université
PARIS-SACLAY

NNT : 2018SACLX049

ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

Approximations par champs de phases pour des problèmes de transport branché

Thèse de doctorat de l'Université Paris-Saclay
préparée à l'Ecole Polytechnique

Ecole doctorale n°573 : Interfaces (approches interdisciplinaires/fondements, applications et innovation)
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 5/10/2018, par

LUCA ALBERTO DAVIDE FERRARI

Composition du Jury :

Filippo Santambrogio Professeur, Université Paris Sud	Président
Ilaria Fragalá Professeur, Politecnico di Milano	Rapporteur
Antoine Lemenant Maître de Conférences, Université Paris Diderot	Rapporteur
Antonin Chambolle Directeur de recherche, École Polytechnique	Directeur de thèse
Benoit Merlet Professeur, Université Lille 1	Co-directeur de thèse
Lucia Scardia Professeur, University of Bath	Examinateur
Blanche Buet Maître de Conférences, Université Paris Sud	Examinateur
Edouard Oudet Professeur, Université Grenoble Alpes	Examinateur

Thèse de doctorat

A Chiara

Remerciements

First of all, I would like to express my deep gratitude to my two supervisors, Antonin Chambolle and Benoit Merlet, for their guidance. I will always be grateful for the suggestions on the subjects and I am grateful to Prof. Chambolle for his inspirations. I will always remember the first time he introduced to me the theory of Gamma convergence and the subject of branched transportation. Seeing him work at a blackboard is something really fascinating.

I am grateful to Prof. Merlet: he introduced me to the complex and fascinating world of research. I learnt from him how to approach a problem, how to write a paper and how to present my results. He was always so kind to go through my writings and suggest how to present my works in a better way and he was always pleased to get his hands dirty and discuss on the most technical difficulties of a proof.

I owe my heartfelt gratitude to Benedikt Wirth e Carolin Rossmanith for their welcome in Munster and for the passionate exchange of ideas that lead to a very nice work written together.

I would like to thank Prof. Lemenant and Prof. Fragala for agreeing to review the current thesis and their suggestions and corrections. I am also grateful to Filippo Santambrogio, Edouard Oudet, Blanche Bouet and Lucia Scardia for accepting to be examinateurs at my Ph.D. defense.

Let me thank all the collegues, Adi, Marco, Valentina, Belhal, Remi, Genevieve, Simona, Matteo, Luca, Vito and Fred at CMAP that have accompanied me during these years of research.

Grazie alle mie due famiglie adottive a Parigi Tommaso, Marta e Gregorio e Martino, Federica e Mattia per tutte quelle volte che vi siete sentiti dire che non avrei mai finito questa tesi ed invece eccoci qui a discuterla.

Un grazie immenso a tutti i miei compagni d'appa passati e presenti Francesco, Tommaso, Matteo, Emanuele, Martino, Giacomo, Pietro, Matteo, Filippo e Pietro. Perch tornare a casa e trovare degli amici con cui si condivide la vita una cosa grandiosa che mi mancher, come mi mancheranno i cartelli stradali appesi a Place d'It e le poltrone trovate per strada.

Un grazie gigante a tutti i miei amici parigini a quelli che son venuti e poi partiti e a quelli che son ancora qui. Hlose, Marta, Lucia, Caterina, Clelia, Paince, Roby, Square, Otto, Laura, Maria, Daniela, Leo, Andrea, Cecilia, Elena, Maria, Pierluca, Sara, Pietro, Chiara, Riccardo, Martino.

Grazie a tutti i miei amici "italiani" Maddalena, Pietro, Elisa, Silvia, Francesca, Elisa che pur da lontano mi hanno fatto compagnia.

Un grazie di cuore alla mia famiglia in particolare mia madre e mio padre per tutto il loro sostegno e per la passione al lavoro che mi hanno insegnato semplicemente mostrandomi il loro modo di lavorare.

L'ultima immensa gratitudine verso Chiara, che in questi tre anni e mezzo di distanza ha comunque trovato il modo di essere sempre presente nel mio cuore e accompagnarmi nella mia strada anche se distante da lei.

Contents

Introduction

When designing a supply-demand distribution network it is convenient to give it a tree structure in which it is preferable to regroup mass in the transportation process. This assumption emerges from numerous observations, for instance the structure of the blood vessels in the cardiovascular system is required to distribute blood from a concentrated source in the heart to a widespread volume or vice-versa, the root system of a tree needs to recollect water from the soil. In these situations we may observe how broad and long vessels are preferable rather than thin spread out ones. The assumption we make is that the actual observed network is optimal with respect to some given cost among all possible networks developing from a source and irrigating a given sink. These structures appear in a wide range of situations (figure 1) and many efforts have been made by the mathematical community in order to give a precise model able to describe all the observable features of these networks.

Figure 1: On the left: root network of a tree. On the right: angiography of an eye in which it is possible to recognize the tree structure of the network of blood vessels.

A first well known approach in the framework of graph theory was proposed by Gilbert in [PST15] where he deals with the *Steiner Minimal Tree* [AT04, PS13] problem. The latter consists in finding the graph connecting a given set of points $\{x_0, \dots, x_N\}$ with minimal total length. More formally, a Steiner minimal tree is the solution of the variational problem

$$\text{argmin}\left\{\mathcal{H}^1(K)\,:\,K \text{ compact, connected and contains } x_0, \dots, x_N\right\}, \tag{1}$$

where $\mathcal{H}^1(K)$ is the Hausdorff 1-dimensional measure of F (the length of K, if it is 1-dimensional and sufficiently smooth). As stated in Courant and Robbins [CR79] the Steiner minimal tree problem can be thought of as a naif model for the network of highways connecting a set of cities. The drawback of the model is that the local intensity of the traffic is not taken into account. Nevertheless it allows to appreciate the issues emerging from these models. As observed in the quoted paper [PST15] in a Steiner minimal tree, differently from the *Minimal Spanning Tree* [Kru56], new vertices

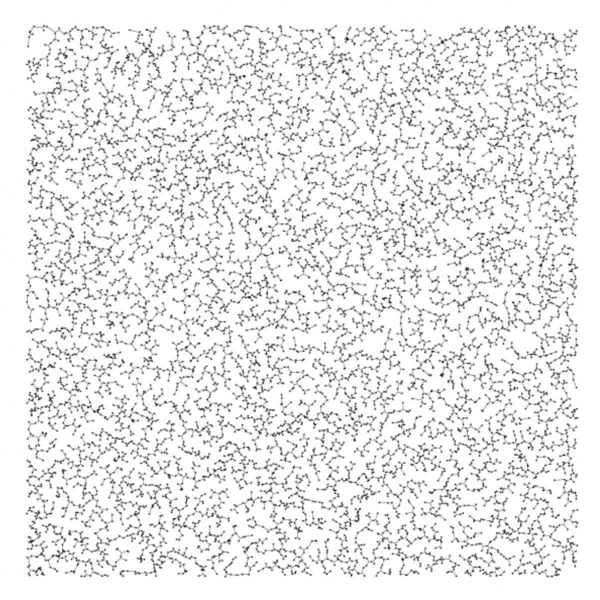


Figure 2: Steiner Minimal Tree connecting 10000 points randomly distributed in the plane. The problem was solved using the GeoSteiner algorithm [WZ97], which is currently the most efficient exact algorithm for computing minimum Steiner trees.

may be added in order to minimize the total length thus, rather than the network itself, the real unknown is its topology. An example of this situation is shown in Figure 3. This feature appears as well in other models in which the cost per unit length depends on the intensity of the traffic flux [Gil67]. In light of this high combinatorial complexity, the problem is in the list of NP-complete problems from Karp [Kar72] and it is still an active field of research even in the operational research community [FMBM16].

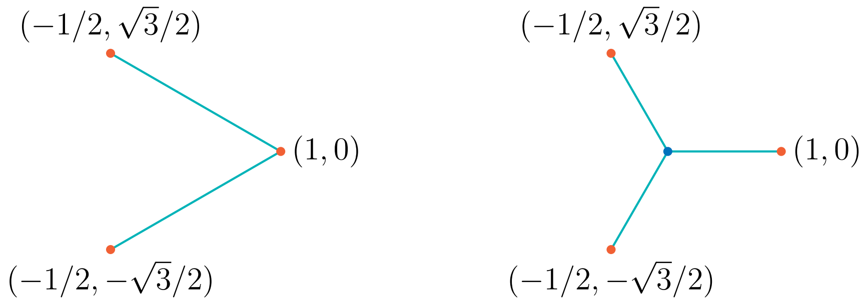


Figure 3: On the left: Minimal Spanning Tree connecting three points situated at the vertices of an equilateral triangle (length $= 2\sqrt{3}$). On the right: Steiner Minimal Tree constrained to connect the same set of points (length $= 3$). In dark blue the additional vertex which allows to decrease the total length.

The purpose of this thesis is to devise approximations of some Branched Transportation problems. Branched Transportation is a mathematical framework for modeling supply-demand distribution networks which is more general than the Steiner problem presented above. In particular the supply factories and the demand locations are modeled as measures supported on points and the network is interpreted as a vector measure, eventally the problem is cast as a constrained optimization problem. Given a function h, the transport cost of a mass m along an edge with length ℓ is $h(m)\,\ell$ and the total cost of a network is defined as the sum of the contributions of all its edges. The branched transportation case corresponds to the specific choice $h(m) = |m|^\alpha$ with $\alpha \in [0, 1)$. The sub-additivity of the cost function, $h(m_1 + m_2) \leq h(m_1) + h(m_2)$, ensures that transporting two masses jointly is cheaper than doing it separately. This formulation shares much of the numerical complexities presented above in the case of the Steiner Minimal tree problem. In this work we introduce various variational approximations by means of elliptic-type functionals to obtain more efficient numerical schemes. Eventually the proposed method is generalized to Plateau-type problems, which is a framework to model soap films spanning a given boundary. In its more general formulation the unknown of this problems is a k-dimensional surface in $\mathbf{R}^n$ spanning a $(k-1)$-dimensional boundary and minimizing a certain cost. Branched transportation corresponds to a Plateau type problem for the choice $k = 1$.

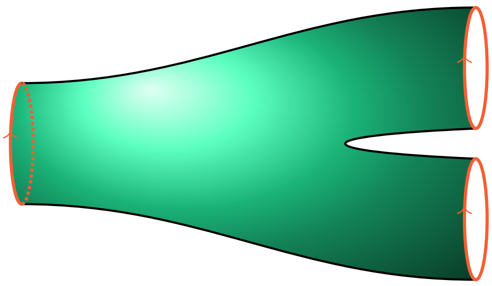


Figure 4: Example of a surface spanning a 1-dimensional boundary consisting of three oriented circles.

Description of the model

Let us introduce precisely the framework for Branched Transportation [BCM09, Vil03]. First we introduce transport networks in a open set $\Omega \in \mathbf{R}^n$, and the associated cost functional. For this purpose consider a segment $\Sigma \subset \Omega$, a positive real number $m \in \mathbf{R}_+$ and the vector $\tau \in \mathbf{S}^{n-1}$ tangent to Σ. The writing

$$m\,\tau\,\mathcal{H}^1 \llcorner \Sigma \tag{2}$$

defines a vector valued measure, where $\mathcal{H}^1 \llcorner \Sigma$ is the Hausdorff 1-dimensional measure in $\mathbf{R}^n$ restricted to the segment Σ. Intuitively, the Radon measure $\mathcal{H}^1 \llcorner \Sigma$ associates to any measurable set A the length of $A \cap \Sigma$. We say that a vector valued measure $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ is *polyhedral* if it is a finite sum of measures of the form (2), namely $\sigma = \sum_i m_i \tau_i \mathcal{H}^1 \llcorner \Sigma_i$. The action of σ on $C_0(\Omega, \mathbf{R}^n)$ is defined by the formula

$$(\sigma, \varphi) = \sum_i \int_{\Sigma_i} m_i\, \varphi \cdot \tau_i \,\mathrm{d}\mathcal{H}^1 \qquad \text{for any } \varphi \in C_0(\Omega, \mathbf{R}^n).$$

A transport cost function $h : \mathbf{R} \to [0, +\infty)$ is an application such that

$$h \text{ is } \begin{cases} \text{even, lower semicontinuous,} \\ \text{sub-additive, with } h(0) = 0. \end{cases} \tag{3}$$

Given a transport cost function h we define the *Gilbert energy* of a polyhedral vector measure σ as

$$\mathscr{E}_h(\sigma) := \sum_i h(m_i)\mathcal{H}^1(\Sigma_i).$$

We endow $\mathcal{M}(\Omega, \mathbf{R}^n)$ with its weak-$*$ topology and extend $\mathscr{E}_h$ by relaxation, namely for a vector measure $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ we let

$$\mathscr{E}_h(\sigma) := \inf \left\{ \liminf_{j \to +\infty} \mathscr{E}_h(\sigma_j) \ : \ \sigma_j \overset{*}{\rightharpoonup} \sigma \text{ and } \sigma_j \text{ polyhedral} \right\}. \tag{4}$$

By White in [Whi99a, 6] conditions (3) are sufficient in order to extend $\mathscr{E}_h$ on $\mathcal{M}(\Omega, \mathbf{R}^n)$. Choosing $h(m) = |m|$ in equation (4) we obtain the *mass functional* which associates to each vector measure σ its total variation

$$|\sigma| = \sup\{(\varphi, \sigma) \ : \ \varphi \in C_0(\Omega, \mathbf{R}^n), \ \|\varphi\|_\infty \leq 1\}.$$

Otherwise, with $h(m) = \chi_{\{m \neq 0\}}$, where χ denotes the characteristic function of a set, $\mathscr{E}_h$ reduces to the *size functional* which measures the length of the support of σ, namely $\sigma \mapsto \mathcal{H}^1(\mathrm{supp}(\sigma))$. Other remarkable choices are represented in Figure 5.

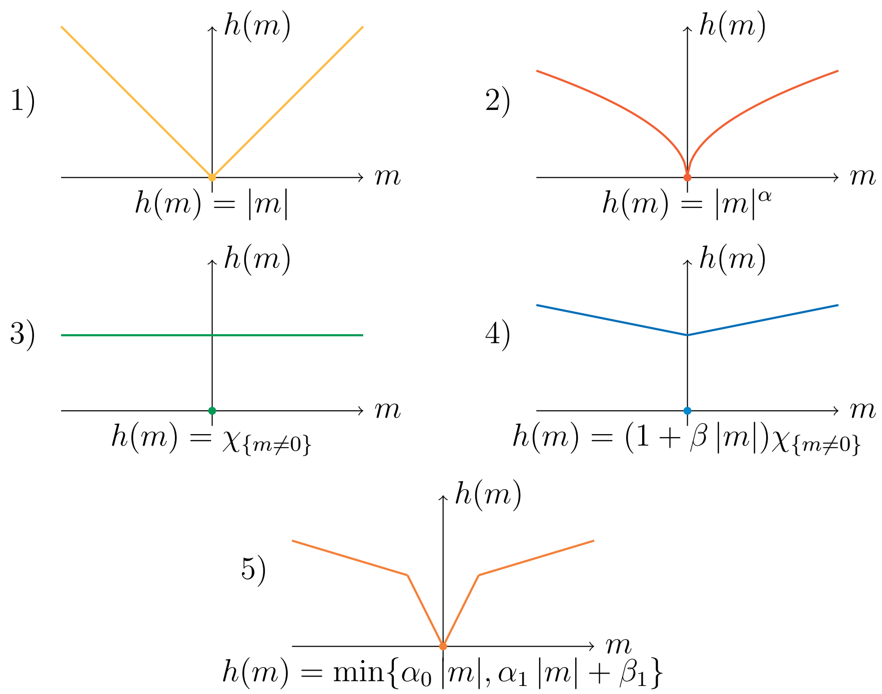


Figure 5: For h as in the graphs we obtain respectively the: 1) Mass, 2) α-Mass, 3) Size, 4) Affine cost, 5) Urban planning functional.

To model the source and the sink of the transport network we introduce two probability measures μ_+, $\mu_- \in \mathcal{P}(\Omega)$ and restrict our attention to the vector space $X^{\mu_+,\mu_-} \subset \mathcal{M}(\Omega, \mathbf{R}^n)$ consisting of those vector measures σ satisfying

$$\operatorname{div} \sigma = \mu_+ - \mu_- \tag{5}$$

in the sense of distributions. As shown in the note [CFM18] if the relaxation is obtained with respect to polyhedral measures in X^{μ_+,μ_-} we still obtain the functional (4).

Finally we are interested in approximating minimizers of the Gilbert energy under the divergence constraint (5), namely:

$$\min \left\{ \mathscr{E}_h(\sigma) : \sigma \in X^{\mu_+,\mu_-} \right\}. \tag{6}$$

The *Branched Transportation* case corresponds to the choice $h(m) = |m|^\alpha$ with $\alpha \in [0,1)$ and has been introduced by Xia who has investigated as well the problem of existence and regularity of solutions. In [Xia03] the author, taking advantage of variationals methods, proves the following

Theorem 0.1 (Existence Theorem)**.** *Given $\alpha \in (1-\frac{1}{n}, 1]$ and two probability measures μ_+, $\mu_- \in \mathcal{P}(\Omega)$, there exists a vector valued measure $\sigma \in X^{\mu_+,\mu_-}$ for which $\mathscr{E}_h(\sigma)$ is minimal. Furthermore for $\hat{\sigma} \in \operatorname{argmin} \mathscr{E}_h(\sigma)$ we have the following estimate*

$$\mathscr{E}_h(\hat{\sigma}) \leq \frac{1}{2^{1-n(1-\alpha)-1}} \frac{\sqrt{n}\,\operatorname{diam}(\Omega)}{2}.$$

In a subsequent result [Xia04, Theorem 2.7] the same author addresses the problem of regularity. To state the result we need to introduce the notion of *rectifiable vector measure*. Namely a vector measure σ is said rectifiable if

$$\sigma = m\,\tau\,\mathcal{H}^1 \llcorner \Sigma \tag{7}$$

where Σ, the support of σ as a distribution, is an $\mathcal{H}^1$-rectifiable set, its $\mathcal{H}^1$-density is the function $m \in L^1(\mathcal{H}^1 \llcorner \Sigma)$ and $\tau : \Sigma \to \mathbf{S}^{n-1}$ spans for $\mathcal{H}^1$-a.e. point in Σ the tangent space to Σ. In the following we denote with (m, τ, Σ) the rectifiable measure σ defined in (7).

Theorem 0.2 (Structure of finite energy networks)**.** *Let μ_+, $\mu_- \in \mathcal{P}(\Omega)$. For $0 \leq \alpha < 1$ if $\sigma \in X^{\mu_+,\mu_-}$ is of finite total variation and finite $\mathscr{E}_h$ energy then it is rectifiable. Furthermore if $\sigma = (m, \tau, \Sigma)$ we have*

$$\mathscr{E}_h(\sigma) = \int_\Sigma |m|^\alpha \, \mathrm{d}\mathcal{H}^1. \tag{8}$$

Equation (8) is of particular relevance since it extends the explicit representation of the functional to any rectifiable measure. The case of general vector valued measures and general transport cost functions h has been taken into consideration by Brancolini and Wirth in [BW18, Proposition 2.32] which shows that

Proposition 0.1 (Generalized Gilbert-Steiner Energy)**.** *Let $\mu_+, \mu_- \in \mathcal{P}(\Omega)$, $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ with finite total variation and such that $\operatorname{div}\sigma = \mu_+ - \mu_-$ then σ can be decomposed as*

$$\sigma = \sigma^\perp + m\,\tau\,\mathcal{H}^1 \llcorner \Sigma$$

where (m, τ, Σ) is the $\mathcal{H}^1$-rectifiable component of σ and $\sigma^\perp$ is the diffused one. Furthermore

$$\mathscr{E}_h(\sigma) = h'(0)|\sigma^\perp|(\Omega) + \int_\Sigma h(m)\,\mathrm{d}\mathcal{H}^1. \tag{9}$$

With an abuse of notation we have denoted $h'(0) = \lim_{m\downarrow 0} h(m)/m$.

Before introducing problems involving surfaces and other higher dimensional objects let us highlight the fact that the Steiner minimal tree problem connecting some points $\{x_0, \ldots, x_N\}$ may be modeled in the context of Branched transportation. Firstly, with the choice $\alpha = 0$, $\mathscr{E}_h$ reduces to the size functional. Secondly the divergence constraint forces any considered vector measure to join the support of μ_+ to the support of μ_- thus, by choosing $\mu_+ = \delta_{x_0}$ and $\mu_- = 1/N \sum_{i=1}^N \delta_{x_i}$ we force x_0 to be connected to each x_i. Gathering all together, with these choices, a minimizer σ of (6) is supported on a set connecting the points in $\{x_1, \ldots, x_N\}$ to x_0 and has support with minimal total length thus is a solution to (1).

The energy introduced above for rectifiable measures supported on 1-dimensional surfaces can be generalized to any dimension $k \in \{1, \ldots, n\}$. To this aim is necessary to introduce the concept of k-currents in $\mathbf{R}^n$. Denote with $\mathcal{D}^k(\Omega)$ the space of smooth differential forms on the open set Ω. The vector space of k-currents, $\mathcal{D}_k(\Omega)$, is the dual to $\mathcal{D}^k(\Omega)$ and it is naturally endowed with its weak-$*$ topology. We mainly follow the notation of [KP08, Fed69] the main difference being the use of σ to denote a k-current instead of a latin capital letter. For a current we can define a notion of boundary by duality as follows

$$\langle \partial\sigma, \omega\rangle = \langle \sigma,\ \mathrm{d}\omega\rangle \quad \text{for all } (k-1)\text{-differential forms } \omega.$$

We call mass of a k-current the supremum of $\langle \sigma, \omega\rangle$ among all k-differential forms with comass bounded by 1, and denote it with $|\sigma|$. In particular by the Radon-Nikodym theorem we can identify a k-current σ with finite mass with the vector valued measure $\tau\mu_\sigma$ where μ_σ is a finite positive valued measure and τ is a μ_σ-measurable map in the set of unitary k-vectors for the mass norm. The relation with vector measure is evident when we consider the fact that the vector spaces $\Lambda_1\mathbf{R}^n$, $\Lambda^1\mathbf{R}^n$ identify with $\mathbf{R}^n$. Thus any vector measure $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ with finite mass identifies with a 1-current with finite mass and vice-versa. Furthermore the divergence operator acting on measures in the sense of distributions is defined by duality as the boundary operator for currents. Thus, in analogy with what has been presented for vector measures, in equation (7), a k-current σ is said to be k-rectifiable if we can associate to it a triplet (θ, τ, Σ) such that

$$\langle \sigma, \omega\rangle = \int_\Sigma \theta\langle \omega, \tau\rangle\,\mathrm{d}\mathcal{H}^k$$

where Σ is a countably k-rectifiable subset of Ω, τ at $\mathcal{H}^k$ a.e. point is a unit simple k-vector that spans the tangent plane to Σ and θ is an $L^1(\Omega, \mathcal{H}^k \llcorner \Sigma)$ function with values in $\mathbf{R}_+$. The vector space of *Rectifiable Currents* is denoted with $R_k(\Omega)$. Among these we single out the subset $P_k(\Omega)$ of rectifiable currents for which Σ is a finite union of polyhedra and θ is constant on each of them, these will be called *Polyhedral Chains*. For any k-current σ such that both σ and $\partial\sigma$ are of finite mass we say that σ is a normal k-current and we write $\sigma \in N_k(\Omega)$. On the space $\mathcal{D}_k(\Omega)$ we can define the *flat norm* by

$$\mathbb{F}(\sigma) = \inf\left\{|\sigma_R| + |\sigma_S| \ : \ \sigma = \sigma_R + \partial\sigma_S \text{ where } \sigma_S \in \mathcal{D}_{k+1}(\Omega) \text{ and } \sigma_R \in \mathcal{D}_k(\Omega)\right\},$$

which metrizes the weak-$*$ topology on currents on compact subsets of $N_k(\Omega)$. Finally the *flat chains* $F_k(\Omega)$ consist of the closure of $P_k(\Omega)$ in the $\mathbb{F}$ topology. By the scheme of Federer [Fed69, 4.1.24] it holds

$$P_k(\Omega) \subset N_k(\Omega) \subset F_k(\Omega).$$

Following the strategy proposed by Fleming [FF60, Fle66] in the context of flat chains with coefficients in groups we now define the energy $\mathscr{E}_h$ on the space of flat chains. Let h be a transport cost function and $\sigma = \sum(m_i\tau_i, \Sigma_i)$ a polyhedral current we let

$$\mathscr{E}_h(\sigma) := \sum_i h(m_i)\mathcal{H}^k(\Sigma_i).$$

In analogy to what has been done before we extend $\mathscr{E}_h$ on the space of flat chains by relaxation. For $\sigma \in F_k(\Omega)$,

$$\mathscr{E}_h(\sigma) := \inf\left\{\liminf_{j\to\infty} \mathscr{E}_h(\sigma_j) \ : \ \sigma_j \text{ polyhedral and } \mathbb{F}(\sigma_j - \sigma) \to 0 \right\}. \tag{10}$$

In Chapter 3 we look for approximations to problems of the type

$$\min\{\mathscr{E}_h(\sigma) \ : \ \partial\sigma = \partial\sigma_0\} \tag{11}$$

where σ_0 is a given polyhedral k-current. These problems have been introduced and studied in [Mor89, DPH03] by Morgan, De Pauw and Hardt among others to propose different models for soap film minimal surfaces. The latter is the k-dimensional generalization to the minimization problem defined in (6). As sketched in [Whi99a, Whi99b] $\mathscr{E}_h$ has an explicit formulation on rectifiable currents, namely for a rectifiable current (m, τ, Σ) we have

$$\mathscr{E}_h(\sigma) := \int_\Sigma h(m)\, \mathrm{d}\mathcal{H}^k.$$

This result has been proved in [CDRMS17, Proposition 2.6], is and is the consequence of the following polyhedral approximation theorem

Theorem 0.3 (Polyhedral approximation)**.** *Let h be a transport cost function and let $\sigma = (m, \tau, \Sigma)$ be a rectifiable k-current. For every $\delta > 0$ there exists a polyhedral k-chain $\hat{\sigma} = \sum(m_i, \tau_i, \Sigma_i)$ such that*

$$\mathbb{F}(\hat{\sigma} - \sigma) \leq \delta, \quad \sum_i h(m_i)\mathcal{H}^k(\Sigma_i) \leq \int_\Sigma h(m)\, \mathrm{d}\mathcal{H}^k + \delta \quad \textit{and} \quad |\hat{\sigma}| \leq |\sigma| + \delta.$$

In addition Colombo et al. in [CDRMS17, Proposition 2.7] have shown that the condition

$$\lim_{m\downarrow 0}\frac{h(m)}{m}=+\infty$$

is equivalent to the fact

$$\mathscr{E}_h(\sigma) \text{ finite if and only if } \sigma \text{ is rectifiable.}$$

This result may be seen in correlation with equation (9) presented above. Let us highlight that the polyhedral approximation result from Colombo et al. does not take into account any boundary constraint for the k-currents. An analogous result with boundary constraint has been proved in the note [CFM18]. We conclude this section with an important sufficient condition for a flat chain to be rectifiable, proved by White in [Whi99a, Corollary 6.1].

Theorem 0.4 (Rectifiability for currents)**.** *Let $\sigma \in N_k(\Omega)$ be a normal k-current supported on a k-rectifiable set; then σ is rectifiable.*

We will take advantage of this theorem even in the context of vector measures. With the notation introduced above it reads as

Theorem 0.5 (Rectifiability for vector-valued measures)**.** *Let $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$. If $|\sigma|(\Omega) + |\nabla\cdot\sigma|(\Omega) < \infty$, $\nabla\cdot\sigma$ is at most a countable sum of Dirac masses and there exists a Borel set Σ with $\mathcal{H}^1(\Sigma) < \infty$ and $\sigma = \sigma \llcorner \Sigma$, then σ is a rectifiable vector measure.*

Variational approximation for minimization problems

We provide approximations to the problems defined in (6) in the sense of Γ-convergence. The latter is a notion of functional convergence introduced by De Giorgi [DG75] to deal with variational problems. Following [DM93, Bra98, AD00, Bra02] we give the operative definition of Γ-convergence.

Definition 1 (Γ-convergence)**.** Let X be a metric space, and for $\varepsilon > 0$ let $\mathscr{F}_\varepsilon : X \to [0, +\infty]$. We say that $\mathscr{F}_\varepsilon$ Γ-converges to $\mathscr{F} : X \to [0, +\infty]$ on X as $\varepsilon \to 0$ and we note $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{F}$ if the following two conditions hold:

(LB) $\Gamma - \liminf$ *inequality:* for any $x \in X$ and any $x_\varepsilon \to x$ it holds

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(x_\varepsilon) \geq \mathscr{F}(x),$$

(UB) $\Gamma - \limsup$ *inequality:* for any $x \in X$ there exists a sequence $(\hat{x}_\varepsilon) \subset X$ such that $\hat{x}_\varepsilon \to x$ and

$$\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\hat{x}_\varepsilon) \leq \mathscr{F}(x).$$

The sequence $(\hat{x}_\varepsilon)$ si called recovery sequence for x. Condition *(UB)* is frequently hard to prove thus it is convenient to find a subset $D \subset X$ such that: for every $x \in X$ there exists an approximating sequence $(x_n) \subset D$ such that $x_n \to x$ and $\mathscr{F}(x_n) \to \mathscr{F}(x)$. If we are able to recover D then a simple diagonal argument shows that it is enough to verify condition *(UB)* for all $x \in D$ rather than for every $x \in X$. In the context of our work the set D corresponds with the set vector space of polyhedral vector measures. Since the definition of Γ-convergence may appear cumbersome let us provide this alternative characterization that allows to appreciate its relevance in the context of the Calculus of Variations.

Theorem 0.6 (Characterization of Γ-convergence)**.** *Let X be a metric space, and for $\varepsilon > 0$ let be given $\mathscr{F}_\varepsilon : X \to [0, +\infty]$ and $\mathscr{F} : X \to [0, +\infty]$. $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{F}$ if and only if for every $\mathscr{G}$ continuous functional, if x_ε minimizes $\mathscr{F}_\varepsilon + \mathscr{G}$ and $x_\varepsilon \to x$ then x minimizes $\mathscr{F} + \mathscr{G}$.*

Our strategy is to replace the singular energy $\mathscr{E}_h$ with a sequence of smoother elliptic type functionals $\mathscr{F}_\varepsilon$ and prove that $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{E}_h$. Then we prove that the family $(\mathscr{F}_\varepsilon)$ is *equicoercive*: any sequence of minima $(\hat{x}_j)$ is precompact in X. This ensures that the sequence of minimizers $\hat{x}_\varepsilon$ converge to a minimum. Finally we look for numerical methods to approximate a minimum $\hat{x}_\varepsilon$.

Let us present three remarkable examples of Γ-convergence: Modica-Mortola, Ambrosio Tortorelli and a variation of the latter. Consider a container $\Omega \subset \mathbf{R}^3$ of unitary volume containing two immiscible liquids modeled by a binary function $\varphi : \Omega \to \{0, 1\}$ so that $\int_\Omega |\varphi| \,\mathrm{d}x = V \in (0, 1)$ represents the percentage of one liquid with respect to the container's volume. We associate to the system an energy depending on the surface tension, by supposing that it is directly proportional to the area of the interface J_φ between the liquids

$$\mathscr{M}(\varphi) = c\mathcal{H}^2(J_\varphi). \tag{12}$$

An alternative way to model this system is to assume that the transition is not given by an infinitesimal separating interface, but is rather a continuous phenomenon occurring in a thin layer of size ε. In view of this Cahn and Hilliard [CH58] consider a continuous phase function $\varphi : \Omega \to [0, 1]$ representing the pointwise mixing between the fluids and postulate an energy of the type

$$\int_\Omega \left[\varepsilon^2 |\nabla\varphi|^2 + \varphi^2 (1 - \varphi)^2\right] \,\mathrm{d}x. \tag{13}$$

The term $\varphi^2 (1 - \varphi)^2$ is called a double well potential and penalizes values far from 0 or 1; inhomogeneity is unfavoured by the gradient term. The link between (12) and (13) was discovered by Modica and Mortola in their papers [MM77a, MM77b]. Their result is more general, as a matter of fact, they prove that a suitable rescaling of the above energy Γ-converges to the perimeter functional in any domain dimension.

Theorem 0.7. *Let $\Omega \subset \mathbf{R}^n$, and let $X = BV(\Omega) \cap L^\infty(\Omega)$. For $\varphi \in X$, $V > 0$ and $\varepsilon > 0$, set*

$$\mathscr{M}_\varepsilon(\varphi) := \begin{cases} \displaystyle\int_\Omega \left[\varepsilon |\nabla\varphi|^2 + \frac{\varphi^2 (1 - \varphi)^2}{\varepsilon}\right] \,\mathrm{d}x, & \text{if } \varphi \in W^{1,2}(\Omega, [0, 1]) \text{ and } \displaystyle\int_\Omega |\varphi| \,\mathrm{d}x = V, \\ +\infty, & \text{otherwise in } X. \end{cases}$$

Let $c = 2\int_0^1 t^2(1-t)^2 \,\mathrm{d}t$ *and*

$$\mathscr{M}(\varphi) := \begin{cases} c\mathcal{H}^{n-1}(J_\varphi), & \textit{if } \varphi = \chi_A \textit{ and } |A| = V, \\ +\infty, & \textit{otherwise in } X. \end{cases}$$

Then $\mathscr{M}_\varepsilon \xrightarrow{\Gamma} \mathscr{M}$ *as* $\varepsilon \to 0$ *in the* L^1 *topology.*

In the above $\mathrm{BV}(\Omega)$ denotes the space of those functions φ such that $\varphi \in L^1(\Omega)$ and the distributional gradient $D\varphi$ is a Radon measure. For *Bounded Variation* functions the distributional gradient can be decomposed into three measures, namely

$$D\varphi = \nabla\varphi + D^c\varphi + [\varphi]\mathcal{H}^{n-1} \llcorner J_\varphi$$

where $\nabla\varphi$ is the component of $D\varphi$ absolutely continuous with respect to the Lebesgue measure, $D^c\varphi$ is a Cantor measure and $[\varphi]\mathcal{H}^{n-1}\llcorner J_\varphi$ is called the jump component of the measure and is absolutely continuous with respect to the measure Hausdorff measure $\mathcal{H}^{n-1}$ restricted to the discontinuity set J_φ. In particular if $\varphi \in \mathrm{BV}(\Omega)$ and $\varphi = \chi_A$ then J_φ is the essential boundary of A contained in Ω and $[\varphi] = 1$. For further results on the theory of functions of Bounded Variations we refer to [AFP00] and the technical introduction of Chapter 1, Section 1.2. Theorem 0.7 is correlated with its respective equicoercivity property.

Corollary 0.1. *If* $\varepsilon \downarrow 0$ *and* φ_ε *minimizes* $\mathscr{M}_\varepsilon$ *then the sequence* (φ_ε) *si pre-compact with respect to the weak-* topology in BV and any limit point minimizes* $\mathscr{M}$*.*

Another example comes from the approximation of the Mumford-Shah functional for image segmentation. In [MS89] the authors consider a function g, defined on a domain Ω, representing the gray scale values of an image of a group of objects given by a camera, with discontinuities along the edges of the objects. The idea is that the segmented image u should be sufficiently smooth outside an $(n-1)$-dimensional set containing the discontinuity set K, namely $u \in W^{1,2}(\Omega \setminus K)$, and the latter should be chosen of minimal $\mathcal{H}^{n-1}$-size. Therefore they propose to optimize in the variables (u, K) the energy

$$\int_{\Omega\setminus K} \left[|\nabla u|^2 + \alpha(u-g)^2\right] \,\mathrm{d}x + \beta\mathcal{H}^{n-1}(K).$$

The parameters α, β control the weight between the fidelity term $|u-g|^2$ and the size of the discontinuity set K. It is convenient to recast the problem in its weak formulation letting $u \in \mathrm{BV}(\Omega)$ and replacing the set K with J_u obtaining the functional

$$\mathscr{S}(u) := \int_\Omega \left[|\nabla u|^2 + \alpha(u-g)^2\right] \,\mathrm{d}x + \beta\mathcal{H}^{n-1}(J_u).$$

To give an approximation of the energy $\mathscr{S}$, Ambrosio and Tortorelli have proposed the family of functionals

$$\mathscr{S}_\varepsilon(u,\varphi) = \int_\Omega |\nabla u|^2\varphi + \frac{\beta}{4}\left[\varepsilon|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{\varepsilon}\right] \,\mathrm{d}x + \alpha\int_\Omega (u-g)^2 \,\mathrm{d}x.$$

In the articles [AT90, AT92] it is proved that $\mathscr{S}_\varepsilon \xrightarrow{\Gamma} \mathscr{S}$. Let us give a heuristic idea behind this result. Since u is close to g, in the event of a strong discontinuity of g the gradient term $|\nabla u|$ explodes. Indeed, high values in the gradient $|\nabla u|$ are controlled by values close to zero in the state function φ. On the other hand the term in square brackets strongly penalizes values of φ far from 1. The competition of the terms in φ results in the fact that $1-\varphi$ represents a smoothed version of the function $1-\chi_{J_u}$. Finally in the limit $\varepsilon \downarrow 0$ the Modica-Mortola term converges to the $\mathcal{H}^{n-1}$ size of the set $\{\varphi \neq 1\}$ which contains the jump set of u. Functionals modeled on the ones from Ambrosio and Tortorelli and the latter functional itself are frequently known under the name of phase-field approximations. This is not only because of the strict relation with the Modica-Mortola functional but even because we may interpret the function φ as a state function, which acquires value 0 on the jump set of u, i.e. on the set of strong discontinuity of the function, and value 1 where u is sufficiently smooth. The two behaviors of u are then interpreted as two possible states and φ models the pointwise state function for the system. This observation has been taken into consideration in the work on fracture theory from Iurlano et al. [CFI16, Iur13]. There φ models the damage state of a material and u is replaced with a displacement function.

To conclude the section we present a variation of the Ambrosio Tortorelli functional proposed by Bonnivard, Lemenant and Santambrogio [LS14, BLS15] to recover in the limit the functional associated to the Steiner minimal tree problem for some points $\{x_0,\dots,x_N\} \subset \Omega \subset \mathbf{R}^2$. Given a continuous function $\varphi : \Omega \to [0,1]$ the authors introduce a geodesic distance weighted on φ, namely

$$d_\varphi(x,y) = \inf\left\{\int_\gamma \varphi \,\mathrm{d}\mathcal{H}^1 \,:\, \gamma \in C([0,1],\Omega),\ \gamma(0)=x,\ \gamma(1)=y\right\}.$$

The distance $d_\varphi(x,y)$ vanishes if and only if the two points x, y are joined by a path on which φ is equal to 0. Now consider the functional

$$\int_\Omega \left[\varepsilon|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{4\varepsilon}\right] \mathrm{d}x + \frac{1}{c_\varepsilon}\sum_{i=1}^N d_\varphi(x_0,x_i)$$

where $c_\varepsilon \to 0$ as $\varepsilon \to 0$. First observe that if

$$\sum_{i=1}^N d_\varphi(x_0,x_i) = 0 \tag{14}$$

then the set $\{\varphi = 0\}$ should include a path-connected subset containing $\{x_0,\dots,x_N\}$. The heuristic argument for the Γ-convergence result follows the ideas presented in the case of the Ambrosio-Tortorelli functional. The exact result in [BLS15] is

Theorem 0.8 (Bonnivard-Lemenant-Santambrogio). *Let $\Omega \subset \mathbf{R}^2$ be an open set, $\{x_0,\dots,x_N\} \subset \Omega$ and $\mu = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. Consider the functional*

$$\mathscr{B}_\varepsilon(\varphi) = \int_\Omega \left[\varepsilon|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{4\varepsilon}\right] \mathrm{d}x + \int_\Omega \frac{1}{c_\varepsilon} d_\varphi(x_0,x)\,\mathrm{d}\mu, \tag{15}$$

and a sequence φ_ε such that

$$\mathscr{B}_\varepsilon(\varphi_\varepsilon) - \inf_\varphi \mathscr{B}_\varepsilon(\varphi) \xrightarrow[\varepsilon\to 0]{} 0.$$

Then the sequence of functions d_{φ_ε} converges uniformly (up to a subsequence) to a function d such that the set $K := \{d = 0\}$ minimizes $\mathcal{H}^1$ among all compact, connected sets containing the points $\{x_0, \ldots, x_n\}$.

A first approach to the problem of approximating the energy $\mathscr{E}_h$ in the case $h = |\cdot|^\alpha$ was proposed by Santambrogio and Oudet in [OS11]. They introduce a functional of the type

$$\int_\Omega \varepsilon^{\alpha+1}|\nabla\sigma|^2 + \varepsilon^{\alpha-1}|\sigma|^\beta \text{ with } \sigma \in W^{1,2}(\Omega, \mathbf{R}^2) \text{ and } \nabla\cdot\sigma = (\mu_+ - \mu_-) * \rho_\varepsilon$$

with $\beta = (4\alpha - 2)/(\alpha + 1)$ and ρ_ε an approximation to the identity. Actually the complete $\Gamma - \limsup$ inequality for the latter result has been provided by Monteil [Mon15, Mon17].

Structure of the thesis

In the **First Chapter** we study a variation of the functional proposed by Lemenant and Santambrogio. Motivated by the observation that

$$d_\varphi(x,y) = \min\left\{\int_\Omega \varphi|\sigma| \,\mathrm{d}x \;:\; \sigma \in \mathcal{M}(\Omega, \mathbf{R}^n) \text{ and } \operatorname{div}\sigma = \delta_x - \delta_y\right\}$$

we replace the term depending on the geodesic distance in (15) with a term depending on the product $\varphi|\sigma|$. The proposed functional is defined on couples (σ, φ) is

$$\int_\Omega \frac{\varphi|\sigma|^2}{2\varepsilon}\,\mathrm{d}x + \int_\Omega \left[\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{2\varepsilon}\right]\,\mathrm{d}x,$$

where σ is a vector-valued function complemented with the constraint

$$\operatorname{div}\sigma = (\mu_+ - \mu_-) * \rho_\varepsilon. \tag{16}$$

In the above ρ_ε is a given approximation of the identity and the phase functions $\varphi \in L^1(\Omega)$ are bounded from below by the quantity $\beta\,\varepsilon$, where $\beta \geq 0$ a given parameter. First, we show that this functional Γ-converges to the energy $\mathscr{E}_h$, for the choice

$$h(m) := \begin{cases} 1 + \beta m, & \text{if } m \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

The proof of the Γ-convergence result is obtained for open convex subsets of $\mathbf{R}^2$. The advantage of choosing a quadratic penalization in σ is that the augmented Lagrangian problem associated to the functional may be explicitly solved in the dual variable. Therefore it is possible to devise an alternate minimization algorithm composed of two smooth elliptic functionals solvable via finite elements methods. The algorithm is proposed and studied at the end of the chapter. We further present and study other algorithms which take advantage of a concept of 'shape derivative' to improve the quality of the approximation.

The generalization to $\Omega \subset \mathbf{R}^n$ is the matter of the **Second Chapter**. To obtain the result in higher dimension the Modica-Mortola component of the functional needs to be rescaled. As observed in [Ghi14] this leads to the introduction of some non linearities in the functional as follows

$$\int_\Omega \frac{\varphi|\sigma|^2}{\varepsilon}\,\mathrm{d}x + \int_\Omega \left[\varepsilon^{p-n+1}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-1}}\right]\,\mathrm{d}x, \tag{18}$$

for some $p > n-1$. Again σ is complemented with the divergence constraint (16) for a suitable choice of ρ_ε and we require a lower bound for the the phase field functions, namely $\varphi \geq \beta\varepsilon^n$. We prove the Γ-convergence of the above functional to $\mathscr{E}_{h_\beta^{n-1}}$ where the cost function h_β^{n-1} is the limit in ε of an optimization problem depending on the co-dimension $n-1$. Namely for a ball $B_r \subset \mathbf{R}^{n-1}$ we let

$$h_{\varepsilon,\beta}^{n-1}(m) = \min\begin{cases} \displaystyle\int_{B_r}\left[\frac{\varphi|\theta|^2}{\varepsilon} + \varepsilon^{p-n+1}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-1}}\right]\,\mathrm{d}x, \\ \displaystyle\varphi \in W^{1,p}(B_r),\ \varphi = 1 \text{ on } \partial B_r \text{ and } \int_{B_r}\theta\,\mathrm{d}x = m. \end{cases}$$

The latter optimization problem corresponds to the 0-dimensional version of (18). We introduce and study $h_{\varepsilon,\beta}^d$ (obtained replacing $n-1$ with d in the latter formula) in the appendix. Some similar phase transition problems with mass constraint which leads to measures concentrated on atoms have been studied by Bouchitté, Dubs and Seppecher in [BDS96] in the context of droplets equilibrium. In particular we show that h_β^d is independent of r and that it is a transport cost function satisfying the conditions (3). We prove as well that there exists a constant $c > 0$ such that

$$\frac{1}{c} \leq \frac{h_\beta^d(m)}{1+\sqrt{\beta}m} \leq c \quad \text{for } m > 0.$$

Remark that the Modica-Mortola component of the functional studied in the second chapter depends on $n-1$, the co-dimension of the problem in the case of rectifiable measures. In **Chapter Three** we investigate a different rescaling to approach minima to (11) defined for k-currents, namely

$$\int_\Omega \frac{\varphi|\sigma|^2}{\varepsilon}\,\mathrm{d}x + \int_\Omega \left[\varepsilon^{p-n+k}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-k}}\right]\,\mathrm{d}x. \tag{19}$$

In this context, σ is no longer a vector measure, to take into account the boundary the constraint needs to be suitably modified. Let σ_0 be a given polyhedral k-current, for ρ_ε a standard approximation of the identity we let σ be a a smooth k-current such that

$$\partial\sigma = \partial\sigma_0 * \rho_\varepsilon.$$

(In equation (19) the current is identified with its density measure.) In the chapter we introduce formally the energy and show that it Γ-converges to the energy $\mathscr{E}_h$ defined in (11) for the transport cost function $h = h_\beta^{n-k}$ studied in the appendix.

Figure 6: Computed mass flux σ and phase fields $\varphi_1, \varphi_2, \varphi_3$ for the cost function shown on the right, $\varepsilon = 0.005$. The color in σ indicates which phase field is active. The result is obtained by optimizing the functional defined in **Chapter Three**.

In the **Fourth and Fifth Chapters** we restrict again our attention to sets $\Omega \subset \mathbf{R}^2$ and develop two functionals for the approximation of any concave and continuous transport cost function h. Note that we say that a transport cost function is concave if it is an even function whose restriction to $[0, +\infty)$ is concave. The first result regards transport cost functions h of the form

$$h(m) = \min\{\alpha_i\,|m| + \beta_i\,:\,0 \le i \le N\}.$$

for $\alpha_0 > \alpha_1 > \ldots > \alpha_N \ge 0$ and $0 \le \beta_0 < \beta_1 < \ldots < \beta_N$. Our approach takes advantage of the result in the First Chapter in which we recovered in the Γ-limit affine cost functions of the form $1 + \beta|m|$. In the case $N = 1$ and $\beta_0 > 0$ the proposed phase-field energy takes the form

$$\int_\Omega \left[\min\left\{\varphi_0^2 + \frac{\alpha_0^2\varepsilon^2}{\beta_0}\,;\,\varphi_1^2 + \frac{\alpha_1^2\varepsilon^2}{\beta_1}\right\}\frac{|\sigma|^2}{2\varepsilon}\right]\,\mathrm{d}x + \beta_0\mathscr{T}_\varepsilon(\varphi_0) + \beta_1\mathscr{T}_\varepsilon(\varphi_1)$$

where $\mathscr{T}_\varepsilon$ is an energy of the Modica-Mortola type defined as

$$\mathscr{T}_\varepsilon(\varphi) = \frac{1}{2}\int_\Omega \left[\varepsilon|\nabla\varphi(x)|^2 + \frac{(\varphi(x)-1)^2}{\varepsilon}\right]\,\mathrm{d}x.$$

Let us highlight the presence of two phase-fields which interact in the constraint component of the functional. Ideally each $1-\varphi_i$ is a smooth indicator function of some subset of the support of the limit rectifiable measure σ. In particular $\varphi_i = 0$ if the choice of the i-th component in the definition of h is optimal with respect to the intensity of the flux of σ. The entire **Fourth Chapter** is devoted to establish the proof of the Γ-convergence result and the study of numerical methods developed in collaboration with Carolin Rossmanith and Benedikt Wirth from Munster University.

In the final chapter of the thesis we study functionals of the form

$$\mathscr{F}_\varepsilon(\sigma,\varphi) := \int_\Omega f(\varphi)|\sigma| + \frac{1}{2}\left[\varepsilon|\nabla\varphi|^2 + \frac{\varphi^2}{\varepsilon}\right]\,\mathrm{d}x$$

The two main differences with respect to the previous models are the linear penalization in $|\sigma|$ and the presence of the term φ^2 instead of $(1-\varphi)^2$. Analogous models with a linear penalization of the $|\sigma|$ component have been studied recently in the case of fracture theory and the generalized Mumford-Shah functional [ABS99, DMOT16]. Our main contribution is to find an explicit form of the weight function f to obtain in the limit the energy $\mathscr{E}_h$. For a continuous and concave transport cost function h, we define f as

$$f(t) = (-h_*)^{-1}(t^2).$$

The function h_* is the (concave) Legendre transform of h. In this model φ takes value 0 and not 1 outside the support of the limit measure σ. By virtue of this general result we address the problem of the numerical approximation of the functional $\mathscr{F}_\varepsilon$. The linear penalization in σ may be seen as a drawback with respect to the methods previously studied which where deeply based on the quadratic cost $|\sigma|^2$. In view of this difference we started investigating new numerical methods based on the Beckman model [Bec52] for transportation. The same result may be obtained with different choice of the well potential. Namely, given a potential W which is an even function, increasing on $[0, +\infty)$ and vanishing in 0 we introduce the transition energy

$$c_W(t) := \int_0^{|t|} 2\sqrt{W(s)}\,\mathrm{d}s.$$

Then choosing $f(t) = (-h_*)^{-1} \circ c_W(t)$ the same Γ-convergence result may be obtained with a family of functionals defined as

$$\mathscr{F}_\varepsilon(\sigma, \varphi) := \int_\Omega f(\varphi)|\sigma| + \frac{1}{2}\left[\varepsilon|\nabla\varphi|^2 + \frac{W(\varphi)}{\varepsilon}\right]\,\mathrm{d}x.$$

In force of this degree of freedom in the choice of the potential W we start analyzing which would be the best choice. These and other questions are the subject of the concluding section which investigates possible developments of the proposed methods.

Chapter 1

Affine cost function

1.1 Introduction

In this chapter we devise an approximation for the minimization problem defined in the introduction in equation (6). In particular we consider the energy $\mathscr{E}_h$ choosing as cost function

$$h(m) := \begin{cases} 1 + \beta|m|, & \text{if } m \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta > 0$ is a fixed positive parameter. Furthermore we will consider only atomic probability measures $\mu_+,\ \mu_- \in \mathcal{P}(\Omega)$ to define the constraint. The results contained in this chapter have been published in the paper [CFM17a]. Let us introduce the precise framework of our approximation. Let $\rho \in C_c^\infty(\mathbf{R}^2, \mathbf{R}_+)$ be a classical radial mollifier with $\operatorname{supp}\rho \subset B_1(0)$ and $\int \rho = 1$. For $\varepsilon \in (0,1]$, we set $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(\varepsilon^{-1}x)$ and we define the space $V_\varepsilon(\Omega)$ of square integrable vector fields with weak divergence satisfying the constraint

$$\nabla \cdot \sigma_\varepsilon = (\mu_+ - \mu_-) * \rho_\varepsilon \quad \text{in } \mathcal{D}'(\mathbf{R}^2). \tag{1.1}$$

For an $\eta = \eta(\varepsilon) > 0$, we denote

$$W_\varepsilon(\Omega) = \left\{\varphi \in W^{1,2}(\Omega) \,:\, \eta \leq \varphi \leq 1 \text{ in } \Omega,\ \varphi \equiv 1 \text{ on } \partial\Omega\right\}.$$

We denote with $X_\varepsilon(\Omega) = V_\varepsilon(\Omega) \times W_\varepsilon(\Omega)$ and define the energy $\mathscr{F}_\varepsilon : \mathcal{M}(\Omega, \mathbf{R}^2) \times L^1(\Omega) \to [0, +\infty]$ as

$$\mathscr{F}_\varepsilon(\sigma,\varphi) := \begin{cases} \displaystyle\int_\Omega \frac{1}{2\varepsilon}\varphi^2|\sigma|^2 \,\mathrm{d}x + \int_\Omega \left[\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{2\varepsilon}\right] \mathrm{d}x, & \text{if } (\sigma,\varphi) \in X_\varepsilon(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.2}$$

From now on, we assume that

$$\frac{\eta}{\varepsilon} \xrightarrow{\varepsilon\downarrow 0} \beta. \tag{1.3}$$

We denote $\mathcal{M}_{\mathscr{S}}(\overline{\Omega})$ the set of $\mathbf{R}^2$-valued measures $\sigma \in \mathcal{M}(\mathbf{R}^2, \mathbf{R}^2)$ with support in $\overline{\Omega}$ such that the constraint

$$\operatorname{div}\sigma = \mu_+ - \mu_- \quad \text{in } \mathcal{D}'(\mathbf{R}^2) \tag{1.4}$$

holds. We define the limit energy $\mathscr{E}_\beta : \mathcal{M}(\overline{\Omega}, \mathbf{R}^2) \times L^1(\Omega) \to [0, +\infty]$ as

$$\mathscr{E}_\beta(\sigma, \varphi) = \begin{cases} \displaystyle\int_\Sigma (1 + \beta\, m)\, \mathrm{d}\mathcal{H}^1 & \text{if } \varphi \equiv 1,\ \sigma \in \mathcal{M}_{\mathscr{S}}(\overline{\Omega}) \text{ and } \sigma = (m, \tau, \Sigma), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

We prove the Γ-convergence of the sequence $(\mathscr{F}_\varepsilon)$ to the energy $\mathscr{E}_\beta$ as $\varepsilon \downarrow 0$. More precisely the convergence holds in $\mathcal{M}(\overline{\Omega}, \mathbf{R}^2) \times L^1(\Omega)$ where $\mathcal{M}(\overline{\Omega}, \mathbf{R}^2)$ is endowed with the weak-$*$ topology and $L^1(\Omega)$ is endowed with its classical strong topology.
We begin by proving the equicoercivity of the sequence $(\mathscr{F}_\varepsilon)$. In this statement and throughout the chapter, we make a small abuse of language by denoting $(a_\varepsilon)_{\varepsilon \in (0,1]}$ and calling sequence a family $\{a_\varepsilon\}$ labeled by a continuous parameter $\varepsilon \in (0, 1]$. In the same spirit, we call subsequence of (a_ε), any sequence (a_{ε_j}) with $\varepsilon_j \to 0$ as $j \to +\infty$.
We establish the following lower bound.

Theorem 1.1 ($\Gamma - \liminf$)**.** *For any sequence $(\sigma_\varepsilon, \varphi_\varepsilon) \subset \mathcal{M}(\Omega, \mathbf{R}^2) \times L^1(\Omega)$ such that $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ and $\varphi_\varepsilon \to \varphi$ in the $L^1(\Omega)$ topology, with $(\sigma, \varphi) \in \mathcal{M}(\Omega, \mathbf{R}^2) \times L^1(\Omega)$,*

$$\liminf_{k \to +\infty} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \geq \mathscr{E}_\beta(\sigma, \varphi).$$

To complete the Γ-convergence analysis, we establish the matching Γ-limsup inequality.

Theorem 1.2 ($\Gamma - \limsup$)**.** *For any $(\sigma, \varphi) \subset \mathcal{M}(\Omega, \mathbf{R}^2) \times L^1(\Omega)$ there exists a sequence $(\sigma_\varepsilon, \varphi_\varepsilon)$ such that $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ and $\varphi_\varepsilon \to \varphi$ in the $L^1(\Omega)$ topology and*

$$\limsup_{k \to +\infty} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \leq \mathscr{E}_\beta(\sigma, \varphi).$$

Theorem 1.3 (Equicoercivity)**.** *Assume $\beta > 0$. For any sequence $(\sigma_\varepsilon, \varphi_\varepsilon)_{\varepsilon \in (0,1]} \subset \mathcal{M}(\Omega, \mathbf{R}^2) \times L^1(\Omega)$ with uniformly bounded energies, i.e.*

$$\sup_\varepsilon \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \ < \ +\infty,$$

there exist a subsequence $\varepsilon_j \downarrow 0$ and a measure $\sigma \in \mathcal{M}_{\mathscr{S}}(\overline{\Omega}, \mathbf{R}^2)$ such that $\sigma_{\varepsilon_j} \to \sigma$ with respect to the weak-$*$ convergence of measures and $\varphi_{\varepsilon_j} \to 1$ in $L^1(\Omega)$. Moreover, σ is a rectifiable measure (i.e., it is of the form $\sigma = (m, \tau, \Sigma)$).

Structure of the chapter: In Section 1.2 we introduce and recall some notation and several tools and notions on SBV functions and introduce an operator acting on vector field measures. In Section 1.3 we study the behavior of the functional $\mathscr{F}_\varepsilon$ on vector fields of the form ∇u (dropping the divergence constraint). In Section 1.4 we prove the equicoercivity result, Theorem 1.3 and we establish the lower bound stated in Theorem 1.1. In Section 1.5 we prove the upper bound of Theorem 1.2. Finally, in the last section, we present and discuss various numerical simulations.

1.2 Preliminaries for the chapter

In the following $\Omega \subset\subset \hat{\Omega} \subset \mathbf{R}^d$ are bounded open convex sets. Given $X \subset \mathbf{R}^d$ (in practice $X = \Omega$ or $X = \hat{\Omega}$), we denote by $\mathcal{A}(X)$ the class of all relatively open subsets of X and by $\mathcal{A}_S(X)$ the subclass of all simply connected relatively open sets $O \subset X$ such that $\overline{O} \cap S = \emptyset$. We denote by $(e_1, \dots, e_d)$ the canonical orthonormal basis of $\mathbf{R}^d$, by $|\cdot|$ the euclidean norm and by $\langle \cdot, \cdot \rangle$ the euclidean scalar product in $\mathbf{R}^d$. The open ball of radius r centered at $x \in \mathbf{R}^d$ is denoted by $B_r(x)$. The $(d-1)$-dimensional Hausdorff measure in $\mathbf{R}^d$ is denoted by $\mathcal{H}^{d-1}$. We write $|E|$ to denote the Lebesgue measure of a measurable set $E \subset \mathbf{R}^d$. When μ is a Borel meaure and $E \subset \mathbf{R}^d$ is a Borel set, we denote by $\mu \llcorner E$ the measure defined as $\mu \llcorner E(F) = \mu(E \cap F)$.
Let us remark that from Section 1.4 onwards, we work in dimension $d = 2$.
For any fixed couple (σ, φ), with $\mathscr{F}_\varepsilon(\sigma, \varphi; O)$ we denote the value of the functional (1.2) on any set $O \in \mathcal{A}(\Omega)$. Similarly we define the with version $\mathscr{E}_\beta(\sigma, \varphi; O)$ the localization of $\mathscr{E}_\beta$ to O.

BV(Ω) is the space of functions $u \in L^1(\Omega)$ having as distributional derivative Du a measure with finite total variation. Following the classical notation as in [AFP00, ABM14] and [Bra98] for $u \in BV(\Omega)$ we have

$$Du = \nabla u \, \mathrm{d}x + (u^+ - u^-)\nu_u \mathcal{H}^{d-1} \llcorner J_u + D^c u,$$

where J_u is the set of "approximate jump points" x where $y \mapsto u(x + \rho y)$ converge as $\rho \to 0$ to $u^+ \chi_{\{y \cdot \nu_u \geq 0\}} + u^- \chi_{\{y \cdot \nu_u < 0\}}$ for some (u^-, u^+, ν_u) and $D^c u$ is the Cantor "part". Let us introduce the space of special functions of bounded variation and a variant:

$$SBV(\Omega) := \{u \in BV(\Omega) : D^c u = 0\},$$

$$GSBV(\Omega) := \{u \in L^1(\Omega) : \max(-T, \min(u, T)) \in SBV(\Omega) \; \forall T > 0\}.$$

Eventually, in Section 1.3, the following space of piecewise constant functions will be useful.

$$PC(\Omega) = \{u \in GSBV(\Omega) : \nabla u = 0\}. \tag{1.6}$$

To conclude this section we recall the slicing method for functions of bounded variation. Let $\tau \in \mathbf{S}^{d-1}$ and let

$$\Pi_\tau := \{y \in \mathbf{R}^d : \langle y, \tau \rangle = 0\}.$$

If $y \in \Pi_\tau$ and $E \subset \mathbf{R}^d$, we define the one dimensional slice

$$E_{\tau,y} := \{t \in \mathbf{R} : y + t\tau \in E\}.$$

For $u : \Omega \to \mathbf{R}$, we define $u_{\tau,y} : \Omega_{\tau,y} \to \mathbf{R}$ as

$$u_{\tau,y}(t) := u(y + t\tau), \quad t \in \Omega_{\tau,y}.$$

Functions in $GSBV(\Omega)$ can be characterized by one-dimensional slices (see [Bra98, Thm. 4.1])

Theorem 1.4. *Let $u \in GSBV(\Omega)$. Then for all $\tau \in \mathbf{S}^{d-1}$ we have*

$$u_{\tau,y} \in GSBV(\Omega_{\tau,y}) \quad \textit{for } \mathcal{H}^{d-1}\textit{-a.e. } y \in \Pi_\tau.$$

Moreover for such y, we have

$$u'_{\tau,y}(t) = \langle \nabla u(y + t\tau), \tau \rangle \quad \textit{for a.e. } t \in \Omega_{\tau,y},$$
$$J_{u_{\tau,y}} = \{t \in \mathbf{R} : y + t\tau \in J_u\},$$

and

$$u_{\tau,y}(t^\pm) = u^\pm(y + t\tau) \quad \textit{or} \quad u_{\tau,y}(t^\pm) = u^\mp(y + t\tau)$$

according to whether $\langle \nu_u, \tau \rangle > 0$ or $\langle \nu_u, \tau \rangle < 0$. Finally, for every Borel function $g : \Omega \to \mathbf{R}$,

$$\int_{\Pi_\tau} \sum_{t \in J_{u_{\tau,y}}} g_{\tau,y}(t) \, \mathrm{d}\mathcal{H}^{d-1}(y) = \int_{J_u} g |\langle \nu_u, \tau \rangle| \, \mathrm{d}\mathcal{H}^{d-1}. \tag{1.7}$$

Conversely if $u \in L^1(\Omega)$ and if for all $\tau \in \{e_1, \dots, e_d\}$ and almost every $y \in \Pi_\tau$ we have $u_{\tau,y} \in SBV(\Omega_{\tau,y})$ and

$$\int_{\Pi_\tau} |Du_{\tau,y}|(\Omega_{\tau,y}) \, \mathrm{d}\mathcal{H}^{d-1}(y) < +\infty$$

then $u \in SBV(\Omega)$.

Let us introduce the linear operator $\perp$ that associates to each vector $v = (v_1, v_2) \in \mathbf{R}^2$ the vector $v^\perp = (-v_2, v_1)$ obtained via a $90°$ counterclockwise rotation of v. Notice that the $\perp$ operator maps divergence-free $\mathbf{R}^2$-valued measures onto curl free $\mathbf{R}^2$-valued measures. Let $O \subset \mathbf{R}^2$ be a simply connected and bounded open set. It is possible to generalize Stokes Theorem to divergence free measures. If μ is a smooth divergence free vector field on O we have $\mu = \nabla u^\perp$ for some smooth function with zero mean value. Then by Poincaré inequality $|u|_{L^1} \le C|\mu|_{L^1}$. The result for μ general divergence free finite vector measure follows by regularization. On the other hand for $u \in PC(\Omega)$, $\sigma := Du^\perp$ is divergence free and,

$$\sigma = (u^+ - u^-)\nu_u^\perp \mathcal{H}^1 = U(J_u, [u], \nu_u^\perp). \tag{1.8}$$

1.3 Local Result

In this section we introduce a localization of the family of functionals $(\mathscr{F}_\varepsilon)$ (see (1.2)). We establish a lower bound and a compactness property for these local energies. In this section we assume $\Omega \subset \mathbf{R}^d$.

Localization. Let $O \in \mathcal{A}_S(\Omega)$ be a simply connected relatively open subset of Ω. For $u_\varepsilon \in W^{1,2}(O)$ and $\varphi_\varepsilon \in W^{1,2}(O)$, we define

$$\mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O) \ := \ \mathscr{F}_\varepsilon(\nabla u_\varepsilon, \varphi_\varepsilon; O)$$

i.e. as the evaluation of the functional $\mathscr{F}_\varepsilon$ on vector fields of the form ∇u with no requirement on the divergence. Notice that for $\varepsilon < d(O, S)$, we have $\nabla \cdot \sigma_\varepsilon \equiv 0$ in O for any $\sigma_\varepsilon \in V_\varepsilon(\Omega)$. By Stokes theorem we have $Du_\varepsilon^\perp = \sigma_\varepsilon$ for some $u_\varepsilon \in W^{1,2}(O)$ and we have

$$\mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; O) = \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O).$$

The rest of the section is devoted to the proof of

Theorem 1.5. *Let $(u_\varepsilon)_{\varepsilon\in(0,1]} \subset W^{1,2}(O)$ be a family of functions with zero mean value and let $(\varphi_\varepsilon) \subset W^{1,2}(O)$ such that $\varphi_\varepsilon \in W^{1,2}(O, [\eta(\varepsilon), 1])$. Assume that $c_0 := \sup_\varepsilon \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O)$ is finite. Then there exist a subsequence ε_j and a function $u \in BV(\Omega)$ such that*

a) $\varphi_{\varepsilon_j} \to 1$ in $L^2(O)$,

b) $u_{\varepsilon_j} \to u$ with respect to the weak-$*$ convergence in BV,

c) $u \in PC(O)$.

