A Terminating Sequent Calculus for Intuitionistic Strong Löb Logic with the Subformula Property

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Provability logics

Provability logics are modal logics where the $\Box$ modality is interpreted as provability.

- GL (Gödel-Löb Logic)

$$\begin{array}{rcll} \mathsf{GL} & = & \mathsf{K} & \text{Classical modal logic} \\ & + & \Box(\Box\varphi \to \varphi) \to \Box\varphi & \text{Gödel-Löb Axiom} \end{array}$$

- GL is the provability logic of PA (Peano Arithmetic):

$$\Box\varphi \quad \rightsquigarrow \quad \varphi \text{ is provable in PA}$$

Open question

What is the provability logic for HA (Heyting Arithmetic), the intuitionistic counterpart of PA?

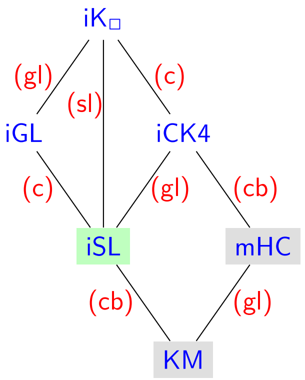
Provability logics: intuitionistic

- In the intuitionistic setting, we only consider the $\Box$ modality. There is no natural interpretation for $\Diamond$ (note that $\Box$ and $\Diamond$ must be independent).
- iGL (Intuitionistic Gödel-Löb Logic)

iGL	$=$	$\mathrm{iK}_\Box$	Intuitionistic Modal Logic ($\Box$-fragment)
	$+$	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	Gödel-Löb Axiom

- iGL is the intuitionistic counterpart of GL
- iGL is sound w.r.t. provability in HA, but not complete.

Provability logics: an overview



Axioms

(c)	$\varphi \to \Box\varphi$	Coreflexion/ Completeness
(gl)	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	Gödel-Löb
(sl)	$(\Box\varphi \to \varphi) \to \varphi$	Strong Löb
(cb)	$\Box\varphi \to ((\psi \to \varphi) \vee \psi)$	Cantor-Benedixson

Logics

iK$_\Box$	Intuitionistic Modal Logic ($\Box$-fragment)
iGL	Intuitionistic Gödel-Löb Logic
iCK4	Minimal Coreflection Logic
iSL	Intuitionistic Strong-Löb Logic
mHC	Modalized-Heyting Calculus
KM	Kuznetsov-Muravitski Logic

⋆ *I. van der Giessen. Ph.D thesis. 2022.*

We focus on iSL, the provability logic of an extension of HA w.r.t. the so-called *slow provability*

⋆ *M. Mojtahedi. On provability logic of HA (2022), under review*

Intuitionistic Strong-Löb logic iSL

$$\mathrm{iSL} = \underbrace{\mathrm{iK}_\Box + \Box(\Box\varphi \to \varphi) \to \Box\varphi}^{\mathrm{iGL}} + \varphi \to \Box\varphi$$

$$= \mathrm{iK}_\Box + (\Box\varphi \to \varphi) \to \varphi$$

$\varphi \to \Box\varphi$	Coreflexion/Completeness
$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	Gödel-Löb
$(\Box\varphi \to \varphi) \to \varphi$	Strong Löb

- iSL has the finite model property with respect to Kripke semantics.

> Kripke models have two ordering relations: $\leq$ (intuitionistic) and R (modal).
> - R is contained in $\leq$
> - R is irreflexive

- Recently, sequent calculi for iSL have been introduced:

Calculus for IPL (Int. Prop. Logic) $+$ rules for $\Box$

The calculus G3i for IPL

- Rules

$$\frac{}{p, \Gamma \Rightarrow p}\,\mathrm{Id} \quad p\text{: prop. var.} \qquad\qquad \frac{}{\bot, \Gamma \Rightarrow \delta}\,L\bot$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta}\,L\wedge \qquad\qquad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\,R\wedge$$

$$\frac{\alpha, \Gamma \Rightarrow \delta \qquad \beta, \Gamma \Rightarrow \delta}{\alpha \vee \beta, \Gamma \Rightarrow \delta}\,L\vee \qquad\qquad \frac{\Gamma \Rightarrow \alpha_k}{\Gamma \Rightarrow \alpha_0 \vee \alpha_1}\,R\vee_k \quad k \in \{0, 1\}$$

$$\frac{\alpha \to \beta, \Gamma \Rightarrow \alpha \qquad \beta, \Gamma \Rightarrow \delta}{\alpha \to \beta, \Gamma \Rightarrow \delta}\,L\!\to \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\,R\!\to$$

Γ: multiset of formulas

- No structural rules
- Subformula property

> A formula occurring in a G3i-tree τ is a subformula of a formula occurring in the root sequent of τ.

