

Problem 1. Let A be a ring. Prove:

(1.1) If $x \in A$ is nilpotent, then $1 - x$ is invertible.

Solution. *Proof.* Suppose that $x^n = 0$ for some $n \in \mathbb{N}$. Then,

$$(1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = 1 - x^n = 1$$

□

(1.2) The nilradical $N(A) = \{x \in A \mid x \text{ nilpotent}\}$ is an ideal of A .

Solution. *Proof.* Let $x, y \in N(A)$, then $x^n = 0$ and $y^m = 0$ for some $n, m \in \mathbb{N}$. Then,

$$(x - y)^{n+m} = \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} x^{n+m-k} y^k = 0$$

since $k \geq n$ or $n + m - k \geq m$ for all k . Thus, $x - y \in N(A)$. We have that $N(A)$ is an additive subgroup of A . Now, let $a \in A$. Then,

$$(ax)^n = a^n x^n = 0a = 0$$

so $ax \in N(A)$. Thus, $N(A)$ is an ideal of A .

□

(1.3) $N(A)$ is contained in all prime ideals of A .

Solution. *Proof.* Let $I \subset A$ be a prime ideal. Let $x \in N(A)$. Then, $x^n = 0 \in I$ for some $n \in \mathbb{N}$. Let us show by induction on n that this implies $x \in I$:

- If $n = 1$, then $x = 0 \in I$.
- Suppose that the statement is true for $n - 1$: $x^{n-1} = 0 \Rightarrow x \in I$. If $0 = x^n = x \cdot x^{n-1}$, since I is prime, $x \in I$ or $x^{n-1} \in I$. In the first case, we are done. The second case is just the inductive hypothesis.

□

(1.4) $N(A)$ is the intersection of all prime ideals of A .

Solution.

Claim. Given $x \notin N(A)$, let Σ_x be the set of all ideals that do not contain any power of x . Then, Σ_x has a maximal element.

Proof. We will use Zorn's lemma. Let us check the conditions:

Claim. Σ_x is a partially ordered set with respect to inclusion.

Claim. Σ_x is not empty.

Proof. Since $x \notin N(A)$, $0 \in \Sigma_x$. □

Claim. Every chain in Σ_x has an upper bound.

Proof. Let $\{I_\alpha\}_{\alpha \in S}$ be a chain in Σ_x . Then, $I = \bigcup_{\alpha \in S} I_\alpha$ is an ideal of A (One can check that if $x, y \in I$, then $x, y \in I_\alpha$ for some $\alpha \in S$, and then check the axioms from there). Let $x^n \in I$ for some $n \in \mathbb{N}$. Then, $x^n \in I_\alpha$ for some $\alpha \in A$. Since I_α is an ideal, $x \in I_\alpha$. Thus, $I \in \Sigma_x$. □

Now that we have verified the conditions of Zorn's lemma, we can conclude that Σ_x has a maximal element. □

Claim. Let $x \notin N(A)$. Then the maximal element $K(x)$ of Σ_x is prime.

Proof. Let $a, b \in A$ such that $ab \in K(x)$. By way of contradiction, suppose that $a \notin K(x)$ and $b \notin K(x)$. Then, $x^n \in (a)$ and $x^m \in (b)$ for some $m, n \in \mathbb{N}$, but $x^{n+m} \notin (ab) = (a)(b)$. Contradiction. □

Finally, we can prove the statement.

Claim. $N(A)$ is the intersection of all prime ideals of A .

Proof. Let J be the intersection of all prime ideals of A . By 1.3, we know that $N(A) \subset J$. We want to prove that if $x \notin N(A)$, then $x \notin J$. Indeed, $J \subset K(x)$ because $K(x)$ is prime and $x \notin K(x)$ because $K(x)$ does not contain any power of x . □

Problem 2. Let A be a ring. Let $a_i \in A$ and $f = a_0 + a_1T + \cdots + a_nT^n \in A[T]$ be a polynomial. Prove:

(2.1) f is a unit in $A[T] \iff a_0$ is a unit in A and a_1, \dots, a_n are nilpotent.

Solution. *Proof.* We will show both implications separately.

(\Leftarrow) Let $a_i^{n_i} = 0$ for all $1 \leq i \leq n$. Consider $s = \sum_{i=1}^n n_i$. Let $h = -a_1T - \cdots - a_nT^n$ be the negative of the polynomial without the constant term. Then,

$$h^s = \left(-\sum_{i=1}^n a_i T^i \right)^s = (-1)^s \sum_{j_1 + \cdots + j_n = s} a_1^{j_1} \cdots a_n^{j_n} T^{j_1 + 2j_2 + \cdots + nj_n}$$

By the pigeonhole principle, in each term of the sum, there is at least one $j_k \geq n_k$ so $a_k^{j_k} = 0$, and thus $h^s = 0$. Then, h is nilpotent, so is $a_0^{-1} \cdot h$ and, by 1.1, $1 - a_0^{-1} \cdot h$ is invertible. Multiplying by a_0 , we get that $f = a_0 - h$ is also invertible.

