Finding Partite Hypergraphs Efficiently

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k-Graphs

Definition

A *k-graph* is a pair $G = (V, E)$ where V is a finite set of *vertices* and $E \subseteq \binom{V}{k}$ is a set of *edges*.

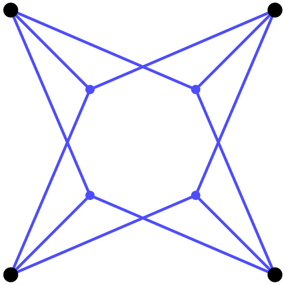


Figure: A complete 3-graph on 4 vertices: $K_4^{(3)}$.

Partite k-Graphs

Definition

A k-graph $G = (V, E)$ is *r-partite* if there exists a partition $V = V_1 \cup \cdots \cup V_r$ such that every edge of G intersects every part V_i in at most one vertex. We write $G = (V_1, \ldots, V_r; E)$.

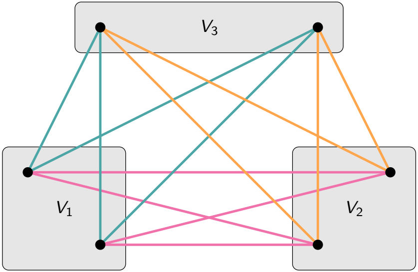


Figure: A complete 3-partite 2-graph: $K^{(3)}(2, 2, 2)$.

Partite k-Graphs

Remark

We may identify E as a subset of $\mathcal{C} = \bigcup_{\{i_1,\dots,i_k\} \in \binom{[r]}{k}} V_{i_1} \times \cdots \times V_{i_k}$.
If $E = \mathcal{C}$, we say that G is a *complete* r-partite k-graph.

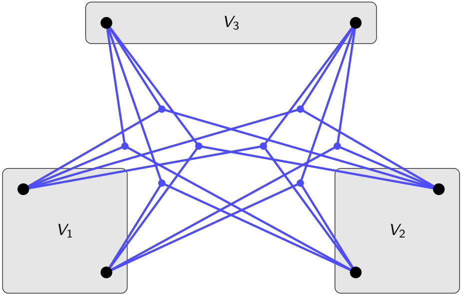


Figure: A complete 3-partite 3-graph: $K^{(2)}(2,2,2)$.

Turán-Type Problems

Definition

Let $G = (V, E)$ be a k-graph and $n \geq |V|$ an integer. The *Turán number* $\operatorname{ex}(G, n)$ is the maximum number of edges in a k-graph on n vertices that does not contain a copy of G as a subgraph.

Determining $\operatorname{ex}(G, n)$ or estimating it as $n \to \infty$ is known as the *Turán problem* for G.

Theorem

For all k-graphs G there exists a constant $\alpha(G) \in [0, 1)$ such that

$$ex(G, n) = (\alpha(G) + o(1)) \cdot \binom{n}{k} \quad \text{as } n \to \infty.$$

Furthermore, $\alpha(G) = 0$ if and only if G is k-partite.

The Kővari–Sós–Turán Theorem

The bound $\operatorname{ex}(G, n) = o(n^k)$ can be improved by a lot.

Definition

Let $1 < t_1 \leq v_1, \ldots, 1 < t_k \leq v_k$ be integers. Then the *generalized Zarankiewicz number* $z(v_1, \ldots, v_k; t_1, \ldots, t_k)$ is the largest integer z for which there exists a k-partite k-graph $H = (V_1, \ldots V_k, F)$ with part sizes $|V_i| = v_i$ and $|F| = z$ edges such that for all choices of $W_i \subset V_i$ of sizes $|W_i| = t_i$, $W_1 \times \cdots \times W_k \not\subset F$.

Theorem (Kővari–Sós–Turán)

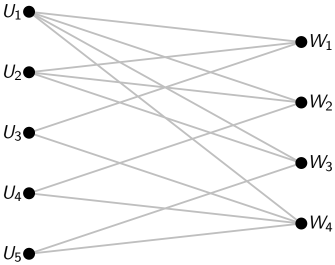
Let $0 < s \leq u$ and $0 < t \leq w$ be integers. Then

$$z(u, w; s, t) \leq (s-1)^{1/t}(w - t + 1)u^{1-1/t} + (t-1)u$$

Standard arguments then show that $\operatorname{ex}(n, K(s, t)) = \mathcal{O}\left(n^{2-1/t}\right)$.

