**Problem 1.** *Let $A$ be a ring. Prove:*

*(1.1) If $x \in A$ is nilpotent, then $1-x$ is invertible.*

**Solution.** *Proof.* Suppose that $x^n = 0$ for some $n \in \mathbb{N}$. Then,

$$(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n = 1$$

□

*(1.2) The nilradical $N(A) = \{x \in A \,|\, x \text{ nilpotent}\}$ is an ideal of $A$ .*

**Solution.** *Proof.* Let $x, y \in N(A)$, then $x^n = 0$ and $y^m = 0$ for some $n, m \in \mathbb{N}$. Then,

$$(x-y)^{n+m} = \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} x^{n+m-k} y^k = 0$$

since $k \geq n$ or $n+m-k \geq m$ for all $k$. Thus, $x - y \in N(A)$. We have that $N(A)$ is an additive subgroup of $A$. Now, let $a \in A$. Then,

$$(ax)^n = a^n x^n = 0a = 0$$

so $ax \in N(A)$. Thus, $N(A)$ is an ideal of $A$. □

*(1.3) $N(A)$ is contained in all prime ideals of $A$ .*

**Solution.** *Proof.* Let $I \subset A$ be a prime ideal. Let $x \in N(A)$. Then, $x^n = 0 \in I$ for some $n \in \mathbb{N}$. Let us show by induction on $n$ that this implies $x \in I$:

- If $n = 1$, then $x = 0 \in I$.
- Suppose that the statement is true for $n-1$: $x^{n-1} = 0 \Rightarrow x \in I$. If $0 = x^n = x \cdot x^{n-1}$, since $I$ is prime, $x \in I$ or $x^{n-1} \in I$. In the first case, we are done. The second case is just the inductive hypothesis.

□

*(1.4) $N(A)$ is the intersection of all prime ideals of $A$ .*

**Solution.**

**Claim.** *Given $x \notin N(A)$, let $\Sigma_x$ be the set of all ideals that do not contain any power of $x$. Then, $\Sigma_x$ has a maximal element.*

*Proof.* We will use Zorn's lemma. Let us check the conditions:

**Claim.** *$\Sigma_x$ is a partially ordered set with respect to inclusion.*

**Claim.** *$\Sigma_x$ is not empty.*

*Proof.* Since $x \notin N(A)$, $0 \in \Sigma_x$. □

**Claim.** *Every chain in $\Sigma_x$ has an upper bound.*

*Proof.* Let $\{I_\alpha\}_{\alpha \in S}$ be a chain in $\Sigma_x$. Then, $I = \bigcup_{\alpha \in S} I_\alpha$ is an ideal of $A$ (One can check that if $x, y \in I$, then $x, y \in I_\alpha$ for some $\alpha \in S$, and then check the axioms from there). Let $x^n \in I$ for some $n \in \mathbb{N}$. Then, $x^n \in I_\alpha$ for some $\alpha \in A$. Since $I_\alpha$ is an ideal, $x \in I_\alpha$. Thus, $I \in \Sigma_x$. □

Now that we have verified the conditions of Zorn's lemma, we can conclude that $\Sigma_x$ has a maximal element. □

**Claim.** *Let $x \notin N(A)$. Then the maximal element $K(x)$ of $\Sigma_x$ is prime.*

*Proof.* Let $a, b \in A$ such that $ab \in K(x)$. By way of contradiction, suppose that $a \notin K(x)$ and $b \notin K(x)$. Then, $x^n \in (a)$ and $x^m \in (b)$ for some $m, n \in \mathbb{N}$, but $x^{n+m} \notin (ab) = (a)(b)$. Contradiction. □

Finally, we can prove the statement.

**Claim.** *$N(A)$ is the intersection of all prime ideals of $A$.*

*Proof.* Let $J$ be the intersection of all prime ideals of $A$. By 1.3, we know that $N(A) \subset J$. We want to prove that if $x \notin N(A)$, then $x \notin J$. Indeed, $J \subset K(x)$ because $K(x)$ is prime and $x \notin K(x)$ because $K(x)$ does not contain any power of $x$. □

**Problem 2.** *Let $A$ be a ring. Let $a_i \in A$ and $f = a_0 + a_1T + \cdots + a_nT^n \in A[T]$ be a polynomial. Prove:*

*(2.1) $f$ is a unit in $A[T] \iff a_0$ is a unit in $A$ and $a_1, \ldots, a_n$ are nilpotent.*