Furthermore for any piecewise function $u \in PC(O)$ and any sequence $(u_\varepsilon, \varphi_\varepsilon)$ such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ and $\varphi_\varepsilon \to 1$, we have the following lower bound of the energy:

$$\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O) \geq \int_{J_u\cap O} (1 + \beta|[u]|)\ \mathrm{d}\mathcal{H}^{d-1}.$$

The proof is achieved in several steps and mostly follows ideas from [Iur13] (see also [CFI16]). In the first step we obtain *(a)* and *(b)*. In step 2 we prove *(c)* and the lower bound for one dimensional slices of $\mathscr{G}_\varepsilon$. Finally in step 3 we prove *(c)* and the lower bound in dimension d. The construction of a recovery sequence that would complete the Γ-limit analysis is postponed to the global model in Section 1.5.

Proof. Step 1: Item *(a)* is a straightforward consequence of the definition of the functional. Indeed, we have

$$\int_O (1 - \varphi_\varepsilon)^2\ \mathrm{d}x \leq 2\,\varepsilon\, \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon) \ \leq\ 2, c_0\, \varepsilon \ \overset{\varepsilon\downarrow 0}{\longrightarrow}\ 0.$$

For *(b)*, since (u_ε) has zero mean value, we only need to show that $\sup_{\varepsilon\in(0,1]}\ |Du_\varepsilon|(O) < +\infty$. Using Cauchy-Schwarz inequality we get

$$[|Du_\varepsilon|(O)]^2 = \left(\int_O |\nabla u_\varepsilon|\right)^2 \leq \left(2\,\varepsilon \int_O \frac{1}{\varphi_\varepsilon^2}\right)\left(\frac{1}{\varepsilon}\int_O \varphi_\varepsilon^2 |\nabla u_\varepsilon|^2\right). \tag{1.9}$$

By assumption, the second therm in the right hand side of (1.9) is bounded by $2c_0$. In order to estimate the first term we split O in the two sets $\{\varphi_\varepsilon < 1/2\}$ and $\{\varphi_\varepsilon \geq 1/2\}$. We have,

$$2\,\varepsilon \int_O \frac{1}{\varphi_\varepsilon^2} \ = \ 2\,\varepsilon \left(\int_{\{\varphi_\varepsilon<1/2\}} \frac{1}{\varphi_\varepsilon^2} + \int_{\{\varphi_\varepsilon\geq 1/2\}} \frac{1}{\varphi_\varepsilon^2}\right).$$

Since $\eta \leq \varphi_\varepsilon \leq 1/2$ on $\{\varphi_\varepsilon \leq 1/2\}$ it holds $\varphi_\varepsilon^2(1-\varphi_\varepsilon)^2 \geq \eta^2(1-1/2)^2$ therefore

$$\begin{aligned}\int_{\{\varphi_\varepsilon<1/2\}} \frac{1}{\varphi_\varepsilon^2} &\leq \frac{2\varepsilon}{\eta^2(1-1/2)^2}\int_{\{\varphi_\varepsilon<1/2\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon} \leq \frac{8\varepsilon}{\eta^2}c_0,\\ \int_{\{\varphi_\varepsilon\geq 1/2\}} \frac{1}{\varphi_\varepsilon^2} &\leq \int_{\{\varphi_\varepsilon\geq 1/2\}} \frac{1}{(1/2)^2} = 4|\{\varphi_\varepsilon \geq 1/2\}|.\end{aligned}$$

Eventually, as $|\{\varphi_\varepsilon \geq 1/2\}| \leq |O|$, combining these estimates with (1.9) we obtain

$$[|Du_\varepsilon|(O)]^2 \leq \frac{\varepsilon^2}{\eta^2}16c_0^2 + 8\varepsilon|O|c_0 \xrightarrow{\varepsilon\downarrow 0} \frac{16c_0^2}{\beta^2} < \infty. \tag{1.10}$$

This establishes *(b)*.

Step 2: In this step we suppose O to be an interval of $\mathbf{R}$, so that u_ε, φ_ε are one-dimensional. We first prove that u is piecewise constant. The idea is that in view of the constraint component of the energy, variations of u_ε are balanced by low values of φ_ε. On the other hand the Modica-Mortola component of the energy implies that $\varphi_\varepsilon \simeq 1$ in most of the domain and that transitions from $\varphi_\varepsilon \simeq 1$ to $\varphi_\varepsilon \simeq 0$ have a constant positive cost (and therefore can occur only finitely many times).

Step 2.1: (Proof of $u \in PC(O)$.) Let us define

$$B_\varepsilon := \left\{x \in O : \varphi_\varepsilon(x) < \frac{3}{4}\right\} \supset A_\varepsilon := \left\{x \in O : \varphi_\varepsilon(x) < \frac{1}{2}\right\}, \tag{1.11}$$

and let

$$C_\varepsilon = \{I \text{ connected component of } B_\varepsilon : I \cap A_\varepsilon \neq \emptyset\}. \tag{1.12}$$

Let us show that the cardinality of C_ε is bounded by a constant independent of ε. Let ε be fixed and consider an interval $I \in C_\varepsilon$. Let $a, b \in \bar{I}$ such that $\{\varphi_\varepsilon(a), \varphi_\varepsilon(b)\} = \{1/2, 3/4\}$. Using the usual Modica-Mortola trick, we have

$$\begin{aligned}\mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; I) &\geq \int_I \frac{\varepsilon|\varphi_\varepsilon'|^2}{2} + \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}\,\mathrm{d}x\\ &\geq \int_{(a,b)} |\varphi_\varepsilon'|(1-\varphi_\varepsilon)\,\mathrm{d}x\\ &\geq \int_{1/2}^{3/4} (1-t)\,dt = \frac{3}{2^5}.\end{aligned}$$

Since all the elements of C_ε are disjoint and $\mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon, \cdot)$ is additive, we deduce from the energy bound that

$$\#C_\varepsilon \leq 2^5 c_0/3,$$

where we denote $\#C_\varepsilon$ the cardinality of C_ε. Next, up to extracting a subsequence we assume that $\#C_\varepsilon = N$ is fixed. The elements of C_ε are of the form $I_i^\varepsilon = (m_i^\varepsilon - w_i^\varepsilon, m_i^\varepsilon + w_i^\varepsilon)$ for $i = 1, \cdots, N$, with $m_i^\varepsilon < m_{i+1}^\varepsilon$. Since $\varphi_\varepsilon \to 1$ in $L^1(O)$ we have

$$\sum_{I_i^\varepsilon \in C_\varepsilon} |I_i^\varepsilon| = \sum_{i=1}^N 2w_i^\varepsilon \to 0. \tag{1.13}$$

Up to extracting a subsequence, we can assume that each sequence (m_i^ε) converges in $\overline{O}$. We call $m_1 \le m_2 \le \cdots \le m_N$ their respective limits. We now prove that

$$|Du|(O \setminus \{m_i\}_{i=1}^N) = 0, \tag{1.14}$$

thus $\mathrm{supp}(|Du|) \subset \{m_1, \cdots, m_N\}$. The latter ensures that u has no Cantor component since Du is supported on a finite number of points and that is a.e. constant outside $\{m_1, \cdots, m_N\}$ so that $u \in PC(O)$, (1.6). To this aim, we fix $x \in O \setminus \{m_i\}_{i=0}^N$ and establish the existence of a neighborhood $B_\delta(x)$ of x for which $|Du|(B_\delta(x)) = 0$. Let $0 < \delta \le \min_i |x - m_i|/2$. Equation (1.13) ensures that for ε small enough $B_\delta(x) \cap C_\varepsilon = \emptyset$. Notice that from the definitions in (1.11) and (1.12) we have that $\varphi_\varepsilon \ge 1/2$ outside the union of the sets in C_ε. Hence, using Cauchy-Schwarz inequality, we have for ε small enough,

$$\begin{aligned}\left(\int_{B_\delta(x)} |u_\varepsilon'| \,\mathrm{d}x\right)^2 &\le 2\delta \int_{B_\delta(x)} |u_\varepsilon'|^2 \,\mathrm{d}x \\ &\le (2\delta)(2\varepsilon)4\left(\frac{1}{2\varepsilon}\int_{B_\delta(x)} \varphi_\varepsilon^2 |u_\varepsilon'|^2 \,\mathrm{d}x\right) \\ &\le 16 c_0 \varepsilon\delta \xrightarrow{\varepsilon\downarrow 0} 0.\end{aligned}$$

By lower semicontinuity of the total variation on open sets we conclude that $|Du|(B_\delta(x)) = 0$, which proves the claim (1.14).

Step 2.2: (Proof of the lower bound for $\mathscr{G}_\varepsilon$.) Without loss of generality we can assume $N = 1$, thus J_u is composed of a single point, otherwise the argument we propose can be applied on each m_i separately. Up to a translation $m_1 = 0$ and we denote $D := u(0^+) = -u(0^-) > 0$. For any $0 < d < D$ there exist six points $y_1 < x_\varepsilon^1 \le \tilde{x}_\varepsilon^1 < \tilde{x}_\varepsilon^2 \le x_\varepsilon^2 < y_2$ such that

$$\begin{aligned}\lim_{\varepsilon\to 0} \varphi_\varepsilon(y_1) &= \lim_{\varepsilon\to 0}\varphi_\varepsilon(y_2) = 1, \\ \lim_{\varepsilon\to 0} \varphi_\varepsilon(x_\varepsilon^1) &= \lim_{\varepsilon\to 0}\varphi_\varepsilon(x_\varepsilon^2) = 0, \\ u_\varepsilon(\tilde{x}_\varepsilon^1) = -D + d, &\qquad u_\varepsilon(\tilde{x}_\varepsilon^2) = D - d.\end{aligned} \tag{1.15}$$

Since $\varphi_\varepsilon \to \varphi$ and $u_\varepsilon \to u$ in L^1 up to a subsequence they converge point-wise almost everywhere and this implies the first and third fact. Let $\inf_{(y_1,y_2)} \varphi_\varepsilon = c_\varepsilon$, then Jensen inequality implies

$$c_0 \ge \int_{y_1}^{y_2} \frac{\varphi_\varepsilon^2 |u_\varepsilon'|^2}{2\varepsilon} \,\mathrm{d}x \ge \frac{c_\varepsilon^2}{2\varepsilon(y_2 - y_1)}\left(\int_{y_1}^{y_2} |u_\varepsilon'| \,\mathrm{d}x\right)^2.$$

Then c_ε must vanish with ε implying statement (1.15). Using the Modica-Mortola trick in the intervals (y_1, x_ε^1) and (x_ε^2, y_2) as above, we compute:

$$\begin{aligned}&\liminf_{\varepsilon\downarrow 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; (y_1, x_\varepsilon^1) \cup (x_\varepsilon^2, y_2)) \ge \\ &\quad\ge \liminf_{\varepsilon\downarrow 0}\left(\int_{y_1}^{x_\varepsilon^1} (1-\varphi_\varepsilon)|\varphi_\varepsilon'| \,\mathrm{d}x + \int_{x_\varepsilon^2}^{y_2} (1-\varphi_\varepsilon)|\varphi_\varepsilon'| \,\mathrm{d}x\right) \ge 1.\end{aligned} \tag{1.16}$$

For the estimate on the interval $I_\varepsilon = (\tilde{x}^1_\varepsilon, \tilde{x}^2_\varepsilon)$ let us introduce:

$$\begin{aligned} G_\varepsilon &:= \left\{ w \in W^{1,2}(I_\varepsilon) : w(\tilde{x}^1_\varepsilon) = -D + d,\, w(\tilde{x}^2_\varepsilon) = D - d \right\}, \\ Z_\varepsilon &:= \left\{ z \in W^{1,2}(I_\varepsilon) : \eta \le z \le 1 \text{ a.e. on } I_\varepsilon \right\}, \\ H_\varepsilon(w,z) &:= \int_{I_\varepsilon} \left(\frac{1}{2\varepsilon} z^2 |w'|^2 + \frac{(1-z)^2}{2\varepsilon} \right) \mathrm{d}x, \\ h_\varepsilon(z) &= \inf_{w \in G_\varepsilon} H_\varepsilon(w,z) \text{ for } z \in Z_\varepsilon. \end{aligned}$$

Note that for $w \in G_\varepsilon$ and $z \in Z_\varepsilon$, we can apply an inequality similar to (1.9). In particular, for z replacing φ_ε and w' taking the place of Du_ε we get

$$\left(\int_{I_\varepsilon} |w'| \,\mathrm{d}x \right)^2 \le \left(\int_{I_\varepsilon} z^2 |w'|^2 \right) \left(\int_{I_\varepsilon} \frac{1}{z^2} \right).$$

Reversing the latter and taking into account the conditions on w obtains

$$\int_{I_\varepsilon} z^2 |w'|^2 \ \ge\ \left(\int_{I_\varepsilon} |w'| \,\mathrm{d}x \right)^2 \left(\int_{I_\varepsilon} \frac{1}{z^2} \right)^{-1} \ \ge\ 4(D-d)^2 \left(\int_{I_\varepsilon} \frac{1}{z^2} \right)^{-1}.$$

From this we deduce the lower bound

$$h_\varepsilon(z) \ \ge\ 4(D-d)^2 \left(2\varepsilon \int_{I_\varepsilon} \frac{1}{z^2} \,\mathrm{d}x \right)^{-1} + \int_{I_\varepsilon} \frac{(1-z)^2}{2\varepsilon} \,\mathrm{d}x. \tag{1.17}$$

Let us remark that optimizing $H_\varepsilon(w,z)$ with respect to $w \in G_\varepsilon$ we see that this inequality is actually an equality. Consider for $0 < \lambda < 1$ the inequalities:

$$\int\limits_{\{x \in I_\varepsilon : \varphi_\varepsilon \ge \lambda\}} \frac{1}{\varphi_\varepsilon^2} \le \frac{|I_\varepsilon|}{\lambda^2}, \qquad \int\limits_{\{x \in I_\varepsilon : \varphi_\varepsilon < \lambda\}} \frac{1}{\varphi_\varepsilon^2} \le \frac{1}{(1-\lambda)^2} \frac{2\varepsilon}{\eta^2} \left(\int_{I_\varepsilon} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon} \,\mathrm{d}x \right).$$

Applying both of them in (1.17) we obtain

$$\begin{aligned} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon, I_\varepsilon) &\ge h_\varepsilon(\varphi_\varepsilon) \\ &\ge \frac{2(D-d)^2}{\frac{\varepsilon |I_\varepsilon|}{\lambda^2} + \frac{1}{(1-\lambda)^2} \frac{2\varepsilon^2}{\eta^2} \left(\int_{I_\varepsilon} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon} \,\mathrm{d}x \right)} + \int_{I_\varepsilon} \left(\frac{(1-\varphi_\varepsilon)^2}{2\varepsilon} \right) \mathrm{d}x \\ &\ge 2(1-\lambda) \frac{\eta}{\varepsilon} (D-d) - (1-\lambda)^2 \frac{\eta^2}{2\varepsilon} \frac{|I_\varepsilon|}{\lambda^2} \end{aligned} \tag{1.18}$$

where the latter inequality is obtained by minimizing the function:

$$t \mapsto \frac{2(D-d)^2}{\frac{\varepsilon |I_\varepsilon|}{\lambda^2} + \frac{1}{(1-\lambda)^2} \frac{2\varepsilon^2}{\eta^2} t} + t.$$

Therefore we can pass to the limit in (1.18) and obtain:

$$\liminf_{\varepsilon \downarrow 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon, I_\varepsilon) \ge (1-\lambda) \beta \, 2(D-d).$$

Sending λ and d to 0 and recalling the estimate in (1.16) we get

$$\liminf_{\varepsilon\downarrow 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon, (y_1, y_2)) \;\geq\; 1 + \beta\, 2D \;=\; 1 + \beta|u(0^+) - u(0^-)|. \tag{1.19}$$

Step 3: Indeed by Fatou's Lemma for any $\tau \in \mathbf{S}^{d-1}$ and $\mathcal{H}^{d-1}$ almost every $y \in \Omega_\tau$ it holds

$$\liminf_{\varepsilon\downarrow 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O) \;\geq$$

$$\int_{\Pi_\tau} \liminf_{\varepsilon\downarrow 0} \left(\int_{O_y^\tau} \frac{1}{2\varepsilon}(\varphi_\varepsilon^2)_y^\tau |(u_\varepsilon')_y^\tau|^2 + \frac{\varepsilon}{2}|(\varphi_\varepsilon')_y^\tau|^2 + \frac{(1-(\varphi_\varepsilon)_y^\tau)^2}{2\varepsilon}\,\mathrm{d}t \right) \mathrm{d}\mathcal{H}^{d-1}(y).$$

Then by the results in *Step 2.1* and *2.2*, in particular inequality (1.19), it holds

$$\liminf_{\varepsilon\downarrow 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O) \geq \int_{\Pi_\tau} \sum_{m_i \in (J_u)_y^\tau} \left[1 + \beta|u_y^\tau(m_i^+) - u_y^\tau(m_i^-)|\right] \mathrm{d}\mathcal{H}^{d-1}(y).$$

Therefore by Theorem 1.4 we have $u \in SBV(O)$. Moreover, since $(u')_y^\tau = 0$ on each slice, we have $u \in PC(O)$. Applying identity (1.7) we get

$$\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O) \geq \int_{J_u \cap O} |\nu_u \cdot \tau|\, [1 + \beta|[u]|]\; \mathrm{d}\mathcal{H}^{d-1}. \tag{1.20}$$

In order to conclude, we use the following localization method stated by Braides in [Bra98, Prop. 1.16].

Lemma 1.1. *Let $\mu : \mathcal{A}(X) \to [0, +\infty)$ be an open-set function superadditive on open sets with disjoint compact closures and let λ be a positive measure on X. For any $i \in \mathbf{N}$ let ψ_i be a Borel function on X such that $\mu(A) \geq \int_A \psi_i \,\mathrm{d}\lambda$ for all $A \in \mathcal{A}(X)$. Then*

$$\mu(A) \geq \int_A \psi \,\mathrm{d}\lambda$$

where $\psi := \sup_i \psi_i$.

For any $u \in PC(O)$ let us introduce the increasing set function μ defined on $\mathcal{A}(O)$ by

$$\mu(A) := \inf_{(\varphi_\varepsilon, u_\varepsilon)\to(1,u)} \left\{ \liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; A) \right\}, \quad \text{for any } A \in \mathcal{A}(O).$$

Observe that for any two open sets A and B with disjoint compact closure and for any $(u_\varepsilon, \varphi_\varepsilon)$ such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ and $\varphi_\varepsilon \to 1$ on $A \cup B$, the restriction of u_ε to A (resp. B) weak-$*$ converges in A (resp. B) to the restriction of u on A (resp. B) and it follows

$$\mu(A + B) \geq \mu(A) + \mu(B).$$

This proves that μ is superadditive on open sets with disjoint compact closures. Let λ be a Radon measure defined as

$$\lambda := [1 + \beta|u(x^+) - u(x^-)|]\mathcal{H}^{d-1} \llcorner J_u.$$

Fix a sequence $(\tau_i)_{i\in\mathbf{N}}$ dense in $\mathbf{S}^{d-1}$. By (1.20) we have

$$\mu(O) \geq \int_O \psi_i \,\mathrm{d}\lambda, \quad i \in \mathbf{N},$$

where

$$\psi_i(x) := \begin{cases} |\langle \nu_u(x), \tau_i\rangle|, & \text{if } x \in J_u, \\ 0, & \text{if } x \in O \setminus J_u. \end{cases}$$

Hence by Lemma 1.1 we finally obtain

$$\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(u_\varepsilon, \varphi_\varepsilon; O) \geq \int_O \sup_i \psi_i(x)\,\mathrm{d}\mu = \int_{J_u\cap O} [1+\beta|[u]|]\ \mathrm{d}\mathcal{H}^{d-1}.$$

□

1.4 Equicoercivity and Γ-liminf

From now till the end of the chapter we assume that $\Omega \subset \mathbf{R}^2$. Let us first produce the following construction.

Lemma 1.2. *Given two probability measures μ_+ and μ_- supported on a finite set of points $S = \{x_0, \dots, x_N\}$, there exists a vector measure $\gamma = U(m_\gamma, \tau_\gamma, \Sigma_\gamma)$ and a finite partition $(\Omega_i) \subset \mathcal{A}(\Omega)$ of Ω such that*

- *a)* $\nabla\cdot\gamma = -\mu_+ + \mu_-$,
- *b) each Ω_i is a polyhedron,*
- *c)* $\Sigma_\gamma \subset \bigcup_i \partial\Omega_i$,
- *d) Ω_i is of finite perimeter for each i and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$,*
- *e)* $|\Omega \setminus \cup_i \Omega_i| = 0$.

Moreover if M is a 1 dimensional countably rectifiable set, we can choose γ and (Ω_i) such that $\mathcal{H}^1(M \cap \bigcup_i \partial\Omega_i) = 0$.

Proof. Let us fix a point $p \in \Omega \setminus S$ and assume

$$\mu_+ - \mu_- = \sum_{i=0}^N a_i \delta_{x_i}.$$

Consider the map $\frac{x_i - p}{|x_i - p|}t + p : [0,1] \to \Omega$, then the measure

$$\gamma_i = \left(\frac{p - x_i}{|p - x_i|}\cdot + x_i\right)_{\#}[0,1]$$

is supported on the segment $[p, x_i] =: \Sigma_\gamma$ and is such that $\nabla\cdot\gamma_i = \delta_{x_i} - \delta_p$ for $i \in \{0, \cdots, N\}$. We define

$$\gamma = \sum_{i=1}^N a_i \gamma_i.$$

By construction (a) holds true. Moreover, up to a small shift of p we may assume that $[p, x_i] \cap [p, x_j] = \{p\}$ for $i \neq j$.
Next, let D_j be the straight line supporting $[p, x_j]$. We define the sets (Ω_i) as the connected components of $\Omega \setminus (D_0 \cup \cdots \cup D_N)$. We see that (c, d, e) hold true.
For the last statement, we observe that by the coarea formula, we have $\mathcal{H}^1(\Sigma \cap \bigcup_i \partial\Omega_i) = 0$ for a.e. choice of p. □

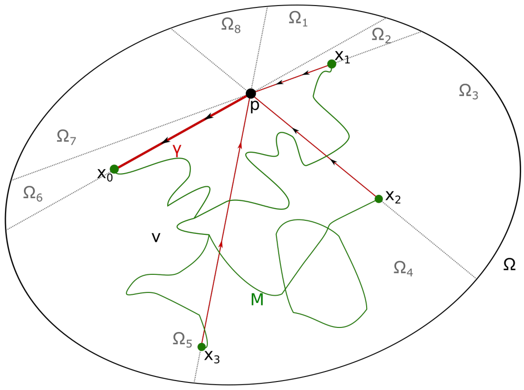


Figure 1.1: Example of the construction of the $\mathcal{H}^1$-rectifiable measure γ (red) and of the partition $\{\Omega_i\}$ (gray) in the case M (green) is being a $\mathcal{H}^1$-rectifiable set. Here $\mu_+ = \delta_{x_0}$ and $\mu_- = 1/3(\delta_{x_1} + \delta_{x_2} + \delta_{x_3})$.

We now prove the compactness property (Theorem 1.3). Let us consider a sequence $(\sigma_\varepsilon, \varphi_\varepsilon) \in \mathcal{M}(\Omega, \mathbf{R}^2)$ uniformly bounded in energy by $c_0 < +\infty$,

$$0 \leq \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \leq c_0 \qquad \text{for } \varepsilon \in (0, 1]. \tag{1.21}$$

Proof of Theorem 1.3. First observe that by definition (1.2) and equation (1.21), we have $\sigma_\varepsilon \in V_\varepsilon(\Omega)$ and $\varphi_\varepsilon \in W_\varepsilon(\Omega)$.

Next, using the arguments of Step 1 of the proof of Theorem 1.5, with $|\sigma_\varepsilon|$ instead of $|\nabla u_\varepsilon|$, inequality (1.10) reads

$$|\sigma_\varepsilon|(\Omega) \leq \sqrt{16\frac{\varepsilon^2}{\eta^2}\, c_0^2 + 8\varepsilon|\Omega|c_0} \xrightarrow{\varepsilon\downarrow 0} \frac{4c_0}{\beta} < \infty. \tag{1.22}$$

Thus the total variation of $(\sigma_\varepsilon)_\varepsilon$ is uniformly bounded as long as $\beta > 0$ and there exists a $\sigma \in \mathcal{M}_{\mathscr{S}}(\overline{\Omega})$ such that up to a subsequence $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}(\overline{\Omega})$.

Now, considering the last term in the energy (1.2) we have

$$\int_\Omega (1 - \varphi_\varepsilon)^2 \,\mathrm{d}x \leq 2\varepsilon\, \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \leq 2\varepsilon\, c_0 \to 0.$$

Hence, $\varphi_\varepsilon \to 1$ in $L^2(\Omega)$.

Let us now study the structure of the limit measure σ. Let us recall that $\hat{\Omega}$ is a bounded convex relatively open set such that $\overline{\Omega} \subset \hat{\Omega}$ and let us extend σ_ε by 0 and

φ_ε by 1 in $\hat{\Omega} \setminus \overline{\Omega}$. Obviously we have $\mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; \hat{\Omega}) = \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; \Omega)$, therefore for any $O \in \mathcal{A}_S(\hat{\Omega})$ applying the localization described in Section 1.3 we can associate to each σ_ε a function $u_\varepsilon \in W^{1,2}(O)$ with mean value 0 such that $\sigma_\varepsilon = \nabla u_\varepsilon^\perp$ in O. Since $|\nabla u_\varepsilon^\perp| = |\nabla u_\varepsilon|$ by Theorem 1.5 there exists a $u \in PC(O)$ such that, up to extracting a subsequence, $u_\varepsilon \overset{*}{\rightharpoonup} u$. Eventually, from formula (1.8), we get

$$\sigma \llcorner O = Du^\perp \llcorner O = -[u]\nu_{J_u}^\perp \mathcal{H}^1 \llcorner (J_u \cap O).$$

Since we can cover $\overline{\Omega} \setminus S$ by countable many sets $O \in \mathcal{A}_S(\hat{\Omega})$, this shows that σ decomposes as

$$\sigma \;=\; (m_\sigma, \tau_\sigma, \Sigma_\sigma) + \omega,$$

where ω is a measure absolutely continuous with respect to $\mathcal{H}^0 \llcorner S$. By Lemma 1.2 there exists a rectifiable measure $\gamma = U(m_\gamma, \tau_\gamma, \Sigma_\gamma)$ such that $\nabla \cdot (\sigma + \gamma) = 0$ and $\mathcal{H}^1(\Sigma_\gamma \cap \Sigma_\sigma) = 0$. Then there exists a $u \in BV(\Omega)$ such that $Du = \sigma^\perp + \gamma^\perp$. Since $u \in BV(\Omega)$ and S is composed by a finite number of points, we deduce $|Du|(S) = 0$ which implies $|\omega|(S) = 0$. Hence σ writes in the form $(m_\sigma, \tau_\sigma, \Sigma_\sigma)$. □

Let us now use the local results of Section 1.3 to prove the $\Gamma - \liminf$ inequality.

Proof of Theorem 1.1. Let $(\sigma_\varepsilon, \varphi_\varepsilon)$ such that $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ and $\varphi_\varepsilon \to \varphi$ as in the statement of the theorem. Without loss of generality, we can suppose that $\mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) < +\infty$. Let $\hat{\Omega}$ be as in the proof of Theorem 1.3 and let us define $\chi = \Gamma - \liminf_\varepsilon \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon)$ and $\lambda = \beta|\sigma| + \mathcal{H}^1 \llcorner \Sigma_\sigma$. Consider the countable family of sets $\{O_i\} \subset \mathcal{A}_S(\hat{\Omega})$ made of the relatively open rectangles $O_i \subset \hat{\Omega} \setminus S$ with vertices in $\mathbf{Q}^2$. The local result stated in Theorem 1.5 gives for any $i \in \mathbf{N}$

$$\chi(A) \geq \mu(O_i \cap A) \geq \lambda(O_i \cap A) = \int_A \psi_i \,\mathrm{d}\lambda,$$

where $\psi_i := \mathbf{1}_{O_i}$. Therefore Lemma 1.1 gives

$$\Gamma - \liminf_{\varepsilon \downarrow 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) = \mu(\hat{\Omega}) \geq \lambda(\hat{\Omega}) = \beta|\sigma|(\overline{\Omega}) + \mathcal{H}^1(\Sigma_\sigma)$$

since $\sup_i \psi_i$ is the constant function 1. □

1.5 Γ-limsup inequality

Let us prove the Γ-limsup inequality stated in Theorem 1.2. Recall that the latter consists in finding a sequence $(\sigma_\varepsilon, \varphi_\varepsilon)$ for any given couple $(\sigma, \varphi) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^2) \times L^1(\Omega)$ such that $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$, $\varphi_\varepsilon \to \varphi$ in $L^1(\Omega)$ and

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \leq \mathscr{E}_\beta(\sigma, \varphi). \tag{1.23}$$

When $\mathscr{E}_\beta(\sigma, \varphi) = +\infty$ the inequality is valid for any sequence therefore by definition (1.5) we can assume $\sigma = (m, \tau, \Sigma)$ and $\varphi = 1$. In view of the results from

White [Whi99b], [Whi99a] and Xia [Xia03] polyhedral vector measures are dense in energy and it is sufficient to consider vector measures of the form

$$\sigma = \sum_{i=1}^{n} (m_i, \tau_i, \Sigma_i), \tag{1.24}$$

where Σ_i is a segment, $m_i \in \mathbf{R}_+$ is $\mathcal{H}^1$-a.e. constant and τ_i is an orientation of Σ_i for each i. We included in appendix A a proof of this result based on BV functions. Without loss of generality we can suppose that for each couple of segments Σ_i, M_j, for $i \neq j$, the intersection $\Sigma_i \cap M_j$ is at most a point (called branching point) not belonging to the relative interior of Σ_i and M_j. We first produce the estimate (1.23) for σ concentrated on a single segment thus let us assume $\sigma = me_1\, \mathcal{H}^1 \llcorner (0,l) \times \{0\}$.

Notation: Let us fix the values

$$a_\varepsilon := \begin{cases} \dfrac{m\beta\,\varepsilon}{2} & \text{if } \beta > 0 \\ \varepsilon & \text{if } \beta = 0 \end{cases}, \quad b_\varepsilon := \varepsilon \ln\left(\frac{1-\eta}{\varepsilon}\right) \quad \text{and} \quad r_\varepsilon = \max\{\varepsilon, a_\varepsilon\}.$$

Let $d_\infty(x,S)$ be the distance function from x to the set $S \subset \Omega$ relative to the infinity norm on $\mathbf{R}^2$ and $Q_r(P) = \{x \in \mathbf{R}^2 : d_\infty(x,P) \leq r\}$ the square centered in P of size $2r$ and sides parallel to the axes. Introduce the sets

$$\begin{aligned}
I_{a_\varepsilon} &:= \{x \in \mathbf{R}^2 : d_\infty(x, [0,l] \times \{0\}) \leq a_\varepsilon\} \cup Q_{r_\varepsilon}(0,0) \cup Q_{r_\varepsilon}(l,0),\\
I_{b_\varepsilon} &:= \{x \in \mathbf{R}^2 : d_\infty(x, I_{a_\varepsilon}) \leq b_\varepsilon\} \setminus I_{a_\varepsilon},\\
I_{c_\varepsilon} &:= \{x \in \mathbf{R}^2 : d_\infty(x, (I_{a_\varepsilon} \cup I_{b_\varepsilon})) \leq \varepsilon\} \setminus (I_{a_\varepsilon} \cup I_{b_\varepsilon}),\\
I_{d_\varepsilon} &:= \Omega \setminus (I_{a_\varepsilon} \cup I_{b_\varepsilon} \cup I_{c_\varepsilon}),\\
\Sigma_\varepsilon(t) &:= \{(t, x_2) \; : \; |x_2| \leq r_\varepsilon\},
\end{aligned}$$

and define $R_\varepsilon = I_{a_\varepsilon} \setminus (Q_{r_\varepsilon}(0,0) \cup Q_{r_\varepsilon}(l,0))$.

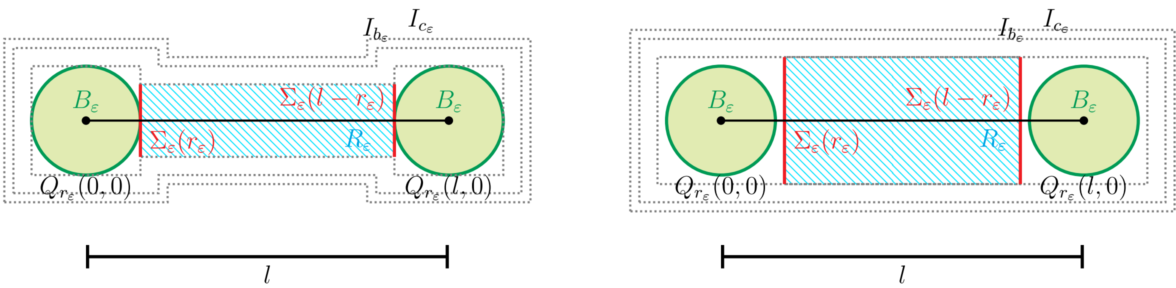
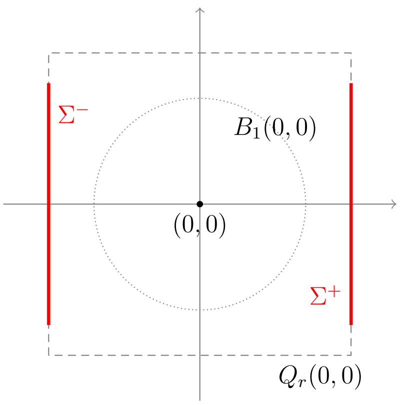


Figure 1.2: Example of the neighborhoods of the segment $[0,l] \times \{0\}$. On the left the case in which $r_\varepsilon = \varepsilon$, on the right the case in which $r_\varepsilon = a_\varepsilon > \varepsilon$. The cyan region is R_ε and $I_{a_\varepsilon} = R_\varepsilon \cup (Q_{r_\varepsilon}(0,0) \cup Q_{r_\varepsilon}(l,0))$. Remark that $\operatorname{supp}(\rho_\varepsilon) = B(0,\varepsilon)$.

Costruction of σ_ε: We build σ_ε as a vector field supported on I_{a_ε}. In particular we add together three different constructions performed respectively on R_ε, $Q_{r_\varepsilon}(0,0)$ and $Q_{r_\varepsilon}(l,0)$. We construct σ_ε on R_ε in order to obtain the Γ-limsup inequality, on the other hand we are forced to modify such construction in a square neighborhood of each ending point of the segment to control $\nabla\cdot\sigma_\varepsilon$. As a matter of fact we need to verify that the piecewise definitions coincide on the sets $\Sigma_\varepsilon(r_\varepsilon)$ and $\Sigma_\varepsilon(l-r_\varepsilon)$ which correspond to the interfaces between $Q_{r_\varepsilon}(0,0)$ and R_ε, and, R_ε and $Q_{r_\varepsilon}(l,0)$. Let $r=r_\varepsilon/\varepsilon$ and consider the problem

$$\begin{cases} \Delta u = \pm m\ (\delta_{x_0} * \rho)\,, & \text{on } Q_r(0,0),\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\pm m}{\mathcal{H}^1(\Sigma)}, & \text{on } \Sigma^\pm = \left\{ x\in \mathbf{R}^2 : x_1=\pm 1,\ |x_2| \le \dfrac{m\beta}{2}\right\},\\ \dfrac{\partial u}{\partial \nu} = 0, & \text{otherwise on } \partial Q_r(0,0)\setminus \Sigma^\pm. \end{cases}$$



In the latter the set Σ^+ (resp. Σ^-) is the image of the set $\Sigma_\varepsilon(r_\varepsilon)$ (resp. $\Sigma_\varepsilon(l-r_\varepsilon)$) via the map

$$x \longmapsto \frac{x}{\varepsilon}, \qquad \left(\text{resp.}\quad x\longmapsto \frac{x-(l,0)}{\varepsilon}\right).$$

Let u^+ be the solution relative to the problem in which every occurrence of $\pm$ is replaced by $+$ and let u^- defined accordingly. Then set

$$\sigma_\varepsilon = \begin{cases} \dfrac{\nabla u^+(x/\varepsilon)}{\varepsilon}, & \text{on } Q_{r_\varepsilon}(0,0),\\ \dfrac{m}{2a_\varepsilon} e_1, & \text{on } R_\varepsilon,\\ \dfrac{\nabla u^-((x-(l,0))/\varepsilon)}{\varepsilon}, & \text{on } Q_{r_\varepsilon}(l,0). \end{cases} \tag{1.25}$$

Indeed, the Neumann Boundary conditions imposed for u^+ (resp. u^-) on Σ^+ (resp. Σ^-) ensure that the latter piecewise definition is continuous on $\Sigma_\varepsilon(r_\varepsilon)$ and $\Sigma_\varepsilon(l-r_\varepsilon)$. By construction we have that $\nabla\cdot\sigma_\varepsilon = m\,[(\delta_{(0,0)}-\delta_{(l,0)}) * \rho_\varepsilon]$ and $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$. Let us point out as well that there exists a constant $c(\alpha,m)$ such that

$$\begin{aligned} c(\alpha,m) &:= \int_{Q_{r_\varepsilon}(l,0)} |\sigma_\varepsilon|^2\,\mathrm{d}x = \int_{Q_{r_\varepsilon}(0,0)} |\sigma_\varepsilon|^2\,\mathrm{d}x \\ &= \int_{Q_r(0,0)} \left|\nabla u^+(x)\right|^2\,\mathrm{d}x = \int_{Q_r(0,0)} \left|\nabla u^-(x)\right|^2\,\mathrm{d}x. \end{aligned} \tag{1.26}$$

Costruction of φ_ε: Most of the properties of φ_ε are a consequence of the inequalities obtained in Theorem 1.5 and the structure of σ_ε. On one hand we need φ_ε to attain the lowest value possible on I_{a_ε} in order to compensate the concentration of σ_ε in this set, on the other, as shown in inequality (1.16), we need to provide the optimal profile for the transition from this low value to 1. For this reasons we are led to consider the following ordinary differential equation associated with the optimal transition

$$\begin{cases} w_\varepsilon' = \dfrac{1}{\varepsilon}(1-w_\varepsilon), \\ w_\varepsilon(0) = \eta. \end{cases} \tag{1.27}$$

Observe that $w_\varepsilon = 1-(1-\eta)\exp\left(\frac{-t}{\varepsilon}\right)$ is the explicit solution of equation (1.27) and set

$$\varphi_\varepsilon(x) := \begin{cases} \eta, & \text{if } x \in I_{a_\varepsilon}, \\ w_\varepsilon(d_\infty(x, I_{a_\varepsilon})), & \text{if } x \in I_{b_\varepsilon}, \\ d_\infty(x, I_{b_\varepsilon}) - \varepsilon + 1, & \text{if } x \in I_{c_\varepsilon}, \\ 1, & \text{otherwise.} \end{cases} \tag{1.28}$$

Evaluation of $\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon)$: (case of a σ concentrated on a line segment)
We prove inequality (1.23) for the sequence we have produced. Since the sets I_{a_ε}, I_{b_ε}, I_{c_ε} and I_{d_ε} are disjoint we can split the energy as follows

$$\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon) = \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; I_{a_\varepsilon}) + \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; I_{b_\varepsilon}) + \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; I_{c_\varepsilon}) + \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; I_{d_\varepsilon}) \tag{1.29}$$

and evaluate each component individually. Since σ_ε is null and φ_ε is constant and equal to 1 in I_{d_ε} we have that $\mathscr{F}_\varepsilon(\sigma,\varphi_\varepsilon; I_{d_\varepsilon}) = 0$. For the other components we strongly use the definitions in (1.25) and (1.28). First we split again the energy on the set I_{a_ε} as following

$$\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; I_{a_\varepsilon}) = \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; R_\varepsilon) + \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; Q_{r_\varepsilon}(0,0)) + \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; Q_{r_\varepsilon}(l,0)).$$

Now identity (1.26) leads to the estimate

$$\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; Q_{r_\varepsilon}(0,0)) = \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; Q_{r_\varepsilon}(l,0)) = \frac{\eta^2}{2\varepsilon}c(\beta,m) + \frac{(1-\eta)^2}{2\varepsilon}\, r_\varepsilon^2$$

and

$$\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; R_\varepsilon) = \left[\frac{1}{2\varepsilon}\eta^2\left|\frac{m}{2a_\varepsilon}\right|^2 + \frac{(1-\eta)^2}{2\varepsilon}\right]|R_\varepsilon| \leq \left[\frac{(m\eta)^2}{8\varepsilon a_\varepsilon^2} + \frac{1}{2\varepsilon}\right]2a_\varepsilon l.$$

Then passing to the limsup we obtain

$$\limsup_{\varepsilon\downarrow 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon; I_{a_\varepsilon}) \leq m\beta l = m\beta\mathcal{H}^1([0,l]\times\{0\}). \tag{1.30}$$

To obtain the inequality on the sets I_{b_ε} and I_{c_ε} we are going to apply the coarea formula therefore let us observe that for both $d_\infty(x, I_{a_\varepsilon})$ and $d_\infty(x, I_{b_\varepsilon})$ there holds $|\nabla d_\infty(x,\cdot)| = 1$ for a.e. $x \in \Omega$ and that there exist a constant $k = k(\beta, m)$ such that the level lines $\{d_\infty(x,\cdot)=t\}$ have $\mathcal{H}^1$ length controlled by $2l+kt$. In view of these remarks we obtain

$$\begin{aligned}
\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon;I_{b_\varepsilon}) &= \int_{I_{b_\varepsilon}} \left[\frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2+\frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}\right]|\nabla d_\infty(x,I_{a_\varepsilon})|\,\mathrm{d}x\\
&= \int_0^{b_\varepsilon}\left[\frac{(1-w_\varepsilon(t))^2}{2\varepsilon}+\frac{\varepsilon}{2}|w_\varepsilon'(t)|^2\right]\mathcal{H}^1(\{d_\infty(\cdot,I_{a_\varepsilon})=t\})\ \mathrm{d}t\\
&\le (2l+k\varepsilon)\left[\frac{1}{2}(1-w_\varepsilon(t))^2\right]_0^{b_\varepsilon}\\
&= \left(l-\frac{k\varepsilon}{2}\right)\left[(1-\eta)^2-\varepsilon^2\right]\xrightarrow[\varepsilon\downarrow 0]{} l=\mathcal{H}^1([0,l]\times\{0\})
\end{aligned}\tag{1.31}$$

and

$$\begin{aligned}
\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon;I_{c_\varepsilon}) &= \int_{I_{c_\varepsilon}} \left[\frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2+\frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}\right]|\nabla d_\infty(x,I_{b_\varepsilon})|\,\mathrm{d}x\\
&= \int_0^{\varepsilon}\left[\frac{(1-t+\varepsilon-1)^2}{2\varepsilon}+\frac{\varepsilon}{2}\right]\mathcal{H}^1(\{d_\infty(\cdot,I_{b_\varepsilon}\cup I_{a_\varepsilon})=t\})\ \mathrm{d}t\\
&\le (2l+k\varepsilon)\left[\frac{(t-\varepsilon)^3}{6\varepsilon}+\frac{\varepsilon}{2}t\right]_0^{\varepsilon}\\
&= (2l+k\varepsilon)\,\varepsilon^2\,\frac{2}{3}\xrightarrow[\varepsilon\downarrow 0]{} 0.
\end{aligned}\tag{1.32}$$

Finally adding up equations (1.29), (1.30), (1.31) and (1.32) we obtain

$$\limsup_{\varepsilon\downarrow 0}\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon)\le(1+\beta\,m)\,\mathcal{H}^1([0,l]\times\{0\}).$$

Case of a generic σ of the form (1.24)*:*

Let us call σ_ε^i, φ_ε^i the functions obtained above for each $\sigma_i=m_i\tau_i\mathcal{H}^1\llcorner\Sigma_i$ and set

$$\sigma_\varepsilon=\sum_{i=1}^n\sigma_\varepsilon^i,\qquad\qquad \varphi_\varepsilon=\min_i\ \varphi_\varepsilon^i.$$

In view of the constraint, it holds

$$(\mu_+-\mu_-)=\nabla\cdot\sigma=\sum_i\nabla\cdot(m_i\tau_i\mathcal{H}^1\llcorner\Sigma_i)=\sum_i m_i(\delta_{P_i^+}-\delta_{P_i^-}),$$

where P_i^+ and P_i^- are the starting and ending point of the segment Σ_i according to its orientation τ_i. Replacing each σ_i with σ_ε^i we have

$$\nabla\cdot\sigma_\varepsilon=\sum_i\nabla\cdot\sigma_\varepsilon^i=\sum_i m_i(\delta_{P_i^+}-\delta_{P_i^-})*\rho_\varepsilon=(\mu_+-\mu_-)*\rho_\varepsilon.$$

Thus σ_ε satisfies constraint (1.1). We now prove inequality (1.23). The following inequality holds true

$$\begin{aligned}
\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon) &= \int_\Omega\frac{1}{2\varepsilon}|\min_i\varphi_\varepsilon^i|^2|\sum_{i=1}^n\sigma_\varepsilon^i|^2+\frac{\varepsilon}{2}|\nabla(\min_i\varphi_\varepsilon^i)|^2+\frac{(1-\min_i\varphi_\varepsilon^i)^2}{2\varepsilon}\,\mathrm{d}x\\
&\le \int_\Omega\frac{1}{2\varepsilon}|\min_i\varphi_\varepsilon^i|^2|\sum_{i=1}^n\sigma_\varepsilon^i|^2\,\mathrm{d}x+\sum_{i=1}^n\int_\Omega\frac{\varepsilon}{2}|\nabla\varphi_\varepsilon^i|^2+\frac{(1-\varphi_\varepsilon^i)^2}{2\varepsilon}\,\mathrm{d}x,
\end{aligned}\tag{1.33}$$

therefore let us estimate the first integral in the latter. Observe that for ε sufficiently small we can assume that all the R^i_ε are pairwise disjoint thus we study the behavior in the squares. Let $M_{i_1}, \dots, M_{i_{m_P}}$ be the segments meeting at a branching point P. For $j = i_1, \dots, i_{m_P}$ let us call $Q_{r^j_\varepsilon}(P)$ the squared neighborhood of P relative to the segment M_j as constructed previously. Let us recall that by definition φ_ε is constant and equal to η on $\cup_{j=i_1}^{m_P} Q_{r^j_\varepsilon}(P)$ then we have the estimate

$$\begin{aligned}
\int\limits_{\cup_{j=i_1}^{m_P}(R^j_\varepsilon \cup Q_{r^j_\varepsilon}(P))} \frac{\varphi_\varepsilon^2}{2\varepsilon}|\sigma_\varepsilon|^2\,\mathrm{d}x &= \sum_{j=i_1}^{m_P}\int\limits_{R^j_\varepsilon}\frac{\varphi_\varepsilon^2}{2\varepsilon}|\sigma_\varepsilon|^2\,\mathrm{d}x + \int\limits_{\cup_{i=i_1}^{m_P} Q_{r^j_\varepsilon}(P)}\frac{\varphi_\varepsilon^2}{2\varepsilon}|\sum_{j=i_1}^{m_P}\sigma^j_\varepsilon|^2\,\mathrm{d}x \\
&\le \sum_{j=i_1}^{m_P}\int\limits_{R^j_\varepsilon}\frac{\varphi_\varepsilon^2}{2\varepsilon}|\sigma_\varepsilon|^2\,\mathrm{d}x + m_P\frac{\eta^2}{2\varepsilon}\sum_{j=i_1}^{m_P}\int\limits_{Q_{r^j_\varepsilon}(P)}|\sigma^j_\varepsilon|^2\,\mathrm{d}x \\
&\le \sum_{j=i_1}^{m_P}\int\limits_{(R^j_\varepsilon\cup Q_{r^j_\varepsilon}(P))}\frac{1}{2\varepsilon}|\varphi^j_\varepsilon|^2|\sigma^j_\varepsilon|^2\,\mathrm{d}x + \underbrace{(m_P-1)\left(\sum_{j=i_1}^{i_{m_P}}c(\beta,m_j)\right)\frac{\eta^2}{2\varepsilon}}_{c(m_P,\beta,m_{i_1},\dots,m_{i_{m_P}})\varepsilon}.
\end{aligned} \tag{1.34}$$

Applying inequality (1.34) on each branching point in equation (1.33) and recomposing the integral gives

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0}\mathscr{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon) &\le \limsup_{\varepsilon\downarrow 0}\sum_{i=1}^{n}\mathscr{F}_\varepsilon(\sigma^i_\varepsilon,\varphi^i_\varepsilon) + n\;c(n,\beta,m_i,\dots,m_n)\varepsilon \\
&\le \sum_{i=1}^{n}(1+\beta\,m_i)\,\mathcal{H}^1(\Sigma_i) \\
&= \int_{\mathrm{supp}(\sigma)}(1+\beta\,m)\,\mathrm{d}\mathcal{H}^1 = \mathscr{E}_\beta(\sigma,1)
\end{aligned}$$

which ends the proof.

1.6 Numerical Approximation

In this section we present numerical evidence of the Γ-convergence result we have shown in the setting of the Steiner Minimal Tree problem. Thus we consider $\mu_+ = \delta_{x_0}$ and $\mu_- = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ for some points $\{x_0, \dots, x_N\} \subset \Omega$. The first issue we address is how to impose the divergence constraint. To this aim it is convenient to introduce the following notation

$$f_\varepsilon = \left(\delta_{x_0} - \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\right) * \rho_\varepsilon,$$

$$G_\varepsilon(\sigma,\varphi) = \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{2\varepsilon}|\varphi|^2|\sigma|^2\right] \,\mathrm{d}x, & \text{if } \sigma \in V_\varepsilon \text{ and } \varphi \in W_\varepsilon, \\ +\infty, & \text{otherwise in } L^2(\Omega,\mathbf{R}^2), \end{cases}$$

$$\Lambda_\varepsilon(\varphi) = \begin{cases} \displaystyle\int_\Omega \left[\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{2\varepsilon}\right] \,\mathrm{d}x, & \text{if } \varphi \in W_\varepsilon, \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases} \tag{1.35}$$

Then let us observe that the following equality holds

$$\min_{\sigma\in L^2(\Omega,\mathbf{R}^2)} G_\varepsilon(\sigma,\varphi) = \inf_{\sigma\in L^2(\Omega,\mathbf{R}^2)} \left\{ \sup_{u\in W^{1,2}(\Omega)} \int_\Omega \frac{1}{2\varepsilon}|\varphi|^2|\sigma|^2 + u(\nabla\cdot\sigma - f_\varepsilon)\,\mathrm{d}x \right\}.$$

By von Neumann's min-max Theorem [ABM14, Thm. 9.7.1] we can exchange inf and sup obtaining for each $\varepsilon > 0$ and $\varphi \in W_\varepsilon$

$$\begin{aligned} \min_\sigma G_\varepsilon(\sigma,\varphi) &= \sup_u \inf_\sigma \int_\Omega \frac{1}{2\varepsilon}|\varphi|^2|\sigma|^2 - (\langle\nabla u,\sigma\rangle + uf_\varepsilon)\,\mathrm{d}x \\ &= -\min_u \int_\Omega \frac{\varepsilon|\nabla u|^2}{2|\varphi|^2} + uf_\varepsilon\,\mathrm{d}x = -\min_u \overline{G}_\varepsilon(u,\varphi), \end{aligned}$$

with $\sigma = \frac{\varepsilon\nabla u}{\varphi^2}$, this naturally leads to the following alternate minimization problem: given an initial guess φ_0 we define

$$\begin{aligned} \sigma_j &:= \frac{\varepsilon\nabla u_j}{\varphi_j^2} \qquad \text{where} \qquad u_j := \operatorname{argmin} \overline{G}_\varepsilon(u,\varphi_j), \\ \varphi_{j+1} &:= \operatorname{argmin} G_\varepsilon(\sigma_j,\varphi) + \Lambda_\varepsilon(\varphi). \end{aligned}$$

This formulation led to Algorithm 1. We define a circular domain Ω containing the points in S endowed with a uniform mesh and four values β, $\varepsilon_{in}, \varepsilon_{end}$ and N_{iter} and a gaussian convolution kernel $\rho_{\varepsilon_{end}}$ in order to define f_ε. We have implemented the algorithm in FREEFEM++ choosing for the discrete spaces for u and φ the space of piecewise polynomials of order 1. To validate Algorithm 1 we tested on the constraint given by four points defining a square inscribed in the unitary circumference, namely $x_0 = (-\sqrt{2}/2, -\sqrt{2}/2)$, $x_1 = (\sqrt{2}/2, -\sqrt{2}/2)$, $x_2 = (\sqrt{2}/2, \sqrt{2}/2)$ and $x_4 = (-\sqrt{2}/2, \sqrt{2}/2)$. Indeed, for such constraint, we can obtain an explicit solution which allows a visual comparison with the one obtained with Algorithm 1. As it is shown in Figure 1.3 the solution is far from being satisfactory. We think the failure of this procedure is due to the relation between the geometry of the space and the one of the solution itself. In particular, to obtain a good approximation it is necessary to

Algorithm 1 Alternate Minimization

Require: $S = \{x_0, \dots, x_N\}$, ε_{in}, ε_{end}, N_{iter}, β, index.
  function ALT. MIN.$(x_0, \dots, x_N, \varepsilon_{in}, \varepsilon_{end}, N_{\text{iter}}, \beta, \rho)$
    set $f_\varepsilon = (N\delta_{x_0} - \sum_{i=1}^{N} \delta_{x_i}) * \rho_{\varepsilon_{end}}$ and $\varphi_0 = 1$
    for $j = 1, \dots, N_{\text{iter}}$ **do**
      $\varepsilon_j = \left(\frac{j - N_{\text{iter}}}{N_{\text{iter}}}\right) \varepsilon_{in} - \left(\frac{j}{N_{\text{iter}}}\right) \varepsilon_{end}$
      $\tilde{\varphi} \leftarrow L^1$-projection of φ_{j-1}^2
      set $u_j := \operatorname{argmin} \overline{G}_{\varepsilon_j}(u, \varphi_{j-1})$
      set $\sigma_j = \frac{\varepsilon_j \nabla u_j}{\tilde{\varphi}_{j-1}}$
      set $\varphi_j := \operatorname{argmin} G_{\varepsilon_j}(\sigma_j, \varphi) + \Lambda_\varepsilon(\varphi)$
      set $\varphi_j = \max\{\eta, \varphi_j\}$
    end for
  end function
  return $\varphi_{N_{\text{iter}}}, \sigma_{N_{\text{iter}}}$.

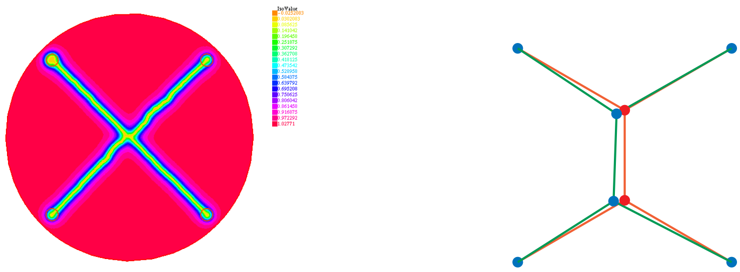


Figure 1.3: On the left: Graph of φ_{Niter} obtained via the Alternate Minimization Algorithm 1. On the right: in red, one of the solutions to the Steiner problem, while in blue, a minimizer of the energy $\mathscr{E}_\beta$.

refine the mesh where φ attains values close to zero but we observed that this restrains the process of approaching the solution. To overcame the problem we propose a modifications to Algorithm (1) to include a step of joint minimization. Let us consider a smooth diffeomorphism $T : \Omega \to \Omega$ and define

$$\varphi_T = \varphi \circ T(x), \qquad \sigma_T = \sigma \circ T(x),$$

and the functional

$$F_\varepsilon(T) = G_\varepsilon(\sigma \circ T, \varphi \circ T) + \Lambda_\varepsilon(\varphi \circ T).$$

Let $dF_\varepsilon(Id)$ be the differential of the the functional F_ε evaluated for $T = Id$. We represent dF as function $V \in W^{1,2}(\Omega, \Omega)$ by solving the elliptic problem

$$\langle V, W\rangle_{W^{1,2}} = \langle dF_\varepsilon(Id), W\rangle_{W^{1,2}} \quad \text{for any test vector field } W.$$

Let $\overline{V}$ be a solution to the latter problem, we perform a gradient descent in the direction $-\overline{V}$. In Algorithm 2, we implemented this joint minimization step. As it is possible to remark from a visual comparison of Figure 1.4 and Figure 1.3 the joint minimization procedure allows to displace the functions. Indeed, as shown in Figure 1.5, the energy decreases during the joint minimization procedure as it. Let us propose a second

Algorithm 2 Joint Minimization

Require: $S = \{x_0, \dots, x_N\}$, ε_{in}, ε_{end}, N_{iter}, β, index, N_{freq}.

  function JOINT MIN.$(x_0, \dots, x_N, \varepsilon_{in}, \varepsilon_{end}, N_{\text{iter}}, \beta, \rho)$

    set $f_\varepsilon = (N\delta_{x_0} - \sum_{i=1}^{N} \delta_{x_i}) * \rho_{\varepsilon_{end}}$ and $\varphi_0 = 1$

    for $j = 1, \dots, N_{\text{iter}}$ **do**

      $\varepsilon_j = \left(\frac{j - N_{\text{iter}}}{N_{\text{iter}}}\right) \varepsilon_{in} - \left(\frac{j}{N_{\text{iter}}}\right) \varepsilon_{end}$

      $\tilde{\varphi} \leftarrow L^1$-projection of φ_{j-1}^2

      set $u_j := \operatorname{argmin} \overline{G}_{\varepsilon_j}(u, \varphi_{j-1})$

      set $\sigma_j = \frac{\varepsilon_j \nabla u_j}{\tilde{\varphi}_{j-1}}$

      set $\varphi_j := \operatorname{argmin} G_{\varepsilon_j}(\sigma_j, \varphi) + \Lambda_\varepsilon(\varphi)$

      if $j \% N_{\text{freq}} == 0$ & $j \geq$ index **then**

        solve $\langle V, W \rangle = \langle dF_{\varepsilon_j}(Id), W \rangle$

        set $\varphi_j = \varphi_j(x - V)$

      end if

      set $\varphi_j = \max\{\eta, \varphi_j\}$

    end for

  end function

  return $\varphi_{N_{\text{iter}}}, \sigma_{N_{\text{iter}}}$.

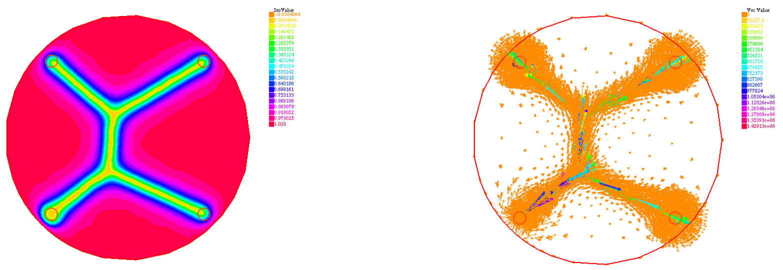


Figure 1.4: On the left the graph of φ_{Niter} on the right the one of σ_{Niter} obtained via the Joint Minimization Algorithm 2.

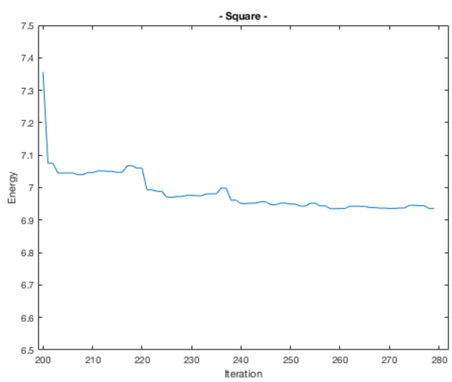


Figure 1.5: Behaviour of the energy during the joint minimization iterations of Algorithm 2.

modification to Algorithm 1. Let us observe that the optimization of the component

Λ_ε defined in equation (1.35) is the one responsible for the length minimization of the support of σ. Therefore it is reasonable to look for a gradient descent in the component Λ_ε. In Algorithm 3 we implement such procedure. This method enhances the length

---

Algorithm 3 Length Minimization

Require: $S = \{x_0, \ldots, x_N\}$, ε_{in}, ε_{end}, N_{iter}, β, index, N_{freq}.
  function LENGTH MIN.$(x_0, \ldots, x_N, \varepsilon_{in}, \varepsilon_{end}, N_{\text{iter}}, \beta, \rho)$
    set $f_\varepsilon = (N\delta_{x_0} - \sum_{i=1}^{N} \delta_{x_i}) * \rho_{\varepsilon_{end}}$ and $\varphi_0 = 1$
    for $j = 1, \ldots, N_{\text{iter}}$ **do**
      $\varepsilon_j = \left(\frac{j - N_{\text{iter}}}{N_{\text{iter}}}\right) \varepsilon_{in} - \left(\frac{j}{N_{\text{iter}}}\right) \varepsilon_{end}$
      $\tilde{\varphi} \leftarrow L^1$-projection of φ_{j-1}^2
      set $u_j := \operatorname{argmin} G'_{\varepsilon_j}(u, \varphi_{j-1})$
      set $\sigma_j = \frac{\varepsilon_j \nabla u_j}{\tilde{\varphi}_{j-1}}$
      set $\varphi_j := \operatorname{argmin} G_{\varepsilon_j}(\sigma_j, \varphi) + \Lambda_\varepsilon(\varphi)$
      if $j \% N_{\text{freq}} == 0$ & $j \geq$ index **then**
        solve $\langle V, W \rangle = \langle d\Lambda_{\varepsilon_j}(Id), W \rangle$
        set $\varphi_j = \varphi_j(x + T)$
      end if
      set $\varphi_j = \max\{\eta, \varphi_j\}$
    end for
  end function
  return $\varphi_{N_{\text{iter}}}, \sigma_{N_{\text{iter}}}$.

---

minimization process since but has the drawback is that displacing φ and σ in the direction $-d\Lambda$ we could loose the divergence constraint. To avoid such eventuality we perform several steps of Alternate Minimization after the displacement. In the next figures we show the graphs obtained for the couple $(\sigma_{N_{\text{iter}}}, \varphi_{N_{\text{iter}}})$ via Algorithm 3 with the choices $\beta = 0.05$, $\varepsilon_{in} = 0.5$, $\varepsilon_{end} = 0.05$, $\beta = 0.05$, $N_{\text{iter}} = 500$ and $index = 300$. We have chosen to make simulations for points located on the vertices of regular polygons of respectively 3, 4, 5 and 6 vertices. A direct visual comparison between the obtained results in Figure 1.6 and the exact solutions in Figure 1.7.

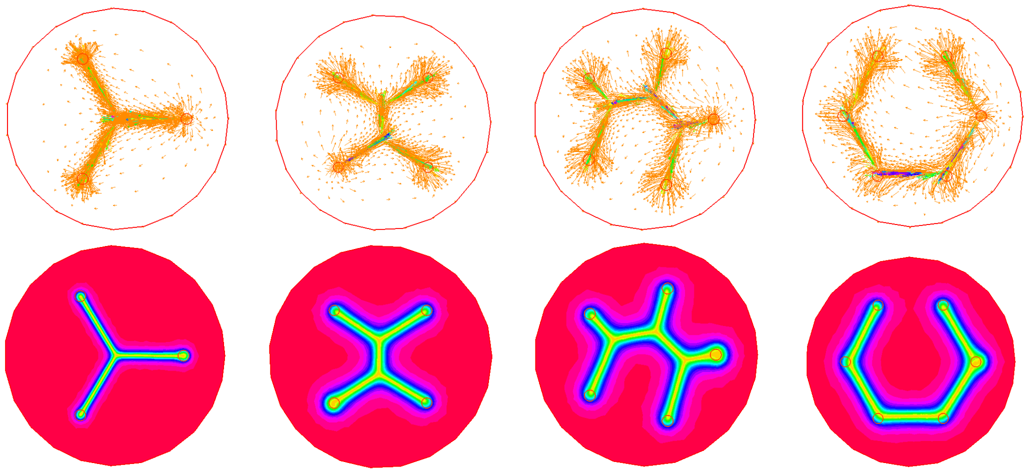


Figure 1.6: Graph of the couple $(\sigma_{N_{\text{iter}}}, \varphi_{N_{\text{iter}}})$ obtained via Algorithm 1 in the case of 3, 4, 5 and 6 points located on the vertices of a regular polygon.

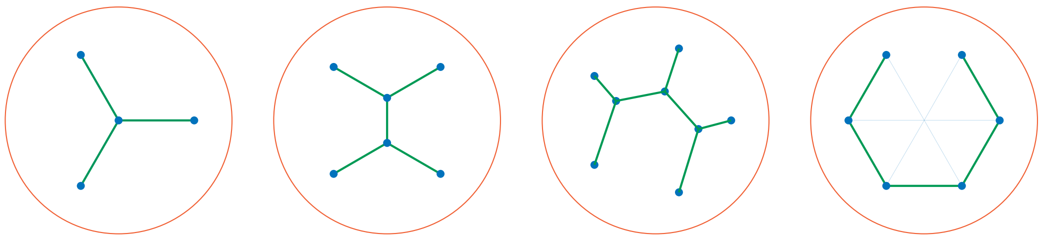


Figure 1.7: Graph of the exact solutions to the Steiner Problem constrained as in the previous figure.

