

# Finding Partite Hypergraphs Efficiently

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# $k$ -Graphs

## Definition

A  $k$ -graph is a pair  $G = (V, E)$  where  $V$  is a finite set of *vertices* and  $E \subseteq \binom{V}{k}$  is a set of *edges*.

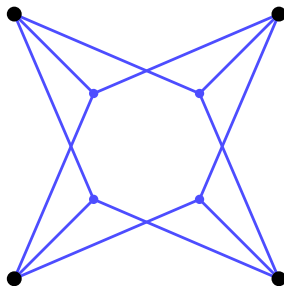


Figure: A complete 3-graph on 4 vertices:  $K_4^{(3)}$ .

# Partite $k$ -Graphs

## Definition

A  $k$ -graph  $G = (V, E)$  is  $r$ -partite if there exists a partition  $V = V_1 \cup \dots \cup V_r$  such that every edge of  $G$  intersects every part  $V_i$  in at most one vertex. We write  $G = (V_1, \dots, V_r; E)$ .

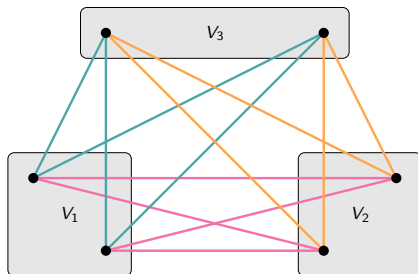


Figure: A complete 3-partite 2-graph:  $K^{(3)}(2, 2, 2)$ .

# Partite $k$ -Graphs

## Remark

We may identify  $E$  as a subset of  $\mathcal{C} = \bigcup_{\{i_1, \dots, i_k\} \in \binom{[r]}{k}} V_{i_1} \times \dots \times V_{i_k}$ .  
If  $E = \mathcal{C}$ , we say that  $G$  is a *complete  $r$ -partite  $k$ -graph*.

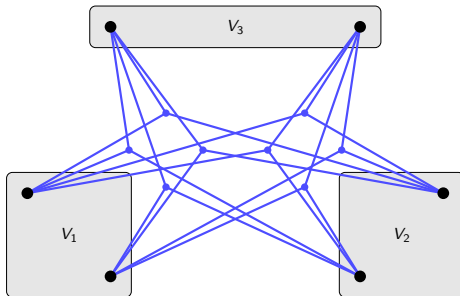


Figure: A complete 3-partite 3-graph:  $K^{(2)}(2, 2, 2)$ .

# Turán-Type Problems

## Definition

Let  $G = (V, E)$  be a  $k$ -graph and  $n \geq |V|$  an integer. The *Turán number*  $\text{ex}(G, n)$  is the maximum number of edges in a  $k$ -graph on  $n$  vertices that does not contain a copy of  $G$  as a subgraph.

Determining  $\text{ex}(G, n)$  or estimating it as  $n \rightarrow \infty$  is known as the *Turán problem* for  $G$ .

## Theorem

For all  $k$ -graphs  $G$  there exists a constant  $\alpha(G) \in [0, 1)$  such that

$$\text{ex}(G, n) = (\alpha(G) + o(1)) \cdot \binom{n}{k} \quad \text{as } n \rightarrow \infty.$$

Furthermore,  $\alpha(G) = 0$  if and only if  $G$  is  $k$ -partite.

# The Kővari–Sós–Turán Theorem

The bound  $\text{ex}(G, n) = o(n^k)$  can be improved by a lot.

## Definition

Let  $1 < t_1 \leq v_1, \dots, 1 < t_k \leq v_k$  be integers. Then the *generalized Zarankiewicz number*  $z(v_1, \dots, v_k; t_1, \dots, t_k)$  is the largest integer  $z$  for which there exists a  $k$ -partite  $k$ -graph  $H = (V_1, \dots, V_k, F)$  with part sizes  $|V_i| = v_i$  and  $|F| = z$  edges such that for all choices of  $W_i \subset V_i$  of sizes  $|W_i| = t_i$ ,  $W_1 \times \dots \times W_k \not\subset F$ .

