Problem 1. *Let A be a ring. Prove:*

(1.1) If $x \in A$ is nilpotent, then $1-x$ is invertible.

Solution. *Proof.* Suppose that $x^n = 0$ for some $n \in \mathbb{N}$. Then,

$$(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n = 1$$

□

(1.2) The nilradical $\mathcal{N}(A) = \{x \in A \,|\, x \text{ nilpotent}\}$ is an ideal of A .

Solution. *Proof.* Let $x, y \in \mathcal{N}(A)$, then $x^n = 0$ and $y^m = 0$ for some $n, m \in \mathbb{N}$. Then,

$$(x-y)^{n+m} = \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} x^{n+m-k} y^k = 0$$

since $k \geq n$ or $n+m-k \geq m$ for all k. Thus, $x - y \in \mathcal{N}(A)$. We have that $\mathcal{N}(A)$ is an additive subgroup of A. Now, let $a \in A$. Then,

$$(ax)^n = a^n x^n = 0a = 0$$

so $ax \in \mathcal{N}(A)$. Thus, $\mathcal{N}(A)$ is an ideal of A. □

(1.3) $\mathcal{N}(A)$ is contained in all prime ideals of A .

Solution. *Proof.* Let $I \subset A$ be a prime ideal. Let $x \in \mathcal{N}(A)$. Then, $x^n = 0 \in I$ for some $n \in \mathbb{N}$. Let us show by induction on n that this implies $x \in I$:

- If $n = 1$, then $x = 0 \in I$.
- Suppose that the statement is true for $n-1$: $x^{n-1} = 0 \Rightarrow x \in I$. If $0 = x^n = x \cdot x^{n-1}$, since I is prime, $x \in I$ or $x^{n-1} \in I$. In the first case, we are done. The second case is just the inductive hypothesis.

□

(1.4) $\mathcal{N}(A)$ is the intersection of all prime ideals of A .

Solution.

Claim. *Given $x \notin \mathcal{N}(A)$, let Σ_x be the set of all ideals that do not contain any power of x. Then, Σ_x has a maximal element.*

Proof. We will use Zorn's lemma. Let us check the conditions:

Claim. *Σ_x is a partially ordered set with respect to inclusion.*

Claim. *Σ_x is not empty.*

Proof. Since $x \notin \mathcal{N}(A)$, $0 \in \Sigma_x$. $\square$

Claim. *Every chain in Σ_x has an upper bound.*

Proof. Let $\{I_\alpha\}_{\alpha \in S}$ be a chain in Σ_x. Then, $I = \bigcup_{\alpha \in S} I_\alpha$ is an ideal of A (One can check that if $x, y \in I$, then $x, y \in I_\alpha$ for some $\alpha \in S$, and then check the axioms from there). Let $x^n \in I$ for some $n \in \mathbb{N}$. Then, $x^n \in I_\alpha$ for some $\alpha \in A$. Since I_α is an ideal, $x \in I_\alpha$. Thus, $I \in \Sigma_x$. $\square$

Now that we have verified the conditions of Zorn's lemma, we can conclude that Σ_x has a maximal element. $\square$

Claim. *Let $x \notin \mathcal{N}(A)$. Then the maximal element $K(x)$ of Σ_x is prime.*

Proof. Let $a, b \in A$ such that $ab \in K(x)$. By way of contradiction, suppose that $a \notin K(x)$ and $b \notin K(x)$. Then, $x^n \in (a)$ and $x^m \in (b)$ for some $m, n \in \mathbb{N}$, but $x^{n+m} \notin (ab) = (a)(b)$. Contradiction. $\square$

Finally, we can prove the statement.

Claim. *$\mathcal{N}(A)$ is the intersection of all prime ideals of A.*

Proof. Let J be the intersection of all prime ideals of A. By 1.3, we know that $\mathcal{N}(A) \subset J$. We want to prove that if $x \notin \mathcal{N}(A)$, then $x \notin J$. Indeed, $J \subset K(x)$ because $K(x)$ is prime and $x \notin K(x)$ because $K(x)$ does not contain any power of x. $\square$

Problem 2. *Let A be a ring. Let $a_i \in A$ and $f = a_0 + a_1 T + \cdots + a_n T^n \in A[T]$ be a polynomial. Prove:*

(2.1) f is a unit in $A[T]$ $\iff$ a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

Solution. *Proof.* We will show both implications separately.

($\Leftarrow$) Let $a_i^{n_i} = 0$ for all $1 \leq i \leq n$. Consider s $= \Sigma_{i=1}^n n_i$. Let h $= -a_1 T - \cdots - a_n T^n$ be the negative of the polynomial without the constant term. Then,

$$h^s = \left(-\sum_{i=1}^n a_i T^i \right)^s = (-1)^s \sum_{j_1 + \cdots + j_n = s} a_1^{j_1} \cdots a_n^{j_n} T^{j_1 + 2 \cdot j_2 \cdots + n \cdot j_n}$$

By the pigeonhole principle, in each term of the sum, there is at least one $j_k \geq n_k$ so $a_k^{j_k} = 0$, and thus $h^s = 0$. Then, h is nilpotent, so is $a_0^{-1} \cdot h$ and, by 1.1, $1 - a_0^{-1} \cdot h$ is invertible. Multiplying by a_0, we get that $f = a_0 - h$ is also invertible.

