Problem 5: Show that if $p(n) = n^\alpha$ for $\alpha < -3/2$, then a.a.s. $\mathbb{G}_{n,p(n)}$ consists of independent edges.

Solution (by Ferran Espuña): Note that the condition stated in the problem is equivalent to all vertices having degree less than 2. For our purposes, we only need to show that it is sufficient: Indeed, if two edges share a vertex, then that vertex has degree at least 2.

Proposition. *If $p(n) = n^\alpha$ for $\alpha < -3/2$, then a.a.s. all vertices in $\mathbb{G}_{n,p(n)}$ have degree less than 2.*

Proof. Fixing n, Let $G = \mathbb{G}_{n,p(n)}$ and X be the number of vertices of degree at least 2 in G. Then,

$$\mathbb{E}(X) = \sum_{v \in (G)} \mathbb{P}(d(v) \geq 2) \tag{1}$$

However, for any $v \in V(G)$, we have that

$$d(v) \geq 2 \iff v \sim s \text{ and } v \sim t \text{ for some } s, t \in V(G) \text{ with } s \neq t;\ s, t \neq v \tag{2}$$

Note that this condition does not depend on the order of s and t. By the union bound, we have that

$$\mathbb{P}(d(v) \geq 2) \leq \sum_{(s,t) \in \binom{V(G) \setminus \{v\}}{2}} \mathbb{P}(v \sim s \text{ and } v \sim t) = \binom{n-1}{2} p(n)^2 < n^2 p(n)^2 \tag{3}$$

Substituting (3) into (1), we get

$$\mathbb{E}(X) < n \cdot n^2 p(n)^2 = n^3 p(n)^2 = n^{3+2\alpha} \tag{4}$$

But since $\alpha < -3/2$, we have that $3 + 2\alpha < 0$ and thus $\mathbb{E}(X) \to 0$ as $n \to \infty$. By Markov's inequality, we have that

$$\mathbb{P}(X \geq 1) \leq \mathbb{E}(X) \to 0 \text{ as } n \to \infty \tag{5}$$

And thus, a.a.s. $X = 0$, that is, all vertices have degree less than 2. $\square$