FEM
Computational Aspects

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Computation of matrices: assembly, reference element and numerical integration

- We want to compute integrals as

$$K_{ij} = a(N_i, N_j) = \int_\Omega \nabla N_i \cdot (\mathbf{A} \nabla N_j) \, d\Omega$$

with element-by-element piece-wise polynomial functions.

$$K_{ij} = \sum_e \int_{\Omega_e} \nabla N_i \cdot (\mathbf{A} \nabla N_j) \, d\Omega = \dots$$

- Gauss quadrature in each element.

Elemental matrices

- Assembly of elemental matrices and vectors

$$\mathbf{K} = \underset{e}{\mathsf{A}}\, \mathbf{K}^e, \quad \mathbf{f} = \underset{e}{\mathsf{A}}\, \mathbf{f}^e$$

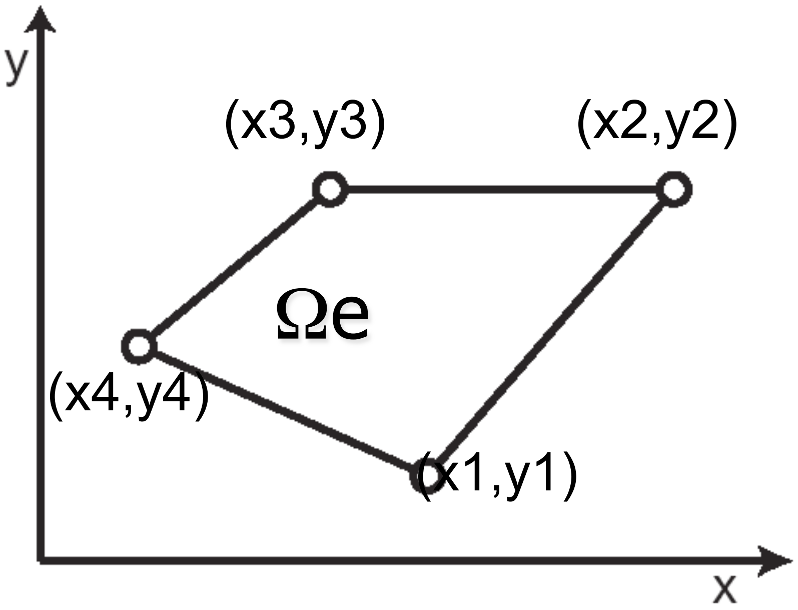
- The elemental matrix $\mathbf{K}^e$ has all the non-null integrals in the element Ω_e

$$K^e_{(i)(j)} = \int_{\Omega_e} \nabla N_{(i)} \cdot \left(\mathbf{A} \nabla N_{(j)} \right) d\Omega \qquad \begin{array}{l} (i) = 1, \ldots, \mathtt{nnode} \\ (j) = 1, \ldots, \mathtt{nnode} \end{array}$$

where $(\cdot)$ denotes the local numbering and `nnode` is the number of nodes in the element. The connectivity matrix gives the equivalence between local numbering and global numbering.

Computation of the elemental matrix

$$K^e_{(i)(j)} = \int_{\Omega_e} \nabla N_{(i)} \cdot \left(\mathbf{A} \nabla N_{(j)}\right) \, d\Omega \qquad \begin{array}{l}(i) = 1, \ldots, \texttt{nnode} \\ (j) = 1, \ldots, \texttt{nnode}\end{array}$$

