Final Year Dissertation

Universitat Pompeu Fabra

From Volatility to Value at Risk: A Comparative Analysis Using Jump and Rough Volatility Models

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Dedication

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I would like to dedicate this work to...

Acknowledgement

(Optional, if used placed on a right page next to an empty left page)

I would like to express my sincere gratitude to:

- My supervisor
- My co-supervisor
- My family

Abstract

The abstract should have at least 200 but not more than 600 words. Placed on a right page next to a blank left page. A list of keywords (approximately 3 to 5) should be just below the abstract, preceded by the word "Keywords". Keywords should be separated by ";".

Keywords: Imaging techniques; Cloud computing; Alzheimer

Chapter 1

Introduction

Volatility is a cornerstone of financial modeling and risk management, capturing the uncertainty inherent in asset prices. Its accurate estimation is crucial for derivative pricing, portfolio optimization, and regulatory compliance. However, conventional models—such as those assuming constant volatility or relying solely on historical data—often fail to capture key empirical features observed in financial markets, including sudden price jumps and persistent volatility clustering. The 2008 financial crisis exposed the fragility of standard risk models and reinforced the need for more realistic frameworks that can better anticipate and quantify extreme events.

In this context, Value at Risk (VaR) remains one of the most widely adopted risk metrics, yet its accuracy heavily depends on the quality of the underlying volatility estimates. Newer models, such as jump diffusion processes and rough volatility models, have emerged as promising alternatives, offering a more faithful representation of market dynamics. These approaches aim to address the limitations of classical models by incorporating discontinuities or long-range dependence, respectively.

This thesis investigates how different volatility modeling approaches affect the computation of VaR. Specifically, it compares three methods: Historical/Realized Volatility (as a benchmark), Jump Diffusion Models, and Rough Volatility Models. The goal is to assess their relative effectiveness in estimating volatility and, consequently, in producing reliable VaR figures. The study aims to identify the trade-offs involved in each method, balancing theoretical rigor with empirical performance and practical feasibility.

The analysis proceeds in two stages. First, the theoretical foundations of each model are examined, focusing on their assumptions, mathematical structure, and relevance in financial literature. Second, the models are implemented in Python using real market data. Each estimated volatility series is input into the Black-Scholes framework to simulate price paths, from which VaR is calculated. The models are evaluated based on their predictive consistency, sensitivity to market shocks, and robustness under stress scenarios.

Chapter 2 presents the theoretical framework, including an overview of VaR, the Black-Scholes model, and the selected volatility models. Chapter 3 details the methodological workflow, from volatility estimation to VaR computation. Chapter 4 contains the empirical analysis and results. Chapter 5 discusses key insights and limitations, and Chapter 6 concludes with a summary and directions for future research. Technical appendices provide code, derivations, and supplementary material.

Chapter 2

Theoretical Framework

2.1 Risk measurement and the concept of VaR

One of the central challenges in financial risk management is quantifying the potential losses a portfolio may incur over a given period of time. This is particularly relevant for financial institutions, asset managers, and regulators, who require reliable tools to assess exposure under normal and stressed market conditions. Market risk, in particular, refers to the risk of losses arising from adverse movements in market prices—such as interest rates, stock prices, exchange rates, or commodity prices. To address this, various risk measures have been developed over time. Among these, Value at Risk (VaR) has emerged as one of the most widely used tools, due to its intuitive appeal and regulatory endorsement (e.g., Basel II and III frameworks) (Jorion, 2007). VaR summarizes the potential loss in value of a financial position or portfolio over a specific time horizon for a given confidence level.

Formally, if ΔV denotes the change in portfolio value over the horizon under analysis, the Value at Risk at confidence level $\alpha \in (0, 1)$ is defined as the smallest value x such that the probability of observing a loss greater than x does not exceed $1 - \alpha$:

$$P(\Delta V < VaR_\alpha) = \alpha$$

In other words, the VaR at level α represents the α-quantile of the loss distribution. For example, a one-day 99% VaR of €1 million indicates that there is only a 1% probability that losses will exceed €1 million in a single day.

