Exercise 1: Let **#CLIQUE** be the problem of counting how many k-cliques exist in a given graph for a given positive integer k. Show that **#CLIQUE** $\in \mathbf{P}^{\mathbf{\#SAT}}$.

Solution: I proceed by direct reduction. Let $G = (V, E)$ be an undirected graph with n vertices and $1 \leq k \leq n$ an integer. I now claim that the number N of *ordered* k-cliques can be found in polynomial time with access to **#SAT** oracle. This is enough because given the number of ordered k-cliques is $N' = k!N$ and $N' \mapsto N'/k! \in \mathbf{FP}$.

Let $T = (u_1, \dots u_k) \in V^k$ be an ordered set of vertices. The elements of T form a k-clique in G if an only if $\{u_i, u_j\} \in E$ for all $1 \leq i < j \leq k$ (note that this implies that all the elements are different, since I are assuming that G is a simple graph, with no loops). To encode this information into a boolean formula I consider the variables x_i^u (for $u \in V$ and $i \in [k]$), representing whether $u_i = u$. The above condition translates to

$$\text{“If } \{w_1, w_2\} \notin E \text{ for } w_1 \neq w_2 \in V, \neg u_i^{w_j} \text{ for all } i \in [k] \text{, for at least one } j \in \{1, 2\}\text{”.}$$

This is, each non-edge $e = \{x, y\}$ of G (of which there are at most $\binom{n}{2}$) introduces a condition on the variables of the form

$$\left(\bigwedge_{i \in [k]} \neg u_i^x \right) \bigvee \left(\bigwedge_{i \in [k]} \neg u_i^y \right) \equiv \bigwedge_{(i,j) \in [k]^2} (\neg u_i^x \vee \neg u_j^y) =: \mathcal{C}_e.$$

The full condition is thus equivalent to

$$\mathcal{C} := \bigwedge_{e \in \binom{V}{2} \setminus E} \mathcal{C}_e,$$

which is in CNF. An assignment $V \times [k] \to \{0, 1\}$ represents a k-tuple of vertices if and only if exactly one vertex is selected for each entry of T. I do this in two steps. First I introduce a formula that ensures *at most* one vertex is selected for each entry:

$$\mathcal{B} := \bigwedge_{\{x,y\} \in \binom{V}{2},\, i \in [k]} (\neg u_i^x \vee \neg u_i^y).$$

Then, I introduce the following other formula, which ensures *at least* one vertex is selected for each entry.

$$\mathcal{A} := \bigwedge_{i \in [k]} \left(\bigvee_{x \in V} u_i^x \right).$$

Formulas $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ can clearly be obtained from G in polynomial time. Furthermore, there is a bijection from assignments $V \times [k] \to \{0, 1\}$ satisfying $\mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C}$ and ordered k-cliques in G. Therefore, a single query to a **#SAT** oracle is enough to count the number N' of ordered k-cliques in G in polynomial time. Then computing $N = N'/k!$ yields the actual number of k-cliques.