

# Commutative Algebra. Theory

Ferran Espuña

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## 1 Rings and Ideals

**Remark 1.1.** Unless otherwise specified, all rings we discuss will be commutative with unit.

**Definition 1.2.** We say that  $I \subseteq A$  is an *ideal* if:

- $(I, +)$  is an abelian group.
- For all  $a \in A$  and for all  $x \in I$ ,  $ax \in I$ .

**Definition 1.3.** We define:

- The *radical* of an ideal  $I$  is  $\text{rad}(I) := \{a \in A \mid a^n \in I, n > 0\}$ .
- (*Colon ideal*)  $(I : J) := \{a \in A \mid aJ \subseteq I\}$ .
- (*Saturation*)  $(I : J^\infty) := \{a \in A \mid \exists n > 0 \text{ s.t. } aJ^n \subseteq I\}$ .

**Definition 1.4.** Let  $f : A \rightarrow B$  be a ring homomorphism, and let  $J \subset B$  be an ideal. Then, the *contraction* of  $J$  to  $A$  is  $J^c := \{a \in A \mid f(a) \in J\}$ .

**Proposition 1.5.** *In the above situation,  $J^c$  is an ideal of  $A$ .*

*Proof.* It is an additive subgroup of  $A$  as  $f$  is an additive group homomorphism. Furthermore, if  $a \in J^c$  and  $r \in A$ , then  $f(a) \in J$  so  $f(ra) = f(r)f(a) \in J$  as  $J$  is an ideal.  $\square$

**Definition 1.6.** Let  $R$  be a ring and  $S \subset R$  a subset. Then, the *ideal generated by  $S$*  is  $\langle S \rangle := \bigcap_{I \supset S} I$ , where  $I$  ranges over all ideals of  $R$ .

**Remark 1.7.**  $\langle S \rangle$  is the smallest ideal of  $R$  containing  $S$ . It can be checked that it is indeed an ideal by noticing that all elements of it belong to all ideals containing  $S$ , and verifying the axioms from there.

**Proposition 1.8.** *In the above situation,*

$$\langle S \rangle = \{a_1 s_1 + \cdots + a_n s_n \mid n \in \mathbb{N}, a_1, \dots, a_n \in A, s_1, \dots, s_n \in S\}$$

*Proof.* We will show both inclusions.

( $\subset$ )  $\langle S \rangle$  is an ideal containing  $S$ , so it contains all elements of the form  $a_1 s_1 + \cdots + a_n s_n$ .

( $\supset$ ) One can easily check that the set on the right is an ideal containing  $S$ , so it contains the intersection of all such ideals, which is  $\langle S \rangle$ .

□

**Definition 1.9.** Let  $f : A \rightarrow B$  be a ring homomorphism, and let  $I \subset A$  be an ideal. Then, the *extension* of  $I$  to  $B$  is  $IB = I^e := \langle f(I) \rangle$ .

**Remark 1.10.**  $I^e$  is by construction, the smallest ideal of  $B$  containing  $f(I)$ .

**Proposition 1.11.** *In the above situation, let  $I^{ec} := (I^e)^c$ , and similarly for the rest. Then:*

1.  $I \subset I^{ec}$ .
2.  $J^{ce} \subset J$
3.  $I^e = I^{ece}$
4.  $J^c = J^{cec}$

*Proof.* Note that if  $K$  is an ideal of  $A$ , then  $f(K) \subset K^e$ . Furthermore, extension and contraction clearly respect inclusions.

1.  $I \subset f^{-1}(f(I)) \subset f^{-1}(I^e) = I^{ec}$ .
2.  $(J^c)^e = \langle f(J^c) \rangle = \langle f(f^{-1}(J)) \rangle \subset \langle J \rangle = J$ .
3. By the two previous points,  $(I^e)^{ce} \subset I^e$  and  $I \subset I^{ec}$  so  $I^e \subset (I^{ec})^e$ .
4. Similarly,  $J^c \subset (J^c)^{ec}$  and  $J^{ce} \subset J$  so  $(J^{ce})^c \subset J^c$ .

