

**Problem 1.** *Let  $A$  be a ring. Prove:*

(1.1) *If  $x \in A$  is nilpotent, then  $1 - x$  is invertible.*

**Solution.** *Proof.* Suppose that  $x^n = 0$  for some  $n \in \mathbb{N}$ . Then,

$$(1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = 1 - x^n = 1$$

□

(1.2) *The nilradical  $N(A) = \{x \in A \mid x \text{ nilpotent}\}$  is an ideal of  $A$ .*

**Solution.** *Proof.* Let  $x, y \in N(A)$ , then  $x^n = 0$  and  $y^m = 0$  for some  $n, m \in \mathbb{N}$ . Then,

$$(x - y)^{n+m} = \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} x^{n+m-k} y^k = 0$$

since  $k \geq n$  or  $n + m - k \geq m$  for all  $k$ . Thus,  $x - y \in N(A)$ . We have that  $N(A)$  is an additive subgroup of  $A$ . Now, let  $a \in A$ . Then,

$$(ax)^n = a^n x^n = 0a = 0$$

so  $ax \in N(A)$ . Thus,  $N(A)$  is an ideal of  $A$ .

□

(1.3)  *$N(A)$  is contained in all prime ideals of  $A$ .*

**Solution.** *Proof.* Let  $I \subset A$  be a prime ideal. Let  $x \in N(A)$ . Then,  $x^n = 0 \in I$  for some  $n \in \mathbb{N}$ . Let us show by induction on  $n$  that this implies  $x \in I$ :

- If  $n = 1$ , then  $x = 0 \in I$ .
- Suppose that the statement is true for  $n - 1$ :  $x^{n-1} = 0 \Rightarrow x \in I$ . If  $0 = x^n = x \cdot x^{n-1}$ , since  $I$  is prime,  $x \in I$  or  $x^{n-1} \in I$ . In the first case, we are done. The second case is just the inductive hypothesis.

□

(1.4)  *$N(A)$  is the intersection of all prime ideals of  $A$ .*

**Solution.**

**Claim.** *Given  $x \notin N(A)$ , let  $\Sigma_x$  be the set of all ideals that do not contain any power of  $x$ . Then,  $\Sigma_x$  has a maximal element.*

*Proof.* We will use Zorn's lemma. Let us check the conditions:

**Claim.**  $\Sigma_x$  is a partially ordered set with respect to inclusion.

**Claim.**  $\Sigma_x$  is not empty.

*Proof.* Since  $x \notin N(A)$ ,  $0 \in \Sigma_x$ . □

**Claim.** Every chain in  $\Sigma_x$  has an upper bound.

*Proof.* Let  $\{I_\alpha\}_{\alpha \in S}$  be a chain in  $\Sigma_x$ . Then,  $I = \bigcup_{\alpha \in S} I_\alpha$  is an ideal of  $A$  (One can check that if  $x, y \in I$ , then  $x, y \in I_\alpha$  for some  $\alpha \in S$ , and then check the axioms from there). Let  $x^n \in I$  for some  $n \in \mathbb{N}$ . Then,  $x^n \in I_\alpha$  for some  $\alpha \in A$ . Since  $I_\alpha$  is an ideal,  $x \in I_\alpha$ . Thus,  $I \in \Sigma_x$ . □

Now that we have verified the conditions of Zorn's lemma, we can conclude that  $\Sigma_x$  has a maximal element. □

**Claim.** Let  $x \notin N(A)$ . Then the maximal element  $K(x)$  of  $\Sigma_x$  is prime.

*Proof.* Let  $a, b \in A$  such that  $ab \in K(x)$ . By way of contradiction, suppose that  $a \notin K(x)$  and  $b \notin K(x)$ . Then,  $x^n \in (a)$  and  $x^m \in (b)$  for some  $m, n \in \mathbb{N}$ , but  $x^{n+m} \notin (ab) = (a)(b)$ . Contradiction. □

Finally, we can prove the statement.

