

Finding Partite Hypergraphs Efficiently

Ferran Espuña Bertomeu

Supervisor: Richard Lang

June 2025

k -Graphs

Definition

A k -graph is a pair $G = (V, E)$ where V is a finite set of *vertices* and $E \subseteq \binom{V}{k}$ is a set of *edges*.

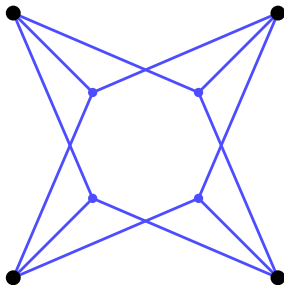


Figure: A complete 3-graph on 4 vertices: $K_4^{(3)}$.

Partite k -Graphs

Definition

A k -graph $G = (V, E)$ is r -partite if there exists a partition $V = V_1 \cup \dots \cup V_r$ such that every edge of G intersects every part V_i in at most one vertex. We write $G = (V_1, \dots, V_r; E)$.

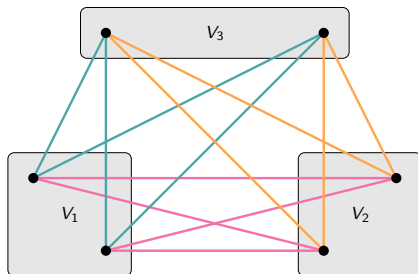


Figure: A complete 3-partite 2-graph: $K^{(3)}(2, 2, 2)$.

Partite k -Graphs

Remark

We may identify E as a subset of $\mathcal{C} = \bigcup_{\{i_1, \dots, i_k\} \in \binom{[r]}{k}} V_{i_1} \times \dots \times V_{i_k}$.
If $E = \mathcal{C}$, we say that G is a *complete r -partite k -graph*.

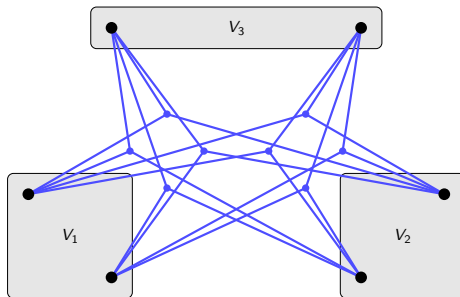


Figure: A complete 3-partite 3-graph: $K^{(2)}(2, 2, 2)$.

Turán-Type Problems

Definition

Let $G = (V, E)$ be a k -graph and $n \geq |V|$ an integer. The *Turán number* $\text{ex}(G, n)$ is the maximum number of edges in a k -graph on n vertices that does not contain a copy of G as a subgraph.

Determining $\text{ex}(G, n)$ or estimating it as $n \rightarrow \infty$ is known as the *Turán problem* for G .

Theorem

For all k -graphs G there exists a constant $\alpha(G) \in [0, 1)$ such that

$$\text{ex}(G, n) = (\alpha(G) + o(1)) \cdot \binom{n}{k} \quad \text{as } n \rightarrow \infty.$$

Furthermore, $\alpha(G) = 0$ if and only if G is k -partite.

The Kővari–Sós–Turán Theorem

The bound $\text{ex}(G, n) = o(n^k)$ can be improved by a lot.

Definition

Let $1 < t_1 \leq v_1, \dots, 1 < t_k \leq v_k$ be integers. Then the *generalized Zarankiewicz number* $z(v_1, \dots, v_k; t_1, \dots, t_k)$ is the largest integer z for which there exists a k -partite k -graph $H = (V_1, \dots, V_k, F)$ with part sizes $|V_i| = v_i$ and $|F| = z$ edges such that for all choices of $W_i \subset V_i$ of sizes $|W_i| = t_i$, $W_1 \times \dots \times W_k \not\subset F$.