Several methodologies exist for estimating VaR, each characterized by different assumptions, advantages, and limitations:

- The **parametric (variance-covariance) method** assumes that asset returns are normally distributed and that portfolio returns can be approximated as a linear combination of the individual components. Under these conditions, VaR can be computed analytically as:

$$VaR_\alpha = \mu + z_\alpha \sigma$$

  where μ and σ are the mean and standard deviation of returns, respectively, and z_α denotes the critical value of the standard normal distribution corresponding to the selected confidence level (Jorion, 2007).
- The **historical simulation method** constructs the empirical distribution of returns based on past data, and computes VaR directly from observed quantiles. This method avoids distributional assumptions but presumes that historical patterns are informative of future risk.
- The **Monte Carlo simulation method** involves generating a large number of future price paths based on stochastic models, thereby producing an empirical distribution of potential portfolio value changes. VaR is then computed as the empirical quantile of simulated losses. This method offers great flexibility, particularly for portfolios with path-dependent or non-linear instruments, albeit at a higher computational cost.

In the present study, a simulation-based VaR estimation framework is adopted. Specifically, Monte Carlo simulations will be conducted under the Black-Scholes model, which remains a standard benchmark in quantitative finance. The key source of variation between the different approaches studied lies in the modeling of the volatility parameter σ, which plays a critical role in shaping the distribution of returns.

The evolution of asset prices will follow the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t$$

where μ denotes the drift term, W_t is a standard Brownian motion, and σ_t represents the instantaneous volatility. While the Black-Scholes framework is retained for consistency, the estimation of σ_t will be carried out using three different methodologies: (i) assuming constant volatility, (ii) introducing jump components, and (iii) applying a rough volatility specification. For each volatility model, a distribution of simulated changes in the portfolio value ΔV will be obtained, and the corresponding VaR will be calculated as:

$$P(\Delta V < VaR_\alpha) = \alpha$$

This procedure allows for a consistent comparison between volatility models, while ensuring that the pricing dynamics remain grounded in a widely accepted theoretical framework.

Despite its extensive use, VaR has been subject to significant criticism in both academic and regulatory contexts. One of its main limitations is that it provides no information regarding the magnitude of losses beyond the VaR threshold, making it insensitive to tail risk (Jorion, 2007). Furthermore, under certain definitions and conditions, VaR does not satisfy the subadditivity property, which is required for a risk measure to be considered coherent according to the axiomatic framework developed by Artzner et al. (1999). Subadditivity ensures that diversification does not increase measured risk—a property that is both theoretically desirable and intuitively compelling.

In light of these shortcomings, alternative risk measures such as Expected Shortfall (ES) have been proposed. Expected Shortfall, also known as Conditional VaR, measures the expected value of losses conditional on exceeding the VaR level. It captures the severity of tail losses and satisfies the four coherence properties: monotonicity, translation invariance, positive homogeneity, and subadditivity (Artzner et al., 1999). ES has gained increasing attention in the context of financial regulation and is now embedded in the Basel III Fundamental Review of the Trading Book.

Nevertheless, due to its simplicity and ease of interpretation, VaR remains a central tool in the practice of risk management. In this thesis, it serves as the principal benchmark for comparing the impact of different volatility estimation approaches. The primary research objective is to assess to what extent the chosen model for volatility—despite being embedded in the same pricing structure—affects the estimation of portfolio risk through the Value at Risk metric.

[1] [2]

2.2 The Black-Scholes Model as Pricing Framework

The Black-Scholes model (Black & Scholes, 1973) represents one of the most influential developments in modern financial theory. Its introduction provided a closed-form solution for pricing European-style options and laid the foundations for the field of quantitative finance. Despite its simplicity, the model remains a cornerstone in both theoretical research and practical applications, including risk management, hedging strategies, and derivative pricing.

At its core, the Black-Scholes framework assumes that the dynamics of the underlying asset price S_t follow a geometric Brownian motion (GBM), which evolves according to the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where:

- $\mu \in \mathbb{R}$ is the constant drift rate (expected return)
- $\sigma > 0$ is the constant volatility of returns
- W_t is a standard Brownian motion

This formulation implies that the logarithmic returns of the asset are normally distributed and that price paths are continuous and nowhere differentiable. The key assumption of constant volatility is particularly relevant for the subsequent valuation of derivatives.

By applying Itô's lemma and the no-arbitrage principle, assuming a completely and frictionless market (zero transaction costs), Black and Scholes derived the option pricing formula for a European call option:

$$C(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

with,

$$d_1 = \frac{ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

where:

- S_t is the current price of the underlying asset,
- K is the strike price,
- T is the time to maturity,
- r is the risk-free interest rate,
- σ is the volatility of the underlying asset,
- $N(\cdot)$ denotes the cumulative distribution function (CDF) of the standard normal distribution.

