Variational approximations of size-mass energies for k-dimensional currents

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Workshop in Calculus of Variations
Paris Diderot 25-26-27 June 2018

Model for vector measures

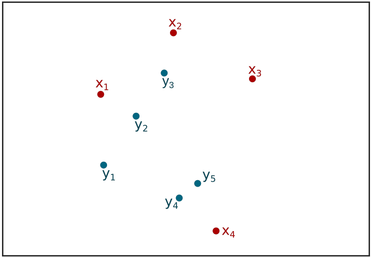
$\omega_+, \omega_- \in \mathcal{P}(\mathbf{R}^n)$ atomic probability measures

- *'Source' measure* $\omega_+ = \sum_i a_i \delta_{x_i}$,
- *'Sink' measure* $\omega_- = \sum_j b_j \delta_{x_j}$.

Polyhedral transport flux

$\sigma \in \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n)$ vector measure

$$\sigma = \sum_i \theta_i \tau_i \mathcal{H}^1 \llcorner \Sigma_i$$



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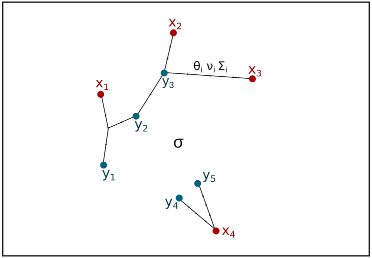

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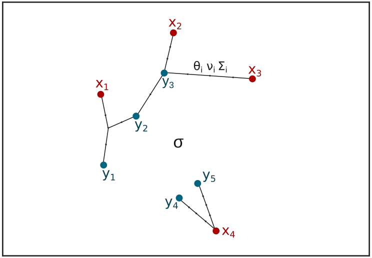
Constraint

$$\operatorname{div} \sigma = \omega_+ - \omega_- \quad \text{in } \mathcal{D}'(\mathbf{R}^n)$$



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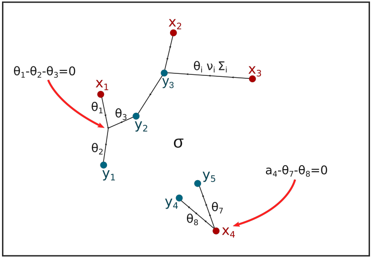

Transport cost function

Cost function

A function $h : \mathbf{R} \to \mathbf{R}^+$ is a *transport cost function* if it is:

1. lower semicontinuous,
2. $h(0) = 0$,
3. sub-additive,
4. even.

Examples:

- Branched transport:

$$h(\theta) = |\theta|^\alpha \quad \text{with } \alpha \in [0, 1)$$

- 'Steiner' cost:

$$h(\theta) = \begin{cases} 0, & \text{if } \theta = 0, \\ 1, & \text{otherwise.} \end{cases}$$

- Urban Planning cost:

$$h(\theta) = \min\{\alpha_0|\theta|, \alpha_1|\theta| + a\}$$
$$a > 0 \text{ and } 0 < \alpha_1 < \alpha_0$$

- Affine cost functional:

$$h(\theta) = \begin{cases} 0, & \text{if } \theta = 0, \\ (1 + \alpha|\theta|), & \text{otherwise.} \end{cases}$$

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For a function h and polyhedral transport flux we set

$$\mathcal{F}(\sigma) = \sum_i h(\theta_i)\mathcal{H}^1(\Sigma_i) \qquad \text{if } \sigma = \sum_i \theta_i \tau_i \mathcal{H}^1 \llcorner \Sigma_i. \tag{1}$$

[White (2000)] If h is a transport cost function then $\mathcal{F}$ extends on $X := \{\sigma \in \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n) \,:\, \operatorname{div}\sigma = \omega_+ - \omega_-\}$ via

Relaxation

$$\overline{\mathcal{F}}(\sigma) := \inf\left\{\liminf_{j\to+\infty} \mathcal{F}(\sigma_j) : \sigma_j \in X \text{ of the form (1) and } \sigma_j \overset{*}{\rightharpoonup} \sigma\right\}.$$

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Model for vector measures

Rectifiable σ

We will say that σ is rectifiable if $\sigma = (\theta, \tau, \Sigma)$ with

1. Σ is $\mathcal{H}^1$-rectifiable,
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[Xia (1997)] in the case $h = |\cdot|^\alpha$ ($\alpha \in [0, 1)$), [Brancolini-Wirth (2017)] for general transport cost functions.

