FINDING PARTITE HYPERGRAPHS EFFICIENTLY

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Abstract. We provide a deterministic polynomial-time algorithm that, for a given k-uniform hypergraph H with n vertices and edge density d, finds a complete k-partite subgraph of H with parts of size at least $c(d,k)(\log n)^{1/(k-1)}$. This generalizes work by Mubayi and Turán on bipartite graphs. The value we obtain for the part size matches the order of magnitude guaranteed by the non-constructive proof due to Erdős and is tight up to a constant factor.

1. Introduction

Hypergraph Turán problems concern how many edges a k-uniform hypergraph $H=(V,E)$ with n vertices can have without containing a specific subgraph G. The maximal such number is known as the *Turán number* $\mathrm{ex}(n,G)$. It is known [5] that $\mathrm{ex}(n,G)$ is sublinear in $\binom{n}{k}$ if and only if G is k-partite, i.e., if its vertex set can be partitioned into k disjoint sets such that each edge contains exactly one vertex from each part. Kővári, Sós, and Turán [6] (for $k=2$) and Erdős [3] (for any $k\geq 2$) established that

$$\mathrm{ex}(n,K(t,\overset{k}{\ldots},t)) \leq \binom{n}{k}\cdot n^{-\frac{1}{t^{(k-1)}}}.$$

This result implies the following.

Remark 1.1. *If H is a k-graph with at least $d\binom{n}{k}$ edges for some constant $d>0$, then it contains $K(t,\overset{k}{\ldots},t)$ as a subgraph, with $t=c(d,k)(\log(n))^{1/(k-1)}$.*

Furthermore, the order of magnitude of t is tight up to a constant factor: For some constant $\hat{c}(d,k)>0$, there are k-graphs with n vertices and edge density d that do not contain $K(\hat{t},\overset{k}{\ldots},\hat{t})$ as a subgraph, where $\hat{t}=\hat{c}(d,k)\log(n)^{1/(k-1)}$. This was already noted by Erdős [3] and can be proved via the random alteration method [1].

Due to the fundamental role of Erdős' result, it is natural to ask whether a fast search algorithm for a complete k-partite subgraph of the size stated in Remark 1.1 exists. A brute-force search for a $K(t,\overset{k}{\ldots},t)$ would involve checking all $\binom{n}{kt}$ vertex subsets, which is superpolynomial in n for $t=\Omega\left((\log n)^{1/(k-1)}\right)$. For $k=2$, Mubayi and Turán [7] developed a deterministic polynomial-time algorithm which reaches the stated order of magnitude for the subgraph part size. This work extends their approach to the general case of k-uniform hypergraphs, reaching analogous results for $k\geq 3$. More concretely, we prove the following.

Theorem 1.2. *There is a deterministic algorithm that, given a k-graph H with n vertices and $m=d\binom{n}{k}$ edges, finds a complete balanced k-partite subgraph $K(t,\overset{k}{\ldots},t)$ in polynomial time, where*

$$t=t(n,d,k)=\left\lfloor\left(\frac{\log n}{\log(16/d)}\right)^{\frac{1}{k-1}}\right\rfloor.$$

This value of t matches the order of magnitude from Remark 1.1.

2. The algorithm

We present a recursive algorithm, `FindPartite`, that finds a $K(t,\overset{k}{\ldots},t)$ in a given k-graph H. The core idea is to reduce the uniformity of the problem from k to $k-1$ in each recursive step. The algorithm takes a k-graph H with n vertices and m edges as input. It first defines the target part size t, a small set size w, and a threshold edge count s for the recursive call, based on the input graph's parameters:

$$t = t(n,d,k) = \left\lfloor \left(\frac{\log n}{\log(16/d)} \right)^{\frac{1}{k-1}} \right\rfloor,$$
$$w = w(n,d,k) = \left\lceil \frac{4t}{d} \right\rceil, \text{ and}$$
$$s = s(n,d,k) = \left\lceil \left(\frac{d}{4} \right)^t \binom{n}{k-1} \right\rceil,$$

where $d = \frac{m}{\binom{n}{k}}$ is the edge density of H. The main steps are:

(1) **Base Case ($k = 1$):** The edge set of a 1-graph is just a collection of singleton sets of vertices. Return the set of all vertices that are "edges".

