

FINDING PARTITE HYPERGRAPHS EFFICIENTLY

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ABSTRACT. We provide a deterministic polynomial-time algorithm that, for a given k -uniform hypergraph H with n vertices and edge density d , finds a complete k -partite subgraph of H with parts of size at least $c(d, k)(\log n)^{1/(k-1)}$. This generalizes work by Mubayi and Turán on bipartite graphs. The value we obtain for the part size matches the order of magnitude guaranteed by the non-constructive proof due to Erdős and is tight up to a constant factor.

1. INTRODUCTION

Hypergraph Turán problems concern how many edges a k -uniform hypergraph $H = (V, E)$ with n vertices can have without containing a specific subgraph G . The maximal such number is known as the *Turán number* $\text{ex}(n, G)$. It is known [5] that $\text{ex}(n, G)$ is sublinear in $\binom{n}{k}$ if and only if G is k -partite, i.e., if its vertex set can be partitioned into k disjoint sets such that each edge contains exactly one vertex from each part. Kővári, Sós, and Turán [6] (for $k = 2$) and Erdős [3] (for any $k \geq 2$) established that

$$\text{ex}(n, K(t, \cdot^k, t)) \leq \binom{n}{k} \cdot n^{-\frac{1}{t^{k-1}}},$$

where $K(t, \cdot^k, t)$ is the hypergraph consisting of k disjoint sets V_1, \dots, V_k of size t and all hyperedges of the form $\{x_1, \dots, x_k\}$ with $x_i \in V_i$ for $i = 1, \dots, k$. This result implies the following.

Remark 1.1. *If H is a k -uniform hypergraph with at least $d\binom{n}{k}$ edges for some constant $d > 0$, then it contains $K(t, \cdot^k, t)$ as a subgraph, with $t = c(d, k)(\log n)^{1/(k-1)}$.*

Furthermore, the order of magnitude of t is tight up to a constant factor: For some constant $\hat{c}(d, k) > 0$, there are k -uniform hypergraphs with n vertices and at least $d\binom{n}{k}$ edges that do not contain $K(\hat{t}, \cdot^k, \hat{t})$ as a subgraph, where $\hat{t} = \hat{c}(d, k)(\log n)^{1/(k-1)}$. This was already noted by Erdős [3] and can be proved via the random alteration method [1].

Due to the fundamental role of Erdős' result, it is natural to ask whether a fast search algorithm for a complete k -partite subgraph of the size stated in Remark 1.1 exists. A brute-force search for a $K(t, \cdot^k, t)$ would involve checking all $\binom{n}{kt}$ vertex subsets, which is super-polynomial in the number of vertices n for $t = \Omega((\log n)^{1/(k-1)})$. For $k = 2$, Mubayi and Turán [7] developed a deterministic polynomial-time algorithm which reaches the stated order of magnitude for the subgraph part size. This work extends their approach to the general case of k -uniform hypergraphs, reaching analogous results for $k \geq 3$. More concretely, we prove the following.

Theorem 1.2. *There is a deterministic algorithm that, given a k -uniform hypergraph H with n vertices and $m = d\binom{n}{k}$ edges, finds a complete balanced k -partite subgraph $K(t, \cdot^k, t)$ in polynomial time, where*

$$t = t(n, d, k) = \left\lfloor \left(\frac{\log n}{\log(16/d)} \right)^{\frac{1}{k-1}} \right\rfloor.$$

This value of t matches the order of magnitude from Remark 1.1.

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2. THE ALGORITHM

We present a recursive algorithm, **FindPartite**, that finds a $K(t, \dots, t)$ in a given k -uniform hypergraph H . The core idea is to reduce the uniformity of the host hypergraph H from k to $k - 1$ in each recursive step. The algorithm takes a k -uniform hypergraph H with n vertices and m edges as input. It first defines the target part size t , a small set size w , and a threshold edge count s for the recursive call, based on the input hypergraph's parameters:

$$\begin{aligned} t &= t(n, d, k) = \left\lfloor \left(\frac{\log n}{\log(16/d)} \right)^{\frac{1}{k-1}} \right\rfloor, \\ w &= w(n, d, k) = \left\lceil \frac{4t}{d} \right\rceil, \text{ and} \\ s &= s(n, d, k) = \left\lceil \left(\frac{d}{4} \right)^t \binom{n}{k-1} \right\rceil, \end{aligned}$$

where $d = \frac{m}{\binom{n}{k}}$ is the edge density of H . The main steps are:

