Exercise 1: Let **#CLIQUE** be the problem of counting how many k-cliques exist in a given graph for a given positive integer k. Show that **#CLIQUE** $\in \mathbf{P}^{\mathbf{\#SAT}}$.

Solution: I proceed by direct reduction. Let $G = (V, E)$ be an undirected graph with n vertices and $1 \leq k \leq n$ an integer. I now claim that the number N' of *ordered* k-cliques can be found in polynomial time with access to a **#SAT** oracle. This is enough because given the number of ordered k-cliques is $N' = k!N$ and $N' \mapsto N'/k! \in \mathbf{FP}$.

Let $T = (u_1, \dots u_k) \in V^k$ be a k-tuple of vertices. The elements of T form a k-clique in G if and only if $\{u_i, u_j\} \in E$ for all $1 \leq i < j \leq k$ (note that this implies that all the elements are different, since I am assuming that G is a simple graph, with no loops). To encode this information into a boolean formula I consider the variables x_i^u (for $u \in V$ and $i \in [k]$), representing whether $u_i = u$. Define

$$F \coloneqq \{(u, w) \in V^2 \mid \{u, w\} \notin E\}.$$

Note that this includes the diagonal entries (u, u). The clique condition can then be expressed as

$$\mathcal{C} \coloneqq \bigwedge_{(u,w) \in F, (i,j) \in \binom{[k]}{2}} (\neg x_i^u \vee \neg x_j^w),$$

which is in CNF. An assignment $V \times [k] \to \{0, 1\}$ represents a k-tuple of vertices if and only if exactly one vertex is selected for each entry of T. I do this in two steps. First I introduce a formula that ensures *at most* one vertex is selected for each entry:

$$\mathcal{B} \coloneqq \bigwedge_{\{u,w\} \in \binom{V}{2},\, i \in [k]} (\neg x_i^u \vee \neg x_i^w).$$

Then, I introduce the following other formula, which ensures *at least* one vertex is selected for each entry.

$$\mathcal{A} \coloneqq \bigwedge_{i \in [k]} \left(\bigvee_{u \in V} x_i^u \right).$$

Formulas $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ can clearly be obtained from G (represented, for example, as an adjacency matrix) in polynomial time. Furthermore, there is a bijection between assignments $V \times [k] \to \{0, 1\}$ satisfying $\mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C}$ and ordered k-cliques in G. Therefore, a single query to a **#SAT** oracle is enough to count the number N' of ordered k-cliques in G in polynomial time. Then computing $N = N'/k!$ yields the actual number of k-cliques.

Exercise 2: We know that $\mathbf{PH} \subseteq \mathbf{P}^{\#\mathbf{P}}$ (Toda's theorem). What would be the consequences of the reverse inclusion $\mathbf{P}^{\#\mathbf{P}} \subseteq \mathbf{PH}$?

Solution: Suppose that $\mathbf{P}^{\#\mathbf{P}} \subseteq \mathbf{PH}$. I will argue that the polynomial hierarchy collapses to a finite level. consider the language

$$L = \{\langle \mathcal{F}, k\rangle \mid \mathcal{F} \text{ is a formula in CNF with less than } k \text{ solutions}\}.$$

Clearly, $L \in \mathbf{P}^{\#\mathbf{P}}$, since given $\langle \mathcal{F}, k\rangle$ one can count the number of solutions to $\mathcal{F}$ with a $\#\mathbf{SAT}$ oracle call and then check whether this number is less than k, all in polynomial time. By assumption, $L \in \mathbf{PH}$ and in particular $L \in \mathbf{\Sigma}_i^p$ for some $i \geq 0$. Consider now another language $A \in \mathbf{PH}$. By Toda's theorem, $A \in \mathbf{P}^{\#\mathbf{P}} = \mathbf{P}^{\#\mathbf{SAT}}$. Suppose that a machine M decides A in polynomial time with access to a $\#\mathbf{SAT}$ oracle. I now argue that each call to the oracle can be replaced by a polynomial number of calls to a machine M' that decides L (which can be done in polynomial time with access to a $\mathbf{\Sigma}_i^p$ oracle), and therefore that $A \in \mathbf{\Delta}_{i+1}^p$, proving that $\mathbf{PH} = \mathbf{\Delta}_{i+1}^p$.