Finding Partite Hypergraphs Efficiently

Ferran Espuña Bertomeu

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Abstract

TODO

1 Introduction

Hypergraph Turán problems study how many edges a k-uniform hypergraph $H = (V, E)$ with n vertices can have without containing a specific subgraph G. The maximal such number is known as the *Turán number* $\mathrm{ex}(n, G)$. It is known [3] that $\mathrm{ex}(n, G) = o\left(\binom{n}{k}\right)$ if and only if G is k-partite, i.e., if its vertex set can be partitioned into k disjoint sets such that each edge contains exactly one vertex from each part. Kővári, Sós, and Turán [4] (for $k = 2$) and Erdős [2] (for any $k \geq 2$) established that

$$\mathrm{ex}(n, K(t, \overset{k}{\ldots}, t)) = \mathcal{O}\left(n^{k-\frac{1}{t^{(k-1)}}}\right), \tag{1}$$

where $K(t, \overset{k}{\ldots}, t)$ is the complete balanced k-partite k-graph with k parts of size t. Furthermore, if H is a k-graph with at least $d\binom{n}{k}$ edges for some constant $d > 0$, then it contains a $K(t, \overset{k}{\ldots}, t)$ with $t = c_d \log(n)^{1/(k-1)}$.

This result is non-constructive, meaning it guarantees the existence of such a subgraph but does not provide an efficient way to find it. Note that a simple brute-force search for a $K(t, \overset{k}{\ldots}, t)$ would involve checking all $\binom{n}{kt}$ vertex subsets, which is superpolynomial in n for $t = \Theta((\log n)^{1/(k-1)})$. Mubayi and Turán [5] developed a polynomial-time algorithm for the case $k = 2$, which reaches the stated order of magnitude for the subgraph part size. This paper extends their approach to the general case of k-uniform hypergraphs, reaching analogous results for $k \geq 3$. More concretely, we prove the following.

Theorem 1. *There is a deterministic algorithm that, given a k-graph H with n vertices and $m = dn^k$ edges, finds a complete balanced k-partite subgraph $K(t, \overset{k}{\ldots}, t)$ in polynomial time, where*

$$t = t(n, d, k) = \ldots$$

This value of t matches the order of magnitude from existence proofs. In fact, a probabilistic argument shows that it is the best possible up to a constant factor.

2 The algorithm

We present a recursive algorithm, `FindPartite`, that finds a $K(t, \overset{k}{\ldots}, t)$ in a given k-graph H. The core idea is to reduce the uniformity of the problem from k to $k - 1$ in each recursive step. The algorithm takes a k-graph H with n vertices and m edges as input. It first defines the target part size t, a small set size w, and a threshold edge count s for the recursive call, based on the input graph's parameters:

$$\begin{aligned} t(n,d,k) &= \dots, \\ w(n,d,k) &= \dots, \text{ and} \\ s(n,d,k) &= \dots, \end{aligned}$$

where $d = \frac{m}{\binom{n}{k}}$ is the edge density of H. The main steps are:

1. **Base Case ($k = 1$):** The edge set of a 1-graph is just a collection of vertices. Return the set of all vertices that are "edges".
2. **Select High-Degree Vertices:** Choose a set $W \subset V$ of w vertices with the highest degrees in H.
3. **Find a Dense Link Graph:** Iterate through all t-subsets $T \subset W$. For each T, consider the set S of all $(k-1)$-subsets of V that form a hyperedge with *every* vertex in T.
4. **Recurse:** As we prove further along using the Kővári–Sós–Turán theorem, for at least one choice of T, the resulting set S will be large ($|S| \geq s$). We form a new $(k-1)$-graph $H' = (V, S)$ and make a recursive call: `FindPartite(`H'`, `$k-1$`)`.
5. **Construct Solution:** The recursive call returns $k-1$ parts $V_1, \dots, V_{k-1}$ of size at least t. By construction, every choice of vertices from these parts forms an edge in H' with every vertex of T. Thus, $(T_1, \dots, T_{k-1}, T)$ form the desired $K(t, \overset{k}{\dots}, t)$ in the original graph H.

The pseudocode is given in Algorithm 1.

