

**Problem 1.** *Let  $A$  be a ring. Prove:*

(1.1) *If  $x \in A$  is nilpotent, then  $1 - x$  is invertible.*

**Solution.** *Proof.* Suppose that  $x^n = 0$  for some  $n \in \mathbb{N}$ . Then,

$$(1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = 1 - x^n = 1$$

□

(1.2) *The nilradical  $\mathcal{N}(A) = \{x \in A \mid x \text{ nilpotent}\}$  is an ideal of  $A$ .*

**Solution.** *Proof.* Let  $x, y \in \mathcal{N}(A)$ , then  $x^n = 0$  and  $y^m = 0$  for some  $n, m \in \mathbb{N}$ . Then,

$$(x - y)^{n+m} = \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} x^{n+m-k} y^k = 0$$

since  $k \geq n$  or  $n + m - k \geq m$  for all  $k$ . Thus,  $x - y \in \mathcal{N}(A)$ . We have that  $\mathcal{N}(A)$  is an additive subgroup of  $A$ . Now, let  $a \in A$ . Then,

$$(ax)^n = a^n x^n = 0a = 0$$

so  $ax \in \mathcal{N}(A)$ . Thus,  $\mathcal{N}(A)$  is an ideal of  $A$ .

□

(1.3)  *$\mathcal{N}(A)$  is contained in all prime ideals of  $A$ .*

**Solution.** *Proof.* Let  $I \subset A$  be a prime ideal. Let  $x \in \mathcal{N}(A)$ . Then,  $x^n = 0 \in I$  for some  $n \in \mathbb{N}$ . Let us show by induction on  $n$  that this implies  $x \in I$ :

- If  $n = 1$ , then  $x = 0 \in I$ .
- Suppose that the statement is true for  $n - 1$ :  $x^{n-1} = 0 \Rightarrow x \in I$ . If  $0 = x^n = x \cdot x^{n-1}$ , since  $I$  is prime,  $x \in I$  or  $x^{n-1} \in I$ . In the first case, we are done. The second case is just the inductive hypothesis.

□

(1.4)  *$\mathcal{N}(A)$  is the intersection of all prime ideals of  $A$ .*

**Solution.**

**Claim.** *Given  $x \notin \mathcal{N}(A)$ , let  $\Sigma_x$  be the set of all ideals that do not contain any power of  $x$ . Then,  $\Sigma_x$  has a maximal element.*

*Proof.* We will use Zorn's lemma. Let us check the conditions:

**Claim.**  $\Sigma_x$  is a partially ordered set with respect to inclusion.

**Claim.**  $\Sigma_x$  is not empty.

*Proof.* Since  $x \notin \mathcal{N}(A)$ ,  $0 \in \Sigma_x$ . □

**Claim.** Every chain in  $\Sigma_x$  has an upper bound.

*Proof.* Let  $\{I_\alpha\}_{\alpha \in S}$  be a chain in  $\Sigma_x$ . Then,  $I = \bigcup_{\alpha \in S} I_\alpha$  is an ideal of  $A$  (One can check that if  $x, y \in I$ , then  $x, y \in I_\alpha$  for some  $\alpha \in S$ , and then check the axioms from there). Let  $x^n \in I$  for some  $n \in \mathbb{N}$ . Then,  $x^n \in I_\alpha$  for some  $\alpha \in A$ . Since  $I_\alpha$  is an ideal,  $x \in I_\alpha$ . Thus,  $I \in \Sigma_x$ . □

Now that we have verified the conditions of Zorn's lemma, we can conclude that  $\Sigma_x$  has a maximal element. □

**Claim.** Let  $x \notin \mathcal{N}(A)$ . Then the maximal element  $K(x)$  of  $\Sigma_x$  is prime.

*Proof.* Let  $a, b \in A$  such that  $ab \in K(x)$ . By way of contradiction, suppose that  $a \notin K(x)$  and  $b \notin K(x)$ . Then,  $x^n \in (a)$  and  $x^m \in (b)$  for some  $m, n \in \mathbb{N}$ , but  $x^{n+m} \notin (ab) = (a)(b)$ . Contradiction. □

Finally, we can prove the statement.

