Analysis and Numerics of the Spatially Homogeneous Fokker-Planck-Landau Equation

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Abstract

We consider the spatially homogeneous Fokker-Planck-Landau equation. This equation is not entirely relevant to physics because the electromagnetic field is not included. The purpose for considering this equation is to study by itself the Landau collision operator. First, analysis is done on this equation to verify the conservation of mass, momentum, and energy. We also verify that the kinetic entropy is a non-increasing function of time. The next step in the analysis is to show that the Fokker-Planck-Landau equation can be expressed in the form of a second-order parabolic-type partial differential equation, with coefficients that involve integrals of the phase space distribution function. In this form, we can approximate the equation using finite difference methods for parabolic-type PDE's. We will then write a Fortran program to numerically approximate the Fokker-Planck-Landau equation in the parabolic form, and verify computationally the conservation and entropy laws.

I Introduction

The Fokker-Planck-Landau equation governs Coulomb interactions between collisional plasma particles. It can be characterized as a modification of the Boltzmann equation, which governs interactions between gas particles. The Boltzmann equation is given in [5, p. 22] as $\dfrac{\partial f}{\partial t} = C[f]$, where $C[f]$ denotes the "collision term." The Landau collision operator $P(f, f)(v)$ can be mathematically derived from $C[f]$. The derivation assumes spatial homogeneity; i.e., dependence only on velocity and time, not position in space. Furthermore, while the Boltzmann collision operators account for neutral hard particle collisions, the Landau collision operator assumes all interactions to be "grazing collisions," which can be characterized as small angle collisions between charged particles. Under these restrictions, $P(f, f)(v)$ is derived from $C[f]$, [1].

We now formally state the spatially homogeneous Fokker-Planck-Landau equation, for a single charged species.

For the spatially homogeneous case, let $f(v, t)$ denote the distribution function of collisional plasma; i.e., f is the number of plasma particles with velocity v at time t per unit volume, where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. We should note that, in the proper context, v is used to denote the vector $v_i\mathbf{i} + v_j\mathbf{j} + v_k\mathbf{k}$.
The spatially homogeneous Fokker-Planck-Landau equation for f is given by:

$$\frac{\partial f}{\partial t} = P(f, f)(v), v \in \mathbb{R}^3, t > 0, f(v, 0) = f_0(v) \tag{1.1}$$

The Landau collision operator, $P(f, f)(v)$, can be expressed:

$$P(f, f)(v) = \nabla_v \cdot p(f, f)(v) \tag{1.2}$$

$$p(f,f)(v) = \int_{\mathbb{R}^3} \phi(v-\bar{v})(\nabla_v f\bar{f} - \nabla_{\bar{v}}\bar{f}f)d\bar{v} \tag{1.3}$$

In (1.3), $\nabla_w f = \nabla f(w)$, and $\bar{f} = f(\bar{v},t)$. In addition, $\phi(w)$ is a positive semi-definite matrix given as follows:

$$\phi(w) = \frac{1}{8\pi}\frac{S(w)}{|w|}; S(w) = I - \frac{w\otimes w}{|w|^2} \tag{1.4}$$

It can be noted that $\phi(w) = \phi(-w)$.

In the following sections, we develop properties of the Landau collision operator, and deduce resulting properties of collisional plasma; namely, the existence of collisional invariants and non-increasing nature of kinetic entropy.

II Constants of Motion; Kinetic Entropy

We are guided in this section by the theory outlined in [3].

We begin by reformulating the Landau collision operator, and the integral of its product with a smooth function $\psi(v)$.

Lemma 2.1. (Reformulation of $P(f,f)(v)$) Given $P(f,f)(v)$, defined in (1.2),(1.3), and f, a differentiable function, $P(f,f)(v)$ can alternatively be expressed as:

$$P(f,f)(v) = \nabla_v \cdot p(f,f)(v) \tag{2.1}$$

$$p(f,f)(v) = \int_{\mathbb{R}^3} \phi(v-\bar{v})f\bar{f}(\nabla_v \log f - \nabla_{\bar{v}} \log \bar{f})d\bar{v} \tag{2.2}$$

Proof. Note that $\nabla_v \log f = \frac{1}{f}\nabla_v f$, or $\nabla_v f = f\nabla_v \log f$. Substituting these expressions into (2.2) and making the appropriate cancellations of f and $\bar{f}$ yields (1.2),(1.3).