□

**Definition 1.12.** Let  $I$  be an ideal of a ring  $A$ . The *quotient ring*  $A/I$  is the ring whose elements are the cosets of  $I$  in  $A$ , and whose operations are defined by

$$\begin{aligned}(a + I) + (b + I) &= (a + b) + I \\ (a + I) \cdot (b + I) &= (a \cdot b) + I\end{aligned}$$

**Remark 1.13.** The sum operation is well defined as it is for all quotient groups. The product operation is well defined as  $I$  is an ideal so if  $p \in (a + I)$ ,  $q \in (b + I)$ , then  $pq \in ab + aI + bI + I^2 \subset (ab + I)$ .

**Proposition 1.14.** Let  $I$  be an ideal of a ring  $A$ . Then, the canonical projection  $\pi : A \rightarrow A/I$  is a ring homomorphism.

*Proof.* It is clearly a group homomorphism. Furthermore,  $\pi(a)\pi(b) = (a + I)(b + I) = ab + I = \pi(ab)$ . Finally,  $\pi(1) = 1 + I$  is clearly the unit of  $A/I$ .  $\square$

**Definition 1.15.** Let  $I \subsetneq R$  be an ideal. Then:

- $I$  is *prime* if  $ab \in I \Rightarrow a \in I$  or  $b \in I$ .
- $I$  is *maximal* there are no ideals  $J$  such that  $I \subsetneq J \subsetneq R$ .

We further define the *spectrum* of  $R$  as

$$\text{Spec}(R) := \{\mathfrak{p} \subset R \text{ ideal} \mid \mathfrak{p} \text{ is prime}\}.$$

and

$$\text{Max}(R) := \{\mathfrak{m} \subset R \text{ ideal} \mid \mathfrak{m} \text{ is maximal}\}.$$

**Proposition 1.16.** Let  $\mathfrak{p} \subset R$  be an ideal. Then,  $\mathfrak{p} \in \text{Spec}(R) \Leftrightarrow R/\mathfrak{p}$  is an integral domain.

*Proof.* Let

$$\begin{aligned}\pi : R &\rightarrow R/\mathfrak{p} \\ a &\mapsto \bar{a} := a + \mathfrak{p}\end{aligned}$$

be the canonical projection. Then,  $\pi$  is a ring homomorphism. Suppose  $a, b \in R$ . Then,

$$ab \in \mathfrak{p} \Leftrightarrow \bar{a}\bar{b} = 0$$

$$(a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}) \Leftrightarrow (\bar{a} = 0 \text{ or } \bar{b} = 0)$$

Therefore, as  $\pi$  is surjective,  $R/\mathfrak{p}$  is an integral domain if and only if  $\mathfrak{p}$  is prime, as both conditions are equivalent to the first condition implying the second.  $\square$

**Lemma 1.17.** *A ring  $S$  is a field if and only if it has no nontrivial ideals  $0 \subsetneq I \subsetneq S$*

*Proof.* We will show both implications.

( $\Rightarrow$ ) Let  $0 \subsetneq I \subsetneq S$  be an ideal. Then, it has a nonzero element  $a$  which must be a unit, so  $1 = aa^{-1} \in I$ , so  $I \subsetneq S = (1) \subset I$ , a contradiction.

( $\Leftarrow$ ) Let  $a \in S$  be nonzero. Then,  $(a) \neq 0$  so  $(a) = S$ , so  $1 \in (a)$ , so  $a$  is a unit.  $\square$

**Proposition 1.18.** *Let  $\mathfrak{m} \subset R$  be an ideal. Then,  $\mathfrak{m} \in \text{Max}(R) \Leftrightarrow R/\mathfrak{m}$  is a field.*

