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From Volatility to Value at Risk: A Comparative Analysis Using Jump, Rough, and SABR Volatility Models

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Abstract

Volatility plays a central role in modern risk management and financial modeling. This thesis investigates how different volatility modeling approaches influence the computation of Value at Risk (VaR), a widely used metric for quantifying downside risk. We compare four methods for estimating volatility: Historical Realized Volatility, Jump Diffusion models (Merton and Kou), Rough Volatility models based on fractional Brownian motion, and the SABR stochastic volatility model. Each volatility estimate is integrated into a Monte Carlo simulation framework based on the Black-Scholes model to compute one-day VaR at the 95% and 99% confidence levels. A sensitivity analysis is also conducted to assess the robustness of each model with respect to key calibration parameters.

Using historical data from the S&P 500 index (2006–2025), we simulate 10,000 price paths for each model and compute the corresponding VaR values. Our results show that jump models produce VaR estimates similar to those obtained using realized volatility, though they differ substantially in kurtosis and tail shape. The Kou model, due to its double-exponential structure, generates more extreme outcomes and higher kurtosis, indicating a stronger representation of tail risk. In contrast, the Rough Volatility and SABR models yield lower VaR estimates and more concentrated return distributions across calibrations, suggesting they may understate extreme losses under typical parameter settings. These findings indicate that, depending on the calibration and market context, both jump and realized volatility models can provide conservative VaR estimates, while models with smoother volatility dynamics may offer a different risk profile.

Keywords: Volatility modeling; Value at Risk; Jump Diffusion; Rough Volatility; Risk Management

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Chapter 1

Introduction

Volatility is a cornerstone of financial modeling and risk management, capturing the uncertainty inherent in asset prices. Its accurate estimation is crucial for derivative pricing, portfolio optimization, and regulatory compliance. However, conventional models—such as those assuming constant volatility or relying solely on historical data—often fail to capture key empirical features observed in financial markets, including sudden price jumps, volatility smiles, and persistent volatility clustering. The 2008 financial crisis exposed the fragility of standard risk models and reinforced the need for more realistic frameworks that can better anticipate and quantify extreme events.

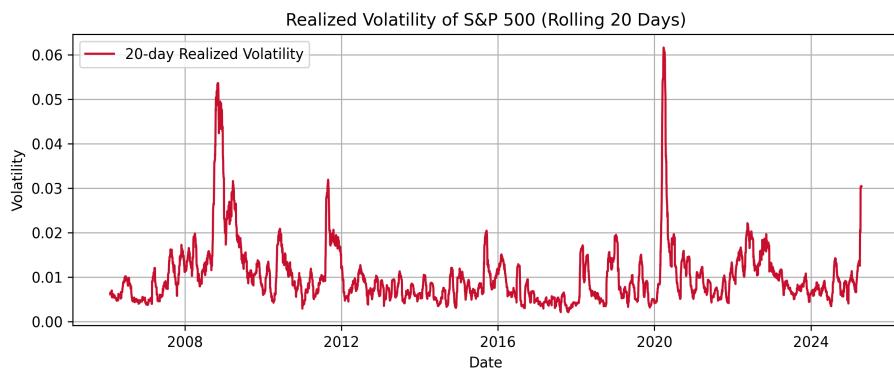


Figure 1: Realized volatility of the S&P 500 index computed over a rolling window of 20 days (2006–2024).

In this context, Value at Risk (VaR) remains one of the most widely adopted risk

metrics, yet its reliability depends heavily on the quality of the underlying volatility estimates. In recent years, several advanced models have been proposed to address the shortcomings of traditional approaches. Among these, jump diffusion processes, rough volatility models, and the SABR (Stochastic Alpha Beta Rho) model stand out as promising alternatives. Each captures distinct market phenomena: jump diffusion models account for abrupt price movements, rough volatility models incorporate long-memory dynamics and irregular paths, and SABR is particularly effective in modeling implied volatility surfaces, especially for interest rate derivatives.

This thesis investigates how different volatility modeling approaches affect the computation of VaR. Specifically, it compares four methods: Historical/Realized Volatility (as a benchmark), Jump Diffusion Models, Rough Volatility Models, and the SABR model. The objective is to assess their relative effectiveness in estimating volatility and, consequently, in producing reliable VaR figures. The study aims to identify the trade-offs involved in each method, balancing theoretical rigor with empirical performance and practical feasibility.

The analysis proceeds in two stages. First, the theoretical foundations of each model are examined, focusing on their assumptions, mathematical structure, and relevance in financial literature. Second, the models are implemented in Python using real market data. Each estimated volatility series is input into the Black-Scholes framework (or the relevant pricing mechanism in the case of SABR) to simulate price paths, from which VaR is computed. The models are evaluated based on their predictive consistency and their sensitivity to changes in model parameters.

Chapter 2 presents the theoretical framework, including an overview of VaR, the Black-Scholes model, and the selected volatility models. Chapter 3 details the methodological workflow, from volatility estimation to VaR computation. Chapter 4 contains the empirical analysis and results. Chapter 5 discusses key insights and limitations, and Chapter 6 concludes with a summary and directions for future research. Technical appendices provide code, derivations, and supplementary material.

Chapter 2

Theoretical Framework

Understanding how volatility modeling affects Value at Risk (VaR) estimation is central to this thesis. While the Black-Scholes framework provides a tractable and widely accepted setting for simulating asset prices, the treatment of volatility within this framework critically shapes the distribution of returns and, consequently, the estimation of risk. This chapter lays out the theoretical underpinnings of the models employed in this study. Beginning with the concept of VaR, it proceeds through the classical Black-Scholes model and its limitations, then presents alternative volatility modeling approaches: jump diffusions, rough volatility, and the SABR model. Each is evaluated in terms of its implications for risk estimation.

2.1 Risk measurement and the concept of VaR

Quantifying potential losses in financial portfolios is a fundamental objective in risk management. Institutions such as banks, asset managers, and regulators rely on robust metrics to assess exposure under both normal and stressed market conditions. Among various measures, Value at Risk (VaR) has become a standard due to its intuitive interpretation and regulatory endorsement (e.g., Basel II and III) (Jorion, 2007). VaR summarizes the potential loss of a portfolio over a specified time horizon at a given confidence level.

Formally, let ΔV denote the change in portfolio value over the horizon of interest. The Value at Risk at confidence level $\alpha \in (0, 1)$ is defined as the smallest value x such that the probability of a loss exceeding x does not exceed $1 - \alpha$:

$$P(\Delta V < VaR_\alpha) = \alpha$$

For example, a one-day 99% VaR of 1 million indicates that there is only a 1% chance that losses will exceed 1 million in a single day.

Several methodologies exist to estimate VaR:

- **Parametric (variance-covariance) method:** Assumes normally distributed returns. If returns are linear and normal, VaR is computed analytically as $VaR_\alpha = \mu + z_\alpha \sigma$.
- **Historical simulation:** Uses past returns to construct the empirical distribution of losses.
- **Monte Carlo simulation:** Generates potential future price paths based on stochastic models, estimating VaR from simulated outcomes.

In this thesis, we adopt the Monte Carlo approach. Asset prices will be simulated using the Black-Scholes framework, with different specifications for the volatility input σ_t . This enables a controlled comparison of how volatility modeling affects the resulting VaR estimates.

2.2 The Black-Scholes Model as Pricing Framework

The Black-Scholes model (Black & Scholes, 1973) remains a cornerstone of modern financial theory. It assumes that asset prices follow geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ is the drift, σ is constant volatility, and W_t is standard Brownian motion.

From this specification, the well-known Black-Scholes option pricing formula is derived. Though based on simplifying assumptions (e.g., frictionless markets, no transaction costs, continuous trading), the model offers an analytically tractable starting point.

In this thesis, the Black-Scholes framework is used as the common structure for all simulations. The core asset price dynamics remain the same across scenarios. The only variable component is the modeling of σ_t , allowing us to isolate the impact of volatility modeling on risk estimation.

2.3 Volatility in the Black-Scholes context

Although the Black-Scholes model assumes constant volatility, empirical data contradict this assumption. Market volatility is time-varying and exhibits features such as clustering, leverage effects, and fat tails (Cont, 2001).

In practice, volatility is often inferred from option prices as implied volatility. Observed volatility smiles and skews suggest that constant volatility fails to capture real market dynamics. Moreover, volatility's term structure is dynamic and frequently mean-reverting.

These limitations motivate the use of alternative volatility models. This thesis examines three such approaches: jump diffusions, rough volatility, and SABR-type dynamics. Each model is integrated into the Black-Scholes price simulation by varying only the volatility process.

2.4 Jump Diffusion Models (Merton and Kou)

Market prices can experience sudden jumps due to events like earnings announcements or geopolitical news. These jumps introduce skewness and excess kurtosis into return distributions.

Merton (1976) introduced a jump diffusion model:

$$dS_t = (\mu - \lambda k)S_t dt + \sigma S_t dW_t + S_t \left(\prod_{j=1}^{dN_t} Y_t - 1 \right)$$

where N_t is a Poisson process and Y_t the jump size. (To see an exhaustive mathematical derivation, consult Appendix D).

Kou (2002) proposed a double-exponential distribution for $\log(Y_t)$, allowing for asymmetric jumps. This model better captures volatility skews and is analytically tractable. The jump size density is given by:

$$f_J(x) = p \eta_1 e^{-\eta_1 x} \mathbf{1}_{x \geq 0} + (1 - p) \eta_2 e^{\eta_2 x} \mathbf{1}_{x < 0}$$

where $p \in (0, 1)$ is the probability of an upward jump, and $\eta_1, \eta_2 > 0$ are the decay parameters for upward and downward jumps, respectively.

In this study, we embed jumps into the volatility process while retaining the Black-Scholes structure for prices. This allows us to analyze how discontinuities affect VaR.

2.5 Fractional/Rough Volatility Models (Rough Bergomi)

High-frequency data reveal that volatility is not only time-varying but also rougher than Brownian motion. Bennedsen et al. (2017) find a Hurst exponent of $H \approx 0.1$ for log-volatility, indicating roughness.

The rough Bergomi model (Bayer et al., 2016) formulates volatility as:

$$\sigma_t = \xi_0(t) \exp \left(\eta \int_0^t (t-s)^{H-1/2} dZ_s - \frac{1}{2} \eta^2 t^{2H} \right)$$

This leads to short-memory, heavy-tailed return distributions.

Simulating asset prices with rough volatility under the Black-Scholes framework allows us to assess the risk implications of this irregular behavior. The resulting VaR estimates will reflect this heightened short-term uncertainty.

2.6 SABR Model (Stochastic Alpha Beta Rho Model)

The SABR model (Hagan et al., 2002) is widely used in interest rate and FX derivatives. It specifies the forward price F_t and volatility σ_t as:

$$\begin{cases} dF_t = \sigma_t F_t^\beta dW_t \\ d\sigma_t = \nu \sigma_t dZ_t \\ \text{with } \mathbb{E}[dW_t dZ_t] = \rho dt \end{cases}$$

In this thesis, we adapt the model to simulate spot prices rather than forward prices, fixing $\beta = 1$ to ensure log-normality. This simplification is reasonable in our context, as we work with equity index data (S&P 500), where the forward and spot prices are nearly equivalent under low interest rate environments and short horizons. Parameters are calibrated from historical data. This provides a third alternative volatility process for evaluating VaR under stochastic and correlated dynamics.