The model is derived under a set of idealized assumptions: constant interest rates, continuous trading, no transaction costs, no dividends, and a frictionless market where short selling is permitted and borrowing/lending occurs at the risk-free rate. While these assumptions may not hold in real-world markets, the framework provides a tractable and analytically elegant starting point for more complex models.

In the context of this thesis, the Black-Scholes model will serve as the structural foundation for simulating asset price paths across all scenarios. The key source of model differentiation lies in the treatment of volatility σ. While the original model assumes constant volatility, empirical studies have consistently shown that market volatility is dynamic and exhibits stylized features such as clustering, mean reversion, and long memory (Cont, 2001). These phenomena motivate the exploration of alternative volatility specifications, which will be discussed in subsequent sections.

By maintaining the Black-Scholes framework for price dynamics and modifying only the volatility component, this study ensures a controlled environment for analyzing the effects of volatility modeling on risk estimation. This strategy preserves the analytical tractability of the model while allowing for meaningful comparisons of risk measures such as Value at Risk.

[3] [4]

2.3 Volatility in the Black-Scholes context

Within the Black-Scholes framework, volatility plays a central role as a key input in option pricing. In its classical formulation, the model assumes that the volatility of the underlying asset, denoted by σ, is both constant and known. This assumption enables analytical tractability but has been consistently challenged by empirical evidence. Financial markets exhibit several features—commonly referred to as stylized facts—that contradict the notion of constant volatility. These include volatility clustering, leverage effects, fat tails, and long memory in return dynamics (Cont, 2001).

In practice, market participants do not observe volatility directly. Instead, they infer it from market prices of options by inverting the Black-Scholes formula. The value of σ that, when inserted into the pricing formula, returns the observed market price of an option is known as the implied volatility. This market-implied measure of expected future volatility has become a standard tool in derivative markets.

However, when implied volatilities are plotted as a function of strike prices or maturities, they tend to display systematic patterns. The most well-known of these is the volatility smile, observed when implied volatility is U-shaped across strikes, with higher values for deep in-the-money and out-of-the-money options than for at-the-money ones. In equity markets, implied volatilities often decrease with the strike price and exhibit a volatility skew (also called "smirk"), particularly following market crashes.

These patterns are inconsistent with the Black-Scholes model, which predicts flat implied volatility surfaces across strikes and maturities. This empirical failure is a strong indication that the assumption of constant volatility does not capture the full dynamics of financial markets.

Moreover, the behavior of implied volatility across maturities—the term structure of volatility—also reveals non-constant and often mean-reverting dynamics. Short-term implied volatilities can react sharply to news or macroeconomic events, while longer-term volatilities tend to move more gradually. These observations suggest the need for more sophisticated models that allow volatility to evolve over time in response to information shocks.

The inadequacy of the constant volatility assumption has prompted the development of alternative modeling approaches. GARCH-type models (Engle, 1982; Bollerslev, 1986) introduced conditional heteroskedasticity in discrete time, while stochastic volatility models (e.g., Hull & White, 1987; Heston, 1993) provided continuous-time formulations where volatility is itself a random process. More recently, models incorporating jumps and rough volatility have been proposed to better match the empirical properties of asset returns and implied volatility surfaces.

In this thesis, we retain the Black-Scholes pricing framework but relax the constant volatility assumption by employing alternative specifications for the volatility input σ. These include models where volatility incorporates jumps or follows a rough stochastic process. The objective is to assess how each of these alternatives affects the risk estimates obtained through Monte Carlo simulation of Value at Risk.

By doing so, we maintain the mathematical consistency and interpretability of the Black-Scholes setting while addressing one of its key limitations: the oversimplified treatment of volatility. Understanding and improving volatility modeling is thus a crucial step in obtaining more realistic and robust measures of financial risk.

[4] [5] [6] [7] [8]

2.4 Jump Diffusion Models (e.g., Merton, Kou)

While the Black-Scholes framework assumes continuous asset price paths driven by Brownian motion, real-world financial markets frequently exhibit sudden, large price movements that are inconsistent with this assumption. These discontinuities, or jumps, are often associated with macroeconomic news, earnings announcements, geopolitical events, or liquidity shocks. Such events produce extreme returns and generate fat tails in the return distribution—phenomena that the classical lognormal model fails to capture adequately.