Chapter 2

Multidimensional case

2.1 Introduction

The content of this and the following chapter are the argument of the published paper [CFM17b]. We generalize the result obtained in the previous chapter to any ambient space $\mathbf{R}^n$. Indeed, since $n > 2$ we cannot take advantage of the localization procedure described previously, namely divergence free vector valued measures may not be regarded as the rotated gradient of a BV function. For this matter we need to introduce different techniques from the previous ones. Furthermore for scaling issues we need to consider a p-laplacian energy rather than the elliptic one in the Modica-Mortola component of the functional similarly to [Ghi14]. Here we take advantage of a result from White [Whi99b, Whi99a] to show that if the family of functionals is equibounded in energy, then up to a subsequence we can extract a sequence of vector measures converging to a 1-rectifiable vector measure. Furthermore the result is based on the study of a dimension reduced problem which is studied in the appendix. This reduced dimension problem has some analogies with the functional studied in [BDS96] in the case of droplets equilibrium featuring measures with fixed total mass which concentrate on atoms.

Let us define the approximating family of functionals precisely. Again let μ_+ and μ_- be two probability measures supported on a countable number of points. Let $\rho : \mathbf{R}^n \to \mathbf{R}_+$ be a classical radial mollifier such that $\operatorname{supp} \rho \subset B_1(0)$ and $\int_{B_1(0)} \rho = 1$. For $\varepsilon > 0$, we set $\rho_\varepsilon = \varepsilon^{-n}\rho(\cdot/\varepsilon)$. Consider vector fields satisfying equation

$$\nabla \cdot \sigma_\varepsilon = (\mu_+ - \mu_-) * \rho_\varepsilon \qquad \text{in } \mathcal{D}'(\mathbf{R}^n). \tag{2.1}$$

We also consider the functions $\varphi \in W^{1,p}(\Omega, [\eta, 1])$ such that $\varphi \equiv 1$ on $\partial\Omega$, where $\eta = \eta(\varepsilon)$ satisfies

$$\eta = \beta\, \varepsilon^n \tag{2.2}$$

for some $\beta \in \mathbf{R}_+$. We denote by $X_\varepsilon(\Omega)$ the set of pairs (σ, φ) such that φ is as stated above and $\sigma \in L^1(\Omega, \mathbf{R}^n)$ satisfies equation (2.1). This set is naturally embedded in $\mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$. For $(\sigma, \varphi) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ and $p > n - 1$ we set

$$\mathscr{F}_{\varepsilon,\beta}(\sigma, \varphi; \Omega) := \begin{cases} \displaystyle\int_\Omega \left[\varepsilon^{p-n+1} |\nabla \varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-1}} + \frac{\varphi |\sigma|^2}{\varepsilon} \right] \mathrm{d}x, & \text{if } (\sigma, \varphi) \in X_\varepsilon(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

Let X be the subset of $\mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ consisting of those couples (σ, φ) such that $\varphi \equiv 1$ and $\sigma = (m, \tau, \Sigma)$ satisfies the constraint

$$\nabla \cdot \sigma_\varepsilon = \mu_+ - \mu_- \qquad \text{in } \mathcal{D}'(\mathbf{R}^n). \tag{2.4}$$

Given any sequence $\varepsilon = (\varepsilon_i)_{i \in \mathbf{N}}$ of positive numbers such that $\varepsilon_i \downarrow 0$, we show that $\mathscr{F}_{\varepsilon,\beta}$ family of functionals Γ-converges to

$$\mathscr{E}_\beta(\sigma, \varphi; \overline{\Omega}) = \begin{cases} \displaystyle\int_{\Sigma \cap \overline{\Omega}} h_\beta(m(x))\, \mathrm{d}\mathcal{H}^1(x), & \text{if } (\sigma, \varphi) \in X \text{ and } \sigma = m\, \tau\, \mathcal{H}^1 \llcorner \Sigma, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.5}$$

The function $h_\beta : \mathbf{R}_+ \to \mathbf{R}_+$ (introduced and studied in Appendix B) is the minimum value of some optimization problem depending on β and on the codimension $n-1$ (we note h_β^d, with $d = n-k$ in the general case $1 \le k \le n-1$). In particular we prove that h_β is lower semicontinuous, subadditive, increasing, $h_\beta(0) = 0$ and that there exists some $c > 0$ such that

$$\frac{1}{c} \le \frac{h_\beta(m)}{1 + \sqrt{\beta}\, m} \le c \quad \text{for } m > 0. \tag{2.6}$$

The Γ-convergence holds for the topology of the weak-$*$ convergence for the sequence of measures (σ_ε) and for the strong L^2 convergence for the phase field (φ_ε). For a sequence $(\sigma_\varepsilon, \varphi_\varepsilon)$ we write $(\sigma_\varepsilon, \varphi_\varepsilon) \to (\sigma, \varphi)$ if $\sigma_\varepsilon \xrightarrow{*} \sigma$ and $\|\varphi_\varepsilon - \varphi\|_{L^2} \to 0$. In the sequel we first establish that the sequence of functionals $(\mathscr{F}_{\varepsilon,\beta})_\varepsilon$ is coercive with respect to this topology.

Theorem 2.1 (Equicoercivity)**.** *Assume that $\beta > 0$. For any sequence $(\sigma_\varepsilon, \varphi_\varepsilon) \subset \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ with $\varepsilon \downarrow 0$, such that*

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \le F_0 < +\infty,$$

there exists $\sigma \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n)$ such that, up to a subsequence, $(\sigma_\varepsilon, \varphi_\varepsilon) \to (\sigma, 1) \in X$.

Then we prove the Γ-liminf inequality

Theorem 2.2 ($\Gamma-\liminf$ inequality)**.** *Assume that $\beta \ge 0$. For any sequence $(\sigma_\varepsilon, \varphi_\varepsilon) \subset \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ that converges to $(\sigma, \varphi) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ as $\varepsilon \downarrow 0$ it holds*

$$\liminf_{\varepsilon \downarrow 0} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \ge \mathscr{E}_\beta(\sigma, \varphi; \overline{\Omega}).$$

We also establish the corresponding Γ-limsup inequality

Theorem 2.3 ($\Gamma-\limsup$ inequality)**.** *Assume that $\beta \ge 0$. For any $(\sigma, \varphi) \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ there exists a sequence $(\sigma_\varepsilon, \varphi_\varepsilon) \subset \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$ such that*

$$(\sigma_\varepsilon, \varphi_\varepsilon) \xrightarrow{\varepsilon \downarrow 0} (\sigma, \varphi) \quad \text{in } \mathcal{M}(\overline{\Omega}, \mathbf{R}^n) \times L^2(\Omega)$$

and

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \le \mathscr{E}_\beta(\sigma, \varphi; \overline{\Omega}).$$

Notice that the coercivity of the family of functionals only holds in the case $\beta > 0$. However, as $\beta \downarrow 0$ we have the important phenomena:

$$h_\beta \overset{\beta\downarrow 0}{\longrightarrow} c\mathbf{1}_{(0,+\infty)} \quad \text{pointwise},$$

for some $c > 0$. As a consequence (2.5) is an approximation of $c\mathcal{H}^1(\Sigma)$ for $\beta > 0$ small.

In the first section of this chapter we anticipate the optimization problem defining the cost function h^d_β and some results which are studied in the Appendix B. This problem is studied independently as it is useful to obtain similar results for k-currents replacing vector valued measures. This extension is studied in the following chapter.

2.2 Reduced problem results in dimension $n-k$

This section is devoted to introducing some notation and results corresponding to the case $k = 1$. In the sequel, these results are used to describe the energetical behaviour of the $(n-k)$-dimensional slices of the configuration $(\sigma_\varepsilon, \varphi_\varepsilon)$. We postpone the proofs to Appendix B.3, B.4 and B.5. We set $d = n-k$, $p > d$ and consider ε to be a sequence such that $\varepsilon \downarrow 0$. Let $B_r(0) \subset \mathbf{R}^d$ be the ball of radius r centered in the origin. We consider the functional

$$\mathscr{G}_{\varepsilon,\beta}(\vartheta, \varphi; B_r) := \int_{B_r} \left[\varepsilon^{p-d} |\nabla \varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^d} + \frac{\varphi |\vartheta|^2}{\varepsilon} \right] \,\mathrm{d}x \tag{2.7}$$

where $\varphi \in W^{1,p}(B_r)$ is constrained to satisfy the lower bound $\varphi \geq \beta\, \varepsilon^{d+1} =: \eta$ and $\vartheta \in L^2(B_r)$ is such that $\mathrm{supp}(\vartheta) \subset B_{\tilde r}$ with $0 < \tilde r < r$, $\|\vartheta\|_1 = m$. This leads to define the set

$$Y_{\varepsilon,\beta}(m, r, \tilde r) = \left\{ (\vartheta, \varphi) \in L^2(B_r) \times W^{1,p}(B_r, [\eta, 1]) \,:\, \|\vartheta\|_1 = m \text{ and } \mathrm{supp}(\vartheta) \subset B_{\tilde r} \right\},$$

and the optimization problem

$$h^d_{\varepsilon,\beta}(m, r, \tilde r) = \inf_{Y_{\varepsilon,\beta}(m,r,\tilde r)} \mathscr{G}_{\varepsilon,\beta}(\vartheta, \varphi; B_r). \tag{2.8}$$

Let $h^d_\beta : [0, +\infty) \longrightarrow \mathbf{R}_+$ be defined as

$$h^d_\beta(m) = \begin{cases} \min\limits_{\hat r > 0} \left\{ \dfrac{\beta\, m^2}{\omega_d\, \hat r^d} + \omega_d\, \hat r^d + (d-1)\, \omega_d\, q^d_\infty(0, \hat r) \right\}, & \text{for } m > 0, \\[2ex] 0, & \text{for } m = 0, \end{cases} \tag{2.9}$$

with

$$q^d_\infty(\xi, \hat r) := \inf \left\{ \int_{\hat r}^{+\infty} t^{d-1} \left[|v'|^p + (1-v)^2 \right] \,\mathrm{d}t \,:\, v(\hat r) = \xi \text{ and } \lim_{t \to +\infty} v(t) = 1 \right\}, \tag{2.10}$$

for $\hat r > 0$, $\xi \geq 0$. For a graph of the profile v realizing the infimum in the latter see Figure B.1. We have the following results

Proposition 2.1. *For any $r > \tilde{r} > 0$, it holds*

$$\liminf_{\varepsilon \downarrow 0} h^d_{\varepsilon,\beta}(m, r, \tilde{r}) \geq h^d_\beta(m). \tag{2.11}$$

There exists a uniform constant $\kappa := \kappa(d, p)$ such that

$$h^d_\beta(m) \geq \kappa \qquad \text{for every } m > 0. \tag{2.12}$$

Proposition 2.2. *For fixed $m > 0$ let r_* be the minimizing radius in the definition of $h^d_\beta(m)$ (2.9). For any $\delta > 0$ and ε small enough there exist a function $\vartheta_\varepsilon = c\mathbf{1}_{B_{r_*\varepsilon}}$ with $c > 0$ such that $\int_{B_r} \vartheta_\varepsilon = m$ and a nondecreasing radial function $\varphi_\varepsilon : B_r \mapsto [\eta, 1]$ such that $\varphi_\varepsilon(0) = \eta$, $\varphi_\varepsilon = 1$ on ∂B_r and*

$$\mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon; B_r) \leq h^d_\beta(m) + \delta. \tag{2.13}$$

Proposition 2.3. *The function h^d_β is continuous in $(0, +\infty)$, increasing, sub-additive and $h^d_\beta(0) = 0$.*

2.3 Compactness

We prove the compactness Theorem 2.1 for the family of functionals $(\mathscr{F}_{\varepsilon,\beta})_\varepsilon$. Let us consider a family of functions $(\sigma_\varepsilon, \varphi_\varepsilon)_{\varepsilon\downarrow 0}$, such that $(\sigma_\varepsilon, \varphi_\varepsilon) \in X_\varepsilon(\Omega)$ and

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \;\leq\; F_0. \tag{2.14}$$

As a first step we prove:

Lemma 2.1. *Assume $\beta > 0$. There exists $C \geq 0$, depending only on Ω, F_0 and β such that*

$$\int_\Omega |\sigma_\varepsilon| \;\leq\; C, \qquad \forall\, \varepsilon. \tag{2.15}$$

As a consequence there exist a positive Radon measure $\mu \in (\mathbf{R}^n, \mathbf{R}_+)$ supported in $\overline{\Omega}$ and a vectorial Radon measure $\sigma \in \mathcal{M}(\overline{\Omega}, \mathbf{R}^n)$ with $\nabla \cdot \sigma = \sum a_j \delta_{x_j}$ and $|\sigma| \leq \mu$ such that up to a subsequence

$$\varphi_\varepsilon \to 1 \text{ in } L^2(\Omega), \qquad |\sigma_\varepsilon| \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\mathbf{R}^n), \qquad \sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma \text{ in } \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n).$$

Proof. We divide the proof into three steps.

Step 1. We start by proving the uniform bound (2.15). Let $\lambda \in (0, 1]$ and let

$$\Omega_\lambda \;:=\; \{x \in \Omega \;:\; \varphi_\varepsilon > \lambda\}\,.$$

Being σ_ε square integrable we identify the measure σ_ε with its density with respect to $\mathscr{L}^n$. Therefore splitting the total variation of σ_ε, we write

$$|\sigma_\varepsilon|(\Omega) = \int_\Omega |\sigma_\varepsilon| \,\mathrm{d}x = \int_{\Omega_\lambda} |\sigma_\varepsilon| \,\mathrm{d}x + \int_{\Omega\setminus\Omega_\lambda} |\sigma_\varepsilon| \,\mathrm{d}x.$$

We estimate each term separately. By the Cauchy-Schwarz inequality we have

$$\int_{\Omega_\lambda} |\sigma_\varepsilon| \leq \left(\int_{\Omega_\lambda} \frac{\varphi_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon}\right)^{1/2} \left(\int_{\Omega_\lambda} \frac{\varepsilon}{\varphi_\varepsilon}\right)^{1/2}.$$

Since $\lambda < \varphi_\varepsilon \leq 1$ on Ω_λ and $\int_{\Omega_\lambda} (\varphi_\varepsilon |\sigma_\varepsilon|^2)/(\varepsilon)\,\mathrm{d}x$ is bounded by $\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega)$, from the previous inequality we get

$$\int_{\Omega_\lambda} |\sigma_\varepsilon| \leq \left(\int_{\Omega_\lambda} \frac{\varphi_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon}\right)^{1/2} \sqrt{\frac{|\Omega|\varepsilon}{\lambda}} \leq \sqrt{\frac{|\Omega|\,\varepsilon\, F_0}{\lambda}}\,.$$

Next, in $\Omega \setminus \Omega_\lambda$, by the Young inequality, we have

$$2\int_{\Omega\setminus\Omega_\lambda} |\sigma_\varepsilon| \leq \int_{\Omega\setminus\Omega_\lambda} \frac{\varphi_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} + \int_{\Omega\setminus\Omega_\lambda} \frac{\varepsilon}{\varphi_\varepsilon}.$$

Using $\varphi_\varepsilon \geq \eta(\varepsilon)$, $\eta/\varepsilon^n = a$ and $(1-\lambda)^2 \leq (1-\varphi_\varepsilon)^2$ in $\Omega \setminus \Omega_\lambda$, we obtain

$$\int_{\Omega\setminus\Omega_\lambda} |\sigma_\varepsilon| \leq \frac{1}{2}\int_{\Omega} \frac{\varphi_\varepsilon |\sigma_\varepsilon|^2}{\varepsilon} + \frac{\varepsilon^n}{2\,\eta\,(1-\lambda)^2}\int_{\Omega} \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^{n-1}} \leq \frac{F_0}{2} + \frac{F_0}{2\,\beta\,(1-\lambda)^2}.$$

Hence

$$|\sigma_\varepsilon|(\Omega) \leq \frac{F_0}{2} + \frac{F_0}{2\,\beta\,(1-\lambda)^2} + \sqrt{\frac{|\Omega|\,\varepsilon\,F_0}{\lambda}}.$$

As $\beta > 0$, this yields (2.15).

Step 2. We easily see from $\int_\Omega (1-\varphi_\varepsilon)^2 \leq F_0 \varepsilon^{n-1}$ that $\varphi_\varepsilon \to 1$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$.

Step 3. The existence of the Radon measures μ and σ such that, up to extraction, $|\sigma_\varepsilon| \overset{*}{\rightharpoonup} \mu$ and $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ follows from (2.15). The properties on the support of μ, on the divergence of σ and the fact that $|\sigma| \leq \mu$ follow from the respective properties of σ_ε. □

We have just showed that the limit σ of a family $(\sigma_\varepsilon, \varphi_\varepsilon)_\varepsilon$ equibounded in energy is bounded in mass. In what follows, we assume $\beta \geq 0$ and that σ_ε is bounded in mass. We show that the limiting σ is rectifiable.

Proposition 2.4. *Assume $\beta \geq 0$ and that the conclusions of Lemma 2.1 hold true. There exists a Borel subset Σ with finite length and a Borel measurable function $\tau : \Sigma \to \mathbf{S}^{n-1}$ such that $\sigma = \tau|\sigma|\llcorner\Sigma$. Moreover, we have the following estimate,*

$$\mathcal{H}^1(\Sigma) \leq C_* F_0,$$

where the constant $C_* \geq 0$ only depends on d and p.

This proposition together with Lemma 2.1 and Theorem 0.5 leads to

Proposition 2.5. *σ is a 1-rectifiable vector measure and in particular Σ is a countably $\mathcal{H}^1$-rectifiable set.*

The latter ensures that the limit couple $(\sigma, 1)$ belongs to X and concludes the proof of Theorem 2.1. We now establish Proposition 2.4

Sketch of the proof: We first define Σ. Then we show in Lemma 2.3 that for $x \in \Sigma$, we have $\liminf_{\varepsilon\downarrow 0} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B(x, r_j)) \geq \kappa r_j$ for a sequence of radii $r_j \downarrow 0$ and $\kappa > 0$. The proof of the lemma is based on slicing and on the results of Appendix B. The proposition then follows from an application of the Besicovitch covering theorem.

First we introduce the Borel set

$$\widetilde{\Sigma} := \left\{ x \in \Omega \,:\, \forall r > 0,\ |\sigma|(B_r(x)) > 0 \text{ and } \exists \tau = \tau(x) \in \mathbf{S}^{n-1} \text{ such that } \tau = \lim_{r\downarrow 0} \tfrac{\sigma(B_r(x))}{|\sigma|(B_r(x))} \right\}.$$

We observe that by Besicovitch derivation theorem,

$$\sigma = \tau|\sigma| \llcorner \widetilde{\Sigma}.$$

Next we fix $\theta \in (0, 1/4^n)$ and define

$$\Gamma := \left\{ x \in \widetilde{\Sigma} \,:\, \exists\, r_0 > 0 \text{ such that, } \frac{|\sigma|(B_{r/4}(x))}{|\sigma|(B_r(x))} \leq \theta \ \text{ for every } r \in (0, r_0] \right\}.$$

We show that this set is $|\sigma|$-negligible.

Lemma 2.2. *We have* $|\sigma|(\Gamma) = 0$.

Proof. Let $x \in \Gamma$. Applying the inequality $|\sigma|(B_{r/4}(x)) \leq \theta|\sigma|(B_r(x))$ with $r = r_k = 4^{-k} r_0$, $k \geq 0$, we get $|\sigma|(B_{r_k}) \leq \theta^k |\sigma|(B_{r_0})$. Hence there exists $C \geq 0$ such that

$$|\sigma|(B_r(x)) \leq C r^{(\ln 1/\theta)/(\ln 4)}.$$

Letting $\lambda = (\ln \frac{1}{\theta})/(\ln 4)$, we have by assumption $\lambda > n$. Therefore, for every $\xi > 0$ there exists $r_\xi = r_\xi(x) \in (0, 1)$ such that

$$|\sigma|(B_{r_\xi}(x)) \leq \xi |B_{r_\xi}(x)|.$$

Now, for $R > 0$, we cover $\Gamma \cap B_R$ with balls of the form $B_{r_\xi(x)}(x)$. Using Besicovitch covering theorem, we have

$$\Gamma \cap B_R \subset \cup_{j=1}^{N(n)} \mathcal{B}_j$$

where $N(n)$ only depends on n and each $\mathcal{B}_j$ is a (finite or countable) disjoint union of balls of the form $B_{r_\xi(x_k)}(x_k)$. Then we get

$$|\sigma|(\Gamma \cap B_R) \leq \sum_{j=1}^{N(n)} |\sigma|(\mathcal{B}_j) \leq N(n)\xi|\mathcal{B}_j| \leq N(n)|B_{R+1}|\xi.$$

Sending ξ to 0 and then R to ∞, we obtain $|\sigma|(\Gamma) = 0$. $\square$

Set $\Sigma := \widetilde{\Sigma} \setminus \Gamma$, from Lemma 2.2, we have $\sigma = \tau|\sigma| \llcorner \Sigma$. Recall that $\mathscr{S} = \operatorname{supp} \mu_+ \cup \operatorname{supp} \mu_-$.

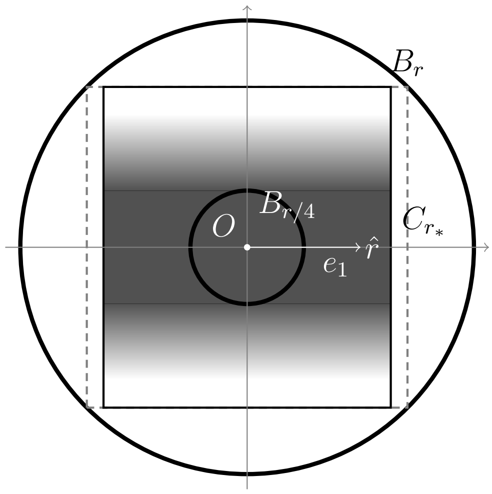


Figure 2.1: Illustration of the sections of B_r, $B_{r/4}$ and C_{r_*}. In grayscale we represent the level sets of the function $\chi_{r_*}(x')\mathbf{1}_{[-\hat{r},\hat{r}]}$.

Lemma 2.3. *For every $x \in \Sigma \setminus \mathscr{S}$, there exists a sequence $(r_j) = (r_j(x)) \subset (0,1)$ with $r_j \downarrow 0$ such that*

$$\liminf_{\varepsilon \downarrow 0} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B(x, r_j)) \geq \sqrt{2}\, \kappa\, r_j,$$

where κ is the constant of Proposition 2.1.

Proof. Let $x \in \Sigma \setminus \mathscr{S}$. Without loss of generality, we assume $x = 0$ and $\tau(x) = e_1$. Let $\xi > 0$ be a small parameter to be fixed later. From the definition of Σ, there exists a sequence $(r_j) = (r_j(x)) \subset (0, d(x, \mathscr{S}))$ such that for every $j \geq 0$,

$$\sigma(B_{r_j}) \cdot e_1 \geq (1-\xi)|\sigma|(B_{r_j}) \qquad \text{and} \qquad |\sigma|(B_{r_j/4}) \geq \theta |\sigma|(B_{r_j}). \tag{2.16}$$

Let us fix $j \geq 0$ and set, to simplify the notation, $r = r_j$ and $r_* = r/\sqrt{2}$. Recall the notation $x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}$ and define the cylinder

$$C_{r_*} := \{x : |x_1| \leq r_* \quad \text{and} \quad |x'| \leq r_*\}$$

so that $C_{r_*} \subset B_r$ and $B_{r/4} \subset C_{r_*/2}$, as shown in figure 2.1. Let $\chi \in C_c^\infty(\mathbf{R}^{n-1}, [0,1])$ be a radial cut-off function such that $\chi(x') = 1$ if $|x'| \leq \frac{1}{2}$ and $\chi(x') = 0$ for $|x'| \geq \frac{3}{4}$. Then, we note $\chi_{r_*}(x') = \chi(x'/r_*)$ and for $s \in [-r, r]$, we set

$$\forall s \in [-r, r], \qquad g_\varepsilon(s) := e_1 \cdot \int_{B'_{r_*}} \sigma_\varepsilon(s, x')\, \chi_{r_*}(x')\, \mathrm{d}x'.$$

Since σ_ε is divergence free, $e_1 \cdot \sigma_\varepsilon(\cdot, s)$ has a meaning on the hyperplane $\{x_1 = s\}$ in the sense of trace, moreover, g_ε is continuous. Now, let us fix $\hat{r} \in [(1-\xi)r_*, r_*]$ such that $\mu(\{-\hat{r}, \hat{r}\} \times B'_r) = 0$ (which holds true for a.e. $\hat{r} \in [(1-\xi)r_*, r_*]$) and let us define the mean value,

$$\bar{g}_\varepsilon := \frac{1}{2\hat{r}} \int_{-\hat{r}}^{\hat{r}} g_\varepsilon(s)\, \mathrm{d}s.$$

From $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$, $|\sigma_\varepsilon| \overset{*}{\rightharpoonup} \mu$, we have

$$\lim_{\varepsilon\downarrow 0} \bar g_\varepsilon \;=\; \left(\frac{1}{2\hat r}\int_{(-\hat r,\hat r)\times B'_{r_*}} \chi_{r_*}(x')\ \mathrm{d}\sigma(s,x')\right)\cdot e_1 \;=:\; \overline{m}. \tag{2.17}$$

From (2.16), we see that $\overline{m} > 0$ for ξ small enough. Indeed, we have

$$\begin{aligned}(1-\xi)|\sigma|(B_r) \;&\le\; 2\hat r\overline{m} + \sigma(B_r)\cdot e_1\\ &= \int_{B_r}\left(1-\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]}\right)\ \mathrm{d}\sigma(s,x')\cdot e_1\\ &\le 2\hat r\overline{m} + \int_{B_r}\left(1-\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]}\right)\ \mathrm{d}|\sigma|(s,x')\\ &\le 2\hat r\overline{m} + |\sigma|(B_r) - \int_{B_r}\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]}\ \mathrm{d}|\sigma|(s,x').\end{aligned}$$

Since by construction $\chi_{r_*}(x')\mathbf{1}_{[-\hat r,\hat r]} \ge \mathbf{1}_{B_{r/4}}$, using the second inequality of (2.16), we have

$$\overline{m} \;\ge\; \frac{1}{2\hat r}(\theta-\xi)|\sigma|(B_r) \;>\; 0,$$

for ξ small enough. Similarly, denoting $\Pi\,:\,\mathbf{R}^n \to \mathbf{R}^{n-1}$, $(t,x')\mapsto x'$ the orthogonal projection onto the last $(n-1)$ coordinates, we deduce again from (2.16) that

$$|\Pi\sigma|(C_{r_*}) \le \frac{\sqrt{\xi}\,\overline{m}}{m-\xi}2\hat r. \tag{2.18}$$

Now, for ε small enough, we have $\nabla\cdot\sigma_\varepsilon = 0$ in C_{r_*}. Using this, we have for almost every $s,t\in[-\hat r,\hat r]$, with $s<t$,

$$g_\varepsilon(t) - g_\varepsilon(s) \;=\; \int_s^t \left[\int_{B'_{r_*}} \sigma_\varepsilon(x',h)\cdot\nabla'\chi_{r_*}(x')\ \mathrm{d}x'\right]\ \mathrm{d}h.$$

Integrating in s over $(-\hat r,\hat r)$, we get for almost every $t\in[-r,r]$,

$$g_\varepsilon(t) - \bar g_\varepsilon \;=\; \frac{1}{2\hat r}\int_{(-\hat r,\hat r)\times B'_{r_*}} \phi_t(h,x')\cdot\sigma_\varepsilon(h,x')\ \mathrm{d}x'\ \mathrm{d}h$$

with

$$\phi_t(h,x') \;=\; \begin{cases}(h+\hat r)\,\nabla'\chi_{r_*}(x') & \text{if } h<t,\\ (h-\hat r)\,\nabla'\chi_{r_*}(x') & \text{if } h>t.\end{cases}$$

We deduce the following convergence

$$g_\varepsilon(t) - \overline{m} \overset{\varepsilon\downarrow 0}{\longrightarrow} \frac{1}{2\hat r}\int_{(-\hat r,\hat r)\times B'_{r_*}} \phi_t(h,x')\cdot\ \mathrm{d}\sigma(h,x') \tag{2.19}$$

in the $L^1(-\hat r,\hat r)$ topology. Using (2.18), we see that the above right hand side is bounded by $c\frac{\sqrt{\xi}}{m-\xi}\overline{m}$. Taking into account (2.18) and the continuity of g_ε, we conclude that

$$\liminf_{\varepsilon\downarrow 0} g_\varepsilon(t) \ge \left(1 - c\frac{\sqrt{\xi}}{m-\xi}\right)\overline{m} \quad \text{for} \quad t\in[-\hat r,\hat r].$$

Next, by decomposing the integral we have

$$\begin{aligned}\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon;B_r) &\geq \int_{-\hat{r}}^{\hat{r}}\int_{B'_{r_*}}\left[\varepsilon^{p-n+1}|\nabla\varphi_\varepsilon|^p+\frac{(1-\varphi_\varepsilon)^2}{\varepsilon^{n-1}}+\frac{\varphi_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon}\right]\,\mathrm{d}x'\,\mathrm{d}t\\ &\geq \int_{-\hat{r}}^{\hat{r}}\int_{B'_{r_*}}\left[\varepsilon^{p-n+1}|\nabla\varphi_\varepsilon|^p+\frac{(1-\varphi_\varepsilon)^2}{\varepsilon^{n-1}}+\frac{\varphi_\varepsilon|\chi_{r_*}(x')\sigma_\varepsilon|^2}{\varepsilon}\right]\,\mathrm{d}x'\,\mathrm{d}t.\end{aligned} \tag{2.20}$$

Let us set

$$\vartheta_\varepsilon^t(x') := |\chi_{r_*}(x')\sigma_\varepsilon(t,x')|.$$

By construction ϑ_ε^t has the properties:

- $\vartheta_\varepsilon^t \in L^1(B'_{r_*})$,
- $\liminf_{\varepsilon\downarrow 0}\int_{B'_{r_*}}\vartheta_\varepsilon^t(x')\,dx' \geq \liminf_{\varepsilon\downarrow 0} g_\varepsilon(t) \geq \left(1-c\frac{\sqrt{\xi}}{m-\xi}\right)\overline{m} = \tilde{m} > 0$,
- $\mathrm{supp}(\vartheta_\varepsilon^t) \subset B'_{\tilde{r}}$ with $\tilde{r} := \frac{3}{4}r_* < r_*$.

By definition of the minimization problem introduced in Section 2.2, we have

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon;B_r) \geq \int_{-\hat{r}}^{\hat{r}}\left[\inf_{(\vartheta,\varphi)\in Y_{\varepsilon,\beta}(\tilde{m},r,\tilde{r})}\mathscr{G}_{\varepsilon,\beta}(\vartheta,\varphi;B_r)\right]\,\mathrm{d}t = \int_{-\hat{r}}^{\hat{r}} h_{\varepsilon,\beta}^{n-1}(\tilde{m},r,\tilde{r})\,\mathrm{d}t. \tag{2.21}$$

Taking the infimum limit, by Fatou's lemma and equation (2.12) of Proposition 2.1 we get

$$\liminf_{\varepsilon\downarrow 0}\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon;B_r) \geq \int_{-\hat{r}}^{\hat{r}}\liminf_{\varepsilon\downarrow 0} h_{\varepsilon,\beta}^{n-1}(\tilde{m},r,\tilde{r})\,\mathrm{d}t \geq 2\,\hat{r}\,\kappa.$$

The latter holds for almost every $\hat{r} \in [(1-\xi)r_*, r_*]$ and eventually, since the $r_* = r/\sqrt{2}$, we conclude

$$\liminf_{\varepsilon\downarrow 0}\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon;B_r) \geq \sqrt{2}\,\kappa\,r.$$

□

The proof of Proposition 2.4 is then obtained via the Besicovitch covering theorem [EG15].

2.4 Γ-liminf inequality

In this section we prove the $\Gamma-\liminf$ inequality stated in Theorem 2.2.

Proof of Theorem 2.2. With no loss of generality we assume that $\liminf_{\varepsilon\downarrow 0}\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon) < +\infty$ otherwise the inequality is trivial. For a Borel set $A \subset \Omega$, we define

$$H(A) := \liminf_{\varepsilon\downarrow 0}\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon;A),$$

so that H is a subadditive set function. By assumption, the limit measure σ is 1-rectifiable; we write $\sigma = m\,\tau\,\mathcal{H}^1\llcorner\Sigma$. Furthermore we can assume σ to be compactly supported in Ω. Consider a convex open set Ω_0 such that $\mathrm{supp}(\nabla\cdot\sigma) = \mathscr{S} \subset\subset \Omega_0 \subset\subset \Omega$

and let $f := [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ be a smooth homotopy of the indentity map on $\mathbf{R}^n$ onto a contraction of $\overline{\Omega}$ into $\overline{\Omega}_0$ such that $f(t,\cdot)$ restricted to Ω_0 is the identity map, for any $t \in [0,1]$. Let $\sigma_t = f(t,\cdot)\sharp\sigma$, indeed $\liminf_{t\downarrow 0} \mathscr{F}(\sigma_t, 1) \geq \mathscr{F}(\sigma, 1)$ as $\sigma_t \overset{*}{\rightharpoonup} \sigma$. Further $\nabla \cdot \sigma_t = \nabla \cdot \sigma$ since $h(t,\cdot)$ is the identity on $\mathscr{S}$. Now we claim that

$$\liminf_{r\downarrow 0} \frac{H\left(\overline{B(x,r)}\right)}{2r} \geq h_\beta(m(x)) \qquad \text{for } \mathcal{H}^1\text{-almost every } x \in \Sigma. \tag{2.22}$$

Let us fix $\lambda \geq 1$ and let us note $h_{\beta,\lambda}(t) := \min(h_\beta(t), \lambda)$. We then introduce the Radon measure

$$H'_\lambda(A) := \int_{\Sigma\cap A} h_{\beta,\lambda}(m)\ \mathrm{d}\mathcal{H}^1.$$

Now, let $\delta \in (0,1)$. Assuming that (2.22) holds true, there exists $\Sigma_0 \subset \Sigma$ with $\mathcal{H}^1(\Sigma\backslash\Sigma_0) = 0$ such that for every $x \in \Sigma_0$, there exists $r_0(x) > 0$ with

$$(1+\delta)H\left(\overline{B(x,r)}\right) \geq 2r h_{\beta,\lambda}(m(x)) \qquad \text{for every } r \in (0, r_0(x)).$$

By the Besicovitch differentiation Theorem, there exists $\Sigma_1 \subset \Sigma$ with $\mathcal{H}^1(\Sigma\backslash\Sigma_1) = 0$ such that for every $x \in \Sigma_1$, there exists $r_1(x) > 0$ with

$$(1+\delta)2r h_\beta(m(x)) \geq H'_\lambda\left(\overline{B(x,r)}\right) \qquad \text{for every } r \in (0, r_1(x)).$$

We consider the familly $\mathcal{B}$ of closed balls $\overline{B(x,r)}$ with $x \in \Sigma_0 \cap \Sigma_1$ and $0 < r < \min(r_0(x), r_1(x))$ and we apply the Vitali-Besicovitch covering theorem [AFP00, Theorem 2.19] to the family $\mathcal{B}$ and to the Radon measure H'_λ. We obtain a disjoint family of closed balls $\mathcal{B}' \subset \mathcal{B}$ such that

$$\begin{aligned} H'_\lambda(\Omega) = H'_\lambda(\Sigma) &= \sum_{\overline{B(x,r)}\in\mathcal{B}'} H'_\lambda\left(\overline{B(x,r)}\right) \\ &\leq (1+\delta)^2 \sum_{\overline{B(x,r)}\in\mathcal{B}'} H\left(\overline{B(x,r)}\right) \leq (1+\delta)^2 H(\Omega). \end{aligned}$$

Sending λ to infinity and then δ to 0, we get the lower bound $H(\Omega) \geq \int_\Sigma h_\beta(m)\, d\mathcal{H}^1$ which proves the theorem.

Let us now establish the claim (2.22). Since σ is a rectifiable measure, we have for $\mathcal{H}^1$-almost every $x \in \Sigma$ and for every $\varphi \in C_c(\mathbf{R}^n)$,

$$\frac{1}{2r}\int \varphi(x+ry)\ \mathrm{d}|\sigma|(y) \xrightarrow{r\downarrow 0} m(x)\int_{\mathbf{R}} \varphi(t\tau(x))\ \mathrm{d}t \tag{2.23}$$

and

$$\frac{1}{2r}\int_{B(x,r)\cap\Sigma} |\tau(y)-\tau(x)|\ \mathrm{d}|\sigma|(y) \xrightarrow{r\downarrow 0} 0. \tag{2.24}$$

Let $x \in \Sigma \setminus \mathcal{S}$ be such a point. Without loss of generality, we assume $x = 0$, $\tau(0) = e_1$ and $\overline{m} := m(0) > 0$. Let $\delta \in (0,1)$. Our goal is to establish a precise lower bound for $\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; C)$ where C is a cylinder of the form

$$C_r^\delta := \{x \in \mathbf{R}^n \,:\, |x_1| < \delta r,\, |x'| < r\}\,.$$

For this we proceed as in the proof of Lemma 2.3, here, the rectifiability of σ simplifies the argument. Let $\chi^\delta \in C_c^\infty(\mathbf{R}^{n-1}, [0,1])$ be a radial cut-off function with $\chi^\delta(x') = 1$ if $|x'| \le \delta/2$, $\chi^\delta(x') = 0$ if $|x'| \ge \delta$. For $\varepsilon > 0$ and $r \in (0, d(0, \partial\Omega))$, we define for $s \in (-r, r)$,

$$g_\varepsilon^{\delta,r}(s) \;:=\; e_1 \cdot \int_{\mathbf{R}^{n-1}} \sigma_\varepsilon(s, x')\, \chi^\delta(x'/r)\; \mathrm{d}x'.$$

We also introduce the mean value

$$\overline{g_\varepsilon^{\delta,r}} \;:=\; \frac{1}{2r} \int_{-r}^{r} g_\varepsilon^{\delta,r}(s)\; \mathrm{d}s.$$

From (2.23), we have for $r > 0$ small enough,

$$\overline{g_0^{\delta,r}} \;:=\; \frac{1}{2r} \int_{-r}^{r} e_1 \cdot \int_{\mathbf{R}^{n-1}} \sigma_\varepsilon(s, x')\, \chi^\delta(x'/r)\; \mathrm{d}x\;\, \mathrm{d}s \;\ge\; (1-\delta)\overline{m}.$$

For such $r > 0$, we deduce from $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$ that for $\varepsilon > 0$ small enough

$$\overline{g_\varepsilon^{\delta,r}} \;:=\; \frac{1}{2r} \int_{-r}^{r} g_\varepsilon^{\delta,r}(s)\; \mathrm{d}s \;\ge\; (1-2\delta)\overline{m}. \tag{2.25}$$

We study the variation of $g_\varepsilon^{\delta,r}(s)$. Using $\nabla \cdot \sigma_\varepsilon = 0$ in C_r^δ, we compute as in the proof of Lemma 2.3,

$$g_\varepsilon^{\delta,r}(t) - \overline{g_\varepsilon^{\delta,r}} \;=\; \frac{1}{2r} \int_{(-r,r)\times B_{\delta r}} \phi_t(x', h) \cdot \sigma_\varepsilon(x', h)\; \mathrm{d}x'\; \mathrm{d}h$$

with

$$\phi_t(h, x') \;=\; \begin{cases} (h + \hat{r})\, \nabla'\chi^\delta(x'/r) & \text{if } h < t, \\ (h - \hat{r})\, \nabla'\chi^\delta(x'/r) & \text{if } h > t. \end{cases}$$

Using again the convergence $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$, we deduce

$$g_\varepsilon^{\delta,r}(t) - \overline{g_\varepsilon^{\delta,r}} \;\stackrel{\varepsilon\downarrow 0}{\longrightarrow}\; \frac{1}{2r} \int_{(-r,r)\times B_{\delta r}} \phi_t(x', h) \cdot\; \mathrm{d}\sigma(x', h),$$

in $L^1(-r, r)$. Now, since $e_1 \cdot \nabla'\chi^\delta \equiv 0$, we deduce from (2.24) that the right hand side goes to 0 as $r \downarrow 0$. Hence, for $r > 0$ small enough,

$$\left| \frac{1}{2r} \int_{(-r,r)\times B_{\delta r}} \phi_t(x', h) \cdot \sigma(x', h)\; \mathrm{d}x'\; \mathrm{d}h \right| \;\le\; \delta\overline{m}.$$

Using (2.25), we conclude that for $r > 0$ small enough and then for $\varepsilon > 0$ small enough, we have

$$g_\varepsilon^{\delta,r}(t) \;\ge\; (1-3\delta)\overline{m}, \qquad \text{for a.e. } t \in (-r, r).$$

By definition of the reduced dimension problem, we conclude that

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; C_r^\delta) \;\ge\; 2r h_{\varepsilon,\beta}^{n-1}\left((1-3\delta)\overline{m}\right).$$

Sending $\varepsilon \downarrow 0$, we obtain

$$H(C_r^\delta) \;\geq\; 2rh_\beta^{n-1}\left((1-3\delta)\overline{m}\right).$$

We notice that $H(B_{\sqrt{1+\delta^2}\,r}) \geq H(C_r^\delta)$. Recall that for the case $n-1$ we omit the superscript in the definition of h, thus dividing by $2\sqrt{1+\delta^2}\,r$ and taking the liminf as $r \downarrow 0$, we get

$$\liminf_{r\downarrow 0} \frac{H(B_{\sqrt{1+\delta^2}\,r})}{2\sqrt{1+\delta^2}\,r} \;\geq\; \frac{h_\beta\left((1-3\delta)\overline{m}\right)}{\sqrt{1+\delta^2}}.$$

Sending δ to 0, we get (2.22) by lower semi-continuity of h_β. □

2.5 Γ-limsup inequality

Proof of Theorem 2.3.
Let us suppose $\mathscr{F}(\sigma,\varphi;\overline{\Omega}) < +\infty$, so that in particular $\varphi \equiv 1$. From Xia [Xia04], we can assume σ to be supported on a finite union of compact segments and to have constant multiplicity on each of them, namely polyhedral vector measures are dense in energy. We first construct a recovery sequence for a measure σ concentrated on a segment with constant multiplicity. Then we show how to deal with the case of a polyhedral vector measures.

Step 1. (σ concentrated on a segment.) Assume that σ is supported on the segment $I = [0,L]\times\{0\}$ and writes as $m\, e_1 \mathcal{H}^1 \llcorner_I$. Consider m constant so that $\nabla\cdot\sigma = m(\delta_{(0,0)} - \delta_{(L,0)})$ and

$$\mathscr{E}_\beta(\sigma,1;\Omega) = h_\beta(m)\,\mathcal{H}^1(I) = L\,h_\beta(m).$$

For $\delta > 0$ fixed, we consider the profiles

$$\overline{\varphi}_\varepsilon(t) := \begin{cases} \eta, & \text{for } 0 \leq t \leq r_*\varepsilon,\\ v_\delta\left(\dfrac{t}{\varepsilon}\right), & \text{for } r_*\varepsilon \leq t \leq r,\\ 1 & \text{for } r \leq t, \end{cases} \qquad \text{and} \qquad \vartheta_\varepsilon = \frac{m\,\chi_{B'_{r_*\varepsilon}}(x')}{\omega_{n-1}\,(\varepsilon r_*)^{n-1}}$$

with r_* and v_δ, defined in Proposition 2.2 with $d = n-1$. Assume $r_* \geq 1$ and let $d(x,I)$ be the distance function from the segment I and introduce the sets

$$I_{r_*\varepsilon} := \{x\in\Omega \;:\; d(x,I) \leq r_*\varepsilon\}, \qquad \text{and} \qquad I_r := \{x\in\Omega \;:\; d(x,I)\leq r\}.$$

Set $\varphi_\varepsilon(x) = \overline{\varphi}_\varepsilon(d(x,I))$ and $\overline{\sigma}_\varepsilon^1 = (m\mathcal{H}^1 \llcorner I) * \rho_\varepsilon$, where ρ_ε is the mollifier of equation (2.1). We first construct the vector measures

$$\sigma_\varepsilon^1 = \overline{\sigma}_\varepsilon^1\, e_1 \qquad \text{and} \qquad \sigma_\varepsilon^2(x_1,x') = \vartheta_\varepsilon(|x'|)\; e_1.$$

Alternatively, $\sigma_\varepsilon^2 = \sigma * \tilde{\rho}_\varepsilon$ for the choice $\tilde{\rho}_\varepsilon(x_1,x') = \chi_{B'_{r_*\varepsilon}}(x')/\,\omega_{n-1}(\varepsilon r_*)^{n-1}$. Let us highlight some properties of σ_ε^1 and σ_ε^2. Both vector measures are radial in x', with an abuse of notation we denote $\overline{\sigma}_\varepsilon^1(x_1,s) = \overline{\sigma}_\varepsilon^1(x_1,|x'|)$. Since, both σ_ε^1 and σ_ε^2 are obtained trough convolution it holds $\mathrm{supp}(\sigma_\varepsilon^1) \cup \mathrm{supp}(\sigma_\varepsilon^2) \subset I_{r_*\varepsilon}$ and they are oriented by the vector e_1 therefore $|\sigma_\varepsilon^1| = \overline{\sigma}_\varepsilon^1$ and $|\sigma_\varepsilon^2| = \vartheta_\varepsilon$. Furthermore for any x_1, it holds

$$\int_{\{x_1\}\times B'_{r_*\varepsilon}} \left[\overline{\sigma}_\varepsilon^1(x_1, x') - \vartheta_\varepsilon(x')\right] \,\mathrm{d}x' = 0. \tag{2.26}$$

We construct σ_ε by interpolating between σ_ε^1 and σ_ε^2. To this aim consider a cutoff function $\zeta_\varepsilon : \mathbf{R} \to \mathbf{R}_+$ satisfying

$$\begin{array}{ll} \zeta_\varepsilon(t) = 1 & \text{for } t \le r_*\varepsilon \text{ or } t \ge L - r_*\varepsilon, \\ \zeta_\varepsilon(t) = 0 & \text{for } 2\, r_*\varepsilon \le t \le L - 2\, r_*\varepsilon, \end{array} \qquad \text{and} \qquad |\zeta_\varepsilon'| \le \frac{1}{r_*\varepsilon}$$

and define σ_ε component-wise as

$$\begin{cases} \sigma_\varepsilon^3 \cdot e_1 = 0, \\ \sigma_\varepsilon^3 \cdot e_i(x_1, x') = -\zeta_\varepsilon'(x_1)\, \dfrac{x_i}{|x'|^{n-1}} \displaystyle\int_0^{|x'|} s^{n-2} \left[\overline{\sigma}_\varepsilon^1(x_1, s) - \vartheta_\varepsilon(s)\right] \,\mathrm{d}s, \quad \text{for } i = 2, \dots, n. \end{cases}$$

The integral corresponds to the difference of the fluxes of σ_ε^1 and σ_ε^2 through the $(n-1)$-dimensional disk $\{x_1\} \times B'$. For σ_ε^3 we have the following

$$\begin{aligned} \nabla \cdot \sigma_\varepsilon^3 = -\zeta_\varepsilon'(x_1) \sum_{i=2}^n \Bigg[\left(\frac{1}{|x'|^{n-1}} - \frac{(n-1)x_i^2}{|x'|^{n+1}} \right) \int_0^{|x'|} s^{n-2} \left[\overline{\sigma}_\varepsilon^1(x_1, s) - \vartheta_\varepsilon(s)\right] \,\mathrm{d}s \\ + \frac{x_i^2}{|x'|^2} \left[\overline{\sigma}_\varepsilon^1(x_1, |x'|) - \vartheta_\varepsilon(|x'|)\right] \Bigg] = -\zeta_\varepsilon'(x_1) \left[\overline{\sigma}_\varepsilon^1(x_1, |x'|) - \vartheta_\varepsilon(|x'|)\right]. \end{aligned} \tag{2.27}$$

Let

$$\sigma_\varepsilon = \zeta_\varepsilon\, \sigma_\varepsilon^1 + (1 - \zeta_\varepsilon)\, \sigma_\varepsilon^2 + \sigma_\varepsilon^3.$$

In force of equation (2.27) and from the construction of σ_ε^1, σ_ε^2 and ζ_ε we have

$$\begin{aligned} \nabla \cdot \sigma_\varepsilon &= \nabla \cdot (\zeta_\varepsilon \sigma_\varepsilon^1) + \nabla \cdot (1 - \zeta_\varepsilon)\sigma_\varepsilon^2 + \nabla \cdot \sigma_\varepsilon^3 \\ &= \zeta_\varepsilon \nabla \cdot \sigma_\varepsilon^1 + \zeta_\varepsilon'(\overline{\sigma}_\varepsilon^1 - \vartheta_\varepsilon) + \nabla \cdot \sigma_\varepsilon^3 \\ &= \zeta_\varepsilon \nabla \cdot \sigma_\varepsilon^1 = \nabla \cdot (\sigma * \rho_\varepsilon). \end{aligned}$$

In addition for any (x_1, x') such that $|x'| \ge r_*\varepsilon$ from (2.26) we derive

$$\sigma_\varepsilon^3 \cdot e_i(x_1, x') = -\zeta_\varepsilon'(x_1)\, \frac{x_i}{|x'|^{n-2}} \int_0^{|x'|} s^{n-1} \left[\overline{\sigma}_\varepsilon^1(x_1, s) - \vartheta_\varepsilon(s)\right] \,\mathrm{d}s = 0$$

which justifies $\mathrm{supp}(\sigma_\varepsilon) \subset I_{r_*\varepsilon}$. Let us now prove

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \le L\, h_\beta(m) + C\delta.$$

We split Ω as the union of $\Omega \setminus I_r$, $C_{r,\varepsilon} := I_r \cap [2\,\varepsilon, L - 2\,\varepsilon] \times \mathbf{R}^{n-1}$ and D_ε and D'_ε, as show in figure 2.2, where $D_\varepsilon = \{x_1 \le 2\, r_*\, \varepsilon\} \cap I_{r_*\varepsilon}$ and $D'_\varepsilon = \{x_1 \ge L - 2\, r_*\, \varepsilon\} \cap I_{r_*\varepsilon}$. On $\Omega \setminus I_r$ we notice that $\sigma_\varepsilon = 0$ and $\varphi_\varepsilon = 1$ therefore

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega \setminus I_r) = 0.$$

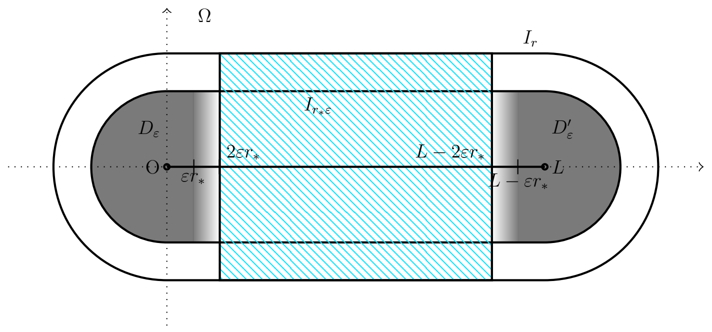


Figure 2.2: Illustration of the interval I and both its r and $(r_*\varepsilon)$-enlargement for $r_* \geq 1$. In grayscale we plot the levels of the function ζ_ε, whilst the striped region corresponds to the cylinder $C_{r,\varepsilon}$.

Observe that $|D_\varepsilon| = |D'_\varepsilon| = C\varepsilon^n$, then we have the upper bound

$$\int_{D_\varepsilon} |\sigma_\varepsilon|^2 \,\mathrm{d}x \leq 2\,\frac{m^2\, r_*^2}{\varepsilon^{n-2}} \left(\int_{B_1} \rho^2 \,\mathrm{d}x + C \right).$$

Taking into consideration this estimate we obtain

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; D_\varepsilon) = \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; D'_\varepsilon) \leq \frac{(1-\eta)^2}{\varepsilon^{n-1}} \mathscr{L}^n(D_\varepsilon) + 2\, m^2\, r_*^2\, \frac{\eta}{\varepsilon^{n-2}}. \tag{2.28}$$

Finally on $C_{r,\varepsilon}$ both σ_ε and φ_ε are independent of x_1 and are radial in x' then by Fubini's theorem and Proposition 2.2 we get

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; C_{r,\varepsilon}) = \int_{2\,\varepsilon r_*}^{L-2\,\varepsilon r_*} \int_{B'_r} \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) \leq\; L\,(h_\beta(m) + C\,\delta).$$

Adding all together gives the desired estimate. It remains to discuss the case $r_* < 1$. From the point of view of the construction of σ_ε we need to replace the functions ζ_ε with $\tilde{\zeta}_\varepsilon$, satifying

$$\begin{aligned} &\tilde{\zeta}_\varepsilon(t) = 1 \quad \text{for } t \leq \varepsilon \text{ or } t \geq L - \varepsilon, \\ &\tilde{\zeta}_\varepsilon(t) = 0 \quad \text{for } 2\,\varepsilon \leq t \leq L - 2\,\varepsilon, \end{aligned} \qquad \text{and} \qquad \left|\tilde{\zeta}'_\varepsilon\right| \leq \frac{1}{\varepsilon}.$$

This choice ensures that σ_ε has all the properties previously obtained with $r_*\,\varepsilon$ replaced by ε accordingly. Define

$$w_\varepsilon(t) := \begin{cases} \eta, & \text{for } t \leq \sqrt{3}\varepsilon, \\ \dfrac{1-\eta}{r-\sqrt{3}}(t-\sqrt{3}) + \eta, & \text{for } \sqrt{3}\varepsilon \leq t \leq r, \end{cases}$$

and set

$$\varphi_\varepsilon = \min\{\overline{\varphi}_\varepsilon(d(x,I)), w_\varepsilon(|x|), w_\varepsilon(|x-(L,0)|)\}.$$

With these choices for φ_ε and σ_ε the estimates follow analogously with small differences in the constants.

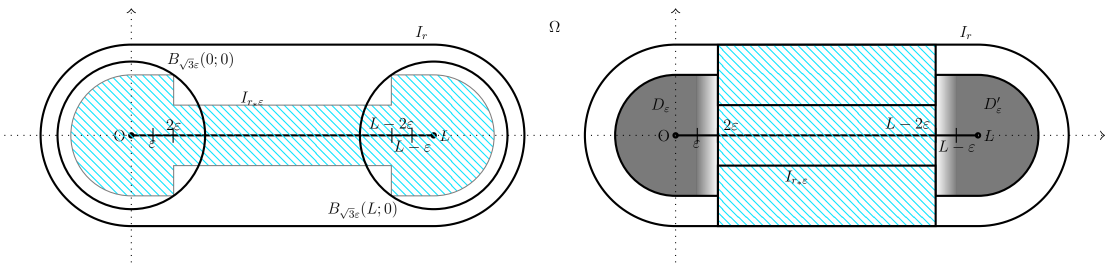


Figure 2.3: On the left the striped region corresponds to $\mathrm{supp}(\sigma_\varepsilon)$, remark that the balls of radius $\sqrt{3}\varepsilon$ centered respectively in $(0;0)$ and $(L;0)$ contain the modifications we have performed to satisfy the constraint. On the right we illustrate the level-lines of the cutoff function $\tilde{\zeta}_\varepsilon$ in grayscale.

Step 2. (Case of a generic σ in polyhedral form.) Indeed, in force of the results quoted in Chapter 1 it is sufficient to show equation (2.3) for a polyhedral vector measure. Following the same notation introduced therein let

$$\sigma = \sum_{j=1}^{N} m_j \mathcal{H}^1 \llcorner \Sigma_j \; \tau_j.$$

With no loss of generality we can assume that the segments Σ_j intersect at most at their extremities. We consider measures σ satisfying constraint (2.4) so that if a point P belongs to $\Sigma_{j_1}, \dots, \Sigma_{j_P}$ it must satisfy of Kirchhoff law,

$$\sum_{j_1}^{j_P} z_j \, m_j = \begin{cases} c_i, & \text{if } P \in \mathscr{S}. \\ 0, & \text{otherwise.} \end{cases} \tag{2.29}$$

where z_j, is $+1$ if P is the ending point of the segment Σ_j with respect to its orientation, and -1 if it is the starting point. Let σ_ε^j and φ_ε^j be the sequences constructed above for each segment Σ_k and define

$$\sigma_\varepsilon = \sum_{j=1}^{N} \sigma_\varepsilon^j \qquad \text{and} \qquad \varphi_\varepsilon = \min_j \left\{ \varphi_\varepsilon^j \right\}.$$

Let P_j and Q_j be respectively the initial and final point of the segment Σ_j and recall that, by the construction made above, for each j

$$\nabla \cdot \sigma_\varepsilon^j = m_j \left(\delta_{P_j} - \delta_{Q_j} \right) * \rho_\varepsilon$$

then by linearity of the divergence operator, it holds

$$\nabla \cdot \sigma_\varepsilon = \sum_{j=1}^{N} \nabla \cdot \sigma_\varepsilon^j = \sum_{j=1}^{N} m_j \, \left(\delta_{P_j} - \delta_{Q_j} \right) * \rho_\varepsilon$$

and the latter satisfies constraint (2.1) in force of equation (2.29). To conclude let us prove that

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \le \sum_{j=1}^{N} h_\beta(m_j) \mathcal{H}^1(\Sigma_j). \tag{2.30}$$

Indeed the following inequality holds true

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \leq \sum_{j=1}^{N} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon^j; \Omega).$$

Suppose

$$\operatorname{supp}(\sigma_\varepsilon^{j_1}) \cap \operatorname{supp}(\sigma_\varepsilon^{j_2}) \cap \cdots \cap \operatorname{supp} \sigma_\varepsilon^{j_P} \neq \varnothing$$

for some $j_1, \ldots, j_P$ and all ε. Let $r_*^{j_1}, \ldots, r_*^{j_P}$ be the radii introduced above for each of these measures, let $\overline{r}_* = \max\{r_*^{j_1}, \ldots, r_*^{j_P}, 1\}$, set $\overline{m} = \max\{m_{j_1}, \ldots, m_{j_P}\}$ and consider $D_{j_1}, \ldots, D_{j_P}$ as defined previously. Since

$$\left| \sum_{k=1}^{j_P} \sigma_\varepsilon^k \right|^2 \leq C \sum_{k=1}^{j_P} \left|\sigma_\varepsilon^k\right|^2$$

and $\varphi_\varepsilon \leq \varphi_\varepsilon^j$ for any j, we have the following inequality

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \operatorname{supp}(\sigma_\varepsilon^{j_1}) \cap \cdots \cap \operatorname{supp}(\sigma_\varepsilon^{j_P})) \leq C \sum_{k=j_1}^{j_P} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon^k, \varphi_\varepsilon^k; D_k).$$

And by inequality (2.28) follows

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \operatorname{supp}(\sigma_\varepsilon^{j_1}) \cap \cdots \cap \operatorname{supp}(\sigma_\varepsilon^{j_P})) \leq C \left(\frac{(1-\eta)^2}{\varepsilon^{n-1}} \sum_{k=j_1}^{j_P} \mathscr{L}^n(D_k) + 2\,\overline{m}^2\,\overline{r}_*^2\,\frac{\eta}{\varepsilon^{n-2}} \right),$$

which vanishes as $\varepsilon \downarrow 0$. Let us remark that the intersection $\operatorname{supp}(\sigma_\varepsilon^{j_1}) \cap \operatorname{supp}(\sigma_\varepsilon^{j_2}) \cap \cdots \cap \operatorname{supp} \sigma_\varepsilon^{j_P}$ is non empty for any ε only if the segments $\Sigma_{j_1}, \ldots, \Sigma_{j_P}$ have a common point. Since we are considering a polyhedral vector measure composed by N segments the worst case scenario is that we have $2N$ intersections in which at most N segments intersects. We conclude

$$\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \leq \sum_{j=1}^{N} \mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon^j, u_\varepsilon^j; \Omega) + C(N) \left(\frac{(1-\eta)^2}{\varepsilon^{n-1}} \sum_{k=j_1}^{j_P} \mathscr{L}^n(D_k) + 2\,\overline{m}^2\,\overline{r}_*^2\,\frac{\eta}{\varepsilon^{n-2}} \right)$$

which, passing to the limit, yields inequality (2.30). □

Chapter 3

The k-dimensional problem

3.1 Introduction

In this chapter we analyze how to address the problem of approximating the k-dimensional Plateau problem. In particular we aim at extending Theorems 2.1, 2.2 and 2.3 in the case where the 1-currents (vector measures) are replaced with k-currents. Let $\sigma_0 \in P_k(\Omega)$ a polyhedral k-current with finite mass and let $\mathscr{S} := \operatorname{supp}(\partial\sigma_0)$ be compactly contained in Ω. We want to minimize a functional of the type (10) where the set of candidates ranges among all currents $\mathcal{D}_k(\overline{\Omega})$ such that

$$\partial\sigma = \partial\sigma_0 \qquad \text{in } \mathcal{D}^k(\mathbf{R}^n). \tag{3.1}$$

Let us introduce a parameter $\eta = \eta(\varepsilon)$ which satisfies

$$\eta(\varepsilon) = \beta\varepsilon^{n-k+1} \quad \text{for} \quad \beta \in \mathbf{R}_+ \tag{3.2}$$

and let $X_\varepsilon(\Omega)$ be the set of pairs $(\sigma_\varepsilon, \varphi_\varepsilon)$ where $\varphi_\varepsilon \in W^{1,p}(\Omega, [\eta, 1])$ and has trace 1 on $\partial\Omega$ and σ_ε is of finite mass with density absolutely continuous with respect to $\mathscr{L}^n$. In this case we identify the current σ_ε with its $L^1(\Omega, \Lambda_k(\mathbf{R}^n))$ density. Furthermore as in equation (2.1) given a convolution kernel ρ_ε we impose the constraint

$$\partial\sigma_\varepsilon = (\partial\sigma_0) * \rho_\varepsilon \qquad \text{in } \mathcal{D}^k(\mathbf{R}^n).$$

For $(\sigma_\varepsilon, \varphi_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ let

$$\mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) := \begin{cases} \displaystyle\int_\Omega \left[\varepsilon^{p-n+k}|\nabla\varphi_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{\varphi_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon}\right] \mathrm{d}x, & \text{if } (\sigma_\varepsilon, \varphi_\varepsilon) \in X_\varepsilon(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.3}$$

Let us denote with X the set of pairs (σ, φ) such that σ is a k-rectifiable current satisfying (3.1) and $\varphi \equiv 1$. In this section we show that for any sequence $\varepsilon \downarrow 0$ the Γ-limit of the family $(\mathscr{F}^k_{\varepsilon,\beta})_{\varepsilon\in\mathbf{R}_+}$ is the functional

$$\mathscr{E}^k_\beta(\sigma, \varphi; \overline{\Omega}) = \begin{cases} \displaystyle\int_{\operatorname{supp}\sigma} h^{n-k}_\beta(m(x))\,\mathrm{d}\mathcal{H}^k(x), & \text{if } (\sigma,\varphi) \in X, \\ +\infty, & \text{otherwise in } \mathcal{M}(\Omega, \mathbf{R}^n) \times L^2(\Omega), \end{cases} \tag{3.4}$$

where the function $h_\beta^{n-k} : \mathbf{R}_+ \to \mathbf{R}_+$ is the function obtained in Appendix B for the choice $d = n - k$ and is endowed with the same properties stated in Chapter 2. In particular under the assumption $p > n - k$ we first prove a compactness theorem.

Theorem 3.1. *Assume that $\beta > 0$. For any sequence $\varepsilon \downarrow 0$, $(\sigma_\varepsilon, \varphi_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ such that*

$$\mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \leq F_0 < +\infty$$

then $\varphi_\varepsilon \to 1$ in $L^2(\Omega)$ and there exists a rectifiable k-current $\sigma \in \mathcal{D}_k(\overline{\Omega})$ such that, up to a subsequence, $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$ and $(\sigma, 1) \in X$.

Then we show the Γ-convergence result, namely

Theorem 3.2. *Assume that $\beta \geq 0$.*

1. *For any $(\sigma, \varphi) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ and any sequence $(\sigma_\varepsilon, \varphi_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ such that $(\sigma_\varepsilon, \varphi_\varepsilon) \to (\sigma, \varphi)$ it holds*

$$\liminf_{\varepsilon \downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \geq \mathscr{E}^k_\beta(\sigma, \varphi; \overline{\Omega}).$$

2. *For any couple $(\sigma, \varphi) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ there exists a sequence $(\sigma_\varepsilon, \varphi_\varepsilon) \in \mathcal{D}_k(\overline{\Omega}) \times L^2(\Omega)$ such that $(\sigma_\varepsilon, \varphi_\varepsilon) \to (\sigma, \varphi)$ and*

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) \leq \mathscr{E}^k_\beta(\sigma, \varphi; \overline{\Omega}).$$

3.2 Compactness and k-rectifiability

Proof of Proposition 3.1. By the same procedure of Lemma 2.1 we obtain

$$|\sigma_\varepsilon|(\Omega) \leq \frac{F_0}{2} + \frac{F_0}{2\,a\,(1-\lambda)^2} + \sqrt{\frac{|\Omega|\,\varepsilon\,F_0}{\lambda}} \tag{3.5}$$

and

$$\int_\Omega (1 - \varphi_\varepsilon)^2 \leq \varepsilon^{n-k}\, F_0.$$

Therefore by the weak compactness of $\mathcal{D}_k(\Omega)$ we obtain the existence of a limit k-current σ a limit measure μ and a subsequence ε such that $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$, $|\sigma_\varepsilon| \stackrel{*}{\rightharpoonup} \mu$. As in the 1-dimensional case it is still necessary to prove the rectifiability of the limit current. This is obtained by showing that the support of σ is of finite size.

