**Exercise 1:** Prove that `DIRECTED DOMINATING SET` is NP-Hard through a series of Karp reductions starting at `3SAT`.

**Solution:** I will first introduce some notation. Let $F$ be a boolean formula with $N$ variables and $M$ CNF clauses with 3 literals each. I will denote the variables as $x_1, x_2, \ldots, x_N$ and the clauses as $c_1, c_2, \ldots, c_M$. For convenience, I will enumerate all possible literals as $l_j = x_j, l_{N+j} = \overline{x_j}$ (there are $2N$ of them). I will denote $c_i = \left(l_{j_{i,1}}, l_{j_{i,2}}, l_{j_{i,3}}\right) = \left(l^i_1, l^i_2, l^i_3\right) \in \{l_1, \ldots, l_{2N}\}^3$.

I will now construct a directed graph $G = (V, E)$ and $k \in \mathbb{N}$ such that $G$ contains a dominating set of size at most $k$ if and only if $F$ is satisfiable. Furthermore, the construction of the graph will clearly be polynomial in time, thus providing the Karp reduction we need directly.

First, I define the vertices $V$ of $G$ as:

- A vertex $L_j$ for each literal $l_j$ ($2N$ in total, which can be created in linear time by scanning $F$). For convenience, I will denote $L^i_u \coloneqq L_{j_{i,u}}$.
- A vertex $C_i$ for each clause $c_j$ ($M$ in total, which similarly can be created in linear time).

Next, I define the edges $E$ of $G$ as:

- $\left(L^i_u, C_i\right)$ for $1 \le u \le 3$, $1 \le i \le M$ (each literal points to the clauses it appears in, which we can construct in linear time).
- $(L_s, L_{s+N})$ and $(L_{s+N}, L_s)$ for $1 \le s \le N$ (each literal points to its negation, which we can construct in linear time).

Finally, I define $k = N$. It remains to be proven that $G$ has a dominating set of size $k \iff F$ is satisfiable:

$\Longleftarrow$ ) Suppose we have an assignment $x_s = B_s \in \{\text{True}, \text{False}\}$ that satisfies $F$. I will show that the set $S \coloneqq \{l_s | B_s = \text{True}\} \cup \{l_{s+N} | B_s = \text{False}\}$, which has size $N = k$, is dominating:

- All $L_j$ are either in $S$ or pointed to by $\overline{L_j} \coloneqq L_{j \pm N} \in S$.
- All $C_i$ are pointed to by their literals, at least one of which is in $S$.

$\Longrightarrow$ ) Suppose there is a dominating set $S$ of size at most $k$. for each variable $x_s$, $L_s$ must either be in $S$ or pointed to by an element of $S$ (that is, one of $L_s, L_{s+N}$ is in $S$). In fact, because there are $N = k$ variables, *exactly* one of them is in $S$, because otherwise $S$ would have more than $k$ elements. Furthermore, $S$ only contains vertices of this form.

This means that a variable assignment $x_s = B_s$ where $B_s = \text{True}$ if $L_s \in S$ and $B_s = \text{False}$ if $L_{s+N} \in S$ can be defined. To show that this assignment satisfies $F$, note that for each clause $c_i$, there is a literal $l_j$ such that $L_j$ points to $C_i$ and $L_j \in S$. If $j \le N$, this means we have assigned $x_j = \text{True}$ and $x_j = l^i_u$ for $1 \le i \le 3$, satisfying the clause. Otherwise, we have assigned $x_{j-N} = \text{False}$ and $\overline{x_{j-N}} = l^i_u$ for $1 \le i \le 3$, satisfying the clause as well.

**Exercise 2:**

## References

[1] Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Łukasz Kaiser, and Illia Polosukhin. Attention is all you need. *Advances in neural information processing systems*, 30, 2017.