**Claim.**  $N(A)$  is the intersection of all prime ideals of  $A$ .

*Proof.* Let  $J$  be the intersection of all prime ideals of  $A$ . By 1.3, we know that  $N(A) \subset J$ . We want to prove that if  $x \notin N(A)$ , then  $x \notin J$ . Indeed,  $J \subset K(x)$  because  $K(x)$  is prime and  $x \notin K(x)$  because  $K(x)$  does not contain any power of  $x$ . □

**Problem 2.** Let  $A$  be a ring. Let  $a_i \in A$  and  $f = a_0 + a_1T + \cdots + a_nT^n \in A[T]$  be a polynomial. Prove:

(2.1)  $f$  is a unit in  $A[T] \iff a_0$  is a unit in  $A$  and  $a_1, \dots, a_n$  are nilpotent.

**Solution.** *Proof.* We will show both implications separately.

( $\Leftarrow$ ) Let  $a_i^{n_i} = 0$  for all  $1 \leq i \leq n$ . Consider  $s = \sum_{i=1}^n n_i$ . Let  $h = -a_1T - \cdots - a_nT^n$  be the negative of the polynomial without the constant term. Then,

$$h^s = \left( - \sum_{i=1}^n a_i T^i \right)^s = (-1)^s \sum_{j_1 + \cdots + j_n = s} a_1^{j_1} \cdots a_n^{j_n} T^{j_1 + 2j_2 + \cdots + nj_n}$$

By the pigeonhole principle, in each term of the sum, there is at least one  $j_k \geq n_k$  so  $a_k^{j_k} = 0$ , and thus  $h^s = 0$ . Then,  $h$  is nilpotent, so is  $a_0^{-1} \cdot h$  and, by 1.1,  $1 - a_0^{-1} \cdot h$  is invertible. Multiplying by  $a_0$ , we get that  $f = a_0 - h$  is also invertible.

( $\Rightarrow$ ) Suppose, there exists  $g = b_0 + b_1T + \cdots + b_mT^m \in A[T]$  such that  $1 = fg = \sum_{i=0}^{n+m} s_i T^i$ , where  $s_i = \sum_{j=0}^n a_j b_{i-j}$  and  $b_k = 0$  for  $k > m$  or  $k < 0$ . We first note that  $1 = s_0 = a_0 b_0 \Rightarrow a_0, b_0 \in A^*$ . Next, suppose  $n > 0$  (Otherwise, there is nothing to show).

**Claim.** Let  $0 \leq k \leq m$ .  $a_n^{k+1}b_{m-k} = 0$

*Proof.* By total induction on  $k$ :

- If  $k = 0$ , then  $0 = s_{n+m} = a_n b_m = (a_n)^{0+1} b_{m-0}$ .
- Suppose that the statement is true for  $0, \dots, k-1$ . Then,  $s_{n+m-k} = 0$  as  $n+m-k \geq n > 0$ . Therefore:

$$0 = a_n^k s_{n+m-k} = \sum_{j=0}^n a_n^k a_j b_{n+m-k-j} = a_n^{k+1} b_{m-k} + \sum_{j=0}^{n-1} a_n^k a_j b_{n+m-k-j}$$

Now, the terms in the sum are zero by the inductive hypothesis, as  $j < n \Rightarrow -(n-k-j) < k \Rightarrow (n-k-j) + 1 \leq k$ . Therefore,  $a_n^{k+1} b_{m-k} = 0$  as we wanted to show. □

Now, by setting  $k = m$ , we get that  $a_n^{m+1} b_0 = 0$ . Since  $b_0$  is a unit,  $a_n^{m+1} = 0$  and  $a_n$  is nilpotent. We are almost done if we realize the following:

**Claim.** Let  $p = c_0 + c_1 T + \dots + c_l T^l \in A[T]$  be an invertible polynomial such that  $c_l$  is nilpotent. Then,  $q = c_0 + c_1 T + \dots + c_{l-1} T^{l-1}$  is also invertible.

*Proof.* Note that  $c_l T^l$  is nilpotent and so is  $c_l T^l p^{-1}$ . Then,  $1 - c_l T^l p^{-1}$  is invertible by 1.1. Finally, because  $p$  is invertible, so is  $q = p - c_l T^l = p(1 - c_l T^l p^{-1})$ . □

We will prove that for  $0 < k \leq n$ ,  $a_0 + \dots + a_k T^k$  is invertible and  $a_k$  is nilpotent by (reverse) induction on  $k$ .