2.7 From volatility estimation to VaR computation

Each volatility model discussed above yields a different shape for the return distribution generated via simulation. These differences, though subtle in modeling, have major implications for the estimation of VaR:

All simulations retain the Black-Scholes price structure and differ only in the volatility process. For each model, we simulate many paths of asset prices, compute

Volatility Model	Return Distribution Shape	Tail Risk Representation	Expected VaR Impact
Realized Volatility	Lognormal (thin-tailed)	Underestimated	Lower
Jump Diffusion	Skewed, leptokurtic	Better captured	Higher
Rough Volatility	Heavy-tailed, short-memory	Very well captured	Higher
SABR Model	Asymmetric, fat-tailed	Well captured (leverage/skew)	Higher

Table 1: Comparison of volatility models and their implications for tail risk and Value at Risk (VaR).

changes in portfolio value ΔV , and estimate VaR as the empirical quantile:

$$P(\Delta V < VaR_\alpha) = \alpha$$

This approach ensures a consistent comparison across models, helping identify whether more sophisticated volatility specifications lead to meaningfully improved risk assessments.

Chapter 3

Methodology

This section outlines the full methodological framework used to estimate Value at Risk (VaR) based on different volatility models. The process follows a structured pipeline: first, volatility is estimated using four different approaches; second, this volatility is used to price one-day-ahead asset values via a modified Black-Scholes model; finally, VaR is computed from the resulting distribution of simulated returns.

3.1 General workflow: from volatility to VaR

The goal of this chapter is to assess how different volatility modeling assumptions impact the resulting Value at Risk (VaR) estimates. The workflow comprises three main stages:

- **Volatility Estimation:** The daily volatility of the S&P 500 is estimated using four models:
 - Historical/Realized volatility
 - Jump Diffusion (Merton and Kou)
 - Rough Volatility (fractional Brownian motion)
 - SABR Stochastic Volatility Model

- **Pricing:** Using each estimated volatility model, the Black-Scholes pricing framework is adapted to simulate 10,000 possible one-day-ahead price paths starting from a normalized price.
- **VaR Calculation:** From the distribution of simulated returns, the 95% and 99% quantiles are computed to derive the 1-day VaR under each volatility specification.

The following sections explain each component in detail.

3.2 Estimation of volatility via different models

3.2.1 Historical/Realized Volatility (benchmark)

As a benchmark, realized volatility is computed using the standard deviation of daily log returns over a rolling 20-day window:

$$\sigma_t^{hist} = \sqrt{\frac{1}{n} \sum_{i=1}^n (r_{t-i} - \bar{r})^2}, \quad n = 20$$

where $r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$

3.2.2 Jump Diffusion

We implement two types of jump models: the classical Merton Jump Diffusion model, and the Double Exponential model proposed by Kou. In both cases, jumps are detected empirically from the log return series as extreme values beyond $\pm 2.5\sigma$.

- **Merton model:**

$$dS_t = \mu S_t dt + \sigma S_t dW_t + J_t S_t dN_t$$

where $dN_t \sim \text{Poisson}(\lambda)$ and $J_t \sim \mathcal{N}(\mu_J, \sigma_J^2)$.

- **Kou model:** Jumps follow a double exponential distribution:

$$f_J(x) = p\eta_1 e^{-\eta_1 x} \mathbf{1}_{x>0} + (1-p)\eta_2 e^{\eta_2 x} \mathbf{1}_{x<0}$$

with parameters p, η_1, η_2 estimated empirically from positive and negative jumps.

3.2.3 Rough Volatility

Following Gatheral et al. (2018), we model volatility as a lognormal process driven by fractional Brownian motion (fBM):

$$\sigma_t = \xi_0 \exp \left(\nu W_t^H - \frac{1}{2} \nu^2 t^{2H} \right)$$

where:

- W_t^H is fractional Brownian motion (fBM) with Hurst exponent,
- ξ_0 is the initial volatility (mean of realized volatility),
- ν is the volatility of volatility (standard deviation of realized volatility),
- H is estimated from the volatility series via the `hurst` method.

In this setup, the volatility process is non-Markovian and rough, capturing empirical features of financial markets not present in classical models.

3.2.4 SABR Stochastic Volatility Model

The SABR (Stochastic Alpha Beta Rho) model, commonly used in interest rate markets, is adapted here to estimate stochastic volatility on spot price series. We assume log-normal dynamics ($\beta = 1$):

$$\begin{cases} dS_t = \sigma_t S_t^\beta dW_t \\ d\sigma_t = \nu \sigma_t dZ_t \\ \text{with } \mathbb{E}[dW_t dZ_t] = \rho dt \end{cases}$$

Parameters are calibrated as follows:

- ξ_0 : initial volatility, taken as the mean of a 20-day rolling window.
- ν : volatility of volatility, computed as the standard deviation of realized volatility.
- ρ : correlation between returns and volatility changes, estimated via Pearson correlation.

This model introduces correlated stochastic shocks to price and volatility, capturing leverage effects and forward-looking uncertainty.

3.3 Pricing via Black-Scholes using estimated volatilities

Once the volatility has been estimated using the three models described above, we proceed to simulate future asset prices over a one-day horizon. To ensure comparability, we normalize the current price as $S_0 = 100$ and use a modified version of the Black-Scholes pricing equation to generate simulated prices:

$$S_T = S_0 \cdot \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} Z \right)$$

where:

- μ is the annualized drift, calibrated from historical log returns,
- σ is the estimated volatility for each model,
- $Z \sim \mathcal{N}(0, 1)$ is a standard normal random variable,
- $T = 1/252$ represents one trading day.

In the case of jump models, an additional jump component J is added inside the exponential, reflecting the impact of rare, large price movements:

$$S_T^{\text{jump}} = S_0 \cdot \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} Z + J \right)$$

For the rough volatility model, the volatility σ_t is simulated path-by-path from the rough stochastic volatility process, making the effective endogenous to each simulation.

In the SABR case, both the volatility path σ_t and its correlation with the asset price are simulated jointly. Euler discretization is used to propagate the coupled SDE system, capturing the co-evolution of price and volatility over the one-day horizon.

In all cases, 10,000 price paths are simulated using the corresponding model parameters, forming the empirical distribution from which returns and VaR are subsequently derived.

3.4 VaR calculation based on simulated prices

With the distribution of one-day-ahead simulated prices $\left\{ S_T^{(i)} \right\}_{i=1}^n$ obtained from each model, we calculate the corresponding distribution of returns as:

$$R^{(i)} = \frac{S_T^{(i)} - S_0}{S_0}$$

Then, the 1-day Value at Risk at a confidence level is defined as:

$$VaR_\alpha = -S_0 \cdot \text{Quantile}_{1-\alpha}(R)$$

For this study, we compute the VaR at the 95% and 99% levels ($\alpha = 0.95$ and $\alpha = 0.99$). The Monte Carlo procedure ensures that the shape of the return distribution reflects the properties of each volatility model, including fat tails and asymmetry.

Comparing the VaR outputs across the three modeling frameworks provides insight

into how volatility assumptions influence downside risk estimates. These results are later evaluated in terms of consistency and relative conservativeness.

3.5 Evaluation criteria: consistency and sensitivity

To assess the performance and robustness of each model, two evaluation criteria are applied to the resulting VaR estimates:

- **Internal Consistency:** We verify that the VaR estimates are coherent with the distributional properties of the underlying simulations. In particular, we examine whether models that introduce jumps, stochastic volatility, or roughness lead to heavier tails and higher VaR, as theoretically expected.
- **Sensitivity Analysis:** The sensitivity of VaR estimates to changes in model inputs (such as window size for volatility, or threshold for jump detection) is assessed. This helps to identify which models are more stable versus those that are more reactive to input variations.

These evaluation tools are crucial not only for comparing models on statistical grounds, but also for assessing their practical usefulness.

Chapter 4

Empirical Application and Results

4.1 Dataset description and preprocessing

The empirical analysis is based on historical price data for the S&P 500 index, retrieved using the `yfinance` Python package. The sample period spans from January 2006 to April 2025, covering a wide range of market conditions, including both tranquil and crisis episodes.

For each trading day, the dataset includes open, high, low, close, and volume prices.

Date	Close	High	Low	Open	Volume
2006-01-03	1268.800049	1270.219971	1245.739990	1248.290039	2554570000
2006-01-04	1273.459961	1275.369995	1267.739990	1268.800049	2515330000
...
2025-04-16	5275.700195	5367.240234	5220.790039	5335.750000	4607750000
2025-04-17	5282.700195	5328.310059	5255.580078	5305.450195	4714880000

Table 2: S&P500 data 2006-2025

From this raw data, the following variables were constructed:

- Logarithmic daily returns, defined as $r_t = \log(P_t/P_{t-1})$.
- Realized volatility, estimated as the rolling standard deviation of daily returns using windows of 5, 20, 60, 120, and 250 days.

Missing values resulting from non-trading days or rolling window truncation were dropped. The resulting cleaned dataset serves as the input for model calibration and Monte Carlo simulation. Figure 2 illustrates the behavior of realized volatility across multiple horizons, providing visual motivation for the need to explore alternative volatility models.

4.2 Implementation of volatility estimators

Each volatility model was calibrated using market data. For the Historical model, volatility is estimated directly from historical returns. For the jump diffusion models, jumps were identified as returns exceeding a 2.5 standard deviation threshold. The estimated parameters for the Merton model are the jump frequency λ , average jump size μ_J , and jump standard deviation σ_J . For the Kou model, upward and downward jumps were separated, and exponential decay parameters η_1 and η_2 were estimated.

The rough volatility model uses the 20-day realized volatility to estimate the Hurst exponent H , mean volatility ξ_0 , and volatility of volatility ν . These parameters are inferred via the `fbm` and `hurst` Python libraries. The SABR model, adapted to the context of spot price simulation, is calibrated using the correlation between log-returns and volatility innovations, producing a value for ρ .

All models are calibrated from the same underlying dataset to allow consistent and fair comparison in the subsequent simulation and VaR analysis.

4.3 Price simulation using Black-Scholes

To isolate the effect of volatility modeling, all price paths are simulated using the Black-Scholes framework:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t$$

where the only difference across simulations lies in the modeling of σ_t .

For each model, 30,000 price paths were generated over a 1-day horizon, starting from $S_0 = 100$. The simulated returns were computed as percentage changes from

S_0 . The simulation results exhibit notable differences:

- The Historical Volatility model produces a near-Gaussian distribution.
- The Merton and Kou models introduce visible skewness and kurtosis through jump components.
- The Rough Volatility and SABR models produce similar results, with symmetric distributions and thinner tails compared to those of the jump and historical models.

These features are consistent with theoretical expectations in the case of jump models. However, the fact that the Rough Volatility and SABR models do not produce heavier tails may be influenced by factors such as the chosen data window, the characteristics of the input series, or the calibration of roughness and correlation parameters.

4.4 VaR results by volatility method

The Value at Risk (VaR) at 95% and 99% confidence levels was computed from the distribution of simulated returns. All VaR figures are reported on a one-day horizon. Table 8 (in the Appendix) summarizes the results.