## Theorem (Kővari–Sós–Turán)

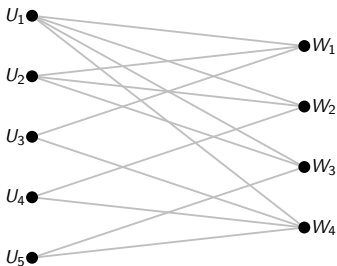
Let  $0 < s \leq u$  and  $0 < t \leq w$  be integers. Then

$$z(u, w; s, t) \leq (s-1)^{1/t}(w-t+1)u^{1-1/t} + (t-1)u$$

Standard arguments then show that  $\text{ex}(n, K(s, t)) = \mathcal{O}(n^{2-1/t})$ .

# Kővari–Sós–Turán: Proof Sketch and Example

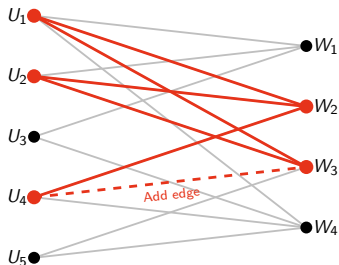
This graph has the maximum number of edges ( $|E| = 13$ ) to be  $K_{3,2}$ -free.



- **Hypothesis:**  $H = (U, W; E)$  is a  $K(s, t)$ -free bipartite  $k$ -graph with  $z = z(u, w; s, t)$  edges, where  $|U| = u$  and  $|W| = w$ .

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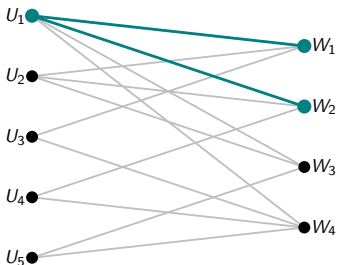
For example, adding the edge  $\{U_4, W_3\}$  creates a  $K_{3,2}$  on vertices  $\{U_1, U_2, U_4\}$  and  $\{W_2, W_3\}$ .

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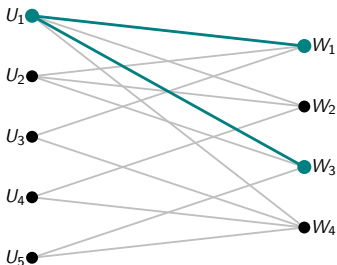


For  $x = U_1$ , we count its  $\binom{4}{2} = 6$  stars.

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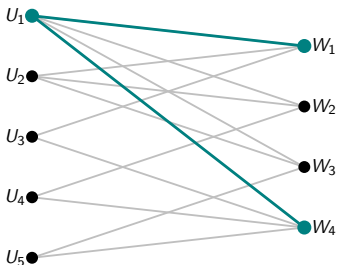


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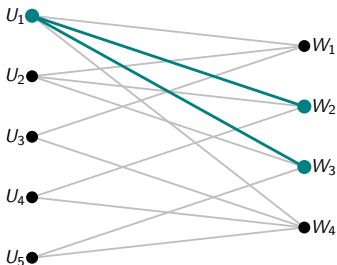


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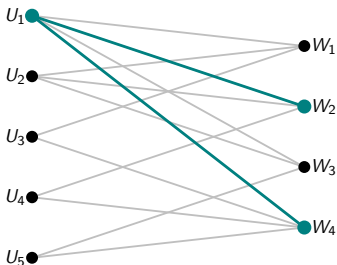


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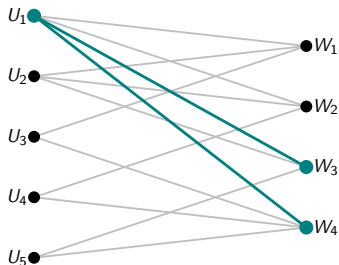


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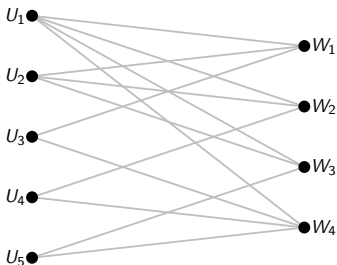


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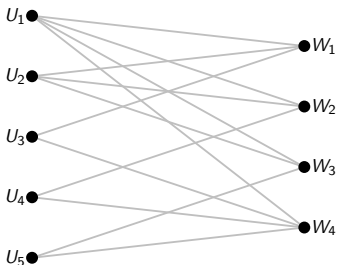


In the example, there are at least  $5 \binom{13/5}{2} = 10.4$  stars (there are actually 12)

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- **Averaging:** By a convexity argument, the number of stars is at least  $u \binom{z/u}{t}$ .

