Problem 1. *Let A be a ring. Prove:*

(1.1) If $x \in A$ is nilpotent, then $1 - x$ is invertible.

Solution. *Proof.* Suppose that $x^n = 0$ for some $n \in \mathbb{N}$. Then,

$$(1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = 1 - x^n = 1$$

□

(1.2) The nilradical $N(A) = \{x \in A \,|\, x \, nilpotent\}$ is an ideal of A .

Solution. *Proof.* Let $x, y \in N(A)$, then $x^n = 0$ and $y^m = 0$ for some $n, m \in \mathbb{N}$. Then,

$$(x - y)^{n+m} = \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} x^{n+m-k} y^k = 0$$

since $k \geq n$ or $n + m - k \geq m$ for all k. Thus, $x - y \in N(A)$. We have that $N(A)$ is an additive subgroup of A. Now, let $a \in A$. Then,

$$(ax)^n = a^n x^n = 0a = 0$$

so $ax \in N(A)$. Thus, $N(A)$ is an ideal of A. □

(1.3) $N(A)$ is contained in all prime ideals of A .

Solution. *Proof.* Let $I \subset A$ be a prime ideal. Let $x \in N(A)$. Then, $x^n = 0 \in I$ for some $n \in \mathbb{N}$. Let us show by induction on n that this implies $x \in I$:

- If $n = 1$, then $x = 0 \in I$.
- Suppose that the statement is true for $n - 1$: $x^{n-1} = 0 \Rightarrow x \in I$. If $0 = x^n = x \cdot x^{n-1}$, since I is prime, $x \in I$ or $x^{n-1} \in I$. In the first case, we are done. The second case is just the inductive hypothesis.

□

(1.4) $N(A)$ is the intersection of all prime ideals of A .

Solution.

Claim. *Given $x \notin N(A)$, let Σ_x be the set of all ideals that do not contain any power of x. Then, Σ_x has a maximal element.*

Proof. We will use Zorn's lemma. Let us check the conditions:

Claim. *Σ_x is a partially ordered set with respect to inclusion.*

Claim. *Σ_x is not empty.*

Proof. Since $x \notin N(A)$, $0 \in \Sigma_x$. $\square$

Claim. *Every chain in Σ_x has an upper bound.*

Proof. Let $\{I_\alpha\}_{\alpha \in S}$ be a chain in Σ_x. Then, $I = \bigcup_{\alpha \in S} I_\alpha$ is an ideal of A (One can check that if $x, y \in I$, then $x, y \in I_\alpha$ for some $\alpha \in S$, and then check the axioms from there). Let $x^n \in I$ for some $n \in \mathbb{N}$. Then, $x^n \in I_\alpha$ for some $\alpha \in A$. Since I_α is an ideal, $x \in I_\alpha$. Thus, $I \in \Sigma_x$. $\square$

Now that we have verified the conditions of Zorn's lemma, we can conclude that Σ_x has a maximal element. $\square$

Claim. *Let $x \notin N(A)$. Then the maximal element $K(x)$ of Σ_x is prime.*

Proof. Let $a, b \in A$ such that $ab \in K(x)$. By way of contradiction, suppose that $a \notin K(x)$ and $b \notin K(x)$. Then, $x^n \in (a)$ and $x^m \in (b)$ for some $m, n \in \mathbb{N}$, but $x^{n+m} \notin (ab) = (a)(b)$. Contradiction. $\square$

Finally, we can prove the statement.

Claim. *$N(A)$ is the intersection of all prime ideals of A.*

Proof. Let J be the intersection of all prime ideals of A. By 1.3, we know that $N(A) \subset J$. We want to prove that if $x \notin N(A)$, then $x \notin J$. Indeed, $J \subset K(x)$ because $K(x)$ is prime and $x \notin K(x)$ because $K(x)$ does not contain any power of x. $\square$

Problem 2. *Let A be a ring. Let $a_i \in A$ and $f = a_0 + a_1 T + \cdots + a_n T^n \in A[T]$ be a polynomial. Prove:*

(2.1) f is a unit in $A[T]$ $\iff$ a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

Solution. *Proof.* We will show both implications separately.

($\Leftarrow$) Let $a_i^{n_i} = 0$ for all $1 \leq i \leq n$. Consider s $= \Sigma_{i=1}^n n_i$. Let h $= -a_1 T - \cdots - a_n T^n$ be the negative of the polynomial without the constant term. Then,

$$h^s = \left(- \sum_{i=1}^{n} a_i T^i \right)^s = (-1)^s \sum_{j_1 + \cdots + j_n = s} a_1^{j_1} \cdots a_n^{j_n} T^{j_1 + 2 \cdot j_2 \cdots + n \cdot j_n}$$

By the pigeonhole principle, in each term of the sum, there is at least one $j_k \geq n_k$ so $a_k^{j_k} = 0$, and thus $h^s = 0$. Then, h is nilpotent, so is $a_0^{-1} \cdot h$ and, by 1.1, $1 - a_0^{-1} \cdot h$ is invertible. Multiplying by a_0, we get that $f = a_0 - h$ is also invertible.

($\Rightarrow$) Suppose, there exists $g = b_0 + b_1 T + \cdots + b_m T^m \in A[T]$ such that $1 = fg = \sum_{i=0}^{n+m} s_i T^i$, where $s_i = \sum_{j=0}^{n} a_j b_{i-j}$ and $b_k = 0$ for $k > m$ or $k < 0$. We first note that $1 = s_0 = a_0 b_0 \Rightarrow a_0, b_0 \in A^*$. Next, suppose $n > 0$ (Otherwise, there is nothing to show).

