

# Extending Mubayi and Turán's Algorithm to $k$ -graphs

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Let  $G$  be an  $r$ -graph with  $n$  vertices and  $m = dn^k$  edges. A polynomial time algorithm is given to find a  $K_{q,\dots,q} \subset G$  for

$$q(k, d) = \left\lfloor \left( \frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor$$

As long as  $d \geq 2^{k+1}n^{-\frac{1}{2^{k-1}}}$ , that is, as long as  $q(k, d) \geq 2$ .

This result is a generalization of the result in 2-graphs by [2], and algorithm will be analogous to the one given there. This algorithm, referred to as **FIND\_PARTITE**( $k, \cdot$ ), involves the following steps:

1. Choose parameters  $q, r, s$  depending on  $n, k$  and  $d$ .
2. Find the set  $R$  of  $r$  vertices with the highest degree in  $G$ .
3. find a subset  $Q \subset R$  with  $q$  vertices and a  $S \subset T := \binom{[n] \setminus Q}{k-1}$  with  $s$  edges satisfying

$$\{x_1, x_2, \dots, x_k\} \in E(G) \forall \{x_2, \dots, x_k\} \in S, x_1 \in Q$$

4. The set  $S$  induces a  $(k-1)$ -graph  $G'$  on  $T$ . Evaluate **FIND\_PARTITE**( $k-1, G'$ ) to find a  $K_{q', \dots, q'}$  in  $G'$  (say,  $H' = \{U_1, \dots, U_{k-1}\}$ ). It will turn out that  $q' \geq q$ , and because of the condition for  $S$  the  $k$ -partite subgraph  $H = \{Q, V_1, \dots, V_{k-1}\}$  is complete in  $G$ , where  $V_i \subset U_i$  is a subset of size  $q$ . Return  $H$ .

For step 1, we will use the following formulas:

$$q(k, d) = \left\lfloor \left( \frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor, \quad r(k, d) = \left\lceil \frac{2q(k, d)}{d} \right\rceil, \quad s(k, d) = \lfloor d^{q(k, d)} n^{k-1} \rfloor$$

The goal is to prove that the algorithm is successful and runs in polynomial time.

**Lemma 1.** *This selection of parameters is sound in the sense that  $q \leq r \leq n$ ,  $k-1 \leq n-r$  and  $s \leq \binom{n-r}{k-1}$ .*

*Proof.*  $q \leq r$  is clear from the definition of  $r$ . Suppose by way of contradiction that  $r > n$ .

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□

**Lemma 2.** *With the above restrictions and choice of parameters, for  $k \geq 3$ , there are at least  $\frac{3}{2}drn^{k-1}$  edges with exactly one vertex in  $R$ .*

*Proof.* The degree sum over  $V(G)$  is  $kdn^k$ . Thus, by the pigeonhole principle, the degree sum over  $R$  is at least  $\frac{r}{n}kdn^k = rkdn^{k-1}$ . For  $2 \leq j \leq n$ , consider the contribution to this sum by edges with exactly  $j$  vertices in  $R$ . Each such edge contributes  $j$  to the sum, and there are at most  $\binom{r}{j} \binom{n-r}{k-j} \leq \frac{r^j n^{k-j}}{j!} \leq \frac{r^j n^{k-j}}{j}$  of them. Thus, the total contrition of these edges is at most  $r^j n^{k-j} \leq r^2 n^{k-2}$ . The number of edges we want is then at least

$$rkdn^{k-1} - (k-1)r^2n^{k-2} = drn^{k-1} \left( k - \frac{(k-1)r}{nd} \right)$$

Suppose, by way of contradiction, that  $k - \frac{(k-1)r}{nd} < \frac{3}{2}$ . Using that  $\frac{k-1}{k-3/2} \leq 2$  for  $k \geq 2$ , we arrive at

$$d < 2rn^{-1} \leq \frac{3q}{nd} \implies nd^2 < 3 \log n$$

Applying our minimum density, this means

$$\sqrt{n} \leq 2^{2k+1} n^{1-\frac{1}{2k-2}} < 3 \log n$$

which is false for all  $n$ .