- $a_n$  has already been done.
- If  $0 < k < n$ , by hypothesis  $a_{k+1}$  is nilpotent and  $a_0 + \dots + a_{k+1} T^{k+1}$  is invertible. Then,  $a_0 + \dots + a_k T^k = (a_0 + \dots + a_{k+1} T^{k+1}) - a_{k+1} T^{k+1}$  is invertible by the claim. Therefore,  $a_k$  is nilpotent.

( $\Rightarrow$ ) (Faster Version) Let  $f$  be a unit in  $A[T]$ .

**Claim.** Let  $\mathfrak{p} \in \text{Spec}(A)$ . Then,  $a_i \in \mathfrak{p}$  for all  $i \in \{1, \dots, n\}$ .

*Proof.*  $\mathfrak{p} \in \text{Spec}(A)$  implies that  $A/\mathfrak{p}$  is an integral domain. Consider the reduction  $\pi : A[T] \rightarrow A/\mathfrak{p}[T]$  that takes each element  $a \in A[T]$  to the class  $\bar{a} \in A/\mathfrak{p}[T]$ . Since  $A/\mathfrak{p}$  is an integral domain,

$$0 = \deg \bar{1} = \deg \bar{f} \cdot \overline{f^{-1}} = \deg \bar{f} + \deg \overline{f^{-1}}$$

In particular,  $\deg \bar{f} = 0$  so, for all  $i \in \{1, \dots, n\}$ ,  $\bar{a}_i = \bar{0} \Rightarrow a_i \in \mathfrak{p}$ . □

Since this holds for all  $\mathfrak{p} \in \text{Spec}(A)$ , we have that  $\forall i \in \{1, \dots, n\}, a_i \in \bigcap_{\mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} = N(A)$  (by 1.4). Finally, suppose that  $f^{-1} = b_0 + b_1 T + \dots + b_m T^m$ . Given that  $f f^{-1} = 1$  we have that  $a_0 b_0 = 1$ , and thus  $a_0$  is invertible. □

(2.2)  $f$  is nilpotent  $\iff a_0, \dots, a_n$  are nilpotent.

**Solution.** *Proof.* We will show both implications separately.

- ( $\Leftarrow$ ) Just use the same argument as in the previous part of the exercise, but directly on  $f$ , not on  $h = a_0 - f$ .
- ( $\Rightarrow$ ) If  $f$  is nilpotent, then  $1 - f$  is invertible by 1.1. Thus, by the previous part,  $-a_1, \dots, -a_n$  are nilpotent. Because  $-a_i$  is nilpotent,  $a_i$  is nilpotent. We only have to prove that  $a_0$  is nilpotent. By the other implication,  $h$  is nilpotent. Then, by 1.2,  $a_0 = f + h$  is nilpotent.

□

(2.3)  $f$  is a zero divisor  $\iff$  there exists  $a \in A$ ,  $a \neq 0$  such that  $af = 0$ .

**Solution.** *Proof.* The backward implication is trivial because of the inclusion  $A \subset A[T]$ . For the forward implication, suppose that  $f$  is a zero divisor. Then, there exists  $g \in A[T]$ ,  $g \neq 0$  such that  $fg = 0$ . Let  $g = b_0 + \dots + b_m T^m$  be the minimum degree polynomial satisfying the condition  $fg = 0$ . Suppose that  $m > 0$ .

**Claim.**  $\exists i \in \{1, \dots, n\}$  s.t.  $a_i g \neq 0$

*Proof.* Suppose not. Then,  $a_i g = 0$  for all  $i \in \{1, \dots, n\}$ . Then,  $a_i b_m = 0$  for all  $i \in \{1, \dots, n\}$ . But then we have  $b_m f = 0$  with  $b_m \in A$ , in contradiction with  $g$  being a polynomial with minimum degree. □

Take  $i$  maximal such that  $a_i g \neq 0$ . Then,  $0 = fg = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m) + \sum_{j=i+1}^n T^j a_j g = (a_0 + \dots + a_i T^i)(b_0 + \dots + b_m T^m)$ , and  $a_i b_m = 0$ . Thus, we have a polynomial  $g' = a_i g \neq 0$  with degree  $m - 1$  satisfying  $f(a_i g) = a_i(fg) = 0$ , in contradiction with  $g$  being of minimal degree. □

**Problem 3.** Let  $A$  be a ring. We define the Jacobson radical  $\mathcal{J}(A)$  as the intersection of all maximal ideals of  $A$ . Prove:

- $x \in \mathcal{J}(A) \iff 1 - xy$  is invertible for all  $y \in A$ .