($\Rightarrow$) Suppose, there exists $g = b_0 + b_1 T + \cdots + b_m T^m \in A[T]$ such that $1 = fg = \sum_{i=0}^{n+m} s_i T^i$, where $s_i = \sum_{j=0}^n a_j b_{i-j}$ and $b_k = 0$ for $k > m$ or $k < 0$. We first note that $1 = s_0 = a_0 b_0 \Rightarrow a_0, b_0 \in A^*$. Next, suppose $n > 0$ (Otherwise, there is nothing to show).

Claim. *Let* $0 \le k \le m$. $a_n^{k+1} b_{m-k} = 0$

Proof. By total induction on k:

- If $k = 0$, then $0 = s_{n+m} = a_n b_m = (a_n)^{0+1} b_{m-0}$.
- Suppose that the statement is true for $0, \dots, k-1$. Then, $s_{n+m-k} = 0$ as $n + m - k \ge n > 0$. Therefore:

$$0 = a_n^k s_{n+m-k} = \sum_{j=0}^{n} a_n^k a_j b_{n+m-k-j} = a_n^{k+1} b_{m-k} + \sum_{j=0}^{n-1} a_n^k a_j b_{n+m-k-j}$$

Now, the terms in the sum are zero by the inductive hypothesis, as $j < n \Rightarrow -(n-k-j) < k \Rightarrow (n-k-j)+1 \le k$. Therefore, $a_n^{k+1} b_{m-k} = 0$ as we wanted to show.

□

Now, by setting $k = m$, we get that $a_n^{m+1} b_0 = 0$. Since b_0 is a unit, $a_n^{m+1} = 0$ and a_n is nilpotent. We are almost done if we realize the following:

Claim. *Let* $p = c_0 + c_1 T + \cdots + c_l T^l \in A[T]$ *be an invertible polynomial such that* c_l *is nilpotent. Then,* $q = c_0 + c_1 T + \cdots + c_{l-1} T^{l-1}$ *is also invertible.*

Proof. Note that $c_l T^l$ is nilpotent and so is $c_l T^l p^{-1}$. Then, $1 - c_l T^l p^{-1}$ is invertible by 1.1. Finally, because p is invertible, so is $q = p - c_l T^l = p(1 - c_l T^l p^{-1})$. □

We will prove that for $0 < k \le n$, $a_0 + \cdots + a_k T^k$ is invertible and a_k is nilpotent by (reverse) induction on k.

- a_n has already been done.
- If $0 < k < n$, by hypothesis a_{k+1} is nilpotent and $a_0 + \cdots + a_{k+1} T^{k+1}$ is invertible. Then, $a_0 + \cdots + a_k T^k = (a_0 + \cdots + a_{k+1} T^{k+1}) - a_{k+1} T^{k+1}$ is invertible by the claim. Therefore, a_k is nilpotent.

($\Rightarrow$) (Faster Version) Let f be a unit in $A[T]$.

Claim. *Let* $\mathfrak{p} \in \operatorname{Spec} A$. *Then,* $a_i \in \mathfrak{p}$ *for all* $i \in \{1, \dots n\}$.

Proof. $\mathfrak{p} \in \operatorname{Spec} A$ implies that $A/\mathfrak{p}$ is an integral domain. Consider the reduction $\pi : A[T] \to A/\mathfrak{p}[T]$ that takes each element $a \in A[T]$ to the class $\overline{a} \in A/\mathfrak{p}[T]$. Since $A/\mathfrak{p}$ is an integral domain,

$$0 = \deg \bar{1} = \deg \bar{f} \cdot \overline{f^{-1}} = \deg \bar{f} + \deg \overline{f^{-1}}$$

In particular, $\deg \bar{f} = 0$ so, for all $i \in \{1, \dots n\}$, $\overline{a_i} = \bar{0} \Rightarrow a_i \in \mathfrak{p}$. □

Since this holds for all $\mathfrak{p} \in \operatorname{Spec} A$, we have that $\forall i \in \{1, \dots n\}, a_i \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \mathcal{N}(A)$ (by 1.4). Finally, suppose that $f^{-1} = b_0 + b_1 T + \cdots + b_m T^m$. Given that $ff^{-1} = 1$ we have that $a_0 b_0 = 1$, and thus a_0 is invertible.

□

(2.2) f is nilpotent $\iff$ $a_0, \ldots, a_n$ are nilpotent.

Solution. *Proof.* We will show both implications separately.

($\Leftarrow$) Just use the same argument as in the previous part of the exercise, but directly on f, not on $h = a_0 - f$.

($\Rightarrow$) If f is nilpotent, then $1 - f$ is invertible by 1.1. Thus, by the previous part, $-a_1, \ldots, -a_n$ are nilpotent. Because $-a_i$ is nilpotent, a_i is nilpotent. We only have to prove that a_0 is nilpotent. By the other implication, h is nilpotent. Then, by 1.2, $a_0 = f + h$ is nilpotent.

□

(2.3) f is a zero divisor $\iff$ there exists $a \in A$, $a \neq 0$ such that $af = 0$.

