A Terminating Sequent Calculus for Intuitionistic Strong Löb Logic with the Subformula Property

Appendix

Camillo Fiorentini[1][0000−0003−2152−7488] and Mauro Ferrari[2][0000−0002−7904−1125]

[1] Dep. of Computer Science, Università degli Studi di Milano, Italy
✉ fiorentini@di.unimi.it
[2] Dep. of Theoretical and Applied Sciences, Università degli Studi dell'Insubria, Italy
mauro.ferrari@uninsubria.it

A Soundness of $\mathsf{GbuSL}_\Box$ and $\mathsf{G3iSL}_\Box^+$

We provide a semantic proof of the soundness of $\mathsf{GbuSL}_\Box$ (Th. 2(iii)); the same reasoning can be applied to prove the soundness of $\mathsf{G3iSL}_\Box^+$ (see Th. 1). We start by proving the following property about forcing of $\Box$-formulas.

Lemma A1 *Let $\mathcal{K} = \langle W, \leq, R, r, V\rangle$ be an iSL-model, let $w \in W$ and assume that $w \nVdash \Box\alpha$. Then, there exists w' such that wRw' and $w' \nVdash \alpha$ and $w' \Vdash \Box\alpha$.*

Proof. Let us assume by contradiction that the assertion does not hold, namely:

(†) for every w' such that wRw', $w' \Vdash \alpha$ or $w' \nVdash \Box\alpha$.

We define an infinite sequence of worlds w_0, w_1, … such that:

(i) $w_k \nVdash \Box\alpha$, for every $k \geq 0$.
(ii) $w_0 R w_1 R w_2 \ldots$.

The definition of the sequence is by induction on k. We set $w_0 = w$. Assume that w_k has been defined. Since $w_k \nVdash \Box\alpha$, there exists w' such that $w_k R w'$ and $w' \nVdash \alpha$. Note that $w_0 R w'$; since $w_0 = w$, by (†) it follows that $w' \nVdash \Box\alpha$, thus we can set $w_{k+1} = w'$. This proves (i) and (ii). Point (ii) yields a contradiction, since R is converse well-founded (see Point (M2) in Sec. 2). ■

We show that the rules of $\mathsf{GbuSL}_\Box$ preserve $\models_{\mathsf{iSL}}$.

Lemma A2 *Let ρ be an application of a rule of $\mathsf{GbuSL}_\Box$ and let $\Gamma \xRightarrow{l} \delta$ be the conclusion of ρ. If, for every premise $\Gamma' \xRightarrow{l'} \delta'$ of ρ, $\Gamma' \models_{\mathsf{iSL}} \delta'$, then $\Gamma \models_{\mathsf{iSL}} \delta$.*

Proof. The proof is by a case analysis on ρ. We only consider the most interesting case, corresponding to an application of rule $R_{\mathrm{b}}^{\Box}$:

$$\frac{\Box\alpha, \Gamma, \Delta \xRightarrow{\mathrm{u}} \alpha}{\Gamma, \Box\Delta \xRightarrow{\mathrm{b}} \Box\alpha}\ R_{\mathrm{b}}^{\Box} \qquad \Gamma \cup \Box\Delta \not\triangleright \Box\alpha$$

We assume $\Gamma \cup \Box\Delta \not\models_{\mathsf{iSL}} \Box\alpha$ and we show $\{\Box\alpha\} \cup \Gamma \cup \Delta \not\models_{\mathsf{iSL}} \alpha$. Since $\Gamma \cup \Box\Delta \not\models_{\mathsf{iSL}} \Box\alpha$, there exists an iSL-model $\mathcal{K} = \langle W, \leq, R, r, V \rangle$ and w in $\mathcal{K}$ such that $w \Vdash \Gamma \cup \Box\Delta$ and $w \nVdash \Box\alpha$. By Lemma A1, there exists w' such that wRw' and $w' \Vdash \Box\alpha$ and $w' \nVdash \alpha$. Note that $w' \Vdash \Gamma \cup \Delta$, thus we conclude $\{\Box\alpha\} \cup \Gamma \cup \Delta \not\models_{\mathsf{iSL}} \alpha$.

