Basics on Finite Element Method (FEM)

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Weighted residuals: strong form → weak form

- Model problem

$$\boxed{\begin{aligned} -\nabla \cdot (\mathbf{A}\nabla u) &= f && \text{en } \Omega \\ u &= u_d && \text{en } \Gamma_d \\ (\mathbf{A}\nabla u)\cdot \boldsymbol{n} &= g_n && \text{en } \Gamma_n \end{aligned}}$$

with

$$\partial\Omega = \Gamma_d \cup \Gamma_n$$

STEP 1. Multiply by a test function v such that v=0 en Γ_d

$$-\int_\Omega v\, \nabla \cdot (\mathbf{A}\nabla u)\; d\Omega = \int_\Omega v f \; d\Omega$$

STEP 2. Integration by parts

$$-\int_\Omega v\,\nabla\cdot(\mathbf{A}\nabla u)\;d\Omega = \int_\Omega vf\;d\Omega$$

"Integration by parts" formula in several dimensions:

$$\int_\Omega g\boldsymbol{\nabla}\cdot\boldsymbol{f}\,d\Omega = -\int_\Omega \boldsymbol{\nabla}g\cdot\boldsymbol{f}\,d\Omega + \int_{\partial\Omega} g\boldsymbol{f}\cdot\boldsymbol{n}\,d\Gamma \qquad (\clubsuit)$$

$$\int_\Omega \nabla v\cdot(\mathbf{A}\nabla u)\;d\Omega - \int_{\partial\Omega} v\,(\mathbf{A}\nabla u)\cdot\boldsymbol{n}\;d\Gamma = \int_\Omega vf\;d\Omega$$

STEP 3. Apply boundary conditions (Neumann BC and v=0 in Γ_d)

$$\boxed{\int_\Omega \nabla v\cdot(\mathbf{A}\nabla u)\;d\Omega = \int_\Omega vf\;d\Omega + \int_{\Gamma_n} vg_n\;d\Gamma}$$

Weak form

- "Find $u \in H^1(\Omega)$ such that $u=u_d$ in Γ_d and

$$a(v,u) = l(v)$$

for all $v \in H^1(\Omega)$ such that $v=0$ en Γ_d", with

$$a(v,u) = \int_\Omega \nabla v \cdot (\mathbf{A} \nabla u) \, d\Omega$$

Bilinear, symmetric & coercive

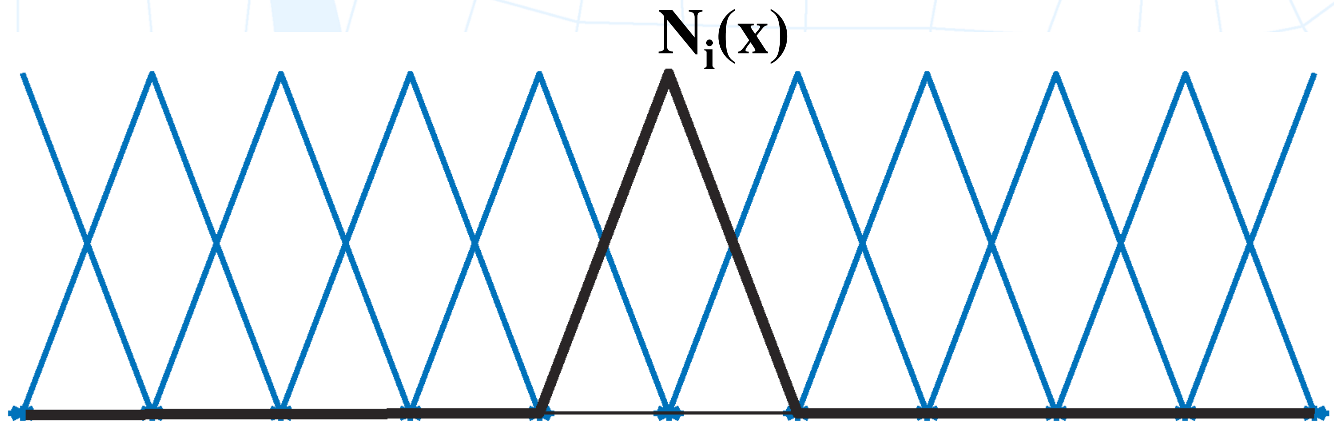
$$l(v) = \int_\Omega v f \, d\Omega + \int_{\Gamma_n} v g_n \, d\Gamma$$

- The equivalence between strong form and weak form can be easily proved.

Piece-wise (element-by-element) polynomial approximation

- In the FEM the solution is approximated with a piece-wise polynomial basis

$$u(x) \simeq u^h(x) = \sum_i u_i N_i(x)$$



- Advantages:
  - compact suport (local basis) $\Rightarrow$ sparse matrices
  - easy computation of integrals
  - physical meaning of coefficients u_i

Prescribed values

- The coefficients corresponding to values known by boundary conditions are set to the prescrived value

$$u^h(x) = \sum_{j \notin \mathcal{B}} u_j N_j(x) + \underbrace{\sum_{i \in \mathcal{B}} u_d(x_i) N_i(x)}_{\psi(x)}$$

  - $u^h(x)$ satisfies (with interpolation error) the Dirichlet boundary condition $u=u_d$ on Γ_d
  - $N_i(x)=0$ on Γ_d for $i \notin B$ (condition for test function v)

- Other techniques: Lagrange multipliers, penalty method, Nitsche method...