□

**Lemma 3.** *For this selection of parameters, there exist sets  $Q, S$  as described in step 3 of the algorithm.*

*Proof.* Consider the bipartite graph with vertex set  $(R, \binom{T}{k-1})$  and edges corresponding to edges in  $G$  with exactly one vertex in  $R$  (and thus all others in  $T$ ). The sets  $Q$  and  $S$  we want to find correspond to a complete bipartite subgraph of this graph with parts of size  $q$  and  $s$  respectively. Suppose that such a subgraph does not exist. [1] tells us then that

$$\begin{aligned} \frac{3}{2}drn^{k-1} &< z \left( \binom{n-r}{k-1}, r; s, q \right) < (s-1)^{1/q} (r-q+1) \binom{n-r}{k-1}^{1-1/q} + (q-1) \binom{n-r}{k-1} \\ &\leq s^{1/q} r \binom{n}{k-1}^{1-1/q} + q \binom{n}{k-1} \leq s^{1/q} r \binom{n}{k-1}^{1-1/q} + \frac{1}{2}drn^{k-1} \end{aligned}$$

Where the last inequality follows from our choice of  $r$ .

Rearranging and approximating the binomial coefficient, we get

$$drn^{k-1} < s^{1/q} r n^{(k-1)(1-1/q)} \iff d < \left( \frac{s}{n^{k-1}} \right)^{1/q}$$

Which is false for the given choice of  $s$ .

□

**Lemma 4.** *For this choice of parameters, the number of edges  $s$  in  $G'$  is such that we can apply the algorithm to  $G'$ . Furthermore, the resulting  $q'$  satisfies  $q' \geq q$ .*

*Proof.* First we calculate a lower bound for the corresponding edge density  $d'$  in  $G'$ :

$$d' = \frac{s}{(n-r)^{k-1}} \geq \frac{(n^{k-1}d^q - 1)}{n^{k-1}} \geq d^q - n^{1-k}$$

Note however that because  $1 \geq d \geq 2^{k+1}n^{-\frac{1}{2^{k-1}}}$  and  $q \geq 2$ , we have

$$d^q \geq 2^{q(k+1)}n^{-\frac{q}{2^{k-1}}} \geq 2^{k+1}n^{-\frac{1}{2^{k-1}}}\left(\frac{\log n}{\log(2^{k+1}/d)}\right)^{1/(k-1)} \geq 2^{k+1}n^{-\frac{1}{2^{k-1}}}\left(\frac{\log n}{2^{1-k}\log(n)}\right)^{1/(k-1)} = 2^{k+1}n^{-\frac{1}{2^{k-2}}}$$

Now, clearly this means that  $n^{1-k} \leq \frac{1}{2}d^q \implies d' \geq \frac{1}{2}d^q \implies d' \geq 2^k n^{-\frac{1}{2^{k-2}}}$ , satisfying our minimum density requirement for  $k-1$ . Furthermore, we can bound

$$\begin{aligned} q' &\geq \left\lfloor \left( \frac{\log n}{\log(2 \cdot 2^{(k-1)+1}/d^q)} \right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left( \frac{\log n}{(k+1)\log 2 - q\log d} \right)^{\frac{1}{k-2}} \right\rfloor \\ &\geq \left\lfloor \left( \frac{\log n}{(k+1)\log 2 - \left(\frac{\log n}{\log(2^{k+1}/d)}\right)^{\frac{1}{k-1}} \log d} \right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left( \frac{(\log n)^{1-\frac{1}{k-1}}}{\frac{(k-1)\log 2}{(\log n)^{\frac{1}{k-1}}} - \frac{\log d}{\log(2^{k+1}/d)^{\frac{1}{k-1}}}} \right)^{\frac{1}{k-2}} \right\rfloor \\ &\geq \left\lfloor \frac{(\log n)^{\frac{1}{k-1}}}{\left(\frac{\log(2^{k+1}/d)}{\log(2^{k+1}/d)^{\frac{1}{k-1}}}\right)^{\frac{1}{k-2}}} \right\rfloor = \left\lfloor \frac{(\log n)^{\frac{1}{k-1}}}{(\log(2^{k+1}/d))^{(1-\frac{1}{k-1})\frac{1}{k-2}}} \right\rfloor = \left\lfloor \left( \frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor = q \end{aligned}$$

where the last inequality follows from the fact that  $n \geq 2^{k+1}/d$ , which is a consequence of our minimum density requirement.

□

## References

- [1] T. Kóvari, V. Sós, and Turán P. On a problem of k. zarankiewicz. *Colloquium Mathematicae*, 3(1):50–57, 1954.
- [2] Dhruv Mubayi and György Turán. Finding bipartite subgraphs efficiently. *Information Processing Letters*, 110(5):174–177, 2010.