Lemma 2.2. (Reformulation of Integral) Let $I = \int_{\mathbb{R}^3} P(f,f)(v)\psi(v)dv$. Then, for a differentiable function $\psi(v)$,

$$I = -\frac{1}{2}\int_{\mathbb{R}^6} f\bar{f}(\nabla_v\psi(v) - \nabla_{\bar{v}}\psi(\bar{v}))^T\phi(v-\bar{v})(\nabla_v \log f - \nabla_{\bar{v}} \log \bar{f})d\bar{v}dv \tag{2.3}$$

Proof. I is equivalent to the following triple integral in $\mathbb{R}^3$:

$$\iiint_{\mathbb{R}^3} \psi(v)(\frac{\partial p}{\partial v_1} + \frac{\partial p}{\partial v_2} + \frac{\partial p}{\partial v_3})dv_1dv_2dv_3 \tag{2.4}$$

We integrate by parts with respect to each component of v, and find that

$$\int \psi(v)\frac{\partial p}{\partial v_i}dv_i = \psi(v)p(f,f)(v)\Big|_{-\infty}^{\infty} - \int \frac{\partial \psi}{\partial v_i}p(f,f)(v)dv_i \tag{2.5}$$

As $t\to\infty$, the product $\psi(v)p(f,f)(v)\to 0$. Integral (2.4) can then be written:

$$-\int_{\mathbb{R}^3}(\frac{\partial\psi}{\partial v_1}p+\frac{\partial\psi}{\partial v_2}p+\frac{\partial\psi}{\partial v_3}p)dv \tag{2.6}$$

$$=-\int_{\mathbb{R}^3}\nabla_v\psi(v)\cdot p(f,f)(v)dv. \tag{2.7}$$

Substituting the expression for $p(f,f)(v)$ given by (2.2), (2.7) can be expressed as:

$$I=-\int_{\mathbb{R}^6} f\bar{f}(\nabla_v\psi(v))^T\phi(v-\bar{v})(\nabla_v\log f-\nabla_{\bar{v}}\log f)d\bar{v}dv \tag{2.8}$$

We note that $\phi(v-\bar{v})=\phi(\bar{v}-v)$, and therefore we can interchange v and $\bar{v}$ in (2.8). Then,

$$I=-\int_{\mathbb{R}^6} f\bar{f}(\nabla_{\bar{v}}\psi(\bar{v}))^T\phi(\bar{v}-v)(\nabla_{\bar{v}}\log f-\nabla_v\log f)dvd\bar{v}$$

$$=\int_{\mathbb{R}^6} f\bar{f}(\nabla_{\bar{v}}\psi(\bar{v}))^T\phi(v-\bar{v})(\nabla_v\log f-\nabla_{\bar{v}}\log f)d\bar{v} \tag{2.9}$$

Therefore, $I=\frac{1}{2}(I_{2.8}+I_{2.9})$, where $I_{2.8}$ and $I_{2.9}$ are the forms of I given in (2.8) and (2.9), respectively. Then,

$$\int_{\mathbb{R}^3}\psi(v)P(f,f)(v)dv=-\frac{1}{2}\int_{\mathbb{R}^6} f\bar{f}(\nabla_v\psi(v)-\nabla_{\bar{v}}\psi(\bar{v}))^T\phi(v-\bar{v})(\nabla_v\log f-\nabla_{\bar{v}}\log f)d\bar{v}dv, \tag{2.10}$$

and (2.3) is verified.

II.1 Kinetic Entropy: The H-Theorem

Definition. Let H, the kinetic entropy of collisional plasma, be defined as follows:

$$H=\int_{\mathbb{R}^3} f(v,t)\log f(v,t)dv. \tag{2.11}$$

Theorem. The kinetic entropy of collisional plasma, H, is a non-increasing function of time; i.e.

$$\frac{dH}{dt}\leq 0, \tag{2.12}$$

Proof. By the product rule,

$$\frac{dH}{dt}=\int(f_t\log(f)+f\cdot\frac{1}{f}f_t)dv. \tag{2.13}$$

Since $f_t=P(f,f)(v)$, this is equivalent to:

$$\frac{dH}{dt}=\int P(f,f)(v)(1+\log f(v,t))dv$$

$$=\int P(f,f)(v)dv+\int P(f,f)(v)\log f(v,t)dv. \tag{2.14}$$

Let $\psi=1$ in (2.10). Then $(\nabla_v\psi(v)-\nabla_{\bar{v}}\psi(\bar{v}))^T=0$, and so $\int_{\mathbb{R}^3}P(f,f)(v)dv=0$. Let $\psi=\log f$ in (2.10). Then $(\nabla_v\psi-\nabla_{\bar{v}}\psi)^T=(\nabla_v\log f-\nabla_{\bar{v}}\log\bar{f})^T$. Since ϕ is positive semi-definite and $f(v)\geq 0$, $\int_{\mathbb{R}^3}P(f,f)(v)\log f(v,t)dv\leq 0$, and $\frac{dH}{dt}\leq 0$. The H-Theorem is then verified. □

II.2 Collisional Invariants: Mass, Momentum, and Kinetic Energy

A collisional invariant is a property of collisional plasma that is preserved through Coulomb interactions between plasma particles.