Discretization of the weak form

Imposing the weak form for $v=N_i(x)$ with $i\notin B$, and replacing the approximation $u^h(x)$

$$a\Big(N_i \;,\; \sum_j u_j N_j + \psi \Big) = l(N_i)$$

$$\Downarrow$$

$$\sum_j a(N_i, N_j)\; u_j = l(N_i) - a(N_i, \psi)$$

Linear system of equations $\boxed{\mathbf{K}\mathbf{u} = \mathbf{f}}$

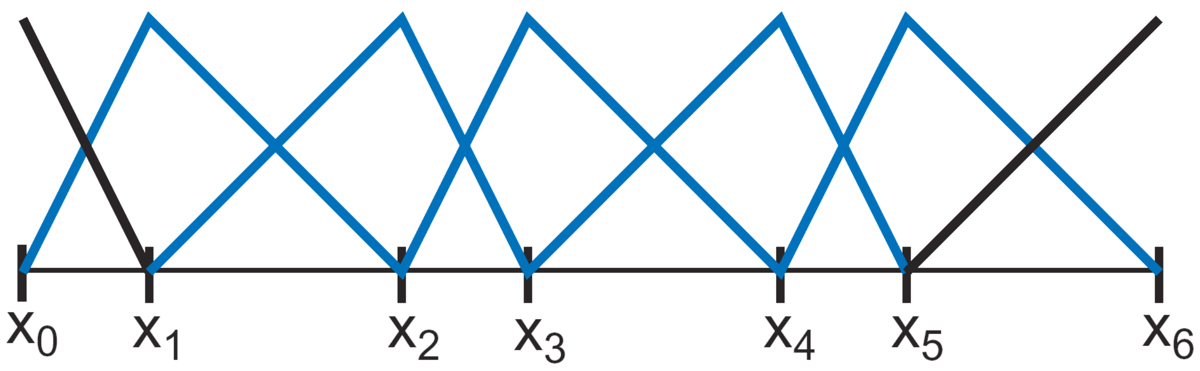
$$K_{ij} = a(N_i, N_j) = \int_\Omega \nabla N_i \cdot (\mathbf{A}\nabla N_j)\; d\Omega$$

$$f_i = l(N_i) - a(N_i, \psi) = \int_\Omega N_i f \; d\Omega + \int_{\Gamma_n} v g_n \; d\Gamma - a(N_i, \psi)$$

1D example (with C^0 linear approximation)

$$-u'' = f \text{ para } x \in [a,b] \qquad u(a) = u(b) = 0 \quad \Rightarrow \quad \int_a^b v'u'\,dx = \int_a^b vf\,dx$$

- Approximation: $$u(x) \simeq u^h(x) = \sum_{j=1}^{5} u_j N_j(x)$$



- Replacing the approximation and v=N_i for i=1...5

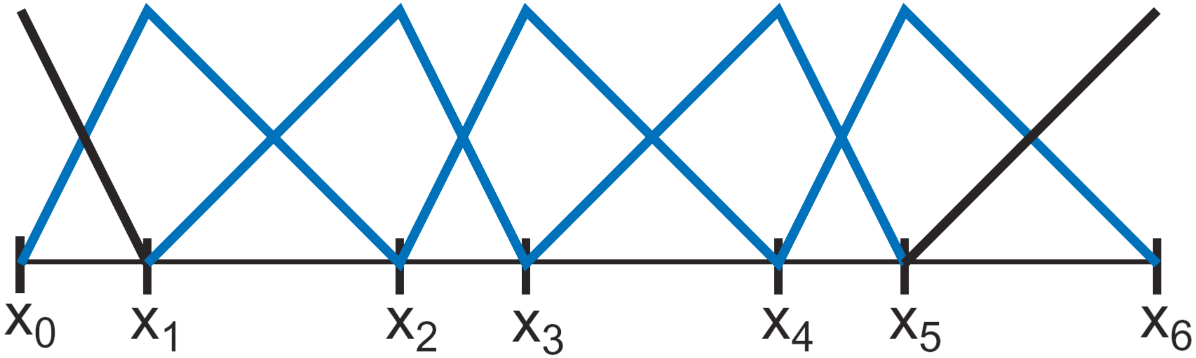
$$\int_a^b N_i' \left(\sum_{j=1}^{5} u_j N_j' \right) dx = \int_a^b N_i f \, dx \quad i = 1, \ldots, 5$$

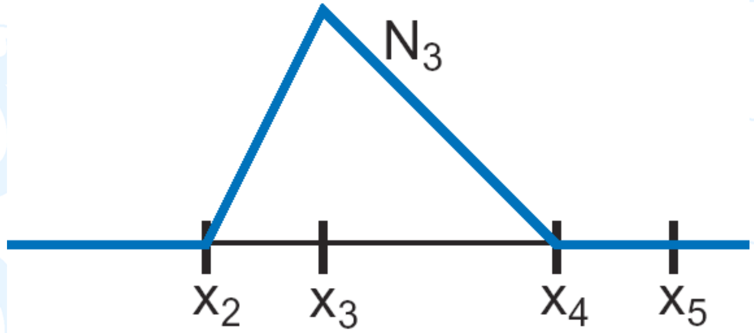
or, equivalently,

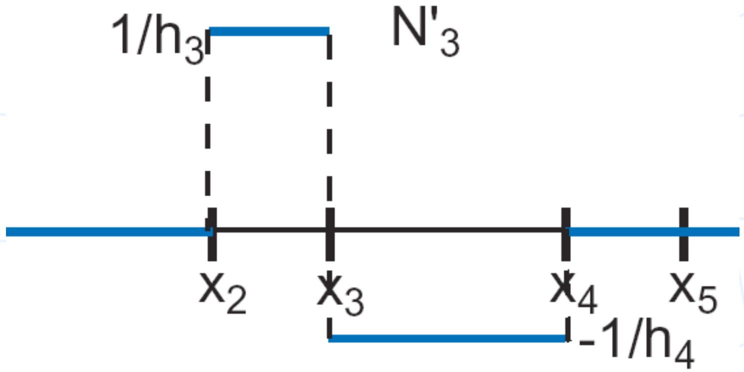
$$\sum_{j=1}^{5} \left(\int_a^b N_i' N_j' \, dx \right) u_j = \int_a^b N_i f \, dx \quad i = 1, \ldots, 5$$

- Linear system 5 × 5: $\boxed{\mathbf{Ku} = \mathbf{f}}$

$$K_{ij} = \int_a^b N_i' N_j' \, dx, \quad f_i = \int_a^b N_i f \, dx$$







- The matrix of the system is **tridiagonal** (sparse in general, with few non-null coefficients)

$$K_{ij} = \int_a^b N_i' N_j' \, dx = 0 \text{ para } |i-j| > 1$$

$$\mathbf{K} = \begin{pmatrix} \left(\frac{1}{h_1} + \frac{1}{h_2}\right) & -\frac{1}{h_2} & & & \\ -\frac{1}{h_2} & \left(\frac{1}{h_2} + \frac{1}{h_3}\right) & -\frac{1}{h_3} & & \\ & -\frac{1}{h_3} & \left(\frac{1}{h_3} + \frac{1}{h_4}\right) & -\frac{1}{h_4} & \\ & & -\frac{1}{h_4} & \left(\frac{1}{h_4} + \frac{1}{h_5}\right) & -\frac{1}{h_5} \\ & & & -\frac{1}{h_5} & \left(\frac{1}{h_5} + \frac{1}{h_6}\right) \end{pmatrix}$$