Theorem (Kővari–Sós–Turán)

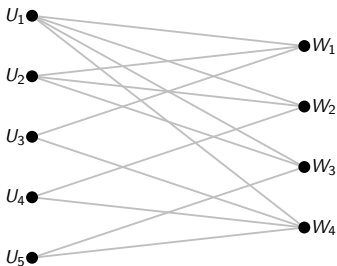
Let $0 < s \leq u$ and $0 < t \leq w$ be integers. Then

$$z(u, w; s, t) \leq (s-1)^{1/t}(w-t+1)u^{1-1/t} + (t-1)u$$

Standard arguments then show that $\text{ex}(n, K(s, t)) = \mathcal{O}(n^{2-1/t})$.

Kővari–Sós–Turán: Proof Sketch and Example

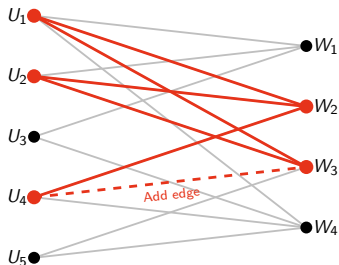
This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.



- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

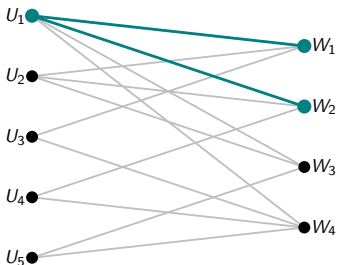


For example, adding the edge $\{U_4, W_3\}$ creates a $K_{3,2}$ on vertices $\{U_1, U_2, U_4\}$ and $\{W_2, W_3\}$.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

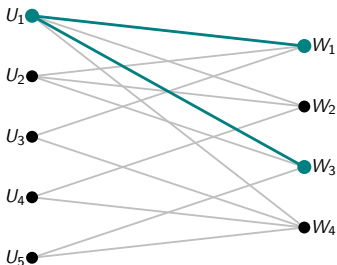


For $x = U_1$, we count its $\binom{4}{2} = 6$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

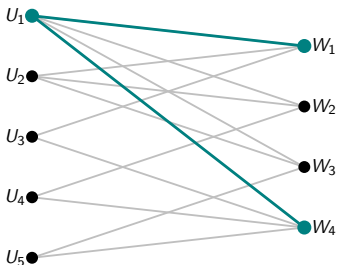


For $x = U_1$, we count its $\binom{4}{2} = 6$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

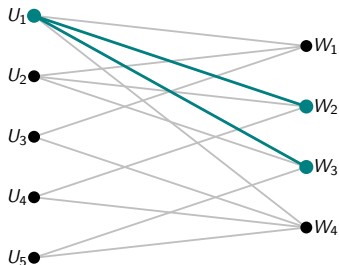


For $x = U_1$, we count its $\binom{4}{2} = 6$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

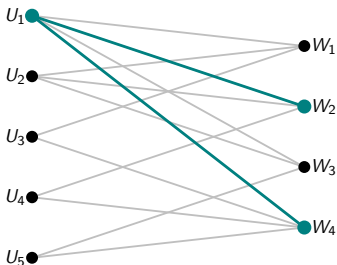


For $x = U_1$, we count its $\binom{4}{2} = 6$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

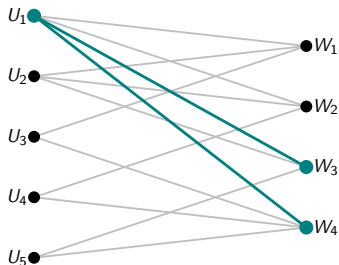


For $x = U_1$, we count its $\binom{4}{2} = 6$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

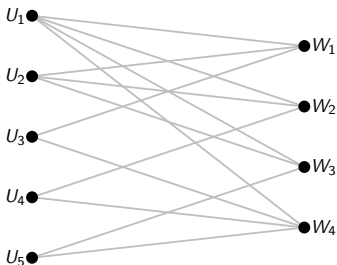


For $x = U_1$, we count its $\binom{4}{2} = 6$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

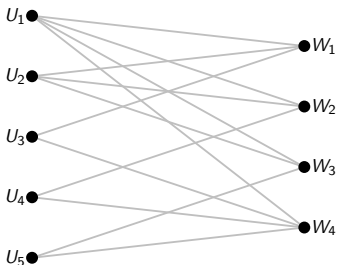


In the example, there are at least $5 \binom{13/5}{2} = 10.4$ stars (there are actually 12)

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .
- **Averaging:** By a convexity argument, the number of stars is at least $u \binom{z/u}{t}$.

