Commutative Algebra. Theory

Ferran Espuña

December 8, 2023

1 Rings and Ideals

Remark 1.1. Unless otherwise specified, all rings we discuss will be commutative with unit.

Definition 1.2. We say that $I \subseteq A$ is and *ideal* if:

- $(I, +)$ is an abelian group.
- For all $a \in A$ and for all $x \in I$, $ax \in I$.

Definition 1.3. We define:

- The *radical* of an ideal I is $rad(I) \coloneqq \{a \in A \mid a^n \in I, n > 0\}$.
- (*Colon ideal*) $(I : J) \coloneqq \{a \in A \mid aJ \subseteq I\}$.
- (*Saturation*) $(I : J^\infty) \coloneqq \{a \in A \mid \exists n > 0 \text{ s.t. } aJ^n \subseteq I\}$.

Definition 1.4. Let $f : A \to B$ be a ring homomorphism, and let $J \subset B$ be an ideal. Then, the *contraction* of J to A is $J^c \coloneqq \{a \in A \mid f(a) \in J\}$.

Proposition 1.5. *In the above situation, J^c is an ideal of A.*

Proof. It is an additive subgroup of A as f is an additive group homomorphism. Furthermore, if $a \in J^c$ and $r \in A$, then $f(a) \in J$ so $f(ra) = f(r)f(a) \in J$ as J is an ideal. $\square$

Definition 1.6. Let R be a ring and $S \subset R$ a subset. Then, the *ideal generated by* S is $\langle S \rangle \coloneqq \bigcap_{I \supset S} I$, where I ranges over all ideals of R.

Remark 1.7. $\langle S \rangle$ is the smallest ideal of R containing S. It can be checked that it is indeed an ideal by noticing that all elements of it belong to all ideals containing S, and veryfying the axioms from there.

Proposition 1.8. *In the above situation,*

$$\langle S\rangle = \{a_1s_1 + \cdots + a_ns_n \ \mid n \in \mathbb{N}, a_1, \ldots, a_n \in A, s_1, \ldots, s_n \in S\}$$

Proof. We will show both inclusions.

($\subset$) $\langle S\rangle$ is an ideal containing S, so it contains all elements of the form $a_1s_1 + \cdots + a_ns_n$.

($\supset$) One can easily check that the set on the right is an ideal containing S, so it contains the intersection of all such ideals, which is $\langle S\rangle$.

$\square$

Definition 1.9. Let $f : A \to B$ be a ring homomorphism, and let $I \subset A$ be an ideal. Then, the *extension* of I to B is $IB = I^e := \langle f(I)\rangle$.

Remark 1.10. I^e is by contstruction, the smallest ideal of B containing $f(I)$.

Proposition 1.11. *In the above situation, let $I^{ec} := (I^e)^c$, and similarly for the rest. Then:*

1. $I \subset I^{ec}$.
2. $J^{ce} \subset J$
3. $I^e = I^{ece}$
4. $J^c = J^{cec}$

Proof. Note that if K is an ideal of A, then $f(K) \subset K^e$. Furthermore, extension and contraction clearly respect inclusions.

1. $I \subset f^{-1}(f(I)) \subset f^{-1}(I^e) = I^{ec}$.
2. $(J^c)^e = \langle f(J^c)\rangle = \langle f(f^{-1}(J))\rangle \subset \langle J\rangle = J$.
3. By the two previous points, $(I^e)^{ce} \subset I^e$ and $I \subset I^{ec}$ so $I^e \subset (I^{ec})^e$.
4. Similarly, $J^c \subset (J^c)^{ec}$ and $J^{ce} \subset J$ so $(J^{ce})^c \subset J^c$.

$\square$

Definition 1.12. Let I be an ideal of a ring A. The *quotient ring* A/I is the ring whose elements are the cosets of I in A, and whose operations are defined by

$$(a+I)+(b+I)=(a+b)+I$$
$$(a+I)\cdot(b+I)=(a\cdot b)+I$$

Remark 1.13. The sum operation is well defined as it is for all quotient groups. The product operation is well defined as I is an ideal so if $p \in (a+I)$, $q \in (b+I)$, then $pq \in ab+aI+bI+I^2 \subset (ab+I)$.

Proposition 1.14. *Let I be an ideal of a ring A. Then, the canonical projection $\pi : A \to A/I$ is a ring homomorphism.*

Proof. It is clearly a group homomorphism. Furthermore, $\pi(a)\pi(b) = (a+I)(b+I) = ab+I = \pi(ab)$. Finally, $\pi(1) = 1+I$ is clearly the unit of A/I. □

Definition 1.15. Let $I \subsetneq R$ be an ideal. Then:

- I is *prime* if $ab \in I \Rightarrow a \in I$ or $b \in I$.
- I is *maximal* there are no ideals J such that $I \subsetneq J \subsetneq R$.