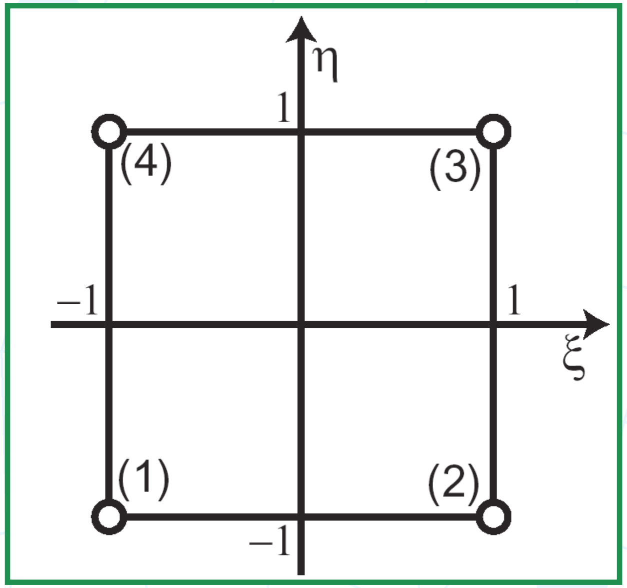


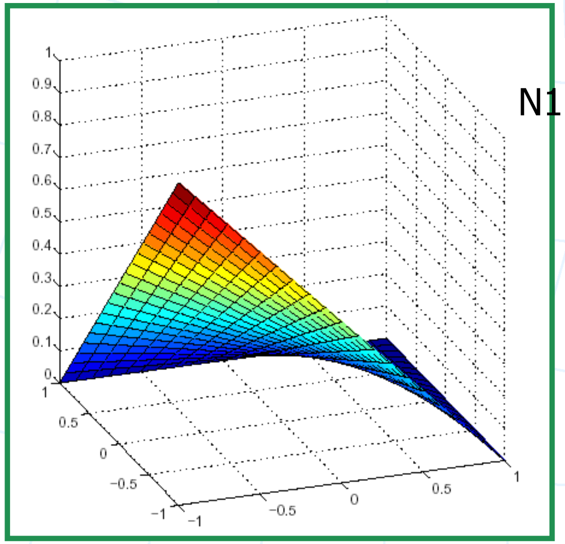
Shape functions

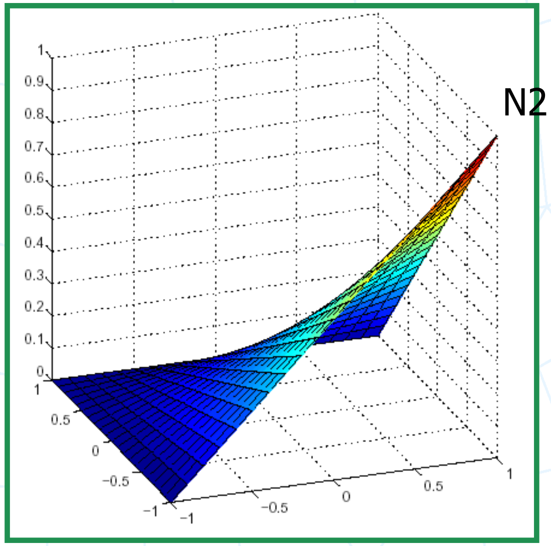
$N_i(x)=?$

Numerical quadrature

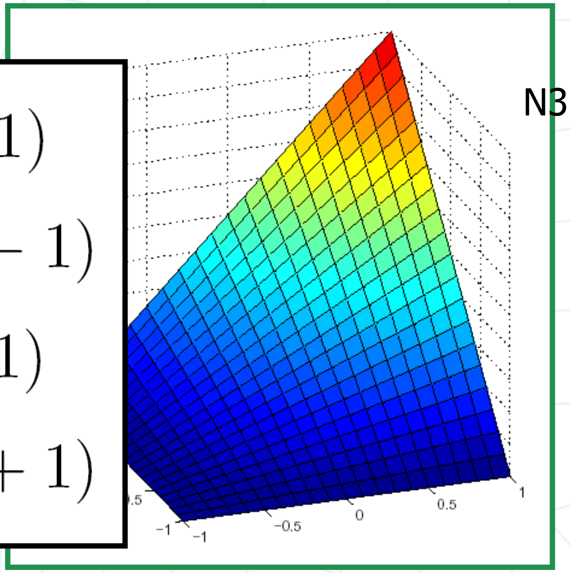
Reference element

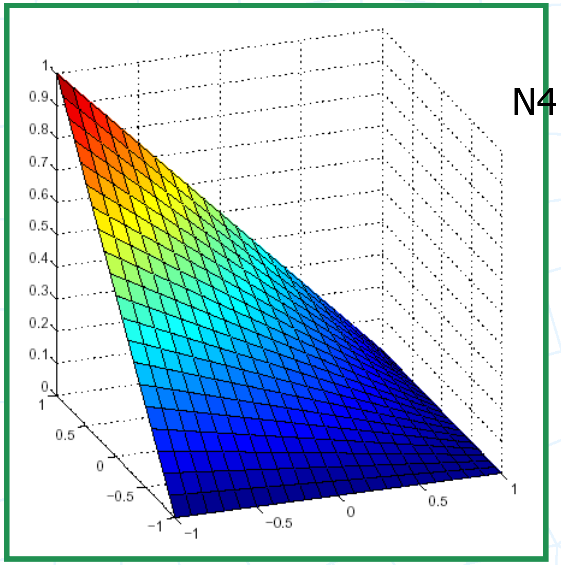




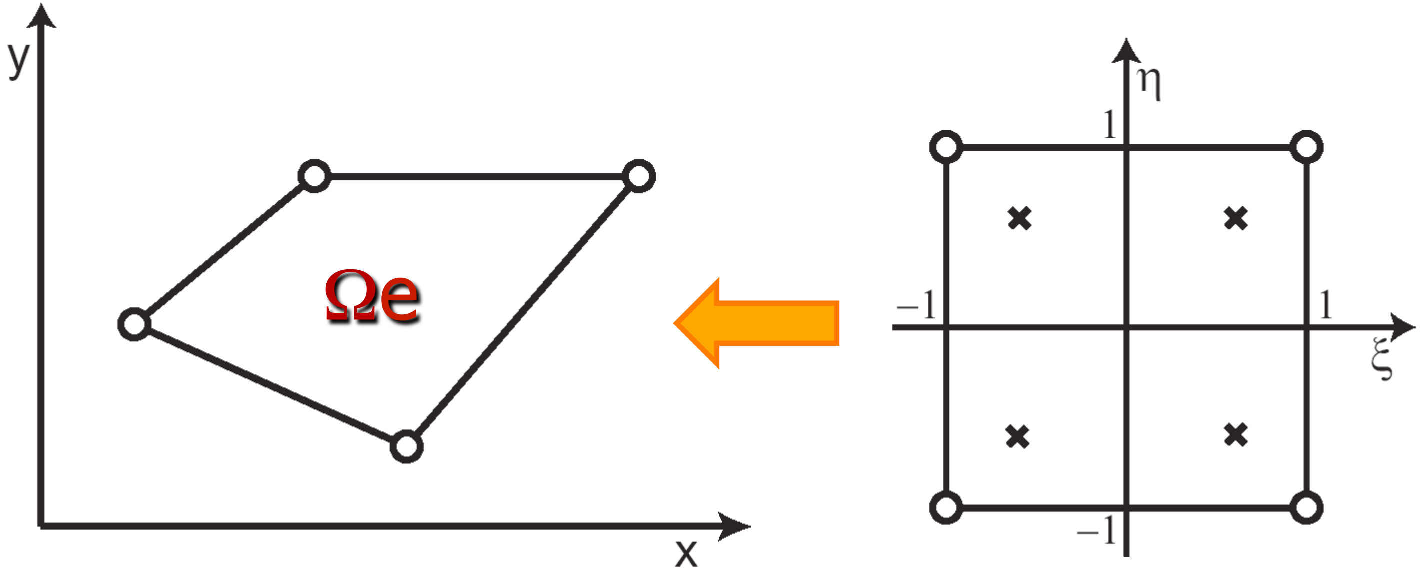


$$N_1(\xi,\eta) = \tfrac{1}{4}(\xi-1)(\eta-1)$$
$$N_2(\xi,\eta) = -\tfrac{1}{4}(\xi+1)(\eta-1)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}(\xi+1)(\eta+1)$$
$$N_4(\xi,\eta) = -\tfrac{1}{4}(\xi-1)(\eta+1)$$