If $\frac{d}{dt}(\int_{\mathbb{R}^3} \psi(v) f(v,t) dv) = 0$, then ψ is a collisional invariant. Since, as given by the FPL equation, $\frac{\partial f}{\partial t} = P(f,f)(v)$, collisional invariants can alternatively be defined in terms of the integral I introduced in the previous subsection.

Definition. Collisional invariants are denoted as ψ, where:

$$I = \int_{\mathbb{R}^3} P(f,f)(v)\psi(v) dv = 0 \tag{2.15}$$

for all f.

In this section, we use the above definition to show that a family of collisional invariants, denoted as ψ, can be expressed as follows:

Theorem. Functions of the form

$$\psi(v) = A + B \cdot v + C|v|^2, \tag{2.16}$$

are collisional invariants, where $A, C \in \mathbb{R}$, and $B \in \mathbb{R}^3$.

Proof. We seek to show that

$$\int_{\mathbb{R}^3} P(f,f)(v)[A + B \cdot v + C|v|^2] dv = 0. \tag{2.17}$$

To do so, consider the integral as a sum $IntA + IntB + IntC$, where:

$$IntA = \int_{\mathbb{R}^3} P(f,f)(v) A dv$$

$$IntB = \int_{\mathbb{R}^3} P(f,f)(v)(B \cdot v) dv$$

$$IntC = \int_{\mathbb{R}^3} P(f,f)(v) C|v|^2 dv.$$

We will show that each term of the sum integrates to 0.

$IntA$**:** For the first term, integrate $A \int_{\mathbb{R}^3} P(f,f)(v) dv = 0$. We have already, in the proof of the

H-Theorem, shown that $\int_{\mathbb{R}^3} P(f,f)(v)dv = 0$, and therefore this term in the integral is 0.

IntB: Let $B = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, b_1, b_2, b_3 \in \mathbb{R}$. Then $IntB = \int_{\mathbb{R}^3} P(f,f)(v)(b_1v_1 + b_2v_2 + b_3v_3)dv$. For the integral $b_i \int_{\mathbb{R}^3} v_iP(f,f)(v)dv, i = 1,2,3$, let $\psi(v) = v_i$ in (2.10). Then $\nabla_v\psi(v) - \nabla_{\bar{v}}\psi(\bar{v}) = \frac{\partial}{\partial v_i}(v_i) - \frac{\partial}{\partial \bar{v}_i}(\bar{v}_i) = 0$, and $b_i \int_{\mathbb{R}^3} v_iP(f,f)(v)dv = 0$. Therefore, $IntB = 0$.

IntC: We integrate $\int_{\mathbb{R}^3} C|v|^2P(f,f)(v)dv = C\int_{\mathbb{R}^3} P(f,f)(v)(v_1^2+v_2^2+v_3^2)dv$. Let $\psi(v) = v_1^2+v_2^2+v_3^2$ in (2.10).Then $\nabla_v\psi(v) - \nabla_{\bar{v}}\psi(\bar{v}) = 2(v-\bar{v})$. From expression (3.3),(3.4) we see that the first component of $2(v-\bar{v})^T\phi(v-\bar{v}) =$

$$2(v_1-\bar{v_1})\frac{[(v_2-\bar{v_2})^2+(v_3-\bar{v_3})^2]}{|v-\bar{v}|^3}+2(v_2-\bar{v_2})[\frac{-(v_1-\bar{v_1})(v_2-\bar{v_2})}{|v-\bar{v}|^3}]+2(v_3-\bar{v_3})[\frac{-(v_1-\bar{v_1})(v_3-\bar{v_3})}{|v-\bar{v}|^3}] = 0,$$

and similarly for the second and third components.
Therefore, $IntC = 0$, and $\psi(v)$ as given in (2.16) constitutes a collisional invariant. □

By specifying values for the constants in (2.16), we can deduce three collisional invariants of collisional plasma:

- Let $A = 1$, and $B, C = 0$. Then $\psi(v) = 1$, and (2.17) gives conservation of mass $\int f(v)dv = K$.
- Let $A, C = 0$, and $B = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$. Then $\psi(v) = v_i, i = 1,2,3$, and and (2.17) gives conservation of momentum $\int v_if(v)dv = K_i$.
- Let $C = 1$, and $A, B = 0$. Then $\psi(v) = |v|^2$, and (2.17) gives conservation of kinetic energy $\int |v|^2f(v)dv = K$.