The calculus $\mathrm{G3iSL}^{+}_{\Box}$ for iSL

$$\mathrm{G3iSL}^{+}_{\Box} \;=\; \mathrm{G3i} \;+\; \dfrac{\Box\alpha, \Gamma, \Delta \Rightarrow \alpha}{\Gamma, \Box\Delta \Rightarrow \Box\alpha}\, R\Box \qquad \Box\alpha\text{: diagonal formula}$$

Polished variant of the calculus $\mathrm{G3iSL}^{a}_{\Box}$ introduced in

- ⋆ *I. van der Giessen, R. Iemhoff. Proof theory for intuitionistic strong Löb logic. Special Volume of the Workshop Proofs! held in Paris in 2017. 2020.*
- ⋆ *I. van der Giessen. Ph.D thesis. 2022.*

- Subformula property
- Main problem

  G3-style calculi are not well-suited for bottom-up proof search, due to the rule for left $\to$

$$\dfrac{\dfrac{\dfrac{\vdots}{p \to q \Rightarrow p} \qquad q \Rightarrow p}{p \to q \Rightarrow p}\, L{\to} \qquad q \Rightarrow p}{p \to q \Rightarrow p}\, L{\to}$$

Bottom-up proof search loops

Beyond G3i

We can replace the looping rule

$$\frac{\alpha \to \beta, \Gamma \Rightarrow \alpha \qquad \beta, \Gamma \Rightarrow \delta}{\alpha \to \beta, \Gamma \Rightarrow \delta} \; L{\to}$$

with specialized rules according to the shape of α in $\alpha \to \beta$:

$$\frac{\beta,\ p,\ \Gamma \Rightarrow \delta}{p \to \beta,\ p,\ \Gamma \Rightarrow \delta} \quad p\text{: prop. var.} \qquad \frac{\alpha_1 \to (\alpha_2 \to \beta),\ \Gamma \Rightarrow \delta}{(\alpha_1 \land \alpha_2) \to \beta,\ \Gamma \Rightarrow \delta}$$

$$\frac{\alpha_1 \to \beta,\ \alpha_2 \to \beta, \Gamma \Rightarrow \delta}{(\alpha_1 \lor \alpha_2) \to \beta,\ \Gamma \Rightarrow \delta} \qquad \frac{\alpha_1,\ \alpha_2 \to \beta, \Gamma \Rightarrow \alpha_2 \qquad \beta,\ \Gamma \Rightarrow \delta}{(\alpha_1 \to \alpha_2) \to \beta,\ \Gamma \Rightarrow \delta}$$

The resulting calculus is known as G4i (alias LJT)

- ⋆ *N. N. Vorob'ev. A new algorithm for derivability in the constructive propositional calculus (Russian). AMS Translations, Series 2, 1970.*
- ⋆ *R. Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. JSL, 1992*
- ⋆ *J. Hudelmaier. Bounds for cut elimination in intuitionistic propositional logic. AML, 1992*

The calculus G4i for IPL

The calculus G4i is (strongly) terminating, namely:

There exists a well-founded relation $\prec$ such that, for every application

$$\frac{\ldots \qquad \sigma' \qquad \ldots}{\sigma}$$

of a rule of G4i, it holds that $\sigma' \prec \sigma$.

Accordingly, backward proof search in G4i always terminates, whatever strategy ones applies.

Note

Any calculus containing rule $L\!\rightarrow$ is not terminating:

$$\frac{\overbrace{p \rightarrow q \Rightarrow p}^{\sigma'} \qquad \ldots}{\underbrace{p \rightarrow q \Rightarrow p}_{\sigma}}\; L\!\rightarrow$$

Since $\sigma' = \sigma$, there is no well-founded relation $\prec$ such that $\sigma' \prec \sigma$.