Kővari–Sós–Turán: Proof Sketch and Example

This graph has the maximum number of edges ($|E| = 13$) to be $K_{3,2}$-free.



- **Hypothesis:** $H = (U, W; E)$ is a $K(s, t)$-free bipartite k-graph with $z = z(u, w; s, t)$ edges, where $|U| = u$ and $|W| = w$.

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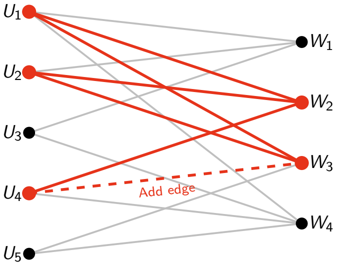

For example, adding the edge $\{U_4, W_3\}$ creates a $K_{3,2}$ on vertices $\{U_1, U_2, U_4\}$ and $\{W_2, W_3\}$.

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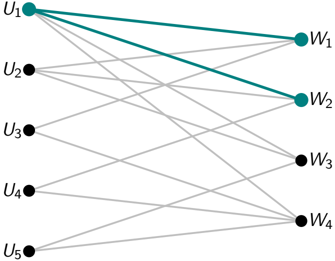

For $x = U_1$, we count its $\binom{4}{2} = 6$ stars.

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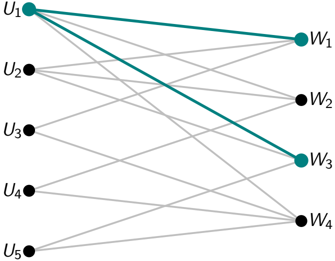

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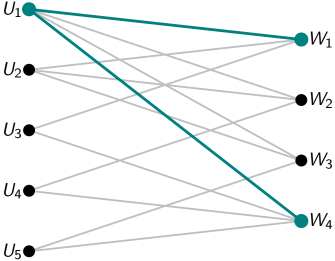

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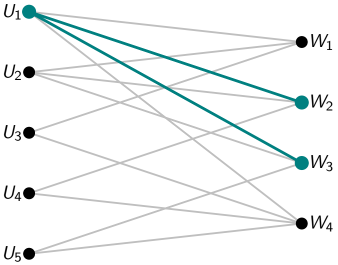

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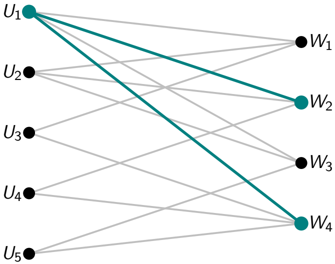

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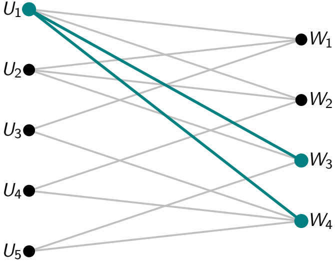

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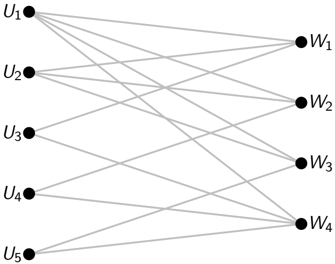

In the example, there are at least $5\binom{13/5}{2} = 10.4$ stars (there are actually 12)

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- **Averaging:** By a convexity argument, the number of stars is at least $u\binom{z/u}{t}$.

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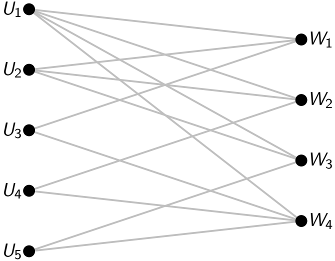

Each set $T \subset W$ (in this case, $T = \{W_2, W_3\}$) is in at most $s - 1 = 3 - 1 = 2$ stars. In total, at most $2\binom{4}{2} = 12$ stars.

