

Problem 5: Show that if $p(n) = n^\alpha$ for $\alpha < -3/2$, then a.a.s. $\mathbb{G}_{n,p(n)}$ consists of independent edges.

Solution (by Ferran Espuña): Note that the condition stated in the problem is equivalent to all vertices having degree less than 2. For our purposes, we only need to show that it is sufficient: Indeed, if two edges share a vertex, then that vertex has degree at least 2.

Proposition. *If $p(n) = n^\alpha$ for $\alpha < -3/2$, then a.a.s. all vertices in $\mathbb{G}_{n,p(n)}$ have degree less than 2.*

Proof. Fixing n , Let $G = \mathbb{G}_{n,p(n)}$ and X be the number of vertices of degree at least 2 in G . Then,

$$(1) \quad \mathbb{E}(X) = \sum_{v \in (G)} \mathbb{P}(d(v) \geq 2)$$

However, for any $v \in V(G)$, we have that

$$(2) \quad d(v) \geq 2 \iff v \sim s \text{ and } v \sim t \text{ for some } s, t \in V(G) \text{ with } s \neq t; s, t \neq v$$

Note that this condition does not depend on the order of s and t . by the union bound, we have that

$$(3) \quad \mathbb{P}(d(v) \geq 2) \leq \sum_{(s,t) \in \binom{V(G) \setminus \{v\}}{2}} \mathbb{P}(v \sim s \text{ and } v \sim t) = \binom{n-1}{2} p(n)^2 < n^2 p(n)^2$$

Substituting (3) into (1), we get

$$(4) \quad \mathbb{E}(X) < n \cdot n^2 p(n)^2 = n^3 p(n)^2 = n^{3+2\alpha}$$

But since $\alpha < -3/2$, we have that $3 + 2\alpha < 0$ and thus $\mathbb{E}(X) \rightarrow 0$ as $n \rightarrow \infty$. By Markov's inequality, we have that

$$(5) \quad \mathbb{P}(X \geq 1) \leq \mathbb{E}(X) \rightarrow 0 \text{ as } n \rightarrow \infty$$

And thus, a.a.s. $X = 0$, that is, all vertices have degree less than 2. □