III Expressing the Landau Equation as a Parabolic-Type PDE

We are guided in this section by the exposition in [6].

In this section, we will reformulate the Landau equation in the form of a parabolic-type partial differential equation. This will enable us to apply finite difference methods to the FPL equation, numerically verifying the H-Theorem and conservation laws.

Let $P(f,f)(v) = term1 - term2$, where,

$$term1 = \nabla_v \cdot \int_{\mathbb{R}^3} \phi(v-\bar{v})[\nabla_v f(v,t)f(\bar{v},t)]d\bar{v} \tag{3.1}$$

$$term2 = \nabla_v \cdot \int_{\mathbb{R}^3} \phi(v-\bar{v})[\nabla_{\bar{v}} f(\bar{v},t)f(v,t)]d\bar{v} \tag{3.2}$$

Let the matrix $\phi(v-\bar{v})$ be denoted as $[a_{ij}], i,j=1,2,3$, where

$$a_{ii} = \frac{(v_k-\bar{v}_k)^2+(v_l-\bar{v}_l)^2}{|v-\bar{v}|^3}, k,l=1,2,3; k,l \neq i \tag{3.3}$$

$$a_{ij} = -\frac{(v_i-\bar{v}_i)(v_j-\bar{v}_j)}{|v-\bar{v}|^3}, i \neq j \tag{3.4}$$

Also,

$$A_{ij}(v,t) = \int_{\mathbb{R}^3} a_{ij}(v,\bar{v})f(\bar{v},t)d\bar{v} \tag{3.5}$$

To begin the reformulation of $term1$, let $\phi(v-\bar{v})\nabla_v f(v,t) = \sum_{j=1}^3 a_{1j}f_{v_j}\mathbf{i}+\sum_{j=1}^3 a_{2j}f_{v_j}\mathbf{j}+\sum_{j=1}^3 a_{3j}f_{v_j}\mathbf{k}$. Then,

$$term1 = \int_{\mathbb{R}^3} \nabla_v \cdot \phi(v-\bar{v})\nabla_v f(v,t)f(\bar{v},t)d\bar{v} \tag{3.6}$$

Applying the product rule to each term of the vector above, results in

$$term1 = \int_{\mathbb{R}^3}[\sum_{i=1}^3\sum_{j=1}^3 a_{ij}\frac{\partial^2 f}{\partial v_i \partial v_j}+\sum_{l=1}^3\sum_{k=1}^3\frac{\partial a_{kl}}{\partial v_k}\frac{\partial f}{\partial v_l}]f(\bar{v},t)d\bar{v}. \tag{3.7}$$

Conforming to the notation previously introduced,

$$term1 = \sum_{i,j} A_{ij}(v,t)\frac{\partial^2 f}{\partial v_i \partial v_j}+\sum_{l=1}^3(\int_{\mathbb{R}^3}\sum_{k=1}^3\frac{\partial a_{kl}}{\partial v_k}f(\bar{v},t)d\bar{v})\frac{\partial f}{\partial v_l} \tag{3.8}$$

For $term2$, again multiply $\phi(v-\bar{v})$ by $\nabla_{\bar{v}} f(\bar{v},t)f(v,t)$, yielding

$$term2 = \int_{\mathbb{R}^3}\nabla_v \cdot [(\sum_{j=1}^3 a_{1j}f_{\bar{v}_j}\mathbf{i}+\sum_{j=1}^3 a_{2j}f_{\bar{v}_j}\mathbf{j}+\sum_{j=1}^3 a_{3j}f_{\bar{v}_j}\mathbf{k})f(v,t)]d\bar{v} \tag{3.9}$$

After taking the divergence, again applying the product rule to each term,

$$term2 = \int_{\mathbb{R}^3}[\sum_{i=1}^3\frac{\partial f}{\partial v_i}(\sum_{j=1}^3 a_{ij}\frac{\partial f}{\partial \bar{v}_j})+f\sum_{l=1}^3\sum_{k=1}^3\frac{\partial a_{kl}}{\partial v_k}\frac{\partial f}{\partial \bar{v}_l}]d\bar{v} \tag{3.10}$$

Consider the first term in the integrand of (3.10). Integrating by parts and letting $a_{ij}f(\bar{v},t) \to 0$ as $t \to \pm\infty$, the term can be rewritten:

$$-\sum_{k=1}^3\frac{\partial f}{\partial v_k}\int_{\mathbb{R}^3}(\sum_{l=1}^3\frac{\partial a_{kl}}{\partial \bar{v}_l}f(\bar{v},t))d\bar{v}$$