(\Rightarrow) Suppose, there exists $g = b_0 + b_1T + \cdots + b_mT^m \in A[T]$ such that $1 = fg = \sum_{i=0}^{n+m} s_i T^i$, where $s_i = \sum_{j=0}^n a_j b_{i-j}$ and $b_k = 0$ for $k > m$ or $k < 0$. We first note that $1 = s_0 = a_0 b_0 \Rightarrow a_0, b_0 \in A^*$. Next, suppose $n > 0$ (Otherwise, there is nothing to show).

Claim. Let $0 \leq k \leq m$. $a_n^{k+1}b_{m-k} = 0$

Proof. By total induction on k :

- If $k = 0$, then $0 = s_{n+m} = a_n b_m = (a_n)^{0+1} b_{m-0}$.
- Suppose that the statement is true for $0, \dots, k-1$. Then, $s_{n+m-k} = 0$ as $n+m-k \geq n > 0$. Therefore:

$$0 = a_n^k s_{n+m-k} = \sum_{j=0}^n a_n^k a_j b_{n+m-k-j} = a_n^{k+1} b_{m-k} + \sum_{j=0}^{n-1} a_n^k a_j b_{n+m-k-j}$$

Now, the terms in the sum are zero by the inductive hypothesis, as $j < n \Rightarrow -(n-k-j) < k \Rightarrow (n-k-j) + 1 \leq k$. Therefore, $a_n^{k+1} b_{m-k} = 0$ as we wanted to show. □

Now, by setting $k = m$, we get that $a_n^{m+1} b_0 = 0$. Since b_0 is a unit, $a_n^{m+1} = 0$ and a_n is nilpotent. We are almost done if we realize the following:

Claim. Let $p = c_0 + c_1 T + \dots + c_l T^l \in A[T]$ be an invertible polynomial such that c_l is nilpotent. Then, $q = c_0 + c_1 T + \dots + c_{l-1} T^{l-1}$ is also invertible.

Proof. Note that $c_l T^l$ is nilpotent and so is $c_l T^l p^{-1}$. Then, $1 - c_l T^l p^{-1}$ is invertible by 1.1. Finally, because p is invertible, so is $q = p - c_l T^l = p(1 - c_l T^l p^{-1})$. □

We will prove that for $0 < k \leq n$, $a_0 + \dots + a_k T^k$ is invertible and a_k is nilpotent by (reverse) induction on k .

- a_n has already been done.
- If $0 < k < n$, by hypothesis a_{k+1} is nilpotent and $a_0 + \dots + a_{k+1} T^{k+1}$ is invertible. Then, $a_0 + \dots + a_k T^k = (a_0 + \dots + a_{k+1} T^{k+1}) - a_{k+1} T^{k+1}$ is invertible by the claim. Therefore, a_k is nilpotent.

(\Rightarrow) (Faster Version) Let f be a unit in $A[T]$.

Claim. Let $\mathfrak{p} \in \text{Spec}(A)$. Then, $a_i \in \mathfrak{p}$ for all $i \in \{1, \dots, n\}$.

Proof. $\mathfrak{p} \in \text{Spec}(A)$ implies that A/\mathfrak{p} is an integral domain. Consider the reduction $\pi : A[T] \rightarrow A/\mathfrak{p}[T]$ that takes each element $a \in A[T]$ to the class $\bar{a} \in A/\mathfrak{p}[T]$. Since A/\mathfrak{p} is an integral domain,

$$0 = \deg \bar{1} = \deg \bar{f} \cdot \overline{f^{-1}} = \deg \bar{f} + \deg \overline{f^{-1}}$$

In particular, $\deg \bar{f} = 0$ so, for all $i \in \{1, \dots, n\}$, $\bar{a}_i = \bar{0} \Rightarrow a_i \in \mathfrak{p}$. □

Since this holds for all $\mathfrak{p} \in \text{Spec}(A)$, we have that $\forall i \in \{1, \dots, n\}, a_i \in \bigcap_{\mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} = N(A)$ (by 1.4). Finally, suppose that $f^{-1} = b_0 + b_1 T + \dots + b_m T^m$. Given that $ff^{-1} = 1$ we have that $a_0 b_0 = 1$, and thus a_0 is invertible. □

(2.2) f is nilpotent $\iff a_0, \dots, a_n$ are nilpotent.