The Rough Volatility and SABR models consistently produce the lowest VaR estimates, indicating lower perceived tail risk. In contrast, the jump models (Merton and Kou) generate higher VaR, particularly at the 99% level, reflecting their sensitivity to extreme events. The Historical Volatility model yields VaR estimates similar to those of the jump models, but clearly differs in kurtosis. This suggests that while the VaR values may be close, the jump models explicitly account for more extreme scenarios, which are not captured by the historical model.

Interestingly, although the Kou and Merton models exhibit higher kurtosis—capturing the presence of fat tails and extreme outcomes—their VaR estimates are not substantially higher than those of other models. This highlights a key insight: higher

kurtosis does not necessarily imply greater Value at Risk. While kurtosis reflects the likelihood of extreme returns in the distribution, VaR focuses on a specific quantile, and the two measures do not always move in tandem. These findings caution against overinterpreting tail metrics when assessing risk through quantile-based measures like VaR.

4.5 Comparative analysis and robustness checks

The models were compared using graphical and statistical tools. Figures 20 and 21 included in the appendix display simulated return densities, overlaid to illustrate tail behavior. Key observations include:

- Jump models display much heavier tails than the other approaches, with pronounced left skew and clear deviations from normality.
- The Rough Volatility and SABR models show narrower return distributions with thinner tails, indicating less extreme simulated outcomes.

Sensitivity analyses were conducted by varying two key calibration inputs: the jump detection threshold (for Merton and Kou) and the volatility estimation window. Changes in the jump threshold had minimal impact on the resulting VaR estimates, suggesting robustness in the jump-based models. In contrast, the volatility window had a significant effect. In the Historical model, longer windows (e.g., 250 days) produced notably lower VaR values, while very short windows (e.g., 5 days) also underestimated risk. Both the Rough Volatility and SABR models showed increasing VaR estimates with longer volatility windows, highlighting their sensitivity to persistent volatility inputs.

These findings reinforce the importance of parameter selection in volatility modeling. While jump models are relatively stable in terms of VaR, models relying on realized volatility estimates—such as Historical, Rough, and SABR—can yield markedly different results depending on the observation window used.

Chapter 5

Discussion

5.1 Interpretation of differences

This section interprets the differences in Value at Risk (VaR) estimates obtained from various volatility modeling approaches applied within the Black-Scholes framework: historical volatility, jump-diffusion models (Merton and Kou), rough volatility, and SABR.

Jump-diffusion models tend to produce VaR estimates similar to, or slightly higher than, those obtained using historical volatility. While the numerical differences in VaR are modest, jump models exhibit significantly higher kurtosis and visibly heavier tails, indicating their superior ability to capture extreme events and asymmetric tail risk.

In contrast, rough volatility and SABR models yield consistently lower VaR estimates in this study, likely due to smoother volatility paths and thinner tails in the simulated return distributions. For example, the rough model—with a relatively high Hurst exponent—generates volatility paths that are less reactive in the short term, resulting in more concentrated return distributions. Similarly, the SABR model produces a return distribution that is concentrated and symmetric, with relatively thin tails—closely resembling those of the rough volatility model.

These differences underscore that similar VaR figures can arise from distributions with very different shapes, and that kurtosis and VaR—while both related to tail risk—do not always align. This reinforces the need for multi-dimensional risk metrics.

5.2 Model limitations and robustness

Each model presents distinct trade-offs. The historical volatility model, while simple and intuitive, is highly sensitive to the chosen rolling window. Short windows (e.g., 20 days) can react strongly to recent volatility shocks—such as the significant increase in market volatility observed following the announcement of new trade tariffs by the Trump administration—leading to elevated VaR estimates despite moderate kurtosis. In contrast, longer windows smooth out recent volatility spikes and may underestimate short-term risk. This responsiveness can be a strength for detecting turbulence, but it also introduces instability in risk forecasts.

Jump-diffusion models explicitly capture tail risk through discontinuities in returns. While their calibration relies on a threshold-based method to detect jumps, the sensitivity analysis shows that their resulting VaR estimates remain relatively stable across different detection thresholds. This robustness, along with the ability to reflect skewness and fat tails (i.e., leptokurtosis), makes them particularly suitable for modeling extreme scenarios—though it also introduces challenges in calibration and interpretation.

Rough volatility and SABR models produce VaR estimates that are strongly influenced by the length of the volatility input window. As the window increases, both models yield progressively higher VaR values—though still below those obtained with jump models. Their simulated return distributions consistently exhibit low kurtosis and high concentration, suggesting that, despite incorporating stochastic features, these models tend to underrepresent short-term tail risk under typical calibrations.

Overall, no single model dominates across all dimensions. Jump-diffusion models

consistently produce higher kurtosis and more extreme simulated returns, with relatively stable VaR estimates across calibration thresholds. However, the comparison with realized volatility is not straightforward. Depending on the chosen window, the historical model can also produce elevated VaR estimates, especially when recent market movements have triggered substantial increases in volatility. Rough volatility and SABR models, while sensitive to the input window, continue to yield lower VaR values across calibrations. Although rough volatility shows an upward trend in VaR with larger windows, its estimates remain below those of the jump and historical models in the data sample studied. These results reflect the distinct risk profiles each approach captures and suggest that model performance depends strongly on market conditions and input choices.

Chapter 6

Conclusions

6.1 Summary of findings

This thesis aimed to analyze how the estimation of volatility impacts the computation of Value at Risk (VaR) under the Black-Scholes framework, by comparing four alternative approaches: historical volatility, jump-diffusion models (Merton and Kou), rough volatility based on fractional Brownian motion, and the SABR stochastic volatility model. The key findings are:

- Volatility modeling has a significant impact on VaR estimates. Jump-diffusion models and historical volatility yield similar VaR levels under certain configurations, but differ substantially in kurtosis and tail structure.
- The Kou model outperforms Merton in capturing asymmetric jumps. This leads to slightly higher VaR estimates and a more accurate reflection of left-tail risk.
- Rough volatility and SABR models consistently produce lower VaR estimates across calibrations. While their values increase with longer input windows, they remain below those of jump and historical models in the sample studied.
- Despite incorporating more complex volatility dynamics, the return distributions generated by rough and SABR models exhibit low kurtosis and strong

concentration, indicating limited ability to reflect short-term tail risk.

- The historical volatility model is sensitive to the selected rolling window. Short windows, in particular, can produce elevated VaR estimates when recent market volatility is high.

6.2 Practical interpretation

The empirical results show that no single model is superior in all respects. Jump-diffusion models are effective at capturing extreme return behavior and are relatively robust to changes in calibration thresholds. However, depending on recent market conditions, historical volatility with short rolling windows can also produce conservative risk estimates. Rough and SABR models tend to produce lower VaR estimates, with outcomes that vary depending on the length of the volatility input window. Their return distributions remain concentrated and low in kurtosis, which may limit their ability to reflect tail risk. These contrasts illustrate the importance of understanding the specific assumptions and sensitivities embedded in each volatility model when interpreting VaR outputs.

6.3 Future research directions

Several avenues for future work emerge from this research:

- Model calibration and estimation. Future work could investigate more efficient or robust methods for calibrating jump and rough volatility models on real-time data, especially in high-frequency environments.
- Extension to other derivative pricing models. This study focused on Black-Scholes; extending the analysis to stochastic volatility frameworks like Heston or multifactor SABR could yield richer insights.
- Stress testing applications. Applying these models in macroeconomic or geopolitical stress testing scenarios could assess their robustness and predictive power under extreme conditions.

- Portfolio-level VaR. While this study analyzed a single asset, extending the framework to a multi-asset portfolio with correlated risk factors would bring the analysis closer to real-world portfolio management.
- Use of Machine Learning techniques. Hybrid approaches that combine traditional financial models with machine learning could be explored to improve volatility forecasting and VaR accuracy.
- Regulatory implications. Further research could also explore how these advanced models align with Basel III/IV regulations and the potential for regulatory acceptance of non-linear VaR estimators.

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Appendix A

GitHub Repository

All the code used for the simulations, data processing, and Value at Risk (VaR) computations presented in this thesis is available at the following GitHub repository:

<https://github.com/ferrangarciarovira/VaR-Volatility-Models>

This repository contains the full set of scripts and resources used to implement the models discussed in the empirical analysis, including data handling, Monte Carlo simulations, and plotting routines.

Appendix B

Figures

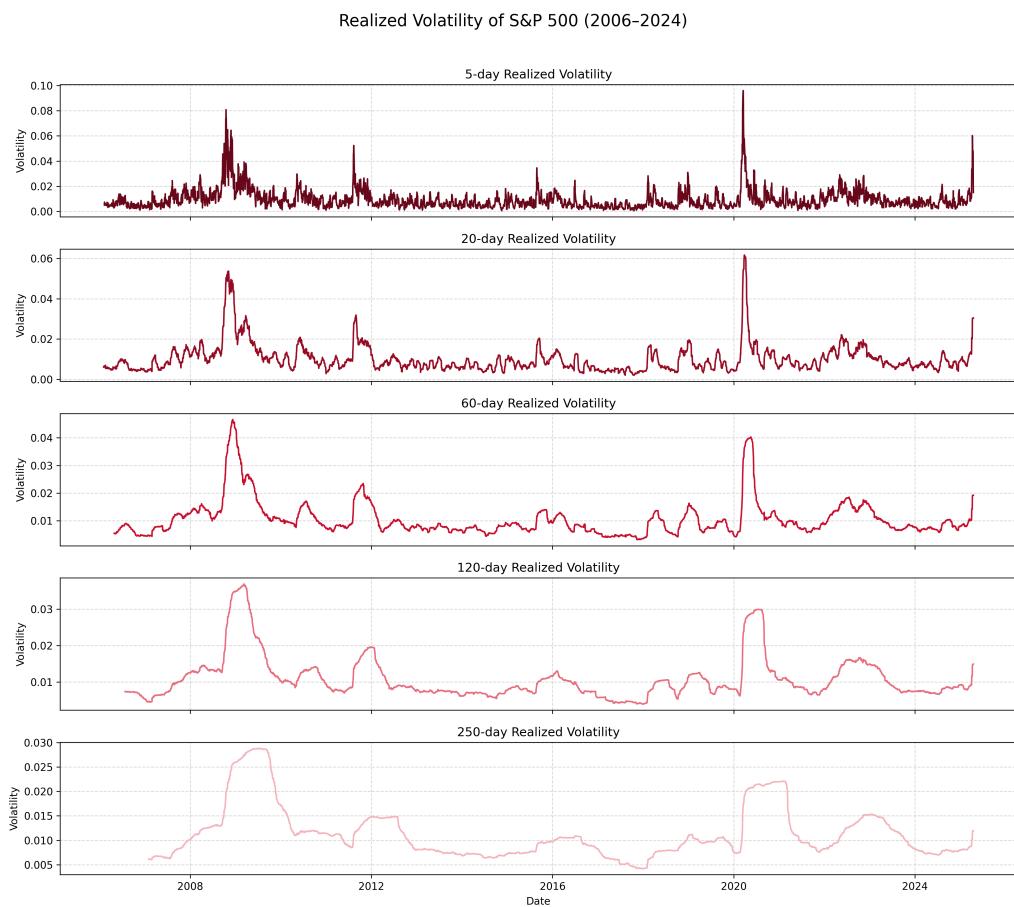


Figure 2: Realized volatility of the S&P 500 index computed over rolling windows of 5, 20, 60, 120, and 250 days (2006–2024).