**Solution.** *Proof.* We will prove the two implications separately:

- ( $\Leftarrow$ ) Suppose  $x \notin \mathcal{J}(A)$ . This means that there exists  $\mathfrak{m} \in \text{Max}(A)$  such that  $x \notin \mathfrak{m}$ . Since  $\mathfrak{m}$  is a maximal ideal,  $(x) + \mathfrak{m} = A$ . Then, there exist  $y' \in A$  and  $m' \in \mathfrak{m}$  such that  $xy' + m' = 1 \in A$ . But this means that  $1 - xy' = m' \in \mathfrak{m}$ , which is invertible. Contradiction with  $\mathfrak{m}$  being an ideal. Thus,  $x \in \mathcal{J}(A)$ .
- ( $\Rightarrow$ ) Suppose that  $\exists y \in A$  such that  $1 - xy$  is not a unit in  $A$ . Notice that:
- \* There exists a maximal ideal  $\mathfrak{M}$  such that  $1 - xy \in \mathfrak{M}$ .
  - \* Since  $x \in \mathfrak{m} \subseteq \mathcal{J}(A)$ , we have that  $xy \in \mathfrak{m}$ .

Finally,  $1 - xy + xy = 1 \in \mathfrak{m}$ , in contradiction with  $\mathfrak{m}$  being an ideal. Thus,  $\forall y \in A$  we have that  $1 - xy$  is a unit in  $A$ .

□

- $\mathcal{J}(A) = A \setminus A^* \Leftrightarrow A$  has only one maximal ideal.

**Solution.** *Proof.* This comes directly from the fact that an ideal  $I \subset A$  cannot contain a unit  $x$ , else  $xx^{-1} = 1 \in I$  and thus  $I = A$ .

$$\begin{aligned} \mathcal{J}(A) = A \setminus A^* &\Leftrightarrow \forall \mathfrak{m} \in \text{Max}(A), \mathfrak{m} \supseteq \{\text{non-units of } A\} = A \setminus A^* \Leftrightarrow \\ &\Leftrightarrow \forall \mathfrak{m} \in \text{Max}(A), \mathfrak{m} = \{\text{non-units of } A\} \Leftrightarrow (A, \mathfrak{m}) \text{ local} \end{aligned}$$

□

- $\mathcal{J}(A[t]) = \mathcal{N}(A[t])$ .

**Solution.** *Proof.* We will prove the two inclusions separately:

( $\supseteq$ )

$$\mathcal{N}(A[T]) = \bigcap \text{Spec}(A[T]) \subseteq \bigcap \text{Max}(A[T]) = \mathcal{J}(A[T])$$

The first equality follows 1.4, while the inclusion is a consequence of  $\text{Max}(A[T]) \subseteq \text{Spec}(A[T])$ .

( $\subseteq$ )

□

**Problem 4.** Let  $A$  be a ring such that every ideal not contained in the nilradical  $\mathcal{N}(A)$  has a non-zero idempotent element (i.e. an element  $e \neq 0$  such that  $e^2 = e$ ). Prove that the nilradical and the Jacobson radical  $\mathcal{J}(A)$  are equal.

**Solution.** *Proof.*

□

**Problem 5.** Let  $A$  be an integral domain. Prove that the ideal  $I = (x^2 - y^3, y^2 - z^3) \subseteq A[x, y, z]$  is prime.

(Hint: Let  $f : R \rightarrow B$  be a ring homomorphism and  $I \subseteq R$  be an ideal such that  $I \subseteq \ker f$ . Then  $f$  factorizes through  $R/I$ . Consider a parametrization  $f : A[x, y, z] \rightarrow A[T]$ .)

**Solution.** *Proof.*

□

**Problem 6.** • Let  $f(T) \in k[T]$  be irreducible where  $k$  is a field. Prove that the ideal  $(f(T))$  is maximal.

**Solution.** *Proof.*

□

- Describe the spectrum of  $\mathbb{R}[T], \mathbb{C}[T], \mathbb{R}[T]/(T^2 + 9), \mathbb{C}[T]/(T^2 + 9)$

**Solution.** *Proof.*

□