In response to these empirical shortcomings, jump diffusion models have been proposed as natural extensions of the Black-Scholes framework. These models incorporate discontinuous movements by combining a continuous diffusion component with a jump component governed by a Poisson process. The most influential formulation is that of Merton (1976), which modifies the stochastic differential equation for the asset price as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_{t^-}(Y_t - 1)dN_t$$

where:

- W_t is a standard Brownian motion,
- N_t is a Poisson process with intensity λ, representing the number of jumps up to time t,
- Y_t is a random variable describing the jump size, typically assumed lognormally distributed, i.e. $log(Y_t) \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$
- S_{t^-} denotes the asset price just before the jump.

The diffusion component captures the continuous price fluctuations, while the jump component accounts for large, abrupt changes. The presence of jumps introduces skewness and excess kurtosis into the return distribution, allowing the model to better reflect the empirical characteristics of financial data.

Although Merton's model significantly improves the fit of theoretical models to observed market prices, its assumption of symmetric and normally distributed jumps limits its flexibility in capturing asymmetric jump behavior, often observed in equity markets.

To address this limitation, Kou (2002) proposed a model in which the logarithmic jump sizes follow a double-exponential distribution. This allows for different rates of increase and decrease in asset prices, improving the model's ability to capture skewed return distributions:

$$f_J(x) = p\eta_1 e^{-\eta_1 x}\mathbf{1}_{x\geq 0} + (1-p)\eta_2 e^{\eta_2 x}\mathbf{1}_{x<0}$$

where:

- $p \in (0,1)$ is the probability of an upward jump,
- $\eta_1, \eta_2 > 0$ are the rate parameters controlling the size of upward and downward jumps.

Kou's formulation preserves analytical tractability and leads to closed-form solutions for European option pricing, making it highly attractive for both theoretical and practical purposes.

The introduction of jumps into asset dynamics has profound implications for option pricing and risk management. In particular, jump diffusion models better explain steep implied volatility skews and smiles, especially for short-dated and out-of-the-money options, which are more sensitive to tail risk. Moreover, in the context of Value at Risk, the inclusion of jumps alters the distribution of returns, increasing the probability of extreme losses and leading to higher, and arguably more realistic, VaR estimates.

In this thesis, we incorporate jump diffusion models into the Monte Carlo simulation framework by modifying the volatility input used in the Black-Scholes equation. While the structure of the model remains formally unchanged, the simulated price paths will now include discontinuities governed by Poisson jumps. This approach allows for a direct comparison between constant-volatility and jump-based volatility scenarios, shedding light on the effect of discontinuities in the context of risk estimation.

Overall, jump diffusion models represent a meaningful step toward reconciling theoretical asset pricing models with observed market behavior, particularly in turbulent environments. Their integration into simulation-based risk measurement frameworks enables a more nuanced and accurate assessment of tail risk, which is of critical importance in modern financial risk management.

[9] [10] [11]

2.5 Fractional/Rough Volatility Models (e.g., rough Bergomi)

In recent years, the traditional modeling of volatility as either constant or driven by smooth stochastic processes has been increasingly challenged by empirical findings from high-frequency financial data. These studies reveal that volatility exhibits a highly irregular, non-Markovian, and persistent structure over time. In particular, the observed sample paths of volatility display properties that are inconsistent with classical diffusion models, suggesting a level of roughness greater than that of Brownian motion.

This empirical insight gave rise to a new class of models known as rough volatility models, which assume that volatility evolves according to a fractional Brownian motion (fBM) with Hurst parameter $H < 0.5$. The foundational empirical study by Bennedsen, Lunde, and Pakkanen (2017) showed that the Hurst exponent of log-volatility is typically around 0.1–0.2 across a variety of financial assets, indicating anti-persistent and extremely rough behavior.