We further define the *spectrum* of R as

$$\mathrm{Spec}(R) \coloneqq \{\mathfrak{p} \subset R \text{ ideal} \mid \mathfrak{p} \text{ is prime}\}.$$

and

$$\mathrm{Max}(R) \coloneqq \{\mathfrak{m} \subset R \text{ ideal} \mid \mathfrak{m} \text{ is maximal}\}.$$

Proposition 1.16. *Let $\mathfrak{p} \subset R$ be an ideal. Then, $\mathfrak{p} \in Spec(R) \Leftrightarrow R/\mathfrak{p}$ is an integral domain.*

Proof. Let

$$\pi : R \to R/\mathfrak{p}$$
$$a \mapsto \bar{a} \coloneqq a+\mathfrak{p}$$

be the canonical projection. Then, π is a ring homomorphism. Suppose $a, b \in R$. Then,

$$ab \in \mathfrak{p} \Leftrightarrow \bar{a}\bar{b} = 0$$
$$(a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}) \Leftrightarrow (\bar{a} = 0 \text{ or } \bar{b} = 0)$$

Therefore, as π is surjective, $R/\mathfrak{p}$ is an integral domain if and only if $\mathfrak{p}$ is prime, as both conditions are equivalent to the first condition implying the second. □

Lemma 1.17. *A ring S is a field if and only if it has no nontrivial ideals $0 \subsetneq I \subsetneq S$*

Proof. We will show both implications.

($\Rightarrow$) Let $0 \subsetneq I \subsetneq S$ be an ideal. Then, it has a nonzero element a which must be a unit, so $1 = aa^{-1} \in I$, so $I \subsetneq S = (1) \subset I$, a contradiction.

($\Leftarrow$) Let $a \in S$ be nonzero. Then, $(a) \neq 0$ so $(a) = S$, so $1 \in (a)$, so a is a unit.

□

Proposition 1.18. *Let $\mathfrak{m} \subset R$ be an ideal. Then, $\mathfrak{m} \in Max(R) \Leftrightarrow R/\mathfrak{m}$ is a field.*

Proof. We will use the above characterization of fields. Because the canonical projection $\pi : R \to R/\mathfrak{m}$ is surjective, Contraction by it respects inequalities, that is $I \subsetneq J \Rightarrow I^c \subsetneq J^c$. On the other hand, for ideals containing $\mathfrak{m}$, extension by π is just the canonical projection, so $I \subsetneq J \Rightarrow I^e \subsetneq J^e$. Finally, $R^e = R/\mathfrak{m}$ and $(R/\mathfrak{m})^c = R$. Therefore, we have a bijection between notrivial ideals of $R/\mathfrak{m}$ and ideals $\mathfrak{m} \subsetneq I \subsetneq R$. One of the sets is empty if and only if the other is. □

Remark 1.19. All fields are integral domains, so $\mathrm{Max}(R) \subset \mathrm{Spec}(R)$. That is, all maximal ideals are prime.

Theorem 1.20. *All rings have maximal ideals.*

Proof. Exercise. Use Zorn's lemma. □

Theorem 1.21. *Let A be a ring and $I \subseteq A$ be an ideal. Then, there exists a maximal ideal $\mathfrak{m} \subseteq A$ such that $I \subseteq \mathfrak{m}$.*

Proof. Exercise. Use Zorn's lemma. □

Definition 1.22. We say that a ring R is *local* if it has a unique maximal ideal $\mathfrak{m}$.

Proposition 1.23. *Let R be a ring. R is local if and only if $R \setminus R^*$ is an ideal (which is then necessarily the maximal ideal).*

Proof. We will show both implications.

($\Rightarrow$) Let R be local with maximal ideal $\mathfrak{m}$. We will show that $\mathfrak{m}$ is exactly $R \setminus R^*$. Let $a \in R \setminus R^*$. Then, $(a) \neq R$ so $(a) \subset \mathfrak{m}$, as it must be contained in some maximal ideal. In particular, $a \in \mathfrak{m}$. On the other hand, if $a \in \mathfrak{m}$, then $(a) \subset \mathfrak{m} \subsetneq R$, so $a \notin R^*$.

($\Leftarrow$) Let $I \subsetneq R$ be an ideal. If it contained a unit, then it would contain 1, so $I = R$. Therefore, $I \subset R \setminus R^*$ so the latter is the unique maximal ideal.