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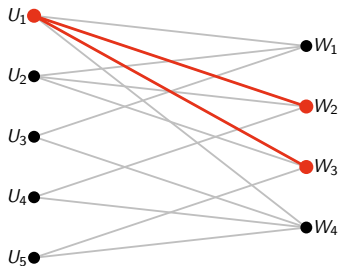
Each set  $T \subset W$  (in this case,  $T = \{W_2, W_3\}$ ) is in at most  $s - 1 = 3 - 1 = 2$  stars. In total, at most  $2\binom{4}{2} = 12$  stars.

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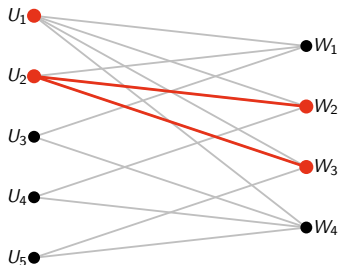


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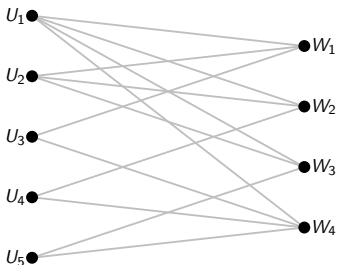


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## Kővari–Sós–Turán: Proof Sketch and Example

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In the example, we conclude that  $10.4 \leq 12$ , which is true. For bigger values of  $z$  this would fail, leading to contradiction and therefore upper bounding  $z$ . In fact,  $z=14$  already fails!

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- **Bounding:** Because  $H$  is  $K(s, t)$ -free, each set  $T \subset W$  is the right component of at most  $(s - 1)$  stars.
- **Conclusion:**  $u \binom{z/u}{t} \leq (s - 1) \binom{w}{t}$ , from which the theorem follows.

# Erdős's Bound for Hypergraphs (1964)

## Theorem (Erdős '64)

For integers  $k \geq 2, t \geq 2$ ,  $ex(n, K(t, \overset{k}{\cdot}, t)) = \mathcal{O}\left(n^{k - \frac{1}{t^{k-1}}}\right)$ .

This generalizes the Kővari–Sós–Turán theorem to  $k$ -graphs. It follows from a similar bound on the corresponding generalized Zarankiewicz number, obtained by induction.

Suppose that  $H = (V_1, \dots, V_k; F)$  is a  $k$ -graph with  $|W_i| = w$ . Let  $H$  have  $z$  edges and no copy of  $K(t, \overset{k}{\cdot}, t)$ .

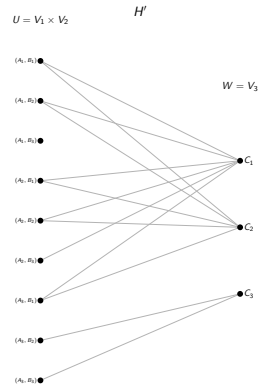
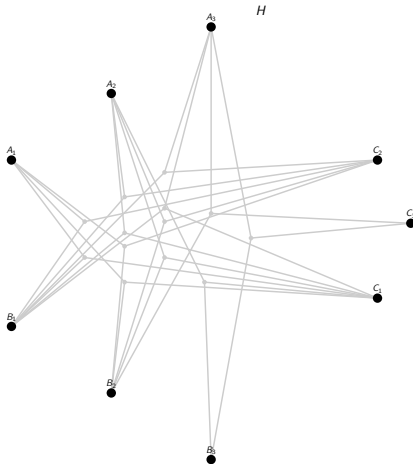
We set up a bipartite  $k$ -graph  $H' = (U, W; F')$  with

$$U = W_1 \times \cdots \times W_{k-1}$$

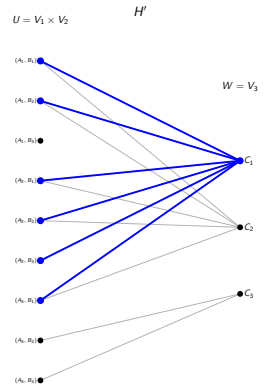
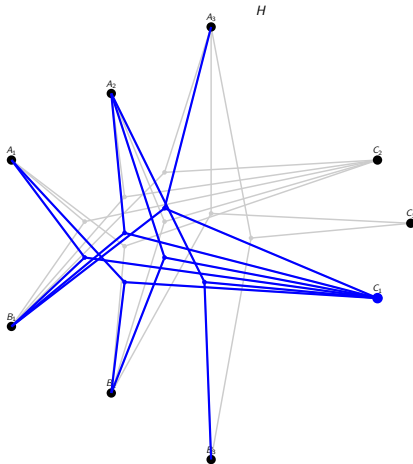
$$W = W_k$$

