Problem 1. *Let A be a ring. Prove:*

(1.1) If $x \in A$ is nilpotent, then $1 - x$ is invertible.

Solution. *Proof.* Suppose that $x^n = 0$ for some $n \in \mathbb{N}$. Then,

$$(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1 - x^n = 1$$

□

(1.2) The nilradical $N(A) = \{x \in A \,|\, x \, nilpotent\}$ is an ideal of A .

Solution. *Proof.* Let $x, y \in N(A)$, then $x^n = 0$ and $y^m = 0$ for some $n, m \in \mathbb{N}$. Then,

$$(x-y)^{n+m} = \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} x^{n+m-k} y^k = 0$$

since $k \geq n$ or $n + m - k \geq m$ for all k. Thus, $x - y \in N(A)$. We have that $N(A)$ is an additive subgroup of A. Now, let $a \in A$. Then,

$$(ax)^n = a^n x^n = 0a = 0$$

so $ax \in N(A)$. Thus, $N(A)$ is an ideal of A. □

(1.3) $N(A)$ is contained in all prime ideals of A .

Solution. *Proof.* Let $I \subset A$ be a prime ideal. Let $x \in N(A)$. Then, $x^n = 0 \in I$ for some $n \in \mathbb{N}$. Let us show by induction on n that this implies $x \in I$:

- If $n = 1$, then $x = 0 \in I$.
- Suppose that the statement is true for $n - 1$: $x^{n-1} = 0 \Rightarrow x \in I$. If $0 = x^n = x \cdot x^{n-1}$, since I is prime, $x \in I$ or $x^{n-1} \in I$. In the first case, we are done. The second case is just the inductive hypothesis.

□

(1.4) $N(A)$ is the intersection of all prime ideals of A .

Solution.

Claim. *Given $x \notin N(A)$, let Σ_x be the set of all ideals that do not contain any power of x. Then, Σ_x has a maximal element.*

Proof. We will use Zorn's lemma. Let us check the conditions:

Claim. *Σ_x is a partially ordered set with respect to inclusion.*

Claim. *Σ_x is not empty.*

Proof. Since $x \notin N(A)$, $0 \in \Sigma_x$. $\square$

Claim. *Every chain in Σ_x has an upper bound.*

Proof. Let $\{I_\alpha\}_{\alpha \in S}$ be a chain in Σ_x. Then, $I = \bigcup_{\alpha \in S} I_\alpha$ is an ideal of A (One can check that if $x, y \in I$, then $x, y \in I_\alpha$ for some $\alpha \in S$, and then check the axioms from there). Let $x^n \in I$ for some $n \in \mathbb{N}$. Then, $x^n \in I_\alpha$ for some $\alpha \in A$. Since I_α is an ideal, $x \in I_\alpha$. Thus, $I \in \Sigma_x$. $\square$

Now that we have verified the conditions of Zorn's lemma, we can conclude that Σ_x has a maximal element. $\square$

Claim. *Let $x \notin N(A)$. Then the maximal element $K(x)$ of Σ_x is prime.*

Proof. Let $a, b \in A$ such that $ab \in K(x)$. By way of contradiction, suppose that $a \notin K(x)$ and $b \notin K(x)$. Then, $x^n \in (a)$ and $x^m \in (b)$ for some $m, n \in \mathbb{N}$, but $x^{n+m} \notin (ab) = (a)(b)$. Contradiction. $\square$

Finally, we can prove the statement.

Claim. *$N(A)$ is the intersection of all prime ideals of A.*

Proof. Let J be the intersection of all prime ideals of A. By 1.3, we know that $N(A) \subset J$. We want to prove that if $x \notin N(A)$, then $x \notin J$. Indeed, $J \subset K(x)$ because $K(x)$ is prime and $x \notin K(x)$ because $K(x)$ does not contain any power of x. $\square$

Problem 2. *Let A be a ring. Let $a_i \in A$ and $f = a_0 + a_1 T + \cdots + a_n T^n \in A[T]$ be a polynomial. Prove:*

(2.1) f is a unit in $A[T]$ $\iff$ a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

Solution. *Proof.* We will show both implications separately.