**Claim.**  $\mathcal{N}(A)$  is the intersection of all prime ideals of  $A$ .

*Proof.* Let  $J$  be the intersection of all prime ideals of  $A$ . By 1.3, we know that  $\mathcal{N}(A) \subset J$ . We want to prove that if  $x \notin \mathcal{N}(A)$ , then  $x \notin J$ . Indeed,  $J \subset K(x)$  because  $K(x)$  is prime and  $x \notin K(x)$  because  $K(x)$  does not contain any power of  $x$ . □

**Problem 2.** Let  $A$  be a ring. Let  $a_i \in A$  and  $f = a_0 + a_1T + \cdots + a_nT^n \in A[T]$  be a polynomial. Prove:

(2.1)  $f$  is a unit in  $A[T] \iff a_0$  is a unit in  $A$  and  $a_1, \dots, a_n$  are nilpotent.

**Solution.** *Proof.* We will show both implications separately.

( $\Leftarrow$ ) Let  $a_i^{n_i} = 0$  for all  $1 \leq i \leq n$ . Consider  $s = \sum_{i=1}^n n_i$ . Let  $h = -a_1T - \cdots - a_nT^n$  be the negative of the polynomial without the constant term. Then,

$$h^s = \left( - \sum_{i=1}^n a_i T^i \right)^s = (-1)^s \sum_{j_1 + \cdots + j_n = s} a_1^{j_1} \cdots a_n^{j_n} T^{j_1 + 2j_2 + \cdots + nj_n}$$

By the pigeonhole principle, in each term of the sum, there is at least one  $j_k \geq n_k$  so  $a_k^{j_k} = 0$ , and thus  $h^s = 0$ . Then,  $h$  is nilpotent, so is  $a_0^{-1} \cdot h$  and, by 1.1,  $1 - a_0^{-1} \cdot h$  is invertible. Multiplying by  $a_0$ , we get that  $f = a_0 - h$  is also invertible.

( $\Rightarrow$ ) Suppose, there exists  $g = b_0 + b_1T + \cdots + b_mT^m \in A[T]$  such that  $1 = fg = \sum_{i=0}^{n+m} s_i T^i$ , where  $s_i = \sum_{j=0}^n a_j b_{i-j}$  and  $b_k = 0$  for  $k > m$  or  $k < 0$ . We first note that  $1 = s_0 = a_0 b_0 \Rightarrow a_0, b_0 \in A^*$ . Next, suppose  $n > 0$  (Otherwise, there is nothing to show).

**Claim.** Let  $0 \leq k \leq m$ .  $a_n^{k+1}b_{m-k} = 0$

*Proof.* By total induction on  $k$ :

- If  $k = 0$ , then  $0 = s_{n+m} = a_n b_m = (a_n)^{0+1} b_{m-0}$ .
- Suppose that the statement is true for  $0, \dots, k-1$ . Then,  $s_{n+m-k} = 0$  as  $n+m-k \geq n > 0$ . Therefore:

$$0 = a_n^k s_{n+m-k} = \sum_{j=0}^n a_n^k a_j b_{n+m-k-j} = a_n^{k+1} b_{m-k} + \sum_{j=0}^{n-1} a_n^k a_j b_{n+m-k-j}$$

Now, the terms in the sum are zero by the inductive hypothesis, as  $j < n \Rightarrow -(n-k-j) < k \Rightarrow (n-k-j) + 1 \leq k$ . Therefore,  $a_n^{k+1} b_{m-k} = 0$  as we wanted to show. □

Now, by setting  $k = m$ , we get that  $a_n^{m+1} b_0 = 0$ . Since  $b_0$  is a unit,  $a_n^{m+1} = 0$  and  $a_n$  is nilpotent. We are almost done if we realize the following:

**Claim.** Let  $p = c_0 + c_1 T + \dots + c_l T^l \in A[T]$  be an invertible polynomial such that  $c_l$  is nilpotent. Then,  $q = c_0 + c_1 T + \dots + c_{l-1} T^{l-1}$  is also invertible.