Symmetric and diagonally dominant matrix:

The matrix is **symmetric and positive definite**

- If the bilinear form a(·,·) is symmetric and coercive, the matrix is symmetric and positive definite.
- The coefficient (*i,j*) of the matrix is non-null only if nodes i and j belong to the same elements: sparse matrices

Computation of integrals: numerical quadrature in each element

- We want to compute integrals as

$$K_{ij} = a(N_i, N_j) = \int_{\Omega} \nabla N_i \cdot (\mathbf{A} \nabla N_j)\, d\Omega$$

with element-by-element piece-wise polynomial functions.

$$K_{ij} = \sum_e \int_{\Omega_e} \nabla N_i \cdot (\mathbf{A} \nabla N_j)\, d\Omega = \ldots$$

- Gauss quadrature in each element.

Elemental matrices

- Assembly of elemental matrices and vectors

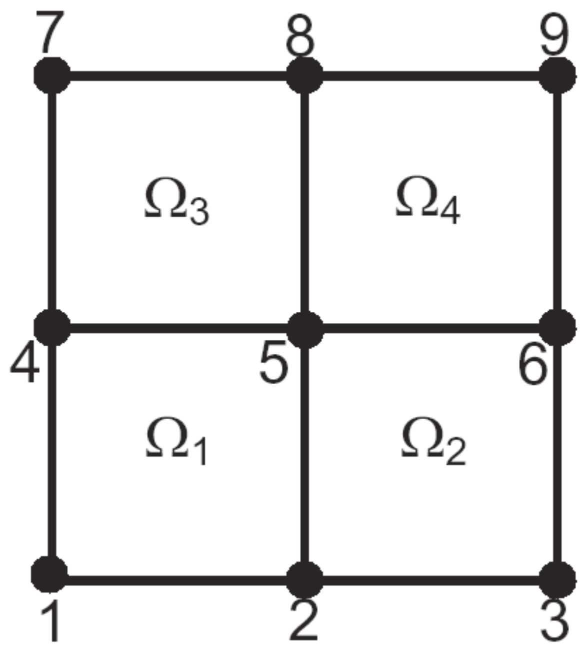
$$\mathbf{K} = \mathop{\mathsf{A}}_{e} \mathbf{K}^{e}, \quad \mathbf{f} = \mathop{\mathsf{A}}_{e} \mathbf{f}^{e}$$

- The elemental matrix **K**^{e} has all the non-null integrals in the element Ω_e

$$K^{e}_{(i)(j)} = \int_{\Omega_e} \nabla N_{(i)} \cdot \left(\mathbf{A} \nabla N_{(j)}\right) \, d\Omega \qquad \begin{array}{l} (i) = 1, \ldots, \texttt{nnode} \\ (j) = 1, \ldots, \texttt{nnode} \end{array}$$

where (·) denotes the local numbering and `nnode` is the number of nodes in the element. The connectivity matrix gives the equivalence between local numbering and global numbering.

Example

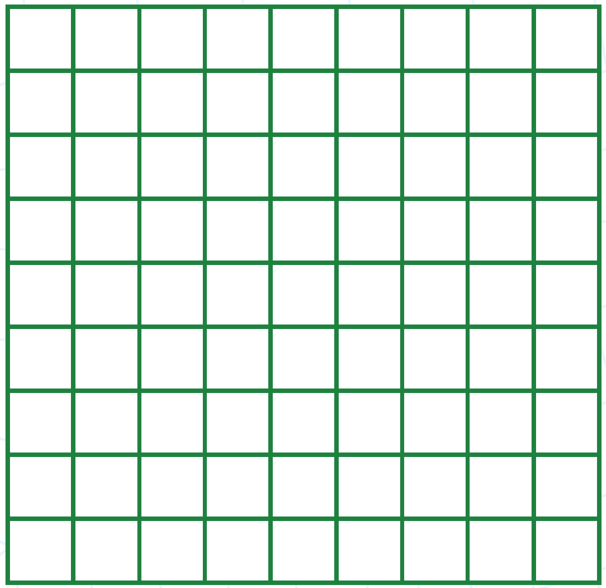


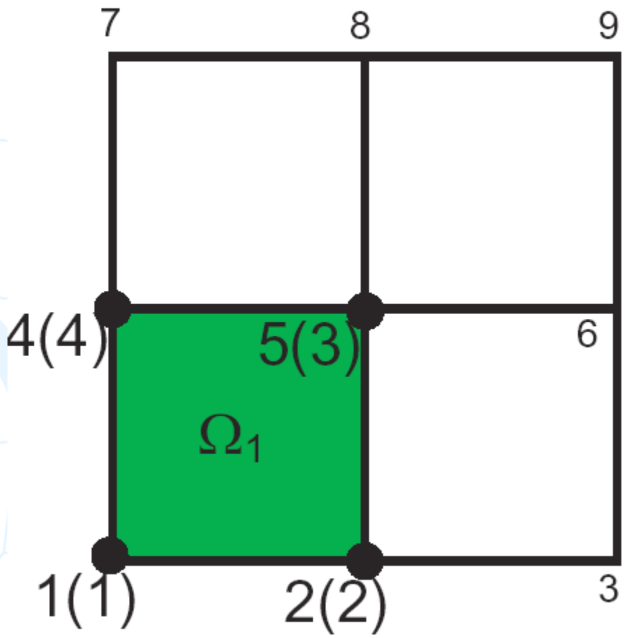
Mesh geometry definition

$$\mathbf{T} = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 2 & 3 & 6 & 5 \\ 4 & 5 & 8 & 7 \\ 5 & 6 & 9 & 8 \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} 0 & 0 \\ 0{,}5 & 0 \\ 1 & 0 \\ 0 & 0{,}5 \\ 0{,}5 & 0{,}5 \\ 1 & 0{,}5 \\ 0 & 1 \\ 0{,}5 & 1 \\ 1 & 1 \end{bmatrix}$$

