Extending Mubayi and Turán's Algorithm to k-graphs

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Let G be an r-graph with n vertices and $m = dn^k$ edges. A polynomial time algorithm is given to find a $K_{q,\ldots,q} \subset G$ for

$$q(k,d) = \left\lfloor \left(\frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor$$

As long as $d \geq 2^{k+1} n^{-\frac{1}{2^{k-1}}}$, that is, as long as $q(k,d) \geq 2$.

This result is a generalization of the result in 2-graphs by [2], and algorithm will be analogous to the one given there. This algorithm, referred to as `FIND_PARTITE`$(k,\cdot)$, is the one described in [2] when $k=2$, and for $k \geq 3$ involves the following steps:

1. Choose parameters q, r, s depending on n, k and d.
2. Find the set R of r vertices with the highest degree in G.
3. find a subset $Q \subset R$ with q vertices and a $S \subset T := \binom{[n]\setminus Q}{k-1}$ with s edges satisfying
$$\{x_1, x_2, \ldots, x_k\} \in E(G) \, \forall \, \{x_2, \ldots, x_k\} \in S, \, x_1 \in Q$$
4. The set S induces a $(k-1)$-graph G' on T. Evaluate `FIND_PARTITE`$(k-1, G')$ to find a $K_{q',\ldots,q'}$ in G' (say, $H' = \{U_1, \ldots, U_{k-1}\}$). It will turn out that $q' \geq q$, and because of the condition for S the k-partite subgraph $H = \{Q, V_1, \ldots V_{k-1}\}$ is complete in G, where $V_i \subset U_i$ is a subset of size q. Return H.

For step 1, we will use the following formulas:

$$q(k,d) = \left\lfloor \left(\frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor, \; r(k,d) = \left\lceil \frac{2q(k,d)}{d} \right\rceil, \; s(k,d) = \left\lfloor d^{q(k,d)} n^{k-1} \right\rfloor$$

The goal is to prove that the algorithm is successful and runs in polynomial time.

Lemma 1. *This selection of parameters, for $k \geq 3$, is sound in the sense that $q \leq r \leq n$, $k-1 \leq n-r$ and $s \leq \binom{n-r}{k-1}$.*

Proof. $q \le r$ is clear from the definition of r. We will show that in fact $r < \frac{n}{2}$. Suppose not:

$$\frac{n}{2} < r \le 1 + \frac{2q}{d} \le 1 + \frac{2\log n}{2^{k+1} n^{-\frac{1}{2^{k-1}}}} \le 1 + \frac{\log n \cdot \sqrt{n}}{8} \le 1 + \frac{n}{8} \implies n < 3$$

Which is a contradicion, as there cannot be any edges and thus $d = 0$. In particular, $r \le n$.

We can similarly show that $k \le \frac{n}{2}$. Suppose not:

$$1 \ge d \ge 2^{\frac{n}{2}} n^{-\frac{1}{2^{n/2}-1}} = n^{\frac{\log(2)n}{2} - \frac{1}{2^{n/2}-1}} \implies \frac{\log(2)n}{2} \le \frac{1}{2^{n/2}-1} \implies n < 3$$

Therefore, $k + r \le n < n+1 \implies k - 1 \le n - r$.

Finally, suppose that $s > \binom{n-r}{k-1}$. Then,

$$\left(\frac{n}{2k}\right)^{k-1} \le \left(\frac{n-r}{k-1}\right)^{k-1} \le \binom{n-r}{k-1} < s \le d^q n^{k-1} \implies \left(\frac{1}{2k}\right)^{k-1} < d^q \le \left(\frac{1}{k!}\right)^2$$

Where in the last inequality we use that $q \ge 2$ and there are at most $\binom{n}{k} \le \frac{n^k}{k!}$ edges in G. We can show that $k!^2 \ge (2k)^{k-1}$ for all k , which means we have reached a contradiction.

$\square$

Lemma 2. *With the above restrictions and choice of parameters, for $k \ge 3$, there are at least $\frac{3}{2} drn^{k-1}$ edges with exactly one vertex in R.*

Proof. The degree sum over $V(G)$ is kdn^k. Thus, by the pigeonhole principle, the degree sum over R is at least $\frac{r}{n} kdn^k = rkdn^{k-1}$. For $2 \le j \le n$, consider the contribution to this sum by edges with exactly j vertices in R. Each such edge contributes j to the sum, and there are at most $\binom{r}{j}\binom{n-r}{k-j} \le \frac{r^j n^{k-j}}{j!} \le \frac{r^j n^{k-j}}{j}$ of them. Thus, the total contrition of these edges is at most $r^j n^{k-j} \le r^2 n^{k-2}$. The number of edges we want is then at least

$$rkdn^{k-1} - (k-1) r^2 n^{k-2} = drn^{k-1} \left(k - \frac{(k-1)r}{nd}\right)$$