($\Leftarrow$) Let $a_i^{n_i} = 0$ for all $1 \le i \le n$. Consider s $= \Sigma_{i=1}^n n_i$. Let h $= -a_1 T - \cdots - a_n T^n$ be the negative of the polynomial without the constant term. Then,

$$h^s = \left(-\sum_{i=1}^n a_i T^i\right)^s = (-1)^s \sum_{j_1 + \cdots + j_n = s} a_1^{j_1} \cdots a_n^{j_n} T^{j_1 + 2 \cdot j_2 \cdots + n \cdot j_n}$$

By the pigeonhole principle, in each term of the sum, there is at least one $j_k \ge n_k$ so $a_k^{j_k} = 0$, and thus $h^s = 0$. Then, h is nilpotent, so is $a_0^{-1} \cdot h$ and, by 1.1, $1 - a_0^{-1} \cdot h$ is invertible. Multiplying by a_0, we get that $f = a_0 - h$ is also invertible.

($\Rightarrow$) Suppose, there exists $g = b_0 + b_1 T + \cdots + b_m T^m \in A[T]$ such that $1 = fg = \sum_{i=0}^{n+m} s_i T^i$, where $s_i = \sum_{j=0}^n a_j b_{i-j}$ and $b_k = 0$ for $k > m$ or $k < 0$. We first note that $1 = s_0 = a_0 b_0 \Rightarrow a_0, b_0 \in A^*$. Next, suppose $n > 0$ (Otherwise, there is nothing to show).

Claim. *Let* $0 \leq k \leq m$. $a_n^{k+1} b_{m-k} = 0$

Proof. By total induction on k:

- If $k = 0$, then $0 = s_{n+m} = a_n b_m = (a_n)^{0+1} b_{m-0}$.
- Suppose that the statement is true for $0, \ldots, k-1$. Then, $s_{n+m-k} = 0$ as $n + m - k \geq n > 0$. Therefore:

$$0 = a_n^k s_{n+m-k} = \sum_{j=0}^{n} a_n^k a_j b_{n+m-k-j} = a_n^{k+1} b_{m-k} + \sum_{j=0}^{n-1} a_n^k a_j b_{n+m-k-j}$$

Now, the terms in the sum are zero by the inductive hypothesis, as $j < n \Rightarrow -(n-k-j) < k \Rightarrow (n-k-j)+1 \leq k$. Therefore, $a_n^{k+1} b_{m-k} = 0$ as we wanted to show.

□

Now, by setting $k = m$, we get that $a_n^{m+1} b_0 = 0$. Since b_0 is a unit, $a_n^{m+1} = 0$ and a_n is nilpotent. We are almost done if we realize the following:

Claim. *Let* $p = c_0 + c_1 T + \cdots + c_l T^l \in A[T]$ *be an invertible polynomial such that* c_l *is nilpotent. Then,* $q = c_0 + c_1 T + \cdots + c_{l-1} T^{l-1}$ *is also invertible.*

Proof. Note that $c_l T^l$ is nilpotent and so is $c_l T^l p^{-1}$. Then, $1 - c_l T^l p^{-1}$ is invertible by 1.1. Finally, because p is invertible, so is $q = p - c_l T^l = p(1 - c_l T^l p^{-1})$. □

We will prove that for $0 < k \leq n$, $a_0 + \cdots + a_k T^k$ is invertible and a_k is nilpotent by (reverse) induction on k.

- a_n has already been done.
- If $0 < k < n$, by hypothesis a_{k+1} is nilpotent and $a_0 + \cdots + a_{k+1} T^{k+1}$ is invertible. Then, $a_0 + \cdots + a_k T^k = (a_0 + \cdots + a_{k+1} T^{k+1}) - a_{k+1} T^{k+1}$ is invertible by the claim. Therefore, a_k is nilpotent.

($\Rightarrow$) (Faster Version) Let f be a unit in $A[T]$.

Claim. *Let* $\mathfrak{p} \in Spec(A)$. *Then,* $a_i \in \mathfrak{p}$ *for all* $i \in \{1, \ldots n\}$.