- (1) **Base Case** ($k = 1$): The edge set of a 1-uniform hypergraph is just a collection of singleton sets of vertices. Return the set of all vertices that are “edges”.
- (2) **Select High-Degree Vertices**: Choose a set $W \subset V$ of w vertices with the highest degrees in H .
- (3) **Find a Dense Link Graph**: Iterate through all subsets $T \subset W$ of size t . For each T , consider the set S of all subsets $y \subset V$ of size $k - 1$ that form a hyperedge with *every* vertex in T .
- (4) **Recurse**: As we prove further along using the Kővári–Sós–Turán theorem, for at least one choice of T , the resulting set S is large ($|S| \geq s$). We form a new $(k - 1)$ -uniform hypergraph $H' = (V, S)$ and make a recursive call: **FindPartite**(H' , $k - 1$).
- (5) **Construct Solution**: The recursive call returns $k - 1$ parts V_1, \dots, V_{k-1} of size at least t . Every choice of one vertex from each of these parts forms an edge in H' . By construction, they form an edge of H with every vertex in T . Thus, V_1, \dots, V_{k-1}, T span the desired $K(t, \dots, t)$ subgraph in the original k -uniform hypergraph.

The pseudocode is given in Algorithm 1. In section 3, we formally prove that this algorithm is correct, in the sense that it returns a tuple (V_1, \dots, V_k) of disjoint sets $V_i \subset V(H)$ of size at least t spanning a complete k -partite subgraph in H . In Section 4, we show that it runs in polynomial time in the number of vertices n of the input hypergraph H . This will conclude the proof of Theorem 1.2.

Algorithm 1 Finding a balanced k -partite k -uniform subgraph

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1: function FINDPARTITE( $H, k$ )
2:   if  $k = 1$  then
3:     return ( $\{x: \{x\} \in E(H)\}$ )
4:   end if
5:    $n \leftarrow |V(H)|$ ,  $m \leftarrow |E(H)|$ ,  $d \leftarrow \frac{m}{\binom{n}{k}}$ 
6:    $t \leftarrow t(n, d, k)$ ,  $w \leftarrow w(n, d, k)$ ,  $s \leftarrow s(n, d, k)$ 
7:   assert ( $t \geq 2$ )
8:    $W \leftarrow$  a set of  $w$  vertices with highest degree in  $H$ 
9:   for all  $T \in \binom{W}{t}$  do
10:     $S \leftarrow \{y \in \binom{V}{k-1}: \forall x \in T, \{x\} \cup y \in E(H)\}$ 
11:    if  $|S| \geq s$  then
12:       $H' \leftarrow (V, S)$   $\triangleright H'$  is a  $(k - 1)$ -uniform hypergraph
13:       $(V_1, \dots, V_{k-1}) \leftarrow \text{FINDPARTITE}(H', k - 1)$ 
14:      return ( $V_1, \dots, V_{k-1}, T$ )
15:    end if
16:  end for
17: end function

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3. PROOF OF CORRECTNESS

We now present the proof that all the steps in Algorithm 1 are well-defined and that it returns a tuple (V_1, \dots, V_k) of disjoint sets $V_i \subset V(H)$ of size at least t spanning a complete k -partite subgraph in H . We assume $t \geq 2$ for our estimates to be easier. If $t < 2$, we may just return the vertices of any single edge in H .

It is not immediately clear that the set W defined in step 2 of the algorithm is well-defined, as for this it is necessary that $w \leq n$. To show this, we first observe that our assumption $t \geq 2$ implies that $1 \geq d \geq \frac{16}{\sqrt{n}}$. Suppose, by way of contradiction, that $w > n$. Then, we have

$$n \leq w - 1 \leq \frac{4t}{d} \leq \frac{4 \log n}{d \log(16/d)} \leq \frac{4\sqrt{n} \log n}{16 \log(16/d)}.$$

Taking the logarithms to be in base e , we note that $\log x \leq \sqrt{x}$ for all positive x , and that $\log(16/d) > 1$. Therefore, we get $n < \frac{n}{4}$, which is a contradiction.