**Solution.** *Proof.* We will show both implications separately.

($\Leftarrow$) Let $a_i^{n_i} = 0$ for all $1 \leq i \leq n$. Consider s $= \Sigma_{i=1}^n n_i$. Let h $= -a_1T - \cdots - a_nT^n$ be the negative of the polynomial without the constant term. Then,

$$h^s = \left(-\sum_{i=1}^{n} a_iT^i\right)^s = (-1)^s \sum_{j_1+\cdots+j_n=s} a_1^{j_1} \cdots a_n^{j_n} T^{j_1+2\cdot j_2 \cdots + n\cdot j_n}$$

By the pigeonhole principle, in each term of the sum, there is at least one $j_k \geq n_k$ so $a_k^{j_k} = 0$, and thus $h^s = 0$. Then, h is nilpotent, so is $a_0^{-1} \cdot h$ and, by 1.1, $1 - a_0^{-1} \cdot h$ is invertible. Multiplying by $a_0$, we get that $f = a_0 - h$ is also invertible.

($\Rightarrow$) Suppose, there exists $g = b_0 + b_1T + \cdots + b_mT^m \in A[T]$ such that $1 = fg = \sum_{i=0}^{n+m} s_iT^i$, where $s_i = \sum_{j=0}^{n} a_jb_{i-j}$ and $b_k = 0$ for $k > m$ or $k < 0$. We first note that $1 = s_0 = a_0b_0 \Rightarrow a_0, b_0 \in A^*$. Next, suppose $n > 0$ (Otherwise, there is nothing to show).

**Claim.** *Let $0 \leq k \leq m$. $a_n^{k+1} b_{m-k} = 0$*

*Proof.* By total induction on $k$:

- If $k = 0$, then $0 = s_{n+m} = a_n b_m = (a_n)^{0+1} b_{m-0}$.
- Suppose that the statement is true for $0, \dots, k-1$. Then, $s_{n+m-k} = 0$ as $n + m - k \geq n > 0$. Therefore:

$$0 = a_n^k s_{n+m-k} = \sum_{j=0}^{n} a_n^k a_j b_{n+m-k-j} = a_n^{k+1} b_{m-k} + \sum_{j=0}^{n-1} a_n^k a_j b_{n+m-k-j}$$

  Now, the terms in the sum are zero by the inductive hypothesis, as $j < n \Rightarrow -(n-k-j) < k \Rightarrow (n-k-j)+1 \leq k$. Therefore, $a_n^{k+1} b_{m-k} = 0$ as we wanted to show.

□

Now, by setting $k = m$, we get that $a_n^{m+1} b_0 = 0$. Since $b_0$ is a unit, $a_n^{m+1} = 0$ and $a_n$ is nilpotent. We are almost done if we realize the following:

**Claim.** *Let $p = c_0 + c_1 T + \cdots + c_l T^l \in A[T]$ be an invertible polynomial such that $c_l$ is nilpotent. Then, $q = c_0 + c_1 T + \cdots + c_{l-1} T^{l-1}$ is also invertible.*

*Proof.* Note that $c_l T^l$ is nilpotent and so is $c_l T^l p^{-1}$. Then, $1 - c_l T^l p^{-1}$ is invertible by 1.1. Finally, because $p$ is invertible, so is $q = p - c_l T^l = p(1 - c_l T^l p^{-1})$. □

We will prove that for $0 < k \leq n$, $a_0 + \cdots + a_k T^k$ is invertible and $a_k$ is nilpotent by (reverse) induction on $k$.

- $a_n$ has already been done.
- If $0 < k < n$, by hypothesis $a_{k+1}$ is nilpotent and $a_0 + \cdots + a_{k+1} T^{k+1}$ is invertible. Then, $a_0 + \cdots + a_k T^k = (a_0 + \cdots + a_{k+1} T^{k+1}) - a_{k+1} T^{k+1}$ is invertible by the claim. Therefore, $a_k$ is nilpotent.

($\Rightarrow$) (Faster Version) Let $f$ be a unit in $A[T]$.

