Appendix of
"Proof search and countermodel construction for iCK4"

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This Appendix includes supplementary material for the paper:

> M. Ferrari, C. Fiorentini, P. Giardini. Proof search and countermodel construction for iCK4, published in Proceedings of *CILC 2025: 40th Italian Conference on Computational Logic*, 2025.

A Proofs of Section 2

Lemma 2. *Let $\mathcal{K} = \langle W, \leq, R, r, V \rangle$ be a strong model and let $w \in W$.*

(i) If w is a reflexive world then, for every formula α, $w \Vdash \alpha \leftrightarrow \alpha^-$.

(ii) If $w \nVdash \Box\alpha$, then there exists $w^\star \in W$ such that $wRw^\star$, $w^\star \nVdash \alpha$ and either (a) $\forall w' \in W : w^\star Rw'$, $w' \Vdash \alpha$ or (b) $w^\star$ is reflexive.

Proof. Point (i). Note that $w \Vdash \varphi \leftrightarrow \Box\varphi$, for every formula φ. Indeed, $w \Vdash \varphi \to \Box\varphi$ follows by strongness of $\mathcal{K}$, $w \Vdash \Box\varphi \to \varphi$ by reflexivity of w. Since α^- is obtained from α by replacing every subformula $\Box\varphi$ with φ, we get , $w \Vdash \alpha \leftrightarrow \alpha^-$.

Point (ii). Since $w \nVdash \Box\alpha$, there exists $w_\alpha \in W$ such that wRw_α and $w_\alpha \nVdash \alpha$. We build a finite sequence $\mathcal{S}$ of pairwise distinct worlds $w_0, \dots, w_n$ of W such that $w_0Rw_1R\dots Rw_n$ and $w_k \nVdash \alpha$ for every $0 \leq k \leq n$. We proceed as follows:

- We set $w_0 = w_\alpha$ (thus, wRw_0).
- Suppose that the last defined world of $\mathcal{S}$ is w_k ($k \geq 0$). If there exists w' such that $w' \notin \{w_0, \dots, w_k\}$ and w_kRw' and $w' \nVdash \alpha$, we set $w_{k+1} = w'$; otherwise, the construction of $\mathcal{S}$ halts and w_k is the last world of $\mathcal{S}$.

Since the worlds in $\mathcal{S}$ are pairwise distinct and W is finite, the construction of $\mathcal{S}$ eventually halts. Let $w^\star$ be the last element of $\mathcal{S}$. We have $w^\star \nVdash \alpha$ and $w_0Rw_1R\dots Rw^\star$ hence, by transitivity of R, $wRw^\star$. If $w^\star$ is reflexive, then $w^\star$ matches (b). Let us assume that $w^\star$ is not reflexive; we show that (a) holds. Let us assume, by contradiction, that there exists w' such that $w^\star Rw'$ and $w' \nVdash \alpha$. Note that $w' \notin \mathcal{S}$, otherwise, by transitivity of R, we would get $w^\star Rw^\star$, against the hypothesis that $w^\star$ is not reflexive. Since $w' \notin \mathcal{S}$ and $w' \nVdash \alpha$, we can extend $\mathcal{S}$ by adding w', a contradiction ($w^\star$ is the last element of $\mathcal{S}$). This proves that, for every w' such that $w^\star R w'$, $w' \Vdash \alpha$; accordingly, $w^\star$ matches (a). ∎

B Proofs of Section 4

To prove Lemma 19, we need the following lemma.

Lemma 20. *Let $\mathcal{T}^{\mathrm{b}}$ be an $\mathcal{R}$-tree only containing* b*-antisequents having root $\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta$; let $\mathcal{K} = \langle W, \leq, R, r, V\rangle$ and $w \in W$ such that:*

(I1) $w \nVdash \delta'$, for every leaf $\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta'$ of $\mathcal{T}^{\mathrm{b}}$;
(I2) $w \Vdash (\Gamma^{\to} \cap \mathrm{Sf}^-(\delta)) \cup \Box\Delta$;
(I3) $V(w) = \Gamma^{\mathrm{at}}$.
Then, $w \nVdash \delta$.

Proof. By induction on depth($\mathcal{T}^{\mathrm{b}}$). The case depth($\mathcal{T}^{\mathrm{b}}$) = 0 is trivial, since the root of $\mathcal{T}^{\mathrm{b}}$ is also a leaf. Let depth($\mathcal{T}^{\mathrm{b}}$) > 0; we only discuss the case where

$$\mathcal{T}^{\mathrm{b}} \;=\; \cfrac{\begin{array}{c}\mathcal{T}^{\mathrm{b}}_0\\ \sigma^{\mathrm{b}}_0 \;=\; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \beta\end{array}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \to \beta}\; R\overset{\triangleright}{\to} \qquad \begin{array}{l}\Gamma \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta\\ \Gamma \triangleright \alpha\end{array}$$

By applying the induction hypothesis to the $\mathcal{R}$-tree $\mathcal{T}^{\mathrm{b}}_0$, having root σ^{b}_0 and the same leaves as $\mathcal{T}^{\mathrm{b}}$, we get $w \nVdash \beta$. Let $\Gamma_\alpha = \Gamma \cap \mathrm{Sf}(\alpha)$; by Lemma 5(iii), $\Gamma_\alpha \triangleright \alpha$. Since $\mathrm{Sf}(\alpha) \subseteq \mathrm{Sf}^-(\alpha \to \beta)$, by hypotheses (I2)–(I3) we get $w \Vdash \Gamma_\alpha$, which implies $w \Vdash \alpha$ (Lemma 5(iv)). This proves $w \nVdash \alpha \to \beta$. ∎