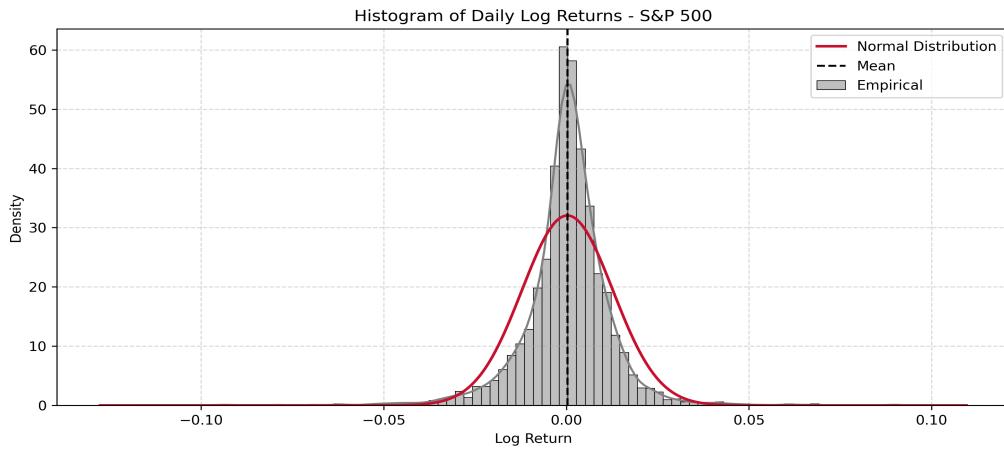


Figure 3: Histogram of daily log returns of the S&P 500 index from 2006 to 2024, compared with the fitted normal distribution.

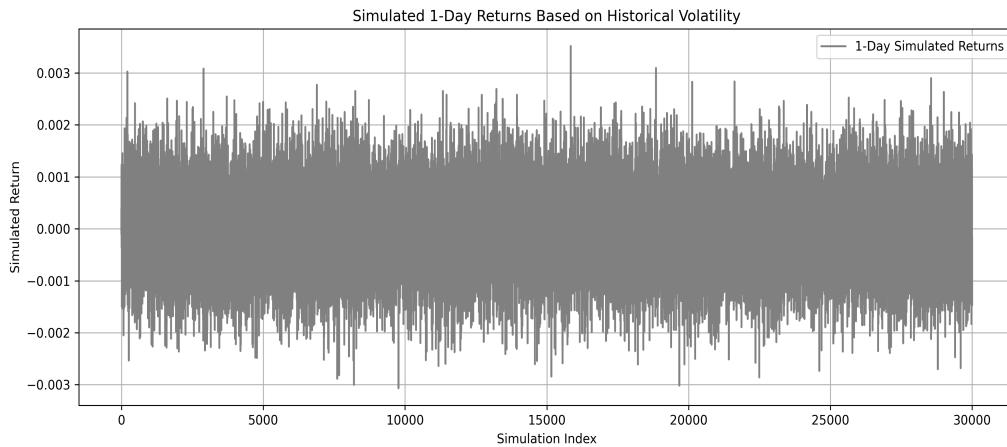


Figure 4: Simulated 1-day returns generated using historical volatility from the S&P 500 index.

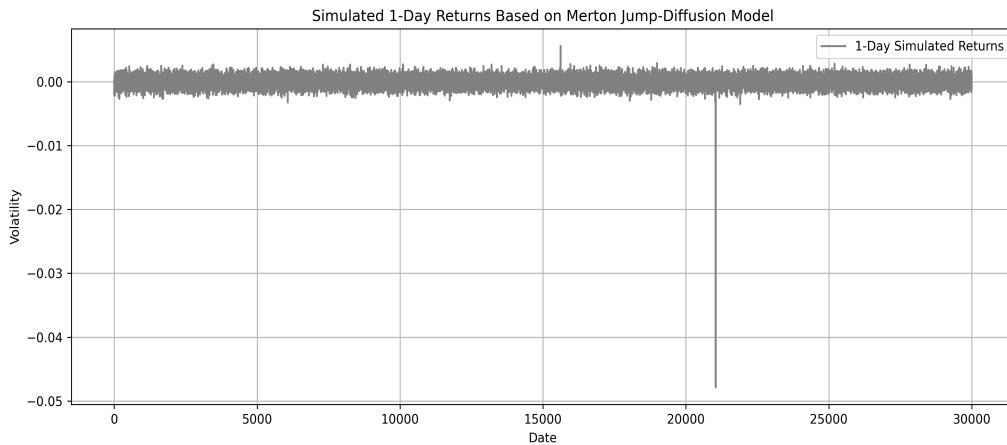


Figure 5: Simulated 1-day returns based on the Merton jump-diffusion model calibrated on historical S&P 500 data.

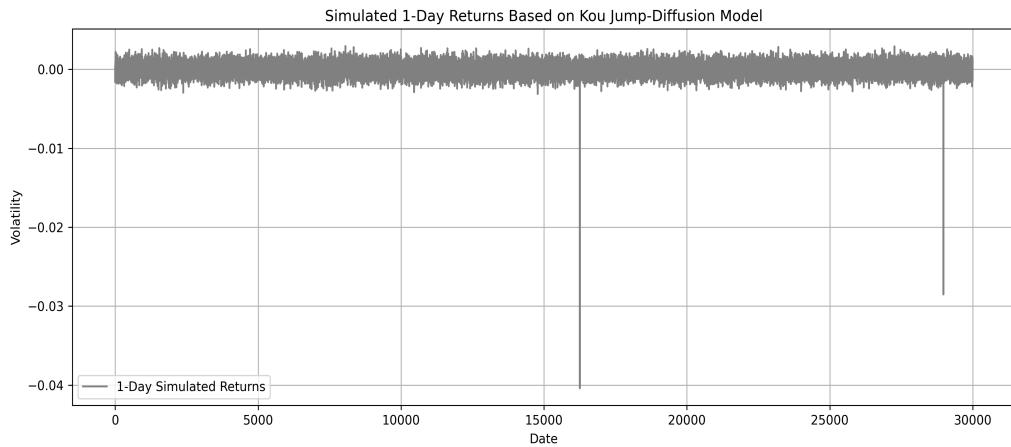


Figure 6: Simulated 1-day returns based on the Kou jump-diffusion model calibrated on historical S&P 500 data.

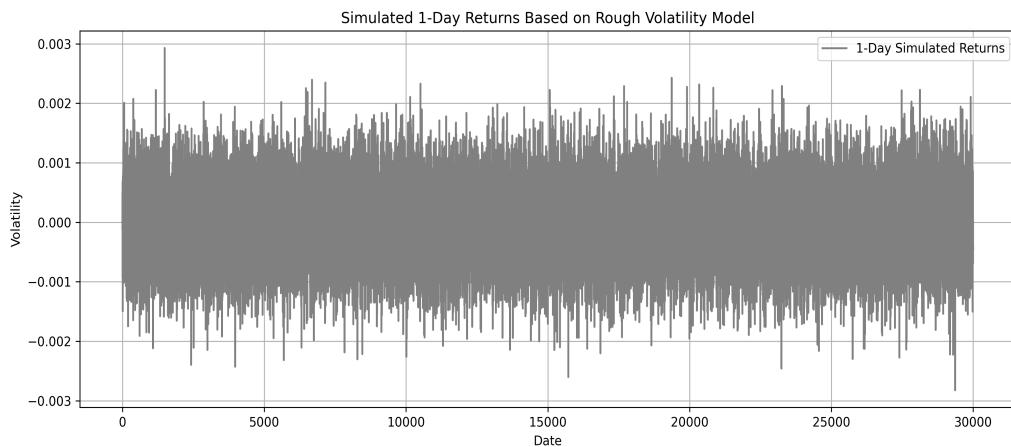


Figure 7: Simulated 1-day returns based on the rough volatility model calibrated to S&P 500 data.

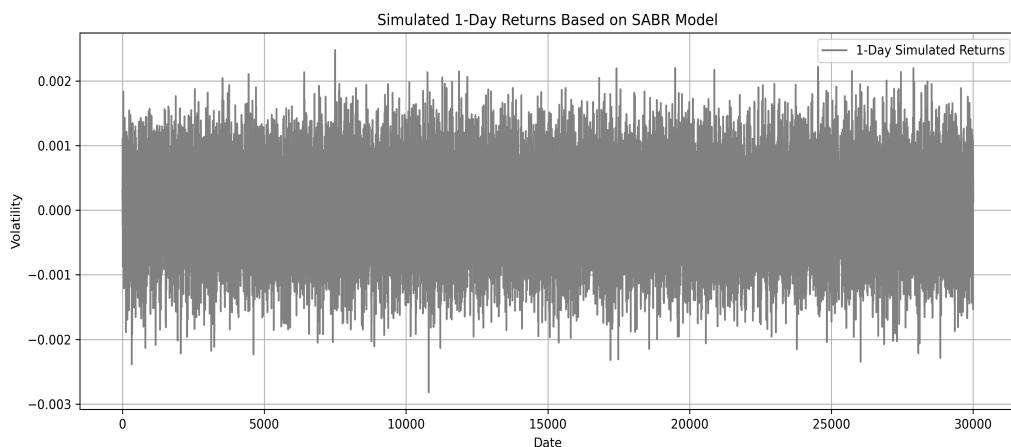


Figure 8: Simulated 1-day returns based on the SABR model calibrated to S&P 500 data.

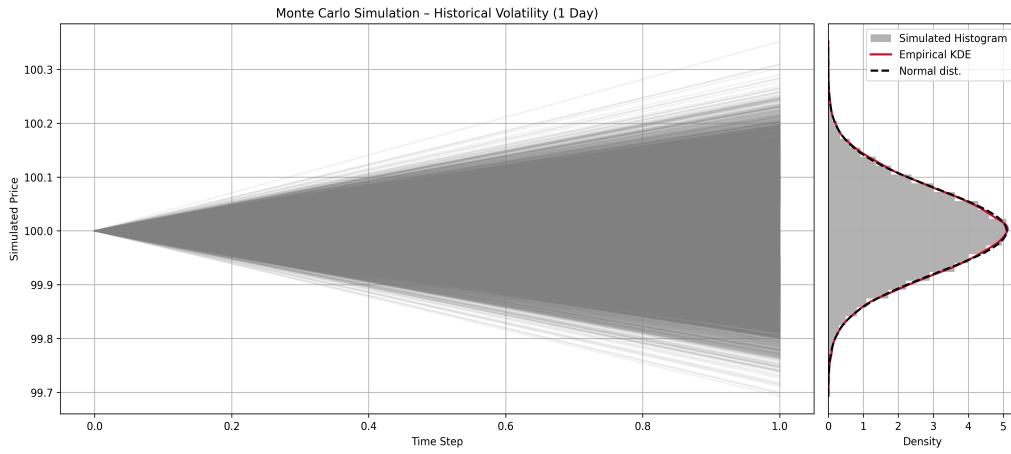


Figure 9: Monte Carlo simulation of 1-day price paths based on historical volatility of the S&P 500 index.