Step 1. (Preliminaries and good representative for $v \in \Lambda_k(\mathbf{R}^n)$.) Let us introduce the set

$$\mathcal{I} := \{I = (i_1, \dots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

and denote $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. So that $\Lambda_k(\mathbf{R}^n)$ is the Euclidean space with basis $\{e_I\}_{I \in \mathcal{I}}$. Let $v \in \Lambda_k(\mathbf{R}^n)$ and consider the problem

$$a_0 = \max\{a \in \mathbf{R} \, : \, v = a f_1 \wedge \cdots \wedge f_k + t \, : \, (f_1, \dots, f_n) \text{ orthonormal basis}, \, t \in (f_1 \wedge \cdots \wedge f_k)^\perp\}.$$

Notice that $a_0 \geq 1/\sqrt{|\mathcal{I}|}$. Assume that the optimum for the preceding problem is obtained with $(f_1,\dots,f_n) = (e_1,\dots,e_n)$. We note

$$v = a_0 e_{I_0} + \sum_{i\in\mathcal{I}_1} a_I e_I + \sum_{I\in\mathcal{J}} a_I e_I$$

with $I_0 = (1,\dots,k)$ and

$$\mathcal{I}_1 := \{I = (i_1,\dots,i_k) \in \mathcal{I} \,:\, 1 \leq i_1 < \cdots < i_{k-1} \leq k < i_k \leq n\},$$
$$\mathcal{J} := \mathcal{I} \setminus (\mathcal{I}_1 \cup \{I_0\}).$$

We claim that $a_I = 0$ for $I \in \mathcal{I}_1$. Indeed, let $I_1 = (e_1,\dots,e_{l-1},e_{l+1},\dots,e_k,e_h) \in \mathcal{I}_1$ and for $\phi \in \mathbf{R}$, let e^ϕ be orthonormal base defined as

$$e_i = e_i^\phi \qquad \text{for } i \neq \{l,h\}, \qquad e_l = \cos(\phi)e_l^\phi - \sin(\phi)e_h^\phi, \qquad e_h = \sin(\phi)e_l^\phi + \cos(\phi)e_h^\phi.$$

In this basis

$$v = \big(a_0\cos(\phi) + a_{I_1}(-1)^{k-l}\sin(\phi)\big)\, e_{I_0}^\phi + t^\phi, \quad \text{with} \quad w^\phi \in (e^\phi)^\perp.$$

By optimality of $(e_1,\dots,e_n)$ we deduce $a_{I_1} = 0$ which proves the claim. Hence we write

$$v = a_0 e_{I_0} + t, \quad \text{with } t \in \operatorname{span}\{e_I \,:\, I \in \mathcal{J}\}. \tag{3.6}$$

Now we let Let $\theta \in (0, 1/4^n)$ and Σ be the set of points for which there exists a sequence $r_j \downarrow 0$ such that

$$\frac{\sigma(B_{r_j}(x))}{|\sigma|(B_{r_j}(x))} \longrightarrow w(x) \in S\Lambda_k(\mathbf{R}^n) \quad \text{and} \quad \frac{|\sigma|(B_{r_j/4}(x))}{|\sigma|(B_{r_j}(x))} \geq \theta.$$

In particular w is a $|\sigma|$-measurable map and we have $\sigma = w\,|\sigma| \llcorner \Sigma$.

Step 2. (Flux of σ_ε trough a small $(n-k)$-disk.) Consider a point $x \in \Sigma \setminus \mathscr{S}$, with no loss of generality we assume $x = 0$. Let $v = w(0)$, up to a change of basis, by equation (3.6) we write

$$v = a_0 e_{I_0} + t, \quad \text{with } t \in \operatorname{span}\{e_I \,:\, I \in \mathcal{J}\}.$$

Let j sufficiently small, such that $B_{r_j} \cap \mathscr{S} = \varnothing$ and

$$\sigma(B_{r_j}) \cdot v \geq (1-\xi)|\sigma|(B_{r_j}). \tag{3.7}$$

Set, to simplify notation, $r_j = r$ and $r_* = r/\sqrt{2}$. For $x \in \mathbf{R}^n$ we write $(x',x'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}$ for the usual decomposition and denote B'_r, B''_r the k-dimensional and the $(n-k)$-dimensional ball respectively. Let $\chi \in C^\infty(B''_1)$ be a radial cut-off function with $\chi(x'') = 1$ for $|x''| \leq 1/2$ and $\chi(x'') = 0$ for $|x''| \geq 3/4$. Set $\chi_{r_*}(x'') = \chi(x''/r_*)$, then since σ_ε is a L^1 function for $\varepsilon > 0$ we can define

$$g_\varepsilon(x') := \int_{B''_{r_*}} \chi_{r_*}(x'')\langle \sigma_\varepsilon, e_{I_0}\rangle \,\mathrm{d}x'' = \int_{B''_{r_*}} \chi_{r_*}(x'')\sigma_\varepsilon^0 \,\mathrm{d}x'' \tag{3.8}$$

for any $x' \in B'_{r_*}$. Let us compute $\partial_l g_\varepsilon(x')$ for $l \in \{1, \dots, k\}$. Since $\partial\sigma_\varepsilon = 0$ in B_r, it holds $\langle \sigma_\varepsilon,\ \mathrm{d}\omega\rangle = 0$ for any smooth $(k-1)$-differential form $\omega \in \mathcal{D}^{k-1}(B_r)$. Choosing ω of the form

$$\omega = \beta(x)\ \mathrm{d}x_1 \wedge \dots\ \mathrm{d}x_{l-1} \wedge\ \mathrm{d}x_{l+1} \wedge \dots \wedge\ \mathrm{d}x_k \tag{3.9}$$

we obtain

$$\begin{aligned}\mathrm{d}\omega = (-1)^{l-1}\partial_l\beta(x)\ \mathrm{d}x_1 \wedge \dots \wedge\ \mathrm{d}x_k +& \\ + (-1)^{k-1}\sum_{h=k+1}^{d} \partial_h\beta(x)\ \mathrm{d}x_1 \wedge \dots\ \mathrm{d}x_{l-1} \wedge\ \mathrm{d}x_{l+1} \wedge \dots \wedge\ \mathrm{d}x_k \wedge\ \mathrm{d}x_h.&\end{aligned}$$

Denote $\sigma_\varepsilon^I = \langle \sigma, e^I\rangle$, then imposing $\langle \sigma_\varepsilon,\ \mathrm{d}\omega\rangle = 0$ for every $\beta \in C_c^\infty(B_r)$ in (3.9) yields

$$(-1)^{k-l}\partial_l\sigma_\varepsilon^0 + \sum_{\substack{h\in\{k+1,\dots,d\}\\ I=(1,\dots,l-1,l+1,\dots,k,h)}} \partial_h\sigma_\varepsilon^I = 0.$$

Hence,

$$\partial_l g_\varepsilon(x') = \frac{(-1)^{k-l}}{r_*} \sum_{\substack{h\in\{k+1,\dots,d\}\\ I=(1,\dots,l-1,l+1,\dots,k,h)}} \int_{B''_{r_*}} \partial_h\chi_{r_*}(x'')\sigma_\varepsilon^I\ \mathrm{d}x''. \tag{3.10}$$

Let us introduce the notation

$$\sigma_\varepsilon^{\mathcal{I}_1} := \sum_{I\in\mathcal{I}_1} \sigma_\varepsilon^I\, e_I,$$

denoting with ∇' the gradient with respect to x', equation (3.10) rewrites as

$$\nabla' g_\varepsilon(x') = \frac{1}{r_*}\int_{B''_{r_*}} Y\left(\frac{x}{r_*}\right)\sigma_\varepsilon^{\mathcal{I}_1}\ \mathrm{d}x''. \tag{3.11}$$

Where Y is smooth and compactly supported in B''_1 and with values into the linear maps : $\mathrm{span}\{e_I : I \in \mathcal{I}_1\} \to \mathbf{R}^k$. Let us prove that, for some $\hat{r}$, the functions g_ε converge in BV-$*$ to some g. First for a.e. choice of $\hat{r} \in [(1-\xi)r_*, r_*]$ it must hold $\mu(\partial B'_{r_*} \times B''_{r_*}) = 0$ so that

$$g_\varepsilon(x') = \int_{B''_{r_*}} \chi_{r_*}(x'')\langle\sigma_\varepsilon, e_{I_0}\rangle\ \mathrm{d}x'' \xrightarrow{\varepsilon\downarrow 0} \int_{B''_{r_*}} \chi_{r_*}(x'')\ \mathrm{d}\langle\sigma, e_{I_0}\rangle =: g(x'). \tag{3.12}$$

Secondly we define the mean value

$$\overline{g} := \frac{1}{|B'_{\hat{r}}|}\int_{B'_{\hat{r}}} g(x')\ \mathrm{d}x' = \frac{1}{|B'_{\hat{r}}|}\int_{B'_{\hat{r}}}\left[\int_{B''_{r_*}} \chi_{r_*}(x'')\ \mathrm{d}\sigma^0\right]\ \mathrm{d}x'.$$

and taking advantage of (3.7) and the definition of Σ, we see that

$$\overline{g} \geq \left(\frac{\theta}{\sqrt{|\mathcal{I}|}} - \xi\right)\frac{|\sigma|(B_r)}{|B'_{\hat{r}}|} > 0.$$

On the other hand, denoting $\Pi : \mathbf{R}^n \to \mathbf{R}^{n-k}$, $x \mapsto x''$, from (3.6), we have

$$|\Pi\sigma|(B'_{\hat r} \times B''_{r_*}) \le \sqrt{3\xi}\left(\frac{\theta}{\sqrt{|\mathcal{I}|}} - \xi\right)\ |B'_{\hat r}|\,\overline{g}.$$

Now from (3.11) - - (3.12) and the latter we obtain

$$\langle D'g, \phi\rangle = \frac{1}{r_*}\int_{B'_{\hat r}\times B''_{r_*}} \phi(x')\ Y\left(\frac{x''}{r_*}\right)\ \mathrm{d}\sigma^{\mathcal{I}_1} \quad \text{and} \quad |D'g|(B'_{\hat r}) \le \frac{C\,|B'_{\hat r}|\,\sqrt{\xi}\,\overline{g}}{r_*}.$$

Finally from Poincaré - Wirtinger inequality and the convergence $g_\varepsilon \to g$ in $L^1(B'_{\hat r})$ is easy to show that for any sufficiently small ε the sets

$$A_\varepsilon = \left\{x \in B_{\hat r}\ :\ g_\varepsilon(x) \ge \frac{\overline{g}}{8}\right\}$$

are such that $|A_\varepsilon| \ge |B'_{\hat r}|/2$.

Step 3. (Conclusion.) Set $\vartheta_\varepsilon(x', x'') = |\chi_{r_*}(x'')\sigma^0_\varepsilon|$ and observe that for fixed x' by construction

$$\int_{B_{r_*}} \vartheta_\varepsilon(x', x'')\ \mathrm{d}x'' = g_\varepsilon(x').$$

Therefore for any $x' \in A_\varepsilon$ it holds $\int_{B_{r_*}} \vartheta_\varepsilon(x', x'')\ \mathrm{d}x'' \ge \overline{g}/8$. Furthermore $\mathrm{supp}(\vartheta_\varepsilon(x')) \subset B'_{\tilde r}$ with $\tilde r := \frac{3}{4}r_* < r_*$. Now, by Fubini

$$\begin{aligned}
\mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B_r) &\ge \int_{A_\varepsilon}\int_{B''_r}\left[\varepsilon^{p-n+k}|\nabla\varphi_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{\varphi_\varepsilon|\sigma_\varepsilon|^2}{\varepsilon}\right]\ \mathrm{d}x''\ \mathrm{d}x' \\
&\ge \int_{A_\varepsilon}\int_{B''_{r_*}}\left[\varepsilon^{p-n+k}|\nabla\varphi_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^{n-k}} + \frac{\varphi_\varepsilon|\vartheta_\varepsilon(x',x'')|^2}{\varepsilon}\right]\ \mathrm{d}x''\ \mathrm{d}x'
\end{aligned} \tag{3.13}$$

With the notation introduced in section 2.2 and by definition of A_ε

$$\begin{aligned}
\mathscr{F}_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B_r) &\ge \int_{A_\varepsilon} \inf_{(\vartheta,\varphi)\in Y_{\varepsilon,\beta}(\overline{g}/8, r, \tilde r)} \mathscr{G}^k_{\varepsilon,\beta}(\vartheta, \varphi)\ \mathrm{d}x' \\
&= \int_{A_\varepsilon} h^{n-k}_{\varepsilon,\beta}\left(\overline{g}/8, r, \tilde r\right)\ \mathrm{d}x' \\
&= h^{n-k}_{\varepsilon,\beta}\left(\overline{g}/8, r, \tilde r\right)|A_\varepsilon|.
\end{aligned}$$

Taking the infimum limit, by Proposition 2.1, in particular equation (2.12) we get

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B_r) \ge \liminf_{\varepsilon\downarrow 0} h^{n-k}_{\varepsilon,\beta}\left(\overline{g}/8, r, \tilde r\right)|A_\varepsilon|\ \ge\ \kappa\,\frac{|B'_{\hat r}|}{2}. \tag{3.14}$$

Recall that the latter stands for a.e. $\hat r \in [(1-\xi)r_*, r_*]$ and $r_* = r/\sqrt{2}$ thus we may rewrite

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B_r)\ \ge\ \kappa\,\frac{\omega_k\, r^k}{2^{1+k/2}}.$$

As in Lemma 2.3 we conclude applying Besicovitch theorem to obtain $\mathcal{H}^k(\Sigma) < +\infty$. Finally, thanks to the latter and equation (3.5), White's rectifiability theorem [Whi99b, Thm 6.1] applies and σ is a k-rectifiable current. $\square$

3.3 Γ-liminf inequality

Proof of item 1) of Theorem 3.2. With no loss of generality we assume that $\liminf_{\varepsilon\downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon) < +\infty$ otherwise the inequality is trivial. For a Borel set $A \subset \Omega$, we define

$$H^k(A) := \liminf_{\varepsilon\downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon,\varphi_\varepsilon; A),$$

so that H^k is a subadditive set function. By assumption, the limit current σ is k-rectifiable; we write $\sigma = m\,\tau\,\mathcal{H}^k \llcorner \Sigma$. We claim that

$$\liminf_{r\downarrow 0} \frac{H^k\left(\overline{B(x,r)}\right)}{\omega_k\, r^k} \geq h^{n-k}_\beta(m(x)) \qquad \text{for } \mathcal{H}^k\text{-almost every } x \in \Sigma. \tag{3.15}$$

Assuming the latter the proof is achieved as in Theorem 2.2. To establish the claim (3.15) we restrict our attention to a single point and we assume $x = 0$, $m = m(0)$ and $\tau(0) = e_1 \wedge \cdots \wedge e_k$ then for any $\xi > 0$ there exists $r_0 = r(\xi)$ such that

$$\begin{aligned} &\langle \sigma, e_1 \wedge \cdots \wedge e_k\rangle(B_r) \geq (1-\xi)|\sigma|(B_r) \quad \text{and} \\ (1-\xi)\, m \leq\ &\frac{|\sigma|(B_r)}{\omega_k r^k} \leq (1+\xi)\, m, \qquad \text{for } r \leq r_0. \end{aligned} \tag{3.16}$$

Let δ be an infinitesimal quantity and set, for $r < r_0$, $\hat{r} = \sqrt{1-\delta^2}\, r$ and $\tilde{r} = \delta r$ and define the cylinder

$$C_{\delta,r}(e_1, \wedge \cdots \wedge e_n) = C_{\delta,r} := \left\{(x'; x'') \in \mathbf{R}^k \times \mathbf{R}^{n-k} : |x'| \leq \hat{r} \text{ and } |x''| \leq \tilde{r}\right\}.$$

Let $\chi(x'')$ be the radial cutoff introduced in the previous proposition and set $\chi_{\tilde{r}}(x'') = \chi(x''/\tilde{r})$, $\sigma^0_\varepsilon = \langle \sigma_\varepsilon, e_1 \wedge \cdots \wedge e_k\rangle$ and for any $x' \in B'_{\hat{r}}$ set

$$g_\varepsilon(x') := \int_{B''_{\tilde{r}}} \chi_{\tilde{r}}(x'')\, \mathrm{d}\langle \sigma_\varepsilon, e_{I_0}\rangle = \int_{B''_{\tilde{r}}} \chi_{\tilde{r}}(x'')\, \mathrm{d}\sigma^0_\varepsilon,$$

as in equation (3.8). Up to a smaller choice for r_0 we can assume $B_r \cap \mathscr{S} = \varnothing$ therefore $\partial\sigma \llcorner B_r = 0$, and from equations (3.8) - (3.11) it holds

$$\nabla' g_\varepsilon(x') = \frac{1}{\tilde{r}} \int_{B''_{\tilde{r}}} Y\left(\frac{x}{\tilde{r}}\right)\, \mathrm{d}\sigma^{\mathcal{I}_1}_\varepsilon.$$

For a.e. choice of δ it holds $|\sigma|(\partial B'_{\hat{r}} \times B''_{\tilde{r}}) = 0$ therefore, for any such choice, γ_ε converges in $BV(B_{\hat{r}})$ to

$$g(x') := \int_{B''_{\tilde{r}}} \chi_{\tilde{r}}(x'')\, \mathrm{d}\sigma^0 \qquad \text{and} \qquad \langle D'g, \phi\rangle = \frac{1}{\tilde{r}} \int_{B'_{\hat{r}} \times B''_{\tilde{r}}} \phi(x')\, Y\left(\frac{x''}{\tilde{r}}\right)\, \mathrm{d}\sigma^{\mathcal{I}_1}.$$

Now we use (3.16) to improve the estimates on $\overline{g}$ and $|D'g|$. Indeed, for δ sufficiently small, $\tilde{r} < \hat{r}/2$ therefore $B_{\tilde{r}} \subset B'_{\hat{r}} \times B''_{\tilde{r}}$ and

$$\lim_{\varepsilon\downarrow 0} \overline{g}_\varepsilon \geq (1-\xi)\frac{1}{|B''_{\hat{r}}|} \int_{B'_{\hat{r}} \times B''_{\tilde{r}}} \chi_{r_*}(x')\, \mathrm{d}|\sigma| \geq (1-\xi)^2 m.$$

and denoting $\Pi : \mathbf{R}^n \to \mathbf{R}^{n-k}$, $x \mapsto x''$ we have

$$|\Pi\sigma|(C_r) \leq (1+\xi)\sqrt{3\xi}\,|B'_{\hat{r}}|\,m \quad \text{and} \quad |D'g|(B'_{\hat{r}}) \leq \frac{C\,|B'_{\hat{r}}|\,\sqrt{\xi}\,m}{\tilde{r}}.$$

Choose r sufficiently small then by Poincaré - Wirtinger inequality there exists a set A of almost full measure in $B_{\hat{r}}$ such that $g_\varepsilon(x') \geq (1-\xi)^2\,m$, and following the proof of the previous lemma (Step 3) up to equation (3.14) we get

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B_r) \geq \liminf_{\varepsilon\downarrow 0} h^{n-k}_{\varepsilon,a}\left((1-\xi)^2\,m, r, \tilde{r}\right)|A|.$$

Since ξ and δ are arbitrary and $|A|$ can be chosen arbitrary close to $|B_{\hat{r}}|$ applying Proposition 2.1 with $d = n-k$ to the latter we conclude

$$\liminf_{\varepsilon\downarrow 0} \mathscr{F}^k_{\varepsilon,\beta}(\sigma_\varepsilon, \varphi_\varepsilon; B_r) \geq h^{n-k}_\beta(m)\,\omega_k r^k.$$

□

3.4 Γ-limsup inequality

For the lim-sup inequality, we start by approximating σ with a polyhedral current: given $\delta > 0$, there exists a k polyhedral current $\tilde{\sigma}$ satisfying $\partial\tilde{\sigma} = \partial\sigma_0$ and with $\mathbb{F}(\tilde{\sigma} - \sigma) < \delta$ and $\mathscr{E}_a(\tilde{\sigma}) < \mathscr{E}_a(\sigma) + \varepsilon$. This result of independent interest is established in [CFM18]. A similar result has been proved recently by Colombo et al. in [CDRMS17, Prop. 2.6] (see also [Whi99b, Section 6]). The authors build an approximation of a k-rectifiable current in flat norm and in energy but their construction creates new boundaries and can not ensure the condition $\partial\sigma = \partial\sigma_0$.

Proof of item 2) of Theorem 3.2:
By [CFM18, Theorem 1.1 and Remark 1.6] we can assume that σ is a polyhedral current. We show how to produce the approximating $(\sigma_\varepsilon, \varphi_\varepsilon)$ for σ supported on a single k-dimensional simplex Q. We assume with no loss of generality that $Q \subset \mathbf{R}^k$, and that σ writes as

$$m\,\mathcal{H}^k \llcorner_Q \wedge (e_1 \wedge \cdots \wedge e_k).$$

For $\delta > 0$ fixed, we consider the optimal profiles

$$\overline{\varphi}_\varepsilon(t) := \begin{cases} \eta, & \text{for } 0 \leq t \leq r_*\varepsilon, \\ v_\delta\left(\dfrac{t}{\varepsilon}\right), & \text{for } r_*\varepsilon \leq t \leq r, \\ 1 & \text{for } r \leq t, \end{cases} \qquad \text{and} \qquad \vartheta_\varepsilon = \frac{m\,\chi_{B''_{r_*\varepsilon}}(x'')}{\omega_{n-k}\,(\varepsilon r_*)^{n-k}}$$

with r_* and v_δ, defined in Proposition 2.2 for the choice $d = n-k$. We denote ∂Q the relative boundary of Q and given a set S we write $d(x, S)$ for the distance function from S. Recall that we use the notation S_t for the t-enlargement of the set S and S'

to denote its projection into $\mathbf{R}^k$. We first assume, as did for the case $k = 1$, $r_* \geq 1$, and introduce ζ_ε a 0-form depending on the first k variables x', satisfying

$$\zeta_\varepsilon(x') = 1, \qquad \text{for } x' \in (\partial Q)'_{r_*\varepsilon} := \{x \in \Omega \;:\; d(x', \partial Q) \leq r_*\varepsilon\},$$
$$\zeta_\varepsilon(x') = 0, \qquad \text{for } x' \in \Omega \setminus (\partial Q)'_{2r_*\varepsilon},$$
$$|\,\mathrm{d}\zeta_\varepsilon| \leq \frac{1}{r_*\varepsilon}.$$

Then we proceed by steps, first set $\overline{\sigma}^1_\varepsilon := (|\sigma| * \rho_\varepsilon)$

$$\sigma^1_\varepsilon = \overline{\sigma}^1_\varepsilon e_1 \wedge \cdots \wedge e_k \qquad \text{and} \qquad \sigma^2_\varepsilon(x', x'') = \vartheta_\varepsilon(|x''|) \wedge (e_1 \wedge \cdots \wedge e_k).$$

and observe that $\mathrm{supp}(\sigma^1_\varepsilon) \cup \mathrm{supp}(\sigma^2_\varepsilon) \subset Q_{r_*\varepsilon}$, both σ^1_ε and σ^2_ε are radial in x'' and with a small abuse of notation we denote $\overline{\sigma}^1_\varepsilon(x', s) = \overline{\sigma}^1_\varepsilon(x', |x''|)$, finally for any x'

$$\int_{\{x'\} \times B''_{r_*\varepsilon}} [\overline{\sigma}^1_\varepsilon(x', |x''|) - \vartheta_\varepsilon(|x''|)]\,\mathrm{d}x'' = 0.$$

Now we take advantage of ζ_ε in order to interpolate between σ^1_ε and σ^2_ε, note that such interpolation may affect the boundary of the new current therefore we first introduce σ^3_ε which corrects this defect. In particular set

$$\sigma^3_\varepsilon(x', x'') = -\sum_{i=k+1}^{n} \left[\frac{x_i}{|x''|^{n-k}} \int_0^{|x''|} s^{n-k-1} \left[\overline{\sigma}^1_\varepsilon(x', s)\vartheta_\varepsilon(s)\right] \llcorner \,\mathrm{d}\zeta_\varepsilon \,\mathrm{d}s \right] \wedge e_i,$$

and

$$\sigma_\varepsilon = \sigma^1_\varepsilon \llcorner \zeta_\varepsilon + \sigma^2_\varepsilon \llcorner (1 - \zeta_\varepsilon) + \sigma^3_\varepsilon.$$

With this choice by a calculation similar to equation (2.27) it holds

$$\partial\sigma_\varepsilon = -\partial\sigma * \rho_\varepsilon \llcorner \zeta_\varepsilon - \sigma^1_\varepsilon \llcorner \,\mathrm{d}\zeta_\varepsilon - \underbrace{\partial\sigma^2_\varepsilon \llcorner (1 - \zeta_\varepsilon)}_{=0} + \sigma^2_\varepsilon \llcorner \,\mathrm{d}\zeta_\varepsilon + \partial\sigma^3_\varepsilon = (\partial\sigma) * \rho_\varepsilon.$$

On the other hand the phase-field is simply defined as $\varphi_\varepsilon(x) = \overline{\varphi}_\varepsilon(d(x, Q))$. In the case $r_* < 1$ we need to modify the construction. For σ_ε it is sufficient to replace every occurrence of ζ_ε with $\tilde{\zeta}_\varepsilon$, which satisfies

$$\tilde{\zeta}_\varepsilon(x') = 1, \qquad \text{for } x' \in (\partial Q)'_\varepsilon := \{x \in \Omega \;:\; d(x', \partial Q) \leq \varepsilon\},$$
$$\tilde{\zeta}_\varepsilon(x') = 0, \qquad \text{for } x' \in \Omega \setminus (\partial Q)'_{2\varepsilon},$$
$$\left|\,\mathrm{d}\tilde{\zeta}_\varepsilon\right| \leq \frac{1}{\varepsilon}.$$

Now let

$$w_\varepsilon(t) := \begin{cases} \eta, & \text{for } t \leq \sqrt{3}\varepsilon, \\ \dfrac{1-\eta}{r - \sqrt{3}}(t - \sqrt{3}) + \eta, & \text{for } \sqrt{3}\varepsilon \leq t \leq r. \end{cases}$$

and set

$$\varphi_\varepsilon = \min\{\overline{\varphi}_\varepsilon(d(x, Q)), w_\varepsilon(d(x, \partial Q))\}.$$

Remark 1. Given a polyhedral current σ such that $\partial\sigma = \partial\sigma_0$ we perform our construction on each simplex and define σ_ε as the sum of these elements. The linearity of the boundary operator grants that $\partial\sigma_\varepsilon = \partial\sigma_0 * \rho_\varepsilon$. The phase field is chosen as the pointwise minimum of the local phase fields. Finally the estimation for the Γ-limsup inequality is achieved in the same manner as Theorem 2.3.

□

3.5 Discussion about the results

By Lemma B.4 for any fixed $d = n - k$ the cost function h^d_β pointwise converges as $\beta \downarrow 0$ to the function

$$h(m) = \begin{cases} \kappa, & \text{for } m > 0, \\ 0, & \text{if } m = 0, \end{cases}$$

where κ is the constant value obtained in Proposition 2.1 and depends on d. This condition is sufficient to prove that the family of functionals $\mathscr{E}^k_\beta$, parametrized in β, Γ-converges to the functional

$$\mathscr{E}^k(\sigma;\Omega) := \begin{cases} \kappa\, \mathcal{H}^k(\Sigma \cap \Omega), & \text{for } \sigma = m\,\tau\,\mathcal{H}^k \llcorner \Sigma, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

As a matter of fact for any sequence $\sigma_\beta \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{D}_k(\Omega)$ it holds

$$\liminf_{\beta\downarrow 0} \mathscr{E}^k_\beta(\sigma;\Omega) \geq \mathscr{E}^k(\sigma;\Omega)$$

since $h^d_\beta(m) \geq \kappa$. On the other hand setting $\sigma_\beta := \sigma$ we construct a recovery sequence for any σ and obtain the Γ-limsup inequality

$$\limsup_{\beta\downarrow 0} \mathscr{E}^k_\beta(\sigma_\beta;\Omega) = \limsup_{\beta\downarrow 0} \mathscr{E}^k_\beta(\sigma;\Omega) = \mathscr{E}^k(\sigma;\Omega).$$

This allows to interpret our result as an approximation of the Plateau problem in any dimension and co-dimension.

Chapter 4

Piecewise affine cost functions

4.1 Introduction

The present chapter is the result of a collaboration with Benedikt Wirth and Carolin Rossmanith from Mnster University. We generalize the approach of Chapter 1 to the case in which the cost function h is piecewise affine. Let $N \in \mathbf{N}$ and $\infty \geq \alpha_0 > \alpha_1 > \cdots > \alpha_N > 0$, $0 = \beta_0 < \beta_1 < \cdots < \beta_N < \infty$, we define the piecewise affine transport cost $h : [0, \infty) \to [0, \infty)$,

$$h(m) = \min_{i=0,\dots,N} \{\alpha_i m + \beta_i\} .$$

If $\alpha_0 = \infty$ we interpret h as

$$h(m) = \begin{cases} 0 & \text{if } m = 0\,, \\ \min_{i=1,\dots,N}\{\alpha_i m + \beta_i\} & \text{else.} \end{cases}$$

We first remark that if $\alpha_0 = \infty$ the right derivative of h in the origin diverges, then $\mathscr{E}_h(\sigma)$ is finite if and only if σ is a rectifiable vector measure as stated in [CDRMS17, Proposition 2.7]. On the contrary, in the case $\alpha_0 < \infty$, the energy $\mathscr{E}_h$ may be finite on more complicated structures. Consider the usual probability measures μ_+ and μ_- which, in the case $\alpha_0 < \infty$ may be considered diffuse, and let σ be a vector measure satisfying the constraint $\nabla \cdot \sigma = \mu_+ - \mu_-$. Recall that in force of the Generalized Gilbert-Steiner formula presented in Proposition 0.1 and our choice of h we have

$$\mathscr{E}_h(\sigma) = \alpha_0 |\sigma^\perp|(\Omega) + \int_\Sigma h(m)\, \mathrm{d}\mathcal{H}^1,$$

in the latter $\alpha_0 = h'(0)$, the right derivative in 0. Above we have decomposed σ as

$$\sigma = \sigma^\perp + m\tau\mathcal{H}^1 \llcorner \Sigma$$

where (m, τ, Σ) is an $\mathcal{H}^1$ rectifiable measure and $\sigma^\perp$ is $\mathcal{H}^1$-diffuse.

The functional proposed in this chapter resembles the one introduced in the first chapter. Let us remark that in the first chapter we were able to recover in the limit any affine cost function of the form $1 + \beta|m|$. Here we rescale the functional presented therein and introduce multiple phase fields, each one responsible for a different component of the piecewise affine cost function. The constraint component is modified to let interact the different phase fields. Let us be more formal. We let $X_\varepsilon^{\mu_+,\mu_-}$ denote the space of $(N+1)$-uples $(\sigma, \varphi_1, \dots, \varphi_N)$ where $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^2)$ such that

$$\nabla \cdot \sigma = (\mu_+ - \mu_-) * \rho_\varepsilon$$

and $\varphi_i \in L^1(\Omega)$ for each $i = \{1, \dots, N\}$. Eventually we let

$$\mathscr{F}_\varepsilon(\sigma, \varphi_1, \dots, \varphi_N) = \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right)\, dx + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon(\varphi_i)\,,$$

where we abbreviated (with some $p > 1$)

$$\begin{aligned}
\mathscr{T}_\varepsilon[\varphi] &= \frac{1}{2}\int_\Omega \left[\varepsilon|\nabla\varphi(x)|^2 + \frac{(\varphi(x)-1)^2}{\varepsilon}\right]\, dx\,,\\
\gamma_\varepsilon(x) &= \min_{i=1,\dots,N}\left\{\varphi_i(x)^2 + \alpha_i^2\varepsilon^2/\beta_i\right\}\,,\\
\omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right) &= \begin{Bmatrix} \frac{\gamma_\varepsilon}{\varepsilon}\frac{|\sigma|^2}{2} & \text{if } |\sigma| \le \frac{\alpha_0}{\gamma_\varepsilon/\varepsilon} \\ \alpha_0(|\sigma| - \frac{\alpha_0}{2\gamma_\varepsilon/\varepsilon}) & \text{if } |\sigma| > \frac{\alpha_0}{\gamma_\varepsilon/\varepsilon}\end{Bmatrix} + \varepsilon^p|\sigma(x)|^2 \quad \text{for } \alpha_0 < \infty\,,\\
\omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right) &= \frac{\gamma_\varepsilon}{\varepsilon}\frac{|\sigma|^2}{2} \qquad\qquad \text{for } \alpha_0 = \infty\,.
\end{aligned} \tag{4.1}$$

Remark 2 (Motivation of $\omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right)$ via relaxation)**.** Keeping the phase fields $\varphi_1, \dots, \varphi_N$ fixed and ignoring the regularizing term $\varepsilon^p|\sigma|^2$, the integrand $\omega_\varepsilon(\alpha_0, \frac{\gamma_\varepsilon}{\varepsilon}, |\sigma|)$ is the convexification in σ of

$$\min\left\{\alpha_0|\sigma|, \frac{(\varphi_1^2 + \alpha_1^2\varepsilon^2/\beta_1)|\sigma|^2}{2\varepsilon}, \dots, \frac{(\varphi_N^2 + \alpha_N^2\varepsilon^2/\beta_N)|\sigma|^2}{2\varepsilon}\right\} = \min\left\{\alpha_0|\sigma|, \frac{\gamma_\varepsilon}{\varepsilon}\frac{|\sigma|^2}{2}\right\}\,,$$

which shows the intuition of the phase field functional much clearer. Indeed, the minimum over $N+1$ terms parallels the minimum in the definition of h, and the i^{th} term for $i = 0, \dots, N$ describes (part of) the transportation cost $\alpha_i m + \beta_i$. However, since the above is not convex with respect to σ, a functional with this integrand would not be weakly lower semi-continuous in σ and consequently possess no minimizers in general. Taking the lower semi-continuous envelope corresponds to replacing the above by $\omega_\varepsilon(\alpha_0, \frac{\gamma_\varepsilon}{\varepsilon}, |\sigma|)$ (note that this only ensures existence of minimizers, but will not change the Γ-limit of the phase field functional).

Note that the pointwise minimum inside γ_ε is well-defined almost everywhere, since all elements of $X_\varepsilon^{\mu_+,\mu_-}$ are Lebesgue-measurable. Note also that for fixed phase fields $\varphi_1, \dots, \varphi_N$ the phase field cost functional $\mathscr{F}_\varepsilon$ is convex in σ. This ensures the existence of minimizers for $\mathscr{F}_\varepsilon$, which follows by a standard application of the direct method as it will be shown in the following section. Eventually we extend the functional on the whole $\mathcal{M}(\Omega, \mathbf{R}^2) \times L^1(\Omega)^N$ letting $(N+1)$-uples $(\sigma, \varphi_1, \dots, \varphi_N)$ we let

$$\mathscr{F}_\varepsilon(\sigma, \varphi_1, \dots, \varphi_N) = \infty$$

if $(\sigma, \varphi_1, \ldots, \varphi_N) \notin X_\varepsilon$. For consistency we introduce the set X consisting of those $(N+1)$-tuples $(\sigma, \varphi_1, \ldots, \varphi_N)$ such that each $\varphi_i = 1$ for each i and $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ is a vector valued measure satifing the constraint

$$\nabla \cdot \sigma = \mu_+ - \mu_-.$$

Then we define the limit energy

$$\mathscr{E}(\sigma, \varphi_1, \ldots, \varphi_N) := \begin{cases} \mathscr{E}_h(\sigma), & \text{if } (\sigma, \varphi_1, \ldots, \varphi_N) \in X, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.2}$$

In particular in this chapter we will prove the following theorems.

Theorem 4.1 (Convergence of phase field cost functional)**.** *For admissible $\mu_+, \mu_- \in \mathcal{P}(\overline{\Omega})$ we have*

$$\Gamma - \lim_{\varepsilon \to 0} \mathscr{F}_\varepsilon = \mathscr{E}\,,$$

where the Γ-limit is with respect to weak-$*$ convergence in $\mathcal{M}(\overline{\Omega}; \mathbf{R}^2)$ and strong convergence in $L^1(\Omega)^N$.

In the above we say admissible since we will implicitly assume that if $\alpha_0 = \infty$ then μ_- and μ_+ are supported on a countable number of points. The proof of this result is provided in the next section. Together with the following equicoercivity statement, whose proof is also deferred to the next section, we have that minimizers of the phase field cost functional $\mathscr{F}_\varepsilon$ approximate minimizers of the original cost functional $\mathscr{E}$.

Theorem 4.2 (Equicoercivity)**.** *For $\varepsilon \to 0$ let $(\sigma^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_N^\varepsilon)$ be a sequence with uniformly bounded phase field cost functional $\mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_N^\varepsilon) < C < \infty$. Then, along a subsequence, $\sigma^\varepsilon \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}(\overline{\Omega}; \mathbf{R}^2)$ for some $\sigma \in \mathcal{M}(\overline{\Omega}; \mathbf{R}^2)$ and $\varphi_i^\varepsilon \to 1$ in $L^1(\Omega)$, $i = 1, \ldots, N$. As a consequence, if $\mu_+, \mu_- \in \mathcal{P}(\overline{\Omega})$ are admissible and such that there exists $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^2)$ with $\mathscr{E}(\sigma, 1, \ldots, 1) < \infty$, then any sequence of minimizers of $\mathscr{F}_\varepsilon$ contains a subsequence converging to a minimizer of $\mathscr{E}$ as $\varepsilon \to 0$.*

4.2 Remarks

Before moving to the proofs of the results previously stated let us stress out some important remarks.

Proposition 4.1 (Existence of minimizers to the phase field functional)**.** *The phase field cost functional $\mathscr{F}_\varepsilon$ has a minimizer $(\sigma, \varphi_1, \ldots, \varphi_N) \in X_\varepsilon$.*

Proof. The functional is bounded below by 0 and has a nonempty domain. Indeed, choose $\hat{\varphi}_1 \equiv \ldots \equiv \hat{\varphi}_N \equiv 1$ and $\hat{\sigma} = \nabla \psi$, where ψ solves $\Delta \psi = \mu_+^\varepsilon - \mu_-^\varepsilon$ in Ω with Neumann boundary conditions $\nabla \psi \cdot \nu_{\partial\Omega} = 0$, $\nu_{\partial\Omega}$ being the unit outward normal to $\partial\Omega$. (Since $\int_\Omega \mu_+^\varepsilon - \mu_-^\varepsilon \, \mathrm{d}x = 0$, a solution ψ exists and lies in $W^{2,2}(\Omega)$ by standard elliptic regularity.) Obviously, $(\hat{\sigma}, \hat{\varphi}_1, \ldots, \hat{\varphi}_N) \in X_\varepsilon^{\mu_+,\mu_-}$ with $\mathscr{F}_\varepsilon(\hat{\sigma}, \hat{\varphi}_1, \ldots, \hat{\varphi}_N) < \infty$.

Now consider a minimizing sequence $(\sigma^k, \varphi_1^k, \ldots, \varphi_N^k) \in X_\varepsilon^{\mu_+,\mu_-}$, $k \in \mathbf{N}$, with

$$\mathscr{F}_\varepsilon(\sigma^k, \varphi_1^k, \ldots, \varphi_N^k) \to \inf \mathscr{F}_\varepsilon$$

monotonically as $k \to \infty$. Since $\mathscr{F}_\varepsilon$ is coercive with respect to $H = L^2(\Omega;\mathbf{R}^2) \times W^{1,2}(\Omega)^N$, $(\sigma^k, \varphi_1^k, \dots, \varphi_N^k)$ is uniformly bounded in H so that we can extract a weakly converging subsequence, still indexed by k for simplicity,

$$(\sigma^k, \varphi_1^k, \dots, \varphi_N^k) \rightharpoonup (\sigma, \varphi_1, \dots, \varphi_N)\,.$$

Due to the closedness of $X_\varepsilon^{\mu_+,\mu_-}$ with respect to weak convergence in H we have $(\sigma, \varphi_1, \dots, \varphi_N) \in X_\varepsilon^{\mu_+,\mu_-}$. Note that the integrand of $\mathscr{F}_\varepsilon$ is convex in $\sigma(x)$ and the $\nabla\varphi_i(x)$ as well as continuous in $\sigma(x)$ and the $\varphi_i(x)$, thus $\mathscr{F}_\varepsilon$ is lower semi-continuous along the sequence. Indeed, consider a subsequence along which each term $\mathscr{T}_\varepsilon[\varphi_i^k]$ converges and along which the φ_i^k converge pointwise almost everywhere (so that also $\gamma_\varepsilon^k(x) = \min_{i=1,\dots,N} \left\{\varphi_i^k(x)^2 + \alpha_i^2\varepsilon^2/\beta_i\right\}$ converges for almost all $x \in \Omega$). By Mazur's lemma, a sequence of convex combinations $\sum_{j=k}^{m_k} \lambda_j^k \sigma^j$ of the σ^k converges strongly (and up to another subsequence again pointwise) so that by Fatou's lemma we have

$$\begin{aligned}
\inf \mathscr{F}_\varepsilon &= \lim_{k\to\infty} \mathscr{F}_\varepsilon(\sigma^k, \varphi_1^k, \dots, \varphi_N^k) \\
&= \lim_{k\to\infty} \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon^k(x)}{\varepsilon}, |\sigma^k(x)|\right)\,\mathrm{d}x + \sum_{i=1}^N \beta_i \lim_{k\to\infty} \mathscr{T}_\varepsilon[\varphi_i^k] \\
&\geq \lim_{k\to\infty} \sum_{j=k}^{m_k} \lambda_j^k \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon^j(x)}{\varepsilon}, |\sigma^j(x)|\right)\,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon(\varphi_i) \\
&\geq \int_\Omega \liminf_{k\to\infty} \sum_{j=k}^{m_k} \lambda_j^k \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon^j(x)}{\varepsilon}, |\sigma^j(x)|\right)\,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon(\varphi_i) \\
&\geq \int_\Omega \liminf_{k\to\infty} \sum_{j=k}^{m_k} \lambda_j^k \omega_\varepsilon\left(\alpha_0, \inf_{i=k,\dots,m_k} \frac{\gamma_\varepsilon^i(x)}{\varepsilon}, |\sigma^j(x)|\right)\,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon(\varphi_i) \\
&\geq \int_\Omega \liminf_{k\to\infty} \omega_\varepsilon\left(\alpha_0, \inf_{i=k,\dots,m_k} \frac{\gamma_\varepsilon^i(x)}{\varepsilon}, \sum_{j=k}^{m_k} \lambda_j^k |\sigma^j(x)|\right)\,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon(\varphi_i) \\
&= \mathscr{F}_\varepsilon(\sigma, \varphi_1, \dots, \varphi_N)\,,
\end{aligned}$$

where we exploited the weak lower semi-continuity of $\mathscr{T}_\varepsilon$, the monotonicity of $\omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right)$ in its second argument, its convexity in its last argument, and its continuity in its latter two arguments. □

Remark 3 (Regularization of σ)**.** Note that the phase field cost functional $\mathscr{F}_\varepsilon$ is $L^2(\Omega;\mathbf{R}^2)$-coercive in σ, which is essential to have sequentially weak compactness of subsets of $X_\varepsilon^{\mu_+,\mu_-}$ with finite cost (and as a consequence existence of minimizers). For $\alpha_0 < \infty$ this is ensured by the regularization term $\varepsilon^p|\sigma|^2$ (which has no other purpose). Without it, the functional would only feature weak-$*$ coercivity for σ in $\mathcal{M}(\overline{\Omega};\mathbf{R}^2)$, however, the integral $\int_\Omega \omega_\varepsilon\big(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\big)\,\mathrm{d}x$ with γ_ε Lebesgue-measurable would in general not be well-defined for $\sigma \in \mathcal{M}(\overline{\Omega};\mathbf{R}^2)$.

Remark 4 (Phase field boundary conditions)**.** Recall that we imposed boundary conditions $\varphi_i = 1$ on $\partial\Omega$. Without those, the recovery sequence from the following section could easily be adapted such that all full phase field profiles near the boundary will be replaced by half, one-sided phase field profiles. It is straightforward to show that the resulting limit functional would become

$$\int_{\Sigma\cap\Omega} \min\{\alpha_0\, m, \alpha_1\, m + \beta_1, \dots, \alpha_N\, m + \beta_N\} \\ + \int_{\Sigma\cap\partial\Omega} \min\{\alpha_0\, m, \alpha_1\, m + \beta_1/2, \dots, \alpha_N\, m + \beta_N/2\} + \alpha_0|\sigma^\perp|(\overline{\Omega})\,,$$

where fluxes along the boundary are cheaper and thus preferred.

Remark 5 (Divergence measure vector fields and flat chains)**.** Any divergence measure vector field can be identified with a flat 1-chain or a locally normal 1-current (see for instance [Š07, Sec. 5] or [BW17, Rem. 2.29(3)]; comprehensive references for flat chains and currents are [Whi57, Fed69]). Furthermore, for a sequence σ^j, $j \in \mathbf{N}$, of divergence measure vector fields with uniformly bounded $\|\nabla\cdot\sigma^j\|_{\mathcal{M}}$, weak-$*$ convergence is equivalent to convergence of the corresponding flat 1-chains with respect to the flat norm [BW17, Rem. 2.29(4)]. Analogously, scalar Radon measures of finite total variation and bounded support can be identified with flat 0-chains or locally normal 0-currents [Whi99b, Thm. 2.2], and for a bounded sequence of compactly supported scalar measures, weak-$*$ convergence is equivalent to convergence with respect to the flat norm of the corresponding flat 0-chains.

From the above it follows that in Theorems 4.1 and 4.2 we may replace weak-$*$ convergence by convergence with respect to the flat norm. Indeed, for both results it suffices to consider sequences $(\sigma^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon)$ with uniformly bounded cost $\mathscr{F}_\varepsilon$. For those we have uniformly bounded $\|\sigma^\varepsilon\|_{\mathcal{M}}$ (by Theorem 4.2) as well as uniformly bounded $\|\nabla\cdot\sigma^\varepsilon\|_{\mathcal{M}} = \|\mu_+^\varepsilon - \mu_-^\varepsilon\|_{\mathcal{M}}$ so that weak-$*$ and flat norm convergence are equivalent.

4.3 The Γ-limit of the phase field functional

In this section we prove the Γ-convergence result. As it is canonical, we begin with the lim inf-inequality, after which we prove the lim sup-inequality as well as equicoercivity.

4.3.1 The $\Gamma-\liminf$ inequality for the dimension-reduced problem

Here we consider the energy reduced to codimension-1 slices of the domain Ω. In our particular case of a two-dimensional domain, each slice is just one-dimensional, which will simplify notation a little (the procedure would be the same for higher codimensions, though). The reduced functional depends on the (scalar) normal flux ϑ through the slice as well as the scalar phase fields $\varphi_1, \dots, \varphi_N$ restricted to the slice. First observe that any measure $\vartheta \in \mathcal{M}(\overline{I})$ can be decomposed into its atoms and a remainder, namely

$$\vartheta = m_\vartheta \mathcal{H}^0 \llcorner S_\vartheta + \vartheta^\perp\,,$$

where $S_\vartheta \subset \overline{I}$ is the set of atoms of ϑ, $m_\vartheta : S_\vartheta \to \mathbf{R}$ is $\mathcal{H}^0\llcorner S_\vartheta$-measurable, and $\vartheta^\perp$ contains no atoms. Analogously to the functional introduced above we define the *reduced cost functional* $\mathscr{G}_h(\cdot; I) : \mathcal{M}(\overline{I}) \to [0, \infty)$,

$$\mathscr{G}_h(\vartheta; I) = \sum_{x \in S_\vartheta \cap I} h(|m_\vartheta(x)|) + h'(0)|\vartheta^\perp|(I)$$

for $\alpha_0 < \infty$ and otherwise

$$\mathscr{G}_h(\vartheta; I) = \begin{cases} \sum_{x \in S_\vartheta \cap I} h(|m_\vartheta(x)|) & \text{if } \vartheta^\perp = 0, \\ \infty & \text{else.} \end{cases}$$

Its extension to $\mathcal{M}(\overline{I}) \times L^1(I)^N$ is $G(\cdot; I) : \mathcal{M}(\overline{I}) \times L^1(I)^N \to [0, \infty)$,

$$\mathscr{G}(\vartheta, \varphi_1, \ldots, \varphi_N; I) = \begin{cases} \mathscr{G}_h(\vartheta; I) & \text{if } \varphi_1 = \ldots = \varphi_N = 1 \text{ almost everywhere,} \\ \infty & \text{else.} \end{cases}$$

For any $(\vartheta, \varphi_1, \ldots, \varphi_N) \in L^2(I) \times W^{1,2}(I)^N$ we define the *reduced phase field functional* on I as

$$\mathscr{G}_\varepsilon(\vartheta, \varphi_1, \ldots, \varphi_N; I) = \int_I \omega_\varepsilon\left(\alpha_0, \tfrac{\gamma_\varepsilon(x)}{\varepsilon}, |\vartheta(x)|\right) \,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon[\varphi_i; I]\,,$$

$$\mathscr{T}_\varepsilon(\varphi; I) = \frac{1}{2} \int_I \left[\varepsilon|\varphi'(x)|^2 + \frac{(\varphi(x) - 1)^2}{\varepsilon}\right] \,\mathrm{d}x\,,$$

with ω_ε and γ_ε from equation (4.1). Eventually we extend the above functional on $\mathcal{M}(I) \times L^1(\mathcal{I})^N$ by setting $\mathscr{G}_\varepsilon(\vartheta, \varphi_1, \ldots, \varphi_N; I) = +\infty$ if $(\vartheta, \varphi_1, \ldots, \varphi_N) \notin L^2(I) \times W^{1,2}(I)^N$.

For notational convenience, we next introduce the sets K_i^ε on which the pointwise minimum inside $\mathscr{G}_\varepsilon$ (or also $\mathscr{F}_\varepsilon$) is realized by the i^{th} element.

Definition 2 (Cost domains)**.** For given $(\vartheta, \varphi_1, \ldots, \varphi_N) \in L^2(I) \times W^{1,2}(I)^N$ we set

$$K_0^\varepsilon = K_0^\varepsilon(\vartheta, \varphi_1, \ldots, \varphi_N; I) = \left\{ x \in I \,\middle|\, |\vartheta(x)| > \frac{\alpha_0 \varepsilon}{\gamma_\varepsilon} \right\}\,,$$

$$K_i^\varepsilon = K_i^\varepsilon(\vartheta, \varphi_1, \ldots, \varphi_N; I) = \left\{ x \in I \setminus \textstyle\bigcup_{j=0}^{i-1} K_j^\varepsilon \,\middle|\, \varphi_i(x)^2 + \frac{\alpha_i^2 \varepsilon^2}{\beta_i} = \gamma_\varepsilon(x) \right\}\,,\; i = 1, \ldots, N\,.$$

The sets are analogously defined for $(\sigma, \varphi_1, \ldots, \varphi_N) \in L^2(\Omega; \mathbf{R}^2) \times W^{1,2}(\Omega)^N$, where we use the same notation (which case is referred to will be clear from the context).

We now show the following lower bound on the energy, from which the $\Gamma - \liminf$ inequality for the dimension-reduced situation will automatically follow.

Proposition 4.2 (Lower bound on reduced phase field functional)**.** *Let $I = (a, b) \subset \mathbf{R}$ and $0 \leq \delta \leq \eta \leq 1$. Let $I_\eta \subset \{x \in I \,|\, \varphi_1(x), \ldots, \varphi_N(x) \geq \eta\}$, and denote the collection of connected components of $I \setminus I_\eta$ by $\mathcal{C}_\eta$. Furthermore define the subcollection of connected components $\mathcal{C}_\eta^\geq = \{C \in \mathcal{C}_\eta \,|\, \inf_C \varphi_1, \ldots, \inf_C \varphi_N \geq \delta\}$ and $C^\geq = \bigcup_{C \in \mathcal{C}_\eta^\geq} C$. Finally assume $\varphi_i(a), \varphi_i(b) \geq \eta$ for all $i = 1, \ldots, N$.*

1. If $\alpha_0 < \infty$ we have

$$\mathscr{G}_\varepsilon(\vartheta, \varphi_1, \ldots, \varphi_N; I) \geq (\eta - \delta)^2 \int_{I_\eta \cup C^{\geq}} \alpha_0 |\vartheta| \, \mathrm{d}x + (\eta - \delta)^2 \sum_{C \in \mathcal{C}_\eta \setminus \mathcal{C}_\eta^{\geq}} \max\left\{\beta_1, h\left(\int_C |\vartheta| \, \mathrm{d}x\right)\right\} - \alpha_0^2 \mathcal{H}^1(I) \frac{\varepsilon}{\delta^2} .$$

2. If $\alpha_0 = \infty$ we have

$$\mathscr{G}_\varepsilon(\vartheta, \varphi_1, \ldots, \varphi_N; I) \geq \frac{\delta^2}{2\varepsilon \mathcal{H}^1(I)} \left(\int_{I_\eta \cup C^{\geq}} |\vartheta| \, \mathrm{d}x\right)^2 + (\eta - \delta)^2 \sum_{C \in \mathcal{C}_\eta \setminus \mathcal{C}_\eta^{\geq}} \max\left\{\beta_1, h\left(\int_C |\vartheta| \, \mathrm{d}x\right)\right\} - \alpha_1^2 \mathcal{H}^1(I) \frac{\varepsilon}{\delta^2} .$$

Proof. 1. ($\alpha_0 < \infty$) We first show that without loss of generality we may assume

$$|\vartheta| \geq \frac{\alpha_0 \varepsilon}{2\gamma_\varepsilon} \frac{1}{1 - (\eta - \delta)^2} \quad \text{on } K_0^\varepsilon . \tag{4.3}$$

The motivation is that there may be regions in which a phase field φ_i is (still) small, but in which we actually have to pay $\alpha_0|\vartheta|$. Thus, in those regions we would like $\omega_\varepsilon(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\vartheta(x)|)$ to approximate $\alpha_0|\vartheta(x)|$ sufficiently well, and the above condition on ϑ ensures

$$\omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\vartheta(x)|\right) = \alpha_0|\vartheta(x)| - \frac{\alpha_0^2 \varepsilon}{2\gamma_\varepsilon(x)} + \varepsilon^p |\vartheta|^2 \geq (\eta - \delta)^2 \alpha_0 |\vartheta(x)| \quad \text{for } x \in K_0^\varepsilon . \tag{4.4}$$

We achieve (4.3) by modifying ϑ while keeping the cost as well as $\int_{I_\eta} |\vartheta| \, \mathrm{d}x$ and $\int_C |\vartheta| \, \mathrm{d}x$ for all $C \in \mathcal{C}_\eta$ the same so that the overall estimate of the proposition is not affected. The modification mimics the relaxation from Remark 2: the modified ϑ oscillates between small and very large values. Indeed, for fixed $C \in \mathcal{C}_\eta \cup \{I_\eta\}$ and $x \in C$ set

$$\hat{\vartheta}(x) = \begin{cases} \max\{\frac{\alpha_0 \varepsilon}{2\gamma_\varepsilon(x)} \frac{1}{1-(\eta-\delta)^2}, \vartheta(x)\} & \text{if } x \in K_0^\varepsilon \cap (-\infty, t_C] , \\ \frac{\alpha_0 \varepsilon}{\gamma_\varepsilon(x)} & \text{if } x \in K_0^\varepsilon \cap (t_C, \infty) , \\ \vartheta(x) & \text{else,} \end{cases}$$

where t_C is chosen such that $\int_C |\hat{\vartheta}| \, \mathrm{d}x = \int_C |\vartheta| \, \mathrm{d}x$ (this is possible, since for $t_C = \infty$ we have $|\hat{\vartheta}| \geq |\vartheta|$ and for $t_C = -\infty$ we have $|\hat{\vartheta}| \leq |\vartheta|$ everywhere on C). The cost did not change by this modification since

$$\begin{aligned}
&\mathscr{G}_\varepsilon(\hat\vartheta,\varphi_1,\dots,\varphi_N;I)-\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I)\\
&=\int_{K_0^\varepsilon}\omega_\varepsilon\left(\alpha_0,\frac{\gamma_\varepsilon(x)}{\varepsilon},|\hat\vartheta(x)|\right)-\omega_\varepsilon\left(\alpha_0,\frac{\gamma_\varepsilon(x)}{\varepsilon},|\vartheta(x)|\right)\,\mathrm{d}x\\
&=\int_{K_0^\varepsilon}\left(\alpha_0|\hat\vartheta(x)|-\frac{\alpha_0^2\varepsilon}{2\gamma_\varepsilon(x)}\right)-\left(\alpha_0|\vartheta(x)|-\frac{\alpha_0^2\varepsilon}{2\gamma_\varepsilon(x)}\right)\,\mathrm{d}x\\
&=\alpha_0\int_{K_0^\varepsilon}|\hat\vartheta(x)|\,\mathrm{d}x-\alpha_0\int_{K_0^\varepsilon}|\vartheta(x)|\,\mathrm{d}x\\
&=0\,.
\end{aligned}$$

Note that the modification $\hat\vartheta$ has a different set $K_0^\varepsilon(\hat\vartheta,\varphi_1,\dots,\varphi_N;I)$ than the original ϑ. Indeed, by definition of $\hat\vartheta$ we have $K_0^\varepsilon(\hat\vartheta,\varphi_1,\dots,\varphi_N;I)\subset K_0^\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I)$ and $|\vartheta(x)|\geq\frac{\alpha_0\varepsilon}{2\gamma_\varepsilon(x)}\frac{1}{1-(\eta-\delta)^2}$ on $K_0^\varepsilon(\hat\vartheta,\varphi_1,\dots,\varphi_N;I)$, as desired.

Let us now abbreviate $m_0=\int_{(I_\eta\cup C^{\geq})\setminus K_0^\varepsilon}|\vartheta|\,\mathrm{d}x$. Using the definition of ω_ε as well as $\gamma_\varepsilon\geq\delta^2$ on $I_\eta\cup C^{\geq}$ we compute

$$\begin{aligned}
\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I_\eta\cup C^{\geq})&\geq\int_{(I_\eta\cup C^{\geq})\cap K_0^\varepsilon}\omega_\varepsilon\left(\alpha_0,\frac{\gamma_\varepsilon}{\varepsilon},|\vartheta|\right)\,\mathrm{d}x+\frac{\delta^2}{2\varepsilon}\int_{(I_\eta\cup C^{\geq})\setminus K_0^\varepsilon}|\vartheta|^2\,\mathrm{d}x\\
&\geq(\eta-\delta)^2\int_{I_\eta\cup C^{\geq}}\alpha_0|\vartheta|\,\mathrm{d}x-\alpha_0m_0+\frac{\delta^2}{2\varepsilon\mathcal{H}^1((I_\eta\cup C^{\geq})\setminus K_0^\varepsilon)}m_0^2\,,
\end{aligned}$$

where we have employed (4.4) and Jensen's inequality. Upon minimizing in m_0, which yields the optimal value $\alpha_0\varepsilon\mathcal{H}^1((I_\eta\cup C^{\geq})\setminus K_0^\varepsilon)/\delta^2$ for m_0, we thus obtain

$$\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I_\eta\cup C^{\geq})\geq(\eta-\delta)^2\int_{I_\eta\cup C^{\geq}}\alpha_0|\vartheta|\,\mathrm{d}x-\frac{\alpha_0^2\varepsilon\mathcal{H}^1(I_\eta\cup C^{\geq})}{2\delta^2}\,.$$

Next consider for each $C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^{\geq}$ and $i=1,\dots,N$ the subsets

$$C_i^{\geq}=C\cap K_i^\varepsilon\cap\{\varphi_i\geq\delta\}\,,\qquad\qquad C_i^{<}=C\cap K_i^\varepsilon\cap\{\varphi_i<\delta\}\,,$$

and abbreviate $m_A=\int_A|\vartheta|\,\mathrm{d}x$ for any $A\subset I$. Using Young's inequality, for $i,j=1,\dots,N$ we have

$$\begin{aligned}
\int_{C_i^{\geq}}\frac{\varphi_i^2+\alpha_i^2\varepsilon^2/\beta_i}{2\varepsilon}|\vartheta|^2\,\mathrm{d}x&\geq\int_{C_i^{\geq}}\frac{\delta^2+\alpha_i^2\varepsilon^2/\beta_i}{2\varepsilon}|\vartheta|^2\,\mathrm{d}x\\
&\geq\int_{C_i^{\geq}}\alpha_i|\vartheta|-\frac{\alpha_i^2\varepsilon/2}{\delta^2+\alpha_i^2\varepsilon^2/\beta_i}\,\mathrm{d}x\geq\alpha_jm_{C_i^{\geq}}-\frac{\alpha_i^2\varepsilon}{2\delta^2}\mathcal{H}^1(C_i^{\geq})\,.
\end{aligned}$$

Similarly, using Jensen's inequality we have

$$\begin{aligned}
\int_{C_i^<}\frac{\varphi_i^2+\alpha_i^2\varepsilon^2/\beta_i}{2\varepsilon}|\vartheta|^2+\frac{\beta_i}{2\varepsilon}(\varphi_i-1)^2\,\mathrm{d}x&\geq\int_{C_i^<}\frac{\alpha_i^2\varepsilon}{2\beta_i}|\vartheta|^2+\frac{\beta_i}{2\varepsilon}(\delta-1)^2\,\mathrm{d}x\\
\geq\frac{\alpha_i^2\varepsilon}{2\beta_i}\frac{1}{\mathcal{H}^1(C_i^<)}\left(\int_{C_i^<}|\vartheta|\,\mathrm{d}x\right)^2&+\frac{\beta_i}{2\varepsilon}(1-\delta)^2\mathcal{H}^1(C_i^<)\geq\alpha_i(1-\delta)m_{C_i^<}\,,
\end{aligned}$$

where in the last step we optimized for $\mathcal{H}^1(C_i^<)$. Finally, if $\inf_C \varphi_i \le \delta$ we have (using Young's inequality)

$$\int_{C\setminus C_i^<} \frac{\beta_i}{2}\left(\varepsilon|\varphi_i'|^2 + \frac{(\varphi_i-1)^2}{\varepsilon}\right)\,\mathrm{d}x \ge \beta_i \int_{C\setminus C_i^<} |\varphi_i'|\,|1-\varphi_i|\,\mathrm{d}x$$
$$\ge \beta_i\left(\int_c^d |\varphi_i'|\,|1-\varphi_i|\,\mathrm{d}x + \int_e^f |\varphi_i'|\,|1-\varphi_i|\,\mathrm{d}x\right) \ge \beta_i\left(\int_\eta^\delta \varphi_i - 1\,\mathrm{d}\varphi_i + \int_\delta^\eta 1-\varphi_i\,\mathrm{d}\varphi_i\right)$$
$$= \beta_i((1-\delta)^2 - (1-\eta)^2) \ge \beta_i(\eta-\delta)^2\,,$$

where (c,d) and (e,f) denote the first and the last connected component of $C \setminus C_i^<$.