*Proof.* Note that  $c_l T^l$  is nilpotent and so is  $c_l T^l p^{-1}$ . Then,  $1 - c_l T^l p^{-1}$  is invertible by 1.1. Finally, because  $p$  is invertible, so is  $q = p - c_l T^l = p(1 - c_l T^l p^{-1})$ . □

We will prove that for  $0 < k \leq n$ ,  $a_0 + \dots + a_k T^k$  is invertible and  $a_k$  is nilpotent by (reverse) induction on  $k$ .

- $a_n$  has already been done.
- If  $0 < k < n$ , by hypothesis  $a_{k+1}$  is nilpotent and  $a_0 + \dots + a_{k+1} T^{k+1}$  is invertible. Then,  $a_0 + \dots + a_k T^k = (a_0 + \dots + a_{k+1} T^{k+1}) - a_{k+1} T^{k+1}$  is invertible by the claim. Therefore,  $a_k$  is nilpotent.

( $\Rightarrow$ ) (Faster Version) Let  $f$  be a unit in  $A[T]$ .

**Claim.** Let  $\mathfrak{p} \in \text{Spec } A$ . Then,  $a_i \in \mathfrak{p}$  for all  $i \in \{1, \dots, n\}$ .

*Proof.*  $\mathfrak{p} \in \text{Spec } A$  implies that  $A/\mathfrak{p}$  is an integral domain. Consider the reduction  $\pi : A[T] \rightarrow A/\mathfrak{p}[T]$  that takes each element  $a \in A[T]$  to the class  $\bar{a} \in A/\mathfrak{p}[T]$ . Since  $A/\mathfrak{p}$  is an integral domain,

$$0 = \deg \bar{1} = \deg \bar{f} \cdot \overline{f^{-1}} = \deg \bar{f} + \deg \overline{f^{-1}}$$

In particular,  $\deg \bar{f} = 0$  so, for all  $i \in \{1, \dots, n\}$ ,  $\bar{a}_i = \bar{0} \Rightarrow a_i \in \mathfrak{p}$ . □

Since this holds for all  $\mathfrak{p} \in \text{Spec } A$ , we have that  $\forall i \in \{1, \dots, n\}, a_i \in \bigcap_{\mathfrak{p} \in \text{Spec } A} \mathfrak{p} = \mathcal{N}(A)$  (by 1.4). Finally, suppose that  $f^{-1} = b_0 + b_1 T + \dots + b_m T^m$ . Given that  $ff^{-1} = 1$  we have that  $a_0 b_0 = 1$ , and thus  $a_0$  is invertible. □

(2.2)  $f$  is nilpotent  $\iff a_0, \dots, a_n$  are nilpotent.

**Solution.** *Proof.* We will show both implications separately.

( $\Leftarrow$ ) Just use the same argument as in the previous part of the exercise, but directly on  $f$ , not on  $h = a_0 - f$ .

( $\Rightarrow$ ) If  $f$  is nilpotent, then  $1 - f$  is invertible by 1.1. Thus, by the previous part,  $-a_1, \dots, -a_n$  are nilpotent. Because  $-a_i$  is nilpotent,  $a_i$  is nilpotent. We only have to prove that  $a_0$  is nilpotent. By the other implication,  $h$  is nilpotent. Then, by 1.2,  $a_0 = f + h$  is nilpotent.

□

(2.3)  $f$  is a zero divisor  $\iff$  there exists  $a \in A$ ,  $a \neq 0$  such that  $af = 0$ .

**Solution.** *Proof.* The backward implication is trivial because of the inclusion  $A \subset A[T]$ . For the forward implication, suppose that  $f$  is a zero divisor. Then, there exists  $g \in A[T]$ ,  $g \neq 0$  such that  $fg = 0$ . Let  $g = b_0 + \dots + b_m T^m$  be the minimum degree polynomial satisfying the condition  $fg = 0$ . Suppose that  $m > 0$ .