(2) **Select High-Degree Vertices:** Choose a set $W \subset V$ of w vertices with the highest degrees in H.

(3) **Find a Dense Link Graph:** Iterate through all t-subsets $T \subset W$. For each T, consider the set S of all $(k-1)$-subsets of V that form a hyperedge with *every* vertex in T.

(4) **Recurse:** As we prove further along using the Kővári–Sós–Turán theorem, for at least one choice of T, the resulting set S is large ($|S| \geq s$). We form a new $(k-1)$-graph $H' = (V, S)$ and make a recursive call: `FindPartite(`H'`, ` $k-1$`)`.

(5) **Construct Solution:** The recursive call returns $k-1$ parts $V_1, \ldots, V_{k-1}$ of size at least t. By construction, every choice of vertices from these parts forms an edge in H' with every vertex of T. Thus, $(V_1, \ldots, V_{k-1}, T)$ form the desired $K(t, \overset{k}{\ldots}, t)$ in the original graph H.

The pseudocode is given in Algorithm 1.

Algorithm 1 Finding a balanced partite k-graph

function FindPartite(H, k)
    if $k = 1$ **then**
        return $(\{x \colon \{x\} \in E(H)\})$
    end if
    $n \leftarrow |V(H)|$, $m \leftarrow |E(H)|$, $d \leftarrow \frac{m}{\binom{n}{k}}$
    $t \leftarrow t(n,d,k)$, $w \leftarrow w(n,d,k)$, $s \leftarrow s(n,d,k)$
    assert $t \geq 2$
    $W \leftarrow$ a set of w vertices with highest degree in H
    for all $T \in \binom{W}{t}$ **do**
        $S \leftarrow \{\, y \in \binom{V}{k-1} \colon \forall x \in T, \{x\} \cup y \in E(H) \,\}$
        if $|S| \geq s$ **then**
            $H' \leftarrow (V, S)$ ▷ H' is a $(k-1)$-graph
            $(V_1, \ldots, V_{k-1}) \leftarrow$ FindPartite$(H', k-1)$
            return $(V_1, \ldots, V_{k-1}, T)$
        end if
    end for
end function

3. Proof of correctness

We now present the proof that Algorithm 1 is correct in the sense that all the steps are well-defined and that it returns a tuple $(V_1, \ldots, V_k)$ of disjoint sets $V_i \subset V(H)$ of size at least t spanning a complete k-partite subgraph in H. We assume $t \geq 2$ for our estimates to be easier. If $t < 2$, we may just return the vertices of any single edge in H.

It is not immediately clear that the set W defined in step 2 of the algorithm is well-defined, as for this it is necessary that $w \leq n$. To show this, we first observe that our assumption $t \geq 2$ implies that $1 \geq d \geq \frac{16}{\sqrt{n}}$. Suppose, by way of contradiction, that $w > n$. Then, we have

$$n \leq w - 1 \leq \frac{4t}{d} \leq \frac{4\log n}{d\log(16/d)} \leq \frac{4\sqrt{n}\log n}{16\log(16/d)}\,.$$

Taking the logarithms to be in base e, we note that $\log x \leq \sqrt{x}$ for all positive x, and that $\log(16/d) > 1$. Therefore, we get $n < \frac{n}{4}$, which is a contradiction.

Next, we prove that in step 3 of the algorithm we indeed find a set $T \in \binom{W}{t}$ such that the associated set $S \subset \binom{V}{k-1}$ has size at least s. That is, Algorithm 1 reaches line 13 at some point in the for loop. For this, consider the bipartite graph B with parts $\binom{V}{k-1}$ and W with edge set

$$\left\{(x,y) \in \binom{V}{k-1} \times W \,\middle|\, x \cup \{y\} \in E\right\}.$$

The edges of B correspond to the edges containing each vertex in W, so there are

$$z = \sum_{y \in W} d_H(y) \geq k \cdot m \cdot \frac{w}{n} = \frac{k \cdot w \cdot d \cdot \binom{n}{k}}{n} = w \cdot d \cdot \binom{n-1}{k-1}$$

of them, where the inequality follows from the fact that we have picked a set of w vertices with highest degree in H. The existence of a set $T \subset W$ as desired is equivalent to there being $T \subset W$ of size t and a set $S \subset \binom{V}{k-1}$ of size s such that the induced bipartite subgraph $B[S,T]$ is complete. To prove that this is the case, we use the following version [4] of the Kővári–Sós–Turán theorem [6].

