Problem 5: Let $T_{d,k}$ be a $(d-1)$-ary rooted tree of height k. Show that

(a) $\lambda_1(T_{d,k}) \le 2\sqrt{d-1}$
(b) $\lim_{k\to\infty} \lambda_1(T_{d,k}) = 2\sqrt{d-1}$

Solution (by Ferran Espuña): Let A be the adjacency matrix of $T_{d,k}$. For example, if $d=3$ and $k=2$,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the first row corresponds to the root of the tree, the second and third rows correspond to its children and the other four rows correspond to the leaves. Now we can rephrase (a) as the following:

Proposition. $\|A\|_2 \le 2\sqrt{d-1}$.

Proof. We will use this famous inequality from linear algebra: $\|B\|_2 \le \sqrt{\|B\|_1 \cdot \|B\|_\infty}$. In this case, we will apply this to powers of A, which is non-negative and symmetric. Therefore, for $B = A^p$

$$\|B\|_1 = \|B\|_\infty = \max_i \sum_j B_{ij}$$

so in particular

$$\|A\|_2^p = \|A^p\|_2 \le \sqrt{\|A^p\|_1 \cdot \|A^p\|_\infty} = \max_i \sum_j A^p_{ij} \tag{1}$$

Which is just a row sum of A^p. This corresponds to the number $P_{i,p}$ of walks of length p from a given vertex v_i. Let's try to approximate this number.

At each step of the walk, the height (distance from the root) either increases by 1 (we say we go *down*) or decreases by 1 (we say we go *up*). Because both at the start and end of the walk the height must be in the range $[0,k]$, the difference between the number of *up* steps s_u and the number of *down* steps s_d must satisfy

$$|s_u - s_d| \le k$$

and obviously

$$s_u + s_d = p$$

In particular,

$$s_d \le \frac{p+k}{2}$$

If we go *up*, there is only one choice (the parent of the current node). If we go *down*, there are $d-1$ choices (the children of the current node). Because of this, if we fix a sequence *up* and *down* steps, there are at most $(d-1)^{(p+k)/2}$ walks from a given node satisfying that sequence. Furthermore, there are 2^p such sequences.

Remark. Not all sequences of *up* and *down* steps are valid, as we should also ensure that the height stays within the range $[0, k]$ during the whole walk. However, because we are only interested in an upper bound, we are free to ignore this.

All in all, for all i,

$$P_{i,p} \leq 2^p \cdot (d-1)^{\frac{p+k}{2}}$$

Putting this together with (1), we get

$$\|A\|_2^p \leq P_{i,p} \leq 2^p \cdot (d-1)^{\frac{p+k}{2}} \implies \|A\|_2 \leq \lim_{p\to\infty} \left(2^p \cdot (d-1)^{\frac{p+k}{2}}\right)^{\frac{1}{p}} = 2\sqrt{d-1}$$

$\square$

Similarly, we can rephrase (b) as the following:

Proposition. $\lim_{k\to\infty} \|A(k)\|_2 = 2\sqrt{d-1}$.

Proof. We would now like to bound *below* the 2-norm of A, so will use the inequality $\|B\|_2 \geq \frac{1}{\sqrt{n}}\|B\|_\infty$, where n is the size of the matrix.

Remark. In our case,

$$n = \sum_{i=0}^{k} (d-1)^i = \frac{(d-1)^{k+1} - 1}{d-2} < (d-1)^{k+1}$$

We repeat the same calculation for the number of walks of length p from a given vertex v_i, but now we need to bound it below. Because of the maximum in the formula in the ∞-norm, we're free to choose the starting vertex v_i. We will choose the root. We're also free to choose $p = 2mk$ for some $m \in \mathbb{N}$.

To make the count easier, we will consider only some of the walks. Specifically, we will consider walks that pass through the root at the $2kj$-th step for all $j \in [0, m]$.

Remark. The number of *down* steps is mk because the walks start and end at the root so $s_u = s_d$.

Remark. The number of valid orderings for the *up* and *down* steps for the steps in between $2jm$ and $2(j+1)m$ is the corresponding *Catalan number*:

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

This is because the walk can go as far down as it wants, as the leaves are at distance k from the root. Therefore, there are C_k^m ways to order the *up* and *down* steps for the whole walk.

All together,

$$\|A\|_2^{2mk} = \|A^{2mk}\|_2 \geq \frac{1}{\sqrt{n}}\|A^{2mk}\|_\infty \geq \frac{1}{\sqrt{n}} C_k^m (d-1)^{mk} = \frac{(d-1)^{mk}}{(k+1)^m \cdot \sqrt{n}} \binom{2k}{k}^m \geq$$
$$\geq \frac{4^{mk} \cdot (d-1)^{mk}}{(k+1)^m \cdot \sqrt{n} \cdot 2^m \cdot k^{m/2}} > \frac{4^{mk} \cdot (d-1)^{mk-(k+1)/2}}{(k+1)^m \cdot 2^m \cdot k^{m/2}}$$

If we now set, for example, $m = k$, we get that

$$\liminf_{k\to\infty} \|A(k)\|_2 \geq \lim_{k\to\infty} \left(\frac{4^{mk} \cdot (d-1)^{mk-(k+1)/2}}{(k+1)^m \cdot 2^m \cdot k^{m/2}}\right)^{\frac{1}{2mk}} = \lim_{k\to\infty} 2\frac{(d-1)^{1/2\,-\,1/4k\,-\,1/4k^2}}{(k+1)^{1/2k} \cdot 2^{1/2k} \cdot k^{1/4k}} = 2\sqrt{d-1}$$

Together with part (a), we have $\lim_{k\to\infty} \|A(k)\|_2 = 2\sqrt{d-1}$

$\square$