$$F' = \{(X, y) \in U \times W \mid X \cup \{y\} \in F\}.$$

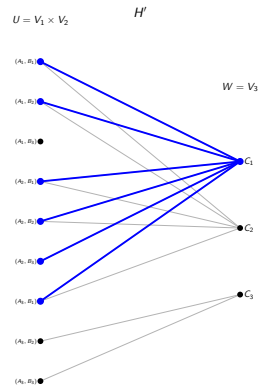
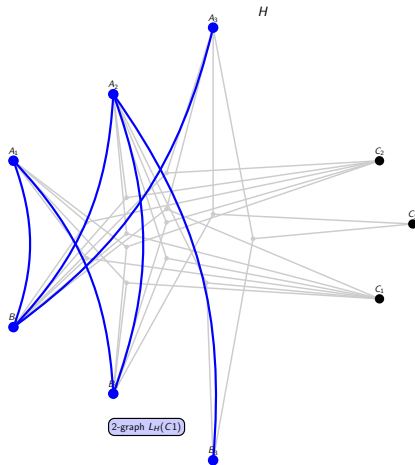
# Erdős's Bound: Proof Sketch ( $k = 3, t = 2$ )



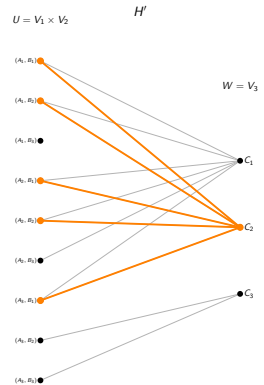
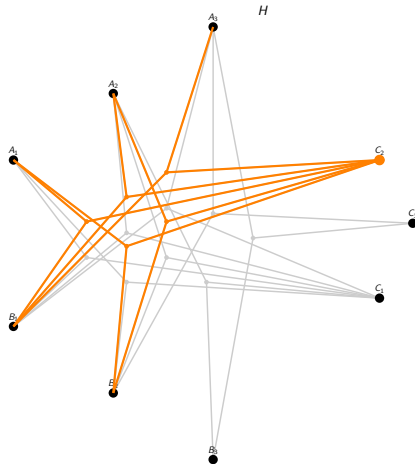
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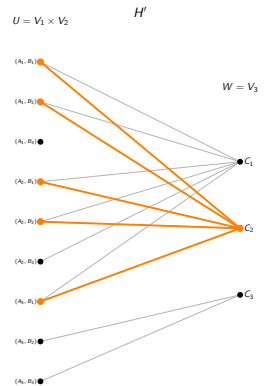
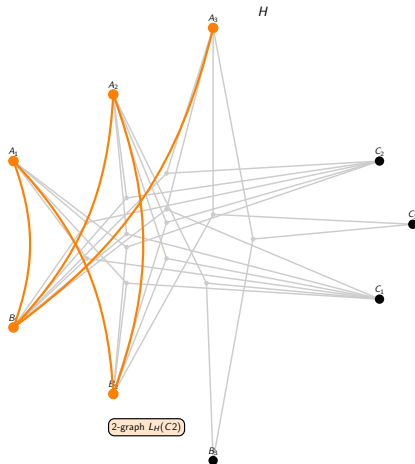


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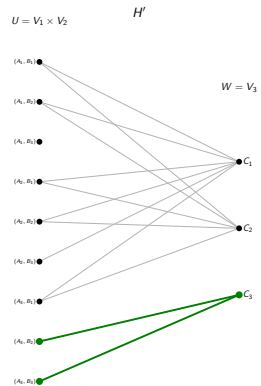
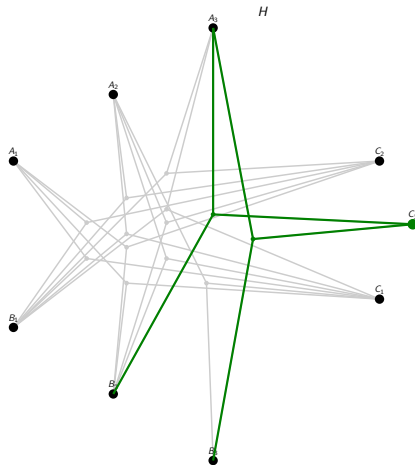