The calculus G4iSLt for iSL

$$\text{G4iSLt} = \text{G4i} + \frac{\Box\alpha, \Gamma, \Delta \Rightarrow \alpha \qquad \beta, \Gamma, \Box\Delta \Rightarrow \delta}{\Gamma, \Box\Delta, \Box\alpha \to \beta \Rightarrow \Box\delta} L\Box\to$$

$$+ \frac{\Box\alpha, \Gamma, \Delta \Rightarrow \alpha}{\Gamma, \Box\Delta \Rightarrow \Box\alpha} R\Box$$

⋆ *I. Shillito, I. van der Giessen, R. Goré, R. Iemhoff. A new calculus for intuitionistic strong Löb logic. Strong termination and cut-elimination, formalised. TABLEAUX 2023. Best paper.*

- Strong terminating
- Proofs of termination and completeness (via cut admissibility) formalized with the Coq proof assistant
- Subformula property is broken!

$$\frac{a \to (b \to c) \Rightarrow d}{(a \wedge b) \to c \Rightarrow d} L\wedge\to$$

$\mathsf{G3iSL}^+_\Box$ vs. G4iSLt

	$\mathsf{G3iSL}^+_\Box$	G4iSLt
Lineage	G3i	G4i
Termination	✗ Weak there exists a terminating proof search strategy	✓ Strong all proof search strategies are terminating
Subformula property	✓ Yes	✗ No

Our challenge

Design a calculus which features the valuable aspects of both calculi:

Strong termination + Subformula property

Make G3i terminating

Get back to the intuitionistic sequent calculus Gbu:

- ⋆ *M. Ferrari, C. Fiorentini, G. Fiorino. A terminating evaluation-driven variant of G3i. TABLEAUX 2013 (Nancy)*
- ⋆ *M. Ferrari, C. Fiorentini, G. Fiorino. An Evaluation-Driven Decision Procedure for G3i. TOCL, 2015.*

- Gbu is a terminating variant of G3i obtained by decorating the sequents with labels.

$$\Gamma \xRightarrow{l} \delta \qquad l \in \{\, \mathrm{b}, \mathrm{u} \,\}$$

b: blocked
u: unblocked

- The bottom-up application of a left rule to a b-sequent is blocked:

$$\dfrac{\dots}{\Gamma \xRightarrow{\mathrm{u}} \delta}\ \text{L rule} \quad ✓ \qquad\qquad \cancel{\dfrac{\dots}{\Gamma \xRightarrow{\mathrm{b}} \delta}\ \text{L rule}}$$

- A Gbu-derivation can be turned into a G3i-derivation by erasing the labels and some slight tweak.

The calculus GbuSL$_\Box$ for iSL

$$\text{GbuSL}_\Box \;=\; \text{Gbu} \;+\; \text{rule } R\Box \text{ with labels}$$

We have to introduce an evaluation relation $\rhd$

$$\Gamma \rhd \varphi \quad \Longleftrightarrow \quad \varphi := \gamma \mid \varphi \wedge \varphi \mid \varphi \vee \_ \mid \_ \vee \varphi \mid \_ \to \varphi \mid \Box\varphi$$
$$\gamma \in \Gamma, \quad \_ \text{ is any formula}$$

- Axiom rule $\mathrm{Ax}^{\rhd}$

$$\frac{}{\Gamma \overset{l}{\Rightarrow} \alpha}\ \mathrm{Ax}^{\rhd} \qquad \text{if } \Gamma \rhd \alpha$$

Examples of instances:

$$\frac{}{\alpha, \Gamma \overset{l}{\Rightarrow} \alpha}\ (= \text{rule Id}) \qquad \frac{}{\alpha, \beta, \Gamma \overset{l}{\Rightarrow} \alpha \wedge \beta}$$

$$\frac{}{\alpha, \Gamma \overset{l}{\Rightarrow} \alpha \vee \beta} \qquad \frac{}{\alpha, \Gamma \overset{l}{\Rightarrow} \beta \to \alpha} \qquad \frac{}{\alpha, \Gamma \overset{l}{\Rightarrow} \Box\alpha} \qquad \cdots$$