Next, we prove that in step 3 of the algorithm we indeed find a set $T \in \binom{W}{t}$ such that the associated set $S \subset \binom{V}{k-1}$ has size at least s . That is, Algorithm 1 reaches line 13 at some point in the for loop. For this, consider the bipartite graph B with parts $\binom{V}{k-1}$ and W with edge set

$$\left\{ (x, y) \in \binom{V}{k-1} \times W \mid x \cup \{y\} \in E \right\}.$$

The edges of B correspond to the edges containing each vertex in W , so there are

$$z = \sum_{y \in W} d_H(y) \geq k \cdot m \cdot \frac{w}{n} = \frac{k \cdot w \cdot d \cdot \binom{n}{k}}{n} = w \cdot d \cdot \binom{n-1}{k-1}$$

of them, where the inequality follows from the fact that we have picked a set of w vertices with highest degree in H . The existence of a set $T \subset W$ as desired is equivalent to there being $T \subset W$ of size t and a set $S \subset \binom{V}{k-1}$ of size s such that the induced bipartite subgraph $B[S, T]$ is complete. To prove that this is the case, we use the following version [4] of the Kővári–Sós–Turán theorem [6].

Lemma 3.1. *Let u, w, s, t be positive integers with $u \geq s$, $w \geq t$, and let B be a bipartite graph with parts W and U such that $|U| = u, |W| = w$. If B has more than*

$$(s-1)^{1/t}(w-t+1)u^{1-1/t} + (t-1)u$$

edges, then there are $T \subset W$ of size t and $S \subset U$ of size s such that the induced bipartite subgraph $B[S, T]$ is complete.

We apply the lemma with $u = \binom{n}{k-1}$. It is clear from the definitions that our parameters satisfy the requirements $u \geq s$ and $w \geq t$. Suppose, by way of contradiction, that

$$w \cdot d \cdot \binom{n-1}{k-1} \leq z \leq (s-1)^{1/t}(w-t+1) \binom{n}{k-1}^{1-1/t} + (t-1) \binom{n}{k-1}.$$

Dividing by $\binom{n}{k-1}$ then shows that

$$\frac{1}{2} \cdot w \cdot d \leq \left(1 - \frac{k-1}{n}\right) \cdot w \cdot d = w \cdot d \cdot \frac{\binom{n-1}{k-1}}{\binom{n}{k-1}} < w \left(\frac{s-1}{\binom{n}{k-1}}\right)^{1/t} + (t-1),$$

where the first inequality follows from $n \geq 16^{2^{k-1}} > 2(k-1)$, which follows from $t \geq 2$ and $d \leq 1$. Finally, since $t \leq \frac{w \cdot d}{4}$ by the definition of w , we obtain

$$\left(\frac{d}{4}\right)^t \binom{n}{k-1} < s-1,$$

in contradiction to the definition of s . So far, we have shown that the sets defined in Algorithm 1 are well-defined and that the recursive call in line 13 is reached. We are now ready to prove that the algorithm returns a $K(t, \dots, t)$ by examining what happens in the recursive call. More precisely, we show the following.

Theorem 3.2. *For $k \geq 2$, if $t \geq 2$, Algorithm 1 returns a tuple (V_1, \dots, V_k) of disjoint sets $V_i \subset V(H)$ such that $|V_i| \geq t$ and $H[V_1, \dots, V_k]$ is complete.*

Proof. We proceed by induction on k . For $k = 2$, the recursive call returns the common neighborhood V_1 of the vertices in T , which is obviously disjoint from T , so it only remains to check that $|V_1| \geq t$. Now, since by construction $|V_1| = |S| \geq s$, it is enough that

$$s = \left\lceil \left(\frac{d}{4}\right)^t \cdot n \right\rceil \geq \left(\frac{d}{4}\right)^{\frac{\log n}{\log(16/d)}} \cdot n = \left(4 \cdot \frac{d}{16}\right)^{\frac{\log n}{\log(16/d)}} \cdot n = 4^{\frac{\log n}{\log(16/d)}} \cdot \frac{1}{n} \cdot n \geq 4^t > t.$$

For $k \geq 3$, we assume the inductive hypothesis holds for $k - 1$. Let d' be the edge density of H' (which is the obtained $(k - 1)$ -uniform hypergraph) and t' be $t(n, d', k - 1)$. As long as $t' \geq 2$, the recursive call returns a tuple (V_1, \dots, V_{k-1}) of disjoint sets $V_i \subset V(H)$ such that $|V_i| \geq t'$ and $H'[V_1, \dots, V_{k-1}]$ is complete.