*Proof.* We will use the above characterization of fields. Because the canonical projection  $\pi : R \rightarrow R/\mathfrak{m}$  is surjective, Contraction by it respects inequalities, that is  $I \subsetneq J \Rightarrow I^c \subsetneq J^c$ . On the other hand, for ideals containing  $\mathfrak{m}$ , extension by  $\pi$  is just the canonical projection, so  $I \subsetneq J \Rightarrow I^e \subsetneq J^e$ . Finally,  $R^e = R/\mathfrak{m}$  and  $(R/\mathfrak{m})^c = R$ . Therefore, we have a bijection between nontrivial ideals of  $R/\mathfrak{m}$  and ideals  $\mathfrak{m} \subsetneq I \subsetneq R$ . One of the sets is empty if and only if the other is.  $\square$

**Remark 1.19.** All fields are integral domains, so  $\text{Max}(R) \subset \text{Spec}(R)$ . That is, all maximal ideals are prime.

**Theorem 1.20.** *All rings have maximal ideals.*

*Proof.* Exercise. Use Zorn's lemma.  $\square$

**Theorem 1.21.** *Let  $A$  be a ring and  $I \subseteq A$  be an ideal. Then, there exists a maximal ideal  $\mathfrak{m} \subseteq A$  such that  $I \subseteq \mathfrak{m}$ .*

*Proof.* Exercise. Use Zorn's lemma.  $\square$

**Definition 1.22.** We say that a ring  $R$  is *local* if it has a unique maximal ideal  $\mathfrak{m}$ .

**Proposition 1.23.** *Let  $R$  be a ring.  $R$  is local if and only if  $R \setminus R^*$  is an ideal (which is then necessarily the maximal ideal).*

*Proof.* We will show both implications.

( $\Rightarrow$ ) Let  $R$  be local with maximal ideal  $\mathfrak{m}$ . We will show that  $\mathfrak{m}$  is exactly  $R \setminus R^*$ . Let  $a \in R \setminus R^*$ . Then,  $(a) \neq R$  so  $(a) \subset \mathfrak{m}$ , as it must be contained in some maximal ideal. In particular,  $a \in \mathfrak{m}$ . On the other hand, if  $a \in \mathfrak{m}$ , then  $(a) \subset \mathfrak{m} \subsetneq R$ , so  $a \notin R^*$ .

( $\Leftarrow$ ) Let  $I \subsetneq R$  be an ideal. If it contained a unit, then it would contain 1, so  $I = R$ . Therefore,  $I \subset R \setminus R^*$  so the latter is the unique maximal ideal.

□

**Definition 1.24.** Let  $R$  be a ring and  $S \subset R$  a subset. We say that  $S$  is *multiplicatively closed* if  $1 \in S$  and  $a, b \in S \Rightarrow ab \in S$ .

**Definition 1.25.** Let  $R$  be a ring and  $S \subset R$  a multiplicatively closed subset. Then,  $S^{-1}R := \{\frac{a}{b} \mid a \in R, b \in S\} / \sim$ , where

$$\frac{a}{s} \sim \frac{b}{t} \Leftrightarrow \exists r \in S: r(at - bs) = 0 \quad (1)$$

**Proposition 1.26.** *In the definition above,  $\sim$  is an equivalence relation.*

*Proof.* The relation is obviously symmetric and reflexive. Let us show that it is transitive. Suppose  $\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \sim \frac{a_3}{s_3}$ :

$$\begin{aligned} t_1(a_1s_2 - a_2s_1) &= 0 \\ t_2(a_2s_3 - a_3s_2) &= 0 \end{aligned}$$

For some  $t_1, t_2 \in S$ . Then, multiply the first equation by  $t_2s_3$  and the second by  $t_1s_1$  to get

$$\begin{aligned} t_1t_2s_3a_1s_2 - t_1t_2s_3a_2s_1 &= 0 \\ t_1t_2s_1a_2s_3 - t_1t_2s_1a_3s_2 &= 0 \end{aligned}$$

Adding both equations and collecting like terms, we get

$$0 = t_1 t_2 s_3 a_1 s_2 - t_1 t_2 s_1 a_3 s_2 = t_1 t_2 s_2 (a_1 s_3 - a_3 s_1) \quad (2)$$