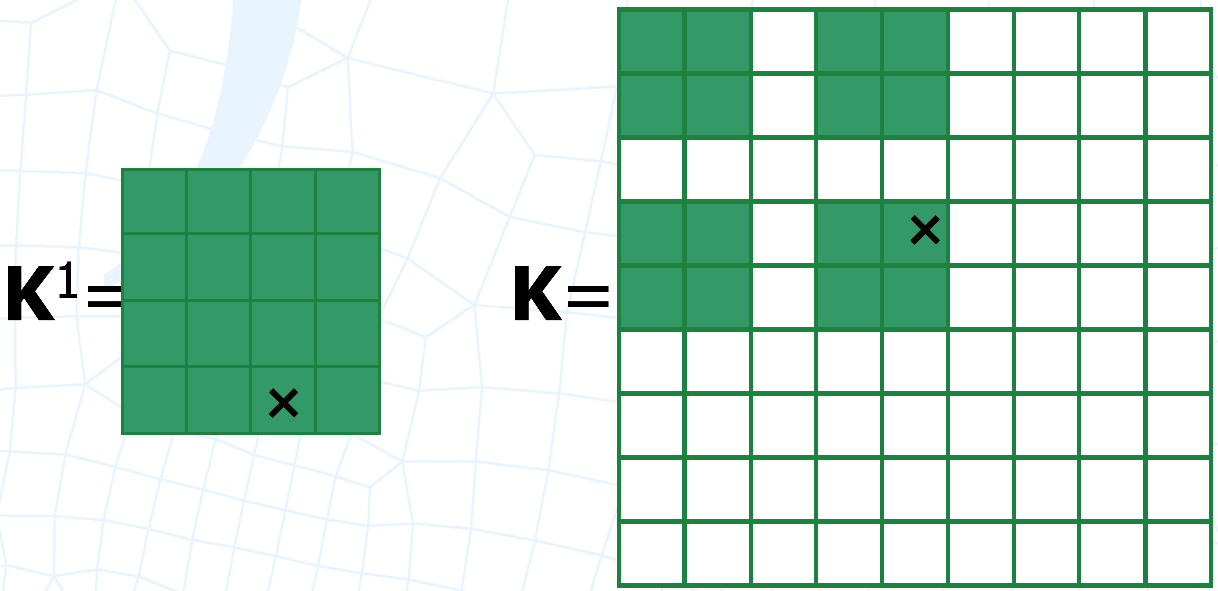
(connectivity matrix)