Proof. $\mathfrak{p} \in Spec(A)$ implies that $A/\mathfrak{p}$ is an integral domain. Consider the reduction $\pi : A[T] \to A/\mathfrak{p}[T]$ that takes each element $a \in A[T]$ to the class $\overline{a} \in A/\mathfrak{p}[T]$. Since $A/\mathfrak{p}$ is an integral domain,

$$0 = \deg \bar{1} = \deg \bar{f} \cdot \overline{f^{-1}} = \deg \bar{f} + \deg \overline{f^{-1}}$$

In particular, $\deg \bar{f} = 0$ so, for all $i \in \{1, \ldots n\}$, $\overline{a_i} = \bar{0} \Rightarrow a_i \in \mathfrak{p}$. □

Since this holds for all $\mathfrak{p} \in Spec(A)$, we have that $\forall i \in \{1, \ldots n\}$, $a_i \in \bigcap_{\mathfrak{p} \in Spec(A)} \mathfrak{p} = N(A)$ (by 1.4). Finally, suppose that $f^{-1} = b_0 + b_1 T + \cdots + b_m T^m$. Given that $f f^{-1} = 1$ we have that $a_0 b_0 = 1$, and thus a_0 is invertible.

□

(2.2) f is nilpotent $\iff$ $a_0, \dots, a_n$ are nilpotent.

Solution. *Proof.* We will show both implications separately.

($\Leftarrow$) Just use the same argument as in the previous part of the exercise, but directly on f, not on $h = a_0 - f$.

($\Rightarrow$) If f is nilpotent, then $1 - f$ is invertible by 1.1. Thus, by the previous part, $-a_1, \dots, -a_n$ are nilpotent. Because $-a_i$ is nilpotent, a_i is nilpotent. We only have to prove that a_0 is nilpotent. By the other implication, h is nilpotent. Then, by 1.2, $a_0 = f + h$ is nilpotent.

□

(2.3) f is a zero divisor $\iff$ there exists $a \in A$, $a \neq 0$ such that $af = 0$.

Solution. *Proof.* The backward implication is trivial because of the inclusion $A \subset A[T]$. For the forward implication, suppose that f is a zero divisor. Then, there exists $g \in A[T]$, $g \neq 0$ such that $fg = 0$. Let $g = b_0 + \dots + b_m T^m$ be the minimum degree polynomial satisfying the condition $fg = 0$. Suppose that $m > 0$.

Claim. *$\exists i \in \{1, \dots, n\}$ s.t. $a_i g \neq 0$*

Proof. Suppose not. Then, $a_i g = 0$ for all $i \in \{1, \dots, n\}$. Then, $a_i b_m = 0$ for all $i \in \{1, \dots, n\}$. But then we have $b_m f = 0$ with $b_m \in A$, in contradiction with g being a polynomial with minimum degree. □

Take i maximal such that $a_i g \neq 0$. Then, $0 = fg = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m) + \sum_{j=i+1}^{n} T^j a_j g = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m)$, and $a_i b_m = 0$. Thus, we have a polynomial $g' = a_i g \neq 0$ with degree $m - 1$ satisfying $f(a_i g) = a_i(fg) = 0$, in contradiction with g being of minimal degree. □

Problem 3. *Let A be a ring. We define the Jacobson radical $\mathcal{J}(A)$ as the intersection of all maximal ideals of A. Prove:*

(3.1) $x \in \mathcal{J}(A) \Leftrightarrow 1 - xy$ is invertible for all $y \in A$.

Solution. *Proof.* We will prove the two implications separately:

($\Leftarrow$) Suppose $x \notin \mathcal{J}(A)$. This means that there exists $\mathfrak{m} \in Max(A)$ such that $x \notin \mathfrak{m}$. Since $\mathfrak{m}$ is a maximal ideal, $(x) + \mathfrak{m} = A$. Then, there exist $y' \in A$ and $m' \in \mathfrak{m}$ such that $xy' + m' = 1 \in A$. But this means that $1 - xy' = m' \in \mathfrak{m}$, which invertible. Contradiction with $\mathfrak{m}$ being an ideal. Thus, $x\mathcal{J}(A)$.

($\Rightarrow$) Suppose that $\exists y \in A$ such that $1 - xy$ is not a unit in A. Notice that:

- There exists a maximal ideal $\mathfrak{M}$ such that $1 - xy \in \mathfrak{m}$.
- Since $x \in \mathfrak{m} \subseteq \mathcal{J}(A)$, we have that $xy \in \mathfrak{m}$.