The calculus GbuSL$_\square$ for iSL

- Left rules

  All of them have an u-sequent as conclusion

$$\frac{}{\bot, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta} L\bot \qquad \text{axiom rule}$$

$$\frac{\alpha, \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha \wedge \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta} L\wedge \qquad \frac{\alpha, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta \qquad \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha \vee \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta} L\vee$$

$$\frac{\alpha \to \beta, \Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha \qquad \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha \to \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta} L\to$$

The label b in the left premise of $L\to$ is crucial to avoid loops:

$$\frac{p \to q \overset{\mathrm{b}}{\Rightarrow} p \qquad q \overset{\mathrm{u}}{\Rightarrow} p}{p \to q \overset{\mathrm{u}}{\Rightarrow} p} L\to$$

The expansion of the left-most branch is blocked!

The calculus GbuSL$_\Box$ for iSL

- Right rules for $\wedge$ and $\vee$

$$\frac{\Gamma \overset{l}{\Rightarrow} \alpha \qquad \Gamma \overset{l}{\Rightarrow} \beta}{\Gamma \overset{l}{\Rightarrow} \alpha \wedge \beta} \, R\wedge \qquad\qquad \frac{\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha_k}{\Gamma \overset{l}{\Rightarrow} \alpha_0 \vee \alpha_1} \, R\vee_k \qquad k \in \{0,1\}$$

- Right rules for $\to$

(1) $\Gamma \triangleright \alpha$

$$\frac{\Gamma \overset{l}{\Rightarrow} \beta}{\Gamma \overset{l}{\Rightarrow} \alpha \to \beta} \, R\overset{\triangleright}{\to}$$

α is not added to Γ
(it would be redundant)

Examples of instances:

$$\frac{\cancel{\alpha}, \alpha, \Gamma \overset{l}{\Rightarrow} \beta}{\alpha, \Gamma \overset{l}{\Rightarrow} \alpha \to \beta} \qquad \frac{\cancel{\alpha \vee \delta}, \alpha, \Gamma \overset{l}{\Rightarrow} \beta}{\alpha, \Gamma \overset{l}{\Rightarrow} (\alpha \vee \delta) \to \beta} \qquad \frac{\cancel{\Box\alpha}, \alpha, \Gamma \overset{l}{\Rightarrow} \beta}{\alpha, \Gamma \overset{l}{\Rightarrow} \Box\alpha \to \beta} \qquad \dots$$

(2) $\Gamma \not\triangleright \alpha$

$$\frac{\alpha, \Gamma \overset{\mathrm{u}}{\Rightarrow} \beta}{\Gamma \overset{l}{\Rightarrow} \alpha \to \beta} \, R\overset{\not\triangleright}{\to}$$

The calculus $\mathrm{GbuSL}_\Box$ for iSL

- Right rules for $\Box$

Original rule $R\Box$

$$\frac{\Box\alpha, \Gamma, \Delta \ \Rightarrow \alpha}{\Gamma, \Box\Delta \ \Rightarrow \Box\alpha}\ R\Box$$

$\Box\alpha$: diagonal formula

Adding labels

(1) The conclusion has label u

$$\frac{\cancel{\Box\alpha}, \Gamma, \Delta \overset{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Delta \overset{\mathrm{u}}{\Rightarrow} \Box\alpha}\ R^\Box_\mathrm{u}$$

The diagonal formula is dropped out

(2) The conclusion has label b

$$\frac{\Box\alpha, \Gamma, \Delta \overset{\mathrm{u}}{\Rightarrow} \alpha}{\underbrace{\Gamma, \Box\Delta}_{\Gamma'} \overset{\mathrm{b}}{\Rightarrow} \Box\alpha}\ R^\Box_\mathrm{b}$$

if $\Gamma' \not\triangleright \Box\alpha$

If $\Gamma' \triangleright \Box\alpha$, the conclusion is an axiom (see the axiom rule $\mathrm{Ax}^\triangleright$)