Solution. *Proof.* The backward implication is trivial because of the inclusion $A \subset A[T]$. For the forward implication, suppose that f is a zero divisor. Then, there exists $g \in A[T]$, $g \neq 0$ such that $fg = 0$. Let $g = b_0 + \cdots + b_m T^m$ be the minimum degree polynomial satisfying the condition $fg = 0$. Suppose that $m > 0$.

Claim. *$\exists i \in \{1, \ldots, n\}$ s.t. $a_i g \neq 0$*

Proof. Suppose not. Then, $a_i g = 0$ for all $i \in \{1, \ldots, n\}$. Then, $a_i b_m = 0$ for all $i \in \{1, \ldots, n\}$. But then we have $b_m f = 0$ with $b_m \in A$, in contradiction with g being a polynomial with minimum degree. □

Take i maximal such that $a_i g \neq 0$. Then, $0 = fg = (a_0 + \cdots + a_i T^i)(b_0 + \cdots + b_m T^m) + \sum_{j=i+1}^{n} T^j a_j g = (a_0 + \cdots + a_i T^i)(b_0 + \cdots + b_m T^m)$, and $a_i b_m = 0$. Thus, we have a polynomial $g' = a_i g \neq 0$ with degree $m - 1$ satisfying $f(a_i g) = a_i(fg) = 0$, in contradiction with g being of minimal degree. □

Problem 3. *Let A be a ring. We define the Jacobson radical $\mathcal{J}(A)$ as the intersection of all maximal ideals of A. Prove:*

(3.1) $x \in \mathcal{J}(A) \Leftrightarrow 1 - xy$ is invertible for all $y \in A$.

Solution. *Proof.* We will prove the two implications separately:

($\Leftarrow$) Suppose $x \notin \mathcal{J}(A)$. This means that there exists $\mathfrak{m} \in \operatorname{Max} A$ such that $x \notin \mathfrak{m}$. Since $\mathfrak{m}$ is a maximal ideal, $(x) + \mathfrak{m} = A$. Then, there exist $y' \in A$ and $m' \in \mathfrak{m}$ such that $xy' + m' = 1 \in A$. But this means that $1 - xy' = m' \in \mathfrak{m}$, which invertible. Contradiction with $\mathfrak{m}$ being an ideal. Thus, $x\mathcal{J}(A)$.

($\Rightarrow$) Suppose that $\exists y \in A$ such that $1 - xy$ is not a unit in A. Notice that:

- There exists a maximal ideal $\mathfrak{M}$ such that $1 - xy \in \mathfrak{m}$.
- Since $x \in \mathfrak{m} \subseteq \mathcal{J}(A)$, we have that $xy \in \mathfrak{m}$.

Finally, $1 - xy + xy = 1 \in \mathfrak{m}$, in contradiction with $\mathfrak{m}$ being an ideal. Thus, $\forall y \in A$ we have that $1 - xy$ is a unit in A.

□

(3.2) $\mathcal{J}(A) = A \setminus A*$ $\Leftrightarrow$ *A has only one maximal ideal.*

Solution. *Proof.* This comes directly from the fact that an ideal $I \subset A$ cannot contain a unit x, else $xx^{-1} = 1 \in I$ and thus $I = A$.

$$\mathcal{J}(A) = A \setminus A^* \Leftrightarrow \forall \mathfrak{m} \in \operatorname{Max} A, \mathfrak{m} \supseteq \{\text{non-units of } A\} = A \setminus A^* \Leftrightarrow$$
$$\Leftrightarrow \forall \mathfrak{m} \in \operatorname{Max} A, \mathfrak{m} = \{\text{non-units of } A\} \Leftrightarrow (A, \mathfrak{m}) \text{ local}$$

□

(3.3) $\mathcal{J}(A[T]) = \mathcal{N}(A[T])$.

Solution. *Proof.* We will prove the two inclusions separately:

($\supseteq$)

$$\mathcal{N}(A[T]) = \bigcap \operatorname{Spec} A[T] \subseteq \bigcap \operatorname{Max} A[T] = \mathcal{J}(A[T])$$

The first equality follows 1.4, while the inclusion is a consequence of $\operatorname{Max} A[T] \subseteq \operatorname{Spec} A[T]$.

($\subseteq$) Take $f = a_0 + a_1T + \cdots + a_nT^n \in \mathcal{J}(A[T])$. Using 3.1 with $y = T$ we know that $1 - fT = 1 - a_0T - a_1T^2 \cdots - a_nT^{n+1}$ is invertible in $A[t]$. But then, from 2.1 we get that $-a_0, \cdots, -a_n$ nilponents, and finally $a_0, \cdots a_n$ are nilponents. We conclude from 2.2 we get that f is nilponent in $A[T]$

□

Problem 4. *Let A be a ring such that every ideal not contained in the nilradical* $\mathcal{N}(A)$ *has a non-zero idempotent element (i.e. an element* $e \neq 0$ *such that* $e^2 = e$*). Prove that the nilradical and the Jacobson radical* $\mathcal{J}(A)$ *are equal.*

Solution. *Proof.* We will prove the two inclusions separately:

($\supseteq$) (Same as in 3.3)

$$\mathcal{N}(A) = \bigcap_{I \in \operatorname{Spec} A} I \subseteq \bigcap_{I \in \operatorname{Max} A} I = \mathcal{J}(A)$$

The first equality follows 1.4, while the inclusion is a consequence of $Max(A) \subseteq Spec(A)$.