Generalized Gilbert-Steiner energy

If $\sigma \in X$ and writes as $\sigma = \sigma^\perp + (\theta, \tau, \Sigma)$ then the relaxation writes as

$$\overline{\mathcal{F}}(\sigma) = h'(0)|\sigma^\perp| + \int_\Sigma h(|\theta|)\, \mathrm{d}\mathcal{H}^1 \quad \text{where} \quad h'(0) = \lim_{t \to 0_+} \frac{h(t)}{t}.$$

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Problem

'Branched transportation' type

For two measures $\omega_+, \omega_- \in \mathcal{P}(\Omega)$ approximate:

$$\overline{\sigma} := \text{argmin}\left\{\overline{\mathcal{F}}(\sigma) : \sigma \in \mathcal{M}(\Omega, \mathbf{R}^n) \text{ and } \operatorname{div}\sigma = \omega_+ - \omega_+\right\}$$

.

Motivation: Solutions to this problems are computationally hard to find. For instance: the case

$$h(\theta) = \begin{cases} 0, & \text{if } \theta = 0, \\ 1, & \text{otherwise.} \end{cases} \qquad \omega_+ = \delta_{x_0}, \quad \omega_- = \frac{1}{N}\sum_1^N \delta_{y_i}$$

corresponds to the Steiner Tree Problem:

find $\overline{K} = \text{argmin}\{\mathcal{H}^1(K) : K \text{ compact, connected and } \{x_0, \ldots, x_N\} \subset K \subset \mathbf{R}^n\}$

Is NP-hard [Karp (1972)], [Leal do Forte et al. (2016)].

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Phase-field approach

[Modica, Mortola (1977)], [Ambrosio, Tortorelli (1990)], [Bonnivard, Lemenant, Santambrogio (2015)]

- Let $\varepsilon \in (0,1]$ and ρ_ε be a convolution kernel
- Relaxed vector measure:

$$\sigma \in L^1(\Omega, \mathbf{R}^n) \quad \text{and} \quad \operatorname{div}\sigma = (\omega_+ - \omega_-) * \rho_\varepsilon \qquad \text{in } \mathcal{D}'(\mathbf{R}^n)$$

- Phase-field:

$$\varphi \in W^{1,2}(\Omega), \quad \varphi \geq \eta \quad \text{and} \quad \varphi \llcorner \partial\Omega \equiv 1$$

Set

$$\mathcal{F}_{\varepsilon,a}(\sigma,\varphi) := \underbrace{\int_\Omega \frac{\varphi|\sigma|^2}{2\,\varepsilon}\,\mathrm{d}x}_{\text{'Constraint'}} + \underbrace{\int_\Omega \varepsilon^{p-n+1}|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{\varepsilon^{n-1}}\,\mathrm{d}x}_{\text{'Rescaled Modica-Mortola'}}.$$

[Chambolle, F, Merlet (2017)]

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Results

Theorem [Chambolle, F., Merlet]

If $\eta = a\varepsilon$ with $a > 0$ then $\{\mathcal{F}_{\varepsilon,a}\}$ is equicoercive with respect to the weak-$*$ convergence of measures and the strong L^2 convergence.