**Claim.** *Let $\mathfrak{p} \in Spec(A)$. Then, $a_i \in \mathfrak{p}$ for all $i \in \{1, \dots n\}$.*

*Proof.* $\mathfrak{p} \in Spec(A)$ implies that $A/\mathfrak{p}$ is an integral domain. Consider the reduction $\pi : A[T] \to A/\mathfrak{p}[T]$ that takes each element $a \in A[T]$ to the class $\overline{a} \in A/\mathfrak{p}[T]$. Since $A/\mathfrak{p}$ is an integral domain,

$$0 = \deg \bar{1} = \deg \bar{f} \cdot \overline{f^{-1}} = \deg \bar{f} + \deg \overline{f^{-1}}$$

In particular, $\deg \bar{f} = 0$ so, for all $i \in \{1, \dots n\}$, $\overline{a_i} = \bar{0} \Rightarrow a_i \in \mathfrak{p}$. □

Since this holds for all $\mathfrak{p} \in Spec(A)$, we have that $\forall i \in \{1, \dots n\}, a_i \in \bigcap_{\mathfrak{p} \in Spec(A)} \mathfrak{p} = N(A)$ (by 1.4). Finally, suppose that $f^{-1} = b_0 + b_1 T + \cdots + b_m T^m$. Given that $ff^{-1} = 1$ we have that $a_0 b_0 = 1$, and thus $a_0$ is invertible.

□

*(2.2) $f$ is nilpotent $\iff$ $a_0, \dots, a_n$ are nilpotent.*

**Solution.** *Proof.* We will show both implications separately.

($\Leftarrow$) Just use the same argument as in the previous part of the exercise, but directly on $f$, not on $h = a_0 - f$.

($\Rightarrow$) If $f$ is nilpotent, then $1 - f$ is invertible by 1.1. Thus, by the previous part, $-a_1, \dots, -a_n$ are nilpotent. Because $-a_i$ is nilpotent, $a_i$ is nilpotent. We only have to prove that $a_0$ is nilpotent. By the other implication, $h$ is nilpotent. Then, by 1.2, $a_0 = f + h$ is nilpotent.

□

*(2.3) $f$ is a zero divisor $\iff$ there exists $a \in A$, $a \neq 0$ such that $af = 0$.*

**Solution.** *Proof.* The backward implication is trivial because of the inclusion $A \subset A[T]$. For the forward implication, suppose that $f$ is a zero divisor. Then, there exists $g \in A[T]$, $g \neq 0$ such that $fg = 0$. Let $g = b_0 + \cdots + b_m T^m$ be the minimum degree polynomial satisfying the condition $fg = 0$. Suppose that $m > 0$.

**Claim.** *$\exists i \in \{1, \dots, n\}$ s.t. $a_i g \neq 0$*

*Proof.* Suppose not. Then, $a_i g = 0$ for all $i \in \{1, \dots, n\}$. Then, $a_i b_m = 0$ for all $i \in \{1, \dots, n\}$. But then we have $b_m f = 0$ with $b_m \in A$, in contradiction with g being a polynomial with minimum degree. □

Take $i$ maximal such that $a_i g \neq 0$. Then, $0 = fg = (a_0 + \cdots + a_i T^i)(b_0 + \cdots + b_m T^m) + \sum_{j=i+1}^{n} T^j a_j g = (a_0 + \cdots + a_i T^i)(b_0 + \cdots + b_m T^m)$, and $a_i b_m = 0$. Thus, we have a polynomial $g' = a_i g \neq 0$ with degree $m - 1$ satisfying $f(a_i g) = a_i(fg) = 0$, in contradiction with $g$ being of minimal degree. □

**Problem 3.** *Let $A$ be a ring. We define the Jacobson radical $\mathcal{J}(A)$ as the intersection of all maximal ideals of $A$. Prove:*

*(3.1) $x \in \mathcal{J}(A) \Leftrightarrow 1 - xy$ is invertible for all $y \in A$.*

**Solution.** *Proof.* We will prove the two implications separately:

($\Leftarrow$) Suppose $x \notin \mathcal{J}(A)$. This means that there exists $\mathfrak{m} \in Max(A)$ such that $x \notin \mathfrak{m}$. Since $\mathfrak{m}$ is a maximal ideal, $(x) + \mathfrak{m} = A$. Then, there exist $y' \in A$ and $m' \in \mathfrak{m}$ such that $xy' + m' = 1 \in A$. But this means that $1 - xy' = m' \in \mathfrak{m}$, which invertible. Contradiction with $\mathfrak{m}$ being an ideal. Thus, $x\mathcal{J}(A)$.