($\subseteq$) Suppose that $\mathcal{J}(A) \nsubseteq \mathcal{N}(A)$. Then, since $\mathcal{J}(A)$ is an intersection of ideals and thus an ideal itself, we have that $\exists e \in \mathcal{J}(A)$ such that $e^2 = e \neq 0$. By 3.1, we have that $1 - e1$ is invertible, so

$$(1+e) = (1-e)^{-1}(1-e)(1+e) = (1-e)^{-1}(1-e^2) = (1-e)^{-1}(1-e) = 1$$

Finally, we get that $e = 0$, in contradiction with the definition of e. Thus, we conclude that $\mathcal{J}(A) \subseteq \mathcal{N}(A)$.

□

Problem 5. *Let A be an integral domain. Prove that the ideal $I = (x^2 - y^3, y^2 - z^3) \subseteq A[x, y, z]$ is prime.*

(Hint: Let $f : R \to B$ be a ring homomorphism and $I \subseteq R$ be an ideal such that $I \subseteq \ker f$. Then f factorizes through R/I. Consider a parametrization $f : A[x, y, z] \to A[T]$.)

Solution. *Proof.* Consider the ring homomorphism

$$\begin{aligned} f\colon A[x,y,z] &\longrightarrow A[T] \\ x &\longmapsto T^9 \\ y &\longmapsto T^6 \\ z &\longmapsto T^4 \end{aligned}$$

Since $f(x^2 - y^3) = (T^9)^2 - (T^6)^3 = 0$ and $f(y^2 - z^3) = (T^6)^2 - (T^4)^3 = 0$, we have that $f(I) = 0$ and $I \subseteq \ker f$. Consider the projection $\pi : A[x, y, z] \to A[x, y, z]/I$. By definition of quotient, $\ker \pi = I$.

Claim. *There exists $g : A[x, y, z]/I \to A[T]$ ring homomorphism such that $f = g \circ \pi$.*

Proof. $g(\bar{a}) = f(a)$.

- $\underline{\bar{a} = \bar{b} \Rightarrow g(\bar{a}) = g(\bar{b})?}$
  Suppose $\bar{a} = \bar{b}$. Then, $a - b \in I$, and since $I = \ker f$, $f(a - b) = 0$. So, $f(a) = f(b)$ and $g(\bar{a}) = g(\bar{b})$.

- $\underline{g(\lambda\bar{a}) = \lambda g(\bar{a})?}$
  $$g(\lambda\bar{a}) = f(\lambda a) = \lambda f(a) = \lambda g(\bar{a})$$

- $\underline{g(\bar{a} - \bar{b}) = g(\bar{a}) - g(\bar{b})?}$

$$g(\bar{a} - \bar{b}) = g(\overline{a - b}) = f(a - b) = f(a) - f(b) = g(\bar{a}) - g(\bar{b})$$

□

Notice that $(z^i)_{i\geq 0}, (xz^i)_{i\geq 0}, (yz^i)_{i\geq 0}, (xyz^i)_{i\geq 0}$ is a set of A-generators of $A[x,y,z]/I$. Also, their images

$$\begin{aligned} g(z^i) &= T^{4i} = T^{4i+0} \\ g(xz^i) &= T^{4i+9} = T^{4(i+2)+1} \\ g(yz^i) &= T^{4i+6} = T^{4(i+1)+2} \\ g(xyz^i) &= T^{4i+15} = T^{4(i+3)+3} \end{aligned}$$

are independent, given the non-congruence of the exponents $mod(4)$. Thus, g is injective. Now, take $\bar{a}, \bar{b} \in A[x,y,z]/I$ such that $\overline{ab} = 0$.

$$0 = g(0) = g(\overline{ab}) = g(\bar{a})g(\bar{b}) \in A[T]$$

Since A is a domain, $A[T]$ is a domain and $g(\bar{a}) = 0$ or $g(\bar{b}) = 0$. But since g is injective, $\bar{a} = 0$ or $\bar{b} = 0$. Thus, $A[x,y,z]/I$ is a domain and we can conclude that I is a prime ideal. □

Problem 6.

(6.1) Let $f(T) \in K[T]$ be irreducible where K is a field. Prove that the ideal $(f(T))$ is maximal.

Solution. *Proof.* Consider $I = (f(T))$ and J such that $I \subseteq J \neq K[T]$.