We claim that $t' \geq t$. This implies that $t' \geq 2$ so we get to apply the inductive hypothesis to H' . Furthermore, the sets V_i that we obtain when applying the algorithm to H' are of size at least $t' \geq t$. By construction of S , all the edges in H' are disjoint from T , hence the sets V_i are also disjoint from T . This means that the sets V_1, \dots, V_{k-1}, T are disjoint. In addition, for all $(x_1, \dots, x_{k-1}, y) \in V_1 \times \dots \times V_{k-1} \times T$, we have that $\{x_1, \dots, x_{k-1}\} \in S$ so $\{x_1, \dots, x_{k-1}, y\} \in E(H)$. Equivalently, $H[V_1, \dots, V_{k-1}, T]$ is complete, finishing the proof.

Let us now prove the claim that $t' \geq t$. By the definition of s , we have $d' \geq \left(\frac{d}{4}\right)^t$. Therefore,

$$t' \geq \left\lceil \left(\frac{\log n}{\log \left(\frac{16}{(d/4)^t} \right)} \right)^{\frac{1}{k-2}} \right\rceil = \left\lfloor \left(\frac{\log n}{\log 16 + t \log(4/d)} \right)^{\frac{1}{k-2}} \right\rfloor.$$

Then, we substitute the definition of t , where removing the floor function maintains the inequality because the right side is decreasing in t :

$$t' \geq \left\lceil \left(\frac{\log n}{\log 16 + \left(\frac{\log n}{\log(16/d)} \right)^{\frac{1}{k-1}} \log(4/d)} \right)^{\frac{1}{k-2}} \right\rceil = \left\lfloor \left(\frac{(\log n)^{(1-\frac{1}{k-1})}}{\frac{\log 16}{(\log n)^{\frac{1}{k-1}}} + \frac{\log(4/d)}{\log(16/d)^{\frac{1}{k-1}}}} \right)^{\frac{1}{k-2}} \right\rfloor.$$

We claim that we can bound the denominator by showing that

$$(1) \quad \frac{\log 16}{(\log n)^{\frac{1}{k-1}}} + \frac{\log(4/d)}{\log(16/d)^{\frac{1}{k-1}}} \leq (\log(16/d))^{(1-\frac{1}{k-1})}.$$

Then, the expression simplifies to

$$t' \geq \left\lceil \left(\frac{(\log n)^{(1-\frac{1}{k-1})}}{(\log(16/d))^{(1-\frac{1}{k-1})}} \right)^{\frac{1}{k-2}} \right\rceil = \left\lfloor \left(\frac{\log n}{\log(16/d)} \right)^{\frac{1}{k-2}(1-\frac{1}{k-1})} \right\rfloor = \left\lfloor \left(\frac{\log n}{\log(16/d)} \right)^{\frac{1}{k-1}} \right\rfloor = t,$$

as desired. Let us prove Inequality (1). Suppose, by way of contradiction, that it does not hold. We can rewrite

$$(\log(16/d))^{(1-\frac{1}{k-1})} = \frac{\log(16/d)}{\log(16/d)^{\frac{1}{k-1}}}$$

and rearrange the inequality to obtain

$$\frac{\log 16}{(\log n)^{\frac{1}{k-1}}} > \frac{\log(16/d) - \log(4/d)}{\log(16/d)^{\frac{1}{k-1}}} = \frac{\log 4}{(\log(16/d))^{\frac{1}{k-1}}}.$$

This implies that

$$t \leq \left(\frac{\log n}{\log(16/d)} \right)^{\frac{1}{k-1}} < \frac{\log 16}{\log 4} = 2,$$

which contradicts the assumption that $t \geq 2$. \square

4. PROOF OF POLYNOMIAL COMPLEXITY

We now analyze the computational complexity of Algorithm 1 to show that it runs in time polynomial in n for any fixed uniformity k . We consider the input hypergraph H to be given as an adjacency k -dimensional tensor. We make the assumption that we have a machine with arbitrarily large random-access memory, and that the operations on integers and floating-point numbers take constant time. Not making these assumptions would only add logarithmic factors to the running time, which we ignore for simplicity.