ISOPARAMETRIC TRANSFORMATION



Reference element
$[-1,1] \times [-1,1]$

- Isoparametric transformation: change of variables from local coordinates (ξ,η) to physical coordinates (x,y)

Isopatametric transformation

$$x(\xi,\eta) = \sum_{i=1}^{\text{nnode}} x_i N_i(\xi,\eta), \qquad y(\xi,\eta) = \sum_{i=1}^{\text{nnode}} y_i N_i(\xi,\eta)$$

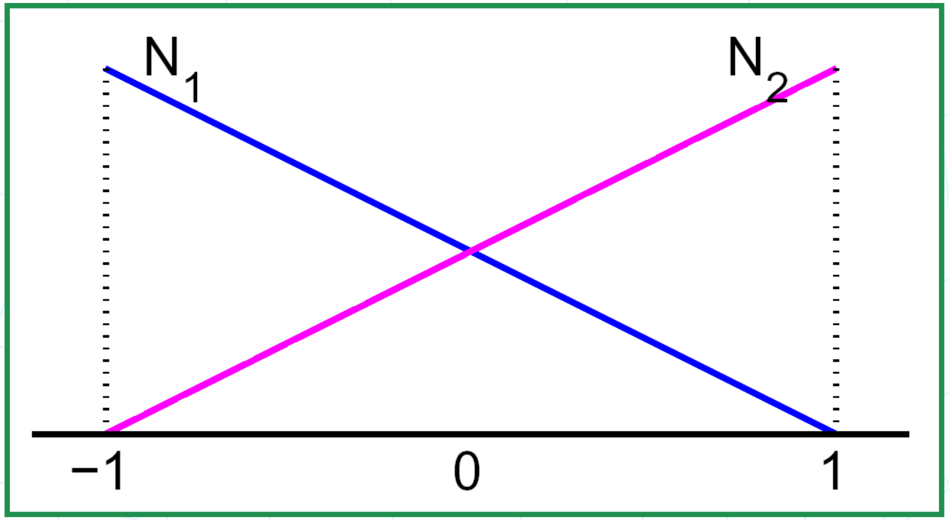
Jacobian matrix

$$\mathbf{J}(\xi,\eta) = \begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2mm] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{i=1}^{\text{nnode}} x_i \frac{\partial N_i}{\partial \xi} & \displaystyle\sum_{i=1}^{\text{nnode}} y_i \frac{\partial N_i}{\partial \xi} \\[4mm] \displaystyle\sum_{i=1}^{\text{nnode}} x_i \frac{\partial N_i}{\partial \eta} & \displaystyle\sum_{i=1}^{\text{nnode}} y_i \frac{\partial N_i}{\partial \eta} \end{pmatrix}$$

Properties: $\boxed{\boldsymbol{\nabla}_{xy} = \mathbf{J}^{-1}\boldsymbol{\nabla}_{\xi\eta}}$ $\boxed{dx\, dy = |\mathbf{J}|\, d\xi\, d\eta}$

Examples

1D linear element



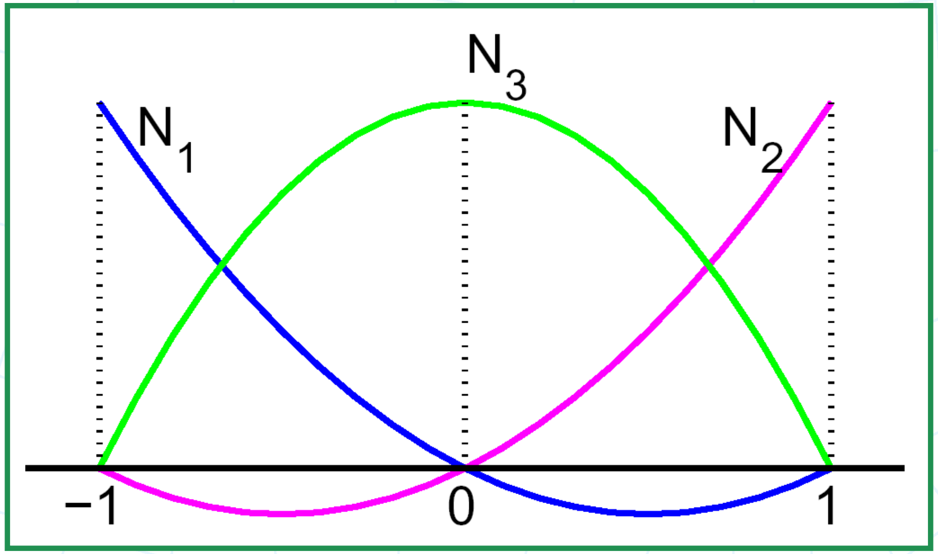
$$u^h = u_1 N_1 + u_2 N_2$$

$$N_1(\xi) = \frac{1}{2}(1-\xi) \quad N_2(\xi) = \frac{1}{2}(1+\xi)$$

Transformación isoparamétrica:

$$x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) = \cdots = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\,\xi$$