The general form of a rough volatility process models the logarithm of volatility as a convolution of a Brownian motion with a fractional kernel:

$$log(\sigma_t) = \int_0^t K(t-s)dW_s, \quad \text{where} \quad K(t) = t^{H-\frac{1}{2}}, \quad H < 0.5$$

This specification implies that the volatility process has short memory, is non-semimartingale, and exhibits pathwise roughness consistent with high-frequency data. Among the different formulations, one of the most tractable and widely used models is the rough Bergomi model, introduced by Bayer, Friz, and Gatheral (2016). It extends the Bergomi forward variance model by incorporating a fractional kernel:

$$S_t = S_0 \ exp\left(\int_0^t \sqrt{V_s}dW_s - \frac{1}{2}\int_0^t V_s ds\right)$$

$$V_t = \xi_0(t) \exp\left(\eta \int_0^t (t-s)^{H-\frac{1}{2}} \, dZ_s - \frac{1}{2}\eta^2 t^{2H}\right)$$

where:

- $\xi_0(t)$ is the initial forward variance curve,
- η is the volatility of volatility parameter,
- Z_s is a Brownian motion correlated with the asset price Brownian motion W_s.

The rough Bergomi model accurately reproduces key features of observed implied volatility surfaces, such as steep short-term skew, smile effects, and term structure dynamics. In contrast to traditional models, which require jumps or local volatility to capture such patterns, rough models achieve this through their inherent memory and irregularity.

In the context of risk management, rough volatility has meaningful implications for the estimation of Value at Risk. Rough processes induce heavier tails and greater variability in short-term return distributions, leading to higher VaR estimates for a given confidence level when compared to constant or smooth stochastic volatility models. Moreover, due to their consistency with observed volatility dynamics, rough models are particularly well-suited for stress testing and for modeling the behavior of markets under sudden shocks.

For this thesis, the rough Bergomi model is implemented within the Monte Carlo simulation framework as a volatility-generating process. Asset prices are simulated under the Black-Scholes formula, with volatility paths sampled from the rough volatility model described above. This methodology allows us to isolate the impact of rough volatility on the distribution of portfolio returns, and to compare the resulting Value at Risk estimates with those obtained under constant and jump-based volatility models.

The incorporation of fractional and rough models represents a frontier in financial modeling. While computationally more demanding, they offer significant advantages in accuracy and realism, both in pricing and in risk estimation. Their growing popularity in the academic literature and among practitioners reflects a paradigm shift in how volatility is understood and modeled in modern finance.

[12] [13] [14]

2.6 From volatility estimation to VaR computation

The preceding sections have presented a range of volatility modeling approaches, each motivated by empirical observations and theoretical limitations of the classical constant-volatility assumption. From the original Black-Scholes model to jump diffusion and rough volatility frameworks, each specification introduces a different structure for the behavior of σ_t, the instantaneous volatility of asset returns. These differences, while often analyzed in the context of option pricing, also have direct implications for the estimation of Value at Risk (VaR).

As established in Section 2.1, VaR represents the quantile of the distribution of portfolio losses over a given time horizon. In a simulation-based framework, such as the one adopted in this thesis, the quality and realism of the VaR estimate depend crucially on the accuracy of the return distribution generated through Monte Carlo simulations. Since volatility governs the dispersion and shape of this distribution, the modeling of σ_t becomes a key determinant of the risk measure.

Under a constant volatility assumption, such as in the classical Black-Scholes framework, simulated price paths exhibit lognormal behavior with a fixed standard deviation. This leads to relatively thin-tailed distributions, which may underestimate the probability and magnitude of extreme losses, especially under turbulent market conditions (Jorion, 2007).

In contrast, models incorporating jumps, such as the Merton or Kou models, introduce discontinuities in the price process. The resulting distributions of returns display skewness and excess kurtosis, thereby increasing the likelihood of extreme outcomes. When used in Monte Carlo simulations, these models tend to produce fatter tails in the empirical distribution of losses, yielding higher VaR estimates that more accurately reflect tail risk (Eraker et al., 2003).

Rough volatility models, such as the rough Bergomi framework, take a different approach. Rather than introducing discontinuities, they embed pathwise irregularity and memory into the volatility process itself. These models generate volatility trajectories that are highly erratic and short-memory, especially at short horizons, leading to return distributions that are non-Gaussian and often heavy-tailed. The roughness of volatility increases short-term uncertainty, which is particularly relevant when calculating VaR over daily or intraday intervals (Gatheral et al., 2018).