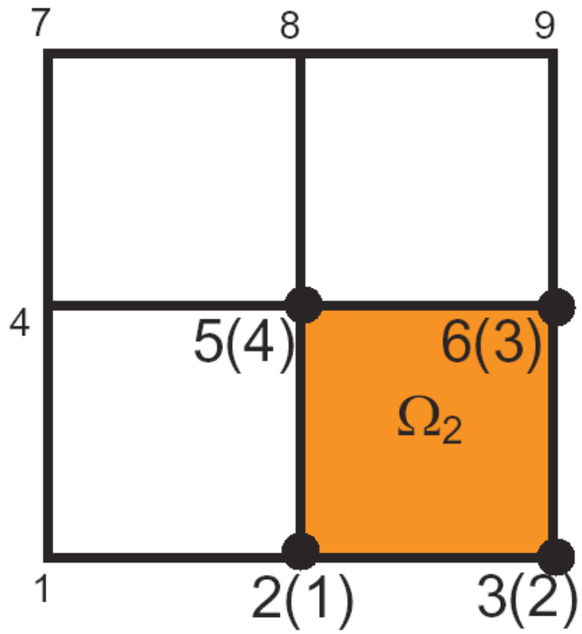
K=

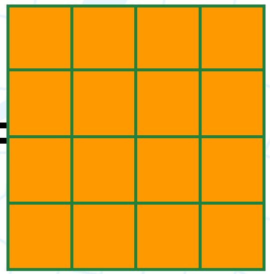


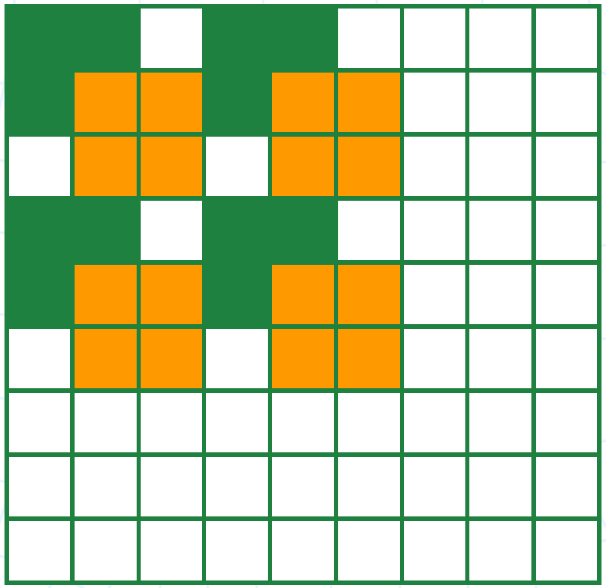


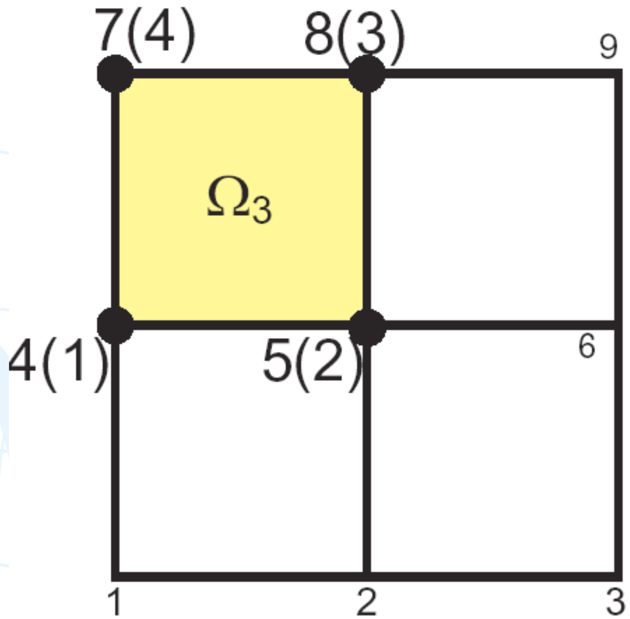
(#) local numbering

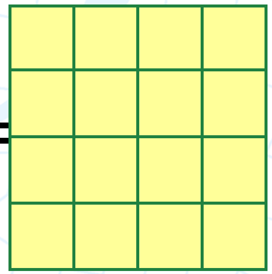


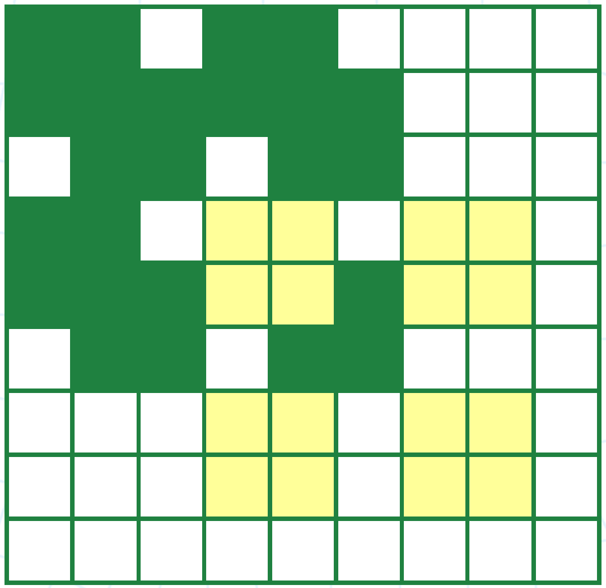


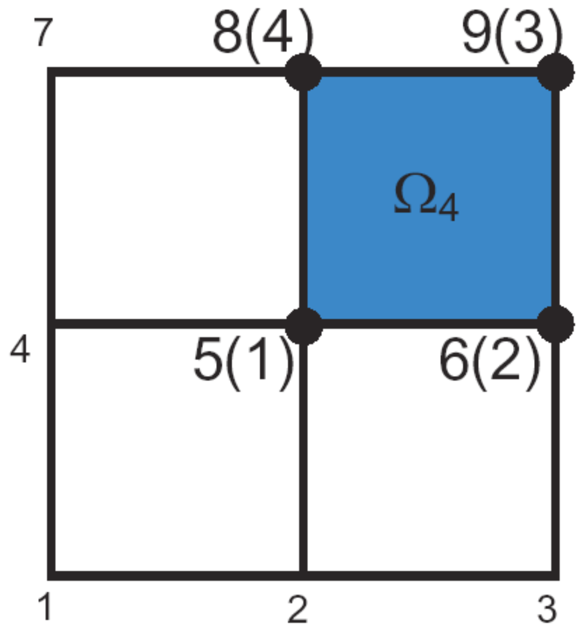
$K^2=$ 

$K=$ 

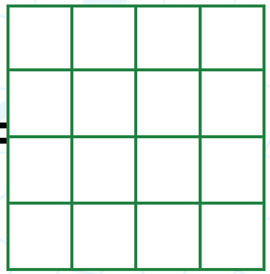


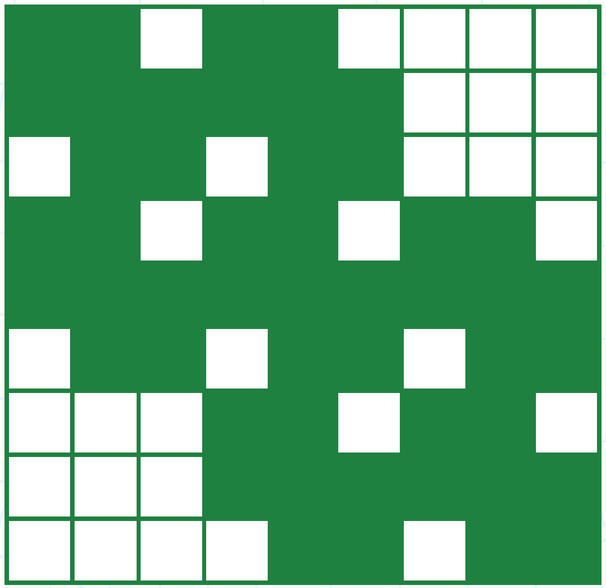
$K^3 =$ 

$K =$ 



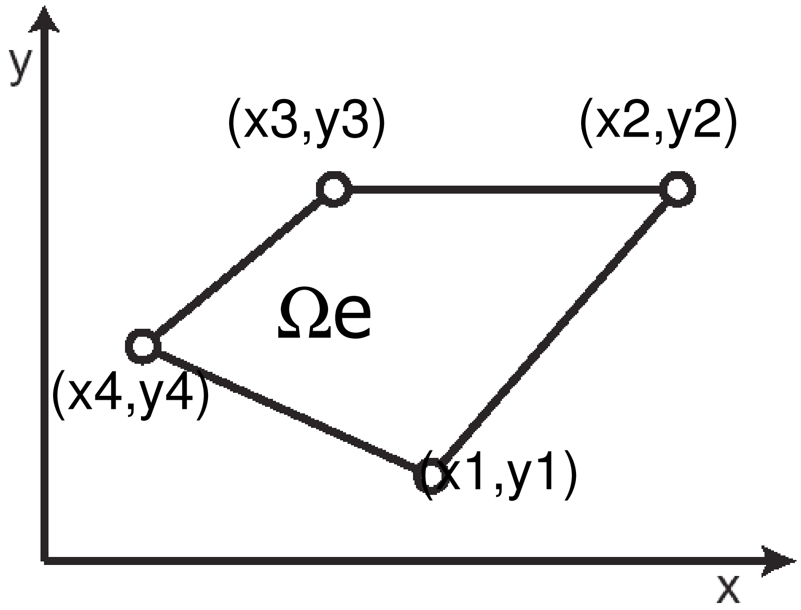
Symmetric and semi-positive matrix (positiveness after imposing boundary conditions)

$\mathbf{K}^4=$ 

$\mathbf{K}=$ 

Computation of the elemental matrix

$$K^e_{(i)(j)} = \int_{\Omega_e} \nabla N_{(i)} \cdot \left(\mathbf{A} \nabla N_{(j)} \right) d\Omega \qquad \begin{array}{l} (i) = 1, \ldots, \mathtt{nnode} \\ (j) = 1, \ldots, \mathtt{nnode} \end{array}$$