Then letting $-\frac{\partial a_{kl}}{\partial \bar{v}_l} = \frac{\partial a_{kl}}{\partial v_l}$, and taking advantage of the symmetry of a_{kl}, this term can be expressed:

$$\sum_{l=1}^3\int_{\mathbb{R}^3}(\sum_{k=1}^3\frac{\partial a_{kl}}{\partial v_k})f(\bar{v},t)\frac{\partial f}{\partial v_l}d\bar{v} \tag{3.11}$$

For the second term in the integrand of (3.10), let $\frac{\partial a_{ij}}{\partial v_i} = -\frac{\partial a_{ij}}{\partial \bar{v}_i}$. Integrating by parts transfers the derivatives on a_{ij} to the partial derivatives of f, and the term can then be written:

$$\sum_{i,j} \int_{\mathbb{R}^3} [a_{ij}(v,\bar{v}) \frac{\partial^2 f}{\partial \bar{v}_i \partial \bar{v}_j} f(v)] d\bar{v} \tag{3.12}$$

Therefore,

$$term2 = (\int_{\mathbb{R}^3} (\sum_{i,j} a_{ij}(v,\bar{v}) \frac{\partial^2 f}{\partial \bar{v}_i \partial \bar{v}_j}) d\bar{v}) f(v,t) + \sum_{l=1}^{3} (\int_{\mathbb{R}^3} (\sum_{k=1}^{3} \frac{\partial a_{kl}}{\partial v_k}) f(\bar{v},t) d\bar{v}) \frac{\partial f}{\partial v_l} \tag{3.13}$$

At this point, we note that $term1$ and $term2$ share the term $\sum_{l=1}^{3} (\int_{\mathbb{R}^3} \sum_{k=1}^{3} \frac{\partial a_{kl}}{\partial v_k} f(\bar{v},t) d\bar{v}) \frac{\partial f}{\partial v_l}$. If we subtract $term1$ and $term2$, the Landau equation can then be written as the difference of two terms:

$$P(f,f)(v) = terma - termb \tag{3.14}$$

$$terma = \sum_{i,j} A_{ij}(v,t) \frac{\partial^2 f}{\partial v_i \partial v_j}$$

$$termb = (\int_{\mathbb{R}^3} (\sum_{i,j} a_{ij}(v,\bar{v}) \frac{\partial^2 f}{\partial \bar{v}_i \bar{v}_j}) d\bar{v}) f(v,t)$$

We can further condense the form of $P(f,f)(v)$ by expressing $terma$ and $termb$ in terms of the Rosenbluth potentials $G(v,t)$ and $H(v,t)$, given in [4], which will be defined as necessary throughout the remainder of the argument.
First, let

$$G(v,t) = \int_{\mathbb{R}^3} |v - \bar{v}| f(\bar{v},t) d\bar{v} \tag{3.15}$$

Noting that $|v - \bar{v}| = \sqrt{(v_1 - \bar{v_1})^2 + (v_2 - \bar{v_2})^2 + (v_3 - \bar{v_3})^2}$, it can be verified that

$$A_{ij} = \frac{\partial^2 G(v,t)}{\partial v_i \partial v_j}$$

For $termb$, we integrate by parts, transferring derivatives from $f(\bar{v},t)$ to a_{ij}, so that

$$termb = -\int_{\mathbb{R}^3} [\sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial a_{ij}}{\partial \bar{v}_j} \frac{\partial f}{\partial \bar{v}_i}] f(v,t) d\bar{v}$$

Letting $-\frac{\partial a_{ij}}{\partial \bar{v}_j} = \frac{\partial a_{ij}}{\partial v_j}$ and factoring out $\frac{\partial f}{\partial \bar{v}_i}$, then,

$$termb = \int_{\mathbb{R}^3} \sum_{i=1}^{3} \frac{\partial f}{\partial \bar{v}_i} [\sum_{j=1}^{3} \frac{\partial a_{ij}}{\partial v_j}] f(v,t) d\bar{v}$$

Now, note that $\sum_{j=1}^{3} \frac{\partial a_{ij}}{\partial v_j} = -\frac{2(v_i - \bar{v}_i)}{|v - \bar{v}|^3}$. To show this, let $i = 1$ and suppress $\bar{v}$. Then,

$$\frac{\partial}{\partial v_1}(\frac{v_2^2+v_3^2}{|v|^3})+\frac{\partial}{\partial v_2}(\frac{-v_1v_2}{|v|^3})+\frac{\partial}{\partial v_3}(\frac{-v_1v_3}{|v|^3})$$

$$=\frac{-3v_1(v_2^2+v_3^2)}{|v|^5}-\frac{v_1^3-2v_1v_2+v_3^2v_1}{|v|^5}-\frac{v_1^2+v_2^2v_1-2v_1v_3^2}{|v|^5}$$