Because  $S$  is multiplicatively closed,  $t_1 t_2 s_2 \in S$ , so  $\frac{a_1}{s_1} \sim \frac{a_3}{s_3}$ . □

**Proposition 1.27.** *The usual operations*

$$\begin{aligned} \frac{a}{s} + \frac{b}{t} &= \frac{at + bs}{st} \\ \frac{a}{s} \cdot \frac{b}{t} &= \frac{ab}{st} \end{aligned}$$

are well defined and make  $S^{-1}R$  into a ring.

*Proof.* Left as an exercise. The arguments are tedious but similar to the above proposition. □

**Proposition 1.28.** *Let  $R$  be a ring and  $S \subset R$  a multiplicatively closed subset. Then, the canonical projection*

$$\begin{aligned} \pi : R &\rightarrow S^{-1}R \\ a &\mapsto \frac{a}{1} \end{aligned}$$

is a ring homomorphism. It is injective if and only if  $S$  contains no zero divisors.

*Proof.*  $\pi(1) = \frac{1}{1}$  is clearly the unit of  $S^{-1}R$ . Furthermore,  $\pi(a)\pi(b) = \frac{a}{1}\frac{b}{1} = \frac{ab}{1} = \pi(ab)$ .  $\pi(a) + \pi(b) = \frac{a}{1} + \frac{b}{1} = \frac{1 \cdot a + 1 \cdot b}{1 \cdot 1} = \frac{a+b}{1} = \pi(a+b)$ . We have that it is a ring homomorphism.  $\text{Ker}(\pi) = \{a \in R \mid \frac{a}{1} \sim \frac{0}{1}\} = \{a \in R \mid \exists s \in S \mid 0 = s(a \cdot 1 + 0 \cdot 1) = sa\}$ . Indeed, the kernel is the set of all elements of  $R$  that are annihilated by some element of  $S$ . In particular, it contains only zero if and only if  $S$  contains no zero divisors. □

**Proposition 1.29.** *Let  $A$  be a ring and  $S \subset A$  a multiplicatively closed subset. Let  $f : A \rightarrow B$  be a ring homomorphism such that  $f(S) \subset B^*$ . Then, there exists a unique ring homomorphism  $g : S^{-1}A \rightarrow B$  such that  $f = g \circ \pi$ , where  $\pi$  is the canonical projection. That is, the following diagram commutes:*

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\pi \downarrow & \nearrow \exists! g & \\
S^{-1}A & & 
\end{array}$$

*Proof.* Suppose that such a  $g$  exists. By definition,

$$g\left(\frac{a}{1}\right) = f(a)$$

For any  $s \in S$ ,

$$1 = f(1) = g\left(\frac{s}{1} \frac{1}{s}\right) = g\left(\frac{s}{1}\right) g\left(\frac{1}{s}\right) \Rightarrow g\left(\frac{1}{s}\right) = g\left(\frac{s}{1}\right)^{-1} = f(s)^{-1}$$

Then,

$$g\left(\frac{a}{s}\right) = g\left(\frac{a}{1} \frac{1}{s}\right) = g\left(\frac{a}{1}\right) g\left(\frac{1}{s}\right) = f(a) f(s)^{-1}$$

Therefore,  $g$  is uniquely determined by  $f$ . The derived expression for  $g$  clearly makes the diagram commute. Finally, we will show that it is well defined and that it is a ring homomorphism. Suppose  $\frac{a}{s} = \frac{b}{t}$ . Then, for some  $u \in S$ ,

$$u(at - bs) = 0 \Rightarrow 0 = f(0) = f(u(at - bs)) = f(u)(f(a)f(t) - f(b)f(s))$$

But  $f(u), f(s), f(t) \in B^*$ , so

$$f(a)f(t) = f(b)f(s) \Rightarrow g\left(\frac{a}{s}\right) = f(a)f(s)^{-1} = f(b)f(t)^{-1} = g\left(\frac{b}{t}\right)$$