Lemma 3.1. *Let u, w, s, t be positive integers with $u \geq s$, $w \geq t$, and let B be a bipartite graph with parts W and U such that $|U| = u, |W| = w$. If B has more than*

$$(s-1)^{1/t}(w-t+1)u^{1-1/t} + (t-1)u$$

edges, then there are $T \subset W$ of size t and $S \subset U$ of size s such that the induced bipartite subgraph $B[S,T]$ is complete.

We apply the lemma with $u = \binom{n}{k-1}$. It is clear from the definitions that our parameter satisfy the requirements $u \geq s$ and $w \geq t$. Suppose, by way of contradiction, that

$$w \cdot d \cdot \binom{n-1}{k-1} \leq (s-1)^{1/t}(w-t+1)\binom{n}{k-1}^{1-1/t} + (t-1)\binom{n}{k-1}.$$

Dividing by $\binom{n}{k-1}$ then shows that

$$\frac{1}{2} \cdot w \cdot d \leq \left(1 - \frac{k-1}{n}\right) \cdot w \cdot d = w \cdot d \cdot \frac{\binom{n-1}{k-1}}{\binom{n}{k-1}} < w\left(\frac{s-1}{\binom{n}{k-1}}\right)^{1/t} + (t-1)\,,$$

where the first inequality follows from $n \geq 16^{2^{k-1}} > 2(k-1)$, which follows from $t \geq 2$ and $d \leq 1$. Finally, since $t \leq \frac{w \cdot d}{4}$ by the definition of w, we obtain

$$\left(\frac{d}{4}\right)^t \binom{n}{k-1} < s - 1\,,$$

in contradiction to the definition of s. So far, we have shown that the sets defined in Algorithm 1 are well-defined and that the recursive call in line 13 is reached. We are now ready to prove that the algorithm returns a $K(t, \overset{k}{\ldots}, t)$ by examining what happens in the recursive call. More precisely, we show the following.

Theorem 3.2. *For $k \geq 2$, if $t \geq 2$, Algorithm 1 returns a tuple $(V_1, \ldots, V_k)$ of disjoint sets $V_i \subset V(H)$ such that $|V_i| \geq t$ and $H[V_1, \ldots, V_k]$ is complete.*

Proof. We proceed by induction on k. For $k=2$, the recursive call returns the common neighborhood V_1 of the vertices in T, which is obviously disjoint from T, so it only remains to check that $|V_1| \geq t$. Now, since by construction $|V_1| = |S| \geq s$, it is enough that

$$s = \left\lceil \left(\frac{d}{4}\right)^t \cdot n \right\rceil \geq \left(\frac{d}{4}\right)^{\frac{\log n}{\log(16/d)}} \cdot n = \left(4 \cdot \frac{d}{16}\right)^{\frac{\log n}{\log(16/d)}} \cdot n = \frac{1}{n} \cdot 4^{\frac{\log n}{\log(16/d)}} \cdot n \geq 4^t > t\,.$$

For $k \geq 3$, we assume the inductive hypothesis holds for $k-1$. If d' is the edge density of the $(k-1)$-graph H' and $t' = t(n, d', k-1)$, as long as $t' \geq 2$, the recursive call returns a tuple $(V_1, \ldots, V_{k-1})$ of disjoint sets $V_i \subset V(H)$ such that $|V_i| \geq t'$ and $H'[V_1, \ldots, V_{k-1}]$ is complete.