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

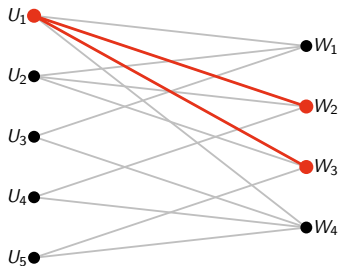


Each set $T \subset W$ (in this case, $T = \{W_2, W_3\}$) is in at most $s - 1 = 3 - 1 = 2$ stars. In total, at most $2\binom{4}{2} = 12$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .
- **Averaging:** By a convexity argument, the number of stars is at least $u\binom{z/u}{t}$.
- **Bounding:** Because H is $K(s, t)$ -free, each set $T \subset W$ is the right component of at most $(s - 1)$ stars.

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

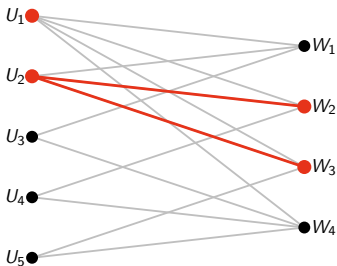


Each set $T \subset W$ (in this case, $T = \{W_2, W_3\}$) is in at most $s - 1 = 3 - 1 = 2$ stars. In total, at most $2\binom{4}{2} = 12$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .
- **Averaging:** By a convexity argument, the number of stars is at least $u\binom{z/u}{t}$.
- **Bounding:** Because H is $K(s, t)$ -free, each set $T \subset W$ is the right component of at most $(s - 1)$ stars.

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.

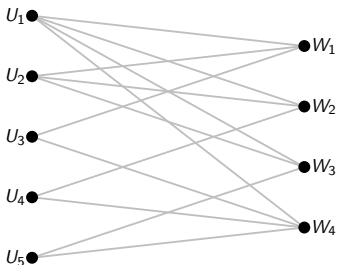


Each set $T \subset W$ (in this case, $T = \{W_2, W_3\}$) is in at most $s - 1 = 3 - 1 = 2$ stars. In total, at most $2 \binom{4}{2} = 12$ stars.

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .
- **Averaging:** By a convexity argument, the number of stars is at least $u \binom{z/u}{t}$.
- **Bounding:** Because H is $K(s, t)$ -free, each set $T \subset W$ is the right component of at most $(s - 1)$ stars.

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$ -free.



In the example, we conclude that $10.4 \leq 12$, which is true. For bigger values of z this would fail, leading to contradiction and therefore upper bounding z . In fact, $z=14$ already fails!

- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$ -free bipartite k -graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.
- **Counting Stars:** For each $x \in U$, there are $\binom{d_H(x)}{t}$ sets $T \subset W$ of t neighbors of x .
- **Averaging:** By a convexity argument, the number of stars is at least $u \binom{z/u}{t}$.
- **Bounding:** Because H is $K(s, t)$ -free, each set $T \subset W$ is the right component of at most $(s - 1)$ stars.
- **Conclusion:** $u \binom{z/u}{t} \leq (s - 1) \binom{w}{t}$, from which the theorem follows.

Erdős's Bound for Hypergraphs (1964)

Theorem (Erdős '64)

For integers $k \geq 2, t \geq 2$, $ex(n, K(t, \overset{k}{\cdot}, t)) = \mathcal{O}\left(n^{k - \frac{1}{t^{k-1}}}\right)$.

