Finding Partite Hypergraphs Efficiently

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Abstract

TODO

1 Introduction

Hypergraph Turán problems study how many edges a k-uniform hypergraph $H = (V, E)$ with n vertices can have without containing a specific subgraph G. The maximal such number is known as the *Turán number* $\mathrm{ex}(n, G)$. It is known that $\mathrm{ex}(n, G) = o\left(\binom{n}{k}\right)$ if and only if G is k-partite, i.e., if its vertex set can be partitioned into k disjoint sets such that each edge contains exactly one vertex from each part. Kővári, Sós, and Turán [17] (for $k = 2$) and Erdős [8] (for any $k \geq 2$) established that

$$\mathrm{ex}(n, K(t, \overset{k}{\ldots}, t)) = \mathcal{O}(n^{k - \frac{1}{t^{(k-1)}}}), \tag{1}$$

where $K(t, \overset{k}{\ldots}, t)$ is the complete balanced k-partite k-graph with k parts of size t. Furthermore, if H is a k-graph with at least $d\binom{n}{k}$ edges for some constant $d > 0$, then it contains a $K(t, \overset{k}{\ldots}, t)$ with $t = c_d \log(n)^{1/(k-1)}$.

This result is non-constructive, meaning it guarantees the existence of such a subgraph but does not provide an efficient way to find it. Note that a simple brute-force search for a $K(t, \overset{k}{\ldots}, t)$ would involve checking all $\binom{n}{kt}$ vertex subsets, which is superpolynomial in n for $t = \Theta((\log n)^{1/(k-1)})$. Mubayi and Turán [19] developed a polynomial-time algorithm for the case $k = 2$, which reaches the stated order of magnitude for the subgraph part size. This paper extends their approach to the general case of k-uniform hypergraphs, reaching analogous results for $k \geq 3$. More concretely, we prove the following.

Theorem 1. *There is a deterministic algorithm that, given a k-graph H with n vertices and $m = dn^k$ edges, finds a complete balanced k-partite subgraph $K(t, \overset{k}{\ldots}, t)$ in polynomial time, where*

$$t = \left\lfloor \left(\frac{\log(n/2^{k-1})}{\log(3/d)} \right)^{\frac{1}{k-1}} \right\rfloor .$$

This value of t matches the order of magnitude from existence proofs. In fact, a probabilistic argument shows that it is the best possible up to a constant factor.

2 Finding a Balanced k-Partite Subgraph

We present a recursive algorithm, `Find_Partite`, that finds a $K(t, \overset{k}{\ldots}, t)$ in a given k-graph H. The core idea is to reduce the uniformity of the problem from k to $k-1$ in each recursive step.

The algorithm takes a k-graph H as input. It first defines the target part size t, a small set size w, and a threshold edge count s for the recursive call, based on the input graph's parameters

$(n,\, d = m/n^k,\, k)$:

$$t(n,d,k) = \left\lfloor \left(\frac{\log\left(n/2^{k-1}\right)}{\log(3/d)} \right)^{\frac{1}{k-1}} \right\rfloor$$

$$w(n,d,k) = \left\lceil \frac{2t(n,d,k)}{d} \right\rceil$$

$$s(n,d,k) = \left\lceil d^{t(n,d,k)} n^{k-1} \right\rceil$$

The main steps are:

1. **Base Case ($k = 1$):** A 1-graph is just a collection of vertices. Return the set of all vertices that are "edges".
2. **Select High-Degree Vertices:** Choose a set $W \subset V$ of w vertices with the highest degrees in H. Let $U = V \setminus W$.
3. **Find a Dense Link Graph:** Iterate through all t-subsets $T \subset W$. For each T, consider the set S of all $(k-1)$-subsets of U that form a hyperedge with *every* vertex in T.
4. **Recurse:** The pigeonhole principle, formalized by the Kővári–Sós–Turán theorem, guarantees that for at least one choice of T, the resulting set S will be large ($|S| \geq s$). We form a new $(k-1)$-graph $H' = (U, S)$ and make a recursive call: `Find_Partite(`H'`, `$k-1$`)`.
5. **Construct Solution:** The recursive call returns $k-1$ parts $V_1, \ldots, V_{k-1}$ of size at least t. By construction, every choice of vertices from these parts forms an edge in H' with T. Thus, $(V_1, \ldots, V_{k-1}, T)$ form the desired $K(t, \overset{k}{\ldots}, t)$ in the original graph H.