Finally, $1 - xy + xy = 1 \in \mathfrak{m}$, in contradiction with $\mathfrak{m}$ being an ideal. Thus, $\forall y \in A$ we have that $1 - xy$ is a unit in A.

□

(3.2) $\mathcal{J}(A) = A \setminus A*$ $\Leftrightarrow$ *A has only one maximal ideal.*

Solution. *Proof.* This comes directly from the fact that an ideal $I \subset A$ cannot contain a unit x, else $xx^{-1} = 1 \in I$ and thus $I = A$.

$$\mathcal{J}(A) = A \setminus A^* \Leftrightarrow \forall \mathfrak{m} \in Max(A), \mathfrak{m} \supseteq \{\text{non-units of } A\} = A \setminus A^* \Leftrightarrow$$
$$\Leftrightarrow \forall \mathfrak{m} \in Max(A), \mathfrak{m} = \{\text{non-units of } A\} \Leftrightarrow (A, \mathfrak{m})\text{local}$$

□

(3.3) $\mathcal{J}(A[T]) = \mathcal{N}(A[T])$.

Solution. *Proof.* We will prove the two inclusions separately:

($\supseteq$)
$$\mathcal{N}(A[T]) = \bigcap Spec(A[T]) \subseteq \bigcap Max(A[T]) = \mathcal{J}(A[T])$$
The first equality follows 1.4, while the inclusion is a consequence of $Max(A[T]) \subseteq Spec(A[T])$.

($\subseteq$) Take $f = a_0 + a_1T + \cdots + a_nT^n \in \mathcal{J}(A[T])$. Using 3.1 with $y = T$ we know that $1 - fT = 1 - a_0T - a_1T^2 \cdots - a_nT^{n+1}$ is invertible in $A[t]$. But then, from 2.1 we get that $-a_0, \cdots, -a_n$ nilponents, and finally $a_0, \cdots a_n$ are nilponents. We conclude from 2.2 we get that f is nilponent in $A[T]$

¿should we write a proof of this or is it overkill?

□

Problem 4. *Let A be a ring such that every ideal not contained in the nilradical $\mathcal{N}(A)$ has a non-zero idempotent element (i.e. an element $e \neq 0$ such that $e^2 = e$). Prove that the nilradical and the Jacobson radical $\mathcal{J}(A)$ are equal.*

Solution. *Proof.* We will prove the two inclusions separately:

($\supseteq$) (Same as in 3.3)
$$\mathcal{N}(A) = \bigcap Spec(A) \subseteq \bigcap Max(A) = \mathcal{J}(A)$$
The first equality follows 1.4, while the inclusion is a consequence of $Max(A) \subseteq Spec(A)$.

($\subseteq$) Suppose that $\mathcal{J}(A) \not\subseteq \mathcal{N}(A)$. Then, since $\mathcal{J}(A)$ is an intersection of ideals and thus an ideal itself, we have that $\exists e \in \mathcal{J}(A)$ such that $e^2 = e \neq 0$. By 3.1, we have that $1 - e1$ is invertible, so
$$(1 + e) = (1 - e)^{-1}(1 - e)(1 + e) = (1 - e)^{-1}(1 - e^2) = (1 - e)^{-1}(1 - e) = 1$$
Finally, we get that $e = 0$, in contradiction with the definition of e. Thus, we conclude that $\mathcal{J}(A) \subseteq \mathcal{N}(A)$.

□

Problem 5. *Let A be an integral domain. Prove that the ideal $I = (x^2 - y^3, y^2 - z^3) \subseteq A[x, y, z]$ is prime.*

(Hint: Let $f : R \to B$ be a ring homomorphism and $I \subseteq R$ be an ideal such that $I \subseteq \ker f$. Then f factorizes through R/I. Consider a parametrization $f : A[x, y, z] \to A[T]$.)

Solution. *Proof.* Consider the ring homomorphism

$$\begin{aligned} f\colon A[x,y,z] &\longrightarrow A[T] \\ x &\longmapsto T^9 \\ y &\longmapsto T^6 \\ z &\longmapsto T^4 \end{aligned}$$

Since $f(x^2 - y^3) = (T^9)^2 - (T^6)^3 = 0$ and $f(y^2 - z^3) = (T^6)^2 - (T^4)^3 = 0$, we have that $f(I) = 0$ and $I \subseteq \ker f$. Consider the projection $\pi : A[x, y, z] \to A[x, y, z]/I$. By definition of quotient, $\ker \pi = I$.