Claim. *If K is a field, then $K[T]$ is a P.I.D.*

Proof. Take I ideal in $K[T]$ and the polynomial of minimal degree $p(T) \neq 0$ in I. Then, $d(p(T)) > 0$, otherwise $p(T) \in K$ and $pp^{-1} = 1 \in I$ (in contradiction with $I \neq K[T]$). Now, consider $g(T) \in I$. By the division algorithm, $\exists q(T), r(T) \in K[T]$ such that $g(T) = p(T)q(T) + r(T)$ with $d(r(T)) < d(p(T))$. Then $r(T) = g(T) - p(T)q(T) \in I$ since $g(T), p(T) \in I$. By minimality of $p(T) \neq 0$, we have that $r(T) = 0$. Thus, $g(T) = p(T)q(T)$, and $I = (p(T))$. □

For the claim we know that $K[T]$ is a P.I.D., thus $\exists h(T) \in K(T)$ such that $J = (h(T))$. Then

$$f(T) \in (f(T)) \subseteq J = (h(T))$$

This implies that $f(T) = h(T)g(T)$, but since $f(T)$ is irreducible, either $h(T)$ or $g(T)$ is an element of K. $h(T) \notin K$, otherwise $J = K[T]$ which is not a field. Then $g(T) \in K$ and $h(T) = f(T)a^{-1}$. We conclude that $J \subset I$, so I is maximal. □

(6.2) Describe the spectrum of $\mathbb{R}[T], \mathbb{C}[T], \mathbb{R}[T]/(T^2+9), \mathbb{C}[T]/(T^2+9)$

Solution. By the first part of the exercise, we know $\mathbb{R}[T]$ and $\mathbb{C}[T]$ are P.I.D.s, and in particular all their nonzero prime ideals are maximal. Thus, the spec of $\mathbb{R}[T]$ and $\mathbb{C}[T]$ consists of the principal ideals generated by irreducible polynomials. The irreducible polynomials in $\mathbb{R}[T]$ are well known, so:

- $\operatorname{Spec}\mathbb{R}[T] = \{(0)\} \cup \{(T-c) \mid c \in \mathbb{R}\} \cup \{(T^2+bT+c) \mid b, c \in \mathbb{R},\, b^2-4c<0\}$
- $\operatorname{Spec}\mathbb{C}[T] = \{(0)\} \cup \{(T-c) \mid c \in \mathbb{C}\}$

This observation also helps us describe the spectrum of the quotients. We know, for any ring R and ideal I, there is a bijection between the ideals of R/I and the ideals of R that contain I, given by the projection $\pi : R \to R/I$. Furthermore, all prime ideals in R/I are of the form $\pi(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R that containing I. T^2+9 is irreducible in $\mathbb{R}[T]$, but it factorizes (uniquely, as P.I.D $\Rightarrow$ U.F.D) as $(T+3i)(T-3i)$ in $\mathbb{C}[T]$. In the first case, the ideal is maximal, so

$$\operatorname{Spec}\mathbb{R}[T]/(T^2+9) = \{(0)\}$$

Equivalently, $\mathbb{R}[T]/(T^2+9)$ is a field. In the second case, the only prime ideals in $\mathbb{C}$ properly containing (T^2+9) are $(T+3i)$ and $(T-3i)$, so

$$\operatorname{Spec}\mathbb{C}[T]/(T^2+9) = \{(0), (T+3i), (T-3i)\}$$

Problem 7. *Describe spec*$(\mathbb{Z}[T])$

Solution. Suppose that $\mathfrak{p} \subset \mathbb{Z}[T]$ is an ideal. We will distinguish three cases:

1. $\mathfrak{p} = (0)$ is a prime ideal of $\mathbb{Z}[T]$ since $\mathbb{Z}[T]$ is an integral domain.

2. $\mathfrak{p} = (f)$ is a principal ideal generated by a non-zero polynomial $f \in \mathbb{Z}[T]$. There are two possibilities:

   - $\deg f = 0 \Leftrightarrow f = n \in \mathbb{Z}$. Suppose (n) is a prime ideal. In this case, it is clear by the inclusion $\varphi : \mathbb{Z} \to \mathbb{Z}[T]$ that $(n) = (n)^c$ is prime in $\mathbb{Z}$. Conversely, let $p \in \mathbb{Z}$ be a prime number. $\mathbb{Z}[T]/(p) \cong \mathbb{Z}/(p)[T]$ is an integral domain since $\mathbb{Z}/(p)$ is a field (in particular, an integral domain). We obtain the ideals

     $$(p) \text{ where } p \in \mathbb{Z} \text{ is prime}$$

   - $\deg f > 0$.

     Claim. (f) *is prime* $\Rightarrow$ f *is primitive.*

     Proof. Otherwise, there is some non-unit $p \in \mathbb{Z}$ that divides all coefficients of f. Then $f = pg$ for some $g \in \mathbb{Z}[T]$, which is a non-trivial factorization of f. f is not irreducible, so it is not prime (since $\mathbb{Z}$ is a domain). □

     Claim. (f) *is prime* $\Rightarrow$ f *is irreducible in* $\mathbb{Q}[T]$.

     Proof. Suppose $f = gh$ is a non-trivial factorization, for some $g, h \in \mathbb{Q}[T]$ (in particular, since all non-zero elements of $\mathbb{Q}$ are units, $\deg g, \deg h > 0 \Rightarrow \deg g, \deg h < \deg f$). Taking a common denominator n and multiplying by it, we obtain $nf = \tilde{g}\tilde{h}$ in $\mathbb{Z}[T]$. By the left side, this belongs to (f), but $\tilde{g}, \tilde{h}$ do not belong to (f) since they have strictly smaller degree than f. □

Claim. *These conditions are also sufficient: if f is irreducible in $\mathbb{Q}[T]$ and primitive, then (f) is prime.*

