Problem 4: Prove Lemma 4.1. from the notes.

Solution (by Ferran Espuña): Let us recall the statement of the lemma: Let $\mathbb{G}_{n,p(n)}$ be the random graph where $p(n) = \frac{c}{n}$ and $c > 1$ is a constant. Let us fix a vertex v and start a BFS from v, exploring the graph in the order $x_1 = v$, $x_2 \in \mathcal{N}(v), \cdots$. Let A_k be the set of vertices in the BFS queue after k steps (that is, those that have appeared as neighbors of some vertex x_t for $t \leq k$ but are not themselves of the form x_t for some $t \leq k$). For example, $A_0 = \{v = x_1\}$, $A_1 = \mathcal{N}(x_1) \setminus \{x_1\} = \mathcal{N}(x_1)$, $A_2 = (\mathcal{N}(x_1) \cup \mathcal{N}(x_2)) \setminus \{x_1, x_2\}, \cdots$

Lemma. *With this notation, a.a.s., either:*

(1) *the process stops before k^- steps, or*
(2) $|A_k| \geq \frac{c-1}{2}k$ *for all* $k \in [k^-, k^+]$ *and thus the process survives until* k^+.

Where $k^- = M(c)\log(n)$ *and* $k^+ = n^{\frac{2}{3}}$.

Proof. We will assume condition (1) does not hold, and we will bound the probability of condition (2) *failing for the first time* at some $k \in [k^-, k^+]$. For this, note that condition (2) holding for all previous steps implies that the process has not stopped before k.

Let Z_k be the set of vertices seen up to step k. That is, $Z_k = A_k \cup \{x_1, \cdots, x_k\}$ so that $|Z_k| = |A_k| + k$. We will shift our focus to Z_k and bound the probability of $|Z_k| < \frac{ck}{2} < \frac{c-1}{2}k + k$. Because we have only conditioned on the total number of seen vertices being large, the number of vertices in Z_k distributed at least as a binomial where each vertex has k independent chances of being seen (once at each step k' up to k). That is,

$$|Z_k| \geq \overline{Z_k} \sim \text{Binom}\left(n, 1 - (1-p)^k\right) \tag{1}$$

with expectation

$$\mathbb{E}\left(\overline{Z_k}\right) = n\left(1 - (1-p)^k\right) \tag{2}$$

To apply Chernoff's bound, we need to bound above

$$1 - \delta = f(k) := \frac{ck/2}{n\left(1-(1-p)^k\right)} = \frac{ck}{2n\left(1-(1-c/n)^k\right)} \leq \frac{ck}{2n\left(1-e^{-ck/n}\right)} \tag{3}$$

Claim. *This bound is increasing in k.*

Proof. Differentiating the right-hand side of (3) with respect to k using the quotient rule, we get the numerator

$$\frac{p}{2}\left(\left(1-e^{-pk}\right) - k\left(pke^{-pk}\right)\right) = \frac{p}{2}\left(1 - e^{-pk}\left(1+pk\right)\right) \tag{4}$$

Which equals 0 when $k = 0$ and itself is increasing in k:

$$\frac{d}{dk}e^{-pk}\left(1+pk\right) = -pe^{-pk}\left(1+pk\right) + pe^{-pk} = -pke^{-pk} < 0 \tag{5}$$

□

Plugging in $k = k^+$, we get, as $n \to \infty$,

$$f(k) \leq f(k^+) = \frac{cn^{2/3}}{2n\left(1-e^{-cn^{-1/3}}\right)} \leq \frac{cn^{-1/3}}{(2-\epsilon)\left(cn^{-1/3}\right)} = \frac{1}{2-\epsilon} < 1 \tag{6}$$

For some small $\epsilon > 0$, where in the middle inequality we have used that $1 - e^{-x} \sim x$ for $x \to 0$ so in particular for any $\epsilon > 0$, for x small enough ($x := cn^{-1/3}$), we have $\frac{2-\epsilon}{2} = 1 - \frac{\epsilon}{2} < \frac{x}{1-e^{-x}}$.

Remark. This bound also works for any $k^+ = o(n)$, not just $n^{\frac{2}{3}}$.

Anyway, the δ in Chernoff's bound is bounded below by a positive constant, and we can apply:

$$\mathbb{P}\left(\overline{Z_k} < \frac{ck}{2}\right) = \mathbb{P}\left(\overline{Z_k} < (1-\delta)\mathbb{E}(\overline{Z_k})\right) \leq e^{-\frac{\delta^2}{2}\mathbb{E}(\overline{Z_k})} \tag{7}$$

Finally, we will union bound over all $k \in [k^-, k^+]$. Because $k \geq k^-$, each term is at most

$$e^{-\frac{\delta^2}{2}\mathbb{E}(\overline{Z_{k^-}})} \leq e^{-\frac{\delta^2 ck^-}{4}} = e^{-\frac{c\delta^2 M(c)\log(n)}{4}} = n^{-\frac{cM(c)\delta^2}{4}} \tag{8}$$

The probability of the condition failing at any point in $[k^-, k^+]$ is at most

$$n^{\frac{2}{3}} n^{-\frac{cM(c)\delta^2}{4}} = n^{\frac{2}{3}-\frac{cM(c)\delta^2}{4}} = n^{\frac{8-3cM(c)\delta^2}{12}} \tag{9}$$

For this to be $o(1)$, we need $8 - 3cM(c)\delta^2$ to be negative. To give a sufficient condition, take $\epsilon = \frac{1}{2}$ Then $\delta = 1 - \frac{2}{3} = \frac{1}{3}$. We need $8 - \frac{cM(c)}{3} < 0$, which is equivalent to $M(c) > \frac{24}{c}$. $\square$