1D quadratic element



$$u^h = u_1 N_1 + u_2 N_2 + u_3 N_3$$

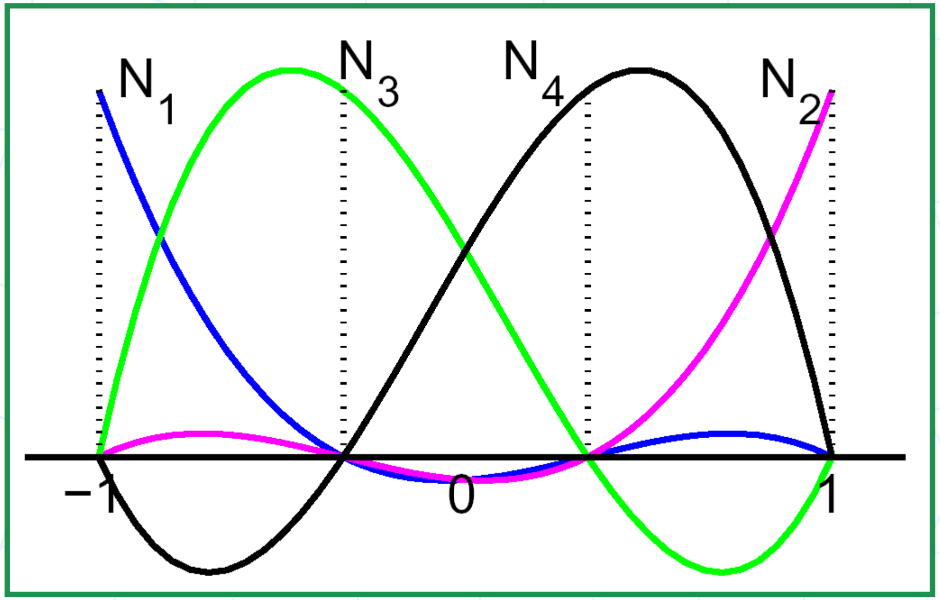
$$N_1(\xi) = \frac{1}{2}\xi(\xi - 1) \quad N_2(\xi) = \frac{1}{2}\xi(\xi + 1)$$

$$N_3(\xi) = (1 + \xi)(1 - \xi)$$

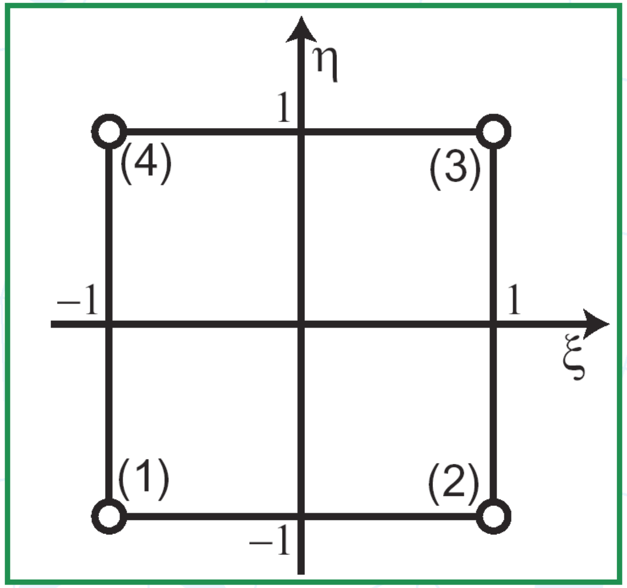
$$x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi) = \dots$$

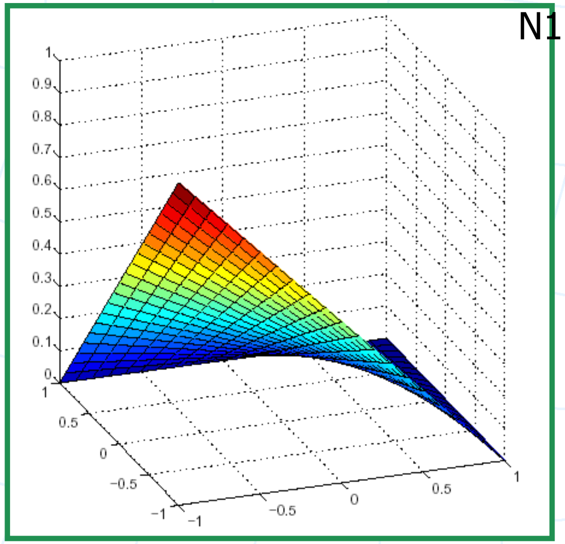
$$= x_3 + \frac{x_2 - x_1}{2}\,\xi + \left(\frac{x_1 + x_2}{2} - x_3\right)\xi^2$$