Therefore,  $g$  is well defined. Finally, let us check that it is a ring homomorphism:

- $g\left(\frac{a}{s}\right) + g\left(\frac{b}{t}\right) = f(a)f(s)^{-1} + f(b)f(t)^{-1} = f(a)f(t)f(t)^{-1}f(s)^{-1} + f(b)f(s)f(t)^{-1}f(s)^{-1} = f(at + bs)f(st)^{-1} = g\left(\frac{at+bs}{st}\right) = g\left(\frac{a}{s} + \frac{b}{t}\right)$
- $g\left(\frac{a}{s}\right)g\left(\frac{b}{t}\right) = f(a)f(s)^{-1}f(b)f(t)^{-1} = f(ab)f(st)^{-1} = g\left(\frac{ab}{st}\right) = g\left(\frac{a}{s} \frac{b}{t}\right)$
- $g\left(\frac{1}{1}\right) = 1$

□

**Proposition 1.30.** *In the above situation, let  $I \subset A$  be an ideal. Consider the canonical projection  $\pi : A \rightarrow S^{-1}A$ . Then we can write the extension of  $I$  by  $\pi$  as  $I^e =: IS^{-1}A = S^{-1}I := \{\frac{j}{s} \mid j \in I, s \in S\}$ . Furthermore, every ideal of  $S^{-1}A$  is of this form.*

*Proof.* It is clear that  $S^{-1}I \subset IS^{-1}A$ . Suppose that  $x = \frac{j_1}{s_1} \frac{a_1}{1} + \dots + \frac{j_n}{s_n} \frac{a_n}{1} \in IS^{-1}A$ , with  $j_i \in I, a_i \in A, s_i \in S$ . Then, by applying the definition of addition repeatedly, we get  $x = \frac{j}{s}$ , with  $j \in I$ . Therefore  $x \in I^e$ .  
Now, take any ideal  $J \subset S^{-1}A$ . We know by 1.11  $J^{ce} \subset J$ . We will show the reverse inclusion in this case. Let  $\frac{a}{s} \in J$ . Then  $\pi(a) = \frac{a}{1} = \frac{a}{s} \frac{s}{1} \in J \Rightarrow a \in J^c \Rightarrow \frac{a}{s} \in J^{ce}$ . In particular,  $J = J^{ce} = J^c S^{-1}A = S^{-1}J^c$ .  $\square$

**Definition 1.31.** (*Total fraction ring*)  $Tot(A) := S^{-1}A$  where  $S = \{a \in A \mid a \text{ is not a zero divisor}\}$ .

**Definition 1.32.** (*Localization at an element*)  $A_f := S^{-1}A$  where  $S = \{f^n \mid n \geq 0\}$  for a given  $f \in A$ .

**Definition 1.33.** Let  $A$  be a ring and  $\mathfrak{p} \in \text{Spec}(A)$  a prime ideal. Then, the *localization of  $A$  at  $\mathfrak{p}$*  is  $A_{\mathfrak{p}} := S^{-1}A$ , where  $S = A \setminus \mathfrak{p}$ .

**Remark 1.34.** Indeed,  $S$  is multiplicatively closed as  $\mathfrak{p}$  is prime.

**Proposition 1.35.** *In the above situation,  $\frac{a}{s} \in A_{\mathfrak{p}}$  is a unit  $\iff a \notin \mathfrak{p}$ .*

*Proof.* We will show both implications.

( $\Rightarrow$ ) Suppose  $\frac{a}{s} \in A_{\mathfrak{p}}$  is a unit. Then, there is some  $\frac{b}{t} \in A_{\mathfrak{p}}$  such that  $\frac{a}{s} \frac{b}{t} = \frac{1}{1}$ . For some  $u \notin \mathfrak{p}, u(ab - st) = 0$ . Rearranging,  $uab = ust$ . But neither of  $u, s, t$  are in  $\mathfrak{p}$ , so their product  $ust$  is not in  $\mathfrak{p}$  (because  $\mathfrak{p}$  is prime). In particular,  $a \notin \mathfrak{p}$ .