$\square$

Definition 1.24. Let R be a ring and $S \subset R$ a subset. We say that S is *multiplicatively closed* if $1 \in S$ and $a, b \in S \Rightarrow ab \in S$.

Definition 1.25. Let R be a ring and $S \subset R$ a multiplicatively closed subset. Then, $S^{-1}R := \{\frac{a}{b} \mid a \in R, b \in S\}/\sim$, where

$$\frac{a}{s} \sim \frac{b}{t} \Leftrightarrow \exists r \in S \colon r(at - bs) = 0 \tag{1}$$

Proposition 1.26. *In the definition above, $\sim$ is an equivalence relation.*

Proof. The relation is obviously symmetric and reflexive. Let us show that it is transitive. Suppose $\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \sim \frac{a_3}{s_3}$:

$$\begin{aligned} t_1(a_1 s_2 - a_2 s_1) = 0 \\ t_2(a_2 s_3 - a_3 s_2) = 0 \end{aligned}$$

For some $t_1, t_2 \in S$. Then, multiply the first equation by $t_2 s_3$ and the second by $t_1 s_1$ to get

$$\begin{aligned} t_1 t_2 s_3 a_1 s_2 - t_1 t_2 s_3 a_2 s_1 = 0 \\ t_1 t_2 s_1 a_2 s_3 - t_1 t_2 s_1 a_3 s_2 = 0 \end{aligned}$$

Adding both equations and collecting like terms, we get

$$0 = t_1 t_2 s_3 a_1 s_2 - t_1 t_2 s_1 a_3 s_2 = t_1 t_2 s_2 (a_1 s_3 - a_3 s_1) \tag{2}$$

Because S is multiplicatively closed, $t_1 t_2 s_2 \in S$, so $\frac{a_1}{s_1} \sim \frac{a_3}{s_3}$.

□

Proposition 1.27. *The usual operations*

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$
$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

are well defined and make $S^{-1}R$ into a ring.

Proof. Left as an exercise. The arguments are tedious but similar to the above proposition. □

Proposition 1.28. *Let R be a ring and $S \subset R$ a multiplicatively closed subset. Then, the canonical projection*

$$\pi : R \to S^{-1}R$$
$$a \mapsto \frac{a}{1}$$

is a ring homomorphism. It is injective if and only if S contains no zero divisors.

Proof. $\pi(1) = \frac{1}{1}$ is clearly the unit of $S^{-1}R$. Furthermore, $\pi(a)\pi(b) = \frac{a}{1}\frac{b}{1} = \frac{ab}{1} = \pi(ab)$. $\pi(a) + \pi(b) = \frac{a}{1} + \frac{b}{1} = \frac{1 \cdot a + 1 \cdot b}{1 \cdot 1} = \frac{a+b}{1} = \pi(a+b)$. We have that it is a ring homomorphism. $\text{Ker}(\pi) = \{a \in R \mid |\frac{a}{1} \sim \frac{0}{1}\} = \{a \in R \mid |\exists s \in S \mid 0 = s(a \cdot 1 + 0 \cdot 1) = sa\}$. Indeed, the kernel is the set of all elements of R that are annihilated by some element of S. In particular, it contains only zero if and only if S contains no zero divisors. □

Proposition 1.29. *Let A be a ring and $S \subset R$ a multiplicatively closed subset. Let $f : A \to B$ be a ring homomorphism such that $f(S) \subset B^*$. Then, there exists a unique ring homomorphism $g : S^{-1}A \to B$ such that $f = g \circ \pi$, where π is the canonical projection. That is, the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\pi"'] & B \\
S^{-1}A \arrow[ur, dashed, "\exists ! g"] &
\end{tikzcd}$$

Proof. Suppose that such a g exists. By definition,

$$g\left(\frac{a}{1}\right) = f(a)$$

For any $s \in S$,

$$1 = f(1) = g\left(\frac{s}{1}\frac{1}{s}\right) = g\left(\frac{s}{1}\right)g\left(\frac{1}{s}\right) \Rightarrow g\left(\frac{1}{s}\right) = g\left(\frac{s}{1}\right)^{-1} = f(s)^{-1}$$

Then,

$$g\left(\frac{a}{s}\right) = g\left(\frac{a}{1}\frac{1}{s}\right) = g\left(\frac{a}{1}\right)g\left(\frac{1}{s}\right) = f(a)f(s)^{-1}$$

Therefore, g is uniquely determined by f. The derived expression for g clearly makes the diagram commute. Finally, we will show that it is well defined and that it is a ring homomorphism. Suppose $\frac{a}{s} = \frac{b}{t}$. Then, for some $u \in S$,

$$u(at - bs) = 0 \Rightarrow 0 = f(0) = f(u(at - bs)) = f(u)(f(a)f(t) - f(b)f(s))$$