The pseudocode is given in Algorithm 1.

Algorithm 1 Finding a balanced partite k-graph

function FIND_PARTITE(H, k)
    if $k = 1$ **then**
        return $(\{x \colon \{x\} \in E(H)\})$
    end if
    $n \leftarrow |V(H)|$, $m \leftarrow |E(H)|$, $d \leftarrow m/n^k$
    $t \leftarrow t(n,d,k)$, $w \leftarrow w(n,d,k)$, $s \leftarrow s(n,d,k)$
    assert $t \geq 2$
    $W \leftarrow$ a set of w vertices with highest degree in H
    for all $T \in \binom{W}{t}$ **do**
        $S \leftarrow \{\, y \in \binom{V \setminus W}{k-1} : \forall x \in T, \{x\} \cup y \in E(H) \,\}$
        if $|S| \geq s$ **then**
            $H' \leftarrow (V \setminus W, S)$ $\triangleright$ H' is a $(k-1)$-graph
            $(V_1, \ldots, V_{k-1}) \leftarrow$ FIND_PARTITE$(H', k-1)$
            return $(V_1, \ldots, V_{k-1}, T)$
        end if
    end for
end function

3 Analysis

We briefly sketch the proof of correctness and polynomial runtime for our algorithm. The full, detailed proofs can be found in the first author's master's thesis [?]. We assume $t \geq 2$, which holds if d is not too small (e.g., $d = \Omega(n^{-1/2^{(k-1)}})$).

3.1 Correctness

The correctness of the algorithm hinges on two key lemmas, which we state here without proof.

1. There always exists a t-subset $T \subseteq W$ for which the set S of common neighbors satisfies $|S| \geq s$.
2. The recursive call `Find_Partite(`H'`,` $k-1$`)` is guaranteed to find parts of size at least t.

The first claim follows from a double-counting argument akin to the proof of the Kővári-Sós-Turán theorem. We define a bipartite graph between the $(k-1)$-subsets of $V \setminus W$ and the vertices in W. By showing this graph is dense, we can guarantee the existence of a complete bipartite subgraph $K(s,t)$, which corresponds to finding the desired set T and a large corresponding set S.

For the second claim, we show that the new hypergraph $H' = (V \setminus W, S)$ is sufficiently dense. Its density d' satisfies $d' \geq d^t$. We then prove that the target part size for the recursive call, $t' = t(n-w, d', k-1)$, satisfies $t' \geq t$. This ensures that the part sizes do not shrink during recursion. For the base case $k = 2 \to k = 1$, we show directly that $|S| \geq t$.

3.2 Complexity

The algorithm runs in polynomial time in n. The dominant cost at each recursive level comes from the loop over all t-subsets of W. The number of iterations is $\binom{w}{t}$. Since $w = \lceil 2t/d \rceil$ and $t = \Theta((\log n)^{1/(k-1)})$, the number of iterations can be bounded:

$$\binom{w}{t} \leq \left(\frac{ew}{t}\right)^t \leq \left(\frac{e(2t/d+1)}{t}\right)^t \approx \left(\frac{3e}{d}\right)^t.$$

Substituting the definition of t, this expression is polynomial in n (e.g., $\mathcal{O}(n^c)$ for some constant c). Inside the loop, constructing the set S and the new hypergraph H' takes polynomial time. Since the recursion depth is fixed at $k-1$, the total runtime is a nested polynomial, which remains polynomial in n. A detailed analysis shows the complexity is roughly $\mathcal{O}(n^{2k+3})$.

4 Conclusion and Future Work

We have presented a deterministic, polynomial-time algorithm to find a large complete balanced k-partite subgraph in any sufficiently dense k-uniform hypergraph. This provides a constructive counterpart to a classical existence result by Erdős in extremal hypergraph theory.

Several avenues for future research remain open.