**Claim.**  $\exists i \in \{1, \dots, n\}$  s.t.  $a_i g \neq 0$

*Proof.* Suppose not. Then,  $a_i g = 0$  for all  $i \in \{1, \dots, n\}$ . Then,  $a_i b_m = 0$  for all  $i \in \{1, \dots, n\}$ . But then we have  $b_m f = 0$  with  $b_m \in A$ , in contradiction with  $g$  being a polynomial with minimum degree. □

Take  $i$  maximal such that  $a_i g \neq 0$ . Then,  $0 = fg = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m) + \sum_{j=i+1}^n T^j a_j g = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m)$ , and  $a_i b_m = 0$ . Thus, we have a polynomial  $g' = a_i g \neq 0$  with degree  $m - 1$  satisfying  $f(a_i g) = a_i(fg) = 0$ , in contradiction with  $g$  being of minimal degree. □

**Problem 3.** Let  $A$  be a ring. We define the Jacobson radical  $\mathcal{J}(A)$  as the intersection of all maximal ideals of  $A$ . Prove:

(3.1)  $x \in \mathcal{J}(A) \iff 1 - xy$  is invertible for all  $y \in A$ .

**Solution.** *Proof.* We will prove the two implications separately:

( $\Leftarrow$ ) Suppose  $x \notin \mathcal{J}(A)$ . This means that there exists  $\mathfrak{m} \in \text{Max } A$  such that  $x \notin \mathfrak{m}$ . Since  $\mathfrak{m}$  is a maximal ideal,  $(x) + \mathfrak{m} = A$ . Then, there exist  $y' \in A$  and  $m' \in \mathfrak{m}$  such that  $xy' + m' = 1 \in A$ . But this means that  $1 - xy' = m' \in \mathfrak{m}$ , which is invertible. Contradiction with  $\mathfrak{m}$  being an ideal. Thus,  $x \in \mathcal{J}(A)$ .

( $\Rightarrow$ ) Suppose that  $\exists y \in A$  such that  $1 - xy$  is not a unit in  $A$ . Notice that:

- There exists a maximal ideal  $\mathfrak{M}$  such that  $1 - xy \in \mathfrak{m}$ .
- Since  $x \in \mathfrak{m} \subseteq \mathcal{J}(A)$ , we have that  $xy \in \mathfrak{m}$ .

Finally,  $1 - xy + xy = 1 \in \mathfrak{m}$ , in contradiction with  $\mathfrak{m}$  being an ideal. Thus,  $\forall y \in A$  we have that  $1 - xy$  is a unit in  $A$ .

□

(3.2)  $\mathcal{J}(A) = A \setminus A^* \Leftrightarrow A$  has only one maximal ideal.

**Solution.** *Proof.* This comes directly from the fact that an ideal  $I \subset A$  cannot contain a unit  $x$ , else  $xx^{-1} = 1 \in I$  and thus  $I = A$ .

$$\begin{aligned} \mathcal{J}(A) = A \setminus A^* &\Leftrightarrow \forall \mathfrak{m} \in \text{Max } A, \mathfrak{m} \supseteq \{\text{non-units of } A\} = A \setminus A^* \Leftrightarrow \\ &\Leftrightarrow \forall \mathfrak{m} \in \text{Max } A, \mathfrak{m} = \{\text{non-units of } A\} \Leftrightarrow (A, \mathfrak{m}) \text{ local} \end{aligned}$$

□

(3.3)  $\mathcal{J}(A[T]) = \mathcal{N}(A[T])$ .

**Solution.** *Proof.* We will prove the two inclusions separately:

( $\supseteq$ )

$$\mathcal{N}(A[T]) = \bigcap \text{Spec } A[T] \subseteq \bigcap \text{Max } A[T] = \mathcal{J}(A[T])$$

The first equality follows 1.4, while the inclusion is a consequence of  $\text{Max } A[T] \subseteq \text{Spec } A[T]$ .

( $\subseteq$ ) Take  $f = a_0 + a_1T + \cdots + a_nT^n \in \mathcal{J}(A[T])$ . Using 3.1 with  $y = T$  we know that  $1 - fT = 1 - a_0T - a_1T^2 - \cdots - a_nT^{n+1}$  is invertible in  $A[t]$ . But then, from 2.1 we get that  $-a_0, \dots, -a_n$  nilpotents, and finally  $a_0, \dots, a_n$  are nilpotents. We conclude from 2.2 we get that  $f$  is nilpotent in  $A[T]$

□

**Problem 4.** Let  $A$  be a ring such that every ideal not contained in the nilradical  $\mathcal{N}(A)$  has a non-zero idempotent element (i.e. an element  $e \neq 0$  such that  $e^2 = e$ ). Prove that the nilradical and the Jacobson radical  $\mathcal{J}(A)$  are equal.