But $f(u), f(s), f(t) \in B^*$, so

$$f(a)f(t) = f(b)f(s) \Rightarrow g\left(\frac{a}{s}\right) = f(a)f(s)^{-1} = f(b)f(t)^{-1} = g\left(\frac{b}{t}\right)$$

Therefore, g is well defined. Finally, let us check that it is a ring homomorphism:

- $g(\frac{a}{s}) + g(\frac{b}{t}) = f(a)f(s)^{-1} + f(b)f(t)^{-1} = f(a)f(t)f(t)^{-1}f(s)^{-1} + f(b)f(s)f(t)^{-1}f(s)^{-1} = f(at + bs)f(st)^{-1} = g(\frac{at+bs}{st}) = g(\frac{a}{s} + \frac{b}{t})$
- $g(\frac{a}{s})g(\frac{b}{t}) = f(a)f(s)^{-1}f(b)f(t)^{-1} = f(ab)f(st)^{-1} = g(\frac{ab}{st}) = g(\frac{a}{s}\frac{b}{t})$
- $g(\frac{1}{1}) = 1$

$\square$

Proposition 1.30. *In the above situation, let $I \subset A$ be an ideal. Consider the canonical projection $\pi : A \to S^{-1}A$. Then we can write the extension of I by π as $I^e =: IS^{-1}A = S^{-1}I := \{\frac{j}{s} \mid j \in I, s \in S\}$. Furthermore, every ideal of $S^{-1}A$ is of this form.*

Proof. It is clear that $S^{-1}I \subset IS^{-1}A$. Suppose that $x = \frac{j_1}{1}\frac{a_1}{s_1} + \cdots + \frac{j_n}{1}\frac{a_n}{s_n} \in IS^{-1}A$, with $j_i \in I, a_i \in A, s_i \in S$. Then, by applying the definition of addition repeatedly, we get $x = \frac{j}{s}$, with $j \in I$. Therefore $x \in I^{-1}B$.
Now, take any ideal $J \subset S^{-1}A$. We know by 1.11 $J^{ce} \subset J$. We will show the reverse inclusion in this case. Let $\frac{a}{s} \in J$. Then $\pi(a) = \frac{a}{1} = \frac{a}{s}\frac{s}{1} \in J \Rightarrow a \in J^c \Rightarrow \frac{a}{s} \in J^{ce}$. In particular, $J = J^{ce} = J^c S^{-1}A = S^{-1}J^c$. □

Definition 1.31. (*Total frection ring*) $Tot(A) := S^{-1}A$ where $S = \{a \in A \mid a \text{ is not a zero divisor }\}$.

Definition 1.32. (*Localization at an element*) $A_f := S^{-1}A$ where $S = \{f^n \mid n \geq 0\}$ for a given $f \in A$.

Definition 1.33. Let A be a ring and $\mathfrak{p} \in \mathrm{Spec}(A)$ a prime ideal. Then, the *localization of A at $\mathfrak{p}$* is $A_\mathfrak{p} := S^{-1}A$, where $S = A \setminus \mathfrak{p}$.

Remark 1.34. Indeed, S is multiplicatively closed as $\mathfrak{p}$ is prime.

Proposition 1.35. *In the above situation, $\frac{a}{s} \in A_\mathfrak{p}$ is a unit $\iff a \notin \mathfrak{p}$.*

Proof. We will show both implications.

(⇒) Suppose $\frac{a}{s} \in A_\mathfrak{p}$ is a unit. Then, there is some $\frac{b}{t} \in A_\mathfrak{p}$ such that $\frac{a}{s}\frac{b}{t} = \frac{1}{1}$. For some $u \notin \mathfrak{p}, u(ab - st) = 0$. Rearanging, $uab = ust$. But neither of u, s, t are in $\mathfrak{p}$, so their product ust is not in $\mathfrak{p}$ (because $\mathfrak{p}$ is prime). In particular, $a \notin \mathfrak{p}$.

(⇐) Suppose $a \notin \mathfrak{p}$. Then, $\frac{a}{s}\frac{s}{a} = 1$

□

Proposition 1.36. *In the above situation, $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$.*

Proof. By Proposition 1.23, We just need to show that $\mathfrak{p}A_\mathfrak{p} = A_\mathfrak{p} \setminus (A_\mathfrak{p})^*$. Indeed, by 1.30 we know $\mathfrak{p}A_\mathfrak{p} = \{\frac{a}{s} \mid a \in \mathfrak{p}, s \notin \mathfrak{p}\}$, which by the previous proposition is exactly the set of nonunits of $A_\mathfrak{p}$.

□