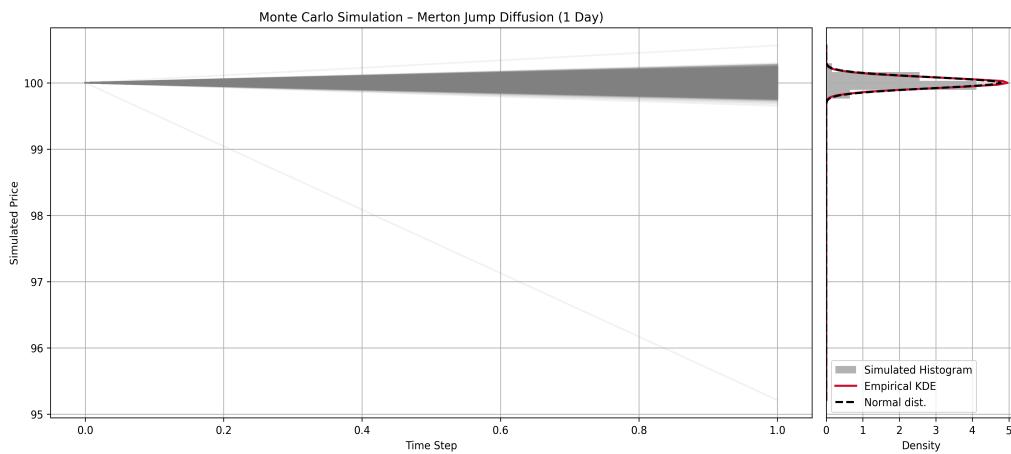


Figure 10: Monte Carlo simulation of 1-day price paths based on the Merton Jump-Diffusion model calibrated to the S&P 500 index.

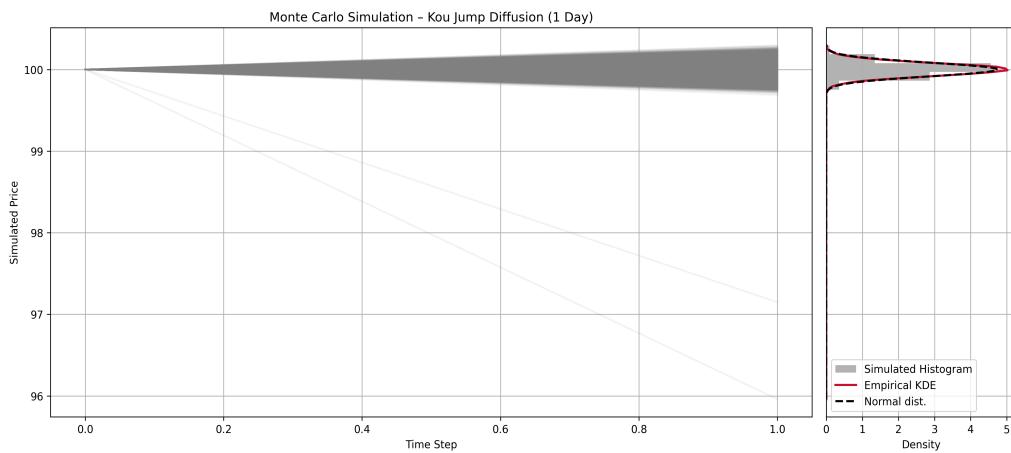


Figure 11: Monte Carlo simulation of 1-day price paths based on the Kou Jump-Diffusion model calibrated to the S&P 500 index.

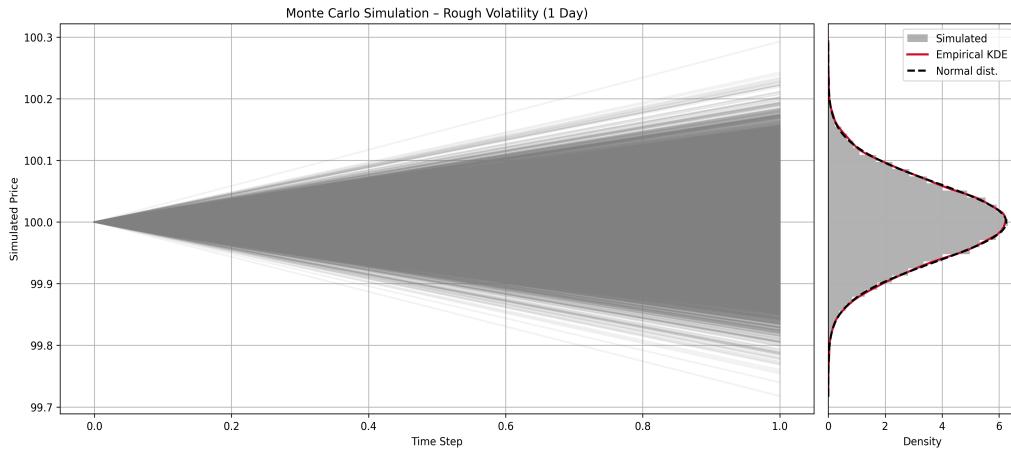


Figure 12: Monte Carlo simulation of 1-day price paths based on the rough volatility model calibrated to the S&P 500 index.

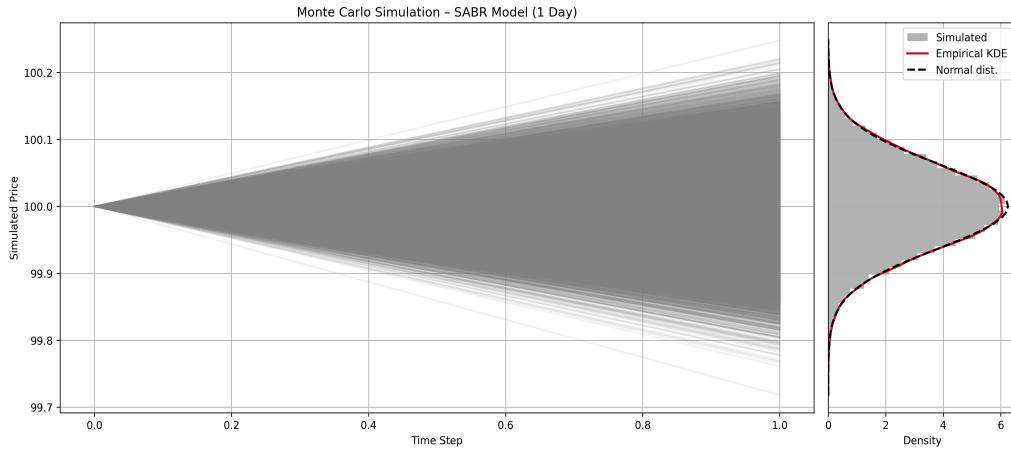


Figure 13: Monte Carlo simulation of 1-day price paths based on the SABR model calibrated to the S&P 500 index.

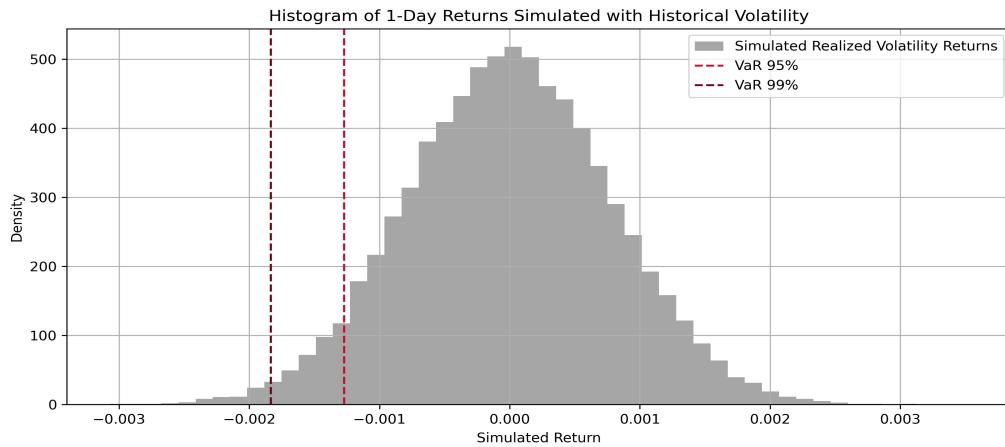


Figure 14: Histogram of 1-day returns simulated using the historical volatility of the S&P 500 index.

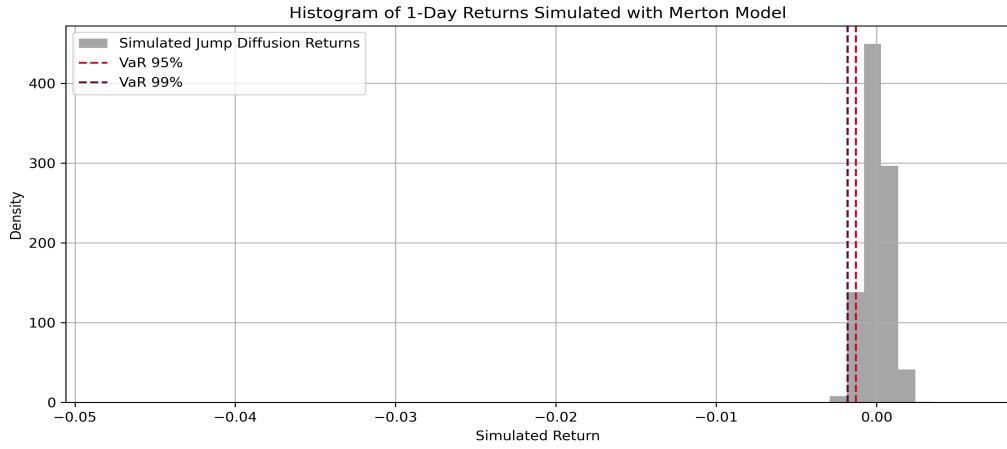


Figure 15: Histogram of 1-day returns simulated using the Merton jump-diffusion model.

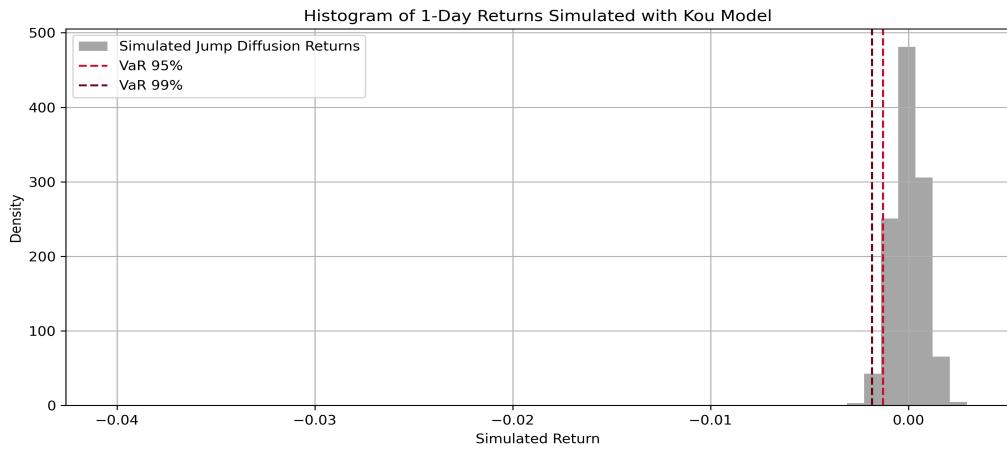


Figure 16: Histogram of 1-day returns simulated using the Kou jump-diffusion model.

