Finding Partite Hypergraphs Efficiently

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k-Graphs

Definition

A *k-graph* is a pair $G = (V, E)$ where V is a finite set of *vertices* and $E \subseteq \binom{V}{k}$ is a set of *edges*.

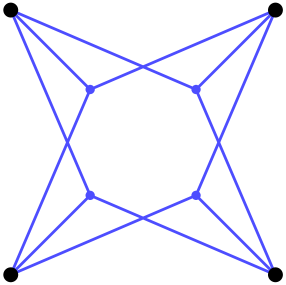

Figure: A complete 3-graph on 4 vertices: $K_4^{(3)}$.

Partite k-Graphs

Definition

A k-graph $G = (V, E)$ is *r-partite* if there exists a partition $V = V_1 \cup \cdots \cup V_r$ such that every edge of G intersects every part V_i in at most one vertex. We write $G = (V_1, \ldots, V_r; E)$.

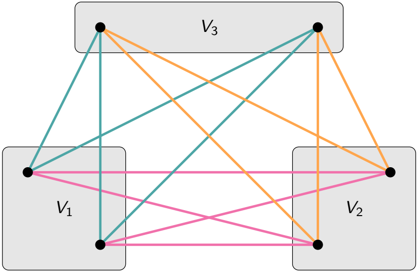

Figure: A complete 3-partite 2-graph: $K^{(3)}(2, 2, 2)$.

Partite k-Graphs

Remark

We may identify E as a subset of $\mathcal{C} = \bigcup_{\{i_1,\ldots,i_k\}\in\binom{[r]}{k}} V_{i_1} \times \cdots \times V_{i_k}$.
If $E = \mathcal{C}$, we say that G is a *complete* r-partite k-graph.

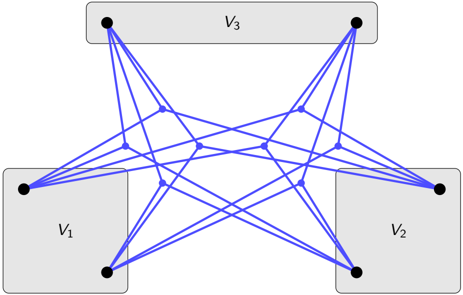

Figure: A complete 3-partite 3-graph: $K^{(2)}(2,2,2)$.

Turán-Type Problems

Definition

Let $G = (V, E)$ be a k-graph and $n \geq |V|$ an integer. The *Turán number* $\operatorname{ex}(G, n)$ is the maximum number of edges in a k-graph on n vertices that does not contain a copy of G as a subgraph.

Determining $\operatorname{ex}(G, n)$ or estimating it as $n \to \infty$ is known as the *Turán problem* for G.

Theorem

For all k-graphs G there exists a constant $\alpha(G) \in [0, 1)$ such that

$$ex(G, n) = (\alpha(G) + o(1)) \cdot \binom{n}{k} \quad \text{as } n \to \infty.$$

Furthermore, $\alpha(G) = 0$ if and only if G is k-partite.

The Kővari–Sós–Turán Theorem

The bound $\mathrm{ex}(G, n) = o(n^k)$ can be improved by a lot.

Definition

Let $1 < t_1 \leq v_1, \ldots, 1 < t_k \leq v_k$ be integers. Then the *generalized Zarankiewicz number* $z(v_1, \ldots, v_k; t_1, \ldots, t_k)$ is the largest integer z for which there exists a k-partite k-graph $H = (V_1, \ldots V_k, F)$ with part sizes $|V_i| = v_i$ and $|F| = z$ edges such that for all choices of $W_i \subset V_i$ of sizes $|W_i| = t_i$, $W_1 \times \cdots \times W_k \not\subset F$.

Theorem (Kővari–Sós–Turán)

Let $0 < s \leq u$ and $0 < t \leq w$ be integers. Then

$$z(u, w; s, t) \leq (s-1)^{1/t}(w - t + 1)u^{1-1/t} + (t-1)u$$

Standard arguments then show that $\mathrm{ex}(n, K(s, t)) = \mathcal{O}\left(n^{2-1/t}\right)$.