Claim. *Let $0 \leq k \leq m$. $a_n^{k+1} b_{m-k} = 0$*

Proof. By total induction on k:

- If $k = 0$, then $0 = s_{n+m} = a_n b_m = (a_n)^{0+1} b_{m-0}$.
- Suppose that the statement is true for $0, \ldots, k-1$. Then, $s_{n+m-k} = 0$ as $n + m - k \geq n > 0$. Therefore:

$$0 = a_n^k s_{n+m-k} = \sum_{j=0}^{n} a_n^k a_j b_{n+m-k-j} = a_n^{k+1} b_{m-k} + \sum_{j=0}^{n-1} a_n^k a_j b_{n+m-k-j}$$

  Now, the terms in the sum are zero by the inductive hypothesis, as $j < n \Rightarrow -(n-k-j) < k \Rightarrow (n-k-j)+1 \leq k$. Therefore, $a_n^{k+1} b_{m-k} = 0$ as we wanted to show.

□

Now, by setting $k = m$, we get that $a_n^{m+1} b_0 = 0$. Since b_0 is a unit, $a_n^{m+1} = 0$ and a_n is nilpotent. We are almost done if we realize the following:

Claim. *Let $p = c_0 + c_1 T + \cdots + c_l T^l \in A[T]$ be an invertible polynomial such that c_l is nilpotent. Then, $q = c_0 + c_1 T + \cdots + c_{l-1} T^{l-1}$ is also invertible.*

Proof. Note that $c_l T^l$ is nilpotent and so is $c_l T^l p^{-1}$. Then, $1 - c_l T^l p^{-1}$ is invertible by 1.1. Finally, because p is invertible, so is $q = p - c_l T^l = p(1 - c_l T^l p^{-1})$. □

We will prove that for $0 < k \leq n$, $a_0 + \cdots + a_k T^k$ is invertible and a_k is nilpotent by (reverse) induction on k.

- a_n has already been done.
- If $0 < k < n$, by hypothesis a_{k+1} is nilpotent and $a_0 + \cdots + a_{k+1} T^{k+1}$ is invertible. Then, $a_0 + \cdots + a_k T^k = (a_0 + \cdots + a_{k+1} T^{k+1}) - a_{k+1} T^{k+1}$ is invertible by the claim. Therefore, a_k is nilpotent.

($\Rightarrow$) (Faster Version) Let f be a unit in $A[T]$.

Claim. *Let $\mathfrak{p} \in Spec(A)$. Then, $a_i \in \mathfrak{p}$ for all $i \in \{1, \ldots n\}$.*

Proof. $\mathfrak{p} \in Spec(A)$ implies that $A/\mathfrak{p}$ is an integral domain. Consider the reduction $\pi : A[T] \to A/\mathfrak{p}[T]$ that takes each element $a \in A[T]$ to the class $\overline{a} \in A/\mathfrak{p}[T]$. Since $A/\mathfrak{p}$ is an integral domain,

$$0 = \deg \bar{1} = \deg \bar{f} \cdot \overline{f^{-1}} = \deg \bar{f} + \deg \overline{f^{-1}}$$

In particular, $\deg \bar{f} = 0$ so, for all $i \in \{1, \ldots n\}$, $\overline{a_i} = \bar{0} \Rightarrow a_i \in \mathfrak{p}$. □

Since this holds for all $\mathfrak{p} \in Spec(A)$, we have that $\forall i \in \{1, \ldots n\}, a_i \in \bigcap_{\mathfrak{p} \in Spec(A)} \mathfrak{p} = N(A)$ (by 1.4). Finally, suppose that $f^{-1} = b_0 + b_1 T + \cdots + b_m T^m$. Given that $f f^{-1} = 1$ we have that $a_0 b_0 = 1$, and thus a_0 is invertible.

□

(2.2) f is nilpotent $\iff$ $a_0, \dots, a_n$ are nilpotent.

Solution. *Proof.* We will show both implications separately.

($\Leftarrow$) Just use the same argument as in the previous part of the exercise, but directly on f, not on $h = a_0 - f$.

($\Rightarrow$) If f is nilpotent, then $1 - f$ is invertible by 1.1. Thus, by the previous part, $-a_1, \dots, -a_n$ are nilpotent. Because $-a_i$ is nilpotent, a_i is nilpotent. We only have to prove that a_0 is nilpotent. By the other implication, h is nilpotent. Then, by 1.2, $a_0 = f + h$ is nilpotent.

□

(2.3) f is a zero divisor $\iff$ there exists $a \in A$, $a \neq 0$ such that $af = 0$.

Solution. *Proof.* The backward implication is trivial because of the inclusion $A \subset A[T]$. For the forward implication, suppose that f is a zero divisor. Then, there exists $g \in A[T]$, $g \neq 0$ such that $fg = 0$. Let $g = b_0 + \dots + b_m T^m$ be the minimum degree polynomial satisfying the condition $fg = 0$. Suppose that $m > 0$.

Claim. $\exists i \in \{1, \dots, n\}$ *s.t.* $a_i g \neq 0$

Proof. Suppose not. Then, $a_i g = 0$ for all $i \in \{1, \dots, n\}$. Then, $a_i b_m = 0$ for all $i \in \{1, \dots, n\}$. But then we have $b_m f = 0$ with $b_m \in A$, in contradiction with g being a polynomial with minimum degree. □

Take i maximal such that $a_i g \neq 0$. Then, $0 = fg = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m) + \sum_{j=i+1}^{n} T^j a_j g = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m)$, and $a_i b_m = 0$. Thus, we have a polynomial $g' = a_i g \neq 0$ with degree $m - 1$ satisfying $f(a_i g) = a_i(fg) = 0$, in contradiction with g being of minimal degree. □