Proof. Consider the natural inclusion $\phi : \mathbb{Z}[T] \to \mathbb{Q}[T]$. We will extend and then contract the ideal (f) of $\mathbb{Z}[T]$. By taking a common denominator and factors:

$$(f)^{ec} = \mathbb{Z}[\mathbb{T}] \cap (f)_{\mathbb{Q}} = \mathbb{Z}[T] \cap \left\{ \frac{n}{m} fh \mid n, m \in \mathbb{Z}, h \in \mathbb{Z}[T] \, \text{primitive} \right\}$$

We know that $(f) \subset (f)^{ec}$. We will prove the other inclusion and therefore equality. Indeed, take $g = \frac{n}{m} fh \in (f)^{ec}$ of the form described above. For any prime $p \mid m$, we have shown that (p) is prime in $\mathbb{Z}[T]$. Furthermore, f, h are primitive, so p does not divide them $\Rightarrow p \mid n$. We can cancel p in the fraction, and repeat the process for all prime factors of m. Therefore, $g \in (f)_{\mathbb{Z}}$. Now, as f is irreducible in $\mathbb{Q}[T]$, which is a P.I.D, $(f)_{\mathbb{Q}} = (f)^e_{\mathbb{Z}}$ is prime in $\mathbb{Q}[T]$. Therefore, so is $(f) = (f)^{ec} \subset \mathbb{Z}[T]$. ☐

With all these claims, this case yields ideals of the form

(f) where $f \in \mathbb{Z}[T]$ is irreducible and primitive

3. $\mathfrak{p}$ is not principal.

Claim. *$\mathfrak{p}$ prime $\Rightarrow p \in \mathfrak{p}$ for some $p \in \mathbb{Z}$ prime.*

Proof. Suppose not. Then, $\mathfrak{p}$ contains no prime number. All its elements are multiples of T. Its extension to $\mathbb{Q}[T]$ must be a principal ideal generated by a multiple of T (say, $\hat{g}(T) = a_k T^k + \cdots + a_m T^m$). Contracting back to $\mathbb{Z}[T]$, we obtain a principal ideal (g), (g is an appropriate multiple of $\hat{g}$ with coprime integer coefficients). We know that $(g) \subset \mathfrak{p}$, but $\mathfrak{p}$ is not principal, so we don't have equality. There exists some nonzero $n \in \mathbb{Z}$ such that $ng \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $n \in \mathfrak{p}$. However, taking the prime factorization of n, we can show by induction on the number of primes that $\mathfrak{p}$ contains a prime number. Indeed:

- If n is prime, we are done.
- If $n = ap$ with p prime, either $p \in \mathfrak{p}$ or $a \in \mathfrak{p}$, reducing the number of primes.

☐

There is a known bijection between the ideals of A containing an ideal I and the ideals of A/I, given by extension and contraction. That is, extending and contracting yields the same ideal. This bijection is also true if we impose that the ideals be prime, because the contraction of a prime ideal is prime. If we apply this to $(p) \subset \mathbb{Z}[T]$, because $\mathbb{Z}[T]/(p) = \mathbb{Z}/(p)[T]$ is a P.I.D, it only contains prime ideals of the form (g) where $g \in \mathbb{Z}/(p)$ is irreducible. Furthermore, because $\mathbb{Z}/(p)$ is a field, we may take g to be monic. we have that the prime ideals of $\mathbb{Z}[T]$ containing (p) are of the form

(p, g) where g is monic and irreducible in $\mathbb{Z}/(p)[T]$

Problem 8. *Which of the following are integral domains? If they are, describe their respective fields of fractions.*

1. $\mathbb{Q}[T]$

   Solution. It is a domain because $\mathbb{Q}$ is a domain. As the field of fractions we get $\mathbb{Q}(T) \coloneqq \{\frac{p}{q} \mid p, q \in Q[T]\}$, that is, the set of rational functions on one variable.

2. $\mathbb{Z}[T]$

   Solution. By the same reasoning, it is an integral domain. Note that the homomorphism by the inclusion $\mathbb{Z} \to \mathbb{Q}$ from its field of fractions (we shal call it $\mathbb{Z}(T)$) into $\mathbb{Q}(T)$ is injective as it is a field homomorphism. Furthermore, it is surjective: by taking common denominators, we can express any element of $\mathbb{Q}(T)$ as $\frac{p'/a}{q'/b} = \frac{p'b}{q'a}$, where $p', q' \in \mathbb{Z}[T]$, $a, b \in \mathbb{Z}$. Therefore, the field of fractions of $\mathbb{Z}[T]$ is also $\mathbb{Q}(T)$.

3. $\mathbb{Z}[T]/(T^n)$

   Solution. This is not a domain for $n > 1$, as $T^{n-1}T = T^n = 0$, but none of the factors is 0. For $n = 1$, it is isomorphic to $\mathbb{Z}$, which is a domain with field of fractions $\mathbb{Q}$.

4. $\mathbb{Z}[T]/(p^n, T), p \in \mathbb{Z}$ *prime.*

   Solution. This is not a domain for $n > 1$, as $p^{n-1}p = p^n = 0$, but none of the factors is 0. For $n = 1$, it is isomorphic to $\mathbb{Z}/(p)$, which is a domain and in fact a field, so its field of fractions is itself.