Let us assume that the sequent $\sigma = \Gamma \stackrel{l}{\Rightarrow} \delta$ is provable in $\mathsf{GbuSL}_\Box$. By applying Lemma A2 to a $\mathsf{GbuSL}_\Box$-derivation of σ, we get $\Gamma \models_{\mathsf{iSL}} \delta$, and this proves the soundness of the calculus $\mathsf{GbuSL}_\Box$.

B The proof search procedure

We recall that a sequent $\Gamma \stackrel{l}{\Rightarrow} \delta$ is *regular* iff $l = \mathrm{u}$ or $\Gamma = \Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Box\Delta$. The procedure `SearchBU` is described in Figs. 1–3 and has the following specification:

- **Input**: A regular sequent $\sigma = \Gamma \stackrel{l}{\Rightarrow} \delta$.
- **Output**: A $\mathsf{GbuSL}_\Box$-derivation of σ or an $\mathsf{RbuSL}_\Box$-derivation of $\Gamma \stackrel{l}{\not\Rightarrow} \delta$.

Proof search is performed by applying backward the rules of $\mathsf{GbuSL}_\Box$, giving priority to the invertible rules. For instance, the recursive call `SearchBU(`$\alpha, \beta, \Gamma' \stackrel{\mathrm{u}}{\Rightarrow} \delta$`)` at line 3 corresponds to the backward application of the (invertible) rule $L\wedge$ to $\sigma = \alpha \wedge \beta, \Gamma' \stackrel{\mathrm{u}}{\Rightarrow} \delta$, where $\Gamma' = \Gamma \setminus \{\alpha \wedge \beta\}$. Let $\mathcal{D}_0$ be the derivation returned by `SearchBU(`$\alpha, \beta, \Gamma' \stackrel{\mathrm{u}}{\Rightarrow} \delta$`)`; according to the type of $\mathcal{D}_0$, at lines 4–5 a $\mathsf{GbuSL}_\Box$-derivation of $\alpha \wedge \beta, \Gamma' \stackrel{\mathrm{u}}{\Rightarrow} \delta$ or an $\mathsf{RbuSL}_\Box$-derivation of $\alpha \wedge \beta, \Gamma' \stackrel{\mathrm{u}}{\not\Rightarrow} \delta$ with root rule $L\wedge$ is returned.

Note that we provide a high-level presentation of the procedure; in a concrete implementation, some low-level details must be further specified (e.g., at line 1, which between rules $\mathrm{Ax}^{\triangleright}$ and $L\bot$ must be applied). Some instructions can be permuted (e.g., the order according which the invertible rules are applied can be changed). The correctness of `SearchBU` can be proved by induction on the well-founded relation $\prec_{\mathrm{bu}}$ (see the proof of Prop. 4).

if $\Gamma \triangleright \delta$ *or* ($l = \mathrm{u}$ *and* $\bot \in \Gamma$) **then return** $\dfrac{}{\Gamma \overset{l}{\Rightarrow} \delta}\,\rho \quad \rho \in \{\, \mathrm{Ax}^{\triangleright},\, L\bot \,\}$

else if $l = u$ *and* $\alpha \wedge \beta \in \Gamma$ **then**

  $\Gamma' \leftarrow \Gamma \setminus \{\alpha \wedge \beta\}$, $\mathcal{D}_0 \leftarrow$ `SearchBU(` $\alpha, \beta, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta$`)`

  if $\mathcal{D}_0$ *is a* $\mathsf{GbuSL}_{\Box}$*-derivation* **then return** $\dfrac{\overset{\mathcal{D}_0}{\alpha, \beta, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta}}{\alpha \wedge \beta, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta}\, L\wedge$

  else return $\dfrac{\overset{\mathcal{D}_0}{\alpha, \beta, \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta}}{\alpha \wedge \beta, \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L\wedge$