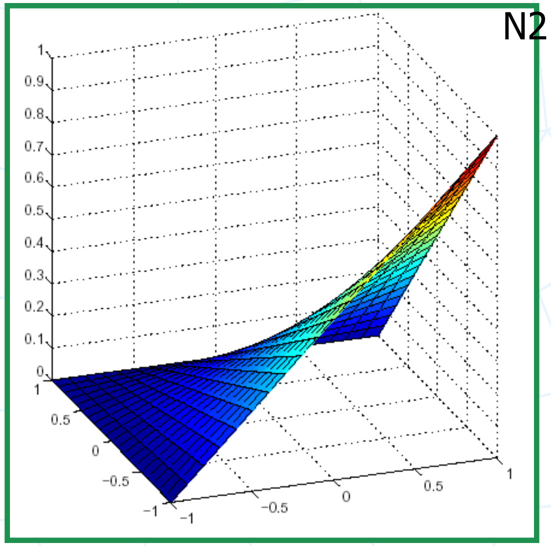
1D cubic element



Q1: bi-linear







$$N_1(\xi,\eta) = \tfrac{1}{4}(\xi-1)(\eta-1)$$

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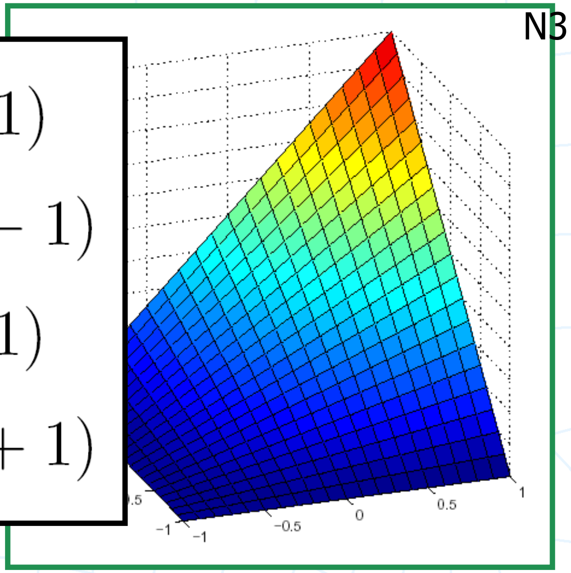
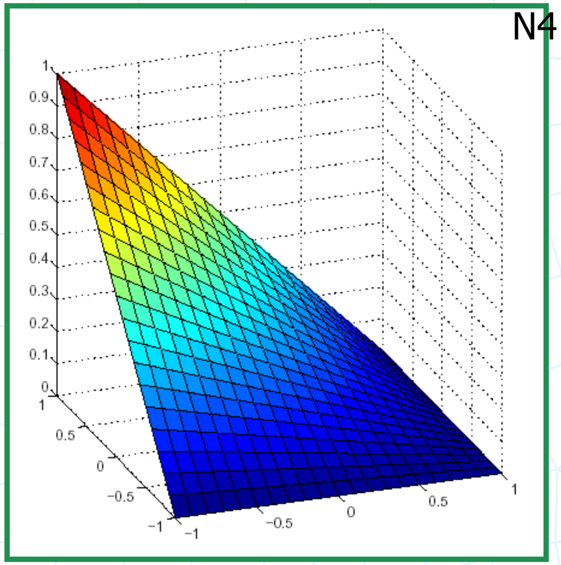
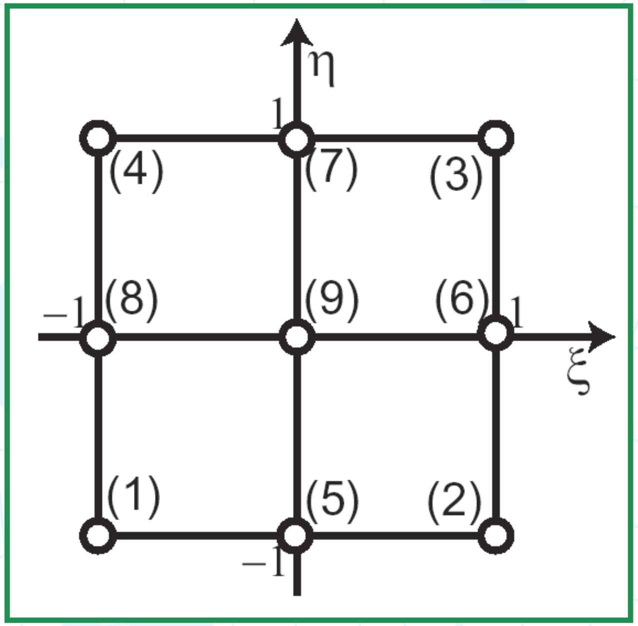
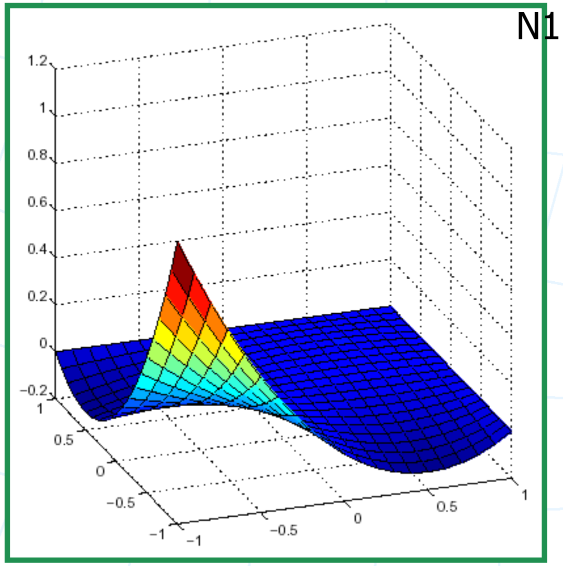
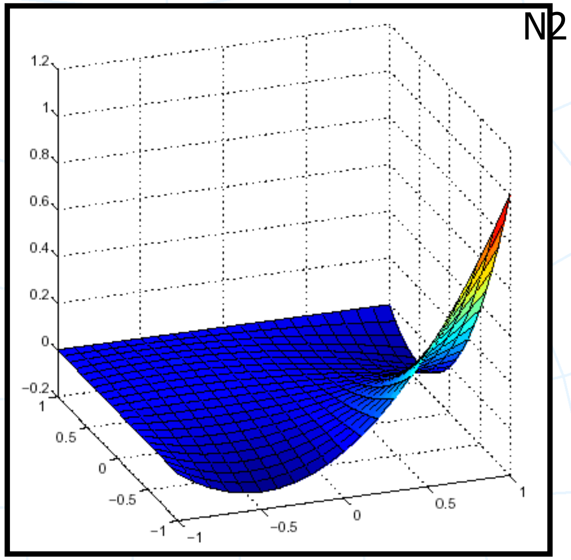


