Exercise 1: Argue that if $\Delta_k^{\mathbf{P}} = \Sigma_k^{\mathbf{P}}$, then $\Delta_k^{\mathbf{P}} = \mathbf{PH}$.

Solution: My plan is to prove that if $\Delta_k^{\mathbf{P}} = \Sigma_k^{\mathbf{P}}$, then $\Delta_k^{\mathbf{P}} = \Pi_k^{\mathbf{P}}$. This implies that $\Sigma_k^{\mathbf{P}} = \Pi_k^{\mathbf{P}}$, and we have seen in class that this implies $\Sigma_k^{\mathbf{P}} = \Pi_k^{\mathbf{P}} = \mathbf{PH}$.

For this, recall the recursive definition of these classes:

$$\Delta_k^{\mathbf{P}} = \mathbf{P}^{\Sigma_i^{\mathbf{P}}}.$$
$$\Sigma_k^{\mathbf{P}} = \mathbf{NP}^{\Sigma_i^{\mathbf{P}}}.$$
$$\Pi_k^{\mathbf{P}} = \mathbf{co\text{-}NP}^{\Sigma_i^{\mathbf{P}}}.$$

The inclusions $\Delta_k^{\mathbf{P}} \subseteq \Sigma_k^{\mathbf{P}}$ and $\Delta_k^{\mathbf{P}} \subseteq \Pi_k^{\mathbf{P}}$ clear from the definitions and have been discussed in class. Therefore, it is enough to prove

$$\Sigma_k^{\mathbf{P}} \subseteq \Delta_k^{\mathbf{P}} \implies \Pi_k^{\mathbf{P}} \subseteq \Delta_k^{\mathbf{P}}. \tag{1}$$

Suppose that $\Sigma_k^{\mathbf{P}} \subseteq \Delta_k^{\mathbf{P}}$ and let $X \in \Pi_k^{\mathbf{P}}$ be a language. Then, $\overline{X} \in \Sigma_k^{\mathbf{P}} \subseteq \Delta_k^{\mathbf{P}}$. However, $\Delta_k^{\mathbf{P}}$ is closed under compliment, because its languages can be decided deterministically in polynomial time with $\Sigma_i^{\mathbf{P}}$ oracles (in particular, the output bit can just be flipped). More formally, let M be a DTM that decides $\overline{X}$ with access to a $\Sigma_i^{\mathbf{P}}$ oracle. Then, a DTM M' that decides X with access to the same oracle can be constructed: Run M on the input and output the opposite of the answer given by M. This implies that $X \in \Delta_k^{\mathbf{P}}$, proving (1).