

Extending Mubayi and Turán's Algorithm to 3-graphs

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Let G be a 3-graph with n vertices and $m = \epsilon n^3$ edges. A polynomial time algorithm is given to find a $K(q, q, q)$ in G for

$$q = c_\epsilon^{(3)} \sqrt{\log n}$$

As long as (insert condition here).

Note that this result is tight up to the constant $c_\epsilon^{(3)}$, as proved in [1]. This result is a generalization of the result in 2-graphs by [3], and algorithm will be analogous to the one given there. The procedure is as follows:

1. Choose parameters $q < r < n$ depending on n and ϵ .
2. Let R be the set of r vertices with the highest degree in G .
3. find a subset $Q \subset R$ with q vertices such that there is a large $S \subset \binom{[n] \setminus Q}{2}$ satisfying $xyz \in E(G) \forall \{x, y\} \in S, z \in Q$. Say, of size s .
4. Apply the algorithm of [3] to find a $K(q, q)$ in the 2-graph induced by S . Say, we find partition $S \supset U \cup V$.

If successful, a $K(q, q, q)$ has been found in G with parts U, V, Q . The problem is now to find parameters q, r such that the above procedure is successful and the algorithm runs in polynomial time.

Lemma 1. *As long as $r < \epsilon n$, there are at least $\epsilon r n^2$ edges in G with exactly one vertex in R .*

Proof. The sum of the degrees in G is $3m$. Therefore, by the pigeonhole principle,

$$\sum_{v \in R} d(v) \geq r \cdot \frac{3m}{n} = 3\epsilon r n^2$$

However, here we are overcounting:

- The edges with only one vertex in R are counted exactly once.
- The edges with two vertices in R are counted twice. The contribution of these is at most $r(r-1)(n-r) < r^2 n$

- The edges with all vertices in R are counted three times. The contribution of these is at most $r(r-1)(r-2) < r^3 < r^2n$

Therefore, the condition will hold as long as $r^2n < \epsilon rn^2 \iff r < \epsilon n$. \square

Next, a counting argument in the style of [2] is used to guarantee the existence of Q and S . The size q of Q will be left as a parameter to be determined later, and the size s of S will be determined by the following lemma:

Lemma 2. *Under the same assumptions as in Lemma 1, and assuming $r \leq n/2$, there is a subset $Q \subset R$ of size q and a subset $S \subset \binom{[n] \setminus Q}{2}$ of size*

$$s := \frac{n^2}{8} \left(\frac{\epsilon}{e} \right)^q$$

such that $xyz \in E(G) \forall \{x, y\} \in S, z \in Q$.

Proof. Let E be the set of edges with exactly one vertex in R . for every $\{x, y\} \in \binom{[n] \setminus R}{2}$, let E_{xy} be the set of edges in E containing x and y . Finally, let

$$T = \left\{ P \subset E_{xy} : x, y \in \binom{[n] \setminus R}{2}, |P| = q \right\}$$

On the one hand, the number of elements in T is

$$\sum_{\{x, y\} \in \binom{[n] \setminus R}{2}} \binom{|E_{xy}|}{q} \geq \binom{n-r}{2} \binom{\epsilon rn^2 / \binom{n-r}{2}}{q} > \binom{n-r}{2} \binom{\epsilon r}{q} \geq \frac{n^2}{8} \binom{\epsilon r}{q}$$

Where the first inequality follows from Jensen's inequality and the third from the fact that $r \leq n/2$.

On the other hand, there are only $\binom{r}{q}$ possible q -subsets of R . By the pigeonhole principle, one of these (say Q) must be the set of vertices in R associated with P_j for k different $P_j \in T$, where k is

$$\frac{|T|}{\binom{r}{q}} > \frac{n^2 \binom{\epsilon r}{q}}{8 \binom{r}{q}} \geq \frac{n^2 \left(\frac{\epsilon r}{q} \right)^q}{8 \left(\frac{er}{q} \right)^q} = \frac{n^2}{8} \left(\frac{\epsilon}{e} \right)^q = s$$

\square

Now, the algorithm of [3] is applied to the graph G' with vertex set $[n]$ and edge set S .

This yields a $K(q', q') \subset G'$ with

$$q' = \left\lfloor \frac{\ln(n/2)}{\ln(2en^2/s)} \right\rfloor = \left\lfloor \frac{\ln(n/2)}{\ln(16e^{q+1}/\epsilon^q)} \right\rfloor$$

For the found subgraph to be a $K(q, q, q)$, it is necessary that $q' \geq q$. A sufficient condition is that

$$q \leq \frac{\ln(n/2)}{\ln(16e^{q+1}/\epsilon^q)} - 1 = \frac{\ln(n/2)}{\ln(16e) - q \ln(e/\epsilon)} - 1$$

This is true for

$$0 \leq q \leq \frac{\ln(16e) + \sqrt{(\ln(16e))^2 + 4 \ln(n/(32e)) \ln(e/\epsilon)}}{2 \ln(e/\epsilon)}$$

so a valid value for q is

$$q = \frac{\sqrt{4 \ln(n/(32e)) \ln(e/\epsilon)}}{\ln(e/\epsilon)} = \frac{\sqrt{\ln(n/(32e))}}{\sqrt{\ln(e/\epsilon)}}$$

References

- [1] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel Journal of Mathematics*, 2(3):183–190, September 1964.
- [2] Sós V. Turán P. Kóvari, T. On a problem of k. zarankiewicz. *Colloquium Mathematicae*, 3(1):50–57, 1954.
- [3] Dhruv Mubayi and György Turán. Finding bipartite subgraphs efficiently. *Information Processing Letters*, 110(5):174–177, 2010.