else if $l = \mathrm{u}$ *and* $\alpha_0 \vee \alpha_1 \in \Gamma$ **then**

  $\Gamma' \leftarrow \Gamma \setminus \{\alpha_0 \vee \alpha_1\}$

  $\mathcal{D}_0 \leftarrow$ `SearchBU(`$\alpha_0, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta$`)`, $\mathcal{D}_1 \leftarrow$ `SearchBU(`$\alpha_1, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta$`)`

  if $\exists k \in \{0,1\}$ *s.t.* $\mathcal{D}_k$ *is an* $\mathsf{RbuSL}_{\Box}$*-derivation* **then**

    return $\dfrac{\overset{\mathcal{D}_k}{\alpha_k, \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta}}{\alpha_0 \vee \alpha_1, \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L\vee_k$

  else return $\dfrac{\overset{\mathcal{D}_0}{\alpha_0, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta} \qquad \overset{\mathcal{D}_1}{\alpha_1, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta}}{\alpha_0 \vee \alpha_1, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta}\, L\vee$

else if $\delta = \alpha \to \beta$ **then**

  if $\Gamma \triangleright \alpha$ **then** $\Gamma_0 \leftarrow \Gamma$, $l_0 = l$, $\rho \leftarrow R\overset{\triangleright}{\to}$

  else $\Gamma_0 \leftarrow \Gamma \cup \{\alpha\}$, $l_0 = \mathrm{u}$, $\rho \leftarrow R\overset{\not\triangleright}{\to}$

  $\mathcal{D}_0 \leftarrow$ `SearchBU(` $\Gamma_0 \overset{l_0}{\Rightarrow} \beta$`)`

  if $\mathcal{D}_0$ *is a* $\mathsf{GbuSL}_{\Box}$*-derivation* **then return** $\dfrac{\overset{\mathcal{D}_0}{\Gamma_0 \overset{l_0}{\Rightarrow} \beta}}{\Gamma \overset{l}{\Rightarrow} \alpha \to \beta}\, \rho$

  else return $\dfrac{\overset{\mathcal{D}_0}{\Gamma_0 \overset{l_0}{\not\Rightarrow} \beta}}{\Gamma \overset{l}{\not\Rightarrow} \alpha \to \beta}\, \rho$

else if $\delta = \delta_0 \wedge \delta_1$ **then**

  $\mathcal{D}_0 \leftarrow$ `SearchBU(`$\Gamma \overset{l}{\Rightarrow} \delta_0$`)`, $\mathcal{D}_1 \leftarrow$ `SearchBU(`$\Gamma \overset{l}{\Rightarrow} \delta_1$`)`

  if $\exists k \in \{0,1\}$ *s.t.* $\mathcal{D}_k$ *is an* $\mathsf{RbuSL}_{\Box}$*-derivation* **then**

    return $\dfrac{\overset{\mathcal{D}_k}{\Gamma \overset{l}{\not\Rightarrow} \delta_k}}{\Gamma \overset{l}{\not\Rightarrow} \delta_0 \wedge \delta_1}\, R\wedge_k$

  else return $\dfrac{\overset{\mathcal{D}_0}{\Gamma \overset{l}{\Rightarrow} \delta_0} \qquad \overset{\mathcal{D}_1}{\Gamma \overset{l}{\Rightarrow} \delta_1}}{\Gamma \overset{l}{\Rightarrow} \delta_0 \wedge \delta_1}\, R\wedge$

Fig. 1. `SearchBU(`$\Gamma \overset{l}{\Rightarrow} \delta$`)` (1)

23 `// Here` $\Gamma = \Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Box\Delta$ `and (` $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ `or` $\delta = \delta_0 \vee \delta_1$ `or` $\delta = \Box\delta_0$ `)`