Claim. *There exists $g : A[x, y, z]/I \to A[T]$ ring homomorphism such that $f = g \circ \pi$.*

Proof. $g(\bar{a}) = f(a)$.

- $\underline{\bar{a} = \bar{b} \Rightarrow g(\bar{a}) = g(\bar{b})?}$
  Suppose $\bar{a} = \bar{b}$. Then, $a - b \in I$, and since $I = \ker f$, $f(a - b) = 0$. So, $f(a) = f(b)$ and $g(\bar{a}) = g(\bar{b})$.

- $\underline{g(\lambda\bar{a}) = \lambda g(\bar{a})?}$
  $$g(\lambda\bar{a}) = f(\lambda a) = \lambda f(a) = \lambda g(\bar{a})$$

- $\underline{g(\bar{a} - \bar{b}) = g(\bar{a}) - g(\bar{b})?}$
  $$g(\bar{a} - \bar{b}) = g(\overline{a - b}) = f(a - b) = f(a) - f(b) = g(\bar{a}) - g(\bar{b})$$

□

Notice that $(z^i)_{i\geq 0}, (xz^i)_{i\geq 0}, (yz^i)_{i\geq 0}, (xyz^i)_{i\geq 0}$ is a set of A-generators of $A[x, y, z]/I$. Also, their images

$$\begin{aligned} g(z^i) &= T^{4i} = T^{4i+0} \\ g(xz^i) &= T^{4i+9} = T^{4(i+2)+1} \\ g(yz^i) &= T^{4i+6} = T^{4(i+1)+2} \\ g(xyz^i) &= T^{4i+15} = T^{4(i+3)+3} \end{aligned}$$

are independent, given the non-congruence of the exponents $mod(4)$. Thus, g is injective. Now, take $\bar{a}, \bar{b} \in A[x, y, z]/I$ such that $\overline{ab} = 0$.

$$0 = g(0) = g(\overline{ab}) = g(\bar{a})g(\bar{b}) \in A[T]$$

Since A is a domain, $A[T]$ is a domain and $g(\bar{a}) = 0$ or $g(\bar{b}) = 0$. But since g is injective, $\bar{a} = 0$ or $\bar{b} = 0$. Thus, $A[x, y, z]/I$ is a domain and we can conclude that I is a prime ideal. □

Problem 6. *(6.1) Let $f(T) \in K[T]$ be irreducible where K is a field. Prove that the ideal $(f(T))$ is maximal.*

Solution. *Proof.* Consider $I = (f(T))$ and J such that $I \subseteq J \neq K[T]$.

Claim. *If K is a field, then $K[T]$ is a P.I.D.*

Proof. Take I ideal in $K[T]$ and the polynomial of minimal degree $p(T) \neq 0$ in I. Then, $d(p(T)) > 0$, otherwise $p(T) \in K$ and $pp^{-1} = 1 \in I$ (in contradiction with $I \neq K[T]$). Now, consider $g(T) \in I$. By the division algorithm, $\exists q(T), r(T) \in K[T]$ such that $g(T) = p(T)q(T) + r(T)$ with $d(r(T)) < d(p(T))$. Then $r(T) = g(T) - p(T)q(T) \in I$ since $g(T), p(T) \in I$. By minimality of $p(T) \neq 0$, we have that $r(T) = 0$. Thus, $g(T) = p(T)q(T)$, and $I = (p(T))$. □

For the claim we know that $K[T]$ is a P.I.D., thus $\exists h(T) \in K(T)$ such that $J = (h(T))$. Then

$$f(T) \in (f(T)) \subseteq J = (h(T))$$

This implies that $f(T) = h(T)g(T)$, but since $f(T)$ is irreducible, either $h(T)$ or $g(T)$ is an element of K. $h(T) \notin K$, otherwise $J = K[T]$ which is not a field. Then $g(T) \in K$ and $h(T) = f(T)a^{-1}$. We conclude that $J \subset I$, so I is maximal. □

(6.2) Describe the spectrum of $\mathbb{R}[T], \mathbb{C}[T], \mathbb{R}[T]/(T^2 + 9), \mathbb{C}[T]/(T^2 + 9)$

Solution. *Proof.* □

Problem 7. *Describe spec*$(\mathbb{Z}[T])$

Solution. Suppose that $\mathfrak{p} \subset \mathbb{Z}[T]$ is an ideal. We will distinguish three cases:

1. $\mathfrak{p} = (0)$. (0) is a prime ideal of $\mathbb{Z}[T]$ since $\mathbb{Z}[T]$ is an integral domain.

