ (x_1 is well-defined since X is finite).

We introduce the following notation:

- $x_{\min}$ is the $\leq$-minimum element of $[x]_\sim$;
- $x_{\max}$ is the $\leq$-maximum element of $[x]_\sim$.

Note that:

- $x_{\max} \nVdash \alpha$ iff $\mathrm{Rn}(x, \alpha) = 0$;
- $x_{\min} \Vdash \alpha$ iff $\mathrm{Rn}(x, \alpha) = 1$.

By conditions (F4) and (F5), if $x_{\min}Sy$, then y is $\leq$-minimal in $[y]_\sim$; similarly, by (F3) and (F5), if $x_{\max}Sy$, then y is $\leq$-maximal in $[y]_\sim$. If x is not $\leq$-minimal, x^- denotes the immediate $\leq$-predecessor of x.

(i). Let $r = \mathrm{Rn}(x, \alpha \wedge \beta)$, $a = \mathrm{Rn}(x, \alpha)$ and $b = \mathrm{Rn}(x, \beta)$; we show that $r = \min(a, b)$. Let us assume that $r = 0$. We have $x_{\max} \nVdash \alpha \wedge \beta$, hence $x_{\max} \nVdash \alpha$ or $x_{\max} \nVdash \beta$. In the former case we get $a = 0$, in the latter $b = 0$; in either case, $r = \min(a, b)$. Assume $r = 1$. We have $x_{\min} \Vdash \alpha \wedge \beta$, hence $x_{\min} \Vdash \alpha$ and $x_{\min} \Vdash \beta$. This implies $a = 1$ and $b = 1$, hence $r = \min(a, b)$. Let us assume $0 < r < 1$. Since $r > 0$, there exists $x_r \in [x]_\sim$ such that $\mathrm{Rn}(x_r) = r$; moreover, since $r < 1$, the world x_r is not $\leq$-minimal, hence $(x_r)^-$ is defined. By definition $x_r \Vdash \alpha \wedge \beta$ and $(x_r)^- \nVdash \alpha \wedge \beta$, hence $(x_r)^- \nVdash \alpha$ or $(x_r)^- \nVdash \beta$. Assume $(x_r)^- \nVdash \alpha$. Since $x_r \Vdash \alpha$, we get $a = \mathrm{Rn}(x_r) = r$; moreover, since $x_r \Vdash \beta$, it holds that $b \geq \mathrm{Rn}(x_r) = r$. This proves $r = \min(a, b)$. Similarly, if $(x_r)^- \nVdash \beta$, we get $b = r$ and $a \geq r$, hence $r = \min(a, b)$.

The proof that $\mathrm{Rn}(x, \alpha \vee \beta)$ is the maximum between $\mathrm{Rn}(x, \alpha)$ and $\mathrm{Rn}(x, \beta)$ is similar.

Let $r = \mathrm{Rn}(x, \alpha \to \beta)$, $a = \mathrm{Rn}(x, \alpha)$ and $b = \mathrm{Rn}(x, \beta)$; we show that $r = a \to b$ (where $\to$ is the algebraic implication). Assume $r = 0$. We have $x_{\max} \nVdash \alpha \to \beta$, hence $x_{\max} \Vdash \alpha$ and $x_{\max} \nVdash \beta$. This implies $a > 0$ and $b = 0$, hence $a \to b = 0 = r$. Let us assume $r = 1$. We have $x_{\min} \Vdash \alpha \to \beta$, hence, for every $y \in [x]_\sim$, if $y \Vdash \alpha$ then $y \Vdash \beta$. Accordingly, $b \leq a$, and this implies $a \to b = 1 = r$. Let us assume $0 < r < 1$. Since $r > 0$, there exists $x_r \in [x]_\sim$ such that $\mathrm{Rn}(x_r) = r$; moreover, since $r < 1$, $(x_r)^-$ is defined. By definition $x_r \Vdash \alpha \to \beta$ and $(x_r)^- \nVdash \alpha \to \beta$; it follows that $(x_r)^- \Vdash \alpha$ and $(x_r)^- \nVdash \beta$. Since $(x_r)^- \Vdash \alpha$, we get $a \geq \mathrm{Rn}((x_r)^-) > r$, namely $a > r$. Note that $(x_r)^- \nVdash \beta$ and $x_r \Vdash \beta$, hence $b = \mathrm{Rn}(x_r) = r$, namely $b = r$. We have proved $a > b$, and this implies $a \to b = b = r$.

(ii). Let $r = \mathrm{Rn}(x, \Box\alpha)$, we show that:

(1) $\mathrm{Rn}(x, \Box\alpha) = \min(\mathcal{R} \cup \{1\})$, where $\mathcal{R} = \{\mathrm{Rn}(y, \alpha) \mid xSy\}$.

Assume $r = 0$. We have $x_{\max} \nVdash \Box\alpha$, hence there exists $y \in X$ such that $x_{\max}Sy$ and $y \nVdash \alpha$. Since y is $\leq$-maximal in $[y]_\sim$, we get $\mathrm{Rn}(y, \alpha) = 0$. This implies $0 \in \mathcal{R}$, and this proves (1).