We claim that $t' \geq t$. This implies that $t' \geq 2$ so we get to apply the inductive hypothesis to H'. Furthermore, the sets V_i that we obtain when applying the algorithm to H' are of size at least $t' \geq t$. By construction of S, all the edges in H' are disjoint from T, and therefore so are the sets V_i. This means that the sets $V_1, \ldots, V_{k-1}, T$ are disjoint. In addition, for all $(x_1, \ldots x_{k-1}, y) \in V_1 \times \cdots \times V_{k-1} \times T$, we have that $\{x_1, \ldots, x_{k-1}\} \in S$ so $\{x_1, \ldots, x_{k-1}, y\} \in E(H)$. Equivalently, $H[V_1, \ldots, V_{k-1}, T]$ is complete, finishing the proof.

Let us now prove the claim that $t' \geq t$. By the definition of s, we have $d' \geq \left(\frac{d}{4}\right)^t$. Therefore,

$$t' \geq \left\lfloor \left(\frac{\log n}{\log\left(\frac{16}{(d/4)^t}\right)}\right)^{\frac{1}{k-2}} \right\rfloor \geq \left\lfloor \left(\frac{\log n}{\log 16 - t\log(d/4)}\right)^{\frac{1}{k-2}} \right\rfloor .$$

Then, we substitute the definition of t, where removing the floor function maintains the inequality because the right side is decreasing in t (recall $d \leq 1$):

$$t' \geq \left\lfloor \left(\frac{\log n}{\log 16 - \left(\frac{\log n}{\log(16/d)}\right)^{\frac{1}{k-1}} \log(d/4)}\right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left(\frac{(\log n)^{\left(1-\frac{1}{k-1}\right)}}{\frac{\log 16}{(\log n)^{\frac{1}{k-1}}} - \frac{\log(d/4)}{\log(16/d)^{\frac{1}{k-1}}}}\right)^{\frac{1}{k-2}} \right\rfloor .$$

We claim that we bound the denominator by showing that

$$\frac{\log 16}{(\log n)^{\frac{1}{k-1}}} - \frac{\log(d/4)}{\log(16/d)^{\frac{1}{k-1}}} \leq (\log(16/d))^{\left(1-\frac{1}{k-1}\right)} . \tag{1}$$

Then, the expression simplifies to

$$t' \geq \left\lfloor \left(\frac{(\log n)^{\left(1-\frac{1}{k-1}\right)}}{(\log(16/d))^{\left(1-\frac{1}{k-1}\right)}}\right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left(\frac{\log n}{\log(16/d)}\right)^{\frac{1}{k-2}\left(1-\frac{1}{k-1}\right)} \right\rfloor = \left\lfloor \left(\frac{\log n}{\log(16/d)}\right)^{\frac{1}{k-1}} \right\rfloor = t\,,$$

as desired. Let us prove Inequality (1). Suppose, by way of contradiction, that it does not hold. We can rewrite

$$(\log(16/d))^{\left(1-\frac{1}{k-1}\right)} = \frac{\log(16/d)}{\log(16/d)^{\frac{1}{k-1}}}$$

and rearrange the inequality to obtain

$$\frac{\log 16}{(\log n)^{\frac{1}{k-1}}} > \frac{\log(16/d) + \log(d/4)}{\log(16/d)^{\frac{1}{k-1}}} = \frac{\log 4}{(\log(16/d))^{\frac{1}{k-1}}} .$$

This implies that

$$t \leq \left(\frac{\log n}{\log(16/d)}\right)^{\frac{1}{k-1}} < \frac{\log 16}{\log 4} = 2\,,$$

which contradicts the assumption that $t \geq 2$. $\square$

4. Proof of Polynomial Complexity

We now analyze the computational complexity of Algorithm 1 to show that it runs in time polynomial in n for any fixed uniformity k. We consider the input hypergraph H to be given as an adjacency k-dimensional tensor. We make the assumption that we have a machine with arbitrarily large random-access memory, and that the operations on integers and floating-point numbers take constant time. Not making these assumptions would only add logarithmic factors to the running time, which we ignore for simplicity.

Let $T_k(n)$ denote the worst-case running time of the function `FindPartite` when called on a k-uniform hypergraph with n vertices. The algorithm's structure gives a recurrence relation for $T_k(n)$. We first analyze the cost of the operations within a single call for a fixed k, excluding the recursive step.