The following table summarizes the expected qualitative effects of each volatility model on the shape of the simulated return distribution and its implications for VaR:

In this thesis, all simulations are conducted under the Black-Scholes asset pricing framework, with variations only in the volatility process. This setup isolates the impact of volatility estimation on the final risk measure. For each model, we generate a large number of asset price paths, compute the corresponding changes in portfolio value ΔV, and estimate the Value at Risk at a given confidence level α by identifying the quantile satisfying:

$$P(\Delta V < VaR_\alpha) = \alpha$$

Volatility Model	Return Distribution Shape	Tail Risk Representation	Expected VaR Impact
Constant (Black-Scholes)	Lognormal (thin-tailed)	Underestimated	Lower
Jump Diffusion	Skewed, leptokurtic	Better captured	Higher
Rough Volatility	Heavy-tailed, short-memory	Very well captured	Higher

Table 1: Comparison of volatility models and their implications for tail risk and Value at Risk (VaR).

This approach enables a controlled and systematic comparison across modeling techniques, providing insights into how the sophistication of volatility estimation affects practical risk assessments. Furthermore, the analysis highlights the importance of using appropriate volatility models in regulatory and risk management contexts, where underestimation of extreme losses can lead to insufficient capital reserves and poor decision-making.

In summary, while VaR is a univariate risk metric, its accuracy and relevance depend critically on the multivariate structure of the underlying model—particularly on the dynamics of volatility. Through this study, we aim to evaluate whether advanced volatility models, such as jump diffusions and rough processes, yield meaningfully improved VaR estimates over the traditional constant-volatility approach, thereby offering a more reliable foundation for risk-sensitive financial applications.

Chapter 3

Methodology

This section outlines the full methodological framework used to estimate Value at Risk (VaR) based on different volatility models. The process follows a structured pipeline: first, volatility is estimated using three different approaches; second, this volatility is used to price one-day-ahead asset values via a modified Black-Scholes model; finally, VaR is computed from the resulting distribution of simulated returns.

3.1 General workflow: from volatility to VaR

The goal of this chapter is to assess how different volatility modeling assumptions impact the resulting Value at Risk (VaR) estimates. The workflow comprises three main stages:

- **Volatility Estimation**: The daily volatility of the S&P 500 is estimated using three models:
  - Historical/Realized volatility
  - Jump Diffusion (Merton and Kou)
  - Rough Volatility (fractional Brownian motion)
- **Pricing**: Using each estimated volatility model, the Black-Scholes pricing framework is adapted to simulate 10,000 possible one-day-ahead price paths

starting from a normalized price .

- **VaR Calculation**: From the distribution of simulated returns, the 95% and 99% quantiles are computed to derive the 1-day VaR under each volatility specification.

The following sections explain each component in detail.

3.2 Estimation of volatility via different models

3.2.1 Historical/Realized Volatility (benchmark)

As a benchmark, realized volatility is computed using the standard deviation of daily log returns over a rolling 20-day window:

$$\sigma_t^{hist} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(r_{t-i} - \bar{r})^2)}, \quad n = 20$$

where $r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$

3.2.2 Jump Diffusion

We implement two types of jump models: the classical Merton Jump Diffusion model, and the Double Exponential model proposed by Kou. In both cases, jumps are detected empirically from the log return series as extreme values beyond $\pm 2.5\sigma$.

- **Merton model**:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t + J_t S_t\, dN_t$$

where $dN_t \sim \text{Poisson}(\lambda)$ and $J_t \sim \mathcal{N}(\mu_J, \sigma_J^2)$.

- **Kou model**: Jumps follow a double exponential distribution:

$$f_J(x) = p\eta_1 e^{-\eta_1 x}\mathbf{1}_{x>0} + (1-p)\eta_2 e^{\eta_2 x}\mathbf{1}_{x<0}$$

with parameters p, η_1, η_2 estimated empirically from positive and negative jumps.

3.2.3 Rough Volatility

Following Gatheral et al. (2018), we model volatility as a lognormal process driven by fractional Brownian motion (fBM):

$$\sigma_t = \xi_0 \exp\left(\nu W_t^H - \frac{1}{2}\nu^2 t^{2H}\right)$$

where:

- W_t^H is fractional Brownian motion (fBM) with Hurst exponent $H < 0.5$,
- ξ_0 is the initial volatility (mean of realized volatility),
- ν is the volatility of volatility (standard deviation of realized volatility),
- H is estimated from the volatility series via the `hurst` method.