5. $\mathbb{Z}[T]/(p, T^n), p \in \mathbb{Z}$ *prime.*

   Solution. This is not a domain for $n > 1$, as $T^{n-1}T = p^n = 0$, but none of the factors is 0. For $n = 1$, it is exactly the same as the previous case.

6. $\mathbb{Z}/n\mathbb{Z}[T]$

   Solution. This is isomorphic to $\mathbb{Z}[T]/(n)$ and therefore it is a domain if and only if (n) is prime. By the previous exercise, the ideal $(n) \subset \mathbb{Z}[T]$ is prime if and only if n is prime. In that case, its field of fractions is $\mathbb{Z}/n\mathbb{Z}(T) = \{\frac{p}{q} \mid p, q \in \mathbb{Z}/n\mathbb{Z}[T]\}$.

7. *$A \times B$, where A and B are any given rings.*

   Solution. This is never domain, as $(1, 0)(0, 1) = (0, 0) = 0$, but none of the factors is 0 (we are working with nontrivial rings, $1 \neq 0$).

Problem 9. *Let A be a ring and let I be an ideal of A. Prove that $S = 1 + I$ is a multiplicatively closed set and describe $\operatorname{Spec} S^{-1}A$.*

Solution.

Claim. *S is multiplicatively closed.*

Proof. $1 \in S$ because $0 \in I$. Let $s_1, s_2 \in S$. Then $s_1 = 1 + i_1$ and $s_2 = 1 + i_2$ for some $i_1, i_2 \in I$. Therefore, $s_1 s_2 = (1 + i_1)(1 + i_2) = 1 + (i_1 + i_2 + i_1 i_2) \in 1 + I = S$. □

Claim.

Problem 10. *Let $I = (4, 2T, T^2)$ be an ideal of $\mathbb{Z}[T]$. Prove that I is primary but not irreducible, checking that $I = (4, T) \cap (2, T^2)$.*

Solution. *Proof.* First of all, let's prove that I is primary.

$$rad(I) = rad((2^2, 2T, T^2)) = (2, 2T, T) = (2, T)$$

By 7, $(2, T)$ is maximal, so I is $(2, T)$-primary. Next, we will see that $I = (4, T) \cap (2, T^2)$ by proving both inclusions:

($\subseteq$) We can trivially see that the generators of $I = (4, 2T, T^2)$ belong to both $(4, T)$ and $(2, T^2)$.

($\supseteq$) Take $x \in (4, T) \cap (2, T^2)$. Since $x \in (2, T^2)$, $\exists q(T), f(T) = a_0 + a_1 T + \cdots + a_n T_n \in \mathbb{Z}[T]$ such that

$$x = 2f(T) + T^2 q(T) = 2(a_0 + a_1 T + \cdots + a_n T_n) + T^2 q(T) = 2a_0 + T(Tq(T) + 2(a_1 + \cdots + a_n T^{n-1}))$$

Then, since $x \in (4, T)$ we have that $4 | 2a_0 \Rightarrow 2 | a_0$ and it exists $a_0' \in \mathbb{Z}$ such that $a_0 = 2a_0'$. We conclude that $x = 4a_0' + T^2 q(T) + 2T(a_1 + \cdots + a_n T^{n-1}) \in (4, 2T, T^2) = I$.

Finally, it is easy to see that:

- $(4, T)$ is $(2, T)$-primary, since $rad((4, T)) = rad((2^2, T)) = (2, T)$ and $(2, T)$ is maximal.
- $(2, T^2)$ is $(2, T)$-primary, since $rad((2, T^2)) = (2, T)$ and $(2, T)$ is maximal.
- $(4, T) \neq (2, T)$, since $2 \notin (4, T)$.
- $(2, T^2) \neq (2, T)$, since $T \notin (2, T^2)$.

Thus, $I = (4, T) \cap (2, T^2)$ is not irreducible. □

Problem 11. *Prove that in the polynomial ring $\mathbb{Z}[T]$ the ideal $\mathfrak{m} = (2, T)$ is maximal and the ideal $\mathfrak{q} = (4, T)$ is $\mathfrak{m}$-primary but not a power of $\mathfrak{m}$.*

Solution. *Proof.* We proved in 7 that $(2, T)$ is maximal and in 10 we proved that $\mathfrak{q} = (4, T)$ is $\mathfrak{m}$-primary. Let's prove that $\mathfrak{q}$ is not a power of $\mathfrak{m}$. First, notice that:

$$\cdots \subseteq \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \cdots \subseteq \mathfrak{m}^2 \subseteq \mathfrak{m}$$

Also:

($\mathfrak{m}^2 \subsetneq \mathfrak{q}$) It is easy to see that the generators of $I = (4, 2T, T^2)$ belong to $\mathfrak{q} = (4, T)$. But $T \notin (4, 2T, T^2)$, so $\mathfrak{m}^2 \neq \mathfrak{q}$.

($\mathfrak{q} \subsetneq \mathfrak{m}$) Since $rad(\mathfrak{q}) = \mathfrak{m}$, $\mathfrak{q} \subseteq \mathfrak{m}$. But $2 \notin (4, 2T, T^2)$, so $\mathfrak{q} \neq \mathfrak{m}$.