Assume $r = 1$, namely $x_{\min} \Vdash \Box\alpha$. To prove (1), we show that, if $\mathcal{R}$ is non-empty, then $\mathcal{R} = \{1\}$. Let us assume $\mathcal{R} \neq \emptyset$ and let $xSy \in \mathcal{R}$. By (F2), there exists $y^\star$ such that $x_{\min}Sy^\star$ and $y^\star \leq y$. Since $x_{\min} \Vdash \Box\alpha$ and $x_{\min}Sy^\star$, we get $y^\star \Vdash \alpha$. Note that $y^\star$ is the $\leq$-minimum element of $[y]_\sim$, hence $\mathrm{Rn}(y, \alpha) = 1$. It follows that $\mathcal{R} = \{1\}$, and this proves (1).

Let us assume $0 < r < 1$. Since $r > 0$, there exists $x_r \in [x]_\sim$ such that $\mathrm{Rn}(x_r) = r$; moreover, since $r < 1$, $(x_r)^-$ is defined. By definition, $x_r \Vdash \Box\alpha$ and $(x_r)^- \nVdash \Box\alpha$. There exists $y^\star$ such that $(x_r)^-Sy^\star$ and $y^\star \nVdash \alpha$. By (F1), there exists y_r such that $y^\star \leq y_r$ and x_rSy_r, thus $\mathrm{Rn}(y_r) = r$. One can easily check that $y^\star$ is the immediate $\leq$-predecessor of y_r; moreover, since $x_r \Vdash \Box\alpha$ and x_rSy_r, it holds that $y_r \Vdash \alpha$. We have $y_r \Vdash \alpha$ and $y^\star \nVdash \alpha$; since $y^\star = (y_r)^-$, we get $\mathrm{Rn}(y_r, \alpha) = r$. Note that $[x]_\sim S[y_r]_\sim$, hence, by property (S1), there exists $y \in [y_r]_\sim$ such that xSy. Since y and y_r are in the same cluster, we get $\mathrm{Rn}(y, \alpha) = \mathrm{Rn}(y_r, \alpha) = r$; accordingly, $r \in \mathcal{R}$. To complete the proof of (1), we have to show that xSy implies $r \leq \mathrm{Rn}(y, \alpha)$. Let xSy. By properties (F1) and (F2), there exists $y_r \in [y]_\sim$ such that x_rSy_r, thus $\mathrm{Rn}(y_r) = \mathrm{Rn}(x_r) = r$. Since $x_r \Vdash \Box\alpha$ and x_rSy_r, we get $y_r \Vdash \alpha$, and this implies $\mathrm{Rn}(y, \alpha) \geq \mathrm{Rn}(y_r) = r$. This concludes the proof of (1). ■

Proposition 13. *Let Φ be a correspondence between $\mathfrak{M}$ and $(\mathcal{K}, \mathrm{Rn})$. For every formula φ and every world w of $\mathfrak{M}$, $e(w, \varphi) = \mathrm{Rn}(\Phi(w), \varphi)$.*

Proof. By induction on φ. If $\varphi \in \mathcal{V}$, the assertion immediately follows from the definition of Φ. The case $\varphi = \bot$ is trivial, since $e(w, \bot) = 0$ and $\mathrm{Rn}(\Phi(w), \bot) = 0$.

Let $\varphi = \alpha \star \beta$, where $\star \in \{\wedge, \vee, \rightarrow\}$. We have:

$$e(w, \alpha \star \beta) \;=\; e(w, \alpha) \star (w, \beta) \stackrel{(\dagger)}{=} \mathrm{Rn}(\Phi(w), \alpha) \star \mathrm{Rn}(\Phi(w), \beta) \stackrel{(\ddagger)}{=} \mathrm{Rn}(\Phi(w), \alpha \star \beta)$$

where (†) follows from the induction hypothesis and (‡) from Lemma 11(i).

Let $\mathfrak{M} = \langle W, R, e \rangle$, let $\mathcal{K} = \langle X, \leq, S, V \rangle$ and $w_r \in \Phi(w)$. Note that:

$$wRw' \quad \Longleftrightarrow \quad \Phi(w)\, S\, \Phi(w') \quad \Longleftrightarrow \quad \exists y \in \Phi(w') \text{ s.t. } w_r S y$$

Accordingly:

$$\{\, \mathrm{Rn}(\Phi(w'), \alpha) \mid w\, R\, w' \,\} \;=\; \{\, \mathrm{Rn}(y, \alpha) \mid w_r\, S\, y \,\} \tag{1}$$

We have:

$$\begin{aligned}
e(w, \Box\alpha) &= \min\left(\{\, e(w', \alpha) \mid w\, R\, w' \,\} \cup \{1\}\right) & \\
&= \min\left(\{\, \mathrm{Rn}(\Phi(w'), \alpha) \mid w\, R\, w' \,\} \cup \{1\}\right) & \text{ind. hyp.} \\
&= \min\left(\{\, \mathrm{Rn}(y, \alpha) \mid w_r\, S\, y \,\} \cup \{1\}\right) & (1) \\
&= \mathrm{Rn}(w_r, \Box\alpha) & \text{Lemma 11(ii)} \\
&= \mathrm{Rn}(\Phi(w), \Box\alpha) & [w_r]_\sim = \Phi(w)
\end{aligned}$$

The case $\varphi = \Diamond\alpha$ is similar. ∎