Problem 6: Let G_1 and G_2 be two Hamiltonian graphs. Show that $G_1 \square G_2$ is Hamiltonian. Show that the hypercube Q^n, for $n \geq 2$, is Hamiltonian.

Solution (by Ferran Espuña): Let $r = |E(G_1)|$, $s = |E(G_2)|$ and let

$$x_1, x_2, \ldots, x_{r-1}, x_r, x_1$$
$$y_1, y_2, \ldots, y_{s-1}, y_s, y_1$$

be Hamiltonian cycles in G_1 and G_2, respectively. We will construct a Hamiltonian cycle in $G_1 \square G_2$ as follows:

- We will course a path moving back and forth between y_1 and y_{s-1} in each copy of G_2, hopping to the next copy using an edge of the form $(x_i, y_1) \sim (x_{i+1}, y_1)$ or $(x_i, y_{s-1}) \sim (x_{i+1}, y_{s-1})$, and always leaving out y_s in each copy so that we can use them later to go back to the original copy (indexed by x_1).
- Then, we will move to (x_r, y_s) and, as said, go back to the original copy of G_2 using only copies of y_s and reaching (x_1, y_s)
- Finally, we can go back to the original copy of x_1, thus completing the cycle.

Proposition. *The cartesian product of two Hamiltonian graphs is Hamiltonian.*

Proof. Let's assume the notation for the cycles in G_1 and G_2 as above. The specifics vary depending on whether r is even or odd, so we will show the full path if it is odd:

$$\begin{array}{l}
(x_1, y_1), \quad (x_1, y_2), \quad \ldots, (x_1, y_{s-1}), \\
(x_2, y_{s-1}), (x_2, y_{s-2}), \ldots, (x_2, y_1), \\
(x_3, y_1), \quad (x_3, y_2), \quad \ldots, (x_3, y_{s-1}), \\
(x_4, y_{s-1}), (x_4, y_{s-2}), \ldots, (x_4, y_1), \\
\qquad\qquad\qquad\quad \ldots \\
(x_r, y_1), \quad (x_r, y_2), \quad \ldots, (x_r, y_{s-1}), \\
(x_r, y_s), \\
(x_{r-1}, y_s), (x_{r-2}, y_s), \ldots, (x_1, y_s), \\
(x_1, y_1)
\end{array}$$

All vertices of the form (x_i, y_j) where $j \neq s$ are covered exactly once in the black or red rows (except for (x_1, y_1), which is both at the beginning and the end) and the rest are in the blue ones. All edges between adjacent nodes in the cycle are in the cartesian product by definition. If r is even, we only need to change the last red row to a black one:

$$(x_r, y_{s-1}), (x_r, y_{s-2}), \ldots, (x_r, y_1)$$

But because both y_1 and y_{s-1} are connected to y_s, this is fine. $\square$

Remark. The case for r even can be made easier by just doing

$$\begin{aligned}
&(x_1, y_1), (x_1, y_2), \quad \dots, (x_1, y_s),\\
&(x_2, y_s), (x_2, y_{s-1}), \dots, (x_2, y_1),\\
&(x_3, y_1), (x_3, y_2), \quad \dots, (x_3, y_s),\\
&\qquad\qquad\qquad\qquad \dots\\
&(x_r, y_s), (x_r, y_{s-1}), \dots, (x_r, y_1),\\
&(x_1, y_1)
\end{aligned}$$

But I did the other method because it works in both cases.

Now, to tackle the hypercube Q^n, we just need to put it as the cartesian product of two hamiltonian graphs:

Proposition. *The hypercube Q^n is Hamiltonian.*

Proof. We proceed by induction:

- For $n \in \{2, 3\}$, just construct the cycles "by hand":
  - $(0,0), (0,1), (1,1), (1,0), (0,0)$
  - $(0,0,0), (0,0,1), (0,1,1), (0,1,0), (1,1,0), (1,1,1), (1,0,1), (1,0,0), (0,0,0)$
- For higher n, we can use the fact that $Q^n = Q^{n-2} \square Q^2$, which is Hamiltonian by the previous proposition.

$\square$

Remark. If we look at the structure of the two cycles we constructed for Q^2 and Q^3, we can find a somewhat simpler method for constructing the cycle for Q^n. In fact, we can show the following:

Claim. *The cartesian product of any graph G with a spanning path and Q^1 is Hamiltonian.*

Proof. We denote $E(Q^1) = \{0, 1\}$ and let

$$y_1, y_2, \dots, y_s$$

be a spanning path in G. We can construct a Hamiltonian cycle in $Q^1 \square G$ as follows:

$$\begin{aligned}
&(0, y_1), (0, y_2), \quad \dots, (0, y_s),\\
&(1, y_s), (1, y_{s-1}), \dots, (1, y_1),\\
&(0, y_1)
\end{aligned}$$

$\square$

Because Q^1 clearly has a spanning path, we can use this claim to show that Q^n is Hamiltonian for all $n \geq 2$ by induction (as we did before).