24 **else if** $\delta = \Box\delta_0$ *and* $l = \mathrm{b}$ **then**

25    $\mathcal{D}_0 \leftarrow$ `SearchBU(`$\Box\delta_0, \Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Delta \overset{\mathrm{u}}{\Rightarrow} \delta_0$`)`

26   **if** $\mathcal{D}_0$ *is a* $\mathsf{GbuSL}_{\Box}$*-derivation* **then**

27   `// Since the condition at line 1 is false,` $\Gamma \not\models_{\mathsf{iSL}} \Box\delta_0$

28   **return** $\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Box\delta_0, \Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Delta \overset{\mathrm{u}}{\Rightarrow} \delta_0\end{array}}{\Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Box\Delta \overset{\mathrm{b}}{\Rightarrow} \Box\delta_0}\, R^{\Box}_{\mathrm{b}}$

29   **else return** $\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Box\delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta_0\end{array}}{\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \Box\delta_0}\, R^{\Box}_{\mathrm{b}}$

30 **else if** $\delta = \Box\delta_0$ *and* $l = \mathrm{u}$ **then**

31    $\mathcal{D}_0 \leftarrow$ `SearchBU(`$\Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Delta \overset{\mathrm{u}}{\Rightarrow} \delta_0$`)`

32   **if** $\mathcal{D}_0$ *is a* $\mathsf{GbuSL}_{\Box}$*-derivation* **then**

33   **return** $\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Delta \overset{\mathrm{u}}{\Rightarrow} \delta_0\end{array}}{\Gamma^{\rightarrow}, \Gamma^{\mathrm{at}}, \Box\Delta \overset{\mathrm{u}}{\Rightarrow} \Box\delta_0}\, R^{\Box}_{\mathrm{u}}$

34   **else** $\mathcal{D}^{\Box}_0 \leftarrow \mathcal{D}_0$ `// proof search continues at line 53`

35   `//` $\mathcal{D}^{\Box}_0$ `is used in the application of rule` $\mathrm{S}^{\Box}_{\mathrm{u}}$ `at line 70`

36 **else if** $\delta = \delta_0 \vee \delta_1$ **then**

37    $\mathcal{D}_0 \leftarrow$ `SearchBU(`$\Gamma \overset{\mathrm{b}}{\Rightarrow} \delta_0$`)`, $\mathcal{D}_1 \leftarrow$ `SearchBU(`$\Gamma \overset{\mathrm{b}}{\Rightarrow} \delta_1$`)`

38   **if** $\exists k \in \{0,1\}$ *s.t.* $\mathcal{D}_k$ *is a* $\mathsf{GbuSL}_{\Box}$*-derivation* **then**

39   **return** $\dfrac{\begin{array}{c}\mathcal{D}_k\\ \Gamma \overset{\mathrm{b}}{\Rightarrow} \delta_k\end{array}}{\Gamma \overset{l}{\Rightarrow} \delta_0 \vee \delta_1}\, R\vee_k$

40   **else if** $l = \mathrm{b}$ **then**

41   **return** $\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_0\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_1\end{array}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_0 \vee \delta_1}\, R\vee$

42   **else**

43      $\mathcal{D}^{\vee}_0 \leftarrow \mathcal{D}_0$, $\mathcal{D}^{\vee}_1 \leftarrow \mathcal{D}_1$ `// proof search continues at line 53`

44   `//` $\mathcal{D}^{\vee}_0$ `and` $\mathcal{D}^{\vee}_1$ `are used in the application of rule` $\mathrm{S}^{\vee}_{\mathrm{u}}$ `at line 68`

Fig. 2. `SearchBU(`$\Gamma \overset{l}{\Rightarrow} \delta$`)` (2)