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- **Bounding:** Because H is $K(s,t)$-free, each set $T \subset W$ is the right component of at most $(s-1)$ stars.

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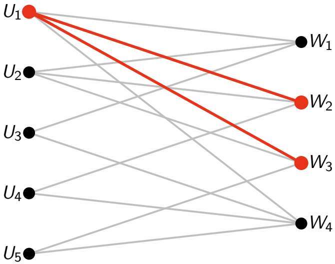

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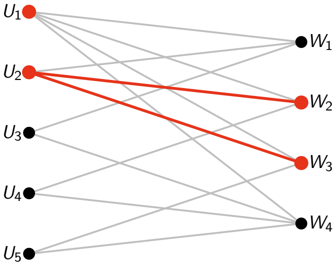

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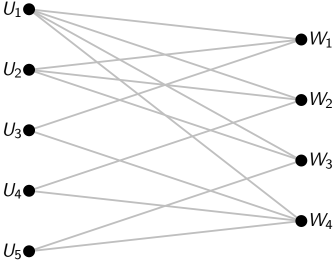

In the example, we conclude that $10.4 \leq 12$, which is true. For bigger values of z this would fail, leading to contradiction and therefore upper bounding z. In fact, z=14 already fails!

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- **Conclusion:** $u\binom{z/u}{t} \leq (s-1)\binom{w}{t}$, from which the theorem follows.

Erdős's Bound for Hypergraphs (1964)

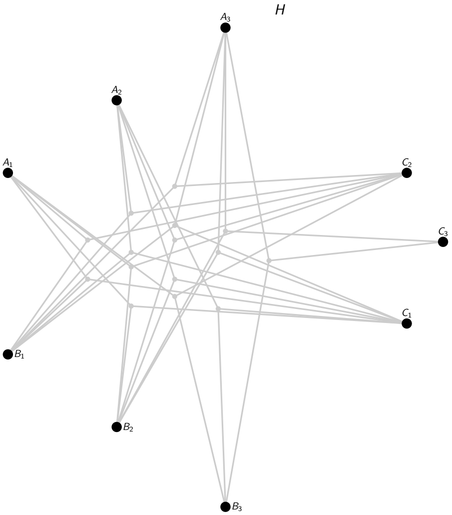
Theorem (Erdős '64)

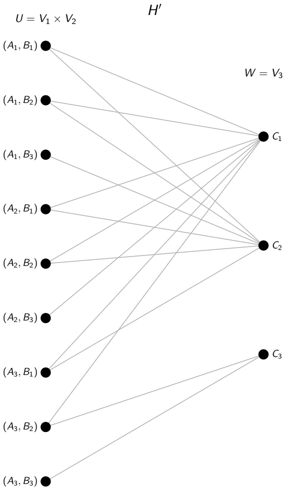
For integers $k \geq 2, t \geq 2$, $ex\left(n, K\left(t, \overset{k}{\ldots}, t\right)\right) = \mathcal{O}\left(n^{k - \frac{1}{t^{k-1}}}\right)$.

This generalizes the Kővari–Sós–Turán theorem to k-graphs.
It follows from a similar bound on the corresponding generalized Zarankiewicz number, obtained by induction.
Suppose that $H = (V_1, \ldots, V_k; F)$ is a k-graph with $|W_i| = w$. Let H have z edges and no copy of $K\left(t, \overset{k}{\ldots}, t\right)$.
We set up a bipartite k-graph $H' = (U, W; F')$ with

$$
\begin{aligned}
U &= W_1 \times \cdots \times W_{k-1} \\
W &= W_k \\
F' &= \{(X, y) \in U \times W \mid X \cup \{y\} \in F\}.
\end{aligned}
$$

Erdős's Bound: Proof Sketch ($k = 3, t = 2$)





