Problem 4: Let $T_{d,k}$ be a $(d-1)$-ary rooted tree of height k. Show that

(a) $\lambda_1(T_{d,k}) \leq 2\sqrt{d-1}$
(b) $\lim_{k\to\infty} \lambda_1(T_{d,k}) = 2\sqrt{d-1}$

Solution (by Ferran Espuña): Let A be the adjacency matrix of $T_{d,k}$. For example, if $d=3$ and $k=2$,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the first row corresponds to the root of the tree, the second and third rows correspond to its children and the other four rows correspond to the leaves. Now we can rephrase (a) as the following:

Proposition. $\|A\|_2 \leq 2\sqrt{d-1}$.

Proof. We will use this famous inequality from linear algebra: $\|B\|_2 \leq \sqrt{\|B\|_1 \cdot \|B\|_\infty}$. In this case, we will apply this to powers of A, which is non-negative and symmetric. Therefore, for $B = A^p$

$$\|B\|_1 = \|B\|_\infty = \max_i \sum_j B_{ij}$$

so in particular

$$\|A\|_2^p = \|A^p\|_2 \leq \sqrt{\|A^p\|_1 \cdot \|A^p\|_\infty} = \max_i \sum_j A^p_{ij} \tag{1}$$

Which is just a row sum of A^p. This corresponds to the number $P_{i,p}$ of paths of length p from a given vertex v_i. Let's try to approximate this number.

At each step of the path, the height (distance from the root) either increases by 1 (we say we go *down*) or decreases by 1 (we say we go *up*). Because both at the start and end of the path the height must be in the range $[0,k]$, the difference between the number of *up* steps s_u and the number of *down* steps s_d must satisfy

$$|s_u - s_d| \leq k$$

and obviously

$$s_u + s_d = p$$

All together,

$$\frac{p-k}{2} \leq s_d \leq \frac{p+k}{2}$$

If we go *up*, there is only one choice (the parent of the current node). If we go *down*, there are at most $d-1$ choices (the children of the current node). Furtheremore, for each choice of s_d there are $\binom{p}{s_d}$ ways to choose the *down* steps. Note that not all the ways of choosing are valid, as we must also ensure that the height stays within the range $[0, k]$ during the whole path. All in all, for all i,

$$P_{i,p} \leq \sum_{s_d=(p-k)/2}^{(p+k)/2} \binom{p}{s_d} (d-1)^{s_d} \leq k \cdot 2^p \cdot (d-1)^{\frac{p}{2}}$$

Putting this together with (1), we get

$$\|A\|_2^p \cdot k \cdot 2^p \cdot (d-1)^{\frac{p}{2}} \implies \|A\|_2 \leq \lim_{p\to\infty} \left(k \cdot 2^p \cdot (d-1)^{\frac{p}{2}}\right)^{\frac{1}{p}} = 2\sqrt{d-1}$$

□

Similarly, we can rephrase (b) as the following:

Proposition. $\lim_{k\to\infty} \|A(k)\| = 2\sqrt{d-1}$.

Proof. We would now like to bound *below* the 2-norm of $A(k)$, so will use the inequality $\|B\|_2 \geq \frac{1}{\sqrt{n}}\|B\|_\infty$, where n is the size of the matrix.

Remark. In our case,

$$n = \sum_{i=0}^{k} (d-1)^i = \frac{(d-1)^{k+1} - 1}{d-2} < (d-1)^{k+1}$$

We repeat the same calculation for the number of paths of length p from a given vertex v_i, but now we need to bound it below. Because of the maximum in the formula in the ∞-norm, we're free to choose the starting vertex v_i. We will choose the root. We're also free to choose $p = 2k$.

To make the count easier, we will consider only some of the paths. Specifically, we will consider paths that start at the root and:

(1) Go straight *down* to a node of depth h in h steps
(2) After that, never go back to a depth lower than h and return to depth h after $2(k-h)$ steps
(3) Finally, go straight to depth $2h$ in h steps

Where we define $h := \lfloor k/2 \rfloor$, so that there are nodes at depth $2h$.

Remark. The number of *down* steps is $h + 2(k-h)/2 + h = k + h$.

Remark. The number of valid orderings for the *up* and *down* steps in (2) is the corresponding *Catalan number*:

$$C_{k-h} = \frac{1}{k-h+1}\binom{2(k-h)}{k-h}$$

This is because the path can go as far down as it wants, as the leaves are at distance $k-h$ from the level h.

All together,

$$\begin{aligned}
\|A(k)\|_2^{2k} &= \|A(k)^{2k}\|_2 \geq \frac{1}{\sqrt{n}}\|A(k)^{2k}\|_\infty \geq \frac{1}{\sqrt{n}} C_{k-h}(d-1)^{k+h} = \\
&= \frac{1}{(k-h+1)\sqrt{n}}\binom{2(k-h)}{k-h}(d-1)^{k+h} \geq \frac{4^{k-h}(d-1)^{k+h}}{(k-h+1)2\sqrt{kn}} > \\
&> \frac{4^{k-h}(d-1)^{k+h-\frac{k+1}{2}}}{(k-h+1)2\sqrt{k}} \geq \frac{2^k(d-1)^{k-1}}{2k\sqrt{k}}
\end{aligned}$$

Which implies

$$\liminf_{k\to\infty}\|A(k)\|_2 \geq \lim_{k\to\infty}\left(\frac{4^k}{(k+1)2\sqrt{kn}}(d-1)^k\right)^{\frac{1}{2k}} = 2\sqrt{d-1}$$

Together with part (a), we have that

$$\lim_{k\to\infty}\|A(k)\|_2 = 2\sqrt{d-1}$$

$\square$