**Solution.** *Proof.* We will prove the two inclusions separately:

( $\supseteq$ ) (Same as in 3.3)

$$\mathcal{N}(A) = \bigcap_{I \in \text{Spec } A} I \subseteq \bigcap_{I \in \text{Max } A} I = \mathcal{J}(A)$$

The first equality follows 1.4, while the inclusion is a consequence of  $\text{Max}(A) \subseteq \text{Spec}(A)$ .

( $\subseteq$ ) Suppose that  $\mathcal{J}(A) \not\subseteq \mathcal{N}(A)$ . Then, since  $\mathcal{J}(A)$  is an intersection of ideals and thus an ideal itself, we have that  $\exists e \in \mathcal{J}(A)$  such that  $e^2 = e \neq 0$ . By 3.1, we have that  $1 - e$  is invertible, so

$$(1 + e) = (1 - e)^{-1}(1 - e)(1 + e) = (1 - e)^{-1}(1 - e^2) = (1 - e)^{-1}(1 - e) = 1$$

Finally, we get that  $e = 0$ , in contradiction with the definition of  $e$ . Thus, we conclude that  $\mathcal{J}(A) \subseteq \mathcal{N}(A)$ . □

**Problem 5.** Let  $A$  be an integral domain. Prove that the ideal  $I = (x^2 - y^3, y^2 - z^3) \subseteq A[x, y, z]$  is prime.

(Hint: Let  $f : R \rightarrow B$  be a ring homomorphism and  $I \subseteq R$  be an ideal such that  $I \subseteq \ker f$ . Then  $f$  factorizes through  $R/I$ . Consider a parametrization  $f : A[x, y, z] \rightarrow A[T]$ .)

**Solution.** *Proof.* Consider the ring homomorphism

$$\begin{aligned} f : A[x, y, z] &\longrightarrow A[T] \\ x &\longmapsto T^9 \\ y &\longmapsto T^6 \\ z &\longmapsto T^4 \end{aligned}$$

Since  $f(x^2 - y^3) = (T^9)^2 - (T^6)^3 = 0$  and  $f(y^2 - z^3) = (T^6)^2 - (T^4)^3 = 0$ , we have that  $f(I) = 0$  and  $I \subseteq \ker f$ . Consider the projection  $\pi : A[x, y, z] \rightarrow A[x, y, z]/I$ . By definition of quotient,  $\ker \pi = I$ .

**Claim.** There exists  $g : A[x, y, z]/I \rightarrow A[T]$  ring homomorphism such that  $f = g \circ \pi$ .

*Proof.*  $g(\bar{a}) = f(a)$ .

- $\bar{a} = \bar{b} \Rightarrow g(\bar{a}) = g(\bar{b})?$

Suppose  $\bar{a} = \bar{b}$ . Then,  $a - b \in I$ , and since  $I = \ker f$ ,  $f(a - b) = 0$ . So,  $f(a) = f(b)$  and  $g(\bar{a}) = g(\bar{b})$ .

- $g(\lambda \bar{a}) = \lambda g(\bar{a})?$

$$g(\lambda \bar{a}) = f(\lambda a) = \lambda f(a) = \lambda g(\bar{a})$$

- $g(\bar{a} - \bar{b}) = g(\bar{a}) - g(\bar{b})?$

$$g(\bar{a} - \bar{b}) = g(\overline{a - b}) = f(a - b) = f(a) - f(b) = g(\bar{a}) - g(\bar{b})$$

□

Notice that  $(z^i)_{i \geq 0}, (xz^i)_{i \geq 0}, (yz^i)_{i \geq 0}, (xyz^i)_{i \geq 0}$  is a set of  $A$ -generators of  $A[x, y, z]/I$ . Also, their images

$$\begin{aligned} g(z^i) &= T^{4i} = T^{4i+0} \\ g(xz^i) &= T^{4i+9} = T^{4(i+2)+1} \\ g(yz^i) &= T^{4i+6} = T^{4(i+1)+2} \\ g(xyz^i) &= T^{4i+15} = T^{4(i+3)+3} \end{aligned}$$

are independent, given the non-congruence of the exponents  $\text{mod}(4)$ . Thus,  $g$  is injective. Now, take  $\bar{a}, \bar{b} \in A[x, y, z]/I$  such that  $\bar{a}\bar{b} = 0$ .