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Box p, \neg\Box p \stackrel{\color{red}{b}}{\Rightarrow} \Box p}\,\mathrm{Ax}^{\rhd}
      \qquad
      \dfrac{}{\Box p, \bot \stackrel{\color{green}{u}}{\Rightarrow} p}\,L\bot
    }{\Box p, \neg\Box p \stackrel{\color{green}{u}}{\Rightarrow} p}\,L{\to}
  }{\neg\Box p \stackrel{\color{red}{b}}{\Rightarrow} \Box p}\,R^{\Box}_{b}
  \qquad
  \dfrac{}{\bot \stackrel{\color{green}{u}}{\Rightarrow} \bot}\,L\bot
}{
  \dfrac{\neg\Box p \stackrel{\color{green}{u}}{\Rightarrow} \bot}{\stackrel{\color{green}{u}}{\Rightarrow} \neg\neg\Box p}\,R^{\not\rhd}_{\to}
}\,{\color{red}{L{\to}}}
$$

This proves that $\neg\neg\Box p \in$ iSL.

√ Crucial point (1)

Suppose the left premise of $\color{red}{L{\to}}$ has label u ...

$$
\dfrac{
  \dfrac{
    \cancel{\neg\Box p \stackrel{\color{red}{b}}{\Rightarrow} \Box p}
    \\
    \neg\Box p \stackrel{\color{green}{u}}{\Rightarrow} \Box p
    \qquad \ldots
  }{
    \cancel{\neg\Box p \stackrel{\color{red}{b}}{\Rightarrow} \Box p}
    \\
    \neg\Box p \stackrel{\color{green}{u}}{\Rightarrow} \Box p
  }\,L{\to}
  \qquad \ldots
}{
  \dfrac{\neg\Box p \stackrel{\color{green}{u}}{\Rightarrow} \bot}{\stackrel{\color{green}{u}}{\Rightarrow} \neg\neg\Box p}\,R^{\not\rhd}_{\to}
}\,{\color{red}{L{\to}}}
$$

Loop!

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box p, \neg\Box p \overset{\mathrm{b}}{\Rightarrow} \Box p}\,\mathrm{Ax}^{\triangleright} \qquad \dfrac{}{\Box p, \bot \overset{\mathrm{u}}{\Rightarrow} p}\,L\bot}{\Box p,\ \neg\Box p \overset{\mathrm{u}}{\Rightarrow} p}\,L\!\to}{\neg\Box p \overset{\mathrm{b}}{\Rightarrow} \Box p}\,R^{\Box}_{\mathrm{b}} \qquad \dfrac{}{\bot \overset{\mathrm{u}}{\Rightarrow} \bot}\,L\bot}{\neg\Box p \overset{\mathrm{u}}{\Rightarrow} \bot}\,L\!\to}{\overset{\mathrm{u}}{\Rightarrow} \neg\neg\Box p}\,R\overset{\ntriangleright}{\to}
$$

√ Crucial point (2)

Suppose that rule $R^{\Box}_{\mathrm{b}}$ omits the diagonal formula $\Box p$ ...

$$
\dfrac{\dfrac{\cancel{\Box p},\ \neg\Box p \overset{\mathrm{b}}{\Rightarrow} \Box p \qquad \dots}{\cancel{\Box p},\ \neg\Box p \overset{\mathrm{u}}{\Rightarrow} p}\,L\!\to}{\neg\Box p \overset{\mathrm{b}}{\Rightarrow} \Box p}\,R^{\Box}_{\mathrm{b}}
$$

Loop!

$$\vdots$$

Properties of $\mathrm{GbuSL}_\Box$

(1) Strong terminating

(2) Sound with respect to iSL

$$\vdash_{\mathrm{GbuSL}_\Box} \Gamma \overset{l}{\Rightarrow} \delta \quad \Longrightarrow \quad \Gamma \models_{\mathrm{iSL}} \delta$$

$\vdash_{\mathrm{GbuSL}_\Box} \sigma$ $\rightsquigarrow$ there exists a $\mathrm{GbuSL}_\Box$-derivation of σ

$\Gamma \models_{\mathrm{iSL}} \delta$ $\rightsquigarrow$ δ is an iSL-consequence of Γ

(3) Complete a with respect to iSL

$$\Gamma \models_{\mathrm{iSL}} \delta \quad \Longrightarrow \quad \vdash_{\mathrm{GbuSL}_\Box} \Gamma \overset{u}{\Rightarrow} \delta$$

(4) We have designed a proof search strategy for $\mathrm{GbuSL}_\Box$ which yields countermodels whenever proof search fails.