This generalizes the Kővari–Sós–Turán theorem to k -graphs. It follows from a similar bound on the corresponding generalized Zarankiewicz number, obtained by induction.

Suppose that $H = (V_1, \dots, V_k; F)$ is a k -graph with $|W_i| = w$. Let H have z edges and no copy of $K(t, \overset{k}{\cdot}, t)$.

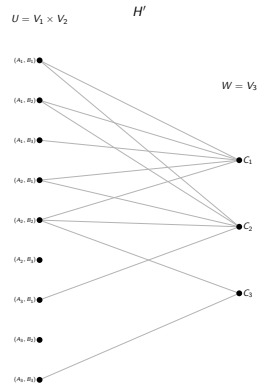
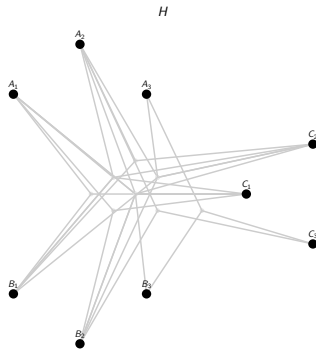
We set up a bipartite k -graph $H' = (U, W; F')$ with

$$U = W_1 \times \cdots \times W_{k-1}$$

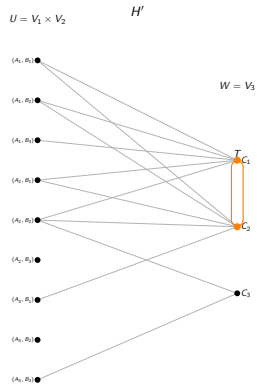
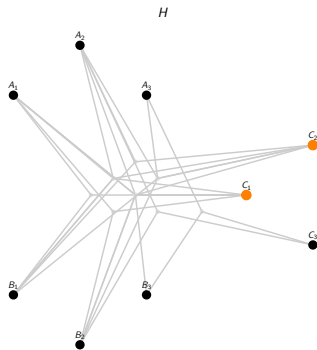
$$W = W_k$$

$$F' = \{(X, y) \in U \times W \mid X \cup \{y\} \in F\}.$$

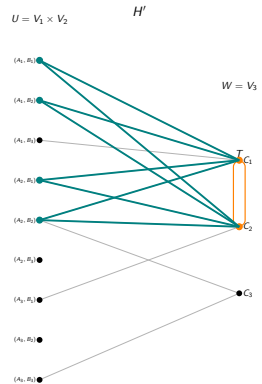
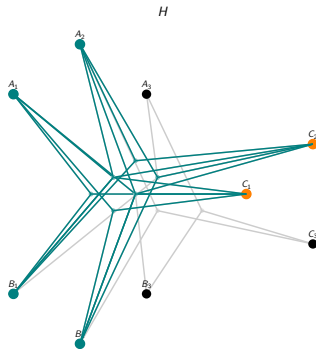
Erdős's Bound: Proof Sketch ($k = 3, t = 2$)



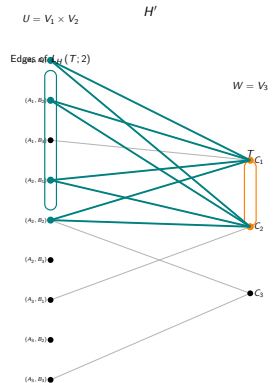
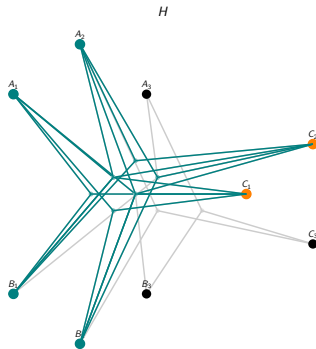
Erdős's Bound: Proof Sketch ($k = 3, t = 2$)



Erdős's Bound: Proof Sketch ($k = 3, t = 2$)



Erdős's Bound: Proof Sketch ($k = 3, t = 2$)



Erdős's Bound: Proof Sketch ($k = 3, t = 2$)