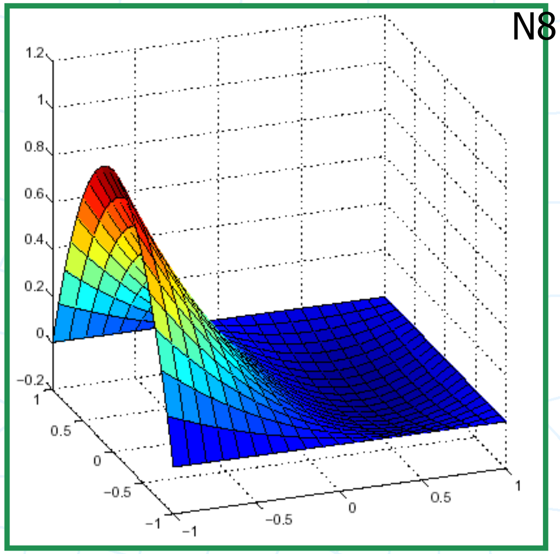


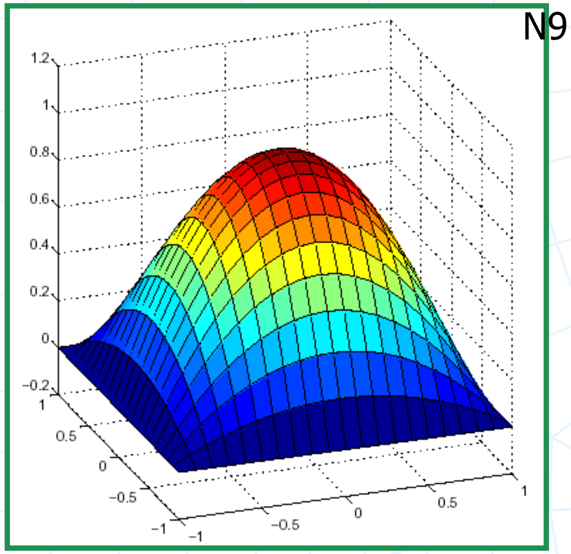
Q2-biquadratic



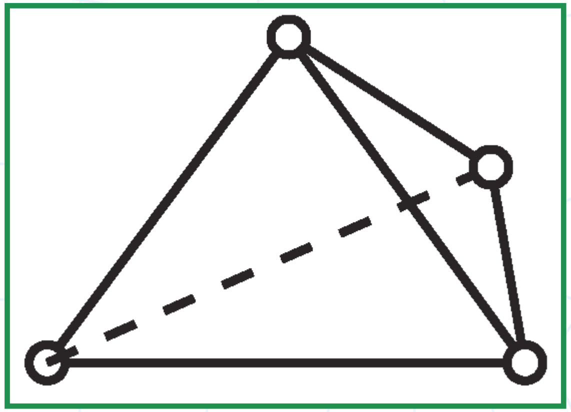




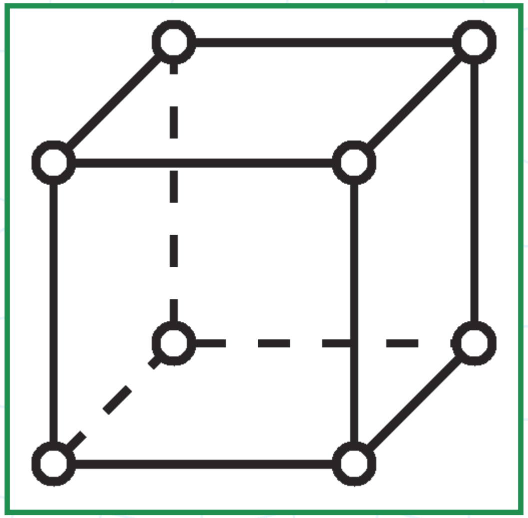




Linear tetrahedra



Linear hexahedra



Change of variables for integration

- Change of variables to go from integrals in the physical element to integrals in the reference element [-1,1]×[-1,1]

$$\boxed{\int_{\Omega_e} N_i f \; dx dy = \int_{\square} N_i f \; |\mathbf{J}| d\xi d\eta}$$

$$\boxed{\begin{gathered}\int_{\Omega_e} \nabla_{xy} N_i \cdot (\mathbf{A} \nabla_{xy} N_j) \; dx dy = \\ \int_{\square} (\mathbf{J}^{-1} \nabla_{\xi\eta} N_i) \cdot (\mathbf{A} (\mathbf{J}^{-1} \nabla_{\xi\eta} N_j)) \; |\mathbf{J}| d\xi d\eta\end{gathered}}$$