($\Rightarrow$) Suppose that $\exists y \in A$ such that $1 - xy$ is not a unit in $A$. Notice that:

- There exists a maximal ideal $\mathfrak{M}$ such that $1 - xy \in \mathfrak{m}$.
- Since $x \in \mathfrak{m} \subseteq \mathcal{J}(A)$, we have that $xy \in \mathfrak{m}$.

Finally, $1 - xy + xy = 1 \in \mathfrak{m}$, in contradiction with $\mathfrak{m}$ being an ideal. Thus, $\forall y \in A$ we have that $1 - xy$ is a unit in $A$.

□

*(3.2)* $\mathcal{J}(A) = A \setminus A*$ $\Leftrightarrow$ *A has only one maximal ideal.*

**Solution.** *Proof.* This comes directly from the fact that an ideal $I \subset A$ cannot contain a unit $x$, else $xx^{-1} = 1 \in I$ and thus $I = A$.

$$\mathcal{J}(A) = A \setminus A^* \Leftrightarrow \forall \mathfrak{m} \in Max(A), \mathfrak{m} \supseteq \{\text{non-units of } A\} = A \setminus A^* \Leftrightarrow$$
$$\Leftrightarrow \forall \mathfrak{m} \in Max(A), \mathfrak{m} = \{\text{non-units of } A\} \Leftrightarrow (A, \mathfrak{m})\text{local}$$

□

*(3.3)* $\mathcal{J}(A[T]) = \mathcal{N}(A[T])$.

**Solution.** *Proof.* We will prove the two inclusions separately:

($\supseteq$)

$$\mathcal{N}(A[T]) = \bigcap Spec(A[T]) \subseteq \bigcap Max(A[T]) = \mathcal{J}(A[T])$$

The first equality follows 1.4, while the inclusion is a consequence of $Max(A[T]) \subseteq Spec(A[T])$.

($\subseteq$) Take $f = a_0 + a_1T + \cdots + a_nT^n \in \mathcal{J}(A[T])$. Using 3.1 with $y = T$ we know that $1 - fT = 1 - a_0T - a_1T^2 \cdots - a_nT^{n+1}$ is invertible in $A[t]$. But then, from 2.1 we get that $-a_0, \cdots, -a_n$ nilponents, and finally $a_0, \cdots a_n$ are nilponents. We conclude from 2.2 we get that $f$ is nilponent in $A[T]$

¿should we write a proof of this or is it overkill?

□

**Problem 4.** *Let A be a ring such that every ideal not contained in the nilradical $\mathcal{N}(A)$ has a non-zero idempotent element (i.e. an element $e \neq 0$ such that $e^2 = e$). Prove that the nilradical and the Jacobson radical $\mathcal{J}(A)$ are equal.*

**Solution.** *Proof.* We will prove the two inclusions separately:

($\supseteq$) (Same as in 3.3)

$$\mathcal{N}(A) = \bigcap Spec(A) \subseteq \bigcap Max(A) = \mathcal{J}(A)$$

The first equality follows 1.4, while the inclusion is a consequence of $Max(A) \subseteq Spec(A)$.

($\subseteq$) Suppose that $\mathcal{J}(A) \nsubseteq \mathcal{N}(A)$. Then, since $\mathcal{J}(A)$ is an intersection of ideals and thus an ideal itself, we have that $\exists e \in \mathcal{J}(A)$ such that $e^2 = e \neq 0$. By 3.1, we have that $1 - e1$ is invertible, so

$$(1 + e) = (1 - e)^{-1}(1 - e)(1 + e) = (1 - e)^{-1}(1 - e^2) = (1 - e)^{-1}(1 - e) = 1$$

Finally, we get that $e = 0$, in contradiction with the definition of $e$. Thus, we conclude that $\mathcal{J}(A) \subseteq \mathcal{N}(A)$.

$\square$

**Problem 5.** *Let $A$ be an integral domain. Prove that the ideal $I = (x^2 - y^3, y^2 - z^3) \subseteq A[x, y, z]$ is prime.*

*(Hint: Let $f : R \to B$ be a ring homomorphism and $I \subseteq R$ be an ideal such that $I \subseteq \ker f$. Then $f$ factorizes through $R/I$. Consider a parametrization $f : A[x, y, z] \to A[T]$.)*

**Solution.** *Proof.* $\square$

**Problem 6.**

*(6.1) Let $f(T) \in K[T]$ be irreducible where $K$ is a field. Prove that the ideal $(f(T))$ is maximal.*