// Here $\Gamma = \Gamma^{\to}, \Gamma^{\mathrm{at}}, \Box\Delta$ and ($\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ or $\delta = \delta_0 \vee \delta_1$ or $\delta = \Box\delta_0$).
// Moreover, if $l = \mathrm{b}$, then $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$
if $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ *and* ($l = \mathrm{b}$ *or* $\Gamma^{\to} = \emptyset$) **then**

  return $\dfrac{}{\Gamma \overset{l}{\not\Rightarrow} \delta}\,\mathrm{Irr}$

else

  // Here $l = \mathrm{u}$ and
  // $\Gamma = \Gamma^{\to}, \Gamma^{\mathrm{at}}, \Box\Delta$ and ($\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ or $\delta = \delta_0 \vee \delta_1$ or $\delta = \Box\delta_0$).
  // Moreover, if $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$, then $\Gamma^{\to} \neq \emptyset$.
  Refs $\leftarrow \emptyset$ // empty set of $\mathsf{RbuSL}_{\Box}$-derivations
  // Refs collects the $\mathsf{RbuSL}_{\Box}$-ders. $\mathcal{D}_0^{\alpha\to\beta}$ of $\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha$, where $\alpha \to \beta \in \Gamma$
  for $\alpha \to \beta \in \Gamma$ **do**

    $\Gamma' \leftarrow \Gamma \setminus \{\alpha \to \beta\}$, $\mathcal{D}_1^{\alpha\to\beta} \leftarrow$ SearchBU($\beta, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta$)
    if $\mathcal{D}_1^{\alpha\to\beta}$ *is an* $\mathsf{RbuSL}_{\Box}$*-derivation* **then**

      return $\dfrac{\dfrac{\mathcal{D}_1^{\alpha\to\beta}}{\beta, \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta}}{\alpha \to \beta, \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta}\, L{\to}$

    else

      $\mathcal{D}_0^{\alpha\to\beta} \leftarrow$ SearchBU($\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha$)
      if $\mathcal{D}_0^{\alpha\to\beta}$ *is a* $\mathsf{GbuSL}_{\Box}$*-derivation* **then**

        return $\dfrac{\dfrac{\mathcal{D}_0^{\alpha\to\beta}}{\alpha \to \beta, \Gamma' \overset{\mathrm{b}}{\Rightarrow} \alpha} \qquad \dfrac{\mathcal{D}_1^{\alpha\to\beta}}{\beta, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta}}{\alpha \to \beta, \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta}\, L{\to}$

      else Refs $\leftarrow$ Refs $\cup \{ \mathcal{D}_0^{\alpha\to\beta} \}$

  // We apply a Succ rule
  if $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ **then**

    return $\dfrac{\cdots\ \dfrac{\mathcal{D}_0^{\alpha\to\beta}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha}\ \cdots}{\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,\mathrm{S}_{\mathrm{u}}^{\mathrm{At}} \qquad \mathcal{D}_0^{\alpha\to\beta} \in \mathsf{Refs}$

  else if $\delta = \delta_0 \vee \delta_1$ **then**

    return $\dfrac{\cdots\ \dfrac{\mathcal{D}_0^{\alpha\to\beta}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha}\ \cdots \qquad \dfrac{\mathcal{D}_0^{\vee}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_0} \qquad \dfrac{\mathcal{D}_1^{\vee}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_1}}{\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta_0 \vee \delta_1}\,\mathrm{S}_{\mathrm{u}}^{\vee} \qquad \mathcal{D}_0^{\alpha\to\beta} \in \mathsf{Refs}$

  else if $\delta = \Box\delta_0$ **then**

    return $\dfrac{\cdots\ \dfrac{\mathcal{D}_0^{\alpha\to\beta}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha}\ \cdots \qquad \dfrac{\mathcal{D}_0^{\Box}}{\Gamma^{\to}, \Gamma^{\mathrm{at}}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta_0}}{\underbrace{\Gamma^{\to}, \Gamma^{\mathrm{at}}, \Box\Delta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \Box\delta_0}\,\mathrm{S}_{\mathrm{u}}^{\Box} \qquad \mathcal{D}_0^{\alpha\to\beta} \in \mathsf{Refs}$

end

Fig. 3. SearchBU($\Gamma \overset{l}{\Rightarrow} \delta$) (3)