$$=\frac{-2v_1v_2^2-2v_1v_3^2-2v_1^3}{|v|^5}=\frac{-2v_1|v|^2}{|v|^5}=\frac{-2v_1}{|v|^3},$$

and similarly for $i=2,3$. Then,

$$termb=-2f(v,t)\int_{\mathbb{R}^3}\frac{\partial f}{\partial \bar{v_1}}\frac{(v_1-\bar{v_1})}{|v-\bar{v}|^3}+\frac{\partial f}{\partial \bar{v_2}}\frac{(v_2-\bar{v_2})}{|v-\bar{v}|^3}+\frac{\partial f}{\partial \bar{v_3}}\frac{(v_3-\bar{v_3})}{|v-\bar{v}|^3}d\bar{v} \tag{3.16}$$

At this point, we seek to express $termb$ in terms of a Laplacian operator. Integration by parts gives,

$$termb=2f(v,t)\int_{\mathbb{R}^3}f(\bar{v},t)[\frac{\partial}{\partial \bar{v_1}}\frac{v_1-\bar{v_1}}{|v-\bar{v}|^3}+\frac{\partial}{\partial \bar{v_2}}\frac{v_2-\bar{v_2}}{|v-\bar{v}|^3}+\frac{\partial}{\partial \bar{v_3}}\frac{v_3-\bar{v_3}}{|v-\bar{v}|^3}]d\bar{v}$$

$$=2f(v,t)\int_{\mathbb{R}^3}f(\bar{v},t)[-\frac{\partial}{\partial v_1}\frac{v_1-\bar{v_1}}{|v-\bar{v}|^3}-\frac{\partial}{\partial v_2}\frac{v_2-\bar{v_2}}{|v-\bar{v}|^3}-\frac{\partial}{\partial v_3}\frac{v_3-\bar{v_3}}{|v-\bar{v}|^3}]d\bar{v}$$

$$=2f(v,t)\int_{\mathbb{R}^3}f(v,t)(\nabla_v\cdot\nabla_v\frac{1}{|v-\bar{v}|}d\bar{v}=2f(v)\nabla_v^2\int_{\mathbb{R}^3}\frac{f(\bar{v},t)}{|v-\bar{v}|}d\bar{v}$$

We note that:

$$\nabla_v\frac{1}{|v-\bar{v}|}=-[\frac{v_1-\bar{v_1}}{|v-\bar{v}|^3}\mathbf{i}+\frac{v_2-\bar{v_2}}{|v-\bar{v}|^3}\mathbf{j}+\frac{v_3-\bar{v_3}}{|v-\bar{v}|^3}\mathbf{k}] \tag{3.17}$$

Let $H(v,t)=\int_{\mathbb{R}^3}\frac{f(\bar{v},t)}{|v-\bar{v}|}d\bar{v}$, another Rosenbluth potential. Then, $termb=2f(v,t)\nabla_v^2H(v,t)$.

Furthermore, $H(v,t)$ is a solution of the Poisson equation. It follows from [2, pp. 245-246] that,

$$\nabla^2H(v,t)=-4\pi f(v,t) \tag{3.18}$$

Therefore, $termb=B(v,t)f(v,t)$, where $B(v,t)=-8\pi f(v,t)$.

In summary, $P(f,f)(v)$ is reformulated as follows:

$$\frac{\partial f}{\partial t}=q[\sum_{i,j=1}^{3}A_{ij}(v,t)\frac{\partial^2 f}{\partial v_i\partial v_j}-B(v,t)f],f(v,0)=f_0(v) \tag{3.19}$$

where

$$A_{ij}(v,t)=\frac{\partial^2 G(v,t)}{\partial v_i\partial v_j},B(v,t)=-8\pi f(v,t) \tag{3.20}$$

IV Numerical Approximation of the Fokker-Planck-Landau Equation

In this section, we outline the procedure for the numerical approximation of the spatially homogeneous Fokker-Planck-Landau equation. Once the initial data are established and normalized, finite difference methods for parabolic-type partial differential equations can be applied.

Step 1. Partition of Phase Space

Consider v, a point in velocity space. Each component in v is bounded below by v_{min}, and above by v_{max}, such that $v_{max} = -v_{min} \geq 0$.