Next, for $C \in \mathcal{C}_\eta \setminus \mathcal{C}_\eta^\ge$ define $j(C) = \max\{j \in \{1,\dots,N\} \mid \inf_C \varphi_j < \delta\}$. Summarizing the previous estimates we obtain

$$\begin{aligned}
&\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N; I\setminus(I_\eta\cup C^\ge))\\
&\ge \sum_{C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^\ge}\Bigg(\int_{C\cap K_0^\varepsilon} \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon}{\varepsilon}, |\vartheta|\right)\,\mathrm{d}x + \sum_{i=1}^N\Bigg(\int_{C_i^\ge} \frac{\varphi_i^2+\alpha_i^2\varepsilon^2/\beta_i}{2\varepsilon}|\vartheta|^2\,\mathrm{d}x\\
&\quad + \int_{C_i^<} \frac{\varphi_i^2+\alpha_i^2\varepsilon^2/\beta_i}{2\varepsilon}|\vartheta|^2 + \frac{\beta_i}{2\varepsilon}(\varphi_i-1)^2\,\mathrm{d}x + \int_{C\setminus C_i^<}\frac{\beta_i}{2}\left(\varepsilon|\varphi_i'|^2+\frac{(\varphi_i-1)^2}{\varepsilon}\right)\,\mathrm{d}x\Bigg)\Bigg)\\
&\ge \sum_{C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^\ge}\Bigg((\eta-\delta)^2\alpha_0 m_{C\cap K_0^\varepsilon} + \sum_{i=1}^N\Bigg(\alpha_{j(C)}m_{C_i^\ge} - \frac{\alpha_{j(C)}^2\varepsilon}{2\delta^2}\mathcal{H}^1(C_i^\ge)\\
&\qquad\qquad + \alpha_i(1-\delta)m_{C_i^<}\Bigg) + \sum_{\substack{i=1\\ \inf_C\varphi_i\le\delta}}^N \beta_i(\eta-\delta)^2\Bigg)\\
&\ge \sum_{C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^\ge}\left((\eta-\delta)^2(\alpha_{j(C)}m_C + \beta_{j(C)}) - \frac{\alpha_{j(C)}^2\varepsilon}{2\delta^2}\mathcal{H}^1(C)\right)\\
&\ge (\eta-\delta)^2\sum_{C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^\ge}\max\left\{\beta_1, h\left(\int_C|\vartheta|\,\mathrm{d}x\right)\right\} - \frac{\alpha_1^2\varepsilon}{2\delta^2}\mathcal{H}^1(I)\,.
\end{aligned}$$

Finally, we obtain the desired estimate,

$$\begin{aligned}
\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I) &= \mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I_\eta\cup C^\ge) + \mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I\setminus(I_\eta\cup C^\ge))\\
&\ge (\eta-\delta)^2\int_{I_\eta\cup C^\ge}\alpha_0|\vartheta|\,\mathrm{d}x - \frac{\alpha_0^2\varepsilon\mathcal{H}^1(I)}{2\delta^2}\\
&\quad + (\eta-\delta)^2\sum_{C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^\ge}\max\left\{\beta_1, h\left(\int_C|\vartheta|\,\mathrm{d}x\right)\right\} - \frac{\alpha_1^2\varepsilon}{2\delta^2}\mathcal{H}^1(I)\,.
\end{aligned}$$

2. ($\alpha_0=\infty$) In this case the set K_0^ε is empty, and the cost functional reduces to

$$\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I) = \int_I \frac{\gamma_\varepsilon(x)|\vartheta(x)|^2}{2\varepsilon} + \sum_{i=1}^N\frac{\beta_i}{2}\left[\varepsilon|\varphi_i'(x)|^2 + \frac{(\varphi_i(x)-1)^2}{\varepsilon}\right]\,\mathrm{d}x.$$

With Jensen's inequality we thus compute

$$\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I_\eta\cup C^{\geq})\geq\int_{I_\eta\cup C^{\geq}}\frac{\gamma_\varepsilon}{2\varepsilon}|\vartheta|^2\ \mathrm{d}x\geq\frac{\delta^2}{2\varepsilon}\int_{I_\eta\cup C^{\geq}}|\vartheta|^2\ \mathrm{d}x$$
$$\geq\frac{\delta^2}{2\varepsilon}\frac{\left(\int_{I_\eta\cup C^{\geq}}|\vartheta|\ \mathrm{d}x\right)^2}{\mathcal{H}^1(I_\eta\cup C^{\geq})}\geq\frac{\delta^2}{2\varepsilon}\frac{\left(\int_{I_\eta\cup C^{\geq}}|\vartheta|\ \mathrm{d}x\right)^2}{\mathcal{H}^1(I)}\,.$$

Furthermore, the same calculation as in the case $\alpha_0<\infty$ yields

$$\mathscr{G}_\varepsilon(\tilde\vartheta,\varphi_1,\dots,\varphi_N;I\setminus(I_\eta\cup C^{\geq}))\geq$$
$$\geq(\eta-\delta)^2\sum_{C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^{\geq}}\max\left\{\beta_1,h\left(\int_C|\vartheta|\ \mathrm{d}x\right)\right\}-\frac{\alpha_1^2\varepsilon}{2\delta^2}\mathcal{H}^1(I)$$

so that we obtain the desired estimate

$$\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I)=\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I_\eta\cup C^{\geq})+\mathscr{G}_\varepsilon(\vartheta,\varphi_1,\dots,\varphi_N;I\setminus(I_\eta\cup C^{\geq}))$$
$$\geq\frac{\delta^2}{2\varepsilon}\frac{\left(\int_{I_\eta\cup C^{\geq}}|\vartheta|\ \mathrm{d}x\right)^2}{\mathcal{H}^1(I)}+(\eta-\delta)^2\sum_{C\in\mathcal{C}_\eta\setminus\mathcal{C}_\eta^{\geq}}\max\left\{\beta_1,h\left(\int_C|\vartheta|\ \mathrm{d}x\right)\right\}-\frac{\alpha_1^2\varepsilon}{\delta^2}\mathcal{H}^1(I)\,.$$

□

Corollary 4.1 ($\Gamma-\liminf$ inequality for reduced functionals)**.** *Let $J\subset\mathbf{R}$ be open and bounded, $\vartheta\in\mathcal{M}(\overline{J})$, and $\varphi_1,\dots,\varphi_N\in L^1(\Omega)$. Then*

$$\Gamma-\liminf_{\varepsilon\to0}\mathscr{G}_\varepsilon/\vartheta,\varphi_1,\dots,\varphi_N;J)\geq\mathscr{G}(\vartheta,\varphi_1,\dots,\varphi_N;J)$$

with respect to weak-$*$ convergence in $\mathcal{M}(\overline{J})$ and strong convergence in $L^1(J)^N$.

Proof. Let $(\vartheta^\varepsilon,\varphi_1^\varepsilon,\dots,\varphi_N^\varepsilon)$ be an arbitrary sequence converging to $(\vartheta,\varphi_1,\dots,\varphi_N)$ in the considered topology as $\varepsilon\to0$, and assume without loss of generality that the limit inferior of $\mathscr{G}_\varepsilon[\vartheta^\varepsilon,\varphi_1^\varepsilon,\dots,\varphi_N^\varepsilon;J]$ is actually a limit and is finite (else there is nothing to show).

It suffices to show the lim inf-inequality for each connected component $\tilde I=(a,b)$ of J separately. Due to $\mathscr{G}_\varepsilon[\vartheta^\varepsilon,\varphi_1^\varepsilon,\dots,\varphi_N^\varepsilon;\tilde I]\geq\frac{\beta_i}{2\varepsilon}\|\varphi_i^\varepsilon-1\|_{L^2}^2$ for $i=1,\dots,N$ we must have $\varphi_i^\varepsilon\to1$ in $L^2(\tilde I)$ and thus also in $L^1(\tilde I)$ so that $\varphi_i=1$. Even more, after passing to another subsequence, by Egorov's Theorem all φ_i^ε converge uniformly to 1 outside a set of arbitrarily small measure. In particular, for any $\xi>0$ we can find an open interval $(a+\xi,b-\xi)\subset I\subset\tilde I$ such that $\varphi_i^\varepsilon\to1$ uniformly on ∂I, and for any $\eta<1$ there is an open set $I_\eta\subset I$ with $\mathcal{H}^1(I\setminus I_\eta)\leq1-\eta$ such that $\varphi_i^\varepsilon\geq\eta$ on $I_\eta\cup\partial I$ for all $i=1,\dots,N$ and all ε small enough.

We now choose $\delta=\varepsilon^{1/3}$ and $\eta=1-\varepsilon$ and denote by $\mathcal{C}_\eta(\varepsilon)$ and $\mathcal{C}_\eta^{\geq}(\varepsilon)$ the collections of connected components of $I\setminus I_\eta$ from Theorem 4.2 (which now depend on ε). Further we abbreviate $\mathcal{C}_\eta^{<}(\varepsilon)=\mathcal{C}_\eta(\varepsilon)\setminus\mathcal{C}_\eta^{\geq}(\varepsilon)$. The bound of Theorem 4.2 implies that the number of elements in $\mathcal{C}_\eta^{<}(\varepsilon)$ is bounded uniformly in ε and η. Passing to another subsequence we may assume $\mathcal{C}^<_\eta(\varepsilon)$ to contain K sets $C_1(\varepsilon), \dots, C_K(\varepsilon)$ whose midpoints converge to $x_1, \dots, x_K \in \overline{I}$, respectively. Thus for an arbitrary $\zeta > 0$ we have that for all ε small enough each $C \in \mathcal{C}^<_\eta(\varepsilon)$ lies inside the closed ζ-neighbourhood $B_\zeta(\{x_1, \dots, x_K\})$ of $\{x_1, \dots, x_K\}$.

Now for $\alpha_0 < \infty$ we obtain from Theorem 4.2

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(\vartheta^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon; I) \\
&\geq \liminf_{\varepsilon\to 0} (\eta-\delta)^2 \int_{I_\eta \cup C^\geq(\varepsilon)} \alpha_0 |\vartheta^\varepsilon| \,\mathrm{d}x \,+\, (\eta-\delta)^2 \sum_{i=1}^K \max\left\{\beta_1, h\left(\int_{C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x\right)\right\} \\
&\geq \liminf_{\varepsilon\to 0} (\eta-\delta)^2 \int_{I\setminus B_\zeta(\{x_1,\dots,x_K\})} \alpha_0 |\vartheta^\varepsilon| \,\mathrm{d}x \,+\, (\eta-\delta)^2 \sum_{i=1}^K h\left(\int_{B_\zeta(\{x_i\})} |\vartheta^\varepsilon| \,\mathrm{d}x\right) \\
&\geq \alpha_0 |\vartheta|(I \setminus B_\zeta(\{x_1, \dots, x_K\})) + \sum_{i=1}^K h\left(|\vartheta|(B_\zeta(\{x_i\}))\right) \,,
\end{aligned}$$

where in the second inequality we used

$$\begin{aligned}
\int_{A\cup B} \alpha_0 |\vartheta| \,\mathrm{d}x + h\left(\int_C |\vartheta| \,\mathrm{d}x\right) &\geq \int_A \alpha_0 |\vartheta| \,\mathrm{d}x + h\left(\int_B |\vartheta| \,\mathrm{d}x\right) + h\left(\int_C |\vartheta| \,\mathrm{d}x\right) \\
&\geq \int_A \alpha_0 |\vartheta| \,\mathrm{d}x + h\left(\int_{B\cup C} |\vartheta| \,\mathrm{d}x\right)
\end{aligned}$$

for all measurable $A, B, C \subset I$ (due to the subadditivity of h) and in the third inequality we used $h(m) \leq \alpha_0 m$ as well as the lower semi-continuity of the mass on an open set. Letting now $\zeta \to 0$ (so that by the σ-continuity of ϑ we have $|\vartheta|(I\setminus B_\zeta(\{x_1, \dots, x_K\})) \to |\vartheta|(I \setminus \{x_1, \dots, x_K\})$) we obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(\vartheta^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon; I) &\geq \alpha_0 |\vartheta|(I \setminus \{x_1, \dots, x_K\}) + \sum_{i=1}^K h(|m_\vartheta(x_i)|) \\
&\geq \alpha_0 |\vartheta^\perp|(I) + \int_{S_\vartheta \cap I} h(|m_\vartheta|) \,\mathrm{d}\mathcal{H}^0 = \mathscr{G}(\vartheta; I)\,.
\end{aligned}$$

If on the other hand $\alpha_0 = \infty$ we obtain from Theorem 4.2

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(\vartheta^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon; I) &\geq \liminf_{\varepsilon\to 0} \frac{\delta^2}{2\varepsilon \mathcal{H}^1(I)} \left(\int_{I_\eta \cup C^\geq(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x\right)^2 \\
&\geq \liminf_{\varepsilon\to 0} \frac{\delta^2}{2\varepsilon \mathcal{H}^1(I)} \left(\int_{I\setminus B_\zeta(\{x_1,\dots,x_K\})} |\vartheta^\varepsilon| \,\mathrm{d}x\right)^2
\end{aligned}$$

which implies $|\vartheta|(I \setminus \{x_1, \dots, x_K\}) = 0$ and thus $|\vartheta^\perp|(\overline{I}) = 0$ as well as $S_\vartheta \cap \overline{I} \subset \{x_1, \dots, x_K\}$. Next note that for all $i \in \{1, \dots, K\}$ with $|m_\vartheta(x_i)| > 0$ we have $\int_{C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x > 0$ for all ε small enough. Indeed,

$$|m_\vartheta(x_i)| \leq \liminf_{\varepsilon\to 0} \int_{B_\zeta(\{x_i\})} |\vartheta^\varepsilon| \,\mathrm{d}x = \liminf_{\varepsilon\to 0} \int_{C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x + \int_{B_\zeta(\{x_i\})\setminus C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x\,,$$

where $\int_{B_\zeta(\{x_i\})\setminus C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x$ decreases to zero by Theorem 4.2. Therefore, Theorem 4.2 implies

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(\vartheta^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_N^\varepsilon; I) \\
&\geq \liminf_{\varepsilon\to 0} \frac{\delta^2}{2\varepsilon\mathcal{H}^1(I)} \left(\int_{I_\eta \cup C^{\geq}(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x\right)^2 + (\eta-\delta)^2 \sum_{\substack{i=1\\ |m_\vartheta(x_i)|>0}}^{K} h\left(\int_{C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x\right) \\
&\geq \liminf_{\varepsilon\to 0} \frac{\delta^2}{2\varepsilon\mathcal{H}^1(I)} \sum_{\substack{i=1\\ |m_\vartheta(x_i)|>0}}^{K} \left(\int_{B_\zeta(\{x_i\})\setminus C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x\right)^2 \\
&\qquad + \sum_{\substack{i=1\\ |m_\vartheta(x_i)|>0}}^{K} \left(h\left(\int_{B_\zeta(\{x_i\})} |\vartheta^\varepsilon| \,\mathrm{d}x\right) - \alpha_1 \int_{B_\zeta(\{x_i\})\setminus C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x\right) \\
&\geq \liminf_{\varepsilon\to 0} \sum_{\substack{i=1\\ |m_\vartheta(x_i)|>0}}^{K} \left(h\left(\int_{B_\zeta(\{x_i\})} |\vartheta^\varepsilon| \,\mathrm{d}x\right) - \frac{\alpha_1^2\varepsilon\mathcal{H}^1(I)}{2\delta^2}\right) \\
&\geq \sum_{\substack{i=1\\ |m_\vartheta(x_i)|>0}}^{K} h\left(|\vartheta|(B_\zeta(\{x_i\}))\right) \\
&= \int_{S_\vartheta} h(|m_\vartheta|) \,\mathrm{d}\mathcal{H}^0 = \mathscr{G}(\vartheta; I)\,,
\end{aligned}$$

where in the second inequality we used $h(m_1+m_2) \leq h(m_1) + \alpha_1 m_2$ for any $m_1 > 0$, $m_2 \geq 0$ and in the third we optimized in $\int_{B_\zeta(\{x_i\})\setminus C_i(\varepsilon)} |\vartheta^\varepsilon| \,\mathrm{d}x$.

The proof is concluded by letting $\xi \to 0$ and noting $\liminf_{\xi\to 0} \mathscr{G}(\vartheta; I) \geq \mathscr{G}(\vartheta; \tilde{I})$. $\square$

4.3.2 The $\Gamma - \liminf$ inequality

We now prove the desired lim inf-inequality, which will be obtained by slicing.

Proposition 4.3 ($\Gamma - \liminf$ of phase field functional)**.** *Let $\mu_+, \mu_- \in \mathcal{P}(\overline{\Omega})$. We have*

$$\Gamma - \liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon = \mathscr{E}$$

with respect to weak-$*$ convergence in $\mathcal{M}(\overline{\Omega}; \mathbf{R}^2)$ and strong convergence in $L^1(\Omega)^N$.

Proof. Let $(\sigma^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_N^\varepsilon)$ converge to $(\sigma, \varphi_1, \ldots, \varphi_N)$ in the considered topology. We first extend σ^ε and σ to $\mathbf{R}^2 \setminus \overline{\Omega}$ by zero and φ_i^ε and φ_i to $\mathbf{R}^2 \setminus \Omega$ by 1 for $i = 1, \ldots, N$. The phase field cost functional and the cost functional are extended to $\mathbf{R}^2$ in the obvious way (their values do not change). Without loss of generality (potentially after extracting a subsequence) we may assume $\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_N^\varepsilon)$ to exist and to be finite (else there is nothing to show). As a consequence we have $\nabla \cdot \sigma^\varepsilon = \mu_+^\varepsilon - \mu_-^\varepsilon$ as well as $\nabla \cdot \sigma = \mu_+ - \mu_-$ and $\varphi_1 \equiv \ldots \equiv \varphi_N \equiv 1$ (since the phase field cost functional is bounded below by $\sum_{i=1}^N \frac{\beta_i}{2\varepsilon} \|\varphi_i^\varepsilon - 1\|_{L^2}^2$).

Now let $A \subset \mathbf{R}^2$ open and bounded; the restriction of the phase field cost functional to a domain A will be denoted $\mathscr{F}_\varepsilon(\cdot; A)$. Choosing some $\xi \in S^1$, by Fubini's decomposition theorem we have

$$\begin{aligned}
&\mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon; A) \\
&= \int_{-\infty}^{\infty} \int_{A_{\xi,t}} \left(\omega_\varepsilon \left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma^\varepsilon(x)| \right) + \sum_{i=1}^N \frac{\beta_i}{2} \left[\varepsilon |\nabla \varphi_i|^2 + \frac{(\varphi_i - 1)^2}{\varepsilon} \right] \right)_{\xi,t} \mathrm{d}x \ \mathrm{d}t \\
&\geq \int_{-\infty}^{\infty} \int_{A_{\xi,t}} \omega_\varepsilon \left(\alpha_0, \frac{(\gamma_\varepsilon)_{\xi,t}}{\varepsilon}, |\sigma_{\xi,t}^\varepsilon| \right) + \sum_{i=1}^N \frac{\beta_i}{2} \left[\varepsilon |(\varphi_i^\varepsilon)_{\xi,t}'|^2 + \frac{((\varphi_i^\varepsilon)_{\xi,t} - 1)^2}{\varepsilon} \right] \mathrm{d}x \ \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \mathscr{G}_\varepsilon(\sigma_{\xi,t}^\varepsilon, (\varphi_1^\varepsilon)_{\xi,t}, \dots, (\varphi_N^\varepsilon)_{\xi,t}; A_{\xi,t}) \ \mathrm{d}t \,,
\end{aligned}$$

where $\mathscr{F}_\varepsilon$ is the dimension-reduced phase field energy and for simplicity we identified the domain $A_{\xi,t}$ of the sliced functions with an open subset of the real line. Fatou's lemma thus implies

$$\liminf_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon; A) \geq \int_{-\infty}^{\infty} \liminf_{\varepsilon \to 0} \mathscr{G}_\varepsilon(\sigma_{\xi,t}^\varepsilon, (\varphi_1^\varepsilon)_{\xi,t}, \dots, (\varphi_N^\varepsilon)_{\xi,t}; A_{\xi,t}) \ \mathrm{d}t \,.$$

By assumption, the left-hand side is finite so that the right-hand side integrand is finite for almost all $t \in \mathbf{R}$ as well. Pick any such t and pass to a subsequence such that lim inf turns into lim. Indeed $\sigma_{\xi,t}^\varepsilon \overset{*}{\rightharpoonup} \sigma_{\xi,t}$ for every ξ and almost all t, as $\sigma^\varepsilon \overset{*}{\rightharpoonup} \sigma$ and Theorem C.1. Thus, Theorem 4.1 on the reduced dimension problem implies

$$\liminf_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\sigma_{\xi,t}^\varepsilon, (\varphi_1^\varepsilon)_{\xi,t}, \dots, (\varphi_N^\varepsilon)_{\xi,t}; A_{\xi,t}) \geq \mathscr{F}(\sigma_{\xi,t}, (\varphi_1)_{\xi,t}, \dots, (\varphi_N)_{\xi,t}; A_{\xi,t})$$

for almost all $t \in \mathbf{R}$ so that

$$\liminf_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon; A) \geq \int_{-\infty}^{\infty} \mathscr{G}(\sigma_{\xi,t}, (\varphi_1)_{\xi,t}, \dots, (\varphi_N)_{\xi,t}; A_{\xi,t}) \ \mathrm{d}t \,.$$

For notational convenience let us now define the auxiliary function κ, defined for open subsets $A \subset \mathbf{R}^2$, as

$$\kappa(A) = \liminf_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon; A) \,.$$

Furthermore, introduce the nonnegative Borel measure

$$\lambda(A) = \alpha_0 |\sigma^\perp|(A) + \int_{S_\sigma \cap A} h(m_\sigma) \ \mathrm{d}\mathcal{H}^1$$

as well as the $|\sigma|$-measureable Borel functions

$$\psi_j : \mathbf{R}^2 \to \mathbf{R}, \qquad \psi_j = \left| \frac{\sigma}{|\sigma|} \cdot \xi^j \right|$$

for some sequence ξ^j, $j \in \mathbf{N}$, dense in S^1.

Since σ is a divergence measure vector field, we have

$$\begin{aligned}\kappa(A) &\geq \int_{-\infty}^{\infty} \mathscr{G}(\sigma_{\xi^j,t}, (\varphi_1)_{\xi^j,t}, \ldots, (\varphi_N)_{\xi^j,t}; A_{\xi^j,t})\ \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \left[\alpha_0|(\sigma_{\xi^j,t})^\perp|(A_{\xi^j,t}) + \int_{S_{\sigma_{\xi^j,t}}\cap A_{\xi^j,t}} h(|m_{\sigma_{\xi^j,t}}|)\ \mathrm{d}\mathcal{H}^0\right]\ \mathrm{d}t \\ &= \alpha_0|\sigma^\perp\cdot\xi^j|(A) + \int_{S_\sigma\cap A} h(|m_\sigma|)|\theta_\sigma\cdot\xi^j|\ \mathrm{d}\mathcal{H}^1 \geq \int_A \psi_j\ \mathrm{d}\lambda\end{aligned}$$

for all $j \in \mathbf{N}$ where we used Remark 9 in the last equality. By [Bra98, Prop. 1.16] the above inequality implies

$$\kappa(A) \geq \int_A \sup_j \psi_j\ \mathrm{d}\lambda$$

for any open $A \subset \mathbf{R}^2$. In particular, choosing A as the 1-neighbourhood of Ω we obtain

$$\begin{aligned}\liminf_{\varepsilon\to0} \mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_N^\varepsilon; \Omega) &= \kappa(A) \\ &\geq \int_A \sup_j \psi_j\ \mathrm{d}\lambda \\ &= \alpha_0|\sigma^\perp|(A) + \int_{S_\sigma\cap A} h(m_\sigma)\ \mathrm{d}\mathcal{H}^1 \\ &= \mathscr{E}(\sigma)\,,\end{aligned}$$

the desired result. □

4.3.3 Equicoercivity

Proof of Theorem 4.2. Due to $C > \mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_N^\varepsilon) \geq \frac{\beta_i}{2\varepsilon}\|\varphi_i^\varepsilon - 1\|_{L^2}^2$ for all $i = 1, \ldots, N$, we have $\varphi_i^\varepsilon \to 1$ in $L^2(\Omega)$ and thus also in $L^1(\Omega)$. Furthermore, we will show further below that $\|\sigma^\varepsilon\|_{L^1} = \|\sigma^\varepsilon\|_{\mathcal{M}}$ is uniformly bounded, which by the Banach–Alaoglu theorem implies existence of a weakly-* converging subsequence (still denoted σ^ε) with limit $\sigma \in \mathcal{M}(\overline{\Omega}; \mathbf{R}^2)$. It is now a standard property of Γ-convergence that, due to the above equicoercivity, any sequence of minimizers of $\mathscr{F}_\varepsilon$ contains a subsequence converging to a minimizer of $\mathscr{E}$.

To finish the proof we show uniform boundedness of σ^ε in $\mathcal{M}(\overline{\Omega}; \mathbf{R}^2)$. Indeed, using $\omega_\varepsilon(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|) \geq \frac{\alpha_0}{2}|\sigma(x)|$ for $x \in K_0^\varepsilon$ (remember that $K_0^\varepsilon = \emptyset$ for $\alpha_0 = \infty$) we obtain

$$\|\sigma^\varepsilon\|_{\mathcal{M}} = \sum_{i=0}^N \int_{K_i^\varepsilon} |\sigma^\varepsilon|\ \mathrm{d}x \leq \frac{2C}{\alpha_0} + \sum_{i=1}^N \int_{K_i^\varepsilon} |\sigma^\varepsilon|\ \mathrm{d}x$$

(the first term is interpreted as zero for $\alpha_0 = \infty$). Furthermore, by Hölder's inequality we have

$$\begin{aligned}\left(\int_{K_i^\varepsilon} |\sigma^\varepsilon|\ \mathrm{d}x\right)^2 &\leq \left(\int_{K_i^\varepsilon} \frac{(\varphi_i^\varepsilon)^2 + \alpha_i^2\varepsilon^2/\beta_i}{2\varepsilon}|\sigma^\varepsilon|^2\ \mathrm{d}x\right)\left(\int_{K_i^\varepsilon} \frac{2\varepsilon}{(\varphi_i^\varepsilon)^2 + \alpha_i^2\varepsilon^2/\beta_i}\ \mathrm{d}x\right) \\ &\leq C\left(\int_{K_i^\varepsilon} \frac{2\varepsilon}{(\varphi_i^\varepsilon)^2 + \alpha_i^2\varepsilon^2/\beta_i}\ \mathrm{d}x\right).\end{aligned}$$

Choosing now some arbitrary $\lambda \in (0,1)$ we can estimate

$$\begin{aligned}\int_{K_i^\varepsilon} \frac{2\varepsilon}{(\varphi_i^\varepsilon)^2 + \alpha_i^2\varepsilon^2/\beta_i} \,\mathrm{d}x &= \int_{K_i^\varepsilon \cap \{\varphi_i^\varepsilon < \lambda\}} \frac{2\varepsilon}{(\varphi_i^\varepsilon)^2 + \alpha_i^2\varepsilon^2/\beta_i} \,\mathrm{d}x \\ &\quad + \int_{K_i^\varepsilon \cap \{\varphi_i^\varepsilon \geq \lambda\}} \frac{2\varepsilon}{(\varphi_i^\varepsilon)^2 + \alpha_i^2\varepsilon^2/\beta_i} \,\mathrm{d}x \\ &\leq \frac{4}{\alpha_i^2(1-\lambda)^2} \int_{K_i^\varepsilon \cap \{\varphi_i^\varepsilon < \lambda\}} \frac{\beta_i(1-\varphi_i^\varepsilon)^2}{2\varepsilon} \,\mathrm{d}x + \frac{2\varepsilon}{\lambda^2}\mathcal{H}^2(\Omega) \\ &\leq \frac{4C}{\alpha_i^2(1-\lambda)^2} + \frac{2\varepsilon}{\lambda^2}\mathcal{H}^2(\Omega)\,.\end{aligned}$$

Summarizing, $\|\sigma^\varepsilon\|_{\mathcal{M}} \leq \frac{C}{2\alpha_0} + \sum_{i=1}^N \sqrt{\frac{4C^2}{\alpha_i^2(1-\lambda)^2} + \frac{2\varepsilon C}{\lambda^2}\mathcal{H}^2(\Omega)}$. □

4.3.4 The $\Gamma - \limsup$ inequality

Proposition 4.4 ($\Gamma - \limsup$ of phase field functional)**.** *Let $\mu_+, \mu_- \in \mathcal{P}(\overline{\Omega})$ be an admissible source and sink. We have*

$$\Gamma - \limsup_{\varepsilon \to 0} \mathscr{F}_\varepsilon = \mathscr{E}$$

with respect to weak-$*$ convergence in $\mathcal{M}(\overline{\Omega}; \mathbf{R}^2)$ and strong convergence in $L^1(\Omega)^N$.

Proof. Consider a mass flux σ between the measures μ_+ and μ_-. We will construct a recovery sequence $(\sigma^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon)$ such that $\sigma^\varepsilon \overset{*}{\rightharpoonup} \sigma$ and $\varphi_1^\varepsilon \to 1, \dots, \varphi_N^\varepsilon \to 1$ in the desired topology as $\varepsilon \to 0$ as well as $\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\sigma^\varepsilon, \varphi_1^\varepsilon, \dots, \varphi_N^\varepsilon) \leq \mathscr{E}(\sigma, 1, \dots, 1)$. Without loss of generality we may restrict our attention to fluxes for which

$$\mathscr{E}_h(\sigma) = \mathscr{E}(\sigma, 1, \dots, 1) \leq C < \infty$$

since otherwise there is nothing to prove. By [BW17, Def. 2.2 & Prop. 2.32] there exists a sequence

$$\sigma_j = \sum_{k=1}^{M_j} m_{k,j}\theta_{k,j}\mathcal{H}^1 \llcorner \Sigma_{k,j}$$

of polyhedral divergence measure vector fields in Ω such that

$$\begin{aligned} &\sigma_j \overset{*}{\rightharpoonup} \sigma, && \nabla \cdot \sigma_j = \mu_+^j - \mu_-^j, \\ &\mu_\pm^j \overset{*}{\rightharpoonup} \mu_\pm, && \mathscr{E}_h^{\mu_+^j, \mu_-^j}(\sigma_j) \to \mathscr{E}_h^{\mu_+, \mu_-}(\sigma). \end{aligned}$$

In the above formula and in the sequel we will specify the constraint measure for the energy $\mathscr{E}_h$ to improve readability. If μ_+ and μ_- are finite linear combinations of Dirac masses (which we have assumed in the case $\alpha_0 = \infty$), we may even choose $\mu_\pm^j = \mu_\pm$. We will construct the recovery sequence based on those polyhedral divergence measure vector fields. In the following we will use the notation

$$\mathscr{F}_\varepsilon(\sigma, \varphi_1, \dots, \varphi_N) = \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right) \,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon(\varphi_i)$$

for the phase field cost functional even without divergence constraints.

Step 1. Initial construction for a single polyhedral segment
In this step we approximate a single line element $m_{k,j}\theta_{k,j}\mathcal{H}^1\llcorner\Sigma_{k,j}$ of σ_j by a phase field version. To this end we fix j and $0 \le k \le M_j$ and drop these indices from now on in the notation for the sake of legibility. Without loss of generality we may assume $\Sigma = [0, L] \times \{0\}$, $m > 0$, and $\theta = e_1$ the standard Euclidean basis vector. Set

$$\bar{\imath} = \operatorname{argmin}\{\alpha_i\, m + \beta_i \,|\, i = 0, \dots, N\}$$

to identify the phase field that will be *active* on Σ ($\bar{\imath} = 0$ means that no phase field is active). We first specify (a preliminary version of) the vector field σ^ε. To this end let d_Σ denote the distance function associated with Σ and define the width

$$a^\varepsilon_{\bar{\imath}} = \frac{\alpha_{\bar{\imath}} m\varepsilon}{2\beta_{\bar{\imath}}} \quad \text{if } \bar{\imath} > 0 \quad \text{and } a^\varepsilon_{\bar{\imath}} = \alpha_0 m\varepsilon \quad \text{otherwise} \tag{4.5}$$

over which the vector field will be diffused. We now set

$$\overline{\sigma}^\varepsilon = \frac{m}{2a^\varepsilon_{\bar{\imath}}}\, \chi_{\{d_\Sigma \le a^\varepsilon_{\bar{\imath}}\}}\, e_1\,,$$

where χ_A shall denote the characteristic function of a set A. Note that this vector field encodes a total mass flux of m that is evenly spread over the $a^\varepsilon_{\bar{\imath}}$-enlargement of Σ. The corresponding active phase field will be zero in that region. Indeed, consider the auxiliary Cauchy problem

$$\phi' = \frac{1}{\varepsilon}(1-\phi), \qquad \phi(0) = 0,$$

whose solution $\phi_\varepsilon(t) = 1 - \exp\left(-\frac{t}{\varepsilon}\right)$ represents the well-known optimal Ambrosio–Tortorelli phase field profile. Then we set $\overline{\varphi}^\varepsilon_i = 0$ for all $i \ne \bar{\imath}$ and, if $\bar{\imath} \ne 0$,

$$\overline{\varphi}^\varepsilon_{\bar{\imath}}(x) = \phi_\varepsilon(\max\{0, d_\Sigma(x) - a^\varepsilon_{\bar{\imath}}\}) = \begin{cases} 0, & \text{if } d_\Sigma(x) < a^\varepsilon_{\bar{\imath}}, \\ 1 - \exp\left(\frac{a^\varepsilon_{\bar{\imath}} - d_\Sigma(x)}{\varepsilon}\right), & \text{if } d_\Sigma(x) \ge a^\varepsilon_{\bar{\imath}}\,. \end{cases}$$

Let us now evaluate the corresponding phase field cost. In the case $\bar{\imath} = 0$ (which can only occur for $\alpha_0 < \infty$) we obtain

$$\begin{aligned} \mathscr{F}_\varepsilon(\overline{\sigma}^\varepsilon(x), \overline{\varphi}^\varepsilon_1, \dots, \overline{\varphi}^\varepsilon_N) &= \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\overline{\gamma}_\varepsilon}{\varepsilon}, |\overline{\sigma}^\varepsilon|\right)\, \mathrm{d}x \le \int_\Omega \alpha_0|\overline{\sigma}^\varepsilon| + \varepsilon^p|\overline{\sigma}^\varepsilon|^2\, \mathrm{d}x \\ = \int_{\{d_\Sigma \le a^\varepsilon_0\}} \frac{1 + \frac{\varepsilon^{p-1}}{2\alpha_0^2}}{2\varepsilon}\, \mathrm{d}x &= (\alpha_0\, m\, L + \pi\, \alpha_0^2\, m^2\, \varepsilon)(1 + \tfrac{\varepsilon^{p-1}}{2\alpha_0^2}) = \alpha_0\, m\, L + C(m, L)\varepsilon^q\,, \end{aligned}$$

where we abbreviated $q = \min\{1, p-1\} > 0$ and $C(m, L) > 0$ denotes a constant depending on m and L. In the case $\bar{\imath} \ne 0$ we have $|\overline{\sigma}^\varepsilon| = \beta_{\bar{\imath}}/(\alpha_{\bar{\imath}}\varepsilon)$ as well as $\overline{\gamma}_\varepsilon = \alpha^2_{\bar{\imath}}\varepsilon^2/\beta_{\bar{\imath}}$ on the support of $|\overline{\sigma}^\varepsilon|$ so that

$$\begin{aligned} \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\overline{\gamma}_\varepsilon}{\varepsilon}, |\overline{\sigma}^\varepsilon|\right)\, \mathrm{d}x &= \left(\frac{\alpha^2_{\bar{\imath}}\varepsilon^2}{\beta_{\bar{\imath}}} \left|\frac{\beta_{\bar{\imath}}}{\alpha_{\bar{\imath}}\varepsilon}\right|^2 \frac{1}{2\varepsilon} + \varepsilon^{p-2}\left|\frac{\beta_{\bar{\imath}}}{\alpha_{\bar{\imath}}}\right|^2\right)(2\, a^\varepsilon_{\bar{\imath}}\, L + \pi\, (a^\varepsilon_{\bar{\imath}})^2) \\ &= \frac{\alpha_{\bar{\imath}}}{2}\, mL + C(m, L)\varepsilon^q\,. \end{aligned}$$

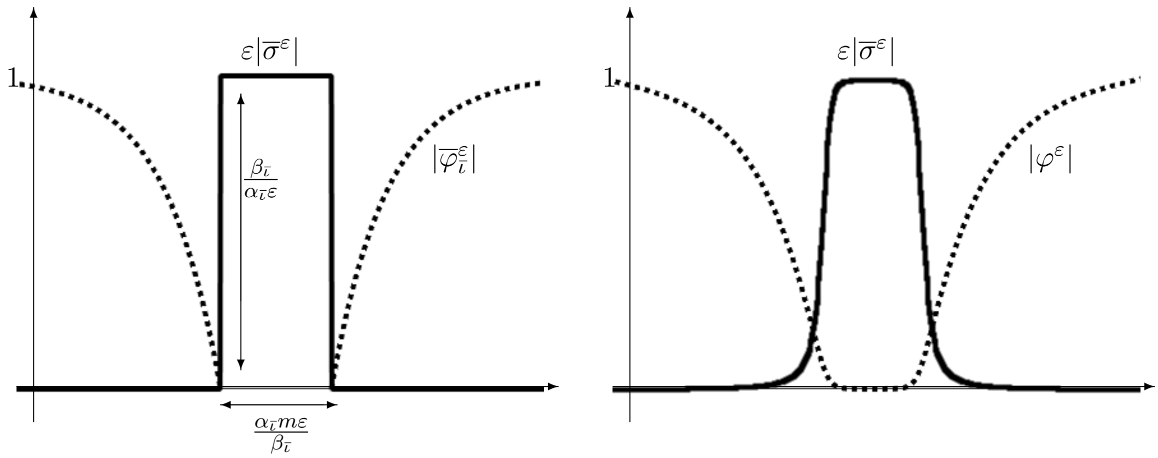


Figure 4.1: Left: Sketch of the optimal profile of $|\overline{\sigma}^\varepsilon|$ and a phase field $\overline{\varphi}_{\overline{\iota}}^\varepsilon$ for some $\overline{\iota} > 0$ with $m = 2$, $\varepsilon = 0.1$, $\alpha_{\overline{\iota}} = 1$, $\beta_{\overline{\iota}} = 1$. Right: Sketch of the numerical solution to the 1D problem with the same parameters.

Furthermore we have $\mathscr{T}_\varepsilon(\overline{\varphi}_i^\varepsilon) = 0$ for $i \neq \overline{\iota}$ and, employing the coarea formula,

$$\begin{aligned}
\beta_{\overline{\iota}}\mathscr{T}_\varepsilon(\overline{\varphi}_{\overline{\iota}}^\varepsilon) &= \frac{\beta_{\overline{\iota}}}{2}\int_\Omega \left[\varepsilon|\nabla\overline{\varphi}_{\overline{\iota}}^\varepsilon|^2 + \frac{(1-\overline{\varphi}_{\overline{\iota}}^\varepsilon)^2}{\varepsilon}\right] \delta x + \\
&= \frac{\beta_{\overline{\iota}}}{2\varepsilon}\left(2La_{\overline{\iota}}^\varepsilon + \pi(a_{\overline{\iota}}^\varepsilon)^2\right) \\
&\qquad + \frac{\beta_{\overline{\iota}}}{2}\int_{a_{\overline{\iota}}^\varepsilon}^\infty \int_{\{d_\Sigma = t\}} \left[\varepsilon|\phi_\varepsilon'(t-a_{\overline{\iota}}^\varepsilon)|^2 + \frac{(\phi_\varepsilon(t-a_{\overline{\iota}}^\varepsilon)-1)^2}{\varepsilon}\right] \, \mathrm{d}\mathcal{H}^1(x) \, \mathrm{d}t \\
&= \frac{\beta_{\overline{\iota}}}{2\varepsilon}\left(2La_{\overline{\iota}}^\varepsilon + \pi(a_{\overline{\iota}}^\varepsilon)^2\right) + \beta_{\overline{\iota}}\int_{a_{\overline{\iota}}^\varepsilon}^\infty \frac{1}{\varepsilon}\exp\left(\frac{2(a_{\overline{\iota}}^\varepsilon - t)}{\varepsilon}\right)(2L + 2\pi t) \, \mathrm{d}t \\
&= \left(\frac{\alpha_{\overline{\iota}}}{2}m + \beta_{\overline{\iota}}\right) L + C(m,L)\varepsilon \, .
\end{aligned}$$

Summarizing,

$$\mathscr{F}_\varepsilon(\overline{\sigma}^\varepsilon, \overline{\varphi}_1^\varepsilon, \ldots, \overline{\varphi}_N^\varepsilon) \leq h(m)\, L + C(m,L)\varepsilon^q \, .$$

Step 2. Adapting sources and sinks of all polyhedral segments
The vector field constructions $\overline{\sigma}_{k,j}^\varepsilon$ from the previous step for each polyhedral segment $\Sigma_{k,j}$ are not compatible with the divergence constraint associated with the measure σ^j, that is,

$$\nabla \cdot \left(\sum_{k=1}^{M_j} \overline{\sigma}_{k,j}^\varepsilon\right) \neq \rho_\varepsilon * \left(\mu_+^j - \mu_-^j\right) \, .$$

We remedy this by adapting the source and sink of each $\overline{\sigma}_{k,j}^\varepsilon$. Set

$$r(j) = \max_{k=1,\ldots,M_j} |m_{k,j}| \cdot \begin{cases} \frac{\alpha_1}{\beta_1} & \text{if } \alpha_0 = \infty, \\ \max\left\{\frac{\alpha_1}{\beta_1}, \alpha_0\right\} & \text{else,} \end{cases}$$

then all vector fields $\overline{\sigma}_{k,j}^\varepsilon$ have support in a band around $\Sigma_{k,j}$ of width no larger than $r(j)\varepsilon$. Without loss of generality we assume $r(j) \geq 1$ (else we just increase it). Again we concentrate on a single segment with fixed j and k and drop these indices in the following (we will also write r instead of $r(j)$). Denote by s^+ and s^- the starting and ending point of the segment Σ with respect to the orientation induced by θ. Consider the elliptic boundary value problems

$$\begin{cases} \Delta u^\pm(x) = \pm m\,(\rho * \delta_0)(x) & \text{on } B_r(0), \\ \nabla u^\pm(x)\cdot\nu(x) = \varepsilon\overline{\sigma}^\varepsilon(\varepsilon x + s^\pm)\cdot\nu(x) & \text{on } \partial B_r(0), \end{cases}$$

where δ_y denotes a Dirac mass centered at y, $B_r(y)$ denotes the open ball of radius r around y, and ν denotes the outer unit normal to $\partial B_r(0)$. Note that the boundary value problems and their solutions u^+ and u^- are independent of ε due to the definition of $\overline{\sigma}^\varepsilon$. Setting

$$\tilde{\sigma}^\varepsilon(x) = \begin{cases} \nabla u^\pm((x-s^\pm)/\varepsilon)/\varepsilon & \text{if } x \in B_{\varepsilon r}(s^\pm), \\ \overline{\sigma}^\varepsilon(x) & \text{else,} \end{cases}$$

(where we assume ε small enough such that $B_{\varepsilon r}(s^+)$ and $B_{\varepsilon r}(s^-)$ do not intersect) it is straightforward to check

$$\nabla\cdot(\tilde{\sigma}^\varepsilon) = m\,\rho_\varepsilon * (\delta_{s^-} - \delta_{s^-})\,.$$

Furthermore, to have at least one phase field zero on the new additional support $B_{\varepsilon r}(s^+)\cup B_{\varepsilon r}(s^-)$ of the vector field we set

$$\varphi_1^\varepsilon(x) = \min\left\{\overline{\varphi}_1^\varepsilon(x), P(|x-s^-|), P(|x-s^+|)\right\} \text{ with } P(t) = \begin{cases} 0 & \text{if } t \le r\,\varepsilon, \\ \frac{t}{r\,\varepsilon} & \text{else} \end{cases}$$

and $\varphi_2^\varepsilon = \overline{\varphi}_2^\varepsilon, \dots, \varphi_N^\varepsilon = \overline{\varphi}_N^\varepsilon$. Reintroducing now the indices k and j, we set

$$\tilde{\sigma}_j^\varepsilon = \sum_{k=1}^{M_j}\tilde{\sigma}_{k,j}^\varepsilon \quad \text{and} \quad (\varphi_i^\varepsilon)_j = \min\{(\varphi_i^\varepsilon)_{j,1},\dots,(\varphi_i^\varepsilon)_{j,M_j}\}$$

for $i = 1,\dots,N$. Obviously, we have, as desired,

$$\nabla\cdot\tilde{\sigma}_j = \nabla\cdot(\rho_\varepsilon * \sigma_j) = \rho_\varepsilon * \left(\mu_+^j - \mu_-^j\right)\,.$$

Let us now estimate the costs. Let us assume that ε is small enough so that all balls $B_{\varepsilon r(j)}(s_{k,j}^\pm)$ are disjoint as are the supports $\operatorname{supp}\tilde{\sigma}_{k,j}^\varepsilon \setminus (B_{\varepsilon r(j)}(s_{k,j}^+)\cup B_{\varepsilon r(j)}(s_{k,j}^-))$ for all k. An upper bound can then be achieved via

$$\begin{aligned} \mathscr{F}_\varepsilon(\tilde{\sigma}_j^\varepsilon, (\varphi_1^\varepsilon)_j,\dots,(\varphi_N^\varepsilon)_j) \le \sum_{k=1}^{M_j}\mathscr{F}_\varepsilon[\overline{\sigma}_{j,k}^\varepsilon, (\overline{\varphi}_1^\varepsilon)_{k,j},\dots,(\overline{\varphi}_N^\varepsilon)_{k,j}] \;+ \\ + \int_{B_{\varepsilon r(j)}(s_{j,k}^+)\cup B_{\varepsilon r(j)}(s_{j,k}^-)} \omega_\varepsilon\left(\alpha_0, \frac{(\tilde{\gamma}_\varepsilon)_j}{\varepsilon}, |\tilde{\sigma}_j^\varepsilon|\right)\,\mathrm{d}x \;+ \\ + \frac{\beta_1}{2}\int_{(B_{2\varepsilon r(j)}(s_{j,k}^+)\cup B_{2\varepsilon r(j)}(s_{j,k}^-))\cap\{(\varphi_1^\varepsilon)_j<(\overline{\varphi}_1^\varepsilon)_j\}} \varepsilon|\nabla(\varphi_1^\varepsilon)_j|^2 + \frac{1}{\varepsilon}((\varphi_1^\varepsilon)_j - 1)^2\,\mathrm{d}x\,. \end{aligned}$$

The last summand can be bounded above by

$$\beta_1 \int_{B_{2\varepsilon r(j)}(0)} \varepsilon |P'(|x|)|^2 + \frac{1}{\varepsilon}(P(|x|)-1)^2 \, \mathrm{d}x \le C\varepsilon$$

for some constant $C > 0$. For the second summand, note that $(\tilde{\gamma}_\varepsilon)_j \le \alpha_1^2 \varepsilon^2/\beta_1$ on $B_{\varepsilon r(j)}(s^{\pm}_{k,j})$ due to $(\varphi_1^\varepsilon)_j = 0$ there; furthermore,

$$\omega_\varepsilon(\alpha_0, (\tilde{\gamma}_\varepsilon)_j/\varepsilon, |\tilde{\sigma}_j^\varepsilon|) \le \left(\frac{(\tilde{\gamma}_\varepsilon)_j}{2\varepsilon} + \varepsilon^p\right) |\tilde{\sigma}_j^\varepsilon|^2 \le \left(\frac{\alpha_1^2 \varepsilon}{2\beta_1} + \varepsilon^p\right) |\tilde{\sigma}_j^\varepsilon|^2 .$$

Thus, if we set $S^\pm = \{l \in \{1,\dots,M_j\} \,|\, s^{\pm}_{j,l} = s\}$ for fixed $s = s^+_{k,j}$ or $s = s^-_{k,j}$ we have

$$\begin{aligned}
&\int_{B_{\varepsilon r(j)}(s)} \omega_\varepsilon\left(\alpha_0, \frac{(\tilde{\gamma}_\varepsilon)_j}{\varepsilon}, |\tilde{\sigma}_j^\varepsilon|\right) \, \mathrm{d}x \le C\varepsilon \int_{B_{\varepsilon r(j)}(s)} |\tilde{\sigma}_j^\varepsilon|^2 \, \mathrm{d}x \\
&= C\varepsilon \int_{B_{\varepsilon r(j)}(s)} \left| \sum_{l\in S^+} \frac{\nabla u^+_{j,l}(\frac{x-s}{\varepsilon})}{\varepsilon} + \sum_{l \in S^-} \frac{\nabla u^-_{j,l}(\frac{x-s}{\varepsilon})}{\varepsilon} \right|^2 \, \mathrm{d}x \\
&\le C\varepsilon M_j \left[\sum_{l\in S^+} \int_{B_{\varepsilon r(j)}(s)} \left| \frac{\nabla u^+_{j,l}(\frac{x-s}{\varepsilon})}{\varepsilon} \right|^2 \, \mathrm{d}x + \sum_{l\in S^-} \int_{B_{\varepsilon r(j)}(s)} \left| \frac{\nabla u^-_{j,l}(\frac{x-s}{\varepsilon})}{\varepsilon} \right|^2 \, \mathrm{d}x \right] \\
&= C\varepsilon M_j \left[\sum_{l\in S^+} \int_{B_{r(j)}(0)} \left|\nabla u^+_{j,l}(x)\right|^2 \, \mathrm{d}x + \sum_{l\in S^-} \int_{B_{r(j)}(s)} \left|\nabla u^-_{j,l}(x)\right|^2 \, \mathrm{d}x \right] \\
&= C(s, \sigma_j)\varepsilon
\end{aligned}$$

for some constant $C(\sigma_j) > 0$ depending on the polyhedral divergence measure vector field σ_j and the considered point s. In summary, we thus have

$$\begin{aligned}
&\mathscr{F}_\varepsilon(\tilde{\sigma}_j^\varepsilon, (\varphi_1^\varepsilon)_j, \dots, (\varphi_N^\varepsilon)_j) \le \\
&\le \sum_{k=1}^{M_j} \left[h(m_{j,k})\mathcal{H}^1(\Sigma_{j,k}) + C(m_{j,k}, L_{j,k})\varepsilon^q + C(s^+_{j,k}, \sigma_j)\varepsilon + C(s^-_{j,k}, \sigma_j)\varepsilon + C\varepsilon \right] \\
&\qquad \le \mathscr{E}_h^{\mu_+^j, \mu_-^j}(\sigma_j) + C(\sigma_j)\varepsilon^q
\end{aligned}$$

for some constant $C(\sigma_j)$ depending on σ_j.

Step 3. Correction of the global divergence
Recall that the vector field σ^ε to be constructed has to satisfy $\nabla \cdot \sigma^\varepsilon = \rho_\varepsilon * (\mu_+ - \mu_-)$. In the case $\alpha_0 = \infty$ the vector field $\tilde{\sigma}_j^\varepsilon$ already has that property due to $\mu^j_\pm = \mu_\pm$ (thus we set $\sigma_j^\varepsilon = \tilde{\sigma}_j^\varepsilon$). However, if $\alpha_0 < \infty$ (so that admissible sources and sinks μ_+ and μ_- do not have to be finite linear combinations of Dirac masses) we still need to adapt the vector field to achieve the correct divergence. To this end, let $\lambda^j_\pm \in \mathcal{M}(\{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) \ge \varepsilon; \mathbf{R}^2)$ be the optimal Wasserstein-1 flux between $\mu^j_\pm$ and $\mu_\pm$, that is, $\lambda_j^\pm$ minimizes $\|\lambda\|_{\mathcal{M}}$ among all vector-valued measures λ with $\nabla \cdot \lambda = \mu^j_\pm - \mu_\pm$. Setting

$$\sigma_j^\varepsilon = \tilde{\sigma}_j^\varepsilon + \rho_\varepsilon * (\lambda^j_+ - \lambda^j_-)$$

we thus obtain $\nabla\cdot\sigma_j^\varepsilon = \rho_\varepsilon * (\mu_+ - \mu_-)$, as desired. The additional cost can be estimated using the fact

$$\omega_\varepsilon\left(\alpha_0, \frac{(\tilde{\gamma}_\varepsilon)_j}{\varepsilon}, |a+b|\right) \leq \omega_\varepsilon\left(\alpha_0, \frac{(\tilde{\gamma}_\varepsilon)_j}{\varepsilon}, |a|\right) + \alpha_0|b| + \varepsilon^p(a^2+2b^2)$$

as well as $\|\rho_\varepsilon * \lambda_\pm^j\|_{L^\infty} \leq C\frac{\|\mu_+^j\|_{\mathcal{M}}+\|\mu_-^j\|_{\mathcal{M}}}{\varepsilon}$, where $\|\mu_\pm^j\|_{\mathcal{M}}$ is an upper bound for the total mass moved by $\lambda_\pm^j$ and the constant $C>0$ depends on ρ. With those ingredients we obtain

$$\begin{aligned}\mathscr{F}_\varepsilon(\sigma_j^\varepsilon, (\varphi_1^\varepsilon)_j, \dots, (\varphi_N^\varepsilon)_j) &\leq \mathscr{F}_\varepsilon[\tilde{\sigma}_j^\varepsilon, (\varphi_1^\varepsilon)_j, \dots, (\varphi_N^\varepsilon)_j] \\ &\quad + \alpha_0\|\rho_\varepsilon * (\lambda_+^j - \lambda_-^j)\|_{L^1} + \varepsilon^p(\|\tilde{\sigma}_j^\varepsilon\|_{L^2}^2 + 2\|\rho_\varepsilon * (\lambda_+^j - \lambda_-^j)\|_{L^2}^2)\,.\end{aligned}$$

Now $\|\rho_\varepsilon*(\lambda_+^j-\lambda_-^j)\|_{L^1} \leq \|\lambda_+^j-\lambda_-^j\|_{\mathcal{M}} \leq \|\lambda_+^j\|_{\mathcal{M}}+\|\lambda_-^j\|_{\mathcal{M}} = W_1(\mu_+^j,\mu_+)+W_1(\mu_-^j,\mu_-) = \kappa_j$ for a constant $\kappa_j>0$ satisfying

$$\kappa(\sigma_j) \to 0 \qquad \text{as } j\to\infty$$

since the Wasserstein-1 distance $W_1(\cdot,\cdot)$ metrizes weak-$*$ convergence. Furthermore, in the previous steps we have already estimated $\varepsilon^p\|\tilde{\sigma}_j^\varepsilon\|_{L^2}^2 \leq C(\sigma_j)\varepsilon^q$. Finally,

$$\begin{aligned}\varepsilon^p\|\rho_\varepsilon * (\lambda_+^j - \lambda_-^j)\|_{L^2}^2 &\leq \varepsilon^p\|\rho_\varepsilon * (\lambda_+^j - \lambda_-^j)\|_{L^\infty}\|\rho_\varepsilon * (\lambda_+^j - \lambda_-^j)\|_{L^1} \\ &\leq 2C\varepsilon^{p-1}(\|\mu_+^j\|_{\mathcal{M}} + \|\mu_-^j\|_{\mathcal{M}})\kappa_j\,.\end{aligned}$$

Summarizing,

$$\begin{aligned}\mathscr{F}_\varepsilon(\sigma_j^\varepsilon, (\varphi_1^\varepsilon)_j, \dots, (\varphi_N^\varepsilon)_j) &\leq \mathscr{E}_h^{\mu_+^j,\mu_-^j}(\sigma_j) + C(\sigma_j)\varepsilon^q + \alpha_0\kappa_j + 2C\varepsilon^{p-1}(\|\mu_+^j\|_{\mathcal{M}}+\|\mu_-^j\|_{\mathcal{M}})\kappa_j \\ &\leq \mathscr{E}_h^{\mu_+^j,\mu_-^j}(\sigma_j) + C\kappa_j + C(\sigma_j)\varepsilon^q\end{aligned}$$

for some constant $C>0$ and $C(\sigma_j)>0$ depending only on σ_j.

Step 4. Extraction of a diagonal sequence
We will set $\sigma^\varepsilon = \sigma_j(\varepsilon)^\varepsilon$, $\varphi_1^\varepsilon = (\varphi_1^\varepsilon)_{j(\varepsilon)}, \dots, \varphi_N^\varepsilon = (\varphi_N^\varepsilon)_{j(\varepsilon)}$ for a suitable choice $j(\varepsilon)$. Indeed, for a monotonic sequence $\varepsilon_1, \varepsilon_2, \dots$ approaching zero we set $j(\varepsilon_1)=1$ and

$$j(\varepsilon_{i+1}) = \begin{cases} j(\varepsilon_i) & \text{if } C(\sigma_{j(\varepsilon_i)+1})\varepsilon_i^q > \frac{1}{j(\varepsilon_i)+1}, \\ j(\varepsilon_i)+1 & \text{else.}\end{cases}$$

Then $j(\varepsilon_i)\to\infty$ and $C(\sigma_{j(\varepsilon_i)})\varepsilon_i^q \to 0$ as $i\to\infty$ so that

$$\begin{aligned}\mathscr{F}_{\varepsilon_i}(\sigma^{\varepsilon_i}, \varphi_1^{\varepsilon_i}, \dots, \varphi_N^{\varepsilon_i}) &= \mathscr{F}_{\varepsilon_i}(\sigma_{j(\varepsilon_i)}^{\varepsilon_i}, (\varphi_1^{\varepsilon_i})_{j(\varepsilon_i)}, \dots, (\varphi_N^{\varepsilon_i})_{j(\varepsilon_i)}) \\ &\leq \mathscr{E}_h^{\mu_+^{j(\varepsilon_i)},\mu_-^{j(\varepsilon_i)}}(\sigma_{j(\varepsilon_i)}) + C\kappa_{j(\varepsilon_i)} + C(\sigma_{j(\varepsilon_i)}){\varepsilon_i}^q \\ &\leq \mathscr{E}_h^{\mu_+^{j(\varepsilon_i)},\mu_-^{j(\varepsilon_i)}}(\sigma_{j(\varepsilon_i)}) + C\kappa_{j(\varepsilon_i)} + \frac{1}{j(\varepsilon_i)} \\ &\to \mathscr{E}_h^{\mu_+,\mu_-}(\sigma) = \mathscr{E}_h(\sigma).\end{aligned}$$

□

4.4 Numerical experiments

Here we describe the numerical discretization with finite elements and the subsequent optimization procedure used in our experiments.

4.4.1 Discretization

The proposed phase field approximation allows a simple numerical discretization with piecewise constant and piecewise linear finite elements. Let $\mathcal{T}_h$ be a triangulation of the space Ω of grid size h such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$. Denoting by $\mathbb{P}^m$ the space of polynomials of degree m, we define the finite element spaces

$$\begin{aligned} X_h^0 &= \{v_h \in L^\infty(\Omega) \mid v_{h\llcorner T} \in \mathbb{P}^0 \;\forall\, T \in \mathcal{T}_h\}\,, \\ X_h^1 &= \{v_h \in C^0(\overline{\Omega}) \mid v_{h\llcorner T} \in \mathbb{P}^m \;\forall\, T \in \mathcal{T}_h\}\,. \end{aligned}$$

Using as before the notation

$$\mathscr{F}_\varepsilon(\sigma, \varphi_1, \ldots, \varphi_N) = \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right)\,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_\varepsilon(\varphi_i)\,,$$

the discretized phase field problem now reads

$$\min_{\substack{(\sigma,\varphi_1,\ldots,\varphi_N)\in X_h^0\times(X_h^1)^N \\ \varphi_1\restriction_{\partial\Omega}=\ldots=\varphi_N\restriction_{\partial\Omega}=1}} \mathscr{F}_\varepsilon(\sigma, \varphi_1, \ldots, \varphi_N)$$

$$\text{such that } \int_\Omega -\sigma\cdot\nabla\lambda\,\mathrm{d}x = \int_\Omega \rho_\varepsilon * (\mu_+ - \mu_-) v_h\,\mathrm{d}x \;\forall \lambda \in X_h^1\,,$$

where all integrals are evaluated using midpoint quadrature and the divergence constraint is enforced in weak form. In our numerical experiments we use $\Omega = (0,1)^2$ with a regular quadrilateral grid whose squares are all divided into two triangles.

4.4.2 Optimization

Here we describe the numerical optimization method used to find a minimizer of the objective functional. We first elaborate the simplest case in which the transport cost h only features a single affine segment, that is, $\alpha_0 = \infty$ and $N = 1$. Afterwards we consider the setting with $N > 1$ and subsequently also with $\alpha_0 < \infty$, which requires more care due to the higher complexity of the energy landscape.

Single phase field and no diffuse mass flux ($N = 1$, $\alpha_0 = \infty$) In this case the energy reads

$$\mathscr{F}_\varepsilon(\sigma, \varphi) = \int_\Omega \frac{\gamma_\varepsilon(x)}{\varepsilon}\frac{|\sigma(x)|^2}{2} + \frac{\beta_1}{2}\left(\varepsilon|\nabla\varphi(x)|^2 + \frac{(\varphi(x)-1)^2}{\varepsilon}\right)\,\mathrm{d}x$$

with $\gamma_\varepsilon(x) = \varphi(x)^2 + \alpha_1^2\varepsilon^2/\beta_1$. The employed optimization method is similar to the one presented in [CFM17b] and updates the variables σ and φ alternatingly.

Let us abbreviate $f_\varepsilon = \rho_\varepsilon * (\mu_+ - \mu_-)$. For minimization with respect to σ, we use the dual variable $\lambda \in X_h^1$ to enforce the divergence constraint and write

$$\begin{aligned}\min_{\substack{\sigma\in X_h^0\\ \int_\Omega \sigma\cdot\nabla\lambda+\lambda f_\varepsilon \,\mathrm{d}x=0\,\forall\lambda\in X_h^1}} \int_\Omega \frac{\gamma_\varepsilon}{\varepsilon}\frac{|\sigma|^2}{2}\,\mathrm{d}x &= \min_{\sigma\in X_h^0}\max_{\lambda\in X_h^1}\int_\Omega \frac{\gamma_\varepsilon}{\varepsilon}\frac{|\sigma|^2}{2} - \sigma\cdot\nabla\lambda - \lambda f_\varepsilon\,\mathrm{d}x\\ &= \max_{\lambda\in X_h^1}\min_{\sigma\in X_h^0}\int_\Omega \frac{\gamma_\varepsilon}{\varepsilon}\frac{|\sigma|^2}{2} - \sigma\cdot\nabla\lambda - \lambda f_\varepsilon\,\mathrm{d}x\,,\end{aligned}$$

where the last step follows by standard convex duality. The minimization in σ can be performed explicitly, yielding $\sigma = \frac{\varepsilon\nabla\lambda}{\gamma_\varepsilon}$. Inserting this solution leads to a maximization problem in λ,

$$\max_\lambda \int_\Omega -\frac{\varepsilon|\nabla\lambda|^2}{2\gamma_\varepsilon} - \lambda f_\varepsilon\,\mathrm{d}x\,.$$

The corresponding optimality conditions,

$$\int_\Omega \frac{\varepsilon\nabla\lambda\cdot\nabla\mu}{\gamma_\varepsilon}\,\mathrm{d}x = -\int_\Omega \mu f_\varepsilon\,\mathrm{d}x \quad \forall\mu\in X_h^1\,, \tag{4.6}$$

represent a linear system of equations that can readily be solved numerically for λ so that subsequently σ can be computed (note that $\int_\Omega f_\varepsilon\,\mathrm{d}x = 0$ so that the above equation has a solution).

Fixing σ, the optimality condition for φ reads

$$\int_\Omega \frac{|\sigma|^2\varphi\psi}{\varepsilon} + \beta_1\varepsilon\nabla\varphi\cdot\nabla\psi + \frac{\beta_1}{\varepsilon}(\varphi-1)\psi\,\mathrm{d}x = 0 \quad \forall\psi\in X_h^1 \text{ with } \psi_{\llcorner\partial\Omega} = 0\,, \tag{4.7}$$

which can again be solved under the constraint $\varphi_{1\llcorner\partial\Omega} = 1$ by a linear system solver.

In addition to the alternating minimization a stepwise decrease of the phase field parameter ε is performed, starting from a large value $\varepsilon_{\text{start}}$ for which the energy landscape shows fewer local minima. Algorithm 4 summarizes the procedure.

Algorithm 4 Minimization for $N = 1$, $\alpha_0 = \infty$

```
function SPFS(ε_start, ε_end, N_iter, α_1, β_1, μ_+, μ_-, ρ_{ε_end})
    set f_ε = (μ_+ − μ_-) * ρ_{ε_end}, σ^0 = 0
    for j = 1, ..., N_iter do
        set ε_j = ε_start − (j − 1) (ε_start − ε_end)/(N_iter − 1)
        set φ^j to the solution of (4.7) for given fixed σ = σ^{j−1}
        set γ_ε^j = (φ^j)^2 + α_1^2 ε_j^2 / β_1
        set λ^j to the solution of (4.6) for given fixed γ_ε = γ_ε^j
        set σ^j = ε_j ∇λ^j / (2 γ_ε^j)
    end for
end function
return σ^{N_iter}, φ^{N_iter}, λ^{N_iter}
```

Multiple phase fields and no diffuse mass flux ($N > 1$, $\alpha_0 = \infty$) Again we aim for an alternating minimization scheme. Fixing $\varphi_1, \ldots, \varphi_N$, the optimization for σ is the same as before, since only γ_ε changes. However, the optimization in the phase fields $\varphi_1, \ldots, \varphi_N$ is strongly nonconvex (due to the minimum in γ_ε) and thus requires a rather good initialization and care in the alternating scheme.

To avoid minimization for phase field φ_i with the min-function inside γ_ε we perform a heuristic operator splitting: in each iteration of the optimization we first identify at each location which term inside $\gamma_\varepsilon = \min(\varphi_1^2 + \alpha_1^2\varepsilon^2/\beta_1, \ldots, \varphi_N^2 + \alpha_N^2\varepsilon^2/\beta_N)$ is the minimizer, which is equivalent to specifying the regions

$$R_i^\varepsilon = \{x \in \Omega \,|\, \gamma_\varepsilon(x) = \varphi_i(x)^2 + \alpha_i^2\varepsilon^2/\beta_i\}\,, \quad i = 1, \ldots, N. \tag{4.8}$$

Afterwards we then optimize the energy $\mathscr{F}_\varepsilon$ separately for each phase field φ_i assuming the regions R_i^ε fixed, that is, we minimize

$$\sum_{i=1}^N \int_\Omega \frac{\varphi_i(x)^2 + \alpha_i^2\varepsilon^2/\beta_i}{\varepsilon} \frac{|\sigma(x)|^2}{2} \chi_{R_i^\varepsilon}(x) + \frac{\beta_i}{2}\varepsilon|\nabla\varphi_i(x)|^2 + \frac{\beta_i}{2}\frac{(\varphi_i(x)-1)^2}{\varepsilon}\,\mathrm{d}x\,,$$

where $\chi_{R_i^\varepsilon}$ is the characteristic function of region R_i^ε. Similarly to the case $N = 1$ of a single phase field, this amounts to solving the linear system

$$\int_\Omega \frac{\varphi_i\psi}{\varepsilon}|\sigma|^2\chi_{R_i^\varepsilon} + \beta_i\varepsilon\nabla\varphi_i\cdot\nabla\psi + \beta_i\frac{\varphi_i\psi}{\varepsilon}\,\mathrm{d}x \quad \forall\psi \in X_h^1 \text{ with } \psi\restriction_{\partial\Omega} = 0 \tag{4.9}$$

for $\varphi_i \in X_h^1$ with $\varphi_i \restriction_{\partial\Omega} = 1$.

Since in the above simple approach the regions R_i^ε and phase fields φ_i can move, but cannot nucleate within a different region (indeed, imagine for instance $R_1^\varepsilon = \Omega$, then $\varphi_2, \ldots, \varphi_N$ will be optimized to equal 1 everywhere so that in the next iteration again $R_1^\varepsilon = \Omega$), a suitable initial guess is crucial. To provide such a guess for the initial regions R_i^ε, we proceed as follows. We first generate some flux network σ^0 consistent with the given source and sink. This can for instance be done using the previously described algorithm for just a single phase field: in our simulations we simply ignore $\varphi_2, \ldots, \varphi_N$ and pretend only the phase field φ_1 would exist (essentially this means we replace the cost function h by $m \mapsto \alpha_1 m + \beta_1$ for $m > 0$; of course, an alternative choice would be to take $m \mapsto \alpha m + \beta$ for some $\alpha, \beta > 0$ that better approximate h for a larger range of values m). We then identify the total mass flowing through each branch of σ^0. To this end we convolve $|\sigma^0|$ with the characteristic function $\chi_{B_r(0)}$ of a disc of radius r. If r is sufficiently large compared to the width of the support of σ^0, we obtain

$$\left(\chi_{B_r(0)} * |\sigma^0|\right)(x) = \int_{B_r(x)} |\sigma^0|(y)\,\mathrm{d}y \approx 2rm(x)\,,$$

where $m(x)$ denotes the mass flux through the nearby branch of σ^0. Now we can compute the regions

$$R_i^\varepsilon = \{x \in \Omega \,|\, i = \mathrm{argmin}_{j=1,\ldots,N}\{\alpha_j m(x) + \beta_j\}\} \tag{4.10}$$

and furthermore use σ^0 as initial guess of the vector field. Algorithm 5 summarizes the full procedure.