Numerical integration

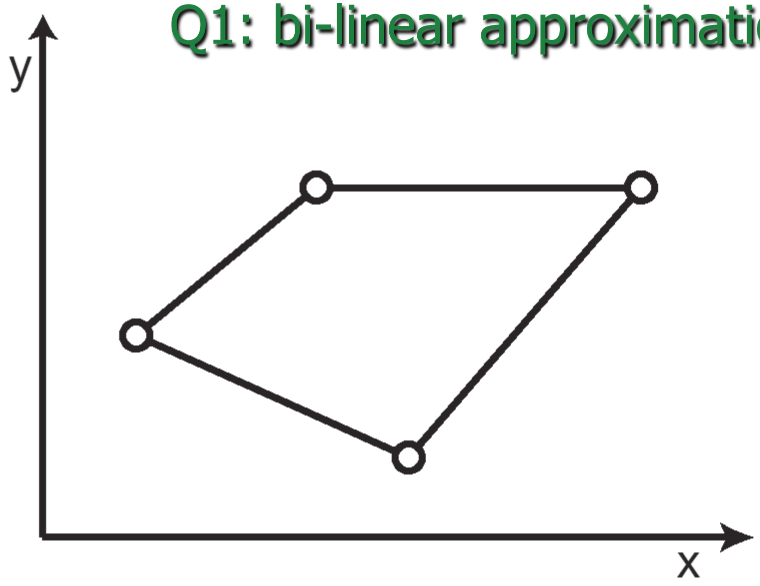
- Numerical quadrature in the reference element (Gauss, quadrature or quadrature for simplexes):
  - Integration points $\boldsymbol{z}_g = (\xi_g, \eta_g)$ and weights ω_g

$$\boxed{\int_{\Omega_e} N_i f \; dx dy \simeq \sum_{g=1}^{\mathrm{n_{gauss}}} N_i(\boldsymbol{z}_g) f(x(\boldsymbol{z}_g), y(\boldsymbol{z}_g)) \; |\mathbf{J}(z_g)| \omega_g}$$

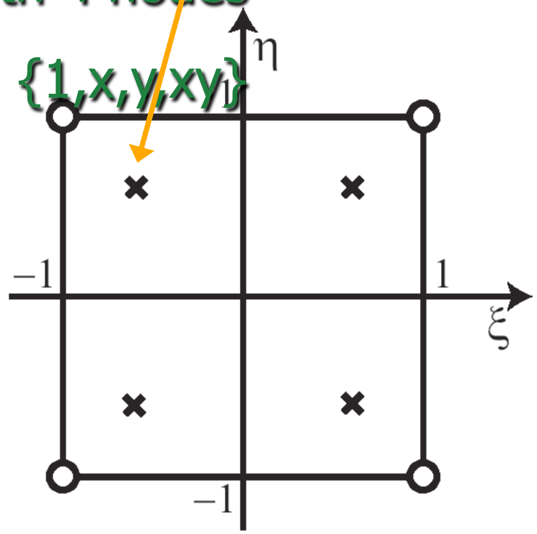
$$\int_{\Omega_e} \nabla_{xy} N_i \cdot (\mathbf{A} \nabla_{xy} N_j) \; dx dy \simeq$$
$$\sum_{g=1}^{\mathrm{n_{gauss}}} (\mathbf{J}^{-1}(\boldsymbol{z}_g) \nabla_{\xi\eta} N_i(\boldsymbol{z}_g)) \cdot \left(\mathbf{A}(\boldsymbol{z}_g) \left(\mathbf{J}^{-1}(\boldsymbol{z}_g) \nabla_{\xi\eta} N_j(\boldsymbol{z}_g) \right) \right) \; |\mathbf{J}(\boldsymbol{z}_g)| \omega_g$$

Integration points

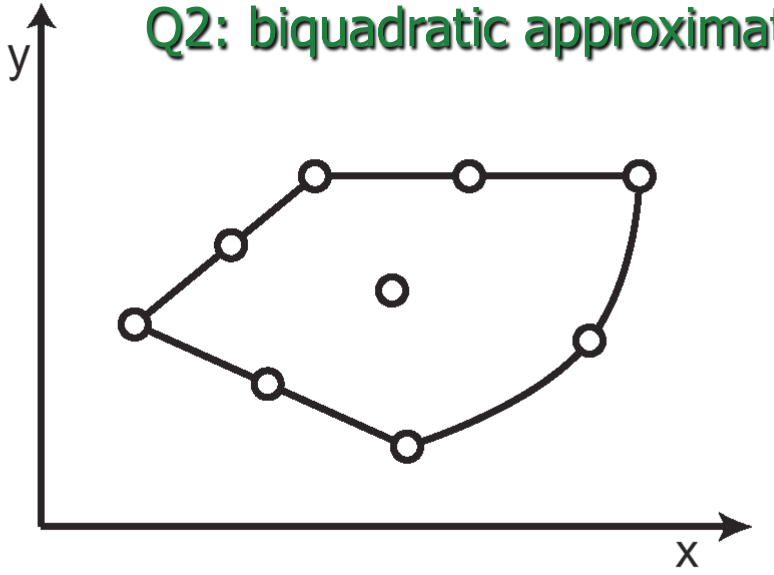
Q1: bi-linear approximation with 4 nodes