**Solution.** *Proof.* Consider $I = (f(T))$ and $J$ such that $I \subseteq J \neq K[T]$.

**Claim.** *If $K$ is a field, then $K[T]$ is a P.I.D.*

*Proof.* Take $I$ ideal in $K[T]$ and the polynomial of minimal degree $p(T) \neq 0$ in $I$. Then, $d(p(T)) > 0$, otherwise $p(T) \in K$ and $pp^{-1} = 1 \in I$ (in contradiction with $I \neq K[T]$). Now, consider $g(T) \in I$. By the division algorithm, $\exists q(T), r(T) \in K[T]$ such that $g(T) = p(T)q(T) + r(T)$ with $d(r(T)) < d(p(T))$. Then $r(T) = g(T) - p(T)q(T) \in I$ since $g(T), p(T) \in I$. By minimality of $p(T) \neq 0$, we have that $r(T) = 0$. Thus, $g(T) = p(T)q(T)$, and $I = (p(T))$. $\square$

For the claim we know that $K[T]$ is a P.I.D., thus $\exists h(T) \in K(T)$ such that $J = (h(T))$. Then

$$f(T) \in (f(T)) \subseteq J = (h(T))$$

This implies that $f(T) = h(T)g(T)$, but since $f(T)$ is irreducible, either $h(T)$ or $g(T)$ is an element of $K$. $h(T) \notin K$, otherwise $J = K[T]$ which is not a field. Then $g(T) \in K$ and $h(T) = f(T)a^{-1}$. We conclude that $J \subset I$, so $I$ is maximal. $\square$

*(6.2) Describe the spectrum of* $\mathbb{R}[T], \mathbb{C}[T], \mathbb{R}[T]/(T^2 + 9), \mathbb{C}[T]/(T^2 + 9)$

**Solution.** By the first part of the exercise, we know $\mathbb{R}[T]$ and $\mathbb{C}[T]$ are P.I.D.s, and in particular all their nonzero prime ideals are maximal. Thus, the spec of $\mathbb{R}[T]$ and $\mathbb{C}[T]$ consists of the principal ideals generated by irreducible polynomials. The irreducible polynomials in $\mathbb{R}[T]$ are well known, so:

- $\operatorname{Spec} \mathbb{R}[T] = \{(0)\} \cup \{(T - c) \mid c \in \mathbb{R}\} \cup \{(T^2 + bT + c) \mid b, c \in \mathbb{R},\, b^2 - 4c < 0\}$
- $\operatorname{Spec} \mathbb{C}[T] = \{(0)\} \cup \{(T - c) \mid c \in \mathbb{C}\}$

This observation also helps us describe the spectrum of the quotients. We know, for any ring $R$ and ideal $I$, there is a bijection between the ideals of $R/I$ and the ideals of $R$ that contain $I$, given by the projection $\pi : R \to R/I$. Furthermore, all prime ideals in $R/I$ are of the form $\pi(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of $R$ that containing $I$. $T^2 + 9$ is irreducible in $\mathbb{R}[T]$, but it factorizes (uniquely, as P.I.D $\Rightarrow$ U.F.D) as $(T+3i)(T-3i)$ in $\mathbb{C}[T]$. In the first case, the ideal is maximal, so

$$\operatorname{Spec}\mathbb{R}[T]/(T^2+9) = \{(0)\}$$

Equivalently, $\mathbb{R}[T]/(T^2+9)$ is a field. In the second case, the only prime ideals in $\mathbb{C}$ properly containing $(T^2+9)$ are $(T+3i)$ and $(T-3i)$, so

$$\operatorname{Spec}\mathbb{C}[T]/(T^2+9) = \{(0), (T+3i), (T-3i)\}$$

**Problem 7.** *Describe spec*$(\mathbb{Z}[T])$

**Solution.** Suppose that $\mathfrak{p} \subset \mathbb{Z}[T]$ is an ideal. We will distinguish three cases:

1. $\mathfrak{p} = (0)$. $(0)$ is a prime ideal of $\mathbb{Z}[T]$ since $\mathbb{Z}[T]$ is an integral domain.