Select $n_v \in \mathbb{Z}$ such that $n_v > 0$ is the desired number of velocity points. We define $v_{\bar{i}} = v_{1,i}\mathbf{i} + v_{2,j}\mathbf{j} + v_{3,k}\mathbf{k}$ for $\bar{i} = (i, j, k)$. Let $i, j, k = 0, \ldots, n_v + 1$, then $v_{\bar{i}} = (v_{min} + i\Delta v)\mathbf{i} + (v_{min} + j\Delta v)\mathbf{j} + (v_{min} + k\Delta v)\mathbf{k}$, where $\Delta v = \dfrac{v_{max} - v_{min}}{n_v + 1}$. Further, $v_{l,0} = v_{min}$ and $v_{l,n_v+1} = v_{max}, l = 1, 2, 3$.

To complete the partition of phase space, we discretize time in addition to velocity. Let t be such that $0 \leq t \leq t_{max}$, and $n_t \in \mathbb{Z}$, $n_t > 0$ be the desired number of time points. Then each time point $t_k = k\Delta t$, where $k = 0, \ldots, n_t$, and $\Delta t = \dfrac{t_{max}}{n_t}$.

We have now established a velocity-time grid over which to numerically approximate the PDE.

Step 2. Initial Data and Normalization

We now establish the notation $f_{est}(i, j, k, t_n)$ as an approximation to the true value of $f(v_{1i}, v_{2j}, v_{3k}, t_n)$. Along the boundary of the velocity grid, we set $f_{est} = 0$; this holds if any of the indices i, j, k are either 0 or $n_v + 1$.

We assume $\int_{\mathbb{R}^3} f_0(v)dv = K$, for $f_0(v)$ given in (1.1).

Let $\bar{f}_0(i, j, k) = f_0(v_{1,i}, v_{2,j}, v_{3,k})$, and

$$\lambda = \frac{1}{K} \sum_{i,j,k=1}^{n_v} \bar{f}_0(i, j, k)(\Delta v)^3.$$

Then let $f_0(i, j, k) = \frac{1}{\lambda}\bar{f}_0(i, j, k)$, and $f_{est}(i, j, k, 0) = f_0(i, j, k)$.

In this case, $\displaystyle\sum_{i,j,k=1}^{n} f_0(i, j, k)(\Delta v)^3 = K$, and the discrete initial data is normalized so that the approximate integral equals K.

Step 3. Compute the Convolution Integral $G(v, t)$ and its Derivatives

The coefficient A_{ij} in (3.21) is given in terms of a second derivative of the Rosenbluth potential G. In order to numerically approximate this derivative, we must approximate G at each (i, j, k). Refer to the expression for $G(v, t)$ given in (3.15).We vary l, m, n from $1, \ldots, n_v$, and i, j, k from $0, \ldots, nv + 1$. G

is again computed using the trapezoid rule.

$$G(i,j,k) = \sum_{l,m,n=1}^{n_v} \sqrt{(v_{1i}-v_{1l})^2+(v_{2j}-v_{2m})^2+(v_{3k}-v_{3n})^2} f_{est}(l,m,n,t_n)(\Delta v)^3 \tag{4.1}$$

For any $i, j, k = 0$ or $n_v + 1$, $G(i, j, k)$ is given values on the boundary at v_{min} and v_{max}.
Once these values for G are computed, difference approximations are used to compute the coefficients A_{ij}. For example,

$$A_{11}(i,j,k,t_n) = \frac{(G(i+1,j,k) - 2G(i,j,k) + G(i-1,j,k))}{(\Delta v)^2}$$

$$\begin{aligned} A_{12}(i,j,k,t_n) &= ((\frac{G(i+1,j+1,k) - G(i+1,j-1,k)}{2\Delta v}) - (\frac{G(i-1,j+1,k) - G(i-1,j-1,k)}{2\Delta v}))/2\Delta v \\ &= \frac{(G(i+1,j+1,k) - G(i+1,j-1,k) - G(i-1,j+1,k) + G(i-1,j-1,k)}{4(\Delta v)^2} \end{aligned}$$

The other derivatives can be computed similarly.

Note also that, due to the symmetry of A_{ij}, the mixed partial derivatives of G need only be computed once.

The coefficient $B(i, j, k, t_n)$ is approximated as $-8\pi f_{est}(i, j, k, t_n)$.