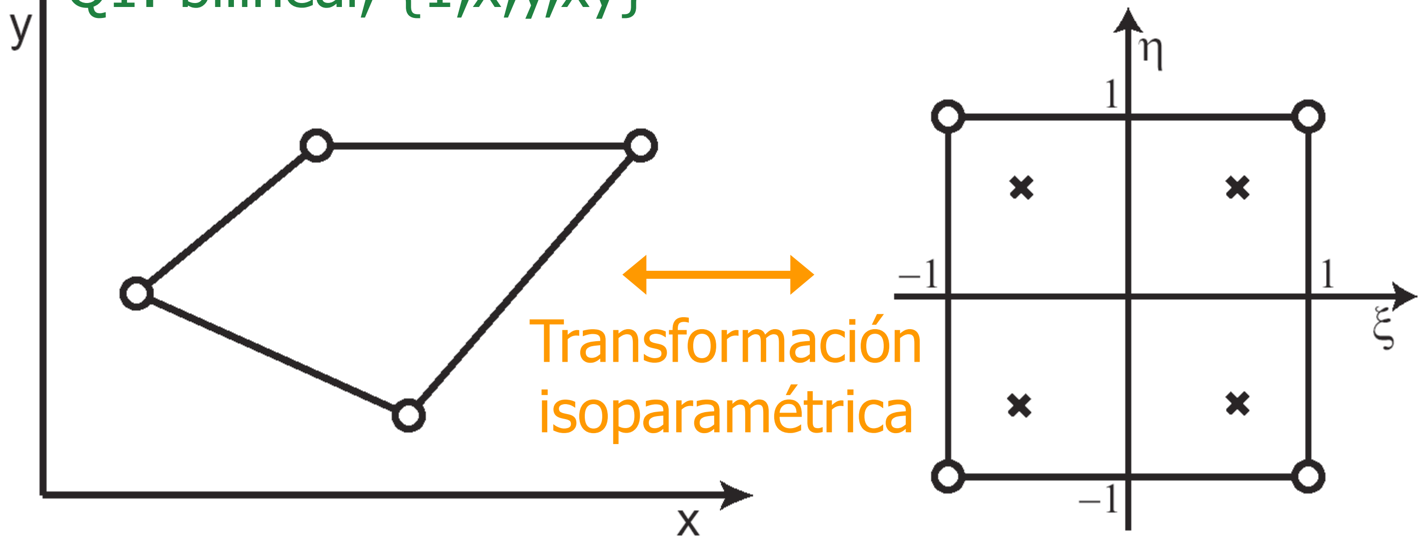
Shape functions

$N_i(x)=?$

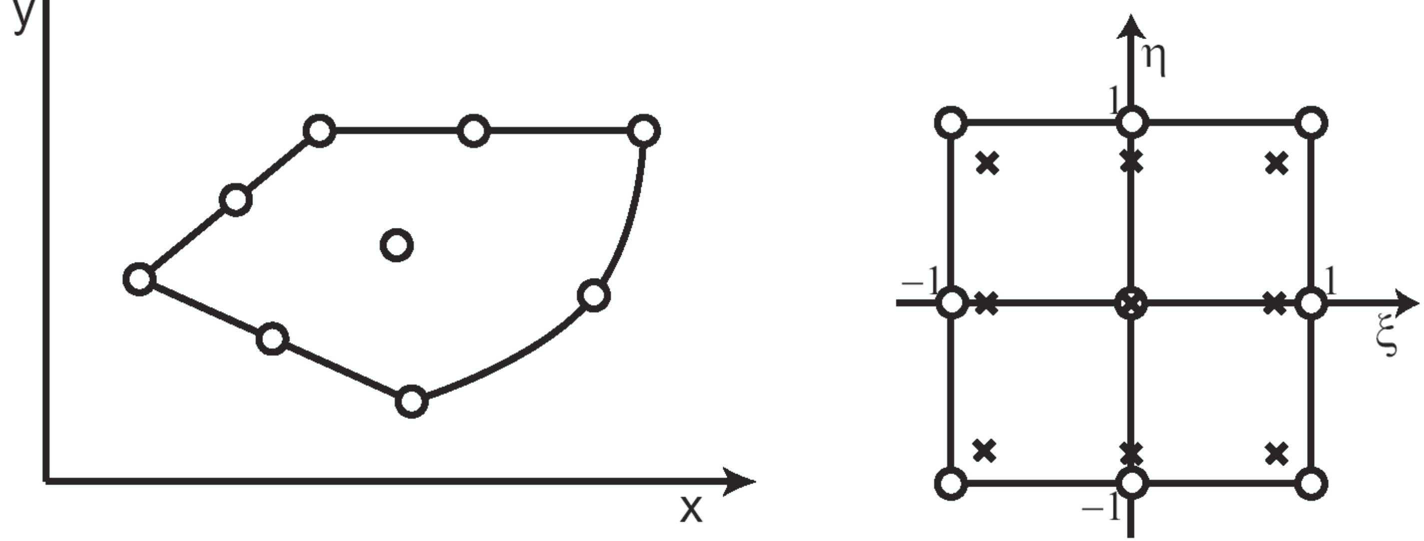
Numerical quadrature

Reference element

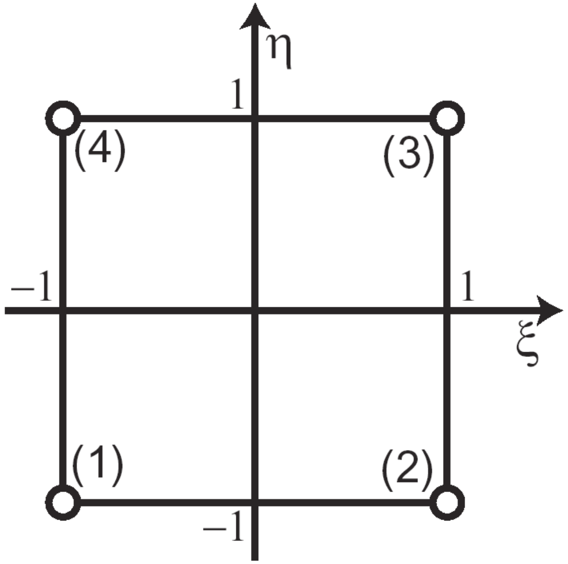
Q1: bilinear, {1,x,y,xy}

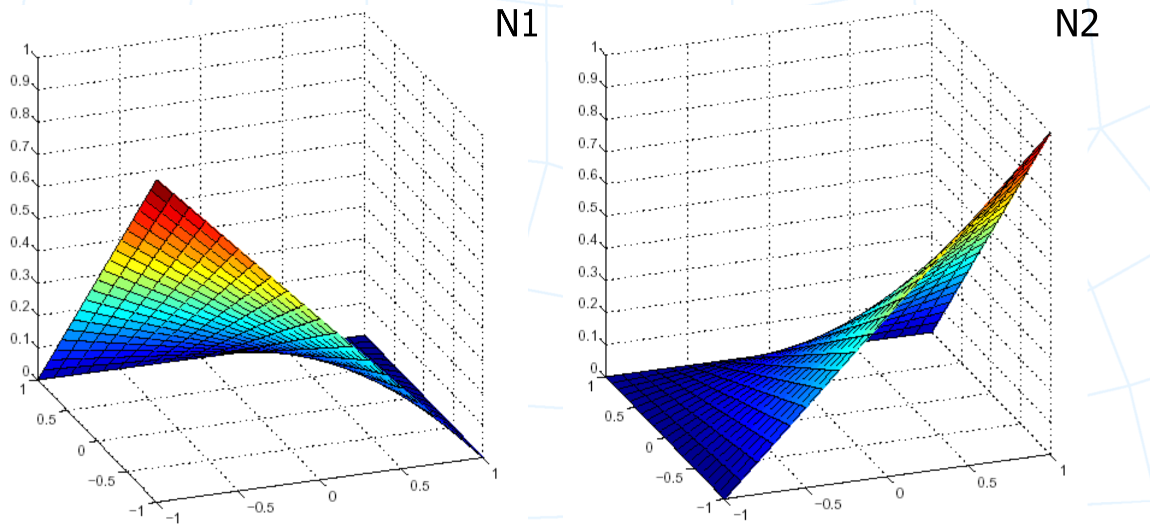


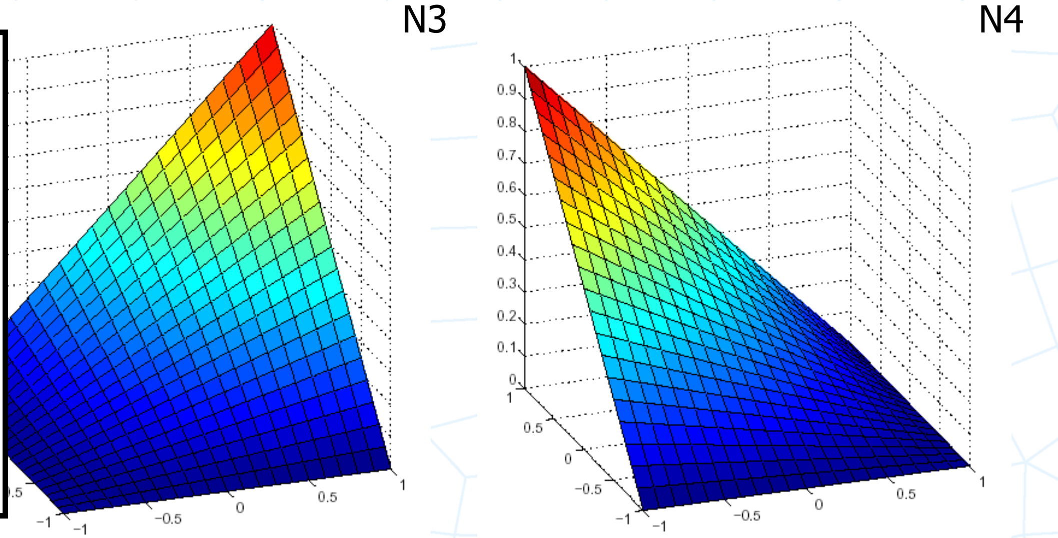
Q2: biquadratic



Q1 element