Assuming that the input tensor consists of a flattened boolean array of size n^k, querying whether a set of k vertices forms an edge in H can be done in constant time. Therefore, calculating the number of edges m can be done in $\mathcal{O}\left(n^k\right)$ time by iterating through all $\binom{n}{k}$ possible edges. The parameters n, m, d, t, w, s are computed in constant time after this step.

The set W of w vertices with highest degrees can be constructed in time $\mathcal{O}\left(n^k\right)$, by, for example, creating an array with the degree of each vertex (in time $\mathcal{O}\left(n \cdot n^{k-1}\right) = \mathcal{O}\left(n^k\right)$), sorting it (in time $\mathcal{O}\left(n \log n\right)$), and taking the first w elements.

The subsets of t elements of W can be iterated through in time $\mathcal{O}\left(\binom{w}{t}\right)$ [8]. Similarly to the argument by Mubayi and Turán [7], we can bound this quantity by a polynomial in n. Indeed,

$$\binom{w}{t} \le \left(\frac{ew}{t}\right)^t \le \left(\frac{e(4t/d+1)}{t}\right)^t \le \left(\frac{4e}{d} + \frac{e}{2}\right)^t \le \left(\frac{4.5 \cdot e}{d}\right)^t < e^{t(2.6+\log(1/d))} = e^{2.6 \cdot t} e^{\log(1/d)\cdot t}.$$

For the first term, we use the fact that $t < (\log n)^{1/(k-1)} \le \log n$ and so $e^{2.6\cdot t} < e^{2.6 \cdot \log n} = n^{2.6}$. For the second term, we use the fact that $t < (\log(n)/\log(1/d))^{1/(k-1)} \le \log(n)/\log(1/d)$, so $e^{\log(1/d)\cdot t} < n$. All in all, we have

$$\binom{w}{t} < n^{2.6} n = n^{3.6}.$$

In each iteration of the loop, the set S can be constructed in the following way. We initialize a flattened boolean array A of size n^{k-1}, with all entries set to `true`. We iterate through all $x \in \binom{[n]}{k-1}$ and $y \in T$ and set the entry corresponding to x to `false` if $x \cup \{y\}$ is not an edge in H. All in all, this takes $\mathcal{O}\left(tn^{k-1}\right) = \mathcal{O}\left(n^k\right)$ steps, and then counting the number of `true` entries in the array takes $\mathcal{O}\left(n^{k-1}\right)$ time. Therefore, the for loop (with the recursive call) takes $\mathcal{O}\left(\binom{w}{t} n^k\right) = \mathcal{O}\left(n^{k+3.6}\right)$ time.

Finally, when the condition $|S| \ge s$ is satisfied, the recursive call to `FindPartite` is made. We can pass the array A to the recursive call directly, and the recursive call takes time $T_{k-1}(n)$. Putting everything together, we have the recurrence relation $T_k(n) = T_{k-1}(n) + \mathcal{O}\left(n^{k+3.6}\right)$. This, together with the base case $T_1(n) = \mathcal{O}\left(n\right)$, gives us $T_k(n) = \mathcal{O}\left(n^{k+3.6}\right)$.

5. Conclusion and Future Work

We have presented a deterministic, polynomial-time algorithm to find a large complete balanced k-partite subgraph in any sufficiently dense k-uniform hypergraph. This provides a constructive counterpart to a classical existence result by Erdős in extremal hypergraph theory.

Several avenues for future research remain open.

- **General Blow-ups:** Our algorithm finds a blow-up of a single edge, $K(t, \overset{k}{\ldots}, t)$. Can this framework be adapted to find a t-blowup of an arbitrary fixed k-graph G when its Turán density is exceeded by a positive constant? For example, for $k = 2$, there are existence results for $t = \Omega\left(\log n\right)$ [2], but directly adapting Algorithm 1 would only yield t $= \mathcal{O}\left((\log n)^{1/(|V(G)|-1)}\right)$.
- **Unbalanced k-Partite Graphs:** The algorithm could be modified to search for unbalanced complete partite graphs $K(t_1, \ldots, t_k)$, where the part sizes may grow at different rates.
- **Optimality:** The bounds on t are asymptotically tight, but the constants can likely be improved with a more refined analysis.

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