Suppose, by way of contradiction, that $k - \frac{(k-1)r}{nd} < \frac{3}{2}$. Using that $\frac{k-1}{k-3/2} \le 2$ for $k \ge 2$, we arrive at

$$d < 2rn^{-1} \le \frac{3q}{nd} \implies nd^2 < 3\log n$$

Applying our minimum density, this means

$$\sqrt{n} \le 2^{2k+1} n^{1-\frac{1}{2^{k-2}}} < 3\log n$$

which is false for all n. $\square$

Lemma 3. *For this selection of parameters, there exist sets Q, S as described in step 3 of the algorithm.*

Proof. Consider the biparite graph with vertex set $\left(R, \binom{T}{k-1}\right)$ and edges corresponding to edges in G with exactly one vertex in R (and thus all others in T). The sets Q and S we want to find correspond to a complete bipartite subgraph of this graph with parts of size q and s respectively. Suppose that such a subgraph does not exist. [1] tells us then that

$$\frac{3}{2}drn^{k-1} < z\left(\binom{n-r}{k-1}, r; s, q\right) < (s-1)^{1/q}(r-q+1)\binom{n-r}{k-1}^{1-1/q} + (q-1)\binom{n-r}{k-1}$$
$$\leq s^{1/q}r\binom{n}{k-1}^{1-1/q} + q\binom{n}{k-1} \leq s^{1/q}r\binom{n}{k-1}^{1-1/q} + \frac{1}{2}drn^{k-1}$$

Where the last inequality follows from our choice of r.

Rearranging and approximating the binomial coefficient, we get

$$drn^{k-1} < s^{1/q}rn^{(k-1)(1-1/q)} \iff d < \left(\frac{s}{n^{k-1}}\right)^{1/q}$$

Which is false for the given choice of s.

$\square$

Lemma 4. *For this choice of parameters, the number of edges s in G' is such that we can apply the algorithm to G'. Furthermore, the resulting q' satisfies $q' \geq q$.*

Proof. First we calculate a lower bond for the corresponding edge density d' in G':

$$d' = \frac{s}{(n-r)^{k-1}} \geq \frac{\left(n^{k-1}d^q - 1\right)}{n^{k-1}} \geq d^q - n^{1-k}$$

Note however that because $1 \geq d \geq 2^{k+1}n^{-\frac{1}{2^{k-1}}}$ and $q \geq 2$, we have

$$d^q \geq 2^{q(k+1)}n^{-\frac{q}{2^{k-1}}} \geq 2^{k+1}n^{-\frac{1}{2^{k-1}}\left(\frac{\log n}{\log(2^{k+1}/d)}\right)^{1/(k-1)}} \geq 2^{k+1}n^{-\frac{1}{2^{k-1}}\left(\frac{\log n}{2^{1-k}\log(n)}\right)^{1/(k-1)}} = 2^{k+1}n^{-\frac{1}{2^{k-2}}}$$

Now, clearly this means that $n^{1-k} \leq \frac{1}{2}d^q \implies d' \geq \frac{1}{2}d^q \implies d' \geq 2^k n^{-\frac{1}{2^{k-2}}}$, satisfying our minimum density requirement for $k-1$. Furthermore, we can bound

$$
\begin{aligned}
q' &\geq \left\lfloor \left(\frac{\log n}{\log(2 \cdot 2^{(k-1)+1}/d^q)} \right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left(\frac{\log n}{(k+1)\log 2 - q \log d} \right)^{\frac{1}{k-2}} \right\rfloor \\
&\geq \left\lfloor \left(\frac{\log n}{(k+1)\log 2 - \left(\frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \log d} \right)^{\frac{1}{k-2}} \right\rfloor = \left\lfloor \left(\frac{(\log n)^{1-\frac{1}{k-1}}}{\frac{(k-1)\log 2}{(\log n)^{\frac{1}{k-1}}} - \frac{\log d}{\log(2^{k+1}/d)^{\frac{1}{k-1}}}} \right)^{\frac{1}{k-2}} \right\rfloor \\
&\geq \left\lfloor \frac{(\log n)^{\frac{1}{k-1}}}{\left(\frac{\log(2^{k+1}/d)}{\log(2^{k+1}/d)^{\frac{1}{k-1}}} \right)^{\frac{1}{k-2}}} \right\rfloor = \left\lfloor \frac{(\log n)^{\frac{1}{k-1}}}{(\log(2^{k+1}/d))^{(1-\frac{1}{k-1})\frac{1}{k-2}}} \right\rfloor = \left\lfloor \left(\frac{\log n}{\log(2^{k+1}/d)} \right)^{\frac{1}{k-1}} \right\rfloor = q
\end{aligned}
$$

where the last inequality follows from the fact that $n \geq 2^{k+1}/d$, which is a consequence of our minimum density requirement.

$\square$

References

[1] T. Kóvari, V. Sós, and Turán P. On a problem of k. zarankiewicz. *Colloquium Mathematicae*, 3(1):50–57, 1954.

[2] Dhruv Mubayi and György Turán. Finding bipartite subgraphs efficiently. *Information Processing Letters*, 110(5):174–177, 2010.