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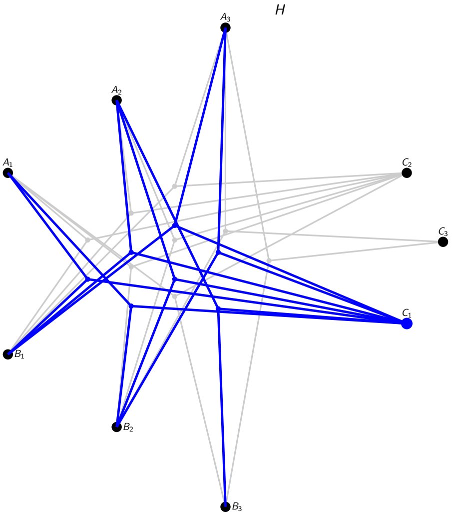

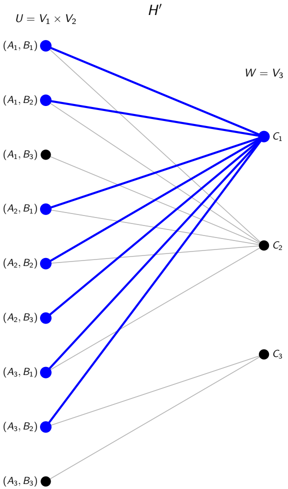

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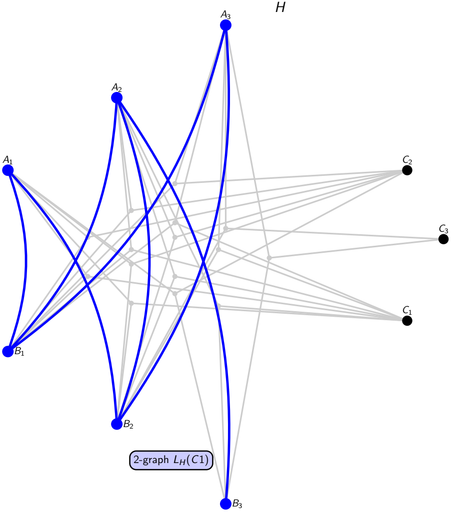

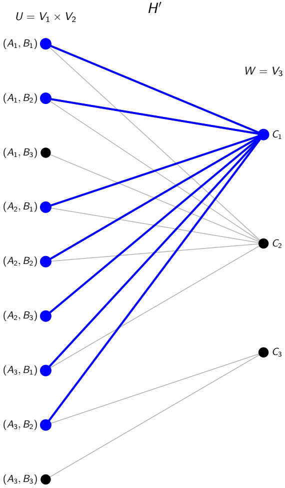

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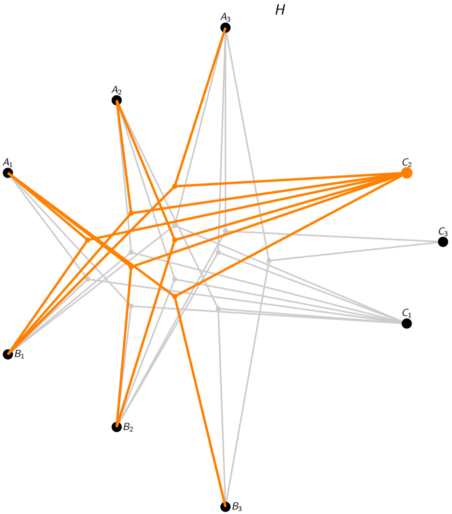