Thus,

$$\cdots \subseteq \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \cdots \subseteq \mathfrak{m}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$$

and $\mathfrak{q}$ is not a power of $\mathfrak{m}$. □

Problem 12. *Let $A = k[x,y,z]$ be a polynomial ring over a field k. Let $\mathfrak{p}_1 = (x,y)$, $\mathfrak{p}_2 = (x,z)$ and $\mathfrak{m} = (x,y,z)$ be ideals of A. Let $I = \mathfrak{p}_1\mathfrak{p}_2$. Prove that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of I and describe the associated primes of I.*

Solution. *Proof.* First of all we need to prove that $I := \mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. We will prove both inclusions:

($\subseteq$) $I = \mathfrak{p}_1\mathfrak{p}_2 = (x^2, xy, xz, yz)$. It is easy to see that all the generators of I, are multiples of either x or y, they are multiples of x or z and they are generators of $\mathfrak{m}^2 = (x^2, y^2, z^2, xy, xz, yz)$. Thus $I \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$

($\supseteq$) Take $f \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Since $f \in \mathfrak{m}^2$, $\exists a_0, a_1, a_2, a_3, a_4, a_5 \in k$ such that

$$f = a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz$$

Now,

$$a_2z^2 = f - (a_0x^2 + a_1y^2 + a_3xy + a_4xz + a_5yz) \in \mathfrak{p}_1$$

so $x|a_2$ or $y|a_2$. Thus $xz|a_2z^2$ or $yz|a_2z^2$ and we can conclude that $a_2z^2 \in I$. Also,

$$a_1y^2 = f - (a_0x^2 + a_2z^2 + a_3xy + a_4xz + a_5yz) \in \mathfrak{p}_2$$

so $x|a_1$ or $z|a_1$. Thus $xy|a_1y^2$ or $yz|a_1y^2$ and we can conclude that $a_1y^2 \in I$.

$$f = \underbrace{a_1y^2}_{\in I} + \underbrace{a_2z^2}_{\in I} + \underbrace{a_0x^2 + a_3xy + a_4xz + a_5yz}_{\in I} \in I$$

Now, it is easy to see that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime in $k[x,y,z]$ and thus they are primary. Also, $rad(\mathfrak{m}^2) = rad(x^2, y^2, z^2, xy, xz, yz) = (x, y, z, xy, xz, yz) = (x, y, z) = \mathfrak{m}$, and since $\mathfrak{m}$ is maximal, $\mathfrak{m}^2$ is $\mathfrak{m}$-primary. So $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a primary decomposition. Now we will prove that it is minimal. First of all, notice that $rad(\mathfrak{p}_1) = \mathfrak{p}_1 = (x,y)$, $rad(\mathfrak{p}_2) = \mathfrak{p}_2 = (x,z)$ and $rad(\mathfrak{m}^2) = \mathfrak{m} = (x,y,z)$ are different. Finally, we see that:

- $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x,y) \cap (x,z) = (x, yz) \nsubseteq (x^2, y^2, z^2, xy, xz, yz) = \mathfrak{m}^2$, since $x \notin \mathfrak{m}^2$.
- $\mathfrak{p}_1 \cap \mathfrak{m}^2 = (x,y) \cap (x^2, y^2, z^2, xy, xz, yz) = (x^2, y^2, xy, xz, yz) \nsubseteq (x,z) = \mathfrak{p}_2$, since $y^2 \notin \mathfrak{p}_2$.
- $\mathfrak{p}_2 \cap \mathfrak{m}^2 = (x,z) \cap (x^2, y^2, z^2, xy, xz, yz) = (x^2, z^2, xy, xz, yz) \nsubseteq (x,y) = \mathfrak{p}_1$, since $z^2 \notin \mathfrak{p}_1$.

This proves that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ s a minimal primary decomposition of I. Also,

$$Ass(A/I) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{m}\}$$

□

Problem 13. *Let A be a ring and let I be an ideal of A. Prove:*

(13.1) If $\mathfrak{p}$ is a minimal prime of I, then $\mathfrak{p}[T]$ is a minimal prime of $I[T]$, and any minimal prime of $I[T]$ is of this form.

Solution. Let $\mathfrak{p}$ be a minimal prime of I. Suppose $J \subseteq A[T]$ is the minimal prime such that $I[T] \subseteq J \subseteq \mathfrak{p}[T]$. By LO QUE HA EXPLICAO FERRAN, $\exists \mathfrak{p}' \in A$ prime such that $J = \mathfrak{p}'[T]$. Then, we take the contraction $I \subseteq \mathfrak{p}' \subseteq \mathfrak{p}$ and by definition of $\mathfrak{p}$ being a minimal of I, we conclude that $\mathfrak{p}' = \mathfrak{p}$ and $\mathfrak{p}[T]' = \mathfrak{p}[T]$. Thus, $\mathfrak{p}[T]$ is a minimal prime of $I[T]$.

(13.2) If $\mathfrak{q}$ is $\mathfrak{p}$-primary ideal of A, then $\mathfrak{q}[T]$ is a $\mathfrak{p}[T]$-primary ideal of $A[T]$.

Solution.

(13.3) If $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ is a minimal decomposition of I, then $I[T] = \mathfrak{q}_1[T] \cap \cdots \cap \mathfrak{q}_n[T]$ is a minimal primary decomposition of $I[T]$.

Solution.