

# Extending Mubayi and Turán's Algorithm to $k$ -graphs

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Let  $G$  be a  $k$ -graph with  $n$  vertices and  $m = dn^k$  edges. A polynomial time algorithm is given to find a  $K_{q, \dots, q} \subset G$  for

$$q(k, d) = \left\lfloor \left( \frac{\log(n/2^k)}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor$$

As long as  $d \geq 4^k n^{-\frac{1}{2^{k-1}}}$ .

*Remark 1.* This minimum density requirement is not a big restriction. For example, if we want  $q \geq 3$ ,

$$n \geq 2^{\frac{3^{k-1}(k-1)-k}{2^{k-1}-1}} \geq 2^{3^{k-1}(k-1)-k+1}$$

This result is a generalization of the result in 2-graphs by [2], and algorithm will be analogous to the one given there. This algorithm, referred to as **FIND\_PARTITE**( $k, \cdot$ ), is the one described in [2] when  $k = 2$ , and for  $k \geq 3$  involves the following steps:

1. Choose parameters  $q, r, s$  depending on  $n, k$  and  $d$ .
2. Find the set  $R$  of  $r$  vertices with the highest degree in  $G$ .
3. find a subset  $Q \subset R$  with  $q$  vertices and a  $S \subset T := \binom{[n] \setminus Q}{k-1}$  with  $s$  edges satisfying

$$\{x_1, x_2, \dots, x_k\} \in E(G) \forall \{x_2, \dots, x_k\} \in S, x_1 \in Q$$

4. The set  $S$  induces a  $(k-1)$ -graph  $G'$  on  $T$ . Evaluate **FIND\_PARTITE**( $k-1, G'$ ) to find a  $K_{q', \dots, q'}$  in  $G'$  (say,  $H' = \{U_1, \dots, U_{k-1}\}$ ). It will turn out that  $q' \geq q$ , and because of the condition for  $S$  the  $k$ -partite subgraph  $H = \{Q, V_1, \dots, V_{k-1}\}$  is complete in  $G$ , where  $V_i \subset U_i$  is a subset of size  $q$ . Return  $H$ .

For step 1, we will use the following formulas:

$$q(k, d) = \left\lfloor \left( \frac{\log(n/2^k)}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor, \quad r(k, d) = \left\lceil \frac{2q(k, d)}{d} \right\rceil, \quad s(k, d) = \lfloor d^{q(k, d)} n^{k-1} \rfloor$$

The goal is to prove that the algorithm is successful and runs in polynomial time.

**Lemma 2.** *This selection of parameters, for  $k \geq 3$ , is sound in the sense that  $q \leq r \leq n$ ,  $k - 1 \leq n - r$  and  $s \leq \binom{n-r}{k-1}$ .*

*Proof.*  $q \leq r$  is clear from the definition of  $r$ . We will show that in fact  $r < \frac{n}{2}$ . Suppose not:

$$\frac{n}{2} \leq r \leq 1 + \frac{2q}{d} \leq 1 + \frac{2 \log(n/2)}{2^{k+1}n^{-\frac{1}{2^{k-1}}}} \leq 1 + \frac{\log n \cdot \sqrt{n}}{8} \leq 1 + \frac{n}{8} \implies n < 3$$

Which is a contradiction, as there cannot be any edges and thus  $d = 0$ . In particular,  $r \leq n$ .

We can similarly show that  $k \leq \frac{n}{2}$ . Suppose not:

$$1 \geq d \geq 2^{\frac{n}{2}} n^{-\frac{1}{2^{n/2}-1}} = n^{\frac{\log(2)n}{2} - \frac{1}{2^{n/2}-1}} \implies \frac{\log(2)n}{2} \leq \frac{1}{2^{n/2}-1} \implies n < 3$$

Therefore,  $k + r \leq n < n + 1 \implies k - 1 \leq n - r$ .

Finally, suppose that  $s > \binom{n-r}{k-1}$ . Then,

$$\left(\frac{n}{2k}\right)^{k-1} \leq \left(\frac{n-r}{k-1}\right)^{k-1} \leq \binom{n-r}{k-1} < s \leq d^q n^{k-1} \implies \left(\frac{1}{2k}\right)^{k-1} < d^q \leq \left(\frac{1}{k!}\right)^2$$

Where in the last inequality we use that  $q \geq 2$  and there are at most  $\binom{n}{k} \leq \frac{n^k}{k!}$  edges in  $G$ .

We can show that  $k!^2 \geq (2k)^{k-1}$  for all  $k$ , which means we have reached a contradiction.  $\square$

**Lemma 3.** *With the above restrictions and choice of parameters, for  $k \geq 3$ , there are at least  $\frac{3}{2}drn^{k-1}$  edges with exactly one vertex in  $R$ .*

*Proof.* The degree sum over  $V(G)$  is  $kdn^k$ . Thus, by the pigeonhole principle, the degree sum over  $R$  is at least  $\frac{r}{n}kdn^k = rkdn^{k-1}$ . For  $2 \leq j \leq n$ , consider the contribution to this sum by edges with exactly  $j$  vertices in  $R$ . Each such edge contributes  $j$  to the sum, and there are at most  $\binom{r}{j} \binom{n-r}{k-j} \leq \frac{r^j n^{k-j}}{j!} \leq \frac{r^j n^{k-j}}{j}$  of them. Thus, the total contribution of these edges is at most  $r^j n^{k-j} \leq r^2 n^{k-2}$ . The number of edges we want is then at least

$$rkdn^{k-1} - (k-1)r^2n^{k-2} = drn^{k-1} \left( k - \frac{(k-1)r}{nd} \right)$$