Step 4. PDE-Solving Algorithm

Again, central difference approximations are used to enumerate estimates for the derivatives of f. For $i, j, k = 1, \ldots, nv$,

$$f_1(t_n) = \frac{f_{est}(i+1,j,k,t_n) - f_{est}(i-1,j,k,t_n)}{2\Delta v}$$

$$f_2(t_n) = \frac{f_{est}(i,j+1,k,t_n) - f_{est}(i,j-1,k,t_n)}{2\Delta v}$$

$$f_3(t_n) = \frac{f_{est}(i,j,k+1,t_n) - f_{est}(i,j,k-1,t_n)}{2\Delta v}$$

$$f_{11}(t_n) = \frac{f_{est}(i+1,j,k,t_n) - 2f_{est}(i,j,k,t_n) + f_{est}(i-1,j,k,t_n)}{(\Delta v)^2}$$

$$f_{22}(t_n) = \frac{f_{est}(i,j+1,k,t_n) - 2f_{est}(i,j,k,t_n) + f_{est}(i,j-1,k,t_n)}{(\Delta v)^2}$$

$$f_{33}(t_n) = \frac{f_{est}(i,j,k+1,t_n) - 2f_{est}(i,j,k,t_n) + f_{est}(i,j,k-1,t_n)}{(\Delta v)^2}$$

$$f_{12}(t_n) = (f_{est}(i+1,j+1,k,t_n) - f_{est}(i+1,j-1,k,t_n) - f_{est}(i-1,j+1,k,t_n) + f_{est}(i-1,j-1,k,t_n))/4(\Delta v)^2$$

$$f_{13}(t_n) = (f_{est}(i{+}1, j, k{+}1, t_n) - f_{est}(i{+}1, j, k{-}1, t_n) - f_{est}(i{-}1, j, k{+}1, t_n) + f_{est}(i{-}1, j, k{-}1, t_n))/4(\Delta v)^2$$

$$f_{23}(t_n) = (f_{est}(i, j{+}1, k{+}1, t_n) - f_{est}(i, j{+}1, k{-}1, t_n) - f_{est}(i, j{-}1, k{+}1, t_n) + f_{est}(i, j{-}1, k{-}1, t_n))/4(\Delta v)^2$$

By taking appropriate approximations and multiplying (3.21) through by Δt, we formulate the following semi-implicit difference equation. Let $A_{ij} = A(i, j, k, t_n)$, $B = B(i, j, k, t_n)$ and $f_{est}(i, j, k, 0) = f_0(i, j, k)$ for $i, j, k = 1, \ldots, n_v$ and for $n = 0, 1, 2, \ldots$

$$f_{est}(i, j, k, t_{n+1}) = f_{est}(i, j, k, t_n) + q\Delta t(A_{11} f_{11}(t_{n+1}) + A_{22} f_{22}(t_{n+1}) + A_{33} f_{33}(t_{n+1})) + q\Delta t(2A_{12} f_{12}(t_n)$$

$$+2A_{13} f_{13}(t_n) + 2A_{23} f_{23}) - q\Delta t B f_{est}(i, j, k, t_n) \tag{4.2}$$

By substituting the expressions for $f_{11}(t_{n+1}), f_{22}(t_{n+1})$, and $f_{33}(t_{n+1})$ into (4.2), and moving each term involving $2f_{est}(i, j, k, n+1)$ to the left, we can express (4.2) as follows.

Let $r = \dfrac{\Delta t}{(\Delta v)^2}$, and $d(i, j, k) = 1 + 2qr(A_{11} + A_{22} + A_{33})$.

$$f_{est}(i, j, k, t_{n+1}) = \frac{qr}{d(i, j, k)}[A_{11}(f_{est}(i+1, j, k, t_{n+1}) - f_{est}(i-1, j, k, t_{n+1})) + A_{22}(f_{est}(i, j+1, k, t_{n+1})$$

$$-f_{est}(i, j-1, k, t_{n+1})) + A_{33}(f_{est}(i, j, k+1, t_{n+1}) - f_{est}(i, j, k-1, t_{n+1})] + \frac{f_{est}(i, j, k, t_n)}{d(i, j, k)}$$

$$+\frac{2q\Delta t(A_{12} f_{12}(t_n) + A_{13} f_{13}(t_n) + A_{23} f_{23}(t_n))}{d(i, j, k)} - \frac{q\Delta t B f_{est}(i, j, k, t_n)}{d(i, j, k)} \tag{4.3}$$

Equation (4.3) is solved iteratively by the Gauss-Seidel method.

Summary:

1. Partition velocity and time space such that $v_{\bar{i}} = (v_{min}, v_{min}, v_{min}) + (i\Delta v + j\Delta v + k\Delta v), i, j, k = 1, \ldots, n_v + 1$, and $t_n = n\Delta t, n = 1, \ldots, n_t$.
2. Approximate the initial data and introduce a normalization parameter λ, so that the approximate integral of f matches its true value K.
3. Compute the Rosenbluth potential $G(i, j, k, t_n)$, and use it to compute the derivatives A_{ij}, which represent coefficients in the PDE, accordingly. Also compute the coefficient B.
4. Use the Gauss-Seidel method to iteratively solve the PDE, as given in (4.3).

The computer program to numerically solve the FPL equation is currently in progress.

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