Algorithm 5 Minimization for $N > 1$, $\alpha_0 = \infty$

function MPFS($\varepsilon_{\text{start}}, \varepsilon_{\text{end}}, N_{\text{iter}}, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N, \mu_+, \mu_-, \rho_{\varepsilon_{\text{end}}}$)
  set $f_\varepsilon = (\mu_+ - \mu_-) * \rho_{\varepsilon_{\text{end}}}$
  set $(\sigma^0, \cdot, \cdot) = SPFS(\varepsilon_{\text{start}}, \varepsilon_{\text{end}}, N_{\text{iter}}, \alpha_1, \beta_1, \mu_+, \mu_-, \rho_{\varepsilon_{\text{end}}})$
  compute regions $R_1^\varepsilon, \dots, R_N^\varepsilon$ via (4.10)
  for $j = 1, \dots, N_{\text{iter}}$ **do**
    set $\varepsilon_j = \varepsilon_{\text{start}} - (j-1)\frac{\varepsilon_{\text{start}} - \varepsilon_{\text{end}}}{N_{\text{iter}} - 1}$
    set φ_i^j to the solution of (4.9) for given fixed $\sigma = \sigma^{j-1}$, $i = 1, \dots, N$
    update regions $R_1^\varepsilon, \dots, R_N^\varepsilon$ via (4.8)
    set $\gamma_\varepsilon^j = \min_{i=1,\dots,N} \left((\varphi_i^j)^2 + \alpha_i^2\varepsilon^2/\beta_i\right)$
    set λ^j to the solution of (4.6) for given fixed $\gamma_\varepsilon = \gamma_\varepsilon^j$
    set $\sigma^j = \frac{\varepsilon_j \nabla\lambda^j}{2\gamma_\varepsilon^j}$
  end for
end function
return $\sigma^{N_{\text{iter}}}, \varphi_1^{N_{\text{iter}}}, \dots, \varphi_N^{N_{\text{iter}}}$

Note that the estimate $m(x)$ of the flowing mass is only valid in close proximity of the support of σ^0 so that the regions R_i^ε are only reliable near σ^0. However, away from σ^0 all phase fields will be close to 1 anyway so that the regions R_i^ε do not play a role there. The effectiveness of the initialization can be further improved by an energy rescaling which we typically perform in our simulations: Recall that the optimal width (4.5) of the support of the vector field σ not only depends on the transported mass $m(x)$, but also on which phase field is active at x. Thus, initializing with some vector field σ that was computed based on preliminary active phase fields may erroneously give slight preference to incorrect phase fields. This can be avoided by a small parameter change which assigns a different ε to each phase field. Indeed, setting $\varepsilon_i = \beta_i\varepsilon/\alpha_i$ to be the phase field parameter associated with phase field φ_i, equation (4.5) shows that the support width of σ becomes $m\varepsilon$ and thus no longer depends on the phase field. Thus, in practice we usually work with the phase field cost

$$\tilde{\mathscr{F}}_\varepsilon(\sigma, \varphi_1, \dots, \varphi_N) = \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\tilde{\gamma}_\varepsilon(x)}{\varepsilon}, |\sigma(x)|\right)\,\mathrm{d}x + \sum_{i=1}^N \beta_i \mathscr{T}_{\varepsilon_i}(\varphi_i) \tag{4.11}$$

with $\tilde{\gamma}_\varepsilon(x) = \min_{i=1,\dots,N}\{\varphi_i(x)^2 + \alpha_i^2\varepsilon\varepsilon_i/\beta_i\} = \min_{i=1,\dots,N}\{\varphi_i(x)^2 + \alpha_i\varepsilon^2\}$. The Γ-convergence result can readily be adapted to this case.

Multiple phase fields and diffuse mass flux ($N > 1$, $\alpha_0 < \infty$) The difference to the previous case is that now there may be nonnegligible mass flux σ in regions where no phase field $\varphi_1, \dots, \varphi_N$ is active. Correspondingly, we adapt the previous alternating minimization scheme by introducing the set

$$R_0^\varepsilon = \left\{x \in \Omega \mid |\sigma(x)| > \frac{\alpha_0}{\gamma_\varepsilon(x)/\varepsilon}\right\},$$

which according to the form of ω_ε in equation (4.1) describes the region in which mass flux σ is penalized by $\alpha_0|\sigma|$. The regions in which the i^{th} phase field is active are thus modified to

$$\tilde{R}_i^\varepsilon = R_i^\varepsilon \setminus R_0^\varepsilon \,. \tag{4.12}$$

As before, we now separately minimize

$$\sum_{i=1}^N \int_\Omega \frac{\varphi_i(x)^2 + \alpha_i^2\varepsilon^2/\beta_i}{\varepsilon}\frac{|\sigma(x)|^2}{2}\chi_{\tilde{R}_i^\varepsilon}(x) + \frac{\beta_i}{2}\varepsilon|\nabla\varphi_i(x)|^2 + \frac{\beta_i}{2}\frac{(\varphi_i(x)-1)^2}{\varepsilon}\;\mathrm{d}x\,, \tag{4.13}$$

for each phase field φ_i. The optimization for σ changes a little compared to the previous case since the problem is no longer quadratic and thus no longer reduces to solving a linear system. Instead we will perform a single step of Newton's method in each iteration. The optimization problem in σ reads

$$\min_{\substack{\sigma\in X_h^0 \\ \int_\Omega \sigma\cdot\nabla\lambda+\lambda f_\varepsilon\;\mathrm{d}x=0\,\forall\lambda\in X_h^1}} \int_\Omega \omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon}{\varepsilon}, |\sigma|\right)\;\mathrm{d}x\,,$$

and its optimality conditions are

$$\begin{aligned} 0 &= \int_\Omega \xi(|\sigma|)\sigma\cdot\psi - \nabla\lambda\cdot\psi\,\delta x \quad &&\text{for all } \psi\in X_h^0\,,\\ 0 &= \int_\Omega \sigma\cdot\nabla\mu + \mu f_\varepsilon\;\mathrm{d}x &&\text{for all } \mu\in X_h^1\,, \end{aligned} \tag{4.14}$$

where

$$\xi(|\sigma|) = \frac{1}{|\sigma|}\partial_3\omega_\varepsilon\left(\alpha_0, \frac{\gamma_\varepsilon}{\varepsilon}, |\sigma|\right) = \min\left(\frac{\gamma_\varepsilon}{\varepsilon}, \frac{\alpha_0}{|\sigma|}\right) + 2\varepsilon^p\,.$$

Letting $\hat{\sigma}$ and $\hat{\lambda}$ be the coefficients of σ and λ in some basis $\{b_i^0\}_i$ of X_h^0 and $\{b_i^1\}_i$ of X_h^1, respectively, the optimality conditions can be expressed as

$$0 = R(\hat{\sigma},\hat{\lambda}) = \begin{pmatrix} M\,[\xi(|\sigma|)] & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix}\hat{\sigma}\\ \hat{\lambda}\end{pmatrix} + \begin{pmatrix}0\\ F\end{pmatrix}\,,$$

where the finite element matrices and vectors are defined as

$$M[\xi]_{ij} = \int_\Omega \xi b_i^0 b_j^0\;\mathrm{d}x\,,\quad B_{ij} = \int_\Omega b_i^0\nabla b_j^1\;\mathrm{d}x\,,\quad F_i = \int_\Omega b_i^1 f_\varepsilon\;\mathrm{d}x\,.$$

In each iteration of the alternating minimization scheme we now take one Newton step for $0 = R(\hat{\sigma},\hat{\lambda})$. As before, the algorithm requires a suitable initial guess, which is determined in the same way as for the case without diffuse component. Algorithm 6 summarizes the alternating scheme.

4.4.3 Experimental results

The algorithms were implemented in MATLAB©; parameters reported in this section refer to the rescaled cost (4.11). We first present simulation results for a single phase field and no diffuse mass flux, $N = 1$ and $\alpha_0 = \infty$. Figure 4.2 and 4.3 show solutions for a source and a number of equal sinks arranged as a regular polygon. If α_1 is small as in Figure 4.2, the solution looks similar to the Steiner tree (which would correspond

Algorithm 6 Minimization for $N > 1$, $\alpha_0 < \infty$

function MPFSD$(\varepsilon_{\text{start}}, \varepsilon_{\text{end}}, N_{\text{iter}}, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N, \mu_+, \mu_-, \rho_{\varepsilon_{\text{end}}})$

  set $f_\varepsilon = (\mu_+ - \mu_-) * \rho_{\varepsilon_{\text{end}}}$

  set $(\sigma^0, \cdot, \lambda^0) = SPFS(\varepsilon_{\text{start}}, \varepsilon_{\text{end}}, N_{\text{iter}}, \alpha_1, \beta_1, \mu_+, \mu_-, \rho_{\varepsilon_{\text{end}}})$

  compute regions $R_0^\varepsilon, \tilde{R}_1^\varepsilon, \dots, \tilde{R}_N^\varepsilon$ via (4.12)

  for $j = 1, \dots, N_{\text{iter}}$ **do**

    set $\varepsilon_j = \varepsilon_{\text{start}} - (j-1)\frac{\varepsilon_{\text{start}} - \varepsilon_{\text{end}}}{N_{\text{iter}} - 1}$

    set φ_i^j to the minimizer of (4.13) for given fixed $\sigma = \sigma^{j-1}$, $i = 1, \dots, N$

    update regions $R_0^\varepsilon, \tilde{R}_1^\varepsilon, \dots, \tilde{R}_N^\varepsilon$ via (4.8) and (4.12)

    set $\gamma_\varepsilon^j = \min_{i=1,\dots,N}\left((\varphi_i^j)^2 + \alpha_i^2\varepsilon^2/\beta_i\right)$

    set $(\hat{\sigma}^j, \hat{\lambda}^j) = (\hat{\sigma}^{j-1}, \hat{\lambda}^{j-1}) - DR(\hat{\sigma}^{j-1}, \hat{\lambda}^{j-1})^{-1} R(\hat{\sigma}^{j-1}, \hat{\lambda}^{j-1})$ for $\gamma_\varepsilon = \gamma_\varepsilon^{j-1}$

  end for

end function

return $\sigma^{N_{\text{iter}}}, \varphi_1^{N_{\text{iter}}}, \dots, \varphi_N^{N_{\text{iter}}}$

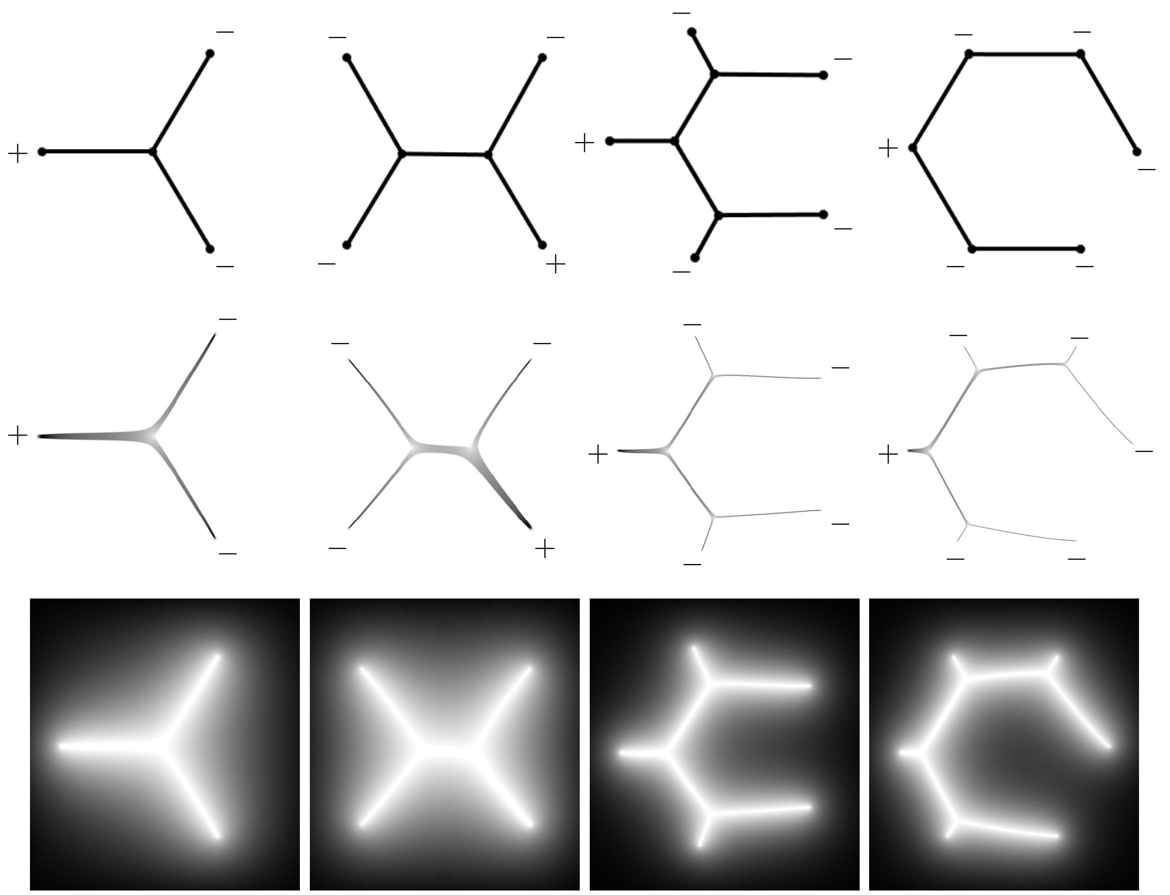


Figure 4.2: Optimal transportation networks for branched transportation from a single source to a number of identical sinks at the corners of regular polygons. The top row shows the ground truth, computed by finite-dimensional optimization of the vertex locations in a network with straight edges, the bottom rows show the computation results from the phase field model, the mass flux σ (middle, only support shown) and phase field φ (bottom). Parameters were $\alpha_1 = 0.05$, $\beta_1 = 1$, $\varepsilon = 0.005$.

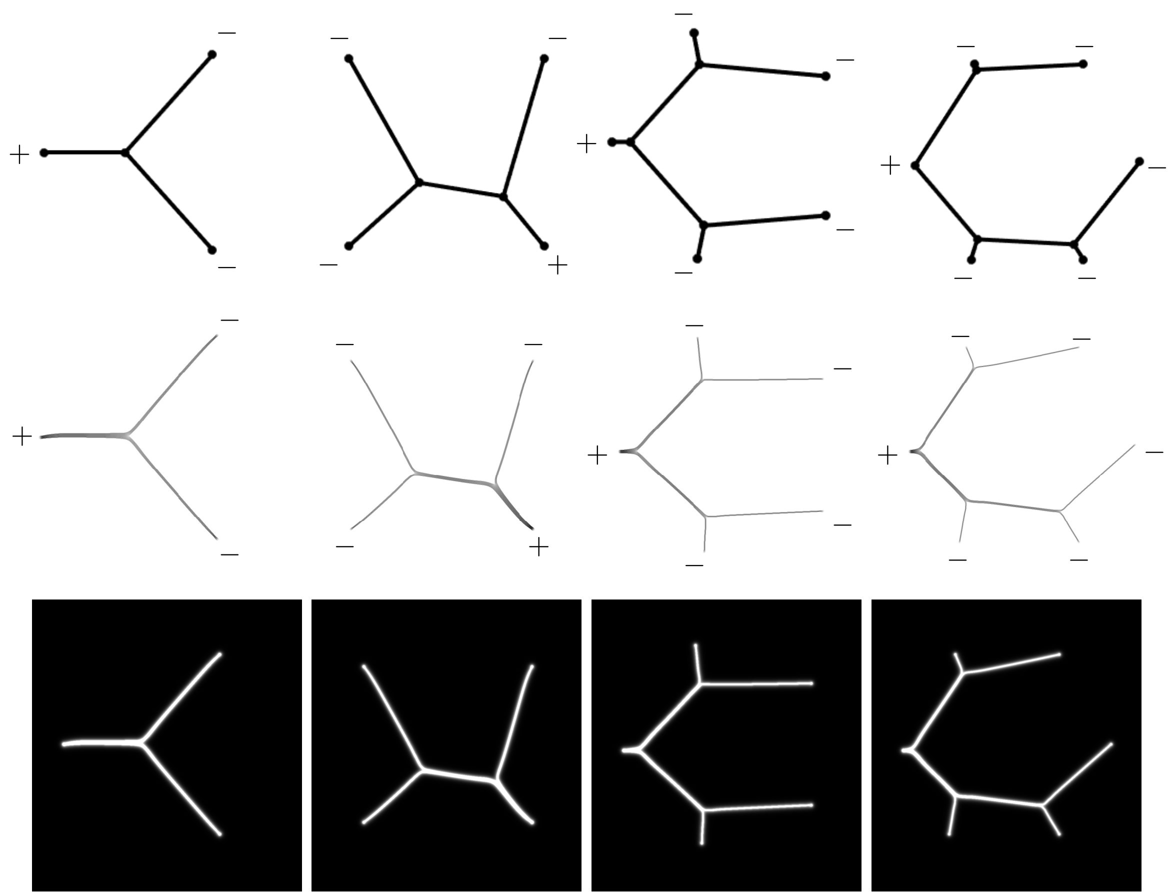


Figure 4.3: Truly optimal network (top), computed mass flux σ (middle), and phase field φ (bottom) for same branched transportation problems as in Figure 4.2, only with $\alpha_1 = 1$, $\beta_1 = 1$, $\varepsilon = 0.005$.

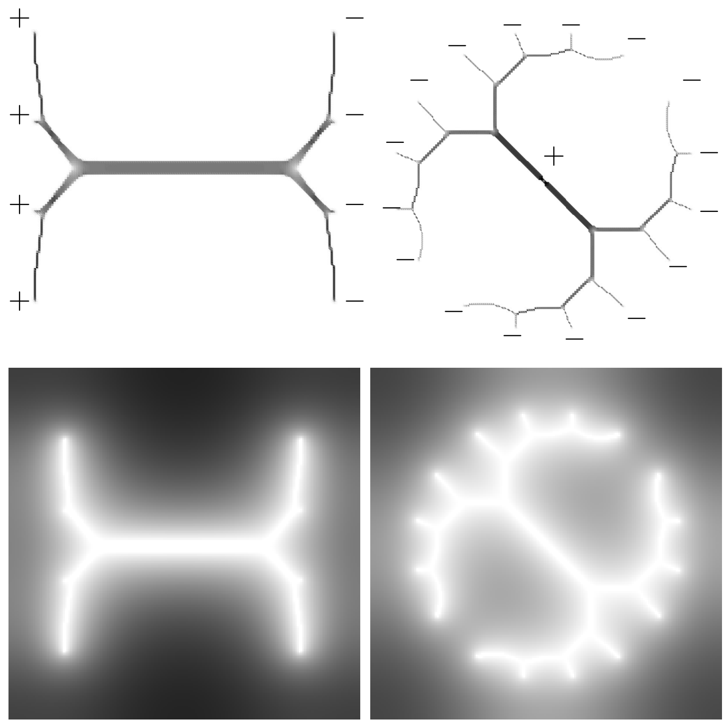


Figure 4.4: Computed mass flux σ and phase field φ for same parameters as in Figure 4.2.

to $\alpha_1 = 0$), while the solutions become much more asymmetric for larger α_1 as in Figure 4.3. More complex examples are displayed in Figure 4.4.

Figure 4.5 shows simulation results for the same source and sink configuration as in Figure 4.4, but this time with $N = 3$ different linear segments in h and corresponding phase fields. It is clear that different phase fields become active on the different network branches according to the mass flux through each branch. This can be interpreted as having streets of three different qualities: the street φ_3 allows faster (cheaper) transport, but requires more maintenance than the others, while street φ_1 requires the least maintenance and only allows expensive transport.

The case $\alpha_0 < \infty$ finally can be interpreted as the situation in which mass can also be transported off-road, that is, part of the transport may happen without a street network, thus having maintenance cost $\beta_0 = 0$, but at the price of large transport expenses α_0 per unit mass. Corresponding results for again the same source and sink configuration are shown in Figure 4.6. In contrast to the case $\alpha_0 = \infty$ it is now also possible to have sources and sinks that are not concentrated in a finite number of points. A corresponding example is shown in Figure 4.7.

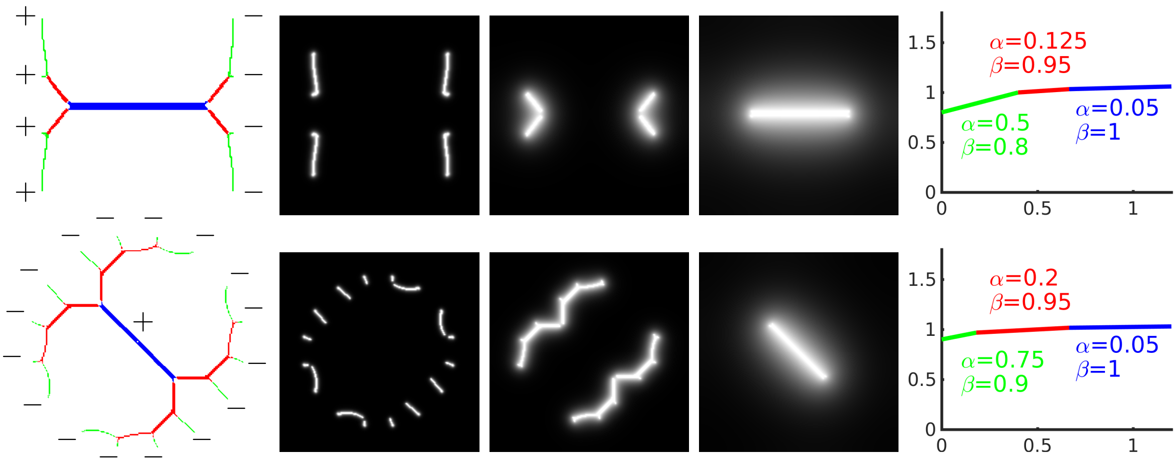


Figure 4.5: Computed mass flux σ and phase fields $\varphi_1, \varphi_2, \varphi_3$ for the same source and sink as in Figure 4.4 and for the cost function shown on the right, $\varepsilon = 0.005$. The color in σ indicates which phase field is active.

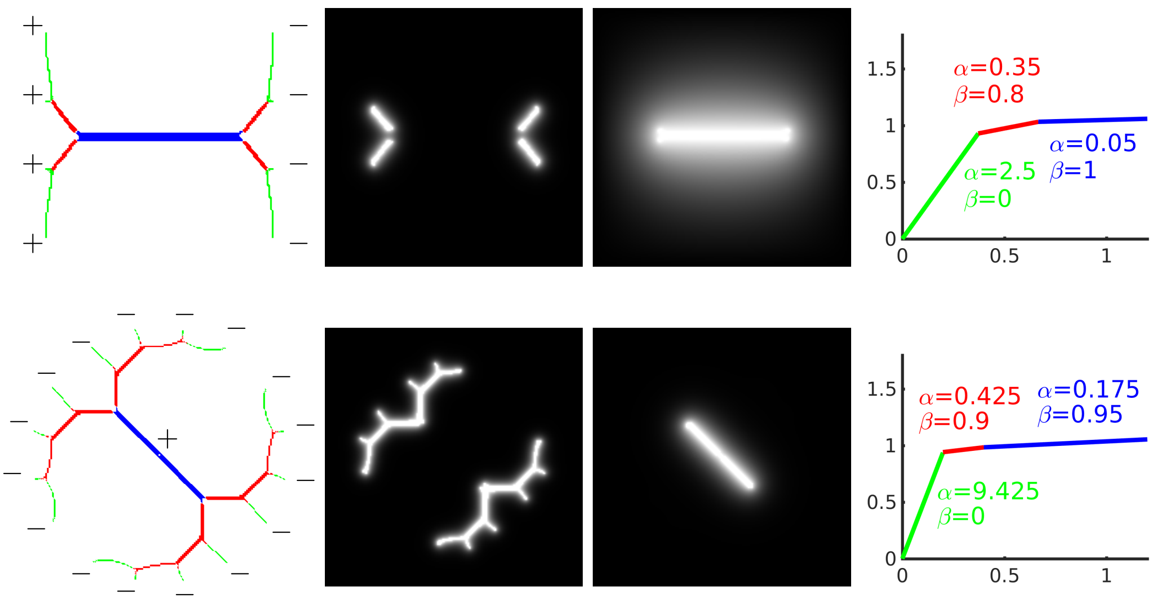


Figure 4.6: Computed mass flux σ and phase fields φ_1, φ_2 for the same source and sink as in Figure4.4 and for the cost function shown on the right, $\varepsilon = 0.005$.

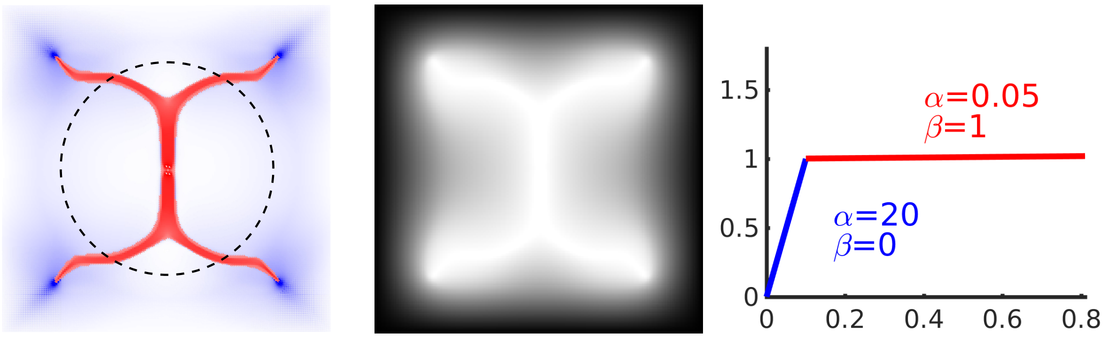


Figure 4.7: Computed mass flux σ and phase field φ_1 for a central point source and a spatially uniform sink outside a circle of radius and for the cost function shown on the right, $\varepsilon = 0.005$.

Chapter 5

Generalized cost functions

5.1 Introduction

In the previous chapters we have considered the phase field function $(1-\varphi)^2$ to model the characteristic function of the support of the vector valued measure σ. The main idea behind this approach was, for a weakly differentiable function φ, to consider the measure

$$\mu_\varphi(A) = \int_{\Omega\cap A} |\nabla\varphi|\,|1-\varphi|\,\mathrm{d}x = \int_{\Omega\cap A} |\nabla W(\varphi)|\,\mathrm{d}x.$$

where A is any Borel set and $W(t) = t^2/2$. As a matter of fact, given a sequence of functions φ_ε for which the quantity

$$\int_\Omega \left(\varepsilon|\nabla\varphi_\varepsilon|^2 + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon}\right)\,\mathrm{d}x$$

is bounded independently of ε then the sequence of Radon measures $\mu_{\varphi_\varepsilon}$ weakly-$*$ converges up to subsequence to the measure $\mathcal{H}^{n-1}\llcorner\{\varphi \neq 1\}$. As already stated in the introduction of this thesis, this fact is essential when approximating the Mumford - Shah functional as in the limit energy one aims at recovering the length of the jump set of some BV function which is contained in the set $\{\varphi \neq 1\}$. Later on some more general functionals have been introduced with different penalization of the jump set [BBB95] to model fractures. Recently [ABS99, DMOT16] other phase-field methods have dealt with the problem of efficiently approaching these energies. The main idea is that the limit φ rather then acquiring only the values $\{0,1\}$ should range over $[0,+\infty)$. In this chapter we follow this method to approximate any functional $\mathscr{E}_h$ where h is a concave cost function.

A way to approximate an energy $\mathscr{E}_h$ is to substitute for h a sequence h_k of piecewise affine functions pointwise converging to h and apply the method described in Chapter 4. The quality of such approximation depends on the number of phase fields used and for an accurate approximation, the numerical complexity of the method may become prohibitive. Here we propose a different model with a single phase function φ its main drawback is that the term in the energy that penalizes σ is linear in σ, it was quadratic or at least strictly convex in the preceding models of the thesis. Let us be more formal. Let us consider a convex open set $\Omega \subset \mathbf{R}^2$. We let $\mu_-, \mu_+ \in \mathcal{P}(\Omega)$ and denote by X_ε

the subset of $\mathcal{M}(\Omega, \mathbf{R}^2) \times L^2(\Omega)$ of those pairs (σ, φ) such that

$$\nabla \cdot \sigma = (\mu_+ - \mu_-) * \rho_\varepsilon$$

for a standard symmetric convolution kernel ρ_ε. For a couple $(\sigma, \varphi) \in \mathcal{M}(\Omega) \times L^2(\Omega)$ we set

$$\mathscr{F}_\varepsilon(\sigma, \varphi) := \begin{cases} \displaystyle\int_\Omega \left[f(\varphi)|\sigma| + \frac{1}{2}\left(\varepsilon |\nabla \varphi|^2 + \frac{\varphi^2}{\varepsilon} \right) \right] \mathrm{d}x & \text{if } (\sigma, \varphi) \in X_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

where the function $f : \mathbf{R} \to \mathbf{R}$ is defined as

$$f(t) := (-h_*)^{-1}(t^2). \tag{5.1}$$

In the above formula h_* is the concave Legendre transform of h (see Section 5.2 for its precise definition). The limit energy $\mathscr{E}$ is defined in equation (4.2) of Chapter 4. We prove:

Theorem 5.1. *Let*

$$\begin{aligned} &h : \mathbf{R} \to [0, +\infty) \textit{ be an even, continuous function} \\ &\textit{such that } h(0) = 0 \textit{ and } h \textit{ is concave on } [0, +\infty). \end{aligned} \tag{5.2}$$

Let f be defined as above then

$$\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{E} \tag{5.3}$$

as $\varepsilon \to 0$.

In Section 5.2 we recall the definition of Legendre transform for a concave function and obtain some properties of the function f which are essential to the Γ-convergence result. The convergence result is obtained again by slicing and we will take advantage of the results in Appendix C following a strategy similar to the one in Chapter 4. We introduce the reduced dimension problem and study the upper and lower bound for the Γ-convergence result in Section 5.3.

5.2 Origin of the model and preliminaries

Let us give a brief idea of the model. Let $\sigma = (m, \tau, \Sigma)$ be a rectifiable vector measure so that the energy may be written as

$$\mathscr{E}_h(\sigma) = \int_\Sigma h(m(x)) \, \mathrm{d}\mathcal{H}^1.$$

Now recall that for a concave function its Legendre transform is defined as

$$h_*(z) := \inf_m \{z\, m - h(m)\}.$$

Furthermore by [ABM14, Theorem 9.3.2] it holds $h_{**} := (h_*)_* = h$ thus we may write

$$\mathscr{E}_h(\sigma) = \int_\Sigma \inf\{z\, m(x) - h_*(z)\} \, \mathrm{d}\mathcal{H}^1.$$

Now letting z be a function we can interchange the integral and the inf signs obtaining

$$\mathscr{E}_h(\sigma) = \inf_z \int_\Sigma z(x)\, m(x) - h_*(z(x))\, \mathrm{d}\mathcal{H}^1.$$

In the above formula we may notice the presence of two measures supported on the rectifiable set Σ with $\mathcal{H}^1 \llcorner \Sigma$-density respectively $z(x)\, m(x)$ and $h_*(z(x))$. We now model our approximating functional. The main idea is to retrieve the measure $h_*(z(x))\, \mathcal{H}^1 \llcorner \Sigma$ by means of a phase field approach. Contrary to the previous approaches we now suppose that the phase field φ takes value 0, not 1, outside an ε-neighborhood of Σ and some value $\varphi(x) \in [0,1]$ if $x \in \Sigma$. Given a potential $W : \mathbf{R} \to \mathbf{R}^+$ we let

$$c_W(t) := 2 \int_0^{|t|} \sqrt{W(s)}\, \mathrm{d}\, s \tag{5.4}$$

be the cost of the transition between 0 and a value t. We set $-h_*(z(x)) = c_W(\varphi(x))$ and suppose that h_* is invertible so that, setting $f(\varphi(x)) := (-h_*)^{-1} \circ c_W(\varphi(x))$, by a change of variables we have

$$\mathscr{E}_h(\sigma) = \inf_\varphi \int_\Sigma f(\varphi(x))\, m(x) + c_W(\varphi(x))\, \mathrm{d}\mathcal{H}^1.$$

Let us observe that the first addend in the latter corresponds to $\int_\Omega f(\varphi(x))\, \mathrm{d}|\sigma|$. Furthermore by reversing the Modica-Mortola arguments used in the previous chapters when dealing with the φ components we know that, up to a small error, $\int_\Sigma c_W(\varphi(x))\, \mathrm{d}\mathcal{H}^1$ is equal to $\int_\Omega \frac{1}{2} \left[\varepsilon |\nabla \varphi|^2 + \frac{W(\varphi)}{\varepsilon}\right]\, \mathrm{d}x$. Considering as potential the function $W(x) = x^2$ and replacing σ with a mollified version of itself we are led to the proposed approximating functional, namely

$$\mathscr{F}_\varepsilon(\sigma, \varphi) = \int_\Omega f(\varphi(x))|\sigma| + \frac{1}{2}\left[\varepsilon |\nabla \varphi|^2 + \frac{\varphi^2}{\varepsilon}\right]\, \mathrm{d}x. \tag{5.5}$$

Let us specify what we will consider when talking about the Legendre transform of h. Since h is an even function first of all consider its restriction to $[0, +\infty)$. Define the quantities

$$\lim_{m \searrow 0^+} \frac{h(m)}{m} = \alpha_0 > 0, \qquad \lim_{m \nearrow +\infty} \frac{h(m)}{m} = \alpha_\infty \geq 0 \quad \text{and} \quad \lim_{m \nearrow +\infty} (\alpha_\infty\, m - h(m)) = \beta_\infty.$$

Being h concave and non decreasing we have

$$\inf_{m \in [0, +\infty)} \{m\, z - h(m)\} = \begin{cases} -\infty, & \text{for } z < \alpha_\infty, \\ 0, & \text{for } z \geq \alpha_0. \end{cases}$$

The first fact follows easily from the inequality $h(m) \leq \alpha_0\, m$. For the second observe that for any m and $t \geq m$ we have

$$h(m) = h\left(\frac{m}{t}\, t + \left(1 - \frac{m}{t}\right) 0\right) \geq \frac{m}{t} h(t) + \left(1 - \frac{m}{t}\right) h(0)$$

thus passing to the limit as $t \to +\infty$ we obtain $h(m) \geq \alpha_\infty m$. We call Legendre transform the function

$$h_*(z) := \inf_{m \in [0,\infty)} \{m\, z - h(m)\}\,.$$

In the following we will always consider the restriction of h_* on the interval in which is well defined and finite, namely $h_* : [\alpha_\infty, \alpha_0] \to [-\beta_\infty, 0]$. In the case $\alpha_\infty = \infty$ or $\beta_\infty = \infty$ the latter intervals are to be considered open. Let us give some of the

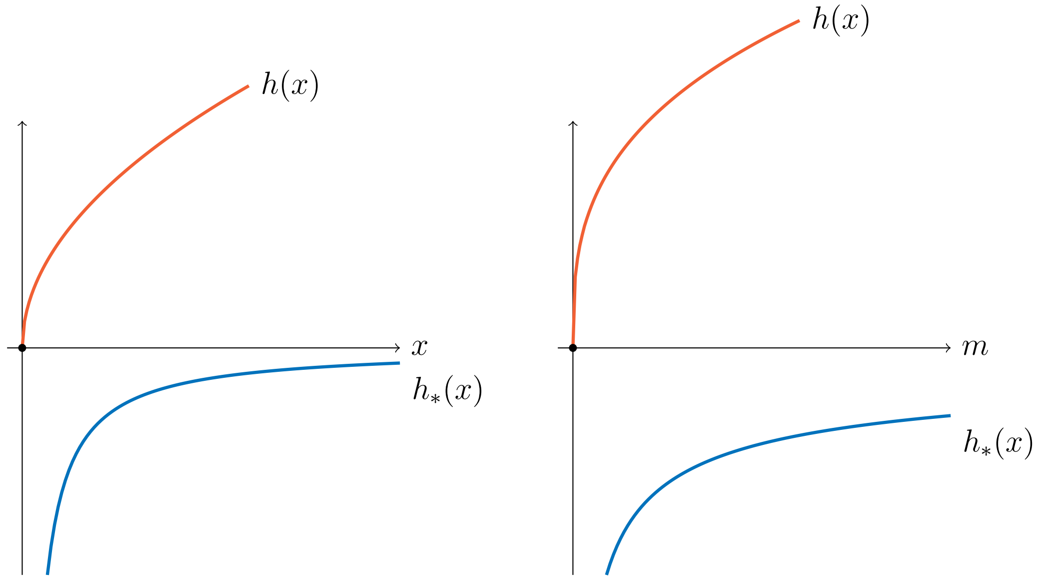


Figure 5.1: Graphs of the function h in red and the corresponding h_* in blue for the choices: $h(x) = 2\sqrt{x}$ on the left and $h(x) = 3x^{1/3}$ on the right.

properties for h.

Lemma 5.1 (Properties for h_*). *We have:*

1. *h_* is continuous,*
2. *h_* is concave,*
3. *h_* is non decreasing,*
4. *$h_*(\alpha_0) = 0$.*

Proof. The function h_* is continuous and concave since is the infimum of a family of affine functions. Let us prove that h_* is monotone non decreasing. By contradiction suppose the existence of two values $z_1 < z_2$ such that $h_*(z_1) > h_*(z_2)$. Therefore there exists an $\varepsilon > 0$ such that for any $x \in [0,\infty)$ it holds

$$z_1\, x - h_*(z_1) + \varepsilon < z_2\, x - h_*(z_2).$$

Now let x_ε be such that $z_2\, x_\varepsilon - h(x_\varepsilon) < h_*(z_2) + \varepsilon/2$ so we obtain

$$h(x_\varepsilon) + \varepsilon \leq z_1\, x_\varepsilon - h_* + \varepsilon < z_2\, x_\varepsilon - h_*(z_2) < h(x_\varepsilon) + \frac{\varepsilon}{2}.$$

The latter is a contradiction thus h_* is monotone non decreasing. Finally, by the inequality $h(m) \leq \alpha_0$ we obtain

$$h_*(\alpha_0) = \inf\{\alpha_0\, m - h(m)\} \geq 0$$

and the latter is actually a minimum as it is evident by choosing $m = 0$. $\square$

The properties stated above ensure that $-h_*$ defines a bijection between the intervals $[\alpha_\infty, \alpha_0]$ and $[0, \beta_\infty]$ and may be inverted. Consider the inverse function $(-h_*)^{-1}$ which, in the case $\beta_\infty < \infty$, we extend constant on $[0, +\infty)$. Recalling the equation (5.4) we set

$$f := (-h_*)^{-1} \circ c_W. \tag{5.6}$$

From the properties of h_* we easily derive:

Lemma 5.2 (Properties for f). *Let $W : \mathbf{R} \to \mathbf{R}_+$ be a non negative, increasing for $x \geq 0$ and even function such that $W(0) = 0$ then the function $f := (-h_*)^{-1} \circ c_W$ is:*

a. continuous on $[0, \infty)$,

b. non decreasing,

c. $f \geq 0$ and $f(0) = \alpha_0$,

Furthermore the following identity holds true

$$\inf_{z\in[0,1]} \{\ f(z)m + c_W(z)\ \} \ = \ h(m).$$

Before moving to the proof of the Γ-convergence result let us produce some examples of function f. In all these cases we will consider the potential function $W(x) = x^2$.

1. The first examples we consider is given by a function with linear growth both at the origin and at in infinity. In facts, for some values $\alpha_0 > \alpha_1 > \ldots > \alpha_N \geq 0$ and $0 \leq \beta_0 < \beta_1 < \ldots < \beta_N$ we consider the piecewise affine functions

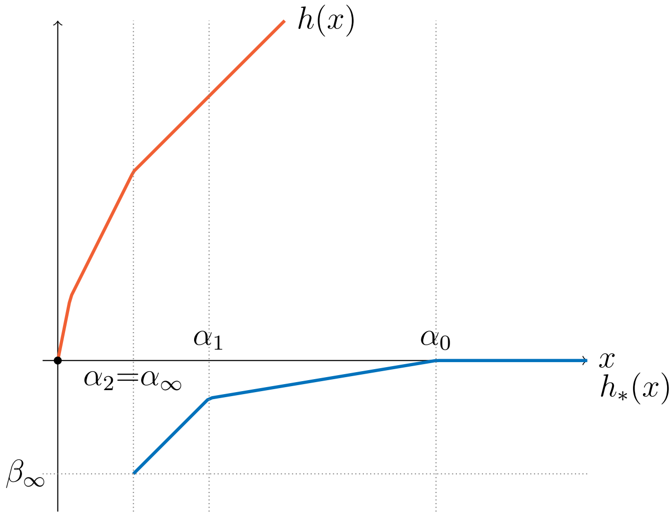
$$h(m) = \min\{\alpha_i\, m + \beta_i\ :\ i \in \{1, \ldots, N\}\}.$$

Indeed, we have

$$\lim_{m\downarrow 0} \frac{h(m)}{m} = \alpha_0, \qquad \alpha_\infty = \alpha_N \qquad \text{and} \qquad \beta_\infty = \beta_N.$$

A direct evaluation gives

$$h_*(x) = \inf_{m\in[0\infty)}\{x\, m - h(m)\} = \begin{cases} -\infty, & x < \alpha_N, \\ -\dfrac{\beta_i - \beta_{i-1}}{\alpha_i - \alpha_{i-1}}(x - \alpha_i) - \beta_i, & \alpha_i \leq x < \alpha_{i-1}, \\ 0, & x \geq \alpha_0. \end{cases}$$



Thus by our notion of Legendre transform h_* is the restriction of the above to the interval $[\alpha_N, \alpha_0]$. Indeed, h_* defines a bijection of $[\alpha_N, \alpha_0]$ onto $[-\beta_\infty, 0]$. Since for our choice of W we have $c_W(x) = x^2$ the function f is given by

$$f(x) := \begin{cases} \dfrac{\alpha_i - \alpha_{i-1}}{\beta_i - \beta_{i-1}}(x^2 - \beta_i) + \alpha_i, & x \in [\sqrt{\beta_{i-1}}, \sqrt{\beta_i}), \\ \alpha_N, & x \geq \sqrt{\beta_N}. \end{cases}$$

Remark that we have extended $(-h_*)^{-1}$ on $[\beta_N, \infty)$ with the value α_N.

2. The second example is a function with linear growth near at the origin namely let $h(m) := \sqrt{1 + |m|} - 1$. We have

$$\alpha_0 = \frac{1}{2}, \qquad \alpha_\infty = 0 \quad \text{and} \quad \beta_\infty = +\infty.$$

For this choice we obtain

$$\inf_{m \in [0,\infty)} \{x\, m - h(m)\} = 1 - x - \frac{1}{4x}$$

which is well defined and invertible on the interval $[0, 1/2]$, namely we have

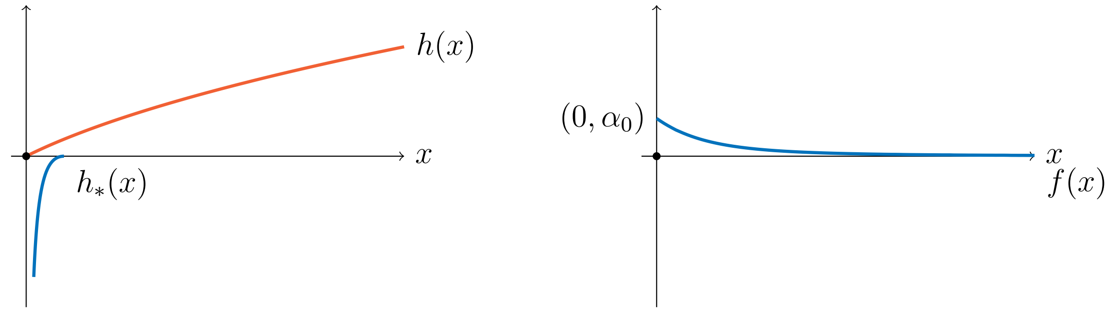
$$(-h_*)^{-1}(x) = \frac{x + 1 - \sqrt{x^2 + 2x}}{2}$$

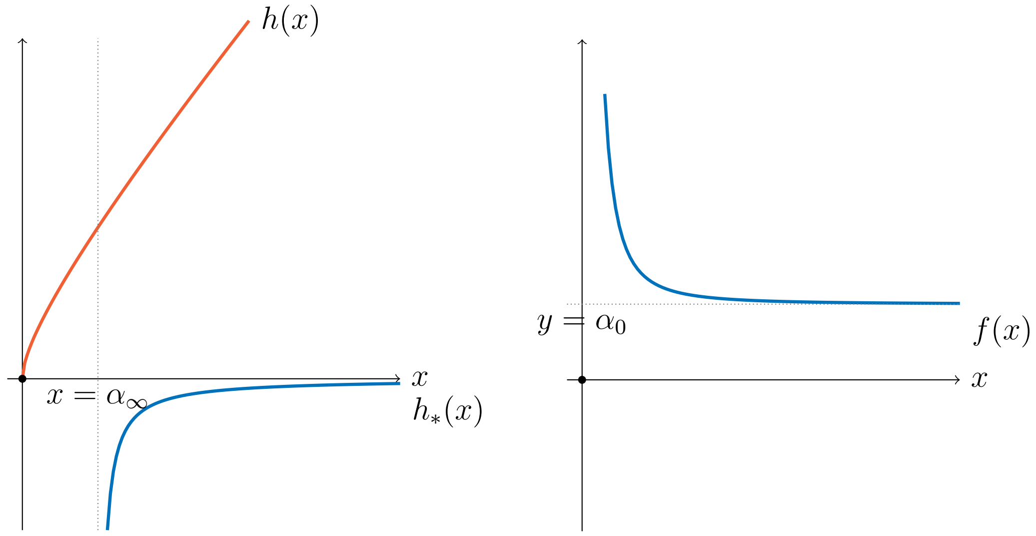
and we set

$$f(x) = \frac{x^2 + 1 - \sqrt{x^4 + 2x^2}}{2}.$$

3. The third example has linear growth at infinity and is given by $h(m) := m + \sqrt{m}$.

$$\alpha_0 = +\infty, \qquad \alpha_\infty = 1 \quad \text{and} \quad \beta_\infty = +\infty.$$





In this case the Legendre transform is given by $h_*(x) = 1/(4-4x)$ and the function f may be defined as

$$f(x) = 1 + \frac{1}{4x^2}.$$

4. The last example we deal with is the branched transport case. For $p > 1$ consider the function $h(m) = p\, m^{1/p}$ for which we have

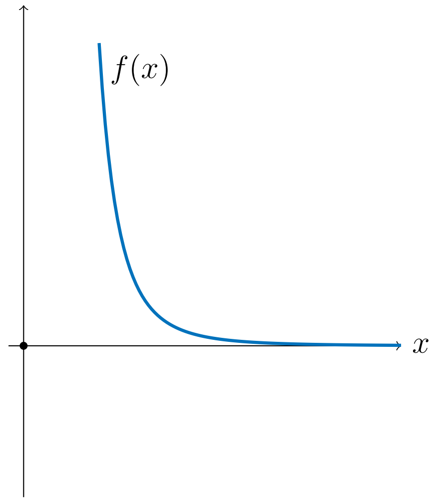
$$\alpha_0 = +\infty, \qquad \alpha_\infty = 0 \quad \text{and} \quad \beta_\infty = +\infty.$$

A direct evaluation gives $h_*(x) = (1-p)\, x^{1/1-p}$ as show in Figure 5.1. Therefore we have

$$f(x) = \left(\frac{x^2}{p-1}\right)^{1-p}.$$

5.3 Proof of Theorem 5.1

This section is devoted to the proof of the Γ-convergence result. Throughout the whole section we will consider the potential $W(t) := t^2$ for simplicity but the result is valid for a wider class of problems. The $\Gamma - \liminf$ inequality is obtained again by slicing.



For this reason following the strategy of Chapter 4, Section 4.3 we define the reduced dimension problem. Given an interval I and a measure $\theta \in \mathcal{M}(I)$ we recall that it may be decomposed into its atomic component and diffuse one so that

$$\theta = \theta^{\perp} + \sum m_\theta \mathcal{H}^0 \llcorner S_\theta$$

where S_θ is a countable set of points. Analogously to what has been already done in the previous chapter let us introduce the functional $\mathscr{G} : \mathcal{M}(I) \times L^2(I) \to [0, \infty)$ defined as

$$\mathscr{G}(\theta, \varphi; I) = \begin{cases} h'(0)|\theta^{\perp}| + \displaystyle\int_{S_\theta} h(m_\theta)\, \mathrm{d}\mathcal{H}^0, & \text{if } \varphi = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

We introduce as well the the reduced phase field functional

$$\mathscr{G}_\varepsilon(\theta, \varphi; I) := \begin{cases} \displaystyle\int_I \left[f(\varphi)|\theta| + \frac{1}{2}\left(\varepsilon|\varphi'|^2 + \frac{\varphi^2}{\varepsilon} \right) \right] \mathrm{d}x, & \text{for } (\theta, \varphi) \in L^1(I) \times L^2(I), \\ +\infty, & \text{otherwise on } \mathcal{M}(I) \times L^2(I). \end{cases}$$

We prove that

Lemma 5.3 ($\Gamma - \liminf$ reduced inequality)**.** *Let $I \subset \mathbf{R}$ be an open set, $(\theta, \varphi_\varepsilon) \in \mathcal{M}(I) \times L^2(I)$. For any $(\theta_\varepsilon, \varphi_\varepsilon)$ such that $\theta_\varepsilon \overset{*}{\rightharpoonup} \theta$ and $\varphi_\varepsilon \to \varphi$ it holds*

$$\liminf_{\varepsilon \downarrow 0} \mathscr{G}_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon; I) \geq \mathscr{G}(\theta, \varphi; I).$$

Proof. With no loss of generality we may assume that I is an interval, that for every $\varepsilon > 0$, $\theta_\varepsilon \in L^1(I)$, $\varphi_\varepsilon \in W^{1,2}(I)$ and

$$\liminf_{\varepsilon \downarrow 0} \mathscr{G}_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon; I) \leq M < +\infty$$

otherwise the inequality is trivial. We further assume that the decomposition for the limit measure θ reads

$$\theta = \theta^{\perp} + \sum_{j \in \mathbf{N}} m_j \delta_{p_j}.$$

Since we have assumed the family $(\theta_\varepsilon, \varphi_\varepsilon)$ to be equibounded in energy it holds

$$\int_\Omega \varphi_\varepsilon^2 \, \mathrm{d}x \leq \varepsilon M.$$

Therefore, up to a subsequence, $\varphi_\varepsilon \to 0$ pointwise almost everywhere. Let $\delta > 0$ be a small value to be chosen later. By Egorov's Theorem φ_ε converge uniformly to 0 on $I \setminus \hat{J}$ for some open set $\hat{J} \subset I$ with $|\hat{J}| < \delta/2$. Now consider

$$J = \hat{J} \cup \bigcup_{j \in \mathbf{N}} (p_j - \delta/2^{j+2}, p_j + \delta/2^{j+2})$$

where the points p_i correspond to the support of the atomic component of θ. For the sake of clarity we may rewrite $J = \cup_{i \in \mathbf{N}} C_i$ with $C_i = (a_i, b_i)$. By uniform convergence we have

$$\lim_{\varepsilon \downarrow 0} \varphi_\varepsilon(a_i) = \lim_{\varepsilon \downarrow 0} \varphi_\varepsilon(b_i) = 1.$$

We set

$$z_i^\varepsilon = \sup_{C_i} |\varphi_\varepsilon|.$$

Now by Young's inequality and a change of variables we have the estimate

$$\begin{aligned} \mathscr{G}_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon; (a_i, b_i)) &\geq \int_{a_i}^{b_i} \left[f(z_i^\varepsilon)|\theta_\varepsilon| + |\varphi'||\varphi| \right] \, \mathrm{d}x \\ &\geq f(z_i^\varepsilon) \int_{a_i}^{b_i} |\theta_\varepsilon| \, \mathrm{d}x + (z_i^\varepsilon)^2. \end{aligned}$$

Applying the latter on each interval C_i we get

$$\mathscr{G}_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon; I) \geq \int_{I \setminus J} f(\varphi_\varepsilon)|\theta_\varepsilon| \, \mathrm{d}x + \sum_{C_i} \left[f(z_i^\varepsilon) \int_{C_i} |\theta_\varepsilon| \, \mathrm{d}x + (z_i^\varepsilon)^2 \right]$$

Let us pass to the liminf in the latter equation taking advantage of Fatou' lemma and the lower semicontinuity of the total variation. Since $\varphi_\varepsilon \to 0$ uniformly on $I \setminus J$ and f is continuous, we have

$$\begin{aligned} \liminf_{\varepsilon \downarrow 0} \mathscr{G}_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon; I) &\geq f(0)|\theta|(I \setminus J) + \sum_i \left[f(z_i)|\theta|(C_i) + (z_i)^2 \right] \\ &\geq f(0)|\theta|(I \setminus J) + \sum_{C_i} \inf_{z \in (0,1)} \left[f(z_i)|\theta|(C_i) + (z_i)^2 \right] \end{aligned}$$

Recalling the properties for f obtained in Lemma 5.2 we have

$$\liminf_{\varepsilon \downarrow 0} \mathscr{G}_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon; I) \geq h'(0)|\theta|(I \setminus J) + \sum_{C_i} h\left(|\theta|(C_i)\right).$$

We conclude observing that $|\theta|(C_i) \geq |m_j|$ if C_i contains some p_j and that θ coincides with $\theta^\perp$ on $I \setminus J$ therefore sending δ to zero we obtain

$$\liminf_{\varepsilon \downarrow 0} \mathscr{G}_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon; I) \geq h'(0)|\theta^\perp|(I) + \sum_{j \in \mathbf{N}} h(|m_j|).$$

□

The latter lemma allows to prove the lower bound for the result in Theorem 5.1.

$\Gamma - \liminf$ *inequality for Theorem 5.1.* Let $(\sigma_\varepsilon, \varphi_\varepsilon)$ converge to (σ, φ) in the considered topology. We first extend σ_ε, σ, φ_ε and φ to $\mathbf{R}^2 \setminus \overline{\Omega}$ by zero. The phasefield cost functional and the cost functional are extended to $\mathbf{R}^2$ in the obvious way (their values do not change). Without loss of generality (potentially after extracting a subsequence) we may assume $\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon)$ to exist and to be finite (else there is nothing to show). As a consequence we have $\operatorname{div} \sigma^\varepsilon = \mu_\varepsilon^+ - \mu_\varepsilon^-$ as well as $\operatorname{div} \sigma = \mu^+ - \mu^-$ and $\varphi \equiv 0$ (since the phasefield cost functional is bounded below by $\frac{1}{2\varepsilon}\int_\Omega \varphi_\varepsilon^2$). Choosing some $\xi \in \mathbf{S}^1$, by Fubini's decomposition theorem we have

$$\begin{aligned}\mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; A) &= \int_{-\infty}^{\infty} \int_{A^{\xi,t}} f(\varphi_\varepsilon^{\xi,t})|\sigma_\varepsilon^{\xi,t}| + \frac{1}{2}\left[\varepsilon|(\varphi_\varepsilon^{\xi,t})'|^2 + \frac{(\varphi_\varepsilon^{\xi,t})^2}{\varepsilon}\right] \delta x\, \delta t \\ &= \int_{-\infty}^{\infty} \mathscr{G}_\varepsilon(\sigma_\varepsilon^{\xi,t}, \varphi_\varepsilon^{\xi,t}; A^{\xi,t})\, \mathrm{d}t.\end{aligned}$$

Fatou's lemma thus implies

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; A) \geq \int_{-\infty}^{\infty} \liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(\sigma_\varepsilon^{\xi,t}, \varphi_\varepsilon^{\xi,t}; A^{\xi,t})\, \delta t\,.$$

By assumption, the left-hand side is finite so that the right-hand side integrand is finite for almost all $t \in \mathbf{R}$ as well. Pick any such t and pass to a subsequence such that $\liminf$ turns into $\lim$. Indeed $\sigma_\varepsilon^{\xi,t} \overset{*}{\rightharpoonup} \sigma^{\xi,t}$ for every ξ and almost all t, as $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$. Thus, the reduced dimension problem 5.3 implies

$$\liminf_{\varepsilon\to 0} \mathscr{G}_\varepsilon(\sigma_\varepsilon^{\xi,t}, \varphi_\varepsilon^{\xi,t}; A^{\xi,t}) \geq \mathscr{G}(\sigma^{\xi,t}, \varphi^{\xi,t}; A^{\xi,t})$$

for almost all $t \in \mathbf{R}$ so that

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; A) \geq \int_{-\infty}^{\infty} \mathscr{G}(\sigma^{\xi,t}, \varphi^{\xi,t}; A^{\xi,t})\, \delta t\,.$$

For notational convenience let us now define the auxiliary function κ, defined for open subsets $A \subset \mathbf{R}^2$, as

$$\kappa(A) = \liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; A)\,.$$

Furthermore, introduce the nonnegative Borel measure

$$\lambda(A) = h'(0)|\sigma^\perp|(A) + \int_{S_\sigma \cap A} h(m_\sigma)\ \mathrm{d}\mathcal{H}^1$$

as well as the $|\sigma|$-measurable Borel functions

$$\psi_j : \mathbf{R}^2 \to \mathbf{R}, \qquad \psi_j = \left|\frac{\sigma}{|\sigma|} \cdot \xi^j\right|$$

for some sequence ξ^j, $j \in \mathbf{N}$, dense in $\mathbf{S}^1$. Since σ is a divergence measure vectorfield, we have

$$\begin{aligned}\kappa(A) &\geq \int_{-\infty}^{\infty} \mathscr{G}(\sigma^{\xi^j,t}, \varphi^{\xi^j,t}; A^{\xi^j,t}) \ \mathrm{d}t \\ &= \int_{-\infty}^{\infty} h'(0)|(\sigma_{\xi^j,t})^{\perp}|(A^{\xi^j,t}) + \int_{S_{\sigma_{\xi^j,t}} \cap A^{\xi^j,t}} h(|m_{\sigma^{\xi^j,t}}|) \ \mathrm{d}\mathcal{H}^0 \ \mathrm{d}t \\ &= h'(0)|\sigma^{\perp} \cdot \xi^j|(A) + \int_{S_\sigma \cap A} h(|m_\sigma|)|\theta_\sigma \cdot \xi^j| \ \mathrm{d}\mathcal{H}^1 \geq \int_A \psi_j \ \mathrm{d}\lambda\end{aligned}$$

for all $j \in \mathbf{N}$ where we have used Remark 9 in the last equality. By [Bra98, Prop. 1.16] the above inequality implies

$$\kappa(A) \geq \int_A \sup_j \psi_j \, \delta\lambda$$

for any open $A \subset \mathbf{R}^2$. In particular, choosing A as the 1-neighborhood of Ω we obtain

$$\begin{aligned}\liminf_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon; \Omega) = \kappa(A) \geq \int_A \sup_j \psi_j \, \delta\lambda \\ = h'(0)|\sigma^{\perp}|(A) + \int_{S_\sigma \cap A} h(m_\sigma) \ \mathrm{d}\mathcal{H}^1 = \mathscr{E}^{\mu^+,\mu^-}[\sigma] \,,\end{aligned}$$

the desired result. $\square$

We now prove the associated upper bound, actually we only construct the recovery sequence for a segment. The general case can be handled as in the previous chapters of the thesis.

$\Gamma - \limsup$ *inequality for Theorem 5.1.* As always we concentrate on a single segment assuming $\sigma = \theta \mathcal{H}^1 \llcorner \Sigma$ with $\Sigma = [0, L] \times \{0\}$ and $\theta = m\, e_1$ with $m > 0$. Let

$$z_m = \operatorname{argmin}\{z \in [0, +\infty) \ : \ f(z)\, m + z^2\}.$$

For the vector field we define

$$\sigma_\varepsilon = \frac{m \ \chi_{\{d_\Sigma \leq \varepsilon^2\}}}{\varepsilon^2} \, e_1$$

where d_Σ is the distance function from the set Σ. For the phase-field, we let Φ be the solution of the following Cauchy problem in $\mathbf{R}$

$$\begin{cases} \varphi' = -|\varphi|, \\ \varphi(0) = z_m, \end{cases}$$

whose solution on $[0, \infty)$ is given by the function $\Phi(t) = z\, e^{-t}$. Let φ_ε be defined as

$$\varphi_\varepsilon(x) := \begin{cases} z_m, & \text{if } d_\Sigma(x) \leq \varepsilon^2, \\ \Phi\left(\dfrac{d_\Sigma(x) - \varepsilon^2}{\varepsilon}\right), & \text{otherwise.} \end{cases}$$

Considering the fact that $\sigma_\varepsilon \equiv 0$ in the set $\{d_\Sigma \geq \varepsilon^2\}$ we have

$$\mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) = \int_{\{d_\Sigma(x) \leq \varepsilon^2\}} f(z_m) \frac{m}{\varepsilon^2}\, \mathrm{d}x + \int_{\{d_\Sigma(x) \geq \varepsilon^2\}} \frac{1}{2} \left(|\nabla \varphi_\varepsilon|^2 + \frac{\varphi_\varepsilon^2}{\varepsilon} \right) \mathrm{d}x$$

Let us remark that in force of the fact $\Omega \subset \mathbf{R}^2$ we have $|\{d_\Sigma(x) \geq \varepsilon^2\}| = \varepsilon^2 L + o(\varepsilon^2)$ and $\mathcal{H}^1(\{d_\Sigma^{-1}(s)\}) = 2L + 2\pi s$. Taking advantage of a change of variables we obtain:

$$\mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) = f(z_m)\, m\, L + o(1) + \int_0^\infty |\Phi'(t)||\Phi(t)| \left[2L + 2\pi(\varepsilon t - \varepsilon^2) \right] \mathrm{d}t.$$

By evaluating the integral on the righthand side directly and passing to the superior limit we conclude

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) = (f(z_m)\, m + z_m^2)\, L = h(m)\, L.$$

Observe that this corresponds exactly with

$$\limsup_{\varepsilon \downarrow 0} \mathscr{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) = \int_\Sigma h(m)\, \mathrm{d}\mathcal{H}^1$$

and the proof in the case of a segment is concluded. □

Conclusion

To conclude this work we highlight some possible developments of the treated thematics. We first focus on a theoretical claim and then we will present some numerical methods which could be implemented in the future.

Let us analyze the h-mass transport problem in $\mathbf{R}^3$. In a recent work [BEZ15] the authors propose to substitute for the gradient of the phase field term in the Ambrosio-Tortorelli functional a term depending on a second order differential operator. This modification enhances the regularity of the phase fields and allows for computational and practical improvements of the existing schemes. Inspired by this idea and those of the last chapter we are led to consider a functional of the form

$$\mathscr{F}_\varepsilon(\sigma, \varphi) := \int_\Omega \left[f(\varphi)|\sigma| + \left(\varepsilon^2 |\Delta\varphi|^2 + \frac{\varphi^2}{\varepsilon^2} \right) \right] \, \mathrm{d}x, \tag{5.7}$$

complemented with the usual divergence constraint $\nabla \cdot \sigma = \mu_+ - \mu_-$. The latter functional resembles closely the one studied in the last Chapter 5 and the heuristic Γ-convergence argument is analogous. The main difference relies in the definition of the penalization function f. In order to define this function we need to consider the transition cost for the phase field which is associated to the following minimization problem

$$T(x) := \begin{cases} \min\limits_v \displaystyle\int_0^\infty \left[\left(v''(r) + \frac{1}{r} v'(r) \right)^2 + v(r)^2 \right] r \, \mathrm{d}r, \\ v \in H^2_{loc}((0, +\infty)), \, v(0) = x, \, v'(0) = 0, \, \lim\limits_{r \to +\infty} v(r) = 0. \end{cases}$$

With this definition for the function T we may follow the same strategy used previously and set $f = (-h_*)^{-1} \circ T$. A first point of investigation would be to prove the following claim

Claim 5.2. *For any sequence $\varepsilon \downarrow 0$ we have*

$$\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{E}.$$

Where $\mathscr{E}$ is defined in equation (4.2).

This result is quite expected and the proof should follow closely the one in the Chapter 5. Indeed we could use the same Modica-Mortola component which has been used in Chapter 3 to obtain a similar result. The advantage of this choice is related to the numerical method presented below.