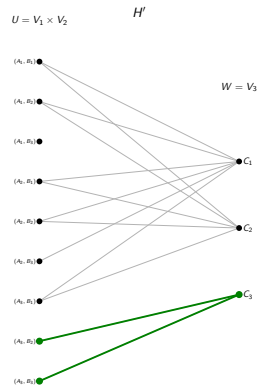
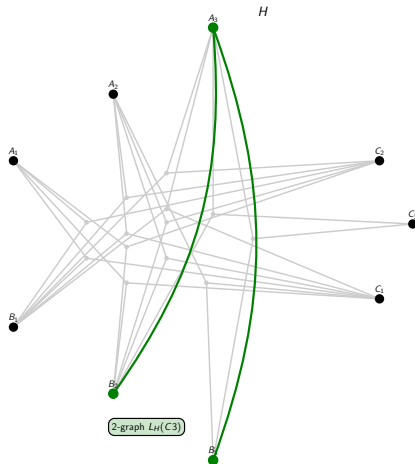
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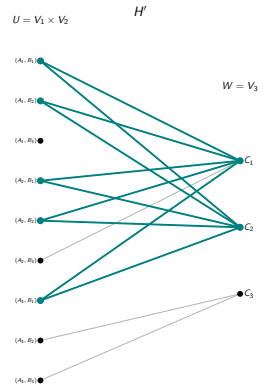
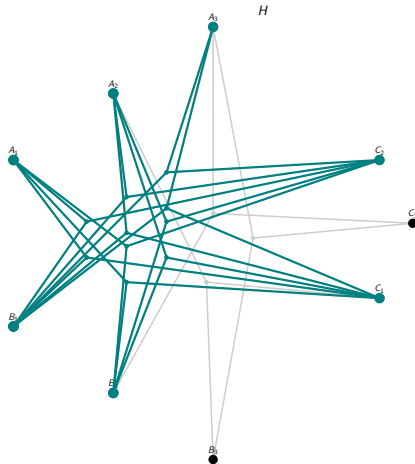
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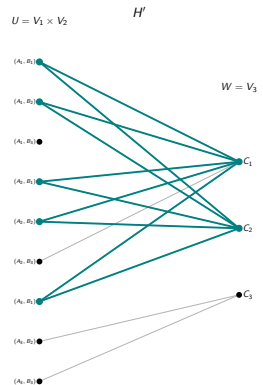
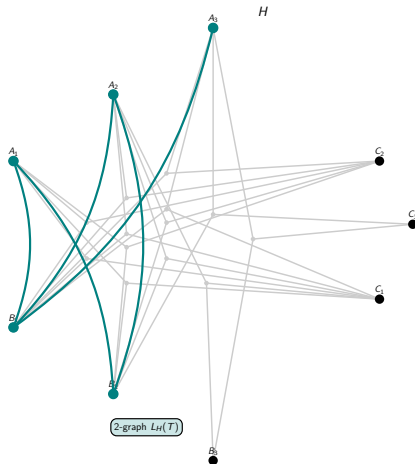
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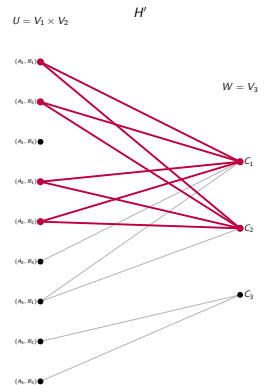
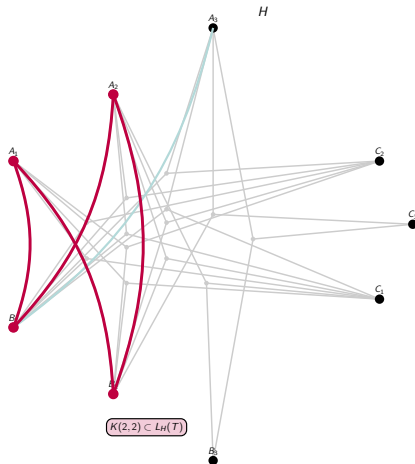
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