Theorem: $\Gamma - \liminf$ [Chambolle, F., Merlet]

$$\Gamma - \lim_{\varepsilon \downarrow 0} \mathcal{F}_{\varepsilon,a} = \mathcal{F}_a(\sigma, \varphi)$$

with respect to the product topology. Where

$$\mathcal{F}_a(\sigma,\varphi) := \begin{cases} \displaystyle\int_\Sigma h_a(\theta)\, \mathrm{d}\mathcal{H}^1, & \text{if } \sigma = \theta\tau\mathcal{H}^1 \llcorner \Sigma \text{ and } \varphi \equiv 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

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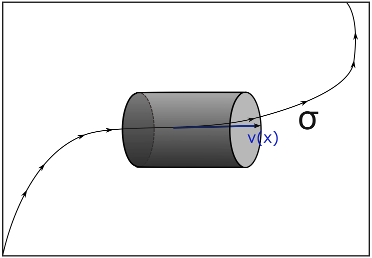
Reduced dimension problem and cost function h_a

We introduce m, the flux of σ trough an $n-1$ dimensional hyperplane and study the energy.



For a ball in $B_r(0) \subset \mathbf{R}^d$ we let

$$E_{\varepsilon,a}(\vartheta, u; B_r) := \int_{B_r} \left[\varepsilon^{p-n+1} |\nabla \varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-1}} + \frac{\varphi |m|^2}{\varepsilon} \right] \mathrm{d}x$$

$$h_{\varepsilon,a}(\theta, r) = \min \begin{cases} E_{\varepsilon,a}(\vartheta, \varphi; B_r) \\ \varphi \in W^{1,p}(B_r), \varphi \llcorner \partial \mathcal{B}_r = 1 \text{ and } \int_{B_r} \vartheta = \theta. \end{cases}$$

Theorem ([Chambolle, F., Merlet])

For every r

$$\lim_{\varepsilon \downarrow 0} h_{\varepsilon,a}(\theta, r) = h_a(\theta),$$

is a transport cost function and for $m \neq 0$ we have the following semi-explicit formulation

$$h_a(m) = \min_{\hat{r}>0} \left\{ \frac{a\, m^2}{\omega_{n-1}\ \hat{r}^{n-1}} + \omega_{n-1}\ \hat{r}^{n-1} + (n-2)\, \omega_{n-1}\ q^{n-1}(\hat{r}) \right\}$$

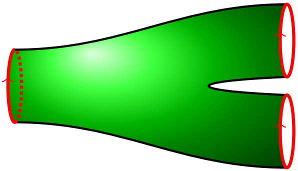
Furthermore $\lim_{a \downarrow 0} h_a^d = c(n,p)$.

In the above ω_{n-1} is the volume of unit $(n-1)$-dimensional ball. And $q^{n-1}(\hat{r})$ is the transition cost

$$q^{n-1}(\hat{r}) := \left\{ \int_{\hat{r}}^{\infty} t^{n-2} \left[|v'|^p + (1-v)^2 \right] \mathrm{d}t \ :\ v(\hat{r}) = 0 \text{ and } \lim_{t\to\infty} v(t) = 1 \right\}$$

Generalization to surfaces

- For $0 < k < n$, we consider the space $D_k(\Omega) = \mathcal{D}^k(\Omega)'$,
- $\sigma = \sum_i \theta_i \tau_i \mathcal{H}^k \llcorner \Sigma_i$ where Σ_i is a k-dimensional polyhedra and τ_i is a k-multivector orienting it,
- the constraint is modeled via the boundary operator ∂,
- $\mathcal{F}(\sigma) = \sum_i h(\theta) \mathcal{H}^k(\Sigma_i)$
- A k-current $\sigma \in \mathcal{D}_k(\Omega)$ is rectifiable, if $\sigma = (\theta, \tau, \Sigma)$ where
  - Σ is an $\mathcal{H}^k$-rectifiable set,
  - τ is a multivector orienting the tangent plane to Σ $\mathcal{H}^k \llcorner \Sigma$-a.e.,
  - $\theta \in L^1(\mathcal{H}^k \llcorner \Sigma)$.



Generalization to surfaces

With the same relaxation procedure proposed above we can define a functional on the space of currents $\sigma \in \mathcal{D}_k(\Omega)$

Theorem ([Colombo et al. (2017)])

If $\sigma \in \mathcal{D}_k(\Omega)$ and it is rectifiable, then

$$\overline{\mathcal{F}}(\sigma) = \int_\Sigma h(\theta)\, \mathrm{d}\mathcal{H}^k.$$

And

$$\overline{\mathcal{F}}(\sigma) < \infty \implies \sigma \text{ rectifiable } \iff h'(0) < \infty.$$

[White (2000)], [Morgan (1989)], [De pauw Hardt (2003)] size functional.