Suppose, by way of contradiction, that  $k - \frac{(k-1)r}{nd} < \frac{3}{2}$ . Using that  $\frac{k-1}{k-3/2} \leq 2$  for  $k \geq 2$ , we arrive at

$$d < 2rn^{-1} \leq \frac{3q}{nd} \implies nd^2 < 3 \log n$$

Applying our minimum density, this means

$$\sqrt{n} \leq 2^{2k+1} n^{1-\frac{1}{2^{k-2}}} < 3 \log n$$

which is false for all  $n$ . □

**Lemma 4.** *For this selection of parameters, there exist sets  $Q, S$  as described in step 3 of the algorithm.*

*Proof.* Consider the bipartite graph with vertex set  $(R, \binom{T}{k-1})$  and edges corresponding to edges in  $G$  with exactly one vertex in  $R$  (and thus all others in  $T$ ). The sets  $Q$  and  $S$  we want to find correspond to a complete bipartite subgraph of this graph with parts of size  $q$  and  $s$  respectively. Suppose that such a subgraph does not exist. [1] tells us then that

$$\begin{aligned} \frac{3}{2} dr n^{k-1} &< z \left( \binom{n-r}{k-1}, r; s, q \right) < (s-1)^{1/q} (r-q+1) \binom{n-r}{k-1}^{1-1/q} + (q-1) \binom{n-r}{k-1} \\ &\leq s^{1/q} r \binom{n}{k-1}^{1-1/q} + q \binom{n}{k-1} \leq s^{1/q} r \binom{n}{k-1}^{1-1/q} + \frac{1}{2} dr n^{k-1} \end{aligned}$$

Where the last inequality follows from our choice of  $r$ .

Rearranging and approximating the binomial coefficient, we get

$$dr n^{k-1} < s^{1/q} r n^{(k-1)(1-1/q)} \iff d < \left( \frac{s}{n^{k-1}} \right)^{1/q}$$

Which is false for the given choice of  $s$ . □

**Lemma 5.** *For this choice of parameters, the number of edges  $s$  in  $G'$  is such that we can apply the algorithm to  $G'$ . Furthermore, the resulting  $q'$  satisfies  $q' \geq q$ .*

*Proof.* First we calculate a lower bond for the corresponding edge density  $d'$  in  $G'$ :

$$d' = \frac{s}{(n-r)^{k-1}} \geq \frac{(n^{k-1} d^q - 1)}{n^{k-1}} \geq d^q - n^{1-k}$$

Note however that because  $1 \geq d \geq 2^{k+1} n^{-\frac{1}{2^{k-1}}}$  and  $q \geq 2$ , we have

$$d^q \geq 2^{q(k+1)} n^{-\frac{q}{2^{k-1}}} \geq 2^{k+1} n^{-\frac{1}{2^{k-1}}} \left( \frac{\log n}{\log(2^{k+1}/d)} \right)^{1/(k-1)} \geq 2^{k+1} n^{-\frac{1}{2^{k-1}}} \left( \frac{\log n}{2^{1-k} \log(n)} \right)^{1/(k-1)} = 2^{k+1} n^{-\frac{1}{2^{k-2}}}$$

Now, clearly this means that  $n^{1-k} \leq \frac{1}{2}d^q \implies d' \geq \frac{1}{2}d^q \implies d' \geq 2^k n^{-\frac{1}{2^{k-2}}}$ , satisfying our minimum density requirement for  $k-1$ . Furthermore, we can bound

$$\begin{aligned}
q' &\geq \left\lfloor \left( \frac{\log((n-r)/2^{k-2})}{\log(2 \cdot 2^{(k-1)+1}/d^q)} \right)^{\frac{1}{k-2}} \right\rfloor \geq \left\lfloor \left( \frac{\log(n/2^{k-1})}{\log(2 \cdot 2^{(k-1)+1}/d^q)} \right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left( \frac{\log(n/2^{k-1})}{(k+1)\log 2 - q\log d} \right)^{\frac{1}{k-2}} \right\rfloor \\
&\geq \left\lfloor \left( \frac{\log(n/2^{k-1})}{(k+1)\log 2 - \left( \frac{\log(n/2^{k-1})}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \log d} \right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left( \frac{\log(n/2^{k-1})^{1-\frac{1}{k-1}}}{\frac{(k+1)\log 2}{\log(n/2^{k-1})^{\frac{1}{k-1}}} - \frac{\log d}{\log(2^{k+1}/d)^{\frac{1}{k-1}}}} \right)^{\frac{1}{k-2}} \right\rfloor \\
&\geq \left\lfloor \frac{(\log n)^{\frac{1}{k-1}}}{\left( \frac{\log(2^{k+1}/d)}{\log(2^{k+1}/d)^{\frac{1}{k-1}}} \right)^{\frac{1}{k-2}}} \right\rfloor = \left\lfloor \frac{(\log n)^{\frac{1}{k-1}}}{(\log(2^{k+1}/d))^{(1-\frac{1}{k-1})\frac{1}{k-2}}} \right\rfloor = \left\lfloor \left( \frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor = q
\end{aligned}$$

where the last inequality follows from the fact that  $n \geq 2^{k+1}/d$ , which is a consequence of our minimum density requirement.

□

## References

- [1] T. Kővari, V. Sós, and Turán P. On a problem of k. zarankiewicz. *Colloquium Mathematicae*, 3(1):50–57, 1954.
- [2] Dhruv Mubayi and György Turán. Finding bipartite subgraphs efficiently. *Information Processing Letters*, 110(5):174–177, 2010.