The proof search strategy has been implemented in Java.

https://github.com/ferram/jtabwb_provers/tree/master/isl_gbuSL

Properties of $\mathsf{GbuSL}_\Box$

We remark that completeness is only guaranteed for u-sequents.

$$a \vee b \models_{\mathsf{iSL}} b \vee a$$

By completeness:

$$\vdash_{\mathsf{GbuSL}_\Box} a \vee b \overset{\mathrm{u}}{\Rightarrow} b \vee a$$

However the corresponding b-sequent is **not** provable in $\mathsf{GbuSL}_\Box$:

$$\dfrac{a \vee b \overset{\mathrm{b}}{\Rightarrow} b}{a \vee b \overset{\mathrm{b}}{\Rightarrow} b \vee a}\, R\vee_0 \qquad\qquad \dfrac{a \vee b \overset{\mathrm{b}}{\Rightarrow} a}{a \vee b \overset{\mathrm{b}}{\Rightarrow} b \vee a}\, R\vee_1$$

$R\vee_0$ $\qquad$ $R\vee_1$

$$a \vee b \overset{\mathrm{b}}{\Rightarrow} b \vee a$$

Actually, to build a $\mathsf{GbuSL}_\Box$-derivation we have to start by applying $L\vee$:

$$\dfrac{\dfrac{\dfrac{}{a \overset{\mathrm{b}}{\Rightarrow} a}\,\mathrm{Ax}^{\triangleright}}{a \overset{\mathrm{u}}{\Rightarrow} b \vee a}\, R\vee_1 \qquad \dfrac{\dfrac{}{b \overset{\mathrm{b}}{\Rightarrow} b}\,\mathrm{Ax}^{\triangleright}}{b \overset{\mathrm{u}}{\Rightarrow} b \vee a}\, R\vee_0}{a \vee b \overset{\mathrm{u}}{\Rightarrow} b \vee a}\, L\vee$$

About termination

- We define a well-founded relations $\prec_{\mathrm{bu}}$ on labelled sequents.
- Critical point:

$$\dfrac{\sigma' = \alpha \to \beta, \Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha \qquad \ldots}{\sigma \;=\; \alpha \to \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\; L{\to}$$

To get $\sigma' \prec_{\mathrm{bu}} \sigma$, we stipulate that $\mathrm{b} < \mathrm{u}$.

- How to accommodate the transition $\mathrm{b} \mapsto \mathrm{u}$?

$$\dfrac{\sigma' \;=\; \alpha, \Gamma \overset{\mathrm{u}}{\Rightarrow} \beta}{\sigma \;=\; \Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha \to \beta}\; R\overset{\not\triangleright}{\to} \quad \text{if } \Gamma \not\triangleright \alpha$$

We observe that:

- α is *new*, namely

$$\Gamma \not\triangleright \alpha$$

- By the subformula property, the number of possible new formulas is finite.

Conclusions: comparison

	G3iSL$_{\Box}^{+}$ [1]	G4iSLt [2]	GbuSL$_{\Box}$
Lineage	G3i	G4i	G3i
Termination	✗ Weak	✓ Strong	✓ Strong
Subformula prop.	✓ Yes	✗ No	✓ Yes
Cut-admissibility	✓ Yes	✓ Yes	?
Countermodels	–	–	✓ Yes

[1] *I. van der Giessen, R. Iemhoff. Proof theory for intuitionistic strong Löb logic. Special Volume of the Workshop Proofs! held in Paris in 2017. 2020.*

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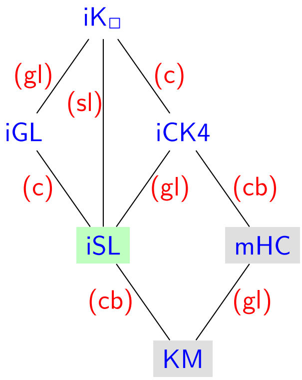
Conclusions: future work



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We aim at extending the presented techniques to other provability logics, in particular to mHC and KM.