Solution. *Proof.* We will show both implications separately.

- (\Leftarrow) Just use the same argument as in the previous part of the exercise, but directly on f , not on $h = a_0 - f$.
- (\Rightarrow) If f is nilpotent, then $1 - f$ is invertible by 1.1. Thus, by the previous part, $-a_1, \dots, -a_n$ are nilpotent. Because $-a_i$ is nilpotent, a_i is nilpotent. We only have to prove that a_0 is nilpotent. By the other implication, h is nilpotent. Then, by 1.2, $a_0 = f + h$ is nilpotent.

□

(2.3) f is a zero divisor \iff there exists $a \in A$, $a \neq 0$ such that $af = 0$.

Solution. *Proof.* The backward implication is trivial because of the inclusion $A \subset A[T]$. For the forward implication, suppose that f is a zero divisor. Then, there exists $g \in A[T]$, $g \neq 0$ such that $fg = 0$. Let $g = b_0 + \dots + b_m T^m$ be the minimum degree polynomial satisfying the condition $fg = 0$. Suppose that $m > 0$.

Claim. $\exists i \in \{1, \dots, n\}$ s.t. $a_i g \neq 0$

Proof. Suppose not. Then, $a_i g = 0$ for all $i \in \{1, \dots, n\}$. Then, $a_i b_m = 0$ for all $i \in \{1, \dots, n\}$. But then we have $b_m f = 0$ with $b_m \in A$, in contradiction with g being a polynomial with minimum degree. □

Take i maximal such that $a_i g \neq 0$. Then, $0 = fg = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m) + \sum_{j=i+1}^n T^j a_j g = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m)$, and $a_i b_m = 0$. Thus, we have a polynomial $g' = a_i g \neq 0$ with degree $m - 1$ satisfying $f(a_i g) = a_i(fg) = 0$, in contradiction with g being of minimal degree. □

Problem 3. Let A be a ring. We define the Jacobson radical $\mathcal{J}(A)$ as the intersection of all maximal ideals of A . Prove:

(3.1) $x \in \mathcal{J}(A) \iff 1 - xy$ is invertible for all $y \in A$.

Solution. *Proof.* We will prove the two implications separately:

- (\Leftarrow) Suppose $x \notin \mathcal{J}(A)$. This means that there exists $\mathfrak{m} \in \text{Max}(A)$ such that $x \notin \mathfrak{m}$. Since \mathfrak{m} is a maximal ideal, $(x) + \mathfrak{m} = A$. Then, there exist $y' \in A$ and $m' \in \mathfrak{m}$ such that $xy' + m' = 1 \in A$. But this means that $1 - xy' = m' \in \mathfrak{m}$, which is invertible. Contradiction with \mathfrak{m} being an ideal. Thus, $x \in \mathcal{J}(A)$.
- (\Rightarrow) Suppose that $\exists y \in A$ such that $1 - xy$ is not a unit in A . Notice that:
- There exists a maximal ideal \mathfrak{M} such that $1 - xy \in \mathfrak{M}$.
 - Since $x \in \mathfrak{M} \subseteq \mathcal{J}(A)$, we have that $xy \in \mathfrak{M}$.

Finally, $1 - xy + xy = 1 \in \mathfrak{m}$, in contradiction with \mathfrak{m} being an ideal. Thus, $\forall y \in A$ we have that $1 - xy$ is a unit in A .

□

(3.2) $\mathcal{J}(A) = A \setminus A^* \Leftrightarrow A$ has only one maximal ideal.

Solution. *Proof.* This comes directly from the fact that an ideal $I \subset A$ cannot contain a unit x , else $xx^{-1} = 1 \in I$ and thus $I = A$.

$$\begin{aligned} \mathcal{J}(A) = A \setminus A^* &\Leftrightarrow \forall \mathfrak{m} \in \text{Max}(A), \mathfrak{m} \supseteq \{\text{non-units of } A\} = A \setminus A^* \Leftrightarrow \\ &\Leftrightarrow \forall \mathfrak{m} \in \text{Max}(A), \mathfrak{m} = \{\text{non-units of } A\} \Leftrightarrow (A, \mathfrak{m}) \text{ local} \end{aligned}$$

□

(3.3) $\mathcal{J}(A[T]) = \mathcal{N}(A[T])$.