In this setup, the volatility process is non-Markovian and rough, capturing empirical features of financial markets not present in classical models.

3.3 Pricing via Black-Scholes using estimated volatilities

Once the volatility has been estimated using the three models described above, we proceed to simulate future asset prices over a one-day horizon. To ensure comparability, we normalize the current price as $S_0 = 100$ and use a modified version of the Black-Scholes pricing equation to generate simulated prices:

$$S_T = S_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right)$$

where:

- μ is the annualized drift, calibrated from historical log returns,
- σ is the estimated volatility for each model,
- $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable,
- $T = 1/252$ represents one trading day.

In the case of jump models, an additional jump component J is added inside the exponential, reflecting the impact of rare, large price movements:

$$S_T^{\text{jump}} = S_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}Z + J\right)$$

For the rough volatility model, the volatility σ_t is simulated path-by-path from the rough stochastic volatility process, making the effective endogenous to each simulation.

In all cases, 10,000 price paths are simulated using the corresponding model parameters, forming the empirical distribution from which returns and VaR are subsequently derived.

3.4 VaR calculation based on simulated prices

With the distribution of one-day-ahead simulated prices $\left\{S_T^{(i)}\right\}_{i=1}^{n}$ obtained from each model, we calculate the corresponding distribution of returns as:

$$R^{(i)} = \frac{S_T^{(i)} - S_0}{S_0}$$

Then, the 1-day Value at Risk at a confidence level is defined as:

$$VaR_\alpha = -S_0 \cdot Quantile_{1-\alpha}(R)$$

For this study, we compute the VaR at the 95% and 99% levels ($\alpha = 0.95$ and $\alpha = 0.99$). The Monte Carlo procedure ensures that the shape of the return distribution reflects the properties of each volatility model, including fat tails and asymmetry.

Comparing the VaR outputs across the three modeling frameworks provides insight into how volatility assumptions influence downside risk estimates. These results are later evaluated in terms of consistency, stress performance, and relative conservativeness.

3.5 Evaluation criteria: consistency, sensitivity, stress tests

To assess the performance and robustness of each model, several evaluation criteria are applied to the resulting VaR estimates:

- **Internal Consistency**: We verify that the VaR estimates are coherent with the distributional properties of the underlying simulations. In particular, we examine whether models that introduce jumps or roughness lead to heavier tails and higher VaR, as theoretically expected.
- **Sensitivity Analysis**: The sensitivity of VaR estimates to changes in model inputs (such as window size for volatility, or threshold for jump detection) is assessed. This helps to identify which models are more stable versus those that are more reactive to input variations.
- **Stress Testing**: To simulate adverse market conditions, we replicate the analysis over periods of known market stress (e.g., 2008–2009, 2020). This allows us to test whether each model adequately captures risk under extreme volatility scenarios.

These evaluation tools are crucial not only for comparing models on statistical grounds, but also for assessing their practical usefulness in risk management contexts.

Chapter 4

Empirical Application and Results

4.1 Dataset description and preprocessing

Date	Close	High	Low	Open	Volume
2006-01-03	1268.800049	1270.219971	1245.739990	1248.290039	2554570000
2006-01-04	1273.459961	1275.369995	1267.739990	1268.800049	2515330000
...	...	...	...	...	...
2025-04-16	5275.700195	5367.240234	5220.790039	5335.750000	4607750000
2025-04-17	5282.700195	5328.310059	5255.580078	5305.450195	4714880000

Table 2: S&P500 data 2006-2025

4.2 Implementation of volatility estimators

4.3 Price simulation using Black-Scholes

4.4 VaR results by volatility method

4.5 Comparative analysis (plots, tables, risk metrics)

Chapter 5

Discussion

5.1 Interpretation of differences

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Chapter 6

Conclusions

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Appendix A

First Appendix

This is an example paragraph. As you can see, the main text uses a font size of 12 pt and a line spacing of 1.5. Neither the paragraphs nor the first lines of paragraphs should be indented.

There is no very strict page limit. Your number of pages will be strongly influenced by the size and total number of your figures and tables. It is recommended staying within 30-50 pages. Do not try to fill as many pages as you can. Longer theses are not necessarily of higher quality and of more non-redundant content than shorter theses. Certainly, a master thesis of 15 pages is too short, and a master thesis of 100 pages is too long.

Appendix B

Second Appendix