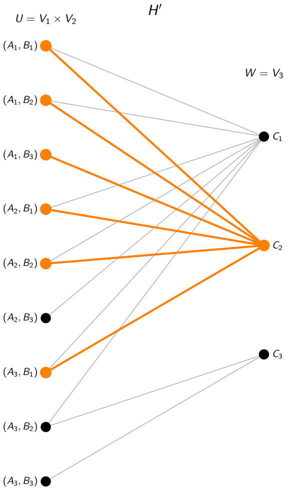

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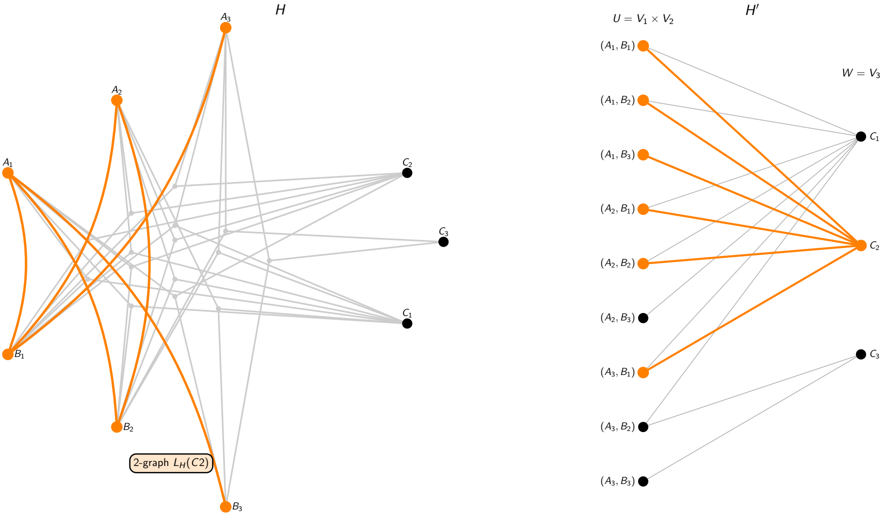

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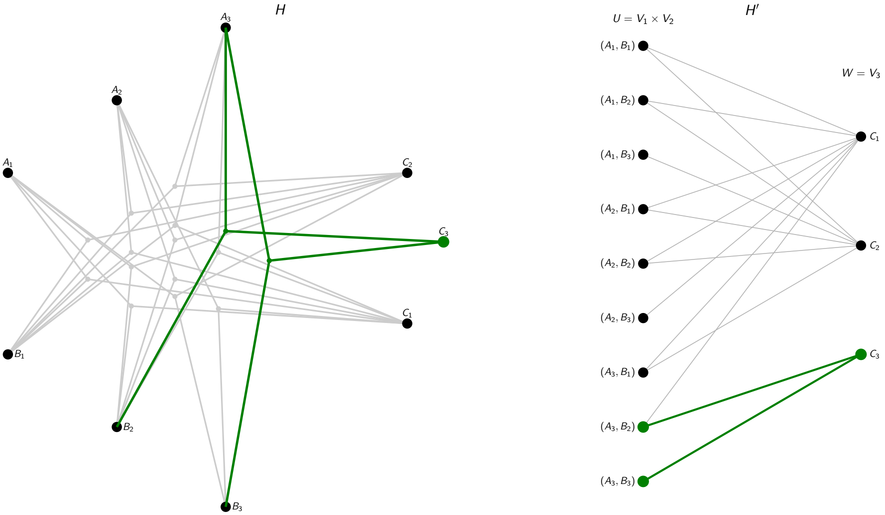

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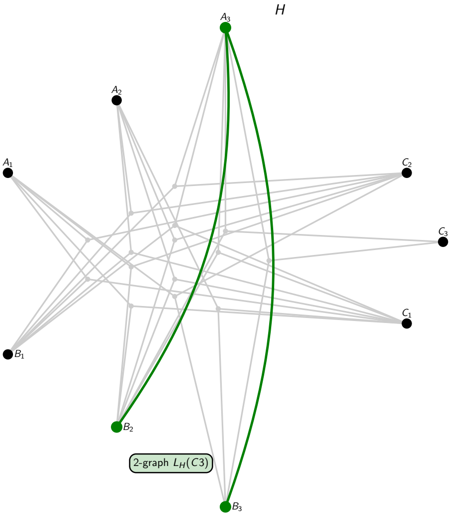