- **General Blow-ups:** Our algorithm finds a blow-up of a single edge, $K(t, \overset{k}{\ldots}, t)$. Can this framework be adapted to find a t_n-blowup of an arbitrary fixed k-graph G? Existence theorems guarantee such structures, but efficient algorithms are lacking.
- **Unbalanced Partite Graphs:** The algorithm could be modified to search for unbalanced complete partite graphs $K(t_1, \ldots, t_k)$, where the part sizes may grow at different rates.
- **Optimality:** The bounds on t are asymptotically tight, but the constants can likely be improved with a more refined analysis. For $k = 2$, it is known that in dense graphs one can find a $t = \Theta(\log n)$ blow-up of any bipartite graph. It is an open question if a constructive proof for this stronger result exists for $k \geq 2$.

References

[1] R. Baber and J. Talbot. Hypergraphs do jump. *Combinatorics, Probability and Computing*, 20(2):161–171, 2011.

[2] S. Ball and V. Pepe. Asymptotic improvements to the lower bound of certain bipartite Turán numbers. *Combinatorics, Probability and Computing*, 21(3):323–329, 2012.

[3] B. Bollobás and P. Erdős. On the structure of edge graphs. *Bulletin of the London Mathematical Society*, 5(3):317–321, 1973.

[4] W. G. Brown. On graphs that do not contain a Thomsen graph. *Canadian Mathematical Bulletin*, 9:281–285, 1966.

[5] M. A. Carvajal, G. Santos, and M. Schacht. Canonical Ramsey numbers for partite hypergraphs. *arXiv preprint arXiv:2411.16218*, 2024.

[6] D. Conlon, C. Pohoata, and D. Zakharov. Random multilinear maps and the Erdős box problem. *Discrete Analysis*, Paper No. 17, 8, 2021.

[7] D. De Caen and Z. Füredi. The maximum size of 3-uniform hypergraphs not containing a Fano plane. *Journal of Combinatorial Theory, Series B*, 78(2):274–276, 2000.

[8] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel Journal of Mathematics*, 2:183–190, 1964.

[9] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bulletin of the American Mathematical Society*, 52:1087–1091, 1946.

[10] P. Erdős, A. Rényi, and V. T. Sós. On a problem of graph theory. *Studia Scientiarum Mathematicarum Hungarica*, 1:215–235, 1966.

[11] P. Erdős and M. Simonovits. Supersaturated graphs and hypergraphs. *Combinatorica*, 3(2):181–192, 1983.

[12] P. Frankl and Z. Füredi. An exact result for 3-graphs. *Discrete Mathematics*, 50(2-3):323–328, 1984.

[13] Z. Füredi. New asymptotics for bipartite Turán numbers. *Journal of Combinatorial Theory, Series A*, 75(1):141–144, 1996.

[14] C. Hyltén-Cavallius. On a combinatorical problem. *Colloquium Mathematicum*, 6:59–65, 1958.

[15] P. Keevash. Hypergraph Turán problems. In *Surveys in combinatorics 2011*, volume 392 of *London Mathematical Society Lecture Note Series*, pages 83–139. Cambridge University Press, 2011.

[16] J. Kollár, L. Rónyai, and T. Szabó. Norm-graphs and bipartite Turán numbers. *Combinatorica*, 16(3):399–406, 1996.

[17] T. Kővári, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Mathematicum*, 3:50–57, 1954.

[18] M. Krivelevich and B. Sudakov. The chromatic numbers of random hypergraphs. *Random Structures & Algorithms*, 12(4):381–403, 1998.

[19] D. Mubayi and G. Turán. Finding bipartite subgraphs efficiently. *arXiv preprint arXiv:0905.2527*, 2009.

[20] A. A. Razborov. Flag algebras. *The Journal of Symbolic Logic*, 72(4):1239–1282, 2007.

[21] A. A. Razborov. On 3-hypergraphs with forbidden 4-vertex configurations. *SIAM Journal on Discrete Mathematics*, 24(3):946–963, 2010.

[22] E. M. Reingold, J. Nievergelt, and N. Deo. *Combinatorial algorithms: theory and practice.* Prentice-Hall, Inc., 1977.

[23] V. R"odl and M. Schacht. Complete partite subgraphs in dense hypergraphs. *Random Structures & Algorithms*, 41(4):557–573, 2012.

[24] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. *Matematikai és Fizikai Lapok*, 48:436–452, 1941.

[25] K. Zarankiewicz. Problem 101. *Colloquium Mathematicae*, 2(3-4):301, 1951.