Regarding the numerical approximations it would be interesting to apply some of the techniques proposed by Bonnivard, Bretin and Lemenant in [BBL18] to our problems. Let us recall that for fixed $\varepsilon \ll 1$ the minimization problem associated to the functional defined in equation (5.5) has the form

$$\min \begin{cases} \int_\Omega \left[f(\varphi)|\sigma| + \frac{1}{2}\left(\varepsilon|\nabla\varphi|^2 + \frac{\varphi^2}{\varepsilon}\right)\right] \, \mathrm{d}x \\ \varphi \in W^{1,2}(\Omega),\ \sigma \in L^1(\Omega) \text{ and } \nabla \cdot \sigma = \mu_+ - \mu_- \end{cases} \tag{5.8}$$

We propose an alternate minimization scheme which is suitable to the case in which either μ_+ or μ_- is atomic. Focus on the minimization in σ for fixed φ, namely

$$\min \left\{ \int_\Omega f(\varphi)|\sigma| \, \mathrm{d}x \,:\, \sigma \in L^1(\Omega) \text{ and } \nabla \cdot \sigma = \mu_+ - \mu_- \right\}. \tag{5.9}$$

The latter is equivalent to the Beckman model for congested transportation [Bec52] in which $f(\varphi)$ models the congestion rate at each point. We reformulate (5.9) as a minimization problem on the set of continuous paths, namely $C([0,1],\Omega)$. We let $\Gamma(\mu_+,\mu_-)$ be the set of measures Q on $C([0,1],\Omega)$ such that

$$e_{0\#}Q = \mu_- \quad \text{and} \quad e_{1\#}Q = \mu_+ \tag{5.10}$$

where $e_0(\gamma) = \gamma(0)$ and $e_1(\gamma) = \gamma(1)$ for any $\gamma \in C([0,1],\Omega)$. For any $Q \in \Gamma(\mu_+,\mu_-)$ the expression

$$\sigma = \int_{C([0,1],\Omega)} \dot\gamma \mathcal{H}^1 \llcorner \gamma([0,1]) \, \mathrm{d}Q$$

defines a vector measure $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ such that $\nabla \cdot \sigma = \mu_+ - \mu_-$, viceversa [Smi93, CS11, San14] to any such σ we may associate a measure $Q \in \Gamma(\mu_+,\mu_-)$. Remark that

$$\begin{aligned} \int_\Omega f(\varphi)|\sigma| \, \mathrm{d}x &= \int_\Omega f(\varphi) \, \mathrm{d}|\sigma| \\ &= \int_\Omega f(\varphi) \, \mathrm{d}\left(\int_{C([0,1],\Omega)} |\dot\gamma| \, \mathrm{d}\mathcal{H}^1 \llcorner \gamma([0,1]) \, \mathrm{d}Q\right) \\ &= \int_{C([0,1],\Omega)} \int_{\gamma([0,1])} f(\varphi) \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}Q \end{aligned}$$

So that the minimization problem (5.9) is equivalent to

$$\min \left\{ \int_{C([0,1],\Omega)} \int_{\gamma([0,1])} f(\varphi) \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}Q \,:\, Q \in \Gamma(\mu_+,\mu_-) \right\}.$$

This equivalence is particularly interesting in the case $\mu_+ = \delta_{x_0}$. As a matter of fact in this case the minimizer in the above is achieved when the measure Q is supported on the geodesics, with respect to the Riemannian metric induced by $f(\varphi)$, joining x_0 to each point in $\mathrm{supp}(\mu_-)$. Therefore the minimization procedure reduces to the problem of finding each one of these geodesics. Eventually this research can be done in a fast and efficient way by means of the Fast Marching Method [Set99]. The minimization of (5.8) with respect to φ can be done by solving via Fast Fourier Transform the associated PDE, namely

$$\varepsilon\Delta\varphi - \frac{\varphi}{\varepsilon} - f'(\varphi)|\sigma| = 0.$$

This approach has two major benefits. Firstly, allows to minimize in the σ variable overcoming the non-differentiability of the norm, secondly it would be quite efficient since it relies on fast algorithms. Furthermore the presented method can be applied as well to the functional defined in equation (5.7). The PDE associated to the φ problem depends on the bilaplacian of φ and takes the form

$$\varepsilon^2\Delta^2\varphi - \frac{\varphi}{\varepsilon^2} - f'(\varphi)|\sigma| = 0.$$

In this the Fast Fourier Transform would provide a better tool with respect to finite elements methods. The same method could be applied to the functional studied in Chapter 3 but would lead to a PDE with worst non-linearities.

Appendix A

Density result for vector measures in $\mathbf{R}^2$

We show that measures which have support contained in a finite union of segments, are dense in energy. Without loss of generality let us assume that $\sigma \in \mathcal{M}_S(\overline{\Omega})$ is such that $\mathscr{E}_\beta(\sigma, 1) < \infty$. In particular $\sigma = U(m_\sigma, \tau_\sigma, \Sigma_\sigma)$ is a $\mathcal{H}^1$-rectifiable measure. Applying Lemma 1.2 we obtain an $\mathcal{H}^1$-rectifiable measure $\gamma = U(m_\gamma, \tau_\gamma, \Sigma_\gamma)$ and a partition of Ω made of polyhedrons $\{\Omega_i\}$ such that $\Sigma_\gamma \subset \cup_i \partial\Omega_i$, $\mathcal{H}^1(\Sigma_\sigma \cap \cup_i \partial\Omega_i) = 0$ and $\sigma + \gamma$ is divergence free.
From the above properties, we can write

$$\sigma^\perp + \gamma^\perp = Du$$

for some $u \in PC(\Omega)$. Our strategy is the following, using existing results [BCG14], we build an approximating sequence for u on each Ω_j whose gradient is supported on a finite union of segments. We then glue these approximations together to obtain a sequence (w_j) approximating u in $\hat{\Omega}$. Where $\hat{\Omega}$ is an open set containing Ω. The main difficulty is to establish that $Dw_j \llcorner [\cup_i \partial\Omega_i]$ is close to $Du \llcorner [\cup_i \partial\Omega_i] = \gamma^\perp$. First let us recall the result in [BCG14]

Lemma A.1. *Let $u \in PC(\Omega)$ be such that*

$$\mathscr{E}_h(u, \Omega) = \int_{\Omega \cap J_u} h([u]) \, \mathrm{d}\mathcal{H}^{d-1} < +\infty.$$

for h a continuous, sub-additive and increasing function on $[0, +\infty)$ such that $h(0) = 0$ and $\lim_{t \to 0} \frac{h(t)}{t} = +\infty$. Then there exists a sequence $(u_l) \subset PC(\Omega)$ with the following properties:

- $\lim_{l \to +\infty} u_l = u$ *in* $L^1(\Omega)$,
- $\lim_{l \to +\infty} \mathscr{E}_h(u_l, \Omega) = \mathscr{E}_h(u, \Omega)$,
- J_{u_l} *is contained in a finite union of facets of polytopes for any $h \in \mathbf{N}$. In particular for any $n \in \mathbf{N}$,*

$$\mathcal{H}^{n-1}(\Omega \cap \overline{J}_{u_l}) = \mathcal{H}^{n-1}(\Omega \cap J_{u_l}) \quad \text{and} \quad \mathcal{H}^{n-1}(\overline{J}_{u_l}) < +\infty.$$

Lemma A.2 (Approximation of u)**.** *There exists a sequence $(w_j) \subset PC(\hat{\Omega})$ with the following properties:*

a) $w_j \to u$ *weakly in* $BV(\hat{\Omega})$,

b) $\operatorname{supp} w_j \subset \overline{\Omega}$,

c) $\limsup_{j\to\infty} \mathscr{E}_\beta(w_j, 1) \leq \mathscr{E}_\beta(u, 1)$,

d) J_{w_j} *is contained in a finite union of segments for any* $j \in \mathbf{N}$,

e) $|Dw_j - Du|(\cup \partial\Omega_i) \to 0$.

Proof. Step 1: In order to apply the results of [BCG14], we first need to modify u and the energy. Let us denote the energy density function $h(t) = 1 + \beta t$ and for $k \geq 0$ and $t \geq 0$ let us introduce the approximation

$$h_k(t) := \begin{cases} (2^{k/2} + \beta 2^{-k/2})\sqrt{t}, & \text{for } t \leq 2^{-k}, \\ h(t), & \text{otherwise.} \end{cases}$$

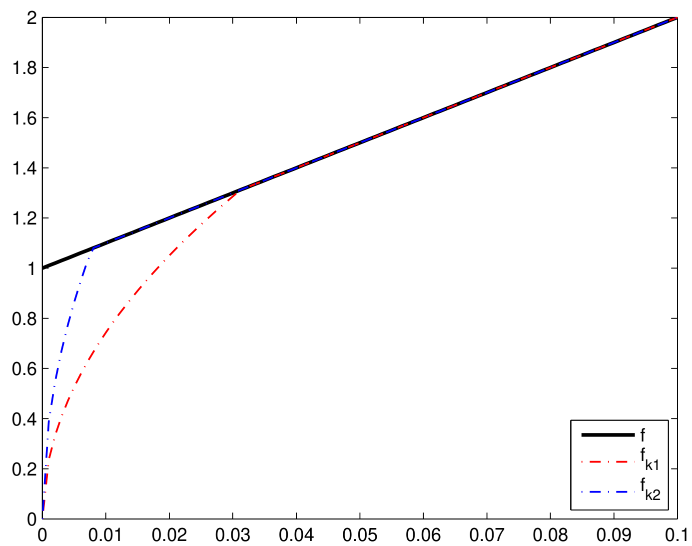


Figure A.1: Graph of h and two of its approximations h_{k_1} and h_{k_2} with $k_1 < k_2$.

We have $0 \leq h_k \leq h$ and $h_k \equiv h$ on $[2^{-k}, +\infty)$. Notice that h_k is continuous, subadditive and increasing on $[0, +\infty)$ and that $h_k(0) = 0$ with $\lim_{t\to 0} \frac{h_k(t)}{t} = +\infty$. We define the associated energy for functions $v \in PC(\hat{\Omega})$ as $\mathscr{E}_{h_k}(v, \hat{\Omega}) := \int_{J_v \cap \hat{\Omega}} h_k([v])\, \mathrm{d}\mathcal{H}^1$. Now we denote $PC_k(\hat{\Omega})$ the set of functions $v \in PC(\hat{\Omega})$, (1.6), such that $v(\hat{\Omega}) \subset 2^{-k}\mathbf{Z}$. For these functions we have $|v^+(x) - v^-(x)| \geq 2^{-k}$ for $\mathcal{H}^1$-almost every $x \in J_v$. Consequently, one has

$$\mathscr{E}_{h_k}(v, \hat{\Omega}) \;=\; \mathscr{E}_h(v, \hat{\Omega}).$$

For each fixed $k \geq 0$, let us introduce the function

$$u_k = 2^{-k} \lfloor 2^k u \rfloor$$

where $\lfloor t \rfloor$ denotes the integer part of the real t. Note that $u_k \in PC_k(\hat{\Omega})$ with $J_{u_k} \subset J_u$ and $\|u - u_k\|_\infty \leq 2^{-k}$. Notice also that in view of $\left|(u_k^+ - u_k^-) - (u^+ - u^-)\right| \leq 2^{-k}$ we have

$$|Du_k - Du|(\hat{\Omega}) \;\leq\; 2^{-k}\mathcal{H}^1(J_u). \tag{A.1}$$

Indeed, $u_k \to u$ strongly in $BV(\hat{\Omega})$, as $\mathcal{H}^1(J_u) < +\infty$. Moreover, we see that

$$\mathscr{E}_{h_k}(u_k, \hat{\Omega}) \;=\; \mathscr{E}_h(u_k, \hat{\Omega}) \;\leq\; \mathscr{E}_h(u, \hat{\Omega}) + \beta 2^{-k}\mathcal{H}^1(J_u). \tag{A.2}$$

Step 2: Let us approximate the function u_k. Let us fix $k \geq 0$ and Ω_i. We can apply Lemma A.1 to the function $u_k \llcorner \Omega_i$ and to the energy $\mathscr{E}_{h_k}(\cdot, \Omega_i)$. We obtain a sequence (w_j^i) which enjoys the following properties:

$$w_j^i(\Omega_i) \subset u_k(\Omega_i) \subset 2^{-k}\mathbf{Z}, \quad \forall j \in \mathbf{N},\ \text{hence } w_j^i \in PC_k(\hat{\Omega}),$$
$$w_i^j \to u_k \text{ in } L^1(\Omega_i) \text{ as } j \to +\infty,$$
$$\lim_{j \to +\infty} \mathscr{E}_{h_k}(w_j^i, \Omega_i) = \lim_{j \to +\infty} \mathscr{E}_h(w_j^i, \Omega_i) = \mathscr{E}_h(u_k, \Omega_i),$$
$$J_{w_i^j} \text{ is contained in a finite union of segments for any } j \in \mathbf{N},$$
$$\int_{\partial\Omega_i} |Tw_j^i - Tu_k| \, \mathrm{d}\mathcal{H}^1 \to 0 \text{ where } T : BV(\Omega_i) \to L^1(\partial\Omega_i) \text{ denotes the trace operator.}$$

Let us now define globally

$$w_j := \sum_j w_j^i 1_{\Omega_i}.$$

From the above properties, we have $w_j \overset{*}{\rightharpoonup} u_k$,

$$\lim \mathscr{E}_h(w_j^i, \hat{\Omega}) = \mathscr{E}_h(u_k, \Omega_i) \tag{A.3}$$

and

$$|Dw_j - Du_k|(\cup_i \partial\Omega_i) \;\to\; 0 \quad \text{as } j \to \infty. \tag{A.4}$$

Eventually, using a diagonal argument, we have proved the existence of a sequence $(w_j) \subset PC(\hat{\Omega})$ satisfying claims *(a)*, *(b)* and *(d)* of the lemma. Moreover, item *(c)* is the consequence of (A.2) and (A.3) and item *(e)* follows from (A.1) and (A.4). □

Going back to the $\mathcal{H}^1$-rectifiable measures $\sigma = U(m_\sigma, \tau_\sigma, \Sigma_\sigma)$, we define the sequence

$$\sigma_j := -Dw_i^\perp - \gamma.$$

We recall that $\gamma = U(m_\gamma, \tau_\gamma, \Sigma_\gamma)$ with $M_\gamma \subset \cup \partial\Omega_i$. In particular $\gamma = -Du^\perp \llcorner (\cup_i \partial\Omega_i)$. We deduce from the previous lemma:

Lemma A.3. *There exists a sequence $(\sigma_j) \in \mathcal{M}_S(\overline{\Omega})$ with the properties:*

- *$\sigma_j \to \sigma$ with respect to weak-$*$ convergence of measures,*
- *$\sigma_j = U(m_{\sigma_j}, \tau_{\sigma_j}, \Sigma_{\sigma_j})$ with M_{σ_j} contained in a finite union of segments,*
- *$\limsup_{j \to \infty} \mathscr{E}_\beta(\sigma_j, 1) \leq \mathscr{E}_\beta(\sigma, 1)$.*

Appendix B

Reduced problem in dimension $n-k$

B.1 Auxiliary problem

In this appendix we show the results previously stated in Section 2.2 of Chapter 2, with the notation introduced therein let us define the auxiliary set

$$\overline{Y}_{\varepsilon,\beta}(m,r) = \left\{(\vartheta,\varphi) \in L^2(B_r) \times W^{1,p}(B_r,[\eta,1]) \,:\, \|\vartheta\|_1 = m \text{ and } \varphi_{|\partial B_r} \equiv 1\right\},$$

and the associated minimization problem

$$\overline{h}^d_{\varepsilon,\beta}(m,r) = \inf_{\overline{Y}_{\varepsilon,\beta}(m,r)} \mathscr{G}_{\varepsilon,\beta}(\vartheta,\varphi;B_r). \tag{B.1}$$

For the sake of clarity let us recall that the functional introduced in equation (2.7) has the expression

$$\mathscr{G}_{\varepsilon,\beta}(\vartheta,\varphi;B_r) := \int_{B_r} \left[\varepsilon^{p-d}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^d} + \frac{\varphi|\vartheta|^2}{\varepsilon}\right] \mathrm{d}x.$$

Analogous optimization problem to (B.1) with mass constraint appears in models of droplets equilibrium. Bouchitté et al. in [BDS96] study a one dimensional smooth version of the problem in which the mass constraint is on the phase field variable φ. Minimizing (B.1) in ϑ we obtain a functional depending only on the variable φ which can be interpreted as a variant in higher dimension of the cited work.

The outline of the appendix is the following. First we show that both $h^d_{\varepsilon,\beta}(m,r,\tilde{r})$ and $\overline{h}^d_{\varepsilon,\beta}(m,r)$ are bounded by the same constant as $\varepsilon \downarrow 0$ and that the value of the second term is achieved by a radially symmetric pair of $\overline{Y}_{\varepsilon,\beta}(m,r)$. These two facts are then used to show that for each m the limit values of $\overline{h}^d_{\varepsilon,\beta}(m,r)$ and $h^d_{\varepsilon,\beta}(m,r,\tilde{r})$ as $\varepsilon \downarrow 0$ are equal and independent of the choices $(r,\tilde{r})$ to the extent that $0 < \tilde{r} < r$. Let us start by showing the first two properties.

Lemma B.1. *For each ε, $m > 0$ and $r > 0$*

a) there exists a constant $C = C(m,\beta) \le C_0(1+\sqrt{\beta}m)$ such that for

$$0 < \varepsilon \le \min\left\{\frac{\tilde{r}}{(\sqrt{\beta}m)^{1/d}}\,,\,\frac{r}{1+(\sqrt{\beta}m)^{1/d}}\right\},$$

there holds,

$$h^d_{\varepsilon,\beta}(m,r,\tilde{r}) < C \qquad \text{and} \qquad \overline{h}^d_{\varepsilon,\beta}(m,r) < C. \tag{B.2}$$

b) Both the problem defined in equation (2.8) *and equation* (B.1) *admit a minimizer. Moreover among the minimizers of $\mathscr{G}_{\varepsilon,\beta}$ in $\overline{Y}_{\varepsilon,\beta}(m,r)$ it is possible to choose a radially symmetric pair $(\vartheta_\varepsilon,\varphi_\varepsilon)$ such that φ_ε is radially non-decreasing and ϑ_ε is radially non-increasing.*

Proof. a) Let $r_1 > 0$ and $\varepsilon > 0$ such that $r_1\varepsilon \le \tilde{r}$, $(1+r_1)\varepsilon \le r$, we define

$$\varphi_\varepsilon(x) := \begin{cases} \eta & \text{if } |x| < r_1\varepsilon, \\ \eta + (1-\eta)(|x|/\varepsilon - r_1) & \text{if } r_1\varepsilon \le |x| < (1+r_1)\varepsilon, \\ 1 & \text{if } (1+r_1)\varepsilon \le |x| < r, \end{cases}$$

and

$$\vartheta_\varepsilon(x) := \begin{cases} \dfrac{m}{|B_{r_1\varepsilon}|} & \text{if } |x| < \varepsilon, \\ 0 & \text{if } \varepsilon \le |x| < r. \end{cases}$$

By construction, $(\varphi_\varepsilon,\vartheta_\varepsilon) \in Y_{\varepsilon,\beta}(m,r,\tilde{r}) \cap \overline{Y}_{\varepsilon,\beta}(m,r)$. We estimate successively the three terms of the energy. First, since $\varepsilon|\nabla\varphi_\varepsilon| = (1-\eta) \le 1$ in $B_{(1+r_1)\varepsilon} \setminus B_{r_1\varepsilon}$ and vanishes outside,

$$\int_{B_r} \varepsilon^{p-d}|\nabla\varphi_\varepsilon|^p \,\mathrm{d}x \le |B_{(1+r_1)\varepsilon} \setminus B_{r_1\varepsilon}|\,\varepsilon^{-d} \le \omega_d(1+r_1)^d.$$

Next, bounding $|1-\varphi_\varepsilon|$ by the characteristic function of $B_{(1+r_1)\varepsilon}$ we have

$$\int_{B_r} \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d}\,\mathrm{d}x \le \ \omega_d(1+r_1)^d.$$

Finally,

$$\int_{B_r} \frac{\varphi_\varepsilon|\vartheta_\varepsilon|^2}{\varepsilon}\,\mathrm{d}x = \frac{1}{\omega_d r_1^d}\frac{\eta\, m^2}{\varepsilon^{d+1}} = \frac{\beta m^2}{\omega_d r_1^d}.$$

Gathering the estimates yields to the bound

$$\max\{h^d_{\varepsilon,\beta}(m,r,\tilde{r}),\overline{h}^d_{\varepsilon,\beta}(m,r)\} \le \mathscr{G}_{\varepsilon,\beta}(\varphi_\varepsilon,\vartheta_\varepsilon) \le 2\omega_d(1+r_1)^d + \frac{am^2}{\omega_d r_1^d}.$$

Then, assuming $(\sqrt{\beta}m)^{1/d}\varepsilon \le \tilde{r}$ and $(1+(\sqrt{\beta}m)^{1/d})\varepsilon \le r$, we can set $r_1 := (\sqrt{\beta}m)^{1/d}$. We obtain,

$$\max\{h^d_{\varepsilon,\beta}(m,r,\tilde{r}),\overline{h}^d_{\varepsilon,\beta}(m,r)\} \le C(1+\sqrt{\beta}m).$$

b) To show the existence of minimizers for both minimization problems we use the direct method of the Calculus of Variation. The lower semicontinuity of the integral with integrand $u|\vartheta|^2$ is ensured by Ioffe's theorem [AFP00, theorem 5.8]. Now given any minimizing pair $(\hat{\vartheta}_\varepsilon, \hat{\varphi}_\varepsilon) \in \overline{Y}_{\varepsilon,\beta}(m,r)$, let ϑ_ε be the decreasing Steiner rearrangement of $\hat{\vartheta}_\varepsilon$ and φ_ε the increasing rearrangement of $\hat{\varphi}_\varepsilon$. Indeed, since $\hat{\varphi}_\varepsilon$ has range in $[\eta, 1]$, we still have $\varphi_{\varepsilon\ |\partial B_r} \equiv 1$. Polya's Szego and Hardy-Littlewood's inequalities [Tal76, LL97] ensure

$$\mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) \leq \mathscr{G}_{\varepsilon,\beta}(\hat{\vartheta}_\varepsilon, \hat{\varphi}_\varepsilon)$$

$\square$

Let us prove the asymptotic equivalence of the values $h^d_{\varepsilon,\beta}(m,r,\tilde{r})$ and $\overline{h}^d_{\varepsilon,\beta}(m,r)$ as $\varepsilon \downarrow 0$.

Lemma B.2 (Equivalence of the two problems)**.** *For any $\tilde{r} < r$ and $m > 0$ it holds*

$$|h^d_{\varepsilon,\beta}(m,r,\tilde{r}) - \overline{h}^d_{\varepsilon,\beta}(m,r)| \xrightarrow{\varepsilon\downarrow 0} 0$$

Proof. Step 1: [$h^d_{\varepsilon,\beta}(m,r,\tilde{r}) \leq \overline{h}^d_{\varepsilon,\beta}(m,r) + O(1)$]
Consider for each ε the radially symmetric and monotone pair $(\vartheta_\varepsilon, \varphi_\varepsilon) \in \overline{Y}_{\varepsilon,\beta}(m,r)$ as introduced in the previous lemma. Take $\xi \in (\eta, 1)$ and let us set

$$r_\xi := \sup\{t \in (0,r) : \varphi_\varepsilon(t) \leq \xi\} \quad \text{with } r_\xi = 0 \text{ if the set is empty.} \tag{B.3}$$

By Cauchy-Schwartz inequality it holds

$$C \geq \frac{\int_{B_r \setminus B_{r_\xi}} \varphi_\varepsilon |\vartheta_\varepsilon|^2 \,\mathrm{d}x}{\varepsilon} \geq \xi \, \frac{\left(\int_{B_r \setminus B_{r_\xi}} |\vartheta_\varepsilon| \,\mathrm{d}x\right)^2}{\omega_d \; r^d \varepsilon}.$$

Let us define $\Delta_\xi := \int_{B_r \setminus B_{r_\xi}} |\vartheta_\varepsilon|$, the latter ensures that $\Delta_\xi \in o(\varepsilon^{1/2})$. Let us now set $\hat{\vartheta}_\varepsilon = \left(\frac{m\vartheta_\varepsilon}{\int_{B_{r_\xi}} \vartheta_\varepsilon}\right) \mathbf{1}_{B_{r_\xi}}$ which is not null for ε small. We have $(\hat{\vartheta}_\varepsilon, \varphi_\varepsilon) \in Y_{\varepsilon,\beta}(m,r,\tilde{r})$ if and only if $r_\xi \leq \tilde{r}$. Indeed, this holds as

$$C \geq \int_{B'_{r_\xi}} \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d} \,\mathrm{d}x \geq \; \omega_d \,(1-\xi)^2 \left(\frac{r_\xi}{\varepsilon}\right)^d, \tag{B.4}$$

which ensures that $r_\xi = O(\varepsilon)$. Finally let us evaluate the energy

$$\begin{aligned}
\mathscr{G}_{\varepsilon,\beta}(\hat{\vartheta}_\varepsilon, \varphi_\varepsilon) &= \int_{B_r} \left[\varepsilon^{p-d} |\nabla \varphi_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d} + \frac{\varphi_\varepsilon |\hat{\vartheta}_\varepsilon|^2}{\varepsilon}\right] \mathrm{d}x \\
&= \int_{B_r} \left[\varepsilon^{p-d} |\nabla \varphi_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d}\right] \mathrm{d}x + \int_{B_{r_\xi}} \frac{\varphi_\varepsilon \, m^2 \, |\vartheta_\varepsilon|^2}{\varepsilon (\int_{B_{r_\xi}} \vartheta_\varepsilon)^2} \,\mathrm{d}x \\
&\leq \frac{m^2 \; \omega_d}{\left(\int_{B_{r_\xi}} \vartheta_\varepsilon\right)^2} \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) = [1 + O(1)] \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon).
\end{aligned}$$

Passing to the infimum we get

$$h^d_{\varepsilon,\beta}(m,r,\tilde{r}) \leq \overline{h}^d_{\varepsilon,\beta}(m,r) + O(1). \tag{B.5}$$

Step 2: [$\overline{h}^d_{\varepsilon,\beta}(m,r) \leq h^d_{\varepsilon,\beta}(m,r,\tilde{r}) + o(1)$]
Consider a minimizing pair $(\vartheta_\varepsilon, \varphi_\varepsilon)$ such that

$$h^d_{\varepsilon,\beta}(m,r,\tilde{r}) = \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon).$$

Let χ be a smooth cutoff function such that $\chi(x) = 1$ if $|x| \leq \tilde{r}$ and $\chi(x) = 0$ if $|x| > \frac{r+\tilde{r}}{2}$ and set $v_\varepsilon = \chi\varphi_\varepsilon + (1-\chi)$. By construction $(\vartheta_\varepsilon, v_\varepsilon) \in \overline{Y}_{\varepsilon,\beta}(m,r)$, furthermore, since $\varphi_\varepsilon \in (0,1]$, it holds that $\varphi_\varepsilon \leq v_\varepsilon$ and $(1-\varphi_\varepsilon)^2 \geq (1-v_\varepsilon)^2$. Moreover as $v_\varepsilon \equiv \varphi_\varepsilon$ on $B_{\tilde{r}}$ we have $\int_{B_r} \varphi_\varepsilon |\vartheta_\varepsilon|^2 \, \mathrm{d}x = \int_{B_r} v_\varepsilon |\vartheta_\varepsilon|^2 \, \mathrm{d}x$. Eventually, we estimate the gradient component of the energy as follows

$$\begin{aligned}
\int_{B_r} \varepsilon^{p-d} |\nabla v_\varepsilon|^p \, \mathrm{d}x &= \int_{B_r} \varepsilon^{p-d} |\chi \nabla\varphi_\varepsilon + (\varphi_\varepsilon - 1)\nabla\chi|^p \, \mathrm{d}x \\
&\leq \int_{B_r} \varepsilon^{p-d} (|\nabla\varphi_\varepsilon| + |\nabla\chi|)^p \, \mathrm{d}x \\
&\leq \int_{B_r} \varepsilon^{p-d} |\nabla\varphi_\varepsilon|^p \, \mathrm{d}x + C(r,\chi)\left(\mathscr{G}^{1-1/p}_{\varepsilon,\beta}(\vartheta_\varepsilon, v_\varepsilon)\varepsilon^{\frac{p-d}{p}} + \varepsilon^{p-d}\right)
\end{aligned}$$

where we have used the inequality $(|a|+|b|)^p \leq |a|^p + C_p(|a|^{p-1}|b| + |b|^p)$ and Holder inequality. We get

$$\overline{h}^d_{\varepsilon,\beta}(m,r) \leq \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, v_\varepsilon) \leq \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) + O(\varepsilon^{\frac{p-d}{p}}) = f^{\tilde{r}}_\varepsilon(m,r) + o(1) \tag{B.6}$$

Step 3: Combining inequalities (B.5) and (B.6) we obtain

$$h^d_{\varepsilon,\beta}(m,r,\tilde{r}) - \overline{h}^d_{\varepsilon,\beta}(m,r) = o(1).$$

□

B.2 Study of the transition energy

Given two values $r_1 < r_2$ let us introduce the functional

$$\mathscr{T}^d(v;(r_1,r_2)) := \int_{r_1}^{r_2} t^{d-1}\left[|v'|^p + (1-v)^2\right] \, \mathrm{d}t$$

and for any triplet $(\xi,\ r_1,\ r_2) \in [0,1] \times \mathbf{R}^+ \times \mathbf{R}^+$ we set

$$q^d(\xi, r_1, r_2) := \inf\left\{\mathscr{T}^d(v;(r_1,r_2)) \ : \ v \in W^{1,p}(r_1,r_2),\ v(r_1) = \xi \text{ and } v(r_2) = 1\right\}. \tag{B.7}$$

This value represents the cost of the transition from ξ to 1 in the ring $B_{r_2} \setminus \overline{B}_{r_1}$. We will say that a function v is admissible for the triplet (ξ, r_1, r_2) if it is a competitor in the above minimization problem. Let us investigate the properties of the function introduced.

Lemma B.3. *For any fixed triplet* $(\xi, r_1, r_2) \in [0,1] \times \mathbf{R}_+ \times \mathbf{R}_+$ *the infimum in equation* (B.7) *is a minimum. Moreover there is a unique function achieving the minimum which is nondecreasing with range in the interval* $[\xi, 1]$*. Finally the function* q^d *satisfies the following properties*

1. $r_2 \mapsto q^d(\xi, r_1, r_2)$ *is nonincreasing,*
2. $r_1 \mapsto q^d(\xi, r_1, r_2)$ *is nondecreasing,*
3. $\xi \ \mapsto q^d(\xi, r_1, r_2)$ *is nonincreasing, and* $g(1, r_1, r_2) = 0$.

Recalling the definition (2.10) *of* q^d_∞*, we have* $q^d_\infty(\xi, \hat{r}) = q^d(\xi, r_1, \infty)$*, and* $q^d_\infty(0,0) > 0$*. Furthermore for any* $r > 0$ *the map* $\xi \mapsto q^d_\infty(\xi, r)$ *is convex and continuous on* $(0, +\infty)$*.*

Proof. Let $(\xi, r_1, r_2) \in [0,1] \times \mathbf{R}_+ \times \mathbf{R}_+$, the infimum is actually a minimum by means of the direct method of the calculus of variations. Such minimum is absolutely continuous on the interval (r_1, r_2) by Morrey's inequality and is unique since $\mathscr{T}^d(v; (r_1, r_2))$ is strictly convex in v. Let $v \in W^{1,p}(r_1, r_2)$ be a minimizer of (B.7) set

$$\overline{v} = \min\{\max(v, \xi), 1\}$$

then $\mathscr{T}^d(\overline{v}; (r_1, r_2)) \leq \mathscr{T}^d(v; (r_1, r_2))$ if $v \neq \overline{v}$. As a consequence for every minimizer of (B.7) we have $\xi \leq v \leq 1$. Similarly setting

$$\overline{v}(s) = \max\{v(t) : r_1 \leq t \leq s\}$$

we have $\mathscr{T}^d(\overline{v}; (r_1, r_2)) \leq \mathscr{T}^d(v; (r_1, r_2))$ if $v \neq \overline{v}$. Hence v is nondecreasing. Let us

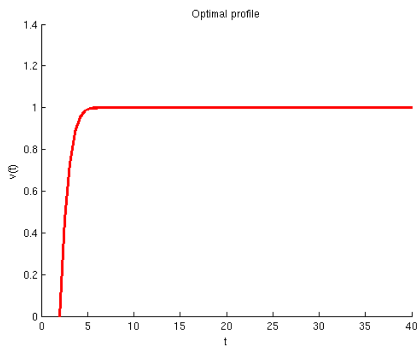


Figure B.1: Profile of the transition function $\overline{v}$ obtained by a numerical optimization (B.7) . Thus $\overline{v} = \operatorname{argmin} \mathscr{T}^d(v, r_1, r_2)$, for the choice of the parameters $p = 3$, $d = 2$, $r_1 = 2$, $r_2 = 40$ and $\xi = 0$.

now study the monotonicity of q^d. To do so let v be the minimizer for (ξ, r_1, r_2):

1. Let $\overline{r}_2 > r_2$ and let us extend v by 1 on the interval $(r_2, \overline{r}_2)$. We have

$$q^d(\xi, r_1, r_2) = \mathscr{T}^d(v; (r_1, r_2)) = \mathscr{T}^d(v; (r_1, \overline{r}_2)) \geq q^d(\xi, r_1, \overline{r}_2).$$

   Hence $r_2 \mapsto q^d$ is nonincreasing.

2. Let $0<\overline{r}_1<r_1$ and set $\Delta = r_1^d - \overline{r}_1^d > 0$ and $\overline{r}_2 = (r_2^d-\Delta)^{\frac{1}{d}} < r_2$. Define the diffeomorphism

$$\begin{aligned} \phi : (r_1, r_2) &\longrightarrow \quad (\overline{r}_1, \overline{r}_2), \\ s \quad &\longmapsto \left[s^d - \Delta\right]^{1/d}. \end{aligned} \tag{B.8}$$

   Let v be the minimizer of (B.7) and $\overline{v}(s) = v\circ\phi(s)$. Let us remark that $\phi'(s) = s^{d-1}/\phi(s)^{d-1}$, thus it holds

$$\begin{aligned} q^d(\xi, r_1, r_2) &= \int_{r_1}^{r_2} t^{d-1}\left[|v'|^p + (1-v)^2\right]\,\mathrm{d}t \\ &= \int_{\overline{r}_1}^{\overline{r}_2} \phi(s)^{d-1}\left[\frac{|\overline{v}'|^p}{|\phi'(s)|^p} + (1-\overline{v})^2\right]\phi(s)'\,\mathrm{d}s \\ &= \int_{\overline{r}_1}^{\overline{r}_2} s^{d-1}\left[\left(1+\frac{\Delta}{s^d-\Delta}\right)^{\frac{pd}{d}}|\overline{v}'|^p + (1-\overline{v})^2\right]\,\mathrm{d}s \\ &\geq q^d(\xi, \overline{r}_1, \overline{r}_2) \geq q^d(\xi, \overline{r}_1, r_2). \end{aligned}$$

   Therefore $r_1 \mapsto q^d$ is nondecreasing.

3. Let $0\leq \xi < \overline{\xi} \leq 1$ and v the absolutely continuous, nondecreasing minimizer of problem $q^d(\xi, r_1, r_2)$. Then there exists $\overline{r}\in(r_1,r_2)$ for which $v(\overline{r}) = \overline{\xi}$. Hence

$$q^d(\xi, r_1, r_2) \geq \mathscr{T}^d(v; (\overline{r}, r_2)) \geq q^d(\overline{\xi}, \overline{r}, r_2) \geq q^d(\overline{\xi}, r_1, r_2).$$

   Hence, $\xi\mapsto q^d$ is nonincreasing. Finally, for $\xi = 1$ consider the constant function $v\equiv 1$ to get $q^d(1, r_1, r_2) = 0$.

Indeed, in view of the monotonicity, for every r_1 and r_2 we have

$$g(0, r_1, r_2) \geq g(0, 0, +\infty) = q^d_\infty(0,0).$$

Let us show $q^d_\infty(0,0) > 0$. As a matter of facts, taken the minimizer v for the problem (2.10), there exists $r\in(0,+\infty)$ such that $v(r) = 1/2$ and we have

$$q^d_\infty(0,0) \geq \int_0^r t^{d-1}\left[|v'|^p + (1-v)^2\right]\,\mathrm{d}t = \int_0^r t^{d-1}|v'|^p\,\mathrm{d}t + \frac{r^d}{4\,d}.$$

A direct evaluation gives

$$\min\left\{\int_0^r t^{d-1}|v'|^p\,\mathrm{d}t \,:\, v(r) = 0 \text{ and } v(r) = 1/2\right\} = \frac{c}{r}$$

and we obtain the estimate

$$q^d_\infty(0,0) \geq \frac{c}{r} + \frac{r^d}{4\,d} > 0.$$

Lastly, let us show that for any r the function $q^d_\infty(\cdot, r)$ is convex. Consider two values ξ_1, $\xi_2\in(0,1)$ and the associated minimizers v_1, v_2 for the respective energy $q^d_\infty(\cdot, r)$.

Indeed, for any $\lambda \in (0,1)$ the function $\lambda v_1 + (1-\lambda)v_2$ is a competitor for the minimization problem $q_\infty^d(\lambda\xi_1 + (1-\lambda)\xi_2, r)$, therefore it holds

$$\begin{aligned} q_\infty^d(\lambda\xi_1 + (1-\lambda)\xi_2, r) &\le \int_r^\infty t^{d-1}\left[|\lambda v_1 - (1-\lambda)v_2|^p + (1-\lambda v_1 + (1-\lambda)v_2)^2\right]\,\mathrm{d}t \\ &\le \lambda\, q_\infty^d(\xi_1, r) + (1-\lambda)\, q_\infty^d(\xi_2, r). \end{aligned}$$

Thus $q_\infty^d(\cdot, r)$ is continuous in the open interval $(0,1)$. To show the continuity in 0 let ξ be small and $v = \operatorname{argmin} q_\infty^d(\xi, r)$. Set

$$f(t) := \begin{cases} \dfrac{1}{1-\sqrt{\xi}}(t-\xi), & t < \sqrt{\xi}, \\ t, & t \ge \sqrt{\xi}. \end{cases}$$

and observe that $f \circ v$ is a competitor for the problem $q_\infty^d(0, r)$. Then

$$\begin{aligned} q_\infty^d(0, r) &\le \int_r^\infty t^{d-1}\left[|(f\circ v)'|^p + (1 - f\circ v^2\right]\,\mathrm{d}t \\ &\le \frac{1}{(1-\sqrt{\xi})^p}\, q_\infty^d(\xi, r) + \int_r^\infty t^{d-1}\left[(1-f\circ v)^2 - (1-v)^2\right]\,\mathrm{d}t \end{aligned}$$

Let us estimate the second addend in the latter. By the definition of f we have

$$\begin{aligned} \int_r^\infty t^{d-1}\left[(1-f\circ v)^2 - (1-v)^2\right]\,\mathrm{d}t &= \int_{\{v<\sqrt{\xi}\}} t^{d-1}\left[(1-f\circ v - v)^2(v - f\circ v)^2\right]\,\mathrm{d}t \\ &\le 4\xi \int_{\{v<\sqrt{\xi}\}} t^{d-1}\,\mathrm{d}t \\ &\le \frac{4\xi}{(1-\sqrt{\xi})^2}\, q_\infty^d(\xi, r). \end{aligned}$$

Since $q_\infty^d(\cdot, r)$ is monotone we have

$$|q_\infty^d(0, r) - q_\infty^d(\xi, r)| \le \max\left\{\frac{1-(1-\sqrt{\xi})^p}{(1-\sqrt{\xi})^p}\ ,\ \frac{4\xi}{(1-\sqrt{\xi})^2}\right\}\kappa,$$

which shows that $q_\infty^d(\cdot, r)$ is continuous in 0. □

B.3 Proof of Proposition 2.1

We show that

$$\liminf_{\varepsilon\downarrow 0} \overline{h}_{\varepsilon,\beta}^d(m, r) \ge h_\beta^d(m)$$

then equation (2.11) easily follows from Lemma B.2. For $m = 0$ set $\vartheta = 0$ and $u = 1$, then $(\vartheta, \varphi) \in Y_{\varepsilon,\beta}(0, r)$ for any radius r and $\mathscr{G}_{\varepsilon,\beta}(\vartheta, \varphi; B_r) = 0$ for each ε. Now suppose $m > 0$ and let $\xi \in (\eta, 1)$. Consider the radially symmetric and monotone minimizing pair $(\vartheta_\varepsilon, \varphi_\varepsilon)$ of Lemma B.1 and r_ξ introduced in equation (B.3). Let us split the set of integration in the two sets B_{r_ξ} and $B_r \setminus B_{r_\xi}$, we obtain

$$\overline{h}^d_{\varepsilon,\beta}(m,r) = \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) \geq \underbrace{\int_{B_r \setminus B_{r_\xi}} \left[\varepsilon^{p-d} |\nabla \varphi_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d}\right] \mathrm{d}x}_{a_\varepsilon} + \underbrace{\int_{B_{r_\xi}} \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d} \mathrm{d}x + \int_{B_r} \frac{\varphi_\varepsilon |\vartheta_\varepsilon|^2}{\varepsilon} \mathrm{d}x}_{b_\varepsilon}. \tag{B.9}$$

We deal with each addend separately. First observe that by Cauchy-Schwarz inequality, it holds

$$\frac{m^2}{\int_{B_r} \frac{1}{\varphi_\varepsilon} \mathrm{d}x} \leq \int_{B_r} \varphi_\varepsilon \vartheta_\varepsilon^2 \, \mathrm{d}x.$$

Plugging the latter in the term b_ε of (B.9) we have

$$b_\varepsilon \geq \int_{B_{r_\xi}} \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d} \mathrm{d}x + \frac{m^2}{\varepsilon \left(\int_{B_r \setminus B_{r_\xi}} \frac{1}{\varphi_\varepsilon} \mathrm{d}x + \int_{B_{r_\xi}} \frac{1}{\varphi_\varepsilon} \mathrm{d}x\right)}$$

taking into account $\eta \leq \varphi_\varepsilon \leq \xi$ in B_{r_ξ}, $\xi \leq \varphi_\varepsilon \leq 1$ in $B_r \setminus B_{r_\xi}$ and $\eta = \beta\, \varepsilon^{d+1}$ we obtain

$$b_\varepsilon \geq \omega_d (1-\xi)^2 \left(\frac{r_\xi}{\varepsilon}\right)^d + \frac{m^2}{\frac{\omega_d}{\beta}\left(\frac{r_\xi}{\varepsilon}\right)^d + \omega_d \frac{\varepsilon r^d}{\xi}}. \tag{B.10}$$

Since $b_\varepsilon \leq \overline{h}^d_{\varepsilon,\beta}(m,r) \leq C(m)$ we deduce that r_ξ/ε belongs to a fixed compact subset $K = K(m,\xi)$ of $(0,+\infty)$. Up to extracting a subsequence, which we do not relabel, we can suppose r_ξ/ε to converge to some $\hat{r} > 0$. Let us now consider the term a_ε. Let v_ε be the radial profile of φ_ε

$$\begin{aligned} a_\varepsilon &= \int_{B_r \setminus B_{r_\xi}} \left[\varepsilon^{p-d} |\nabla \varphi_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon^d}\right] \mathrm{d}x \\ &= (d-1)\, \omega_d \int_{r_\xi/\varepsilon}^{r/\varepsilon} t^{d-1} \left[|v_\varepsilon'|^p + (1-v_\varepsilon)^2\right] \mathrm{d}t. \end{aligned}$$

With the notation introduced in section B.2 and Lemma B.3 therein we deduce

$$\liminf_{\varepsilon \downarrow 0} a_\varepsilon \geq (d-1)\, \omega_d \liminf_{\varepsilon \downarrow 0} q^d\left(\xi; (r_\xi/\varepsilon, r/\varepsilon)\right) \geq (d-1)\, \omega_d\, q^d_\infty(\xi, \hat{r}),$$

where q^d_∞ has been defined in (2.10). Combining inequality (B.10) and the latter we get

$$\lim_{\varepsilon \downarrow 0} \overline{h}^d_{\varepsilon,\beta}(m,r) \geq (d-1)\, \omega_d\, q^d_\infty(\xi, \hat{r}) + (1-\xi)^2\, \omega_d\, \hat{r}^d + \frac{\beta\, m^2}{\omega_d\, \hat{r}^d}.$$

Sending ξ to 0 we have, by continuity (Lemma B.3) $q^d_\infty(\xi, \hat{r}) \to q^d_\infty(0, \hat{r})$. Then taking the infimum in $\hat{r}$, we obtain

$$\liminf_{\varepsilon \downarrow 0} \overline{h}^d_{\varepsilon,\beta}(m,r) \geq \min_{\hat{r}} \left\{ (d-1)\, \omega_d\, q^d_\infty(0, \hat{r}) + \omega_d\, \hat{r}^d + \frac{\beta\, m^2}{\omega_d\, \hat{r}^d} \right\}.$$

Again by Lemma B.3 the function $q^d_\infty(0,\hat{r})$ is nondecreasing in $\hat{r}$, and $q^d_\infty(0,0) > 0$ therefore setting

$$\kappa := (d-1)\,\omega_d\, q^d_\infty(0,0) \leq h^d_\beta(m)$$

we conclude the proof of Proposition 2.1.

B.4 Proof of Proposition 2.2

Let $\delta > 0$, by Lemma B.3 for ε sufficiently small

$$q^d(\eta;(r_*, r/\varepsilon)) \leq q^d_\infty(0,r_*) + \delta.$$

Let

$$v_\delta(t) = \operatorname{argmin}\left\{ \mathscr{T}^d \left(v; \left(r_*, \frac{r}{\varepsilon}\right)\right)\ \mathrm{d}t \ :\ v\,(r_*) = \eta \text{ and } v\left(\frac{r}{\varepsilon}\right) = 1 \right\}.$$

and set

$$\varphi_\varepsilon(t) := \begin{cases} \eta & \text{for } 0 \leq t \leq r_*\varepsilon \\ v_\delta\left(\dfrac{t}{\varepsilon}\right) & \text{for } r_*\varepsilon \leq t \leq r \end{cases}$$

Set $\vartheta_\varepsilon(s)$ to be constant equal to $\frac{m}{\omega_d\,(\varepsilon r_*)^d}$ on the ball $B_{\varepsilon r_*}$ and zero outside. Indeed, the pair $(\vartheta_\varepsilon, \varphi_\varepsilon(|x|))$ belongs to $\overline{Y}_{\varepsilon,\beta}(m,r)$. That is because φ_ε is greater then η and attains value 1 at the border of B_r and

$$\int_{B_r} \vartheta_\varepsilon(x)\ \mathrm{d}x = \frac{m}{\omega_d(\varepsilon r_*)^d}\ \omega_d(\varepsilon r_*)^d = m.$$

Let us show that the pair $(\vartheta_\varepsilon, \varphi_\varepsilon)$ defined satisfy inequality (2.13). Taking advantage of the radial symmetry of the functions we get

$$\begin{aligned} \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) = \int_{\varepsilon r_*}^{r} t^{d-1} \left[\varepsilon^{p+d} |\varphi'_\varepsilon|^p + \frac{(1-\varphi_\varepsilon)}{\varepsilon^d} \right]\ \mathrm{d}t\ + \\ + \frac{(1-\eta)^2}{\varepsilon^d}\ \omega_d\ (\varepsilon r_*)^d + \frac{\eta}{\varepsilon} \left(\frac{m}{\omega_d\ (\varepsilon r_*)^d} \right)^2\ \omega_d\ (\varepsilon r_*)^d. \end{aligned}$$

By simplifying the expression and considering the change of variable $s = \frac{t}{\varepsilon}$ in the latter it holds

$$\begin{aligned} \mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) &= (d-1)\,\omega_d \int_{r_*}^{\frac{r}{\varepsilon}} s^{d-1}\left[|v'_\delta|^p + (1-v_\delta)\right]\ \mathrm{d}s + (1-\eta)^2\,\omega_d\, r^d_\delta + \frac{\eta\, m^2}{\omega_d\, r^d_*\, \varepsilon^{d+1}} \\ &\leq (d-1)\ \omega_d\ q^d\,(\eta;(r_*, r/\varepsilon)) + (1-\eta)^2\ \omega_d\ r^d_* + \frac{\eta}{\varepsilon}\frac{m^2}{\omega_d\ r^d_*} \end{aligned}$$

Then, by Lemma B.3 for ε sufficiently small we have

$$\mathscr{G}_{\varepsilon,\beta}(\vartheta_\varepsilon, \varphi_\varepsilon) \leq \frac{\beta\, m^2}{\omega_d\ r^d_*} +\ \omega_d\ r^d_* + (d-1)\ \omega_d\ q^d_\infty(0,r_*) + (d-1)\omega_{d-1}\delta = h^d_\beta(m) + C\delta,$$

which ends the proof of Proposition 2.2.

B.5 Proof of Proposition 2.3

Propositions 2.1, 2.2 and lemma B.2 ensure that

$$h_\beta^d(m) = \lim_{\varepsilon \downarrow 0} \overline{h}_{\varepsilon,\beta}^d(m,r) = \lim_{\varepsilon \downarrow 0} h_{\varepsilon,\beta}^d(m,r,\tilde{r}) \tag{B.11}$$

independently of the choices for r and $\tilde{r} < r$. For the sake of clarity we introduce

$$T(m,r) := \left\{ \frac{\beta\, m^2}{\omega_d\, r^d} + \ \omega_d\, r^d + (d-1)\, \omega_d\, q_\infty^d(0,r) \right\}$$

and recall that $h_\beta^d(m) = \min_r T(m,r)$ for $m > 0$ and $h_\beta^d(0) = 0$, see (2.9).

Proof.
Let us prove the continuity of h_β^d on $(0,+\infty)$. For $m_1, m_2 \in (0,+\infty)$ and for $i = 1,2$ let r_i be such that $h_\beta^d(m_i) = T(m_i, r_i)$. On one hand comparing with $r = 1$ it holds

$$\frac{m_i^2}{\omega_{d-1}\, r_i^d} \le h_\beta^d(m_i) \le T(m_i, 1) \tag{B.12}$$

on the other hand analougusly we have

$$\omega_{d-1}\, r_i^d \le h_\beta^d(m_i) \le T(m_i, 1). \tag{B.13}$$

Consequently $\omega_{d-1}\, r_i^d$ belongs to the compact set $[m_i/T(m_i,1), T(m_i,1)]$. Now remark that

$$h_\beta^d(m_1) \le T(m_1, r_2) = h_\beta^d(m_2) + T(m_1,r_2) - T(m_2,r_2)$$

thus

$$|h_\beta^d(m_1) - h_\beta^d(m_2)| \le |T(m_1,r_2) - T(m_2,r_2)| \le \frac{|m_1^2 - m_2^2|}{\omega_{d-1} \min\{r_1^d, r_2^d\}}$$

and taking into account inequality (B.12) we have

$$|h_\beta^d(m_1) - h_\beta^d(m_2)| \le (m_1 + m_2) \max\left\{ \frac{T(m_1,1)}{m_1^2}, \frac{T(m_2,1)}{m_2^2} \right\} |m_1 - m_2|.$$

Observing that $T(\cdot,1)$ is continuous we conclude that h_β^d is continuous on $(0,+\infty)$. Next, we see that h_β^d is non decreasing. Let $0 < m_1 < m_2$ and $r > 0$. Let $(\vartheta,\varphi) \in \overline{Y}_{\varepsilon,\beta}(m_2,r)$ such that $\mathscr{G}_{\varepsilon,\beta}(\vartheta,\varphi;B_r) = \overline{h}_{\varepsilon,\beta}^d(m_2,r)$. Set $\overline{\vartheta} = m_1\vartheta/m_2$ and remark that the pair $(\overline{\vartheta}, u)$ belongs to $\overline{Y}_{\varepsilon,\beta}(m_1,r)$. Therefore we have the following set of inequalities

$$\overline{h}_{\varepsilon,\beta}^d(m_1,r) \le \mathscr{G}_{\varepsilon,\beta}(\overline{\vartheta},\varphi;B_r) = \mathscr{G}_{\varepsilon,\beta}\left(\frac{m_1\vartheta}{m_2},\varphi;B_r\right) < \mathscr{G}_{\varepsilon,\beta}(\vartheta,\varphi;B_r) = \overline{h}_{\varepsilon,\beta}^d(m_2,r).$$

Passing to the limit as $\varepsilon \downarrow 0$ we obtain

$$h_\beta^d(m_1) \le h_\beta^d(m_2).$$

Let us now prove the sub-additivity. For a radius r consider the competitors $(\vartheta_j, u_j) \in \overline{Y}_{\varepsilon,\beta}(m_j, r)$ for $j = 1, 2$. Consider the ball B_{2r+1} centered in the origin and two points x_1, x_2 such that the balls $B_r(x_1)$, $B_r(x_2)$ are disjoint and contained in B_{2r+1}. Set

$$\overline{\vartheta}(x) := \begin{cases} \vartheta_1(x - x_1), & x \in B_r(x_1), \\ \vartheta_2(x - x_2), & x \in B_r(x_2), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\overline{u}(x) := \begin{cases} u_1(x - x_1), & x \in B_r(x_1), \\ u_2(x - x_2), & x \in B_r(x_2), \\ 1, & \text{otherwise}, \end{cases}$$

and observe that the pair $(\overline{\vartheta}, \overline{u})$ belongs to $\overline{Y}(m_1 + m_2, 2r + 1)$. Being the balls $B_r(x_j)$ disjoint we have

$$\begin{aligned} \overline{h}^d_{\varepsilon,\beta}(m_1 + m_2, r_1 + r_2) &\leq \mathscr{G}_{\varepsilon,\beta}(\vartheta_1(x - x_1), u_1(x - x_1); B_r(x_1)) + \\ &\quad + \mathscr{G}_{\varepsilon,\beta}(\vartheta_2(x - x_2), u_2(x - x_2); B_r(x_2)) \\ &= \overline{h}^d_{\varepsilon,\beta}(m_1, r) + h^d_{\varepsilon,\beta}(m_2, r). \end{aligned}$$

Passing to the limit as $\varepsilon \downarrow 0$, and recalling that it is independent of the choice of the radius, we get

$$h^d_a(m_1 + m_2) \leq h^d_a(m_1) + h^d_a(m_2).$$

□

We conclude the appendix by showing that

Lemma B.4. *For any sequence $\beta_i \downarrow 0$ it holds*

$$h^d_{\beta_i} \longrightarrow \kappa \mathbf{1}_{(0,\infty)}$$

pointwise.

Proof. We have already shown that $h^d_\beta(m) \geq \kappa$ for $m > 0$. For $m > 0$ choose $\hat{r} = (\sqrt{\beta} m)^{1/d}$, then by definition it holds

$$\kappa \leq h^d_\beta(m) \leq (d - 1)\, \omega_d\, q^d_\infty(0, (\sqrt{\beta} m)^{1/d}) + \omega_d \sqrt{\beta} m + \frac{\sqrt{\beta} m}{\omega_d}.$$

Finally simply recall that $(d - 1)\, \omega_d\, q^d_\infty(0, 0) = \kappa$ and that $q^d_\infty(0, \cdot)$ is continuous. □

Appendix C

Slicing of measures

We derive now some technical construction for divergence measure vector fields which are needed to reduce the $\Gamma - \liminf$ inequality of Chapters 4 and 5 to the lower-dimensional setting. In particular, we will introduce slices of a divergence measure vector field, which in the language of geometric measure theory correspond to slices of currents. We will slice in the direction of a unitary vector $\xi \in \mathbf{S}^{n-1}$ with orthogonal hyperplanes of the form

$$H_{\xi,t} = \pi_\xi^{-1}(t) \qquad \text{for the projection } \pi_\xi : \mathbf{R}^n \to \mathbf{R},\ \pi_\xi(x) = x \cdot \xi\,.$$

The orthogonal projection onto $H_{\xi,t}$ is denoted

$$\pi_{H_{\xi,t}}(x) = (I - \xi \otimes \xi)x + t\xi\,.$$

The slicing will essentially be performed via disintegration. Let σ be a compactly supported divergence measure vector field. By the Disintegration Theorem [AFP00, Thm. 2.28], for all $\xi \in \mathbf{S}^{n-1}$ and almost all $t \in \mathbf{R}$ there exists a unique measure $\nu_{\xi,t} \in \mathcal{M}(H_{\xi,t})$ such that

$$\|\nu_{\xi,t}\|_{\mathcal{M}} = 1 \quad \text{ and } \quad \sigma \cdot \xi = \nu_{\xi,t} \otimes {\pi_\xi}_\# |\sigma \cdot \xi|(t)\,.$$

We decompose ${\pi_\xi}_\# |\sigma \cdot \xi|$ into its absolutely continuous and singular part according to

$$ {\pi_\xi}_\# |\sigma \cdot \xi| = \sigma_\xi(t)\ \mathrm{d}t + \sigma_\xi^\perp$$

for $\mathrm{d}t$ the Lebesgue measure on $\mathbf{R}$.

Lemma C.1. *For any $\xi \in \mathbf{S}^{n-1}$ and any compactly supported divergence measure vector field σ we have $\sigma_\xi^\perp = 0$, that is, the measure ${\pi_\xi}_\# |\sigma \cdot \xi| = \sigma_\xi(t)\ \mathrm{d}t$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$. Moreover, for almost all $t \in \mathbf{R}$ and any compactly supported $\theta \in \mathcal{C}^\infty(\mathbf{R}^n)$ we have*

$$\sigma_\xi(t) \int_{H_{\xi,t}} \theta\ \mathrm{d}\nu_{\xi,t} = \int_{\{\xi \cdot x < t\}} \nabla\theta \cdot\ \mathrm{d}\sigma + \int_{\{\xi \cdot x < t\}} \theta\ \mathrm{d}\,\mathrm{div}\,\sigma\,. \tag{C.1}$$

Proof. Abbreviate $H = \xi^\perp = H_{\xi,0}$ with corresponding orthogonal projection π_H, let $\phi \in \mathcal{C}^\infty(H)$ and $\psi \in \mathcal{C}^\infty(\mathbf{R})$ be compactly supported, and define

$$I(\phi, \psi) = \int_{\mathbf{R}^n} \phi(\pi_H(x))\psi(\pi_\xi(x))\ \mathrm{d}(\sigma \cdot \xi)(x)\,.$$

Introducing $\Psi(t) = \int_{-\infty}^{t} \psi(s)\ \mathrm{d}s$ we obtain via the chain and product rule

$$(\phi \circ \pi_H)(\psi \circ \pi_\xi)\xi = (\phi \circ \pi_H)\nabla[\Psi \circ \pi_\xi] = \nabla[(\phi \circ \pi_H)(\Psi \circ \pi_\xi)] - \nabla[\phi \circ \pi_H](\Psi \circ \pi_\xi)$$

so that (denoting by χ_A the characteristic function of a set A)

$$\begin{aligned}
I(\phi,\psi) &= \int_{\mathbf{R}^n} \nabla[(\phi \circ \pi_H)(\Psi \circ \pi_\xi)] \cdot\ \mathrm{d}\sigma - \int_{\mathbf{R}^n} \nabla[\phi \circ \pi_H](\Psi \circ \pi_\xi) \cdot\ \mathrm{d}\sigma \\
&= -\int_{\mathbf{R}^n} (\phi \circ \pi_H)(\Psi \circ \pi_\xi)\ \mathrm{d}\operatorname{div}\sigma - \int_{\mathbf{R}^n} \nabla[\phi \circ \pi_H](\Psi \circ \pi_\xi) \cdot\ \mathrm{d}\sigma \\
&= -\int_{\mathbf{R}^n} \phi(\pi_H(x)) \left(\int_{-\infty}^{\pi_\xi(x)} \psi(s)\ \mathrm{d}s \right) \mathrm{d}\operatorname{div}\sigma(x) \\
&\quad - \int_{\mathbf{R}^n} \nabla[\phi \circ \pi_H](x) \left(\int_{-\infty}^{\pi_\xi(x)} \psi(s)\ \mathrm{d}s \right) \cdot\ \mathrm{d}\sigma \\
&= -\int_{\mathbf{R}^n} \phi(\pi_H(x)) \left(\int_{\mathbf{R}} \chi_{\{\xi\cdot x \geq s\}} \psi(s)\ \mathrm{d}s \right) \mathrm{d}\operatorname{div}\sigma(x) \\
&\quad - \int_{\mathbf{R}^n} \nabla[\phi \circ \pi_H](x) \left(\int_{\mathbf{R}} \chi_{\{\xi\cdot x \geq s\}} \psi(s)\ \mathrm{d}s \right) \cdot\ \mathrm{d}\sigma\,.
\end{aligned}$$

(Note that we could just as well have used $\chi_{\{\xi\cdot x > s\}}$ instead of $\chi_{\{\xi\cdot x \geq s\}}$, which would ultimately lead to integration domains $\{\xi \cdot x \leq t\}$ in (C.1); for almost all t this will be the same.) Applying the Fubini–Tonelli Theorem we obtain

$$\begin{aligned}
I(\phi,\psi) &= -\int_{\mathbf{R}} \psi(s) \left[\int_{\{\xi\cdot x \geq s\}} \nabla[\phi \circ \pi_H](x) \cdot\ \mathrm{d}\sigma(x) + \int_{\{\xi\cdot x \geq s\}} \phi(\pi_H(x))\ \mathrm{d}\operatorname{div}\sigma(x) \right] \mathrm{d}s \\
&= \int_{\mathbf{R}} \psi(s) \left[\int_{\{\xi\cdot x < s\}} \nabla[\phi \circ \pi_H](x) \cdot\ \mathrm{d}\sigma(x) + \int_{\{\xi\cdot x < s\}} \phi(\pi_H(x))\ \mathrm{d}\operatorname{div}\sigma(x) \right] \mathrm{d}s\,,
\end{aligned}$$

where in the second step we just added $0 = \int_{\mathbf{R}^n} \nabla[\phi \circ \pi_H] \cdot\ \mathrm{d}\sigma + \int_{\mathbf{R}^n} \phi \circ \pi_H\ \mathrm{d}\operatorname{div}\sigma$ in the square brackets. On the other hand, using the disintegration of $\sigma \cdot \xi$ we also have

$$\begin{aligned}
I(\phi,\psi) &= \int_{\mathbf{R}^n} \phi(\pi_H(x))\psi(\pi_\xi(x))\ \mathrm{d}(\sigma\cdot\xi)(x) \\
&= \int_{\mathbf{R}} \left[\psi(s) \int_{H_{\xi,s}} \phi(\pi_H(y))\ \mathrm{d}\nu_{\xi,s}(y) \right] (\sigma_\xi(s)\ \mathrm{d}s + \ \mathrm{d}\sigma_\xi^\perp(s))\,.
\end{aligned}$$

Comparing both expressions for $I(\phi,\psi)$ we can identify

$$\begin{aligned}
&\left[\int_{H_{\xi,s}} \phi(\pi_H(y))\ \mathrm{d}\nu_{\xi,s}(y) \right] (\sigma_\xi(s)\ \mathrm{d}s + \ \mathrm{d}\sigma_\xi^\perp(s)) \\
&\qquad = \left[\int_{\{\xi\cdot x < s\}} \nabla[\phi \circ \pi_H](x) \cdot\ \mathrm{d}\sigma(x) + \int_{\{\xi\cdot x < s\}} \phi(\pi_H(x))\ \mathrm{d}\operatorname{div}\sigma(x) \right] \mathrm{d}s\,.
\end{aligned}$$

Since the right-hand side has no singular component with respect to the Lebesgue measure, we deduce $\left[\int_{H_{\xi,s}} \phi(\pi_H(y))\ \mathrm{d}\nu_{\xi,s}(y) \right] \sigma_\xi^\perp(s) = 0$. Now note that any compactly supported function in $\mathcal{C}^0(\mathbf{R}^n)$ or $\mathcal{C}^1(\mathbf{R}^n)$ can be arbitrarily well approximated (in the respective norm) by finite linear combinations of tensor products $(\phi \circ \pi_H)(\psi \circ \pi_\xi)$ with $\phi \in \mathcal{C}^\infty(H)$ and $\psi \in \mathcal{C}^\infty(\mathbf{R})$ with compact support. Thus, the above implies $\int_{\mathbf{R}} \left[\int_{H_{\xi,s}} \theta(y)\ \mathrm{d}\nu_{\xi,s}(y)\right]\ \mathrm{d}\sigma_\xi^\perp(s) = 0$ for any compactly supported $\theta \in \mathcal{C}^0(\mathbf{R}^n)$ so that

$$\nu_{\xi,s} \otimes \sigma_\xi^\perp(s) = 0 \qquad \text{and thus} \qquad \sigma_\xi^\perp(s) = 0\,.$$

Summarizing, we have $\sigma \cdot \xi = \sigma_\xi(s)\nu_{\xi,s} \otimes\ \mathrm{d}s$ and

$$\begin{aligned}\int_{H_{\xi,s}} & \phi(\pi_H(x))\sigma_\xi(s)\ \mathrm{d}\nu_{\xi,s}(x)\ \mathrm{d}s \\ &= \int_{\{\xi\cdot x<s\}} \nabla[\phi\circ\pi_H](x)\cdot\ \mathrm{d}\sigma(x) + \int_{\{\xi\cdot x<s\}} \phi(\pi_H(x))\,\mathrm{d}\,\mathrm{div}\,\sigma(x)\end{aligned}$$

for all compactly supported $\phi \in \mathcal{C}^\infty(H)$. Note that the right-hand side is left-continuous in s so that the left-hand side is as well. Consequently, $\sigma_\xi(s)\nu_{\xi,s}$ is left-continuous in s with respect to weak-* convergence. Now let $\chi \in \mathcal{C}^\infty(\mathbf{R})$ with $\chi = 1$ on $(-\infty, 0]$, $\chi = 0$ on $[1,\infty)$, and $0 \le \chi \le 1$, and define for $\rho > 0$

$$\chi^\rho(x) = \chi\left(\tfrac{\pi_\xi(x)-t}{\rho}\right), \quad \sigma^\rho = \chi^\rho\sigma, \quad \mu^\rho = \chi^\rho\,\mathrm{div}\,\sigma, \quad \sigma^\rho_{\xi,s} = \tfrac{1}{\rho}\chi'\left(\tfrac{\pi_\xi(\cdot)-t}{\rho}\right)\sigma_\xi(s)\nu_{\xi,s}.$$

In the distributional sense we have

$$\mathrm{div}\,\sigma^\rho = \mu^\rho + \sigma^\rho_{\xi,s} \otimes\ \mathrm{d}s$$

so that for any compactly supported $\theta \in \mathcal{C}^\infty(\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^n} \nabla\theta\cdot\ \mathrm{d}\sigma^\rho + \int_{\mathbf{R}^n} \theta\ \mathrm{d}\mu^\rho = -\int_{\mathbf{R}}\int_{H_{\xi,s}} \theta\ \mathrm{d}\sigma^\rho_{\xi,s}\ \mathrm{d}s\,.$$

Letting $\rho \to 0$ and using the left-continuity of $\sigma_\xi(s)\nu_{\xi,s}$ in s we arrive at (C.1). □

We now define the slice of a divergence measure vector field as the measure obtained via disintegration with respect to the one-dimensional Lebesgue measure.