( $\Leftarrow$ ) Suppose  $a \notin \mathfrak{p}$ . Then,  $\frac{a}{s} \frac{s}{a} = 1$

$\square$

**Proposition 1.36.** *In the above situation,  $A_{\mathfrak{p}}$  is a local ring with maximal ideal  $\mathfrak{p}A_{\mathfrak{p}}$ .*

*Proof.* By Proposition 1.23, We just need to show that  $\mathfrak{p}A_{\mathfrak{p}} = A_{\mathfrak{p}} \setminus (A_{\mathfrak{p}})^*$ . Indeed, by 1.30 we know  $\mathfrak{p}A_{\mathfrak{p}} = \{\frac{a}{s} \mid a \in \mathfrak{p}, s \notin \mathfrak{p}\}$ , which by the previous proposition is exactly the set of nonunits of  $A_{\mathfrak{p}}$ .

$\square$

**Theorem 1.37.** *We have:*



- $S^{-1}I = S^{-1}A \Leftrightarrow I \cap S \neq \emptyset$ .
- $\mathfrak{p} \in \text{Spec}(A)$  s.t.  $\mathfrak{p} \cap S = \emptyset \Leftrightarrow S^{-1}\mathfrak{p} \in \text{Spec}S^{-1}A$
- There is a bijection

$$\begin{aligned} \{\mathfrak{p} \in \text{Spec}A \mid \mathfrak{p} \cap S = \emptyset\} &\longleftrightarrow \text{Spec}S^{-1}A \\ \mathfrak{p} &\longmapsto S^{-1}\mathfrak{p} \\ \mathfrak{q} = \mathfrak{q} \cap A &\longleftarrow \mathfrak{q} \end{aligned}$$

*Proof.* TODO

□

**Proposition 1.38.** Let  $f : A \longrightarrow B$  be a ring homomorphism. Let  $S \subseteq A$  and  $T \subseteq B$  be multiplicatively closed subsets such that  $f(S) \subseteq T$ . Then there exist a unique  $g : S^{-1}A \longrightarrow T^{-1}B$  such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \varphi \downarrow & & \downarrow \psi \\ S^{-1}A & \xrightarrow{g} & T^{-1}B \end{array}$$

*Proof.* TODO

□

**Corollary 1.38.1.** Let  $f : A \longrightarrow A/I$  with a given ideal  $I \subseteq A$ . Let  $S \subseteq A$  be a multiplicatively closed set and  $\overline{S} \subseteq A/I$  the associated one. Then

$$\overline{S}^{-1}A/I \cong S^{-1}A/S^{-1}I$$

**Definition 1.39.** The *residue field* of a ring  $A$  w.r.t. a prime ideal  $\mathfrak{p} \in \text{Spec}A$  is

$$k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$$

**Definition 1.40.** Let  $f : A \longrightarrow B$  be a ring homomorphism. This induces the morphism  $f^* : \text{Spec}B \longrightarrow \text{Spec}A$ , with  $f^*(\mathfrak{q}) = \mathfrak{q}^c$ . Then, the *fiber* of  $\mathfrak{p} \in \text{Spec}A$  is defined as

$$(f^*)^{-1}(\mathfrak{p}) := \{\mathfrak{q} \in \text{Spec}B \mid \mathfrak{q}^c = \mathfrak{p}\}$$

**Proposition 1.41.** *In the conditions of the previous definition, we have that*

$$(f^*)^{-1}(\mathfrak{p}) = \operatorname{Spec}(T^{-1}B/\mathfrak{p}T^{-1}B) \cong \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$$

where  $T = f(A \setminus \mathfrak{p})$ .

*Proof.* TODO. La primera igualdad la hemos demostrado en clase. La segunda solo se enuncia en los apuntes.

□