Problem 8: Let P be a ranked poset with the LYM property. A *regular covering of chains* of P is a family $\mathcal{C}$ of maximal chains such that any two elements of P with the same rank are in the same number of chains in $\mathcal{C}$. Prove that, for any real valued function $f : P \longrightarrow \mathbb{R}$ and every subset $X \subset P$,

$$\sum_{x\in X} \frac{f(x)}{|P_{r(x)}|} \leq \max_{c\in\mathcal{C}} \sum_{x\in X\cap c} f(x) \tag{1}$$

Deduce the following theorem of Erdős: If $\mathcal{A}$ is a family of subsets of $[n]$ such that the longest chain in $\mathcal{A}$ has length at most k, then

$$|\mathcal{A}| \leq \sum_{i=0}^{k-1} \binom{n}{\lfloor (n+i)/2 \rfloor} \tag{2}$$

[Hint: For the first inequality consider the function F: $\mathcal{C} \to \mathbb{R}$ definded as $F(c) = \sum_{x\in c} f(x)$ and use double counting. For the second inequality, chose $f(x) = |P_{r(x)}|$.]

Solution (by Ferran Espuña): Every element of P_k is in the same number of chains of $\mathcal{C}$ for all k. Conversely, every chain in $\mathcal{C}$ is maximal, so it contains one element of each P_k. Therefore, every element $x \in P_{r(x)}$ is contained in exactly $\frac{|\mathcal{C}|}{|P_{r(x)}|}$ chains of $\mathcal{C}$. Therefore,

$$\sum_{c\in\mathcal{C}} \sum_{x\in X\cap c} f(x) = \sum_{x\in X} \frac{|\mathcal{C}| f(x)}{|P_{r(x)}|} = |\mathcal{C}| \sum_{x\in X} \frac{f(x)}{|P_{r(x)}|} \tag{3}$$

However,

$$\sum_{c\in\mathcal{C}} \sum_{x\in X\cap c} f(x) \leq |\mathcal{C}| \max_{c\in\mathcal{C}} \sum_{x\in X\cap c} f(x) \tag{4}$$

Putting these two equations together, and dividing by $|\mathcal{C}|$, we get (1).

To get Erdős' theorem, we take P to be the poset of subsets of $[n]$ ordered by inclusion. To get a regular covering of P, we just take $\mathcal{C}$ to be the family of all maximal chains. This is a regular covering because every element of P_k is in exactly $k!(n-k)!$ chains. We will apply (1) to the set $X = \mathcal{A}$. We define $f(x) = |P_{r(x)}|$, so that the left hand side of (1) is just $\sum_{x\in X} 1 = |\mathcal{A}|$. Because $|P_j| = \binom{n}{j}$, and chains don't have elements of the same rank, the right hand side of (1) is the sum of at most k binomial coefficients $\binom{n}{j}$ for *different* j. Taking the k largest ones (and recalling that $\binom{n}{j} = \binom{n}{n-j}$), we get Erdős' theorem.

Remark 1. (2) is a generalization of Sperner's theorem, which we can recover by taking $k = 1$ so that $\mathcal{A}$ is an antichain.

Remark 2. The LYM property wasn't used in the proof, so (1) holds for any ranked poset with a regular covering. In fact, we can recover the LYM property by repeating the same argument with $f(x) \equiv 1$ and letting $\mathcal{A}$ be an antichain so that the right hand side of (1) is 1. let us state this formally:

Proposition 0.1. *Let P be a ranked poset. If there exists a regular covering of P, then P has the LYM property.*