$$0 = g(0) = g(\bar{a}\bar{b}) = g(\bar{a})g(\bar{b}) \in A[T]$$

Since  $A$  is a domain,  $A[T]$  is a domain and  $g(\bar{a}) = 0$  or  $g(\bar{b}) = 0$ . But since  $g$  is injective,  $\bar{a} = 0$  or  $\bar{b} = 0$ . Thus,  $A[x, y, z]/I$  is a domain and we can conclude that  $I$  is a prime ideal.  $\square$

### Problem 6.

(6.1) Let  $f(T) \in K[T]$  be irreducible where  $K$  is a field. Prove that the ideal  $(f(T))$  is maximal.

**Solution.** *Proof.* Consider  $I = (f(T))$  and  $J$  such that  $I \subseteq J \neq K[T]$ .

**Claim.** If  $K$  is a field, then  $K[T]$  is a P.I.D.

*Proof.* Take  $I$  ideal in  $K[T]$  and the polynomial of minimal degree  $p(T) \neq 0$  in  $I$ . Then,  $d(p(T)) > 0$ , otherwise  $p(T) \in K$  and  $pp^{-1} = 1 \in I$  (in contradiction with  $I \neq K[T]$ ). Now, consider  $g(T) \in I$ . By the division algorithm,  $\exists q(T), r(T) \in K[T]$  such that  $g(T) = p(T)q(T) + r(T)$  with  $d(r(T)) < d(p(T))$ . Then  $r(T) = g(T) - p(T)q(T) \in I$  since  $g(T), p(T) \in I$ . By minimality of  $p(T) \neq 0$ , we have that  $r(T) = 0$ . Thus,  $g(T) = p(T)q(T)$ , and  $I = (p(T))$ .  $\square$

For the claim we know that  $K[T]$  is a P.I.D., thus  $\exists h(T) \in K(T)$  such that  $J = (h(T))$ . Then

$$f(T) \in (f(T)) \subseteq J = (h(T))$$

This implies that  $f(T) = h(T)g(T)$ , but since  $f(T)$  is irreducible, either  $h(T)$  or  $g(T)$  is an element of  $K$ .  $h(T) \notin K$ , otherwise  $J = K[T]$  which is not a field. Then  $g(T) \in K$  and  $h(T) = f(T)g^{-1}$ . We conclude that  $J \subset I$ , so  $I$  is maximal.  $\square$

(6.2) Describe the spectrum of  $\mathbb{R}[T], \mathbb{C}[T], \mathbb{R}[T]/(T^2 + 9), \mathbb{C}[T]/(T^2 + 9)$

**Solution.** By the first part of the exercise, we know  $\mathbb{R}[T]$  and  $\mathbb{C}[T]$  are P.I.D.s, and in particular all their nonzero prime ideals are maximal. Thus, the spec of  $\mathbb{R}[T]$  and  $\mathbb{C}[T]$  consists of the principal ideals generated by irreducible polynomials. The irreducible polynomials in  $\mathbb{R}[T]$  are well known, so:

- $\text{Spec } \mathbb{R}[T] = \{(0)\} \cup \{(T - c) \mid c \in \mathbb{R}\} \cup \{(T^2 + bT + c) \mid b, c \in \mathbb{R}, b^2 - 4c < 0\}$
- $\text{Spec } \mathbb{C}[T] = \{(0)\} \cup \{(T - c) \mid c \in \mathbb{C}\}$