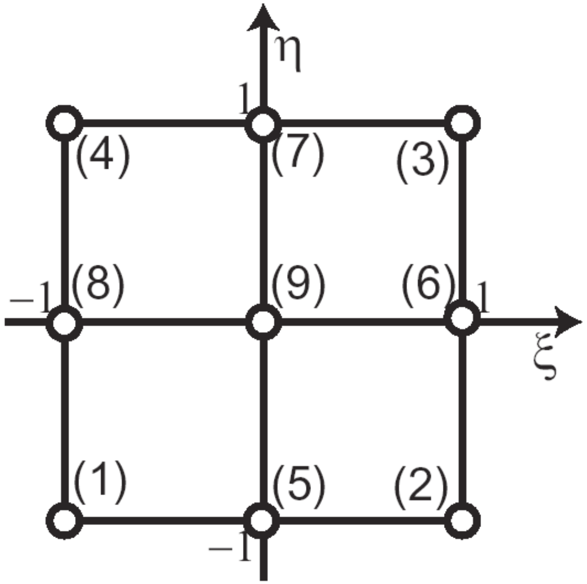


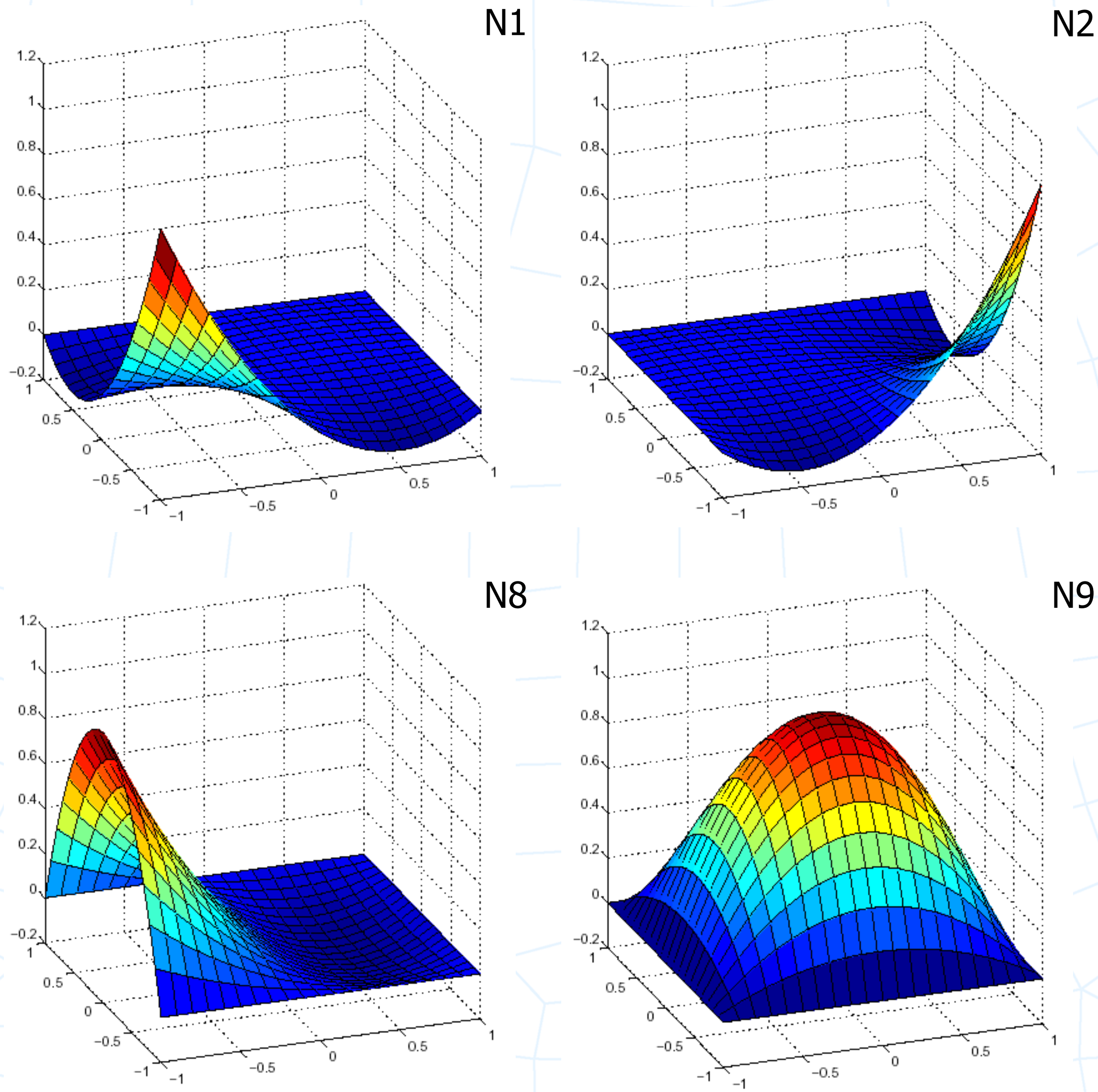




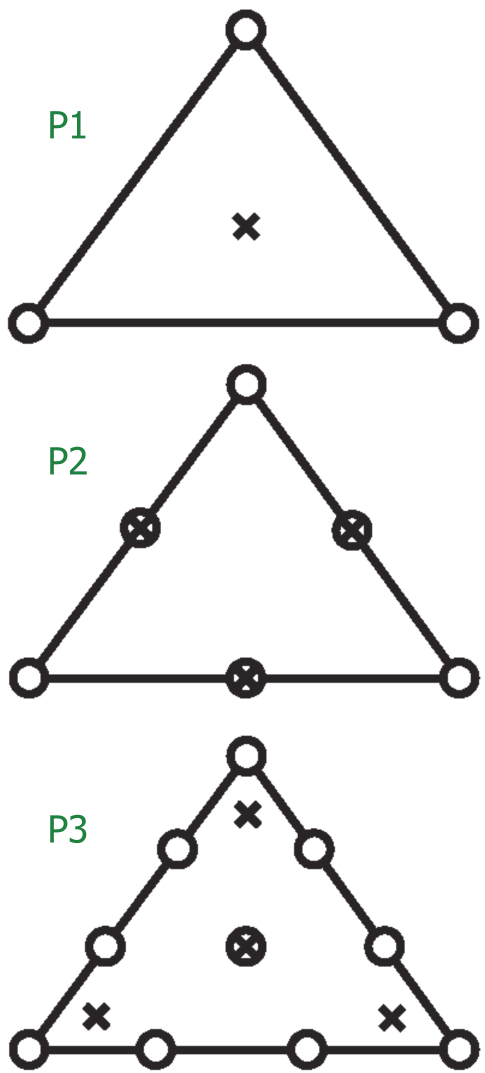
$$N_1(\xi,\eta) = \tfrac{1}{4}(\xi-1)(\eta-1)$$
$$N_2(\xi,\eta) = -\tfrac{1}{4}(\xi+1)(\eta-1)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}(\xi+1)(\eta+1)$$
$$N_4(\xi,\eta) = -\tfrac{1}{4}(\xi-1)(\eta+1)$$