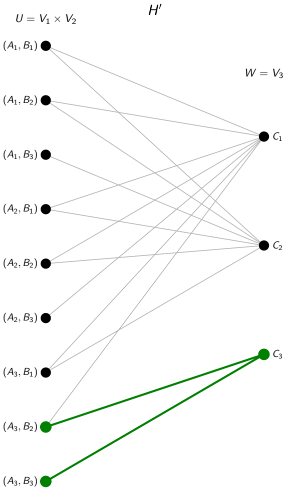

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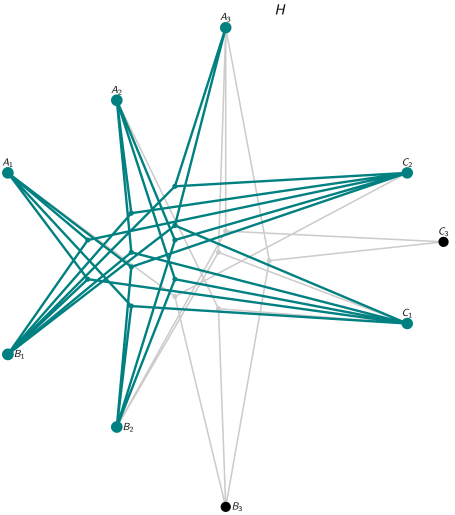

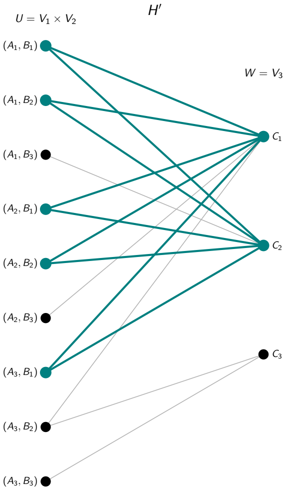

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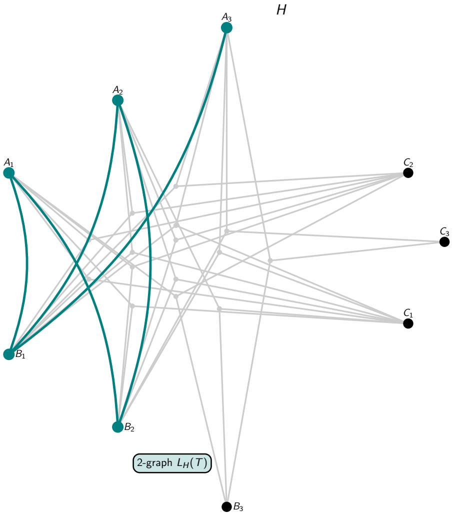

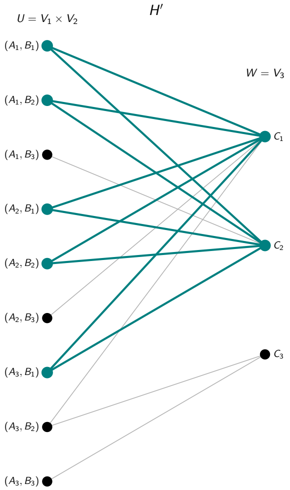

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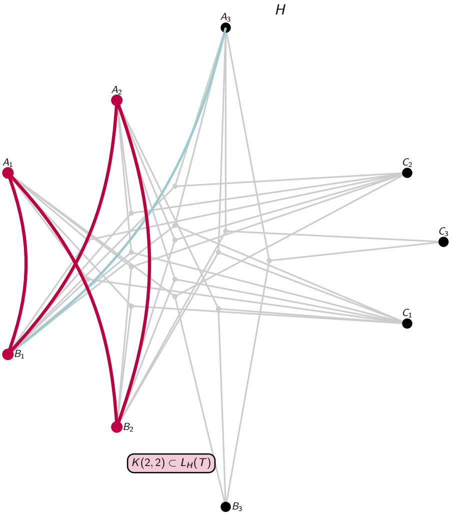

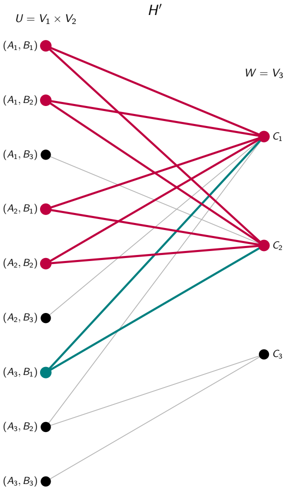

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