

Exercise 1: Let $\#\text{CLIQUE}$ be the problem of counting how many k -cliques exist in a given graph for a given positive integer k . Show that $\#\text{CLIQUE} \in \mathbf{P}^{\#\text{SAT}}$.

Solution: I proceed by direct reduction. Let $G = (V, E)$ be an undirected graph with n vertices and $1 \leq k \leq n$ an integer. I now claim that the number N' of *ordered* k -cliques can be found in polynomial time with access to a $\#\text{SAT}$ oracle. This is enough because given the number of ordered k -cliques is $N' = k!N$ and $N' \mapsto N'/k! \in \mathbf{FP}$.

Let $T = (u_1, \dots, u_k) \in V^k$ be a k -tuple of vertices. The elements of T form a k -clique in G if and only if $\{u_i, u_j\} \in E$ for all $1 \leq i < j \leq k$ (note that this implies that all the elements are different, since I am assuming that G is a simple graph, with no loops). To encode this information into a boolean formula I consider the variables x_i^u (for $u \in V$ and $i \in [k]$), representing whether $u_i = u$. Define

$$F := \{(u, w) \in V^2 \mid \{u, w\} \notin E\}.$$

Note that this includes the diagonal entries (u, u) . The clique condition can then be expressed as

$$\mathcal{C} := \bigwedge_{(u,w) \in F, (i,j) \in \binom{[k]}{2}} (\neg x_i^u \vee \neg x_j^w),$$

which is in CNF. An assignment $V \times [k] \rightarrow \{0, 1\}$ represents a k -tuple of vertices if and only if exactly one vertex is selected for each entry of T . I do this in two steps. First I introduce a formula that ensures *at most* one vertex is selected for each entry:

$$\mathcal{B} := \bigwedge_{\{u,w\} \in \binom{V}{2}, i \in [k]} (\neg x_i^u \vee \neg x_i^w).$$

Then, I introduce the following other formula, which ensures *at least* one vertex is selected for each entry.

$$\mathcal{A} := \bigwedge_{i \in [k]} \left(\bigvee_{u \in V} x_i^u \right).$$

Formulas \mathcal{A} , \mathcal{B} and \mathcal{C} can clearly be obtained from G (represented, for example, as an adjacency matrix) in polynomial time. Furthermore, there is a bijection between assignments $V \times [k] \rightarrow \{0, 1\}$ satisfying $\mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C}$ and ordered k -cliques in G . Therefore, a single query to a $\#\text{SAT}$ oracle is enough to count the number N' of ordered k -cliques in G in polynomial time. Then computing $N = N'/k!$ yields the actual number of k -cliques.