Q2 element

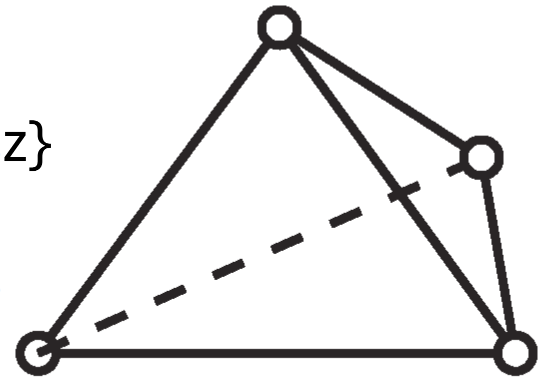




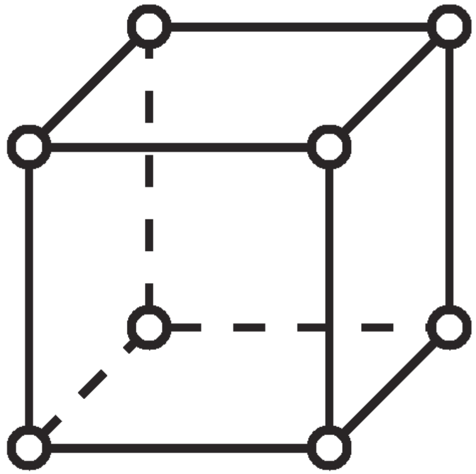
- **Triangles**:
  - P1: linear, {1, x, y}
  - P2: quadratic, {1, x, y, xy, x^2, y^2}
  - …



Tetrahedra {1, x, y, z}



Hexahedra {1, x, y, z, xy, xz, yz, xyz}



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