Plateau type problems

Given a polyhedral normal chain $\sigma_0 \in \mathcal{D}_k(\Omega)$ approach

$$\overline{\sigma} := \text{argmin}\left\{\overline{\mathcal{F}}(\sigma) : \sigma \in \mathcal{D}_k(\Omega) \text{ and } \partial\sigma = \partial\sigma_0\right\}.$$

Consider the functional

$$\mathcal{F}^k_{\varepsilon,a}(\sigma,\varphi) := \underbrace{\int_\Omega \frac{\varphi|\sigma|^2}{2\,\varepsilon}\,\mathrm{d}x}_{\text{'Constraint'}} + \underbrace{\int_\Omega \varepsilon^{p-n+k}|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{\varepsilon^{n-k}}\,\mathrm{d}x}_{\text{'Rescaled Modica-Mortola'}}.$$

Where σ is a smoothed k-current:

$$\sigma \in \mathcal{D}_k(\Omega), \quad \partial\sigma = \partial\sigma_0 * \rho_\varepsilon \quad \sigma \ll \mathcal{L}^1$$

φ is a phase-field function:

$$\varphi \in W^{1,2}(\Omega), \quad \varphi \geq \eta \quad \text{and} \quad \varphi \llcorner \partial\Omega \equiv 1$$

Ghilardin (2014)

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Theorem [Chambolle, F., Merlet]

If $\eta = a\varepsilon$ with $a > 0$ then $\{\mathcal{F}^k_{\varepsilon,a}\}$ is equicoercive with respect to the Flat norm for currents and the strong L^2 convergence.

Theorem: $\Gamma - \liminf$ [Chambolle, F., Merlet]

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with respect to the product topology. Where

$$\mathcal{F}^k_a(\sigma, \varphi) := \begin{cases} \displaystyle\int_\Sigma h_a^{n-k}(\theta)\, \mathrm{d}\mathcal{H}^k, & \text{if } \sigma = \theta\tau\mathcal{H}^1 \llcorner \Sigma \text{ and } \varphi \equiv 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

Results

Theorem ([Chambolle, F., Merlet])

For $a = 0$ we have

$$h_0^d(\theta) = \begin{cases} c, & \text{if } \theta \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

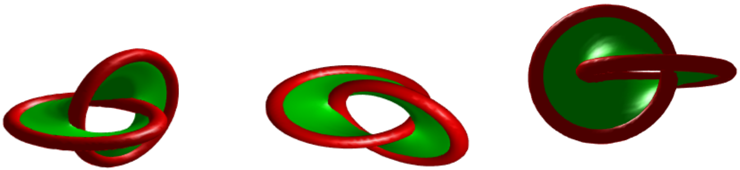
for some $c = c(n, k, p) > 0$. Furthermore

$$\Gamma \lim_{a \downarrow 0} \mathcal{F}_a^k = \mathcal{F}_0^k := \begin{cases} c\mathcal{H}^k(\Sigma), & \text{if } \varphi = 1 \text{ a.e. } \sigma = \theta\tau\mathcal{H}^k \llcorner \Sigma \partial\sigma = \partial\sigma_0, \\ +\infty & \text{otherwise.} \end{cases}$$

Numerics in collaboration with Elie Bretin

Remarks

In the case $n = 3$ and $k = 2$ we may identify the space $\mathcal{D}_2(\Omega)$ with $\mathcal{M}(\Omega, \mathbf{R}^3)$ and the boundary operator corresponds to the curl operator as a distribution.



- *A phase-field approximation of the Steiner problem in dimension two* (published in **Advances in Calculus of Variations**),
- *Variational approximations of size-mass energies for k-dimensional currents* (to appear in **ESAIM: cocv**).

Thank you for your attention!