Let $T_k(n)$ denote the worst-case running time of the function `FindPartite` when called on a k -uniform hypergraph with n vertices. The algorithm's structure gives a recurrence relation for $T_k(n)$. We first analyze the cost of the operations within a single call for a fixed k , excluding the recursive step.

Assuming that the input tensor consists of a flattened boolean array of size n^k , querying whether a set of k vertices forms an edge in H can be done in constant time. Therefore, calculating the number of edges m can be done in $\mathcal{O}(n^k)$ time by iterating through all $\binom{n}{k}$ possible edges. The parameters n, m, d, t, w, s are computed in constant time after this step.

The set W of w vertices with highest degrees can be constructed in time $\mathcal{O}(n^k)$, for instance, by creating an array with the degree of each vertex (in time $\mathcal{O}(n \cdot n^{k-1}) = \mathcal{O}(n^k)$), sorting it in descending order (in time $\mathcal{O}(n \log n)$), and taking the first w elements.

The subsets of t elements of W can be iterated through in time $\mathcal{O}(\binom{w}{t})$ [8]. Similarly to the argument by Mubayi and Turán [7], we can bound this quantity by a polynomial in n . Indeed,

$$\binom{w}{t} \leq \left(\frac{ew}{t}\right)^t \leq \left(\frac{e(4t/d+1)}{t}\right)^t \leq \left(\frac{4e}{d} + \frac{e}{2}\right)^t \leq \left(\frac{4.5 \cdot e}{d}\right)^t < e^{t(2.6+\log(1/d))} = e^{2.6 \cdot t} e^{\log(1/d) \cdot t}.$$

For the first term, we use the fact that $t < (\log n)^{1/(k-1)} \leq \log n$ and so $e^{2.6 \cdot t} < e^{2.6 \cdot \log n} = n^{2.6}$. For the second term, we use that $t < ((\log n)/\log(1/d))^{1/(k-1)} \leq (\log n)/\log(1/d)$, so $e^{\log(1/d) \cdot t} < n$. All in all, we have

$$\binom{w}{t} < n^{2.6} n = n^{3.6}.$$

In each iteration of the loop, the set S can be constructed in the following way. We initialize a flattened boolean array A of size n^{k-1} , with all entries set to `true`. We iterate through all $x \in \binom{[n]}{k-1}$ and $y \in T$ and set the entry corresponding to x to `false` if $x \cup \{y\}$ is not an edge in H . All in all, this takes $\mathcal{O}(tn^{k-1}) = \mathcal{O}(n^k)$ steps, and then counting the number of `true` entries in the array takes $\mathcal{O}(n^{k-1})$ time. Therefore, the for loop (with the recursive call) takes $\mathcal{O}(\binom{w}{t} n^k) = \mathcal{O}(n^{k+3.6})$ time.

Finally, when the condition $|S| \geq s$ is satisfied, the recursive call to `FindPartite` is made. We can pass the array A to the recursive call directly, and the recursive call takes time $T_{k-1}(n)$. Putting everything together, we have the recurrence relation $T_k(n) = T_{k-1}(n) + \mathcal{O}(n^{k+3.6})$. This, together with the base case $T_1(n) = \mathcal{O}(n)$, gives us $T_k(n) = \mathcal{O}(n^{k+3.6})$.

5. CONCLUDING REMARKS AND FUTURE WORK

We have presented a deterministic, polynomial-time algorithm to find a large complete balanced k -partite subgraph in any sufficiently dense k -uniform hypergraph. This provides a constructive counterpart to a classical existence result by Erdős in extremal hypergraph theory.

Several avenues for future research remain open.

- **General Blow-ups:** Our algorithm finds a blow-up of a single edge, $K(t, \dots, t)$. Can this framework be adapted to find a t -blowup of an arbitrary fixed k -uniform hypergraph G when its Turán density is exceeded by a positive constant? For example, for $k = 2$, there are existence results for $t = \Omega(\log n)$ [2], but directly adapting Algorithm 1 would only yield $t = \mathcal{O}((\log n)^{1/(|V(G)|-1)})$.
- **Unbalanced k-Partite Graphs:** The algorithm could be modified to search for unbalanced complete partite graphs $K(t_1, \dots, t_k)$, where the part sizes may grow at different rates.
- **Optimality:** The bounds on t are asymptotically tight, but the constants can be improved significantly with a more refined analysis.

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