Solution. *Proof.* We will prove the two inclusions separately:

(\supseteq)

$$\mathcal{N}(A[T]) = \bigcap \text{Spec}(A[T]) \subseteq \bigcap \text{Max}(A[T]) = \mathcal{J}(A[T])$$

The first equality follows 1.4, while the inclusion is a consequence of $\text{Max}(A[T]) \subseteq \text{Spec}(A[T])$.

(\subseteq) Take $f = a_0 + a_1T + \cdots + a_nT^n \in \mathcal{J}(A[T])$. Using 3.1 with $y = T$ we know that $1 - fT = 1 - a_0T - a_1T^2 - \cdots - a_nT^{n+1}$ is invertible in $A[t]$. But then, from 2.1 we get that $-a_0, \dots, -a_n$ nilpotents, and finally a_0, \dots, a_n are nilpotents. We conclude from 2.2 we get that f is nilpotent in $A[T]$

□

Problem 4. Let A be a ring such that every ideal not contained in the nilradical $\mathcal{N}(A)$ has a non-zero idempotent element (i.e. an element $e \neq 0$ such that $e^2 = e$). Prove that the nilradical and the Jacobson radical $\mathcal{J}(A)$ are equal.

Solution. *Proof.* We will prove the two inclusions separately:

(\supseteq) (Same as in 3.3)

$$\mathcal{N}(A) = \bigcap \text{Spec}(A) \subseteq \bigcap \text{Max}(A) = \mathcal{J}(A)$$

The first equality follows 1.4, while the inclusion is a consequence of $\text{Max}(A) \subseteq \text{Spec}(A)$.

(\subseteq) Suppose that $\mathcal{J}(A) \not\subseteq \mathcal{N}(A)$. Then, since $\mathcal{J}(A)$ is an intersection of ideals and thus an ideal itself, we have that $\exists e \in \mathcal{J}(A)$ such that $e^2 = e \neq 0$. By 3.1, we have that $1 - e1$ is invertible, so

$$(1 + e) = (1 - e)^{-1}(1 - e)(1 + e) = (1 - e)^{-1}(1 - e^2) = (1 - e)^{-1}(1 - e) = 1$$

Finally, we get that $e = 0$, in contradiction with the definition of e . Thus, we conclude that $\mathcal{J}(A) \subseteq \mathcal{N}(A)$.

¿should we write a proof of this or is it overkill?

□

Problem 5. Let A be an integral domain. Prove that the ideal $I = (x^2 - y^3, y^2 - z^3) \subseteq A[x, y, z]$ is prime.

(Hint: Let $f : R \rightarrow B$ be a ring homomorphism and $I \subseteq R$ be an ideal such that $I \subseteq \ker f$. Then f factorizes through R/I . Consider a parametrization $f : A[x, y, z] \rightarrow A[T]$.)

Solution. Proof.

□

Problem 6. (6.1) Let $f(T) \in K[T]$ be irreducible where K is a field. Prove that the ideal $(f(T))$ is maximal.

Solution. Proof. Consider $I = (f(T))$ and J such that $I \subseteq J \neq K[T]$.

Claim. If K is a field, then $K[T]$ is a P.I.D.

Proof. Take I ideal in $K[T]$ and the polynomial of minimal degree $p(T) \neq 0$ in I . Then, $d(p(T)) > 0$, otherwise $p(T) \in K$ and $pp^{-1} = 1 \in I$ (in contradiction with $I \neq K[T]$). Now, consider $g(T) \in I$. By the division algorithm, $\exists q(T), r(T) \in K[T]$ such that $g(T) = p(T)q(T) + r(T)$ with $d(r(T)) < d(p(T))$. Then $r(T) = g(T) - p(T)q(T) \in I$ since $g(T), p(T) \in I$. By minimality of $p(T) \neq 0$, we have that $r(T) = 0$. Thus, $g(T) = p(T)q(T)$, and $I = (p(T))$. □

For the claim we know that $K[T]$ is a P.I.D., thus $\exists h(T) \in K(T)$ such that $J = (h(T))$. Then

$$f(T) \in (f(T)) \subseteq J = (h(T))$$

This implies that $f(T) = h(T)g(T)$, but since $f(T)$ is irreducible, either $h(T)$ or $g(T)$ is an element of K . $h(T) \notin K$, otherwise $J = K[T]$ which is not a field. Then $g(T) \in K$ and $h(T) = f(T)g(T)^{-1}$. We conclude that $J \subset I$, so I is maximal. □

(6.2) Describe the spectrum of $\mathbb{R}[T], \mathbb{C}[T], \mathbb{R}[T]/(T^2 + 9), \mathbb{C}[T]/(T^2 + 9)$

Solution. Proof.

□