Lemma 19. *Let $\mathcal{D}$ be an $\mathcal{R}$-derivation of $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ having form (1) where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$; let $\mathcal{K} = \langle W, \leq, R, r, V\rangle$ and $w \in W$ such that:*

(J1) for every $w' \in W$ such that $w < w'$, it holds that $w' \Vdash \Gamma^{\to}$.
(J2) For every $w' \in W$ such that wRw', it holds that $w' \Vdash \Delta$.
(J3) For every $\sigma' = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$ such that $\sigma^{\mathrm{u}} \ll \sigma'$, there exists $w' \in W$ such that $w \leq w'$ and $w' \Vdash \alpha$ and $w' \nVdash \beta$.
(J4) For every $\sigma' = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ such that $\sigma^{\mathrm{u}} \ll_R \sigma'$, there exists $w' \in W$ such that wRw' and $w' \nVdash \alpha$.
(J5) For every $\sigma' = \Gamma^- \overset{\mathrm{u}}{\not\Rightarrow} \alpha^-$ such that $\sigma^{\mathrm{u}} \ll^*_R \sigma'$, there exists $w' \in W$ such that wRw', w' is reflexive and $w' \nVdash \alpha^-$.
(J6) $V(w) = \Gamma^{\mathrm{at}}$.
Then, $w \Vdash \Gamma$ and $w \nVdash \delta$.

Proof. We show that:

(P1) $w \nVdash \chi$, for every premise $\sigma^{\mathrm{b}}_\chi = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \chi$ of Succ;
(P2) $w \Vdash \alpha \to \beta$, for every $\alpha \to \beta \in \Gamma^{\to}$.

We introduce the following induction hypothesis:

(IH1) to prove Point (P1) for a formula χ, we inductively assume that Point (P2) holds for every formula $\alpha \to \beta$ such that $|\alpha \to \beta| < |\chi|$;

(IH2) to prove Point (P2) for a formula $\alpha \to \beta$, we inductively assume that Point (P1) holds for every formula χ such that $|\chi| < |\alpha \to \beta|$.

We prove Point (P1). Let $\sigma^{\mathrm{b}}_{\chi}$ be the premise of Succ displayed in schema (1). We show that the $\mathsf{RbuSL}_{\Box}$-tree $\mathcal{T}^{\mathrm{b}}_{X}$ and w match the hypotheses (I1)–(I3) of Lemma 20, so that we can apply the lemma to infer $w \nVdash \chi$.

We prove (I1). Let $\sigma^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta$ any leaf of $\mathcal{T}^{\mathrm{b}}_{X}$; we show that $w \nVdash \delta$. By definition of schema (1), one of the following cases holds.

(a) $\sigma^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \to \beta$ and $\sigma^{\mathrm{u}} = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$ and $\sigma^{\mathrm{u}} \ll \sigma^{\mathrm{u}}$;

(b) $\sigma^{\mathrm{b}} = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha$ and $\sigma^{\mathrm{u}} = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ and $\sigma^{\mathrm{u}} \ll_R \sigma^{\mathrm{u}}$;

(c) $\sigma^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha$ and $\sigma^{\mathrm{u}} = \Gamma^{-} \overset{\mathrm{u}}{\not\Rightarrow} \alpha^{-}$ and $\sigma^{\mathrm{u}} \ll^{*}_{R} \sigma^{\mathrm{u}}_{i}$;

(d) $\sigma^{\mathrm{b}} = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta$ is irreducible.

In case (a), by hypothesis (J3) there is $w' \in W$ such that $w \leq w'$ and $w' \Vdash \alpha$ and $w' \nVdash \beta$, hence $w \nVdash \alpha \to \beta$. In case (b), by hypothesis (J4) there is w' such that wRw' and $w' \nVdash \alpha$, hence $w \nVdash \Box\alpha$. Let us consider case (c). By hypothesis (J5) there exists a reflexive world w' such that wRw' and $w' \nVdash \alpha^{-}$. By Lemma 2, $w' \Vdash \alpha \leftrightarrow \alpha^{-}$; it follows that $w' \nVdash \alpha$, hence $w \nVdash \Box\alpha$. In case (d), we have $\delta \in \mathcal{V} \cup \{\bot\}$ and $\delta \notin \Gamma^{\mathrm{at}}$. Since $V(w) = \Gamma^{\mathrm{at}}$ (hypothesis (J6)), we get $w \nVdash \delta$. This proves that hypothesis (I1) holds.

We prove (I2). Let $\gamma \in \Gamma^{\to} \cap \mathrm{Sf}^{-}(\chi)$; since $|\gamma| < |\chi|$, by (IH1) we get $w \Vdash \gamma$. Moreover, $w \Vdash \Box\Delta$ by (J2), thus (I2) holds. Finally, (I3) coincides with (J6). We can apply Lemma 20 and conclude $w \nVdash \chi$, and this proves Point (P1).

We prove Point (P2). Let $\alpha \to \beta \in \Gamma^{\to}$, let $w' \in W$ be such that $w \leq w'$ and $w' \Vdash \alpha$; we show that $w' \Vdash \beta$. Note that $\sigma^{\mathrm{b}}_{\alpha} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha$ is a premise of Succ; since $|\alpha| < |\alpha \to \beta|$, by (IH2) we get $w \nVdash \alpha$. This implies that $w < w'$. By hypothesis (J1), $w' \Vdash \alpha \to \beta$, hence $w' \Vdash \beta$; this proves (P2).

We prove the assertion of the lemma. By (P2) and hypotheses (J2) and (J6), we get $w \Vdash \Gamma$. The proof that $w \nVdash \delta$ depends on the specific rule Succ at hand and follows from Point (P1) and hypothesis (J6). ■