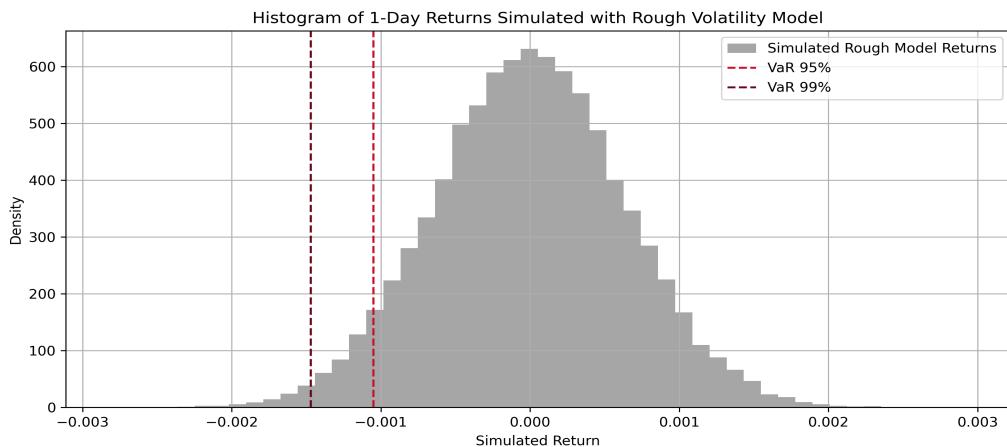


Figure 17: Histogram of 1-day returns simulated using the rough volatility model.

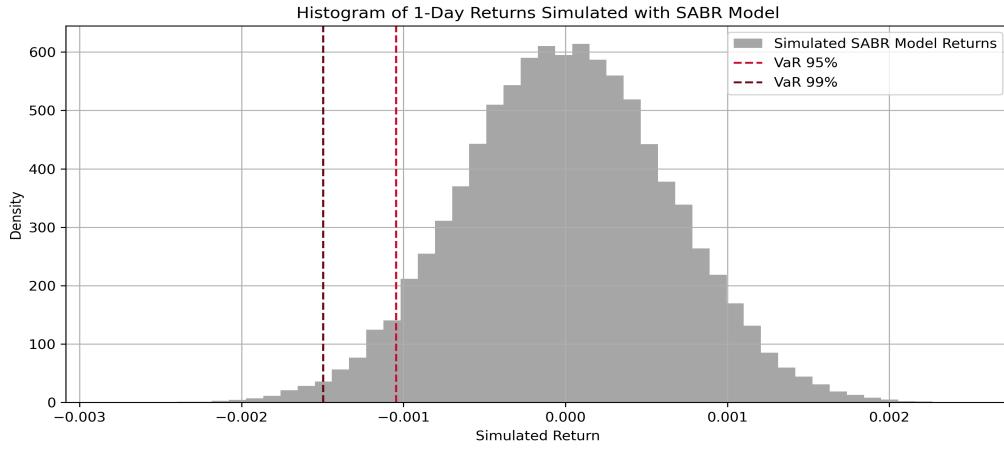


Figure 18: Histogram of 1-day returns simulated using the SABR (Stochastic Alpha Beta Rho) model.

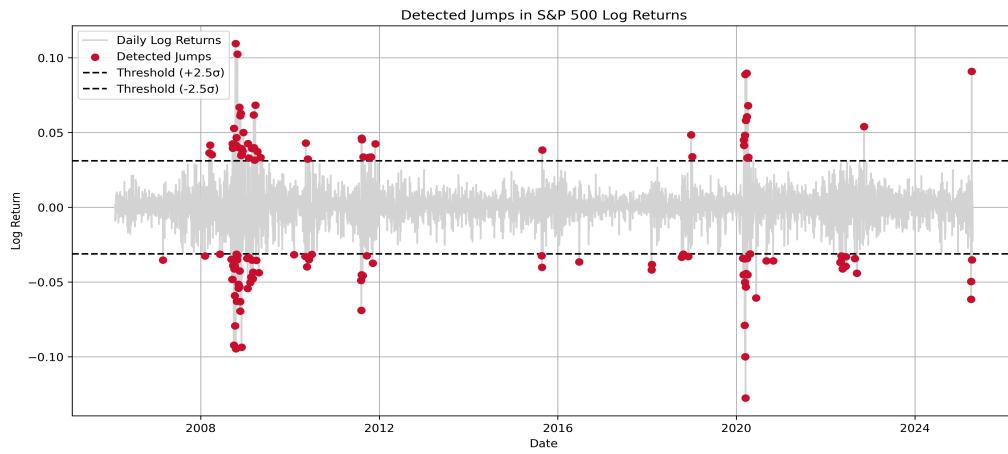


Figure 19: Detected jumps in daily log returns of the S&P 500 index from 2006 to 2024.

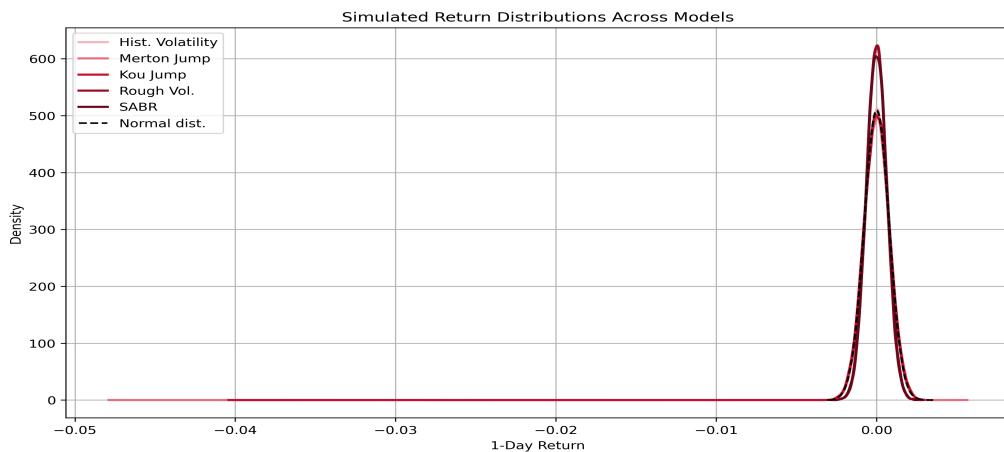


Figure 20: Comparison of simulated 1-day return distributions across models.

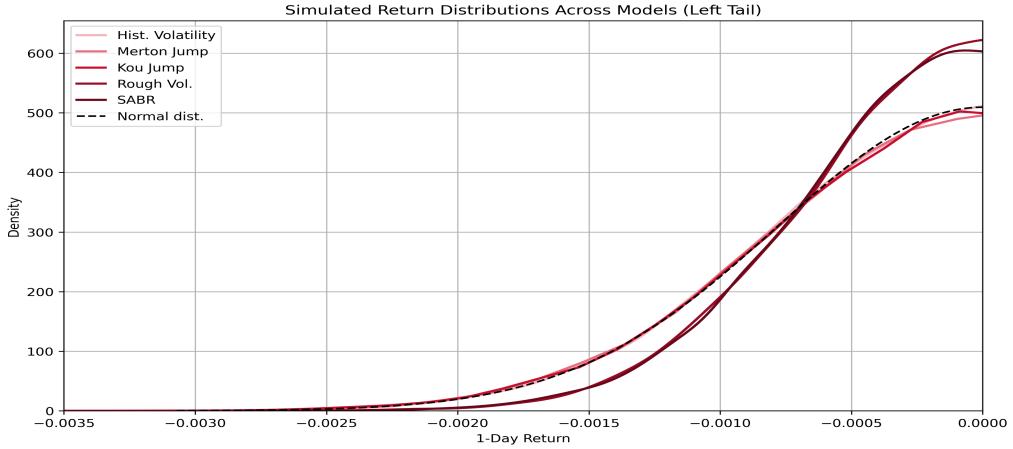


Figure 21: Zoom on the left tail of the simulated 1-day return distributions across models.

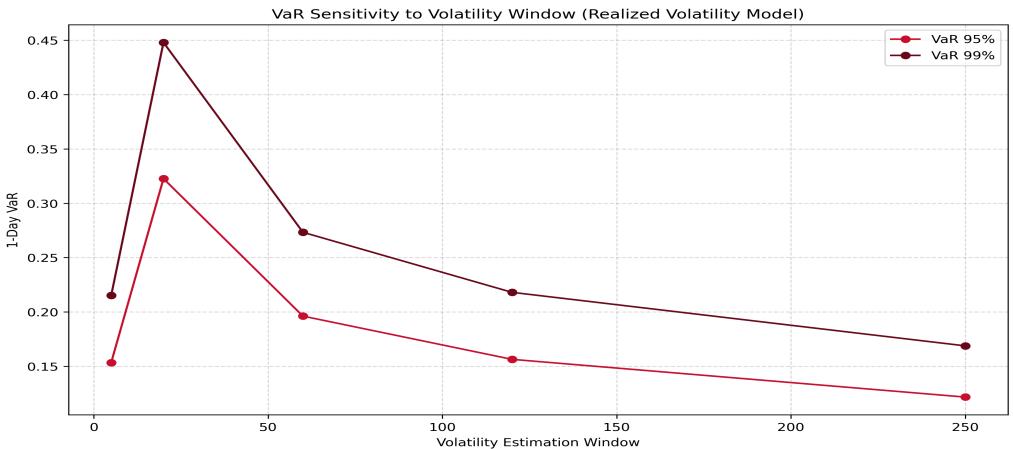


Figure 22: Sensitivity of 1-day Value at Risk (VaR) at the 95% and 99% confidence levels to the length of the rolling volatility window.

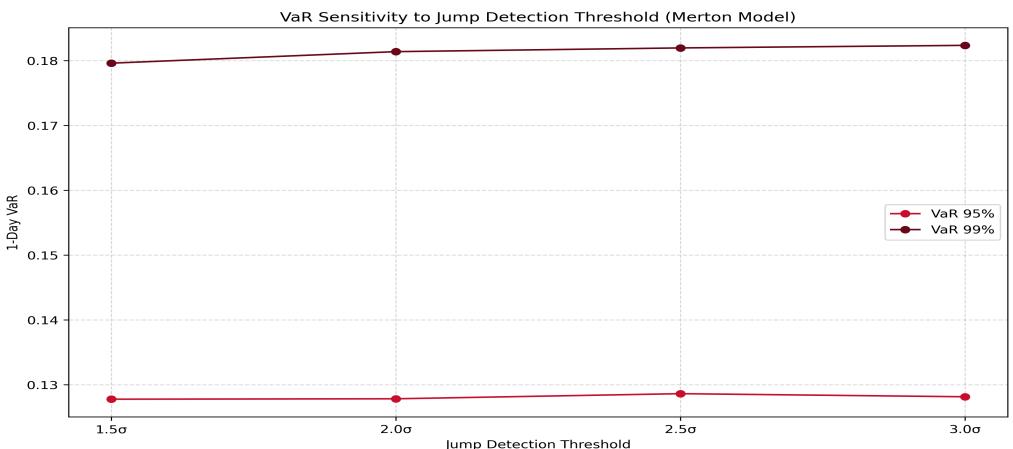


Figure 23: Sensitivity of 1-day Value at Risk (VaR) at the 95% and 99% confidence levels to the jump detection threshold in the Merton jump-diffusion model.