2. $\mathfrak{p} = (f)$ is a principal ideal generated by a non-zero polynomial $f \in \mathbb{Z}[T]$. There are two possibilities:

   - $\deg f = 0 \Leftrightarrow f = n \in \mathbb{Z}$. Suppose (n) is a prime ideal. In this case, it is clear by the inclusion $\varphi : \mathbb{Z} \to \mathbb{Z}[T]$ that $(n) = (n)^c$ is prime in $\mathbb{Z}$. Conversely, let $p \in \mathbb{Z}$ be a prime number. $\mathbb{Z}[T]/(p) \cong \mathbb{Z}/(p)[T]$ is an integral domain since $\mathbb{Z}/(p)$ is a field (in particular, an integral domain). We obtain the ideals (p) where $p \in \mathbb{Z}$ is prime.
   - $\deg f > 0$.

     Claim. (f) *is prime* $\Rightarrow f$ *is primitive.*

     Proof. Otherwise, there is some non-unit $p \in \mathbb{Z}$ that divides all coefficients of f. Then $f = pg$ for some $g \in \mathbb{Z}[T]$, which is a non-trivial factorization of f. f is not irreducible, so it is not prime (since $\mathbb{Z}$ is a domain). □

     Claim. (f) *is prime* $\Rightarrow f$ *is irreducible in* $\mathbb{Q}[T]$.

     Proof. Suppose $f = gh$ is a non-trivial factorization, for some $g, h \in \mathbb{Q}[T]$ (in particular, since all non-zero elements of $\mathbb{Q}$ are units, $\deg g, \deg h > 0 \Rightarrow \deg g, \deg h < \deg f$). Taking a common denominator n and multiplying by it, we obtain $nf = \tilde{g}\tilde{h}$ in $\mathbb{Z}[T]$. By the left side, this belongs to (f), but $\tilde{g}, \tilde{h}$ do not belong to (f) since they have strictly smaller degree than f. □

     Claim. *These conditions are also sufficient: if f is irreducible in $\mathbb{Q}[T]$ and primitive, then (f) is prime.*

     Proof. Consider the natural inclusion $\phi : \mathbb{Z}[T] \to \mathbb{Q}[T]$. We will extend and then contract the ideal (f) of $\mathbb{Z}[T]$. By taking a common denominator and factors:

     $$(f)^{ec} = \mathbb{Z} \cap (f)_{\mathbb{Q}} = \mathbb{Z}[T] \cap \left\{ \frac{n}{m} fh \mid n, m \in \mathbb{Z}, h \in \mathbb{Z}[T] \, \text{primitive} \right\}$$

     We know that $(f) \subset (f)^{ec}$. We will prove the other inclusion and therefore equality. Indeed, take $g = \frac{n}{m} fh \in (f)^{ec}$ of the form described above. For any prime $p \mid m$, we have shown that (p) is prime in $\mathbb{Z}[T]$. Furthermore, f, h are primitive, so p does not divide them $\Rightarrow p \mid n$. We can cancel p in the fraction, and repeat the process for all prime factors of m. Therefore, $g \in (f)_{\mathbb{Z}}$. Now, as f is irreducible in $\mathbb{Q}[T]$, which is a PID, $(f)_{\mathbb{Q}} = (f)_{\mathbb{Z}}^e$ is prime in $\mathbb{Q}[T]$. Therefore, so is $(f) = (f)^{ec} \subset \mathbb{Z}[T]$. □

     With all these claims, this case yields ideals of the form (f) where $f \in \mathbb{Z}[T]$ is irreducible and primitive.

3. $\mathfrak{p}$ is not principal. I don't know how to do this case. Idea: Consider the gcd of all the generators in $\mathbb{Q}[T]$, we have bezout coefficients, so we can write it as a linear combination of them with rational coefficients.