Problem 7: A family $\mathcal{A}$ of m-subsets of [n] is a *k-covering* if every $i \in [n]$ is contained in at least k sets of $\mathcal{A}$. A k-covering $\mathcal{A}$ is *decomposable* if there exists a partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are both 1-coverings. Show that if each point is in at most $t \leq 2^{k-4}/m$ sets then $\mathcal{A}$ is decomposable.

Solution (by Ferran Espuña): We will work with a random partition of $\mathcal{A}$ into $\mathcal{A}_1$ and $\mathcal{A}_2$. That is, each set of $\mathcal{A}$ is independently assigned to $\mathcal{A}_1$ with probability 1/2, and to $\mathcal{A}_2$ with probability 1/2. For convenience, we will define

$$U_i \coloneqq \bigcup_{S \in \mathcal{A}_i} S$$

Our goal will be to prove that the event

$$B \coloneqq \{U_1 = U_2 = [n]\}$$

occurs with positive probability. For this, we will define the events

$$B_i \coloneqq \{i \notin U_1 \text{ or } i \notin U_2\}, \quad i \in [n]$$

and note that

$$B = \cap_{i\in[n]} \overline{B_i}$$

We will see that these events satisfy the conditions of the Lovász Local Lemma and we will be done. First of all, because each $i \in [n]$ is in $c_i \geq k$ sets of $\mathcal{A}$, the probability that all of them end up in the same set is

$$2^{-c_i} + 2^{-c_i} = 2^{1-c_i} \leq 2^{1-k} \eqqcolon p$$

where each summand corresponds to all sets ending up in $\mathcal{A}_1$ or $\mathcal{A}_2$. Next, note that if we define

$$D_i \coloneqq \{i \neq j \in [n] \mid \{i,j\} \subseteq S \text{ for some } S \in \mathcal{A}\}$$

then B_i is independent of all B_j with $j \notin D_i$. Let us now calculate a bound for all $|D_i|$. For this, we will double count the number of pairs in

$$P_i \coloneqq \{(S,j) \mid i \neq j \in [n], \{i,j\} \subset S \in \mathcal{A}\}$$

On the one hand, if we start counting on the first coordinate this is clearly at most $(m-1)t$. On the other hand, it is at least $|D_i|$ (one for each possible j). Therefore,

$$|D_i| \leq (m-1)t \eqqcolon d$$

All together,

$$ep(d+1) = e2^{1-k}((m-1)t+1) \leq e2^{1-k}mt \leq e2^{1-k}2^{k-4} = e/8 < 1$$

□