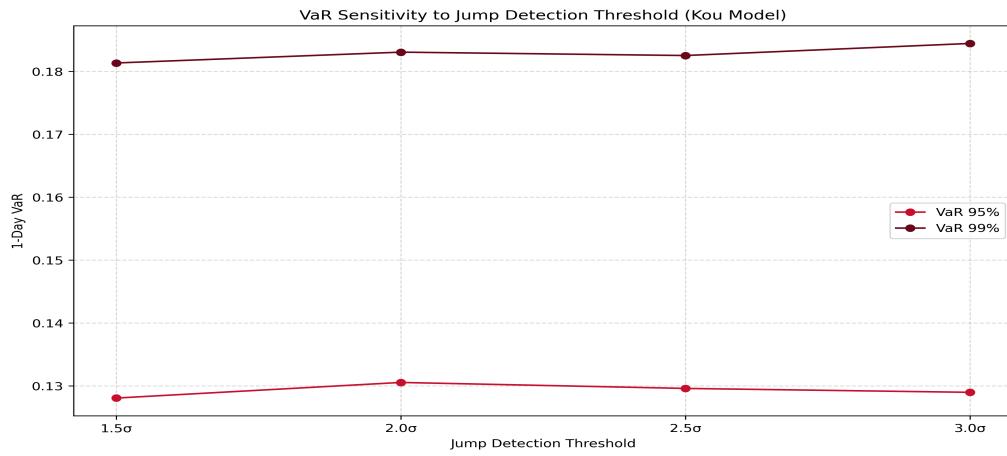


Figure 24: Sensitivity of 1-day Value at Risk (VaR) at the 95% and 99% confidence levels to the jump detection threshold in the Kou jump-diffusion model.

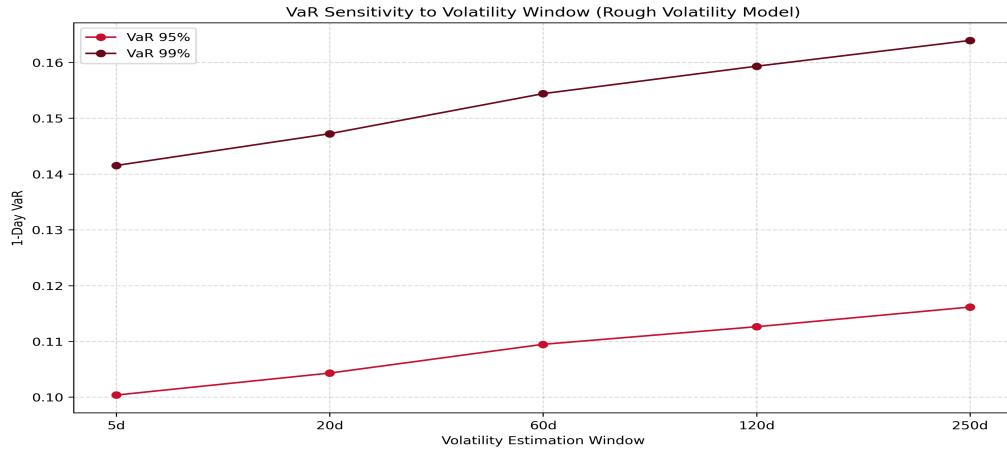


Figure 25: Sensitivity of 1-day Value at Risk (VaR) at the 95% and 99% confidence levels to the volatility estimation window under the rough volatility model.

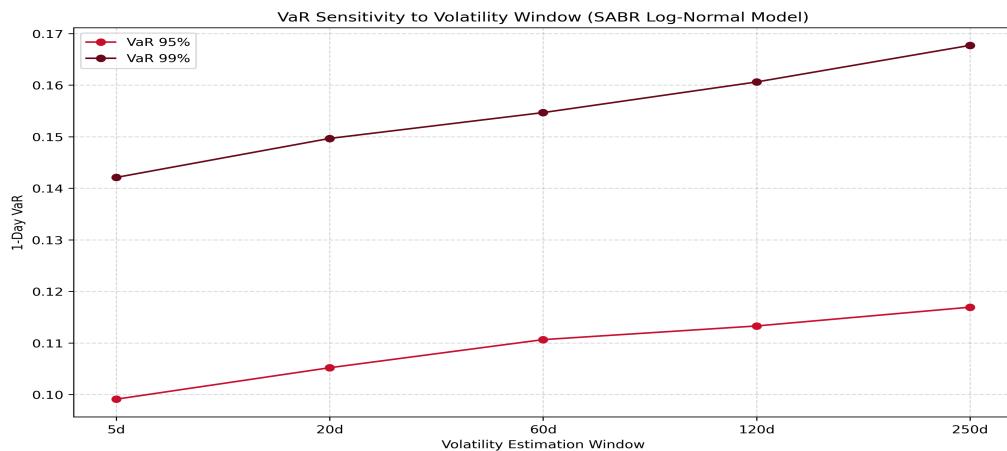


Figure 26: Sensitivity of 1-day Value at Risk (VaR) at the 95% and 99% confidence levels to the volatility estimation window under the SABR log-normal model.

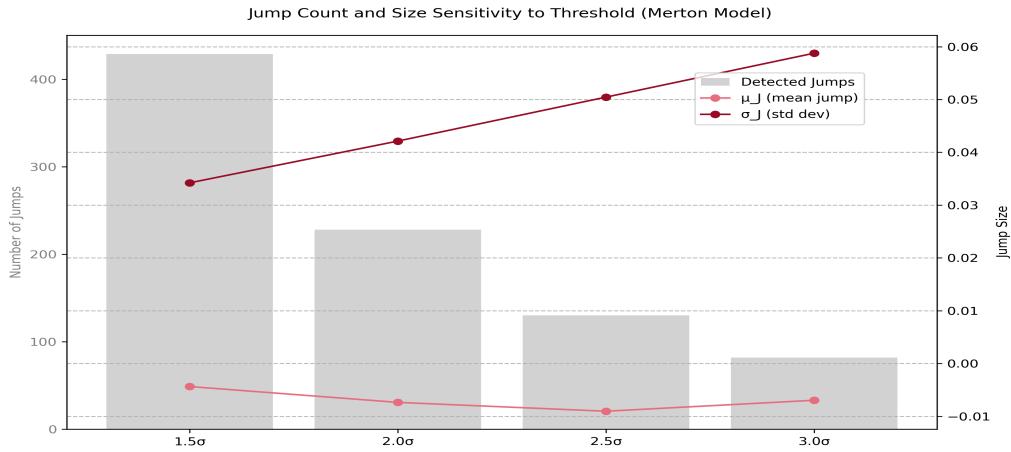


Figure 27: Sensitivity of estimated jump parameters in the Merton model to the detection threshold.

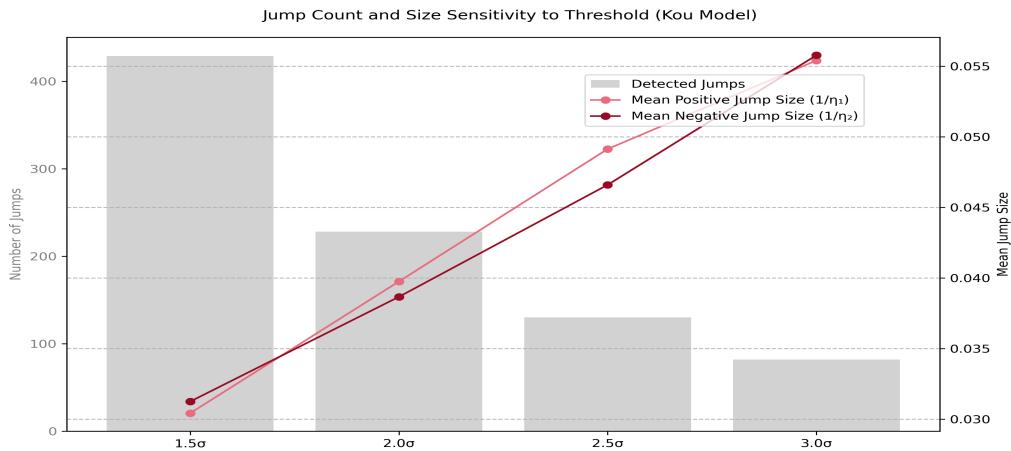


Figure 28: Sensitivity of estimated jump parameters in the Kou model to the detection threshold.

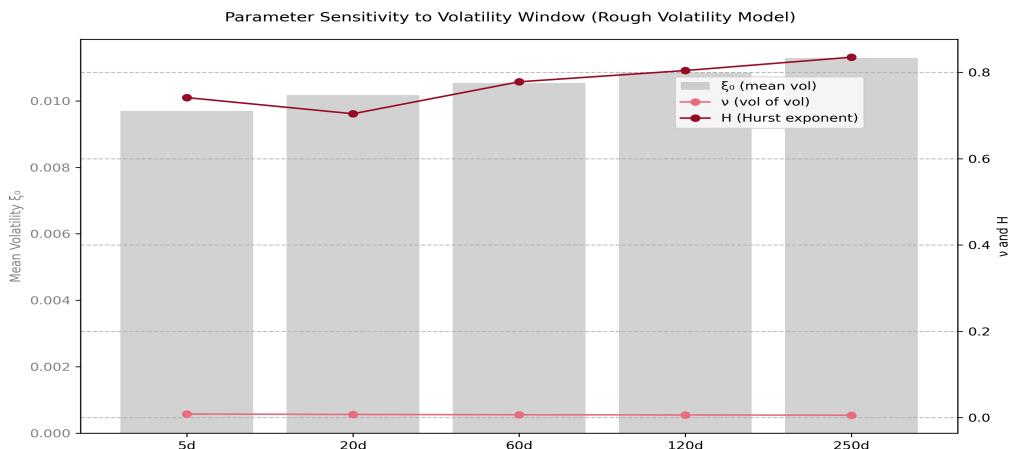


Figure 29: Sensitivity of rough volatility model parameters to the volatility estimation window.

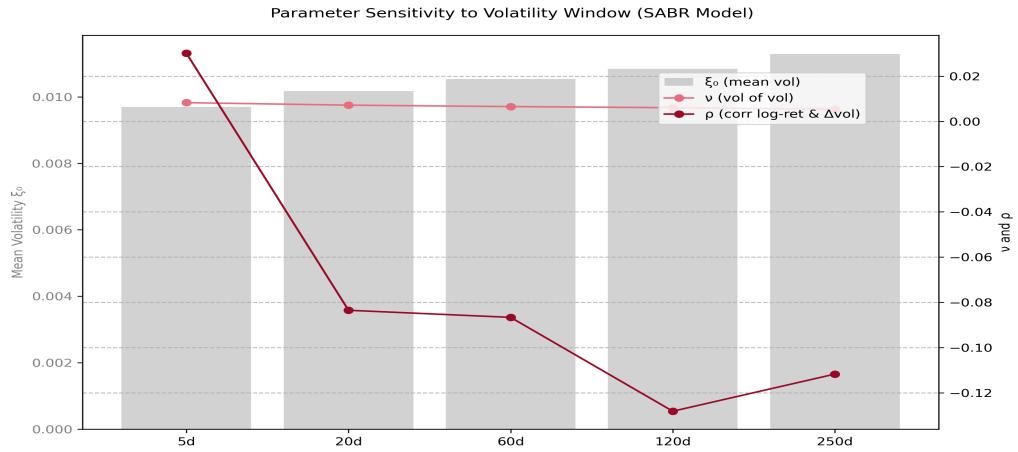


Figure 30: Sensitivity of SABR model parameters to the volatility estimation window.