Algorithm 1 Finding a balanced partite k-graph

```
function FindPartite($H, k$)
    if $k = 1$ then
        return $(\{x \colon \{x\} \in E(H)\})$
    end if
    $n \leftarrow |V(H)|$, $m \leftarrow |E(H)|$, $d \leftarrow \frac{m}{\binom{n}{k}}$
    $t \leftarrow t(n,d,k)$, $w \leftarrow w(n,d,k)$, $s \leftarrow s(n,d,k)$
    assert $t \geq 2$
    $W \leftarrow$ a set of $w$ vertices with highest degree in $H$
    for all $T \in \binom{W}{t}$ do
        $S \leftarrow \{\, y \in \binom{V}{k-1} \colon \forall x \in T, \{x\} \cup y \in E(H) \,\}$
        if $|S| \geq s$ then
            $H' \leftarrow (V, S)$                                ▷ $H'$ is a $(k-1)$-graph
            $(V_1, \dots, V_{k-1}) \leftarrow$ FindPartite($H', k-1$)
            return $(V_1, \dots, V_{k-1}, T)$
        end if
    end for
end function
```

3 Analysis

We now presnt the proof of correctness and polynomial runtime for our algorithm. We assume $t \geq 2$ for our estimates to be easier. If $t < 2$, we may just return the vertices of any single edge in H.

3.1 Correctness

We will prove the correctness of the algorithm in two steps. First, we will prove that in step 3 of the algorithm we indeed find a set $T \in \binom{W}{t}$ such that the associated set $S \subset \binom{V}{k-1}$ has size at least s. For this, consider the bipartite graph B with parts W and $\binom{V}{k-1}$ with edge set

$$\left\{(x, y) \in W \times \binom{V}{k-1} \middle| y \cup \{x\} \in E\right\}.$$

The existance of a set $T \subset W$ as desired is equivalent to there being $T \subset W$ of size t and a set $S \subset \binom{V}{k-1}$ of size s such that the induced bipartite subgraph is complete. For this, we use a version the Kővári–Sós–Turán theorem [4], which we state and prove here for completeness.

Lemma 2. *Let u, w, s, t be positive integers with $u \geq s$, $w \geq t$, and let B be a bipartite graph with parts W and U such that $|U| = u, |W| = w$. If B has more than*

$$(s-1)^{1/t}(w-t+1)u^{1-1/t} + (t-1)u$$

edges, then there are $T \subset W$ of size t and $S \subset U$ of size s such that the induced bipartite subgraph $B[T, S]$ is complete.

Proof. TODO □

Note that in our case, $u = \binom{n}{k-1}$. The edges of B correspond to the the edges containing each vertex in W, so there are

$$z = \sum_{y \in W} d_H(y) \geq k \cdot m \cdot \frac{w}{n} \geq \frac{k \cdot w \cdot d \cdot \binom{n}{k}}{n}$$

of them, where the first inequality follows from the fact that we have picked a set of w vertices with highest degree in H. . . . TODO

3.2 Complexity

TODO re-evaluate the complexity analysis.

4 Conclusion and Future Work

We have presented a deterministic, polynomial-time algorithm to find a large complete balanced k-partite subgraph in any sufficiently dense k-uniform hypergraph. This provides a constructive counterpart to a classical existence result by Erdős in extremal hypergraph theory.

Several avenues for future research remain open.

- **General Blow-ups:** Our algorithm finds a blow-up of a single edge, $K(t, \overset{k}{\ldots}, t)$. Can this framework be adapted to find a t_n-blowup of an arbitrary fixed k-graph G? Existence theorems guarantee such structures, but efficient algorithms are lacking.
- **Unbalanced Partite Graphs:** The algorithm could be modified to search for unbalanced complete partite graphs $K(t_1, \ldots, t_k)$, where the part sizes may grow at different rates.
- **Optimality:** The bounds on t are asymptotically tight, but the constants can likely be improved with a more refined analysis. For $k = 2$, it is known that in dense graphs one can find a $t = \Theta(\log n)$ blow-up of any bipartite graph. It is an open question if a constructive proof for this stronger result exists for $k \geq 2$.

References

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