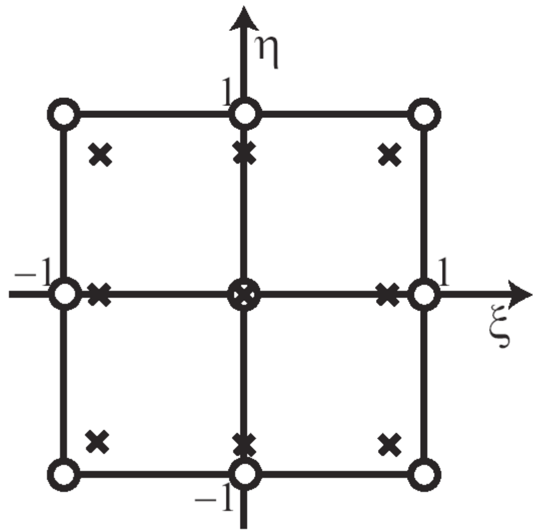


{1,x,y,xy}

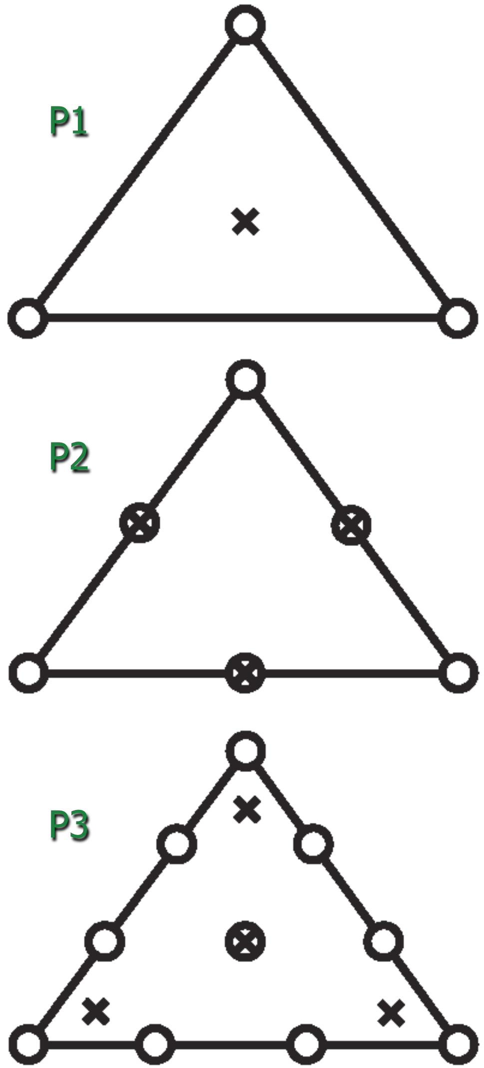


Q2: biquadratic approximation with 9 nodes





- **Triangles**:
  - Cubature rules for integration
  - Linear (P1, {1, x, y}), quadratic (P2, {1, x, y, xy, x^2, y^2}) ...



FEM for Computational Mechanics

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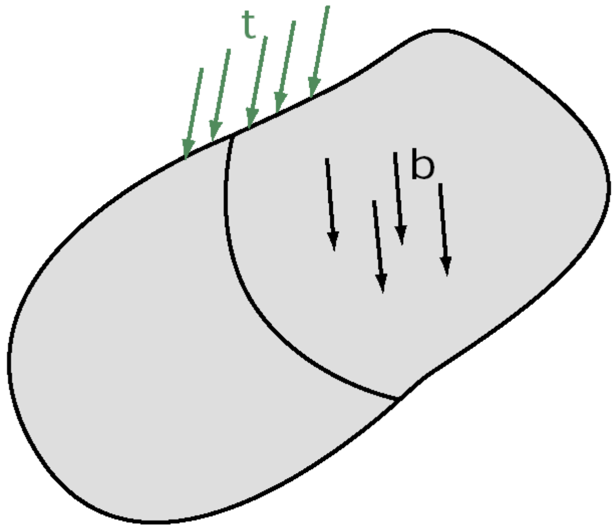
Mechanical problem: principle of virtual work

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} = \boldsymbol{0} & \qquad \text{en } \Omega \\ \boldsymbol{u} = \boldsymbol{u}_d & \qquad \text{en } \Gamma_d \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} & \qquad \text{en } \Gamma_n \end{aligned}$$

$$\Gamma_d \cup \Gamma_n = \partial\Omega$$



$$\boxed{\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\sigma}(\boldsymbol{u}) \, d\Omega = \int_\Omega \rho \boldsymbol{v} \cdot \boldsymbol{b} + \int_{\Gamma_n} \boldsymbol{v} \cdot \boldsymbol{t} \, d\Gamma}$$

for all virtual displacement v (with v=0 on Γd)

Deduction with weighted residuals

- PDE

$$\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} = \mathbf{0}$$

- Multipliying by v such taht v=0 on Γ_d

$$-\int_\Omega \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{\sigma}) \, d\Omega = \int_\Omega \rho \boldsymbol{v} \cdot \boldsymbol{b} \, d\Omega$$

- Using $\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma}) = \nabla \boldsymbol{v} : \boldsymbol{\sigma} + \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{\sigma})$

$$\int_\Omega \nabla \boldsymbol{v} : \boldsymbol{\sigma} \, d\Omega - \int_\Omega \nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma}) \, d\Omega = \int_\Omega \rho \boldsymbol{v} \cdot \boldsymbol{b} \, d\Omega$$

and the Gauss divergence theorem (integration by parts)

$$\int_{\Omega} \nabla \boldsymbol{v} : \boldsymbol{\sigma} \, d\Omega - \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, d\Gamma = \int_{\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{b} \, d\Omega$$

$$\int_{\Omega} \nabla \boldsymbol{v} : \boldsymbol{\sigma} \, d\Omega - \int_{\Gamma_d} \boldsymbol{v} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, d\Gamma = \int_{\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{b} \, d\Omega + \int_{\Gamma_n} \boldsymbol{v} \cdot \boldsymbol{t} \, d\Gamma$$

- Given that v=0 on Γ_d, $\sigma \cdot$n=t on Γ_n and σ is a symmetric tensor

$$\int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\sigma} \, d\Omega = \int_{\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{b} \, d\Omega + \int_{\Gamma_n} \boldsymbol{v} \cdot \boldsymbol{t} \, d\Gamma$$