This observation also helps us describe the spectrum of the quotients. We know, for any ring  $R$  and ideal  $I$ , there is a bijection between the ideals of  $R/I$  and the ideals of  $R$  that contain  $I$ , given by the projection  $\pi : R \rightarrow R/I$ . Furthermore, all prime ideals in  $R/I$  are of the form  $\pi(\mathfrak{p})$  for some prime ideal  $\mathfrak{p}$  of  $R$  that containing  $I$ .  $T^2 + 9$  is irreducible in  $\mathbb{R}[T]$ , but it factorizes (uniquely, as P.I.D  $\Rightarrow$  U.F.D) as  $(T + 3i)(T - 3i)$  in  $\mathbb{C}[T]$ . In the first case, the ideal is maximal, so

$$\text{Spec } \mathbb{R}[T]/(T^2 + 9) = \{(0)\}$$

Equivalently,  $\mathbb{R}[T]/(T^2 + 9)$  is a field. In the second case, the only prime ideals in  $\mathbb{C}$  properly containing  $(T^2 + 9)$  are  $(T + 3i)$  and  $(T - 3i)$ , so

$$\text{Spec } \mathbb{C}[T]/(T^2 + 9) = \{(0), (T + 3i), (T - 3i)\}$$

**Problem 7.** Describe  $\text{spec}(\mathbb{Z}[T])$

**Solution.** Suppose that  $\mathfrak{p} \subset \mathbb{Z}[T]$  is an ideal. We will distinguish three cases:

1.  $\mathfrak{p} = (0)$  is a prime ideal of  $\mathbb{Z}[T]$  since  $\mathbb{Z}[T]$  is an integral domain.
2.  $\mathfrak{p} = (f)$  is a principal ideal generated by a non-zero polynomial  $f \in \mathbb{Z}[T]$ . There are two possibilities:
  - $\deg f = 0 \Leftrightarrow f = n \in \mathbb{Z}$ . Suppose  $(n)$  is a prime ideal. In this case, it is clear by the inclusion  $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}[T]$  that  $(n) = (n)^c$  is prime in  $\mathbb{Z}$ . Conversely, let  $p \in \mathbb{Z}$  be a prime number.  $\mathbb{Z}[T]/(p) \cong \mathbb{Z}/(p)[T]$  is an integral domain since  $\mathbb{Z}/(p)$  is a field (in particular, an integral domain). We obtain the ideals

$$(p) \text{ where } p \in \mathbb{Z} \text{ is prime}$$

- $\deg f > 0$ .

**Claim.**  $(f)$  is prime  $\Rightarrow f$  is primitive.

*Proof.* Otherwise, there is some non-unit  $p \in \mathbb{Z}$  that divides all coefficients of  $f$ . Then  $f = pg$  for some  $g \in \mathbb{Z}[T]$ , which is a non-trivial factorization of  $f$ .  $f$  is not irreducible, so it is not prime (since  $\mathbb{Z}$  is a domain).  $\square$

**Claim.**  $(f)$  is prime  $\Rightarrow f$  is irreducible in  $\mathbb{Q}[T]$ .

*Proof.* Suppose  $f = gh$  is a non-trivial factorization, for some  $g, h \in \mathbb{Q}[T]$  (in particular, since all non-zero elements of  $\mathbb{Q}$  are units,  $\deg g, \deg h > 0 \Rightarrow \deg g, \deg h < \deg f$ ). Taking a common denominator  $n$  and multiplying by it, we obtain  $nf = \tilde{g}\tilde{h}$  in  $\mathbb{Z}[T]$ . By the left side, this belongs to  $(f)$ , but  $\tilde{g}, \tilde{h}$  do not belong to  $(f)$  since they have strictly smaller degree than  $f$ .  $\square$



**Claim.** *These conditions are also sufficient: if  $f$  is irreducible in  $\mathbb{Q}[T]$  and primitive, then  $(f)$  is prime.*

*Proof.* Consider the natural inclusion  $\phi : \mathbb{Z}[T] \rightarrow \mathbb{Q}[T]$ . We will extend and then contract the ideal  $(f)$  of  $\mathbb{Z}[T]$ . By taking a common denominator and factors:

$$(f)^{ec} = \mathbb{Z} \cap (f)_{\mathbb{Q}} = \mathbb{Z}[T] \cap \left\{ \frac{n}{m}fh \mid n, m \in \mathbb{Z}, h \in \mathbb{Z}[T] \text{ primitive} \right\}$$