Appendix C

Tables

Parameter	Value	Description
μ	0.0003	Mean of returns
σ	0.0124	Standard deviation of returns
VaR 95%	0.1270	1-day Value at Risk (95% confidence)
VaR 99%	0.1834	1-day Value at Risk (99% confidence)

Table 3: Estimated parameters and simulated 1-day Value at Risk using Historical Volatility.

Parameter	Value	Description
λ	0.0269	Jumps per day
μ_J	-0.0090	Average jump size
σ_J	0.0505	Std. dev. of jumps
Detected jumps	130 / 4834	Total jumps in dataset
VaR 95%	0.1291	1-day Value at Risk (95% confidence)
VaR 99%	0.1810	1-day Value at Risk (99% confidence)

Table 4: Estimated parameters and simulated 1-day VaR using the Merton Jump Diffusion model.

Parameter	Value	Description
p	0.3923	Probability of positive jump
η_1	20.35	Positive jump intensity
η_2	21.46	Negative jump intensity
VaR 95%	0.1286	1-day Value at Risk (95% confidence)
VaR 99%	0.1839	1-day Value at Risk (99% confidence)

Table 5: Estimated parameters and simulated 1-day VaR using the Kou Jump Diffusion model.

Parameter	Value	Description
μ	0.0003	Annualized drift
ξ_0	0.0102	Mean volatility
ν	0.0072	Volatility of volatility
H	0.7041	Hurst exponent
VaR 95%	0.1050	1-day Value at Risk (95% confidence)
VaR 99%	0.1470	1-day Value at Risk (99% confidence)

Table 6: Estimated parameters and simulated 1-day VaR using the Rough Volatility model.

Parameter	Value	Description
μ	0.0003	Annualized drift
ξ_0	0.0102	Mean volatility
ν	0.0072	Volatility of volatility
β	1	Elasticity parameter
ρ	-0.0835	Correlation between asset and volatility
VaR 95%	0.1044	1-day Value at Risk (95% confidence)
VaR 99%	0.1495	1-day Value at Risk (99% confidence)

Table 7: Estimated parameters and simulated 1-day VaR using the SABR model.

Model	VaR 95%	VaR 99%	Kurtosis
Historical Volatility	0.1270	0.1834	3.01
Merton Jump Diffusion	0.1291	0.1810	362.93
Kou Jump Diffusion	0.1286	0.1839	228.23
Rough Volatility	0.1050	0.1470	3.04
SABR Model	0.1044	0.1495	2.96

Table 8: Comparison of 1-day VaR (95% and 99%) and kurtosis across different volatility models.

Rolling Window (days)	σ (Realized Volatility)	VaR 95% (Hist.)	VaR 99% (Hist.)
5	0.0150	0.1533	0.2152
20	0.0304	0.3227	0.4480
60	0.0193	0.1961	0.2733
120	0.0149	0.1563	0.2180
250	0.0120	0.1216	0.1687

Table 9: Sensitivity analysis of the parametric volatility model to the rolling window size used for realized volatility estimation.

Threshold	Detected Jumps	λ (jumps/-day)	μ_J	σ_J	VaR 95%	VaR 99%
1.5σ	429	0.0887	-0.0044	0.0342	0.1277	0.1796
2.0σ	228	0.0472	-0.0074	0.0421	0.1278	0.1814
2.5σ	130	0.0269	-0.0090	0.0505	0.1286	0.1820
3.0σ	82	0.0170	-0.0070	0.0588	0.1281	0.1824

Table 10: Sensitivity analysis of the Merton Jump Diffusion model to jump detection thresholds.

Threshold	Detected Jumps	λ (jump-s/day)	p	η_1 (pos. intensity)	η_2 (neg. intensity)	VaR 95%	VaR 99%
1.5σ	429	0.0887	0.4359	32.88	32.01	0.1281	0.1813
2.0σ	228	0.0472	0.3991	25.16	25.87	0.1306	0.1831
2.5σ	130	0.0269	0.3923	20.35	21.46	0.1296	0.1825
3.0σ	82	0.0170	0.4390	18.05	17.93	0.1290	0.1844

Table 11: Sensitivity analysis of the Kou Jump Diffusion model to the threshold used for jump detection.

Volatility Window	ξ_0 (mean vol)	ν (vol of vol)	H	VaR 95%	VaR 99%
5d	0.0097	0.0083	0.7415	0.1004	0.1415
20d	0.0102	0.0072	0.7041	0.1043	0.1472
60d	0.0105	0.0065	0.7781	0.1095	0.1544
120d	0.0108	0.0061	0.8042	0.1126	0.1593
250d	0.0113	0.0054	0.8352	0.1161	0.1639

Table 12: Sensitivity analysis of the Rough Volatility model to the window size used for volatility estimation.

Volatility Window	ξ_0 (mean vol)	ν (vol of vol)	ρ	VaR 95%	VaR 99%
5d	0.0097	0.0083	0.0302	0.0991	0.1421
20d	0.0102	0.0072	-0.0835	0.1052	0.1497
60d	0.0105	0.0065	-0.0867	0.1106	0.1547
120d	0.0108	0.0061	-0.1282	0.1133	0.1606
250d	0.0113	0.0054	-0.1117	0.1169	0.1677

Table 13: Sensitivity analysis of the SABR model to the window size used for volatility estimation.

Model	Sensitivity	VaR 95%	VaR 99%	Kurtosis	Parameters
Realized Volatility	5	0.1533	0.2152	3.01	$\sigma = 0.015$
	20	0.3227	0.4480	3.01	$\sigma = 0.0304$
	60	0.1961	0.2733	3.01	$\sigma = 0.0193$
	120	0.1563	0.2180	3.01	$\sigma = 0.0149$
	250	0.1216	0.1687	3.01	$\sigma = 0.012$
Merton Jump	1.5σ	0.1277	0.1796	1,575.63	$\lambda = 0.0887, \mu_J = -0.0044, \sigma_J = 0.0342$
	2.0σ	0.1278	0.1814	1,575.63	$\lambda = 0.0472, \mu_J = -0.0074, \sigma_J = 0.0421$
	2.5σ	0.1286	0.1820	1,575.63	$\lambda = 0.0269, \mu_J = -0.0090, \sigma_J = 0.0505$
	3.0σ	0.1281	0.1824	1,575.63	$\lambda = 0.0170, \mu_J = -0.0070, \sigma_J = 0.0588$
Kou Jump	1.5σ	0.1281	0.1813	16,711.39	$\lambda = 0.0887, p = 0.4359, \eta_1 = 32.88, \eta_2 = 32.01$
	2.0σ	0.1306	0.1831	16,711.39	$\lambda = 0.0472, p = 0.3991, \eta_1 = 25.16, \eta_2 = 25.87$
	2.5σ	0.1296	0.1825	16,711.39	$\lambda = 0.0269, p = 0.3923, \eta_1 = 20.35, \eta_2 = 21.46$
	3.0σ	0.1290	0.1844	16,711.39	$\lambda = 0.0170, p = 0.4390, \eta_1 = 18.05, \eta_2 = 17.93$
Rough Volatility	5d	0.1004	0.1415	2.99	$\xi_0 = 0.0097, \nu = 0.0083, H = 0.7415$
	20d	0.1043	0.1472	2.99	$\xi_0 = 0.0102, \nu = 0.0072, H = 0.7041$
	60d	0.1095	0.1544	2.99	$\xi_0 = 0.0105, \nu = 0.0065, H = 0.7781$
	120d	0.1126	0.1593	2.99	$\xi_0 = 0.0108, \nu = 0.0061, H = 0.8042$
	250d	0.1161	0.1639	2.99	$\xi_0 = 0.0113, \nu = 0.0054, H = 0.8352$
SABR	5d	0.0991	0.1421	3.05	$\xi_0 = 0.0097, \nu = 0.0083, \rho = 0.0302$
	20d	0.1052	0.1497	3.05	$\xi_0 = 0.0102, \nu = 0.0072, \rho = -0.0835$
	60d	0.1106	0.1547	3.05	$\xi_0 = 0.0105, \nu = 0.0065, \rho = -0.0867$
	120d	0.1133	0.1606	3.05	$\xi_0 = 0.0108, \nu = 0.0061, \rho = -0.1282$
	250d	0.1169	0.1677	3.05	$\xi_0 = 0.0113, \nu = 0.0054, \rho = -0.1117$

Table 14: Sensitivity analysis across all volatility models with varying calibration inputs.

Appendix D

Mathematical Derivations

We start from the BSM with an added “Jump” component:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \text{Jump}$$

This jump component has two sources of randomness: the *arrival*, which is modeled like a Poisson process, and the *magnitude*, which is assumed to follow a log-normal distribution.

The magnitude of a jump $\ln(Y_t) \sim N(\mu, \sigma^2)$ can be thought of as a “scaling factor.”

Right after a jump, the stock price moves to:

$$S_t \rightarrow Y_t S_t,$$

where if $0 < Y_t < 1$, it will be a *negative* jump, and otherwise it will be a *positive* jump

Hence,

$$dS_t = Y_t S_t - S_t \rightarrow \frac{dS_t}{S_t} = Y_t - 1.$$

We can add this to the BS dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (Y_t - 1).$$

Note: for the arrival of the jumps, we assume they follow a Poisson process with parameter λ :

$$dN_t = \begin{cases} 1, & \text{with probability } \lambda dt, \\ 0, & \text{with probability } 1 - \lambda dt. \end{cases}$$

Thus we can update the dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (Y_t - 1) dN_t.$$

We want to know what the “expected” jump will be at any given time, to absorb it into a drift-correction parameter. Note that

$$\mathbb{E}[(Y_t - 1) dN_t] = \mathbb{E}[Y_t - 1] \mathbb{E}[dN_t] = k \lambda dt,$$

where $k = \mathbb{E}[Y_t - 1]$. Thus the jump process introduces an extra “drift,” and we subtract it off for a martingale adjustment:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (Y_t - 1) dN_t - \lambda k dt.$$

We must also account for the possibility of multiple jumps at the same instant. If $dN_t = 3$, say, with jump-sizes Y_1, Y_2, Y_3 , then

$$S_t \longrightarrow S_t Y_1 Y_2 Y_3 = S_t \prod_{j=1}^3 Y_j,$$

and in general

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \left(\prod_{j=1}^{dN_t} Y_j - 1 \right) - \lambda k dt.$$

Putting all terms together in the usual SDE form we obtain our final equation,

$$dS_t = S_t(\mu - \lambda k) dt + \sigma S_t dW_t + S_t \left(\prod_{j=1}^{dN_t} Y_j - 1 \right)$$