2. $\mathfrak{p} = (f)$ is a principal ideal generated by a non-zero polynomial $f \in \mathbb{Z}[T]$. There are two possibilities:

   - $\deg f = 0 \Leftrightarrow f = n \in \mathbb{Z}$. Suppose $(n)$ is a prime ideal. In this case, it is clear by the inclusion $\varphi : \mathbb{Z} \to \mathbb{Z}[T]$ that $(n) = (n)^c$ is prime in $\mathbb{Z}$. Conversely, let $p \in \mathbb{Z}$ be a prime number. $\mathbb{Z}[T]/(p) \cong \mathbb{Z}/(p)[T]$ is an integral domain since $\mathbb{Z}/(p)$ is a field (in particular, an integral domain). We obtain the ideals $(p)$ where $p \in \mathbb{Z}$ is prime.
   - $\deg f > 0$.

     **Claim.** $(f)$ *is prime* $\Rightarrow$ $f$ *is primitive.*

     *Proof.* Otherwise, there is some non-unit $p \in \mathbb{Z}$ that divides all coefficients of $f$. Then $f = pg$ for some $g \in \mathbb{Z}[T]$, which is a non-trivial factorization of $f$. $f$ is not irreducible, so it is not prime (since $\mathbb{Z}$ is a domain). □

     **Claim.** $(f)$ *is prime* $\Rightarrow$ $f$ *is irreducible in* $\mathbb{Q}[T]$.

     *Proof.* Suppose $f = gh$ is a non-trivial factorization, for some $g, h \in \mathbb{Q}[T]$ (in particular, since all non-zero elements of $\mathbb{Q}$ are units, $\deg g, \deg h > 0 \Rightarrow \deg g, \deg h < \deg f$). Taking a common denominator $n$ and multiplying by it, we obtain $nf = \tilde{g}\tilde{h}$ in $\mathbb{Z}[T]$. By the left side, this belongs to $(f)$, but $\tilde{g}, \tilde{h}$ do not belong to $(f)$ since they have strictly smaller degree than $f$. □

     **Claim.** *These conditions are also sufficient: if $f$ is irreducible in* $\mathbb{Q}[T]$ *and primitive, then* $(f)$ *is prime.*

     *Proof.* Consider the natural inclusion $\phi : \mathbb{Z}[T] \to \mathbb{Q}[T]$. We will extend and then contract the ideal $(f)$ of $\mathbb{Z}[T]$. By taking a common denominator and factors:

     $$(f)^{ec} = \mathbb{Z} \cap (f)_{\mathbb{Q}} = \mathbb{Z}[T] \cap \left\{ \frac{n}{m} fh \mid n, m \in \mathbb{Z}, h \in \mathbb{Z}[T] \,\text{primitive} \right\}$$

     We know that $(f) \subset (f)^{ec}$. We will prove the other inclusion and therefore equality. Indeed, take $g = \frac{n}{m} fh \in (f)^{ec}$ of the form described above. For any prime $p \mid m$, we have shown that $(p)$ is prime in $\mathbb{Z}[T]$. Furthermore, $f, h$ are primitive, so $p$ does not divide them $\Rightarrow p \mid n$. We can cancel $p$ in the fraction, and repeat the process for all prime factors of $m$. Therefore, $g \in (f)_{\mathbb{Z}}$. Now, as $f$ is irreducible in $\mathbb{Q}[T]$, which is a PID, $(f)_{\mathbb{Q}} = (f)_{\mathbb{Z}}^{e}$ is prime in $\mathbb{Q}[T]$. Therefore, so is $(f) = (f)^{ec} \subset \mathbb{Z}[T]$. □

With all these claims, this case yields ideals of the form $(f)$ where $f \in \mathbb{Z}[T]$ is irreducible and primitive.

3. $\mathfrak{p}$ is not principal. I don't know how to do this case. Idea: Consider the gcd of all the generators in $\mathbb{Q}[T]$, we have bezout coefficients, so we can write it as a linear combination of them with rational coefficients.

**Problem 8.**

**Problem 9.** *Let $A$ be a ring and let $I$ be an ideal of $A$. Prove that $S = 1 + I$ is a multiplicatively closed set and describe Spec$S^{-1}A$.*

**Solution.**

**Claim.** *$S$ is multiplicatively closed.*

*Proof.* $1 \in S$ because $0 \in I$. Let $s_1, s_2 \in S$. Then $s_1 = 1 + i_1$ and $s_2 = 1 + i_2$ for some $i_1, i_2 \in I$. Therefore, $s_1 s_2 = (1 + i_1)(1 + i_2) = 1 + (i_1 + i_2 + i_1 i_2) \in 1 + I = S$. □

**Claim.**

**Problem 10.**

**Problem 11.**

**Problem 12.**

**Problem 13.**