We know that  $(f) \subset (f)^{ec}$ . We will prove the other inclusion and therefore equality. Indeed, take  $g = \frac{n}{m}fh \in (f)^{ec}$  of the form described above. For any prime  $p \mid m$ , we have shown that  $(p)$  is prime in  $\mathbb{Z}[T]$ . Furthermore,  $f, h$  are primitive, so  $p$  does not divide them  $\Rightarrow p \mid n$ . We can cancel  $p$  in the fraction, and repeat the process for all prime factors of  $m$ . Therefore,  $g \in (f)_{\mathbb{Z}}$ . Now, as  $f$  is irreducible in  $\mathbb{Q}[T]$ , which is a P.I.D,  $(f)_{\mathbb{Q}} = (f)_{\mathbb{Z}}^e$  is prime in  $\mathbb{Q}[T]$ . Therefore, so is  $(f) = (f)^{ec} \subset \mathbb{Z}[T]$ .  $\square$

With all these claims, this case yields ideals of the form

$$(f) \text{ where } f \in \mathbb{Z}[T] \text{ is irreducible and primitive}$$

3.  $\mathfrak{p}$  is not principal.

**Claim.**  $\mathfrak{p} \text{ prime} \Rightarrow p \in \mathfrak{p} \text{ for some } p \in \mathbb{Z} \text{ prime.}$

*Proof.* Suppose not. Then,  $\mathfrak{p}$  contains no prime number. All its elements are multiples of  $T$ . Its extension to  $\mathbb{Q}[T]$  must be a principal ideal generated by a multiple of  $T$  (say,  $\hat{g}(T) = a_k T^k + \dots + a_m T^m$ ). Contracting back to  $\mathbb{Z}[T]$ , we obtain a principal ideal  $(g)$ , ( $g$  is an appropriate multiple of  $\hat{g}$  with coprime integer coefficients). We know that  $(g) \subset \mathfrak{p}$ , but  $\mathfrak{p}$  is not principal, so we don't have equality. There exists some nonzero  $n \in \mathbb{Z}$  such that  $ng \in \mathfrak{p}$ . Since  $\mathfrak{p}$  is prime,  $n \in \mathfrak{p}$ . However, taking the prime factorization of  $n$ , we can show by induction on the number of primes that  $\mathfrak{p}$  contains a prime number. Indeed:

- If  $n$  is prime, we are done.
- If  $n = ap$  with  $p$  prime, either  $p \in \mathfrak{p}$  or  $a \in \mathfrak{p}$ , reducing the number of primes.

$\square$

There is a known bijection between the ideals of  $A$  containing an ideal  $I$  and the ideals of  $A/I$ , given by extension and contraction. That is, extending and contracting yields the same ideal. This bijection is also true if we impose that the ideals be prime, because the contraction of a prime ideal is prime. If we apply this to  $(p) \subset \mathbb{Z}[T]$ , because  $\mathbb{Z}[T]/(p) = \mathbb{Z}/(p)[T]$  is a P.I.D, it only contains prime ideals of the form  $(g)$  where  $g \in \mathbb{Z}/(p)$  is irreducible. Furthermore, because  $\mathbb{Z}/(p)$  is a field, we may take  $g$  to be monic. we have that the prime ideals of  $\mathbb{Z}[T]$  containing  $(p)$  are of the form

$$(p, g) \text{ where } g \text{ is monic and irreducible in } \mathbb{Z}/(p)[T]$$

**Problem 8.**

**Problem 9.** *Let  $A$  be a ring and let  $I$  be an ideal of  $A$ . Prove that  $S = 1 + I$  is a multiplicatively closed set and describe  $\text{Spec } S^{-1}A$ .*

**Solution.**

**Claim.**  *$S$  is multiplicatively closed.*

*Proof.*  $1 \in S$  because  $0 \in I$ . Let  $s_1, s_2 \in S$ . Then  $s_1 = 1 + i_1$  and  $s_2 = 1 + i_2$  for some  $i_1, i_2 \in I$ . Therefore,  $s_1 s_2 = (1 + i_1)(1 + i_2) = 1 + (i_1 + i_2 + i_1 i_2) \in 1 + I = S$ .  $\square$

**Claim.****Problem 10.****Problem 11.****Problem 12.****Problem 13.**