Definition 3 (Sliced sets, functions, and measures)**.** Let $\xi \in \mathbf{S}^{n-1}$ and $t \in \mathbf{R}$.

1. For $A \subset \mathbf{R}^n$ we define the *sliced set* $A_{\xi,t} = A \cap H_{\xi,t}$.
2. For $f : A \to \mathbf{R}$ we define the *sliced function* $f_{\xi,t} : A_{\xi,t} \to \mathbf{R}$, $f_{\xi,t} = f|_{A_{\xi,t}}$. For $f : A \to \mathbf{R}^n$ we define $f_{\xi,t} : A_{\xi,t} \to \mathbf{R}^n$, $f_{\xi,t} = \xi \cdot f|_{A_{\xi,t}}$.
3. We define the *sliced measure* of a compactly supported divergence measure vector field σ as
$$\sigma_{\xi,t} = \sigma_\xi(t)\ \nu_{\xi,t}\,.$$
By Lemma C.1 it holds $\sigma\cdot\xi = \sigma_{\xi,t} \otimes\ \mathrm{d}t$.

Remark 6 (Properties of sliced functions and measures)**.** 1. By Fubini's theorem it follows that for any function f of Sobolev-type $W^{m,p}$ the corresponding sliced function $f_{\xi,t}$ is well-defined and also of Sobolev-type $W^{m,p}$ for almost all $\xi \in \mathbf{S}^{n-1}$ and $t \in \mathbf{R}$. For the same reason, strong convergence $f_j \to_{j\to\infty} f$ in $W^{m,p}$ implies strong convergence $(f_j)_{\xi,t} \to f_{\xi,t}$ in $W^{m,p}$ on the sliced domain.

2. The definitions of sliced functions and measures are consistent in the following sense. If we identify a Lebesgue function f with the measure $\chi = f\mathcal{L}$ for $\mathcal{L}$ the Lebesgue measure, then the same identification holds between $f_{\xi,t}$ and $\chi_{\xi,t}$ for almost all $\xi \in \mathbf{S}^{n-1}$ and $t \in \mathbf{R}$.

3. Let σ be a divergence measure vector field, then the properties [AFP00, Thm. 2.28] of the disintegration $\sigma \cdot \xi = \nu_{\xi,t} \otimes \pi_{\xi_\#}|\sigma \cdot \xi|(t) = \nu_{\xi,t} \otimes \sigma_\xi(t)\ \mathrm{d}t = \sigma_{\xi,t} \otimes\ \mathrm{d}t$ immediately imply the following. The map $t \mapsto \|\sigma_{\xi,t}\|_{\mathcal{M}}$ is integrable and satisfies

$$\int_{\mathbf{R}} \|\sigma_{\xi,t}\|_{\mathcal{M}}\ \mathrm{d}t = \int_{\mathbf{R}} \sigma_\xi(t)\ \mathrm{d}t = \|\sigma \cdot \xi\|_{\mathcal{M}}\,.$$

Furthermore, for any measurable function $f : \mathbf{R}^n \to \mathbf{R}$, absolutely integrable with respect to $|\sigma \cdot \xi|$, it holds

$$\int_{\mathbf{R}^n} f(x)\ \mathrm{d}\sigma{\cdot}\xi = \int_{\mathbf{R}} \int_{H_{\xi,t}} f(x)\ \mathrm{d}\nu_{\xi,t}(x)\ \mathrm{d}\pi_{\xi_\#}|\sigma{\cdot}\xi|(t) = \int_{\mathbf{R}} \int_{H_{\xi,t}} f(x)\ \mathrm{d}\sigma_{\xi,t}(x)\ \mathrm{d}t\,.$$

We briefly relate our definition of sliced measures to other notions of slices from the literature.

Remark 7 (Notions of slices)**.** 1. Let $\mathrm{Lip}(A)$ denote the set of bounded Lipschitz functions on $A \subset \mathbf{R}^n$. An alternative definition of the slice of a divergence measure vector field σ was introduced by Šilhavý [Š07] as the linear operator

$$\sigma_{\xi,t} : \mathrm{Lip}(H_{\xi,t}) \to \mathbf{R}, \quad \sigma_{\xi,t}(\varphi|_{H_{\xi,t}}) = \lim_{\delta \searrow 0} \frac{1}{\delta} \int_{\{x\in\mathbf{R}^n \,|\, t-\delta<x\cdot\xi<t\}} \varphi\xi \cdot\ \mathrm{d}\sigma \tag{C.2}$$

for all $\varphi \in \mathrm{Lip}(\mathbf{R}^n)$ (the right-hand side is well-defined and only depends on $\varphi|_{H_{\xi,t}}$ [Š07, Thm. 3.5 & Thm. 3.6]). This $\sigma_{\xi,t}$ equals the so-called normal trace of σ on $H_{\xi,t}$ (see [Š07] for its definition and properties). In general it is not a measure but continuous on $\mathrm{Lip}(H_{\xi,t})$ in the sense

$$\sigma_{\xi,t}(\varphi) \le (\|\sigma\|_{\mathcal{M}} + \|\operatorname{div}\sigma\|_{\mathcal{M}})\|\varphi\|_{W^{1,\infty}} \quad \text{for all } \varphi \in \mathrm{Lip}(H_{\xi,t})\,.$$

2. Interpreting a divergence measure vector field as a 1-current or a flat 1-chain, Šilhavý's definition of $\sigma_{\xi,t}$ is identical to the classical slice of σ on $H_{\xi,t}$ as for instance defined in [Whi99b] or [Fed69, 4.3.1] (note that Šilhavý's definition corresponds to [Š07, (3.8)], whose analogue for currents is [Fed69, 4.3.2(5)]).

3. Our notion of a sliced measure from Definition 3 is equivalent to both abovementioned notions. Indeed, (C.1) implies

$$\sigma_{\xi,t} = (\operatorname{div}\sigma)\llcorner\{x\cdot\xi<t\} - \operatorname{div}(\sigma\llcorner\{x\cdot\xi<t\})\,,$$

which shows that the sliced measure represents the normal flux through the hyperplane $H_{\xi,t} = \{x \cdot \xi = t\}$. This, however, is the same characterization as given in [Š07, (3.6)] and [Fed69, 4.2.1] for both above notions of slices.

We conclude the section with several properties needed for the $\Gamma-\liminf$ inequality. The following result makes use of the Kantorovich–Rubinstein norm (see for instance [LLSV14, eq. (2) & (5)]; in geometric measure theory it is known as the flat norm) on $\mathcal{M}(\mathbf{R}^n)$, defined by

$$\begin{aligned}\|\mu\|_{\mathrm{KR}} &= \inf\{\|\mu_1\|_{\mathcal{M}} + \|\mu_2\|_{\mathcal{M}} \,|\, \mu_1 \in \mathcal{M}(\mathbf{R}^n),\ \mu_2 \in \mathcal{M}(\mathbf{R}^n;\mathbf{R}^n),\ \mu = \mu_1 + \operatorname{div}\mu_2\} \\ &= \sup\left\{\int_\Omega f\ \mathrm{d}\mu \,\middle|\, f \text{ Lipschitz with constant 1, } |f| \le 1\right\}.\end{aligned}$$

For measures of uniformly bounded support and uniformly bounded mass it is known to metrize weak-$*$ convergence (see for instance [BW17, Rem. 2.29(3)-(4)]). We will furthermore make use of the following fact. Let $T_s : x \mapsto x - s\xi$ be the translation by s in direction $-\xi$. It is straightforward to check that for any divergence measure vector field $\mu \in \mathcal{M}(\mathbf{R}^n;\mathbf{R}^n)$ we have

$$\operatorname{div}(\pi_{H_{\xi,t}\#}(\mu - \mu\cdot\xi\,\xi)) = \pi_{H_{\xi,t}\#}(\operatorname{div}\mu)\,.$$

As a consequence, for any $\mu \in \mathcal{M}(H_{\xi,t})$ and $\nu \in \mathcal{M}(H_{\xi,t+s})$ we have

$$\|\mu - \nu\|_{\mathrm{KR}} \ge \|\mu - T_{s\#}\nu\|_{\mathrm{KR}}\,.$$

Indeed, let $\delta > 0$ arbitrary and $\mu_1 \in \mathcal{M}(\mathbf{R}^n)$, $\mu_2 \in \mathcal{M}(\mathbf{R}^n;\mathbf{R}^n)$ with $\mu-\nu = \mu_1+\operatorname{div}\mu_2$ such that $\|\mu-\nu\|_{\mathrm{KR}} \ge \|\mu_1\|_{\mathcal{M}} + \|\mu_2\|_{\mathcal{M}} - \delta$, then $\tilde\mu_1 = \pi_{H_{\xi,t}\#}\mu_1$ and $\tilde\mu_2 = \pi_{H_{\xi,t}\#}(\mu_2 - \mu_2\cdot\xi\,\xi)$ satisfy $\mu - T_{s\#}\nu = \tilde\mu_1 + \operatorname{div}\tilde\mu_2$ and thus

$$\|\mu - T_{s\#}\nu\|_{\mathrm{KR}} \le \|\tilde\mu_1\|_{\mathcal{M}} + \|\tilde\mu_2\|_{\mathcal{M}} \le \|\mu_1\|_{\mathcal{M}} + \|\mu_2\|_{\mathcal{M}} \le \|\mu-\nu\|_{\mathrm{KR}} + \delta\,.$$

Theorem C.1 (Weak convergence of sliced measures)**.** *Let $\sigma^j \overset{*}{\rightharpoonup} \sigma$ as $j \to \infty$ for a sequence $\{\sigma^j\}$ of compactly supported divergence measure vector fields with uniformly bounded $\|\operatorname{div}\sigma^j\|_{\mathcal{M}}$. Then for almost all $\xi \in \mathbf{S}^{n-1}$ and $t \in \mathbf{R}$ we have*

$$\sigma^j_{\xi,t} \overset{*}{\rightharpoonup} \sigma_{\xi,t}\,.$$

Proof. It suffices to show $\sigma^j_{\xi,t} \overset{*}{\rightharpoonup} \sigma_{\xi,t}$ for a subsequence.

Consider the measures $\nu^j = |\sigma^j| + |\operatorname{div}\sigma^j|$. Since $\|\nu^j\|_{\mathcal{M}}$ is uniformly bounded, a subsequence converges weakly-$*$ to some compactly supported nonnegative $\nu \in \mathcal{M}(\mathbf{R}^n)$ (the subsequence is still indexed by j). For $I \subset \mathbf{R}$ introduce the notation $H_{\xi,I} = \bigcup_{t\in I} H_{\xi,t}$. Then for almost all $t \in \mathbf{R}$, $\nu(H_{\xi,[t-s,t+s]}) \to 0$ as well as $(|\sigma|+|\operatorname{div}\sigma|)(H_{\xi,[t-s,t+s]}) \to 0$ as $s \searrow 0$. For such a t we show convergence of $\sigma^j_{\xi,t} - \sigma_{\xi,t}$ to zero in the Kantorovich–Rubinstein norm which implies weak-$*$ convergence. To this end fix some arbitrary $\delta > 0$. Given $\zeta > 0$ let $\rho_\zeta = \rho(\cdot/\zeta)/\zeta$ for a nonnegative smoothing kernel $\rho \in \mathcal{C}^\infty(\mathbf{R})$ with support in $[-1,1]$ and $\int_{\mathbf{R}}\rho\ \mathrm{d}t = 1$. For any compactly supported divergence measure vector field λ we now define the convolved slice $\lambda_{\xi,\zeta,t}$ by

$$\begin{aligned}\int_{H_{\xi,t}} g\ \mathrm{d}\lambda_{\xi,\zeta,t} &= \int_{\mathbf{R}} \rho_\zeta(-s)\int_{H_{\xi,t}} g\ \mathrm{d}T_{s\#}\lambda_{\xi,t+s}\ \mathrm{d}s \\ &= \int_{\mathbf{R}} \rho_\zeta(-s)\int_{H_{\xi,t}} g\circ T_s\ \mathrm{d}\lambda_{\xi,t+s}\ \mathrm{d}s \quad \forall g \in \mathcal{C}(H_{\xi,t})\,,\end{aligned}$$

where $T_s : x \mapsto x - s\xi$ is the translation by s in direction $-\xi$. By Remark 6(3) we have $\sigma^j_{\xi,\zeta,t} \stackrel{*}{\rightharpoonup} \sigma_{\xi,\zeta,t}$. Furthermore, there exist $\zeta > 0$ and $J \in \mathbf{N}$ such that $\|\sigma_{\xi,t} - \sigma_{\xi,\zeta,t}\|_{\mathrm{KR}} \leq \frac{\delta}{3}$ and $\|\sigma^j_{\xi,t} - \sigma^j_{\xi,\zeta,t}\|_{\mathrm{KR}} \leq \frac{\delta}{3}$ for all $j \geq J$. Indeed, for a compactly supported divergence measure vector field λ we have

$$\begin{aligned}
\|\lambda_{\xi,t} - \lambda_{\xi,\zeta,t}\|_{\mathrm{KR}} &\leq \int_{\mathbf{R}} \rho_\zeta(-s)\|\lambda_{\xi,t} - T_{s\#}\lambda_{\xi,t+s}\|_{\mathrm{KR}} \;\mathrm{d}s \\
&\leq \int_{\mathbf{R}} \rho_\zeta(-s)\|\lambda_{\xi,t} - \lambda_{\xi,t+s}\|_{\mathrm{KR}} \;\mathrm{d}s \\
&= \int_{\mathbf{R}} \rho_\zeta(-s)\|\operatorname{div}(\lambda \llcorner H_{\xi,[t,t+s)}) - (\operatorname{div}\lambda)\llcorner H_{\xi,[t,t+s)}\|_{\mathrm{KR}} \;\mathrm{d}s \\
&\leq \int_{\mathbf{R}} \rho_\zeta(-s)\left[|\lambda|(H_{\xi,[t,t+s)}) + |\operatorname{div}\lambda|(H_{\xi,[t,t+s)})\right] \;\mathrm{d}s \\
&\leq |\lambda|(H_{\xi,[t-\zeta,t+\zeta]}) + |\operatorname{div}\lambda|(H_{\xi,[t-\zeta,t+\zeta]})\,,
\end{aligned}$$

where in the equality we employed Remark 7(3). Thus, we can simply pick ζ such that $|\sigma|(H_{\xi,[t-\zeta,t+\zeta]}) + |\operatorname{div}\sigma|(H_{\xi,[t-\zeta,t+\zeta]}) \leq \frac{\delta}{3}$ and $\nu(H_{\xi,[t-\zeta,t+\zeta]}) \leq \frac{\delta}{6}$, while we choose J such that $(\nu^j - \nu)(H_{\xi,[t-\zeta,t+\zeta]}) \leq \frac{\delta}{6}$ for all $j > J$. Now let $\bar{J} \geq J$ such that $\|\sigma^j_{\xi,\zeta,t} - \sigma_{\xi,\zeta,t}\|_{\mathrm{KR}} \leq \frac{\delta}{3}$ for all $j \geq \bar{J}$, then we obtain

$$\|\sigma^j_{\xi,t} - \sigma_{\xi,t}\|_{\mathrm{KR}} \leq \|\sigma^j_{\xi,t} - \sigma^j_{\xi,\zeta,t}\|_{\mathrm{KR}} + \|\sigma^j_{\xi,\zeta,t} - \sigma_{\xi,\zeta,t}\|_{\mathrm{KR}} + \|\sigma_{\xi,\zeta,t} - \sigma_{\xi,t}\|_{\mathrm{KR}} \leq \delta$$

for all $j > \bar{J}$. The arbitrariness of δ concludes the proof. □

Remark 8 (Flat convergence of sliced currents)**.** The convergence from Theorem C.1 is consistent with the following property of slices of 1-currents: If σ^j, $j \in \mathbf{N}$, is a sequence of 1-currents of finite mass with $\sigma^j \to \sigma$ in the flat norm, then (potentially after choosing a subsequence) $\sigma^j_{\xi,t} \to \sigma_{\xi,t}$ in the flat norm for almost every $\xi \in \mathbf{S}^{n-1}$, $t \in \mathbf{R}$ (see [CDRMS17, step 2 in proof of Prop. 2.5] or [Whi99b, Sec. 3]).

Remark 9 (Characterization of sliced measures)**.** 1. Let the compactly supported divergence measure vector field σ be countably 1-rectifiable, that is, $\sigma = \theta m \mathcal{H}^1 \llcorner S$ for a countably 1-rectifiable set $S \subset \mathbf{R}^n$ and $\mathcal{H}^1\llcorner S$-measurable functions $m : S \to [0,\infty)$ and $\theta : S \to \mathbf{S}^{n-1}$, tangent to S $\mathcal{H}^1$-almost everywhere. Then the coarea formula for rectifiable sets [Fed69, Thm. 3.2.22] implies $|\theta \cdot \xi|\mathcal{H}^1 \llcorner S = \mathcal{H}^0 \llcorner S_{\xi,t} \otimes \mathcal{H}^1(t)$ so that

$$\int_{\mathbf{R}^n} f \;\mathrm{d}\sigma \cdot \xi = \int_S f m \theta \cdot \xi \;\mathrm{d}\mathcal{H}^1 = \int_{\mathbf{R}} \int_{S_{\xi,t}} f m \operatorname{sgn}(\xi \cdot \theta) \;\mathrm{d}\mathcal{H}^0 \;\mathrm{d}t$$

for any Borel function f. Hence, for almost all t,

$$\sigma_{\xi,t} = \operatorname{sgn}(\xi \cdot \theta)\, m \mathcal{H}^0 \llcorner S_{\xi,t}\,.$$

The choice $f = \frac{\tau(m)}{m}\operatorname{sgn}(\xi \cdot \theta)$ yields

$$\int_S \tau(m)|\theta \cdot \xi| \;\mathrm{d}\mathcal{H}^1 = \int_{\mathbf{R}} \int_{S_{\xi,t}} \tau(m) \;\mathrm{d}\mathcal{H}^0 \;\mathrm{d}t\,.$$

2. Let the compactly supported divergence measure vector field σ be $\mathcal{H}^1$-diffuse, that is, it is singular with respect to the one-dimensional Hausdorff measure on any countably 1-rectifiable set. Then for almost all $\xi \in \mathbf{S}^{n-1}$ and $t \in \mathbf{R}$, $\sigma_{\xi,t}$ is $\mathcal{H}^0$-diffuse, that is, it does not contain any atoms. Indeed, let $\sigma_{\xi,t}$ have an atom at $x \in H_{\xi,t}$, then

$$x \in \Theta(\sigma) = \left\{ x \in \mathbf{R}^n \,\middle|\, \liminf_{\rho \searrow 0} |\sigma|(B_\rho(x))/\rho > 0 \right\} ,$$

where $B_\rho(x)$ denotes the open ball of radius ρ centred at x. This can be deduced as follows. Let $\phi \in \mathcal{C}^\infty(\mathbf{R})$ be smooth and even with support in $(-1,1)$ and $\phi(0) = \operatorname{sgn}(\sigma_{\xi,t}(\{x\}))$. Further abbreviate $K = \max_{x\in\mathbf{R}} |\phi'(x)| > 0$ and $\phi_\rho = \phi(|\cdot - x|/\rho)$ for any $\rho > 0$. Equation (C.1) now implies

$$\begin{aligned}\int_{H_{\xi,t}} \phi_\rho \;\mathrm{d}\sigma_{\xi,t} &= \int_{\{\xi\cdot x<t\}} \phi_\rho \;\mathrm{d}\operatorname{div}\sigma + \int_{\{\xi\cdot x<t\}} \nabla\phi_\rho \cdot \;\mathrm{d}\sigma \\ &\le \int_{\{\xi\cdot x<t\}} \phi_\rho \;\mathrm{d}\operatorname{div}\sigma + K\frac{|\sigma|(B_\rho(x))}{\rho} .\end{aligned}$$

Taking on both sides the limit inferior as $\rho \to 0$ we obtain

$$|\sigma_{\xi,t}|(\{x\}) \le K \liminf_{\rho\searrow 0} |\sigma|(B_\rho(x))/\rho,$$

as desired. As a result, for a given ξ the set of t such that $\sigma_{\xi,t}$ is not $\mathcal{H}^0$-diffuse is a subset of $\pi_\xi(\Theta)$. Thus it remains to show that for almost all $\xi \in \mathbf{S}^{n-1}$ the set $\pi_\xi(\Theta)$ is a Lebesgue-nullset. Writing

$$\Theta = \bigcup_{p\in\mathbf{N}} \Theta_p \quad \text{for } \Theta_p = \left\{ x \in \mathbf{R}^n \,\middle|\, \liminf_{\rho\searrow 0} \sigma(B_\rho(x))/\rho \ge \frac{1}{p} \right\} ,$$

it actually suffices to show that $\pi_\xi(\Theta)$ is a Lebesgue-nullset for any $p \in \mathbf{N}$. Now by the properties of the 1-dimensional density of a measure [AFP00, Thm. 256],

$$\mathcal{H}^1(\Theta_p) \le \frac{p}{2}|\sigma|(\Theta_p)$$

so that Θ_p can be decomposed into a countably 1-rectifiable and a purely 1-unrectifiable set [AFP00, p. 83],

$$\Theta_p = \Theta_p^r \cup \Theta_p^u$$

(Θ_p^u purely 1-unrectifiable means $\mathcal{H}^1(\Theta_p^u \cap f(\mathbf{R})) = 0$ for any Lipschitz $f : \mathbf{R} \to \mathbf{R}^n$). By the $\mathcal{H}^1$-diffusivity assumption on σ we have (abbreviating the Lebesgue measure by $\mathcal{L}$)

$$\mathcal{L}(\pi_\xi(\Theta_p^r)) \le \mathcal{H}^1(\Theta_p^r) \le \frac{p}{2}|\sigma|(\Theta_p^r) = 0 ,$$

and by a result due to Besicovitch [AFP00, Thm. 2.65] we have

$$\mathcal{L}(\pi_\xi(\Theta_p^u)) = 0$$

for almost all $\xi \in \mathbf{S}^{n-1}$. Thus, for almost all $\xi \in \mathbf{S}^{n-1}$ we have $\mathcal{L}(\pi_\xi(\Theta_p)) = 0$, as desired.

Remark 10 (Characterization of divergence measure vector fields)**.** By a result due to Smirnov [Smi93], any divergence measure vector field σ can be decomposed into simple oriented curves $\sigma_\gamma = \gamma_\# \dot{\gamma}\ \mathrm{d}s \llcorner [0,1]$ with $\gamma : [0,1] \to \mathbf{R}^n$ a Lipschitz curve and $\mathrm{d}s$ the Lebesgue measure, that is,

$$\sigma = \int_J \sigma_\gamma \ \mathrm{d}\mu_\sigma(\gamma)$$

with J the set of Lipschitz curves and μ_σ a nonnegative Borel measure. The results of this section can alternatively be derived by resorting to this characterization, since the slice of a simple oriented curve σ_γ can be explicitly calculated.

Appendix D

Résumé substantiel en langue française

Lors de la conception d'un réseau de distribution offre-demande, il convient de lui donner une structure d'arbre dans laquelle il est préférable de regrouper la masse dans le processus de transport. Cette hypothèse émerge de nombreuses observations, par exemple, la structure des vaisseaux sanguins dans le système cardiovasculaire est requise pour distribuer le sang d'une source concentrée dans le cœur à un volume répandu ou vice-versa, le système racinaire d'un arbre a besoin de récupérer l'eau du sol. Dans ces situations, nous pouvons observer à quel point des vaisseaux larges et longs sont préférables plutôt que des vaisseaux éparpillés. L'hypothèse que nous faisons est que le réseau observé est optimal par rapport à un coût donné parmi tous les réseaux possibles se développant à partir d'une source et irriguant un puits donné. Ces structures apparaissent dans un large gamme de situations et de nombreux efforts ont été faits par la communauté mathématique afin de donner un modèle précis capable de décrire toutes les caractéristiques observables de ces réseaux.

Figure D.1: A gauche: réseau de racines d'un arbre. A droite: angiographie d'un oeil dans lequel il est possible de reconnaître la structure d'arbre du réseau des vaisseaux sanguins.

Une première approche bien connue dans le cadre de la théorie des graphes a été proposée par Gilbert dans [PST15] où il s'occupe du problème de *l'arbre minimal de Steiner* [AT04, PS13]. Ce dernier consiste à trouver le graphique reliant un ensemble donné de points $\{x_0, \dots, x_N\}$ avec une longueur totale minimale. Plus formellement, un arbre minimal Steiner est la solution du problème variationnel

$$\operatorname{argmin}\left\{\mathcal{H}^1(K) \,:\, K \text{ compact, connecté et contient } x_0, \dots, x_N\right\}, \tag{D.1}$$

où $\mathcal{H}^1(K)$ est la mesure de Hausdorff 1-dimensionnelle de k. (la longueur de K, si il est 1-dimensionnelle et suffisamment lisse.). Comme indiqué dans Courant and Robbins [CR79], le problème de l'arbre minimal de Steiner peut être considéré comme un modèle naàf pour le réseau d'autoroutes reliant un ensemble de villes. L'inconvénient

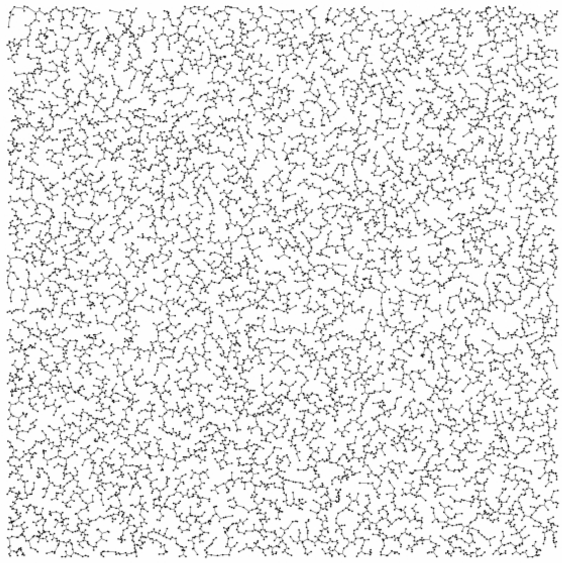


Figure D.2: Steiner Minimal Tree reliant 10000 points répartis aléatoirement dans le plan. Le problème a été résolu en utilisant l'algorithme GeoSteiner [WZ97], qui est actuellement l'algorithme exact le plus efficace pour calculer les arbres Steiner minimums.

du modèle est que l'intensité locale du trafic n'est pas prise en compte. Néanmoins il permet d'apprécier la problématique de ces modèles. Comme observé dans le document cité [PST15] dans un arbre minimal Steiner, différemment du *Minimal Spanning Tree* [Kru56], de nouveaux sommets peuvent être ajoutés afin de minimiser la longueur totale ainsi, plutôt que le réseau lui-même, la vraie inconnue est la topologie. Un exemple de cette situation est illustré dans la Figure D.3. Cette caractéristique apparaît également dans d'autres modèles dans lesquels le coût par unité de longueur dépend de l'intensité du flux [Gil67]. A la lumière de cette complexité combinatoire élevée, le problème se trouve dans la liste des problèmes NP-complets de Karp [Kar72] et c'est toujours l'argument de [FMBM16].

Le but de cette thèse est de concevoir des approximations de certains problèmes de Transport Branché. Le transport branché est un cadre mathématique de modélisation des réseaux de distribution offre-demande qui est plus général que le problème Steiner présenté ci-dessus. En particulier les usines d'approvisionnement et les lieux de demande sont modélisés comme des mesures supportées sur des points et le réseau est interprété comme une mesure vectorielle, enfin le problème est présenté comme un problème d'optimisation sous contraintes. Le coût de transport d'une masse m le long d'un bord de longueur ℓ est $h(m)\,\ell$ et le coût total d'un réseau est défini comme la somme de la contribution sur tous ses bords. Le cas de transport branché consiste dans le choix spécifique $h(m) = |m|^\alpha$ avec $\alpha \in [0, 1)$. La sous-additivité de la fonction

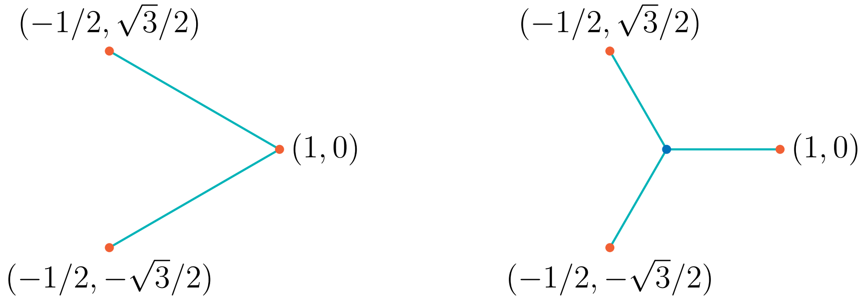


Figure D.3: On the left: Minimal Spanning Tree connecting three points situated at the vertices of an equilateral triangle (longueur $= 2\sqrt{3}$). Sur la droite: Steiner Minimal Tree contraint de connecter le même ensemble de points (longueur $= 3$). En bleu foncé le sommet supplémentaire qui permet de diminuer la longueur totale.

de coût, $h(m_1 + m_2) \leq h(m_1) + h(m_2)$, assure que transporter deux masses conjointement est moins cher que de le faire séparément. Cette formulation partage la plupart des complexités numériques présentées ci-dessus dans le cas du problème de l'arbre minimal de Steiner. Dans ce travail, nous introduisons diverses approximations variationnelles au moyen de fonctions de type elliptique pour obtenir des schémas numériques plus efficaces. Finalement, la méthode proposée est généralisée aux problèmes de type Plateau qui est un cadre pour modéliser les films de savon couvrant une frontière donnée. Dans sa formulation plus générale, l'inconnu de ces problèmes est une surface k-dimensionnelle en $\mathbf{R}^n$ enjambant une frontière $(k-1)$-dimensionnelle et minimisant un certain coût. Le transport branché correspond à un problème de type Plateau pour le choix $k = 1$.

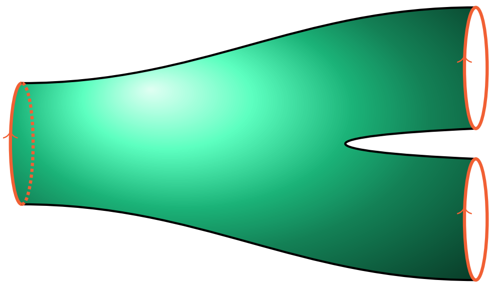


Figure D.4: Exemple d'une surface enjambant une frontière 1-dimensionnelle composée de trois cercles orientés.

Description du modèle

Présentons précisément le schéma du Transport Branché [BCM09, Vil03]. Tout d'abord, nous introduisons les réseaux de transport dans un ensemble ouvert $\Omega \in \mathbf{R}^n$, et le fonctionnelle de coût associé. Pour cela, considérez un segment $\Sigma \subset \Omega$, un nombre réel positif $m \in \mathbf{R}_+$ et un vecteur $\tau \in \mathbf{S}^{n-1}$ tangent à Σ, l'écriture

$$m\,\tau\,\mathcal{H}^1 \llcorner \Sigma \tag{D.2}$$

définit une mesure à valeur vectorielle, où $\mathcal{H}^1 \llcorner \Sigma$ est la mesure de Hausdorff 1-dimensionnelle en $\mathbf{R}^n$ limitée au segment Σ. Intuitivement, la mesure Radon $\mathcal{H}^1 \llcorner \Sigma$ associe à tout ensemble mesurable A la longueur de $A \cap \Sigma$. Nous disons qu'une mesure vectorielle $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ est *polyhedral* si c'est une somme finie de mesures de la forme (D.2), à savoir $\sigma = \sum_i m_i \tau_i \mathcal{H}^1 \llcorner \Sigma_i$. L'action de σ sur $C_0(\Omega, \mathbf{R}^n)$ est définie par la formule suivante

$$(\sigma, \varphi) = \sum_i \int_{\Sigma_i} m_i \, \varphi \cdot \tau_i \, \mathrm{d}\mathcal{H}^1 \qquad \text{pour tout } \varphi \in C_0(\Omega, \mathbf{R}^n).$$

Une fonction de coût de transport $h : \mathbf{RR} \to [0, +\infty)$ est une application telle que

$$h \text{ est } \begin{cases} \text{pair, semi-continu inférieurements,} \\ \text{sous-additif, avec } h(0) = 0. \end{cases} \tag{D.3}$$

Étant donné une fonction de coût de transport h, nous définissons le *énergie de Gilbert* sur la mesure vectorielle polyédrique comme suit

$$\mathscr{E}_h(\sigma) := \sum_i h(m_i) \mathcal{H}^1(\Sigma_i).$$

Nous dotons $\mathcal{M}(\Omega, \mathbf{R}^n)$ avec sa topologie faible-$*$ et étendons $\mathscr{E}_h$ sur cet espace par relaxation, à savoir pour une mesure vectorielle σ nous fixons

$$\mathscr{E}_h(\sigma) := \inf \left\{ \liminf_{j \to +\infty} \mathscr{E}_h(\sigma_j) \, : \, \sigma_j \overset{*}{\rightharpoonup} \sigma \text{ et } \sigma_j \text{ polyhedral} \right\}. \tag{D.4}$$

Par White dans [Whi99a, 6] les conditions (D.3) sont suffisantes pour étendre $\mathscr{E}_h$ sur $\mathcal{M}(\Omega, \mathbf{R}^n)$. En choisissant $h(m) = |m|$ dans l'équation (D.4) on obtient le *fonctionnelle de masse* qui associe à chaque vecteur σ sa variation totale

$$|\sigma| = \sup\{(\varphi, \sigma) \, : \, \varphi \in C_0(\Omega, \mathbf{R}^n), \, \|\varphi\|_\infty \leq 1\}.$$

Sinon, avec $h(m) = \chi_{\{m \neq 0\}}$ où χ désigne la fonction caractéristique d'un ensemble, $\mathscr{E}_h$ se réduit au *fonctionnelle de taille* qui mesure la longueur du support de σ, à savoir $\sigma \mapsto \mathcal{H}^1(\mathrm{supp}(\sigma))$. D'autres choix remarquables sont représentés dans la Figure D.5.

Pour modéliser la source et le puits du réseau de transport, nous introduisons deux mesures de probabilité $\mu_+, \mu_- \in \mathcal{P}(\Omega)$ et limitons notre attention à l'espace vectoriel $X^{\mu_+,\mu_-} \subset \mathcal{M}(\Omega, \mathbf{R}^n)$ composé de ces mesures vectorielles σ satisfaisant

$$\operatorname{div} \sigma = \mu_+ - \mu_- \tag{D.5}$$

dans le sens de distributions. Comme est montre dans la note [CFM18] si la relaxation est obtenu par rapport aux mesures polyédriques en X^{μ_+,μ_-} nous obtenons toujours le fonctionnel (D.4).

Enfin, nous sommes intéressés à approcher les minimiseurs de l'énergie de Gilbert sous la contrainte de divergence (D.5), à savoir:

$$\min \left\{ \mathscr{E}_h(\sigma) : \sigma \in X^{\mu_+,\mu_-} \right\}. \tag{D.6}$$

Le cas du *Transport branché* correspond au choix $h(m) = |m|^\alpha$ avec $\alpha \in [0, 1)$ et a été introduit par Xia qui a également étudié le problème de l'existence et de la régularité des solutions. Dans [Xia03] l'auteur, profitant des méthodes variationnelles, prouve ce qui suit

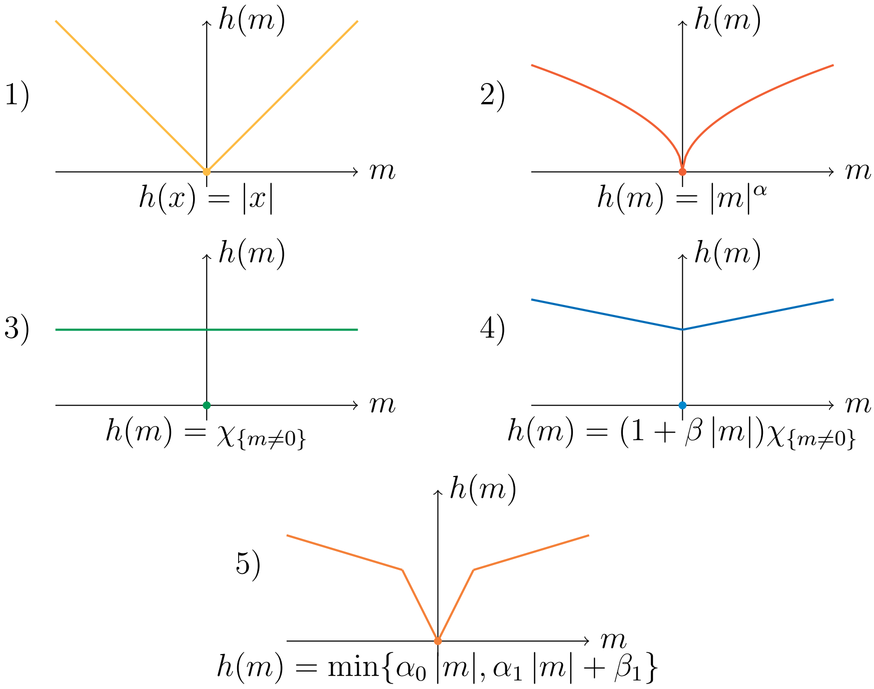


Figure D.5: Pour h comme dans les graphes nous obtenons respectivement le : 1) Masse, 2) α-Masse, 3) Taille, 4) Coût affine, 5) Planification urbaine fonctionnelle.

Theorem D.1 (Théorème de l'existence)**.** *Donné $\alpha \in (1 - \frac{1}{n}, 1]$ et deux mesures de probabilité $\mu_+, \mu_- \in \mathcal{P}(\Omega)$, il existe une mesure a valeurs vectorielle $\sigma \in X^{\mu_+,\mu_-}$ pour laquelle $\mathscr{E}_h(\sigma)$ est minimal. De plus, nous avons l'estimation suivante*

$$\mathscr{E}_h(\sigma) \leq \frac{1}{2^{1-n(1-\alpha)-1}} \frac{\sqrt{n}\,\mathrm{diam}(\Omega)}{2}.$$

Dans un résultat subséquent [Xia04, Théorème 2.7] le même auteur analyse le problème de la régularité. Pour énoncer le résultat, nous devons introduire la notion de *rectifiable vector measure.* A savoir une mesure vectorielle σ est dit rectifiable si

$$\sigma = m\,\tau\,\mathcal{H}^1 \llcorner \Sigma \tag{D.7}$$

où Σ, le support de σ comme distribution, est un ensemble $\mathcal{H}^1$-rectifiable, sa densité $\mathcal{H}^1$ est la fonction $m \in L^1(\mathcal{H}^1 \llcorner \Sigma)$ et $\tau : \Sigma \to \mathbf{S}^{n-1}$ génère pour $\mathcal{H}^1$-a.e. point dans Σ l'espace tangent à Σ. Dans ce qui suit, nous dénotons avec (m, τ, Σ) la mesure rectifiable σ définie dans (D.7).

Theorem D.2 (Structure des réseaux d'énergie finie)**.** *Pour $0 \leq \alpha < 1$ si $\sigma \in X^{\mu_+,\mu_-}$ est de variation totale finie et d'énergie $\mathscr{E}_h$ finie alors il est rectifiable. De plus si $\sigma = (m, \tau, \Sigma)$ nous avons*

$$\mathscr{E}_h(\sigma) = \int_\Sigma |m|^\alpha \, \mathrm{d}\mathcal{H}^1. \tag{D.8}$$

L'équation (D.8) est particulièrement significative puisqu'elle étend la représentation explicite de la fonctionnelle à toute mesure rectifiable. Le cas des fonctions génériques de coût de transport a été pris en considération par Brancolini et Wirth in [BW18, Proposition 2.32] qui montre que

Proposition D.1 (Énergie Gilbert-Steiner généralisée). *Soit* $\mu_+, \mu_- \in \mathcal{P}(\Omega)$, $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ *à variation totale finie et telle que* $\operatorname{div} \sigma = \mu_+ - \mu_-$ *alors* σ *peut être décomposé en tant que*

$$\sigma = \sigma^\perp + m\,\tau\,\mathcal{H}^1 \llcorner \Sigma$$

où (m, τ, Σ) *est le composant* $\mathcal{H}^1$*-rectifiable de* σ *et* $\sigma^\perp$ *est le composant diffus. De plus*

$$\mathscr{E}_h(\sigma) = h'(0)|\sigma^\perp| + \int_\Sigma h(m)\,\mathrm{d}\mathcal{H}^1. \tag{D.9}$$

Lorsqu'avec un abus de notation, nous avons dénoté $h'(0) = \lim_{m \downarrow 0} h(m)/m$.

Avant d'introduire des problèmes impliquant des surfaces et d'autres objets de dimensions supérieures, soulignons le fait que le problème de l'arbre minimal Steiner reliant certains points $\{x_0, \ldots, x_N\}$ peut être modélisé dans le contexte du transport Branché. Tout d'abord, avec le choix $\alpha = 0$, $\mathscr{E}_h$ se réduit à le fonctionnelle de taille. Deuxièmement, la contrainte de divergence oblige toute mesure vectorielle considérée à joindre le support de μ_+ au support de μ_- donc, en choisissant $\mu_+ = \delta_{x_0}$ et $\mu_- = 1/N \sum_{i=1}^N \delta_{x_i}$ nous forçons x_0 à être connecté à chaque x_i. En rassemblant tous ensemble, avec ces choix, un minimiseur σ of (D.6) est supporté sur un ensemble reliant chaque couple de points dans $\{x_0, \ldots, x_N\}$ et a un support avec une longueur totale minimale donc est une solution du problème (D.1).

L'énergie introduite ci-dessus pour les mesures rectifiables supportées sur des surfaces 1-dimensionnelles peut être généralisée à n'importe quelle dimension $k \in \{1, \ldots, n\}$. Pour ce faire, il est nécessaire d'introduire le concept de k-courants dans $\mathbf{R}^n$. Dénotez avec $\mathcal{D}^k(\Omega)$ l'espace des formes différentielles lisses sur l'ensemble ouvert Ω. L'espace vectoriel de k-courants, $\mathcal{D}_k(\Omega)$, est le dual de $\mathcal{D}^k(\Omega)$ et il est naturellement doté de sa topologie faible-$*$. Nous suivons principalement la notation de [KP08, Fed69] la principale différence étant l'utilisation de σ pour désigner un k-courant au lieu d'une lettre alphabétique majuscule. Pour un courant on peut définir une notion de frontière par dualité comme suit

$$\langle \partial\sigma, \omega \rangle = \langle \sigma,\, \mathrm{d}\omega \rangle \quad \text{pour tous les } (k-1) \text{ formes différentiels } \omega.$$

Nous appelons masse d'un k-courant la borne supérieure $\langle \sigma, \omega \rangle$ parmi toutes les formes différentielles k avec comass délimité par 1, et on le désigne avec $|\sigma|$. En particulier par le théorème de Radon-Nikodym, nous pouvons identifier un k-courant σ à masse finie avec la mesure à valeurs vectorielle $\tau\mu_\sigma$ où μ_σ est une mesure finie et positive, et τ est une fonction μ_σ-mesurable a valeurs dans l'ensemble de k-vecteurs unitaires pour la norme de masse. La relation avec la mesure vectorielle est évidente quand on considère le fait que les espaces vectoriels $\Lambda_1\mathbf{R}^n$, $\Lambda^1\mathbf{R}^n$ s'identifient avec $\mathbf{R}^n$. Ainsi, chaque measure vectorielles $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$ de masse finie s'identifie avec un 1-courant de masse finie et vice-versa. De plus, l'opérateur de divergence agissant sur les mesures au sens de distribution est défini par la dualité comme l'opérateur de frontière pour les courants. Donc, en analogie avec ce qui a été présenté pour les mesures vectorielles, dans l'équation (D.7), un k-courant σ est dit k-rectifiable si nous pouvons lui associer un triplet (θ, τ, Σ) de telle sorte que

$$\langle \sigma, \omega \rangle = \int_{\Sigma} \theta \langle \omega, \tau \rangle \, \mathrm{d}\mathcal{H}^k$$

où Σ est un sous-ensemble k-rectifiable de Ω, τ à $\mathcal{H}^k$ a.e. point est un simple k-vecteur unitaire qui enjambe le plan tangent à Σ et θ est une fonction $L^1(\Omega, \mathcal{H}^k \llcorner \Sigma)$ qui peut être supposée positive. L'espace vectoriel de *Courants rectifiables* est indiqué par $R_k(\Omega)$. Parmi ceux-ci nous nommons le sous-ensemble $P_k(\Omega)$ de courants rectifiables pour lesquels Σ est une union finie de polyèdres et θ est constant sur chacun d'eux, ceux-ci seront appelés *Chaînes polyédriques*. Pour tout k-courant σ tel que σ et $\partial\sigma$ sont de masse finie, nous disons que σ est un k-courant normal et nous écrivons $\sigma \in N_k(\Omega)$. Sur l'espace $\mathcal{D}_k(\Omega)$ nous pouvons définir la norme *flat* par

$$\mathbb{F}(\sigma) = \inf \left\{ |\sigma_R| + |\sigma_S| \ : \ \sigma = \sigma_R + \partial\sigma_S \text{ où } \sigma_S \in \mathcal{D}_{k+1}(\Omega) \text{ et } \sigma_R \in \mathcal{D}_k(\Omega) \right\},$$

qui métrifie la topologie faible-$*$ pour les courants de $N_k(\Omega)$ avec support compact. Enfin, l'espace de *flat chains* $F_k(\Omega)$ consiste en la fermeture de $P_k(\Omega)$ dans la topologie $\mathbb{F}$. Par le schéma de Federer [Fed69, 4.1.24] nous avons la suivante chaîne d'inclusions

$$P_k(\Omega) \subset N_k(\Omega) \subset F_k(\Omega).$$

Suivant la stratégie proposée par Fleming [FF60, Fle66] dans le contexte des flat chains à coefficients en groupes, nous définissons maintenant l'énergie $\mathscr{E}_h$ sur l'espace des flat chains. Soit h est une fonction de coût de transport et $\sigma = \sum (m_i \tau_i, \Sigma_i)$ un courant polyédrique on impose

$$\mathscr{E}_h(\sigma) := \sum_i h(m_i) \mathcal{H}^k(\Sigma_i).$$

Par analogie à ce qui a été fait avant, nous étendions $\mathscr{E}_h$ sur l'espace des chaînes plates par relaxation

$$\mathscr{E}_h(\sigma) := \inf \left\{ \liminf_{j \to \infty} \mathscr{E}_h(\sigma_j) \ : \ \sigma_j \text{ polyhedral et } \mathbb{F}(\sigma_j - \sigma) \to 0 \right\}.$$

Dans le chapitre 3 nous cherchons des approximations aux problèmes de type

$$\min\{\mathscr{E}_h(\sigma) \ : \ \partial\sigma = \partial\sigma_0\} \tag{D.10}$$

où σ_0 est une$k$-courant polyédrique donnée. Ces problèmes ont été introduits et étudiés dans [Mor89, DPH03] par Morgan, De Pauw et Hardt entre autres pour proposer différents modèles pour des surfaces minimales de film de savon. Ce dernier est la généralisation k-dimensionnelle du problème de minimisation défini dans (D.6). Comme indiqué dans [Whi99a, Whi99b] $\mathscr{E}_h$ a une formulation explicite pour les courants rectifiables, à savoir pour un courant rectifiable (m, τ, Σ), nous avons

$$\mathscr{E}_h(\sigma) := \int_{\Sigma} h(m) \, \mathrm{d}\mathcal{H}^k.$$

Ce résultat a été prouvé en [CDRMS17, Proposition 2.6], et est la conséquence du théorème d'approximation polyédrique suivant

Theorem D.3 (Approximation polyédrique). *Soit h est une fonction de coût de transport et $= (m, \tau, \Sigma)$ un k-courant rectifiable. Pour chaque $\delta > 0$ il existe une k-chain polyédrique $\hat{\sigma} = \sum(m_i\tau_i, \Sigma_i)$ telle que*

$$\mathbb{F}(\hat{\sigma} - \sigma) \leq \delta, \quad \sum_i h(m_i)\mathcal{H}^k(\Sigma_i) \leq \int_\Sigma h(m)\, \mathrm{d}\mathcal{H}^k + \delta \quad \text{et} \quad |\hat{\sigma}| \leq |\sigma| + \delta.$$

En outre, Colombo et al. in [CDRMS17, Proposition 2.7] ont montré que la condition

$$\lim_{m\downarrow 0} \frac{h(m)}{m} = +\infty$$

est l'équivalent au fait que

$$\mathscr{E}_h(\sigma) \text{ est fini si et seulement si } \sigma \text{ est rectifiable.}$$

Ce résultat peut être vu en corrélation avec l'équation (D.9) présentée précédemment. Il faut souligner que le résultat d'approximation polyédrique de Colombo et al. ne tient compte aucune contrainte de frontière pour les k-courants. Un résultat analogue avec contrainte de frontière a été prouvé dans la note [CFM18]. Nous concluons cette section avec une condition suffisante pour qu'une flat chain soit rectifiable, prouvée par White in [Whi99a, Corollaire 6.1].

Theorem D.4 (Rectifiabilité pour les courants). *Soit $\sigma \in N_k(\Omega)$ une k-courant normal supporté sur un ensemble k-rectifiable alors σ est rectifiable.*

Nous allons profiter de ce théorème dans le contexte des mesures vectorielles, avec la notation introduite ci-dessus le même théorème se lit comme suit

Theorem D.5 (Rectifiability for vector valued measures). *Soit $\sigma \in \mathcal{M}(\Omega, \mathbf{R}^n)$. Si $|\sigma|(\Omega) + |\nabla \cdot \sigma|(\Omega) < \infty$, $\nabla \cdot \sigma$ est une somme dénombrable de masses Dirac et il existe un ensemble de Borel Σ avec $\mathcal{H}^1(\Sigma) < \infty$ et $\sigma = \sigma\sigma \llcorner \Sigma$ alors σ est une mesure vectorielle rectifiable.*

Approximation variationnelle pour les problèmes de minimisation

Nous fournissons des approximations aux problèmes définis dans (D.6) dans le sens de Γ-convergence. Cette dernière est une notion de convergence fonctionnelle introduite par De Giorgi [DG75] pour traiter des problèmes variationnels. En suivant [DM93, Bra98, AD00, Bra02] on donne la définition opérationnelle de Γ-convergence.

Definition 4 (Γ-convergence). Soit X un espace métrique, et pour $\varepsilon > 0$ on donne $\mathscr{F}_\varepsilon : X \to [0, +\infty]$. Nous disons que $\mathscr{F}_\varepsilon$ Γ converge vers F sur X en tant que $\varepsilon \to 0$ et nous notons $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{F}$ si les deux conditions suivantes sont satisfaites :

(LB) *Inégalité* $\Gamma - \liminf$*:* pour tout $x \in X$ et tout $x_\varepsilon \to x$ on a

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(x_\varepsilon) \geq \mathscr{F}(x),$$

(UP) *Inégalité* $\Gamma - \limsup$*:* pour tout $x \in X$ il existe une séquence $(\hat{x}_\varepsilon) \subset X$ telle que $\hat{x}_\varepsilon \to x$ et

$$\limsup_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\hat{x}_\varepsilon) \leq \mathscr{F}(x).$$

La séquence $\hat{x}_\varepsilon$ est appelée *recovery sequence* pour x. La condition *(UB)* est souvent difficile à prouver donc il est pratique de trouver un sous-ensemble $D \subset X$ tel que : pour chaque $x \in X$ il existe une séquence approchante $(x_n) \subset X$ tel que $x_n \to x$ et $\mathscr{F}(x_n) \to \mathscr{F}(x)$. Si nous sommes capables de retrouver D alors un simple argument diagonal montre qu'il suffit de vérifier la condition *(UB)* pour tous les $x \in D$ plutôt que pour chaque $x \in X$. Dans le contexte de notre travail, l'ensemble D correspond à l'espace vectoriel des mesures vectorielles polyédriques. Puisque la définition de Γ-convergence peut paraître encombrante, nous donnons une caractérisation alternative qui permet de goûter sa pertinence dans le contexte du Calcul des Variations.

Theorem D.6 (Caractérisation pour la Γ-convergence)**.** *Soit X un espace métrique, et $\varepsilon > 0$, soit $\mathscr{F}_\varepsilon : X \to [0, +\infty]$ et $\mathscr{F} : X \to [0, +\infty]$. $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{F}$ si et seulement si pour chaque $\mathscr{G}$ fonctionnel continu, si x_ε minimise $\mathscr{F}_\varepsilon + \mathscr{G}$ et $x_\varepsilon \to x$ alors x minimise $\mathscr{F} + \mathscr{G}$.*

Notre stratégie est de remplacer l'énergie singulière $\mathscr{E}_h$ par une séquence de fonctionnels de type elliptique plus lisse $\mathscr{F}_\varepsilon$ et de prouver que $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{E}_h$. Puis nous prouvons que la famille $(\mathscr{F}_\varepsilon)$ est *equicoercive* : toute séquence de minima $(\hat{x}_j)$ est précompacte en X. Ceci assure que la séquence de minimiseurs $\hat{x}_\varepsilon$ convergent vers un minimum. Enfin, nous cherchons des méthodes numériques pour approcher un minimum $\hat{x}_\varepsilon$.

On va exposer trois exemples remarquables de convergence Γ. Considérons un conteneur $\Omega \subset \mathbf{R}^3$ de volume unitaire contenant deux liquides non miscibles modélisés par une fonction binaire $\varphi : \Omega \to \Omega \to \{0, 1\}$ de sorte que $\int_\Omega |\varphi| \, \mathrm{d}x = V \in (0, 1)$ représente le pourcentage d'un liquide par rapport au volume du récipient. On associe au système une énergie en fonction de la tension superficielle, en supposant qu'elle est directement proportionnelle à la surface de l'interface J_φ entre les liquides.

$$\mathscr{M}(\varphi) = c\mathcal{H}^2(J_\varphi).$$

Une autre façon d'étudier les systèmes est de supposer que la transition n'est pas donnée par une interface de séparation infinitésimale, mais qu'il s'agit plutôt d'un phénomène continu se produisant dans une fine épaisseur de taille ε. Compte tenu de cela, Cahn et Hilliard [CH58] envisagent une fonction de phase continue $\varphi : \Omega \to [0, 1]$ représentant le mélange ponctuel entre les fluides et supposons une énergie du type

$$\int_\Omega \left[\varepsilon^2 |\nabla \varphi|^2 + \varphi^2 (1 - \varphi)^2\right] \, \mathrm{d}x.$$

Le terme $\varphi^2 (1 - \varphi)^2$ est appelé un potentiel double puits et pénalise les valeurs loin de 0 ou 1 et l'inhomogénéité n'est pas favorisée par le terme de gradient. Le lien entre les deux énergies a été découvert par Modica et Mortola dans leurs papiers [MM77a, MM77b]. Leur résultat est plus général, ils prouvent qu'un rescaling approprié de l'énergie ci-dessus Γ converge vers le périmètre fonctionnel pour toutes le dimension de domaine.

Theorem D.7. *Soit $\Omega \subset \mathbf{R}^n$, et soit $X = BV(\Omega) \cap L^\infty(\Omega)$. Pour $\varphi \in X$ et $\varepsilon > 0$ on pose*

$$\mathscr{M}_\varepsilon(\varphi) := \begin{cases} \int_\Omega \left[\varepsilon|\nabla\varphi|^2 + \frac{\varphi^2(1-\varphi)^2}{\varepsilon}\right] \mathrm{d}x, & \text{if } \varphi \in W^{1,2}(\Omega,[0,1]) \text{ et } \int_\Omega |\varphi| \,\mathrm{d}x = V, \\ +\infty, & \text{autrement dans } BV(\Omega,[0,1]). \end{cases}$$

Soit $c = 2\int_0^1 t^2(1-t)^2 \,\mathrm{d}t$ et

$$\mathscr{M}(\varphi) := \begin{cases} c\mathcal{H}^{n-1}(J_\varphi), & \text{if } \varphi = \chi_A \text{ et } |A| = V, \\ +\infty, & \text{autrement dans } BV(\Omega). \end{cases}$$

Alors $\mathscr{M}_\varepsilon \xrightarrow{\Gamma} \mathscr{M}$ quand $\varepsilon \to 0$.

Dans ce qui précède BV(Ω) indique l'espace de ces fonctions φ tel que $\varphi \in L^1(\Omega)$ et le gradient dans le sense des distributions $D\varphi$ est une mesure de Radon. Pour les fonctions à *Variations Bornées*, le gradient distributionnel peut être décomposé en trois mesures, à savoir

$$D\varphi = \nabla\varphi + D^c\varphi + [\varphi]\mathcal{H}^{n-1} \llcorner J_\varphi$$

o $\nabla\varphi$ est la composante de $D\varphi$ absolument continue par rapport à la mesure de Lebesgue, $D^c\varphi$ est une mesure Cantor et $[\varphi]\mathcal{H}^{n-1} \llcorner J_\varphi$ est appelé la composante de saut de la mesure et est absolument continu par rapport à la mesure Hausdorff $\mathcal{H}^{n-1}$ restreinte à l'ensemble de discontinuité J_φ. En particulier si $\varphi \in$ BV(Ω) et $\varphi = \chi_A$ alors J_φ est la frontière essentielle de A contenue dans Ω et $[\varphi] = 1$. Pour d'autres résultats sur la théorie des fonctions à variations bornée, nous nous référons à [AFP00] et à l'introduction technique du Chapitre 1, Section 1.2. Le résultat est corrélé avec sa respective propriété d'équicoercivité.

Corollary D.1. *Si $\varepsilon \downarrow 0$ et φ_ε minimise $\mathscr{M}_\varepsilon$ alors la séquence (φ_ε) est pre-compact et tout point limite minimise $\mathscr{M}$.*

Un autre exemple vient de l'approximation du fonctionnelle de Mumford-Shah pour la segmentation d'images. Dans [MS89] les auteurs considèrent une fonction g, définie sur un domaine Ω, représentant le niveau de gris d'une image d'un groupe d'objets donnée par une caméra, avec des discontinuités long les bords des objets. L'idée est que l'image segmentée u devrait être suffisamment lisse à l'extérieur d'un ensemble $(n-1)$-dimensionnel contenant l'ensemble de discontinuité K, à savoir $u \in W^{1,2}(\Omega\backslash K)$, et ce dernier devrait être choisi de $\mathcal{H}^{n-1}$-size minimal. C'est pourquoi ils proposent d'optimiser dans les variables (u, K) l'énergie

$$\int_{\Omega\backslash K} \left[|\nabla u|^2 + \alpha(u-g)^2\right] \,\mathrm{d}x + \beta\mathcal{H}^{n-1}(K).$$

The parameters α, β control the weight between the fidelity term $|u-g|^2$ and the size of the discontinuity set K. It is convenient to recast the problem in its weak formulation letting $u \in$ BV(Ω) and replacing the set K with J_u obtaining the functional

$$\mathscr{S}(u) := \int_\Omega \left[|\nabla u|^2 + \alpha(u-g)^2\right] \,\mathrm{d}x + \beta\mathcal{H}^{n-1}(J_u).$$

Pour approcher l'énergie $\mathscr{S}$, Ambrosio et Tortorelli ont proposé la famille des fonctionnelles

$$\mathscr{S}_\varepsilon(u,\varphi) = \int_\Omega |\nabla u|^2 \varphi + \frac{\beta}{4}\left[\varepsilon|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{\varepsilon}\right]\,\mathrm{d}x + \alpha\int_\Omega (u-g)^2\,\mathrm{d}x.$$

Dans les articles [AT90, AT92] il est prouvé que $\mathscr{S} \xrightarrow{\Gamma} \mathscr{S}$. Donnons une idée heuristique derrière ce résultat. Puisque u est proche de g dans le cas d'une discontinuité forte de g, le terme de gradient $|\nabla u|$ explose. En effet, valeurs élevées dans le gradient $|\nabla u|$ sont contrôlées par des valeurs proches de zéro dans la fonction d'état φ. D'autre part, le terme entre crochets pénalise fortement les valeurs de φ loin de 1. La concurrence des termes en φ se traduit par le fait que $1-\varphi$ représente une version lissée de la fonction $1-\chi_{J_u}$. Enfin dans la limite $\varepsilon \downarrow 0$ le terme Modica-Mortola converge vers la taille $\mathcal{H}^{n-1}$ de l'ensemble $\{\varphi \neq 1\}$ qui contient l'ensemble de saut de u. Les fonctionnelles modelées sur celles d'Ambrosio et de Tortorelli et cette dernière fonctionnelle elle-même sont souvent connues sous le nom d'approximations de champ de phase. Ce n'est pas seulement à cause de la relation stricte avec la fonctionnelle de Modica-Mortola mais aussi parce que nous pouvons interpréter la fonction φ comme une fonction d'état, elle acquiert la valeur 0 sur l'ensemble de saut de u, c'est-à-dire sur l'ensemble de discontinuité forte de la fonction, et la valeur 1 oú u est suffisamment lisse. Les deux comportements de u sont alors interprétés comme deux états possibles et φ modélise la fonction d'état dans chaque point pour le système. Cette observation a été prise en considération dans les travaux sur la théorie de la fracture d'Iurlano et al. [CFI16, Iur13]. Lá, φ modélise l'état de détérioration d'un matériau et u est remplacé par une fonction de déplacement.

Pour conclure la section, nous présentons une variation de la fonctionnelle de Ambrosio-Tortorelli proposée par Bonnivard, Lemenant et Santambrogio [LS14, BLS15] pour récupérer dans la limite la fonctionnelle associée au problème de l'arbre minimal Steiner pour certains points $\{x_0,\dots,x_N\} \subset \Omega \subset \mathbf{R}^2$. Étant donné une fonction continue $\varphi : \Omega \to [0,1]$ les auteurs introduisent une distance géodésique dépendant de φ, à savoir

$$d_\varphi(x,y) = \inf\left\{\int_\gamma \varphi\,\mathrm{d}\mathcal{H}^1 \;:\; \gamma \in C([0,1],\Omega),\, \gamma(0)=x,\, \gamma(1)=y\right\}.$$

La distance $d_\varphi(x,y)$ est nulle si et seulement si les deux points x, y sont reliés par un chemin sur lequel φ est égal à 0. Considérez la fonctionelle

$$\int_\Omega \left[\varepsilon|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{4\varepsilon}\right]\,\mathrm{d}x + \frac{1}{c_\varepsilon}\sum_{i=1}^N d_\varphi(x_0,x_i)$$

oú $c_\varepsilon \to 0$ quand $\varepsilon \to 0$. Remarquez d'abord que si

$$\sum_{i=1}^N d_\varphi(x_0,x_i) = 0 \tag{D.11}$$

alors l'ensemble $\{\varphi = 0\}$ devrait inclure un sous-ensemble connecté pour chemins contenant $\{x_0,\dots,x_N\}$. L'argument heuristique pour le résultat de Γ -convergence suit les idées présentées dans le cas de la fonctionnelle de Ambrosio-Tortorelli. Le résultat exact dans [BLS15] est

Theorem D.8 (Bonnivard-Lemenant-Santambrogio)**.** *Soit $\Omega \subset \mathbf{R}^2$ un ensemble ouvert, $\{x_0, \dots, x_N\} \in \Omega$ et $\mu = \frac{1}{N}\sum_{i=0}^{N} \delta_{x_i}$. Considérons la fonctionnelle*

$$\mathscr{B}_\varepsilon(\varphi) = \int_\Omega \left[\varepsilon|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{4\varepsilon}\right] \mathrm{d}x + \int_\Omega \frac{1}{c_\varepsilon} d_\varphi(x_0, x)\, \mathrm{d}\mu, \tag{D.12}$$

et une séquence φ_ε telle que

$$\mathscr{B}_\varepsilon(\varphi_\varepsilon) - \inf_\varphi \mathscr{B}_\varepsilon(\varphi) \xrightarrow[\varepsilon\to 0]{} 0.$$

Alors la séquence de fonctions d_{φ_ε} converge uniformément (a moins d'une sous-séquence) vers une fonction d telle que l'ensemble $K := \{d = 0\}$ minimise $\mathcal{H}^1$ parmi tous les ensembles compacts et connectés contenant les points $\{x_0, \dots, x_n\}$.

Une première approche au problème de l'approximation de l'énergie $\mathscr{E}_h$ dans le cas $h = |\cdot|^\alpha$ a été proposée par Santambrogio et Oudet dans [OS11]. Ils ont introduit un fonctionnel du type

$$\int_\Omega \varepsilon^{\alpha+1}|\nabla\sigma|^2 + \varepsilon^{\alpha-1}|\sigma|^\beta \text{ with } \sigma \in W^{1,2}(\Omega, \mathbf{R}^2) \text{ and } \nabla\cdot\sigma = (\mu_+ - \mu_-) * \rho_\varepsilon$$

avec $\beta = (4\alpha - 2)/(\alpha + 1)$ et ρ_ε une suite régularisante. En fait, l'inégalité $\Gamma - \limsup$ complète pour ce dernier résultat a été fournie par Monteil dans [Mon15, Mon17].

Structure de la thèse

Dans le **Premier Chapitre** nous étudions une variation de la fonctionnelle proposée par Lemenant et Santambrogio. Motivé par l'observation selon laquelle

$$d_\varphi(x, y) = \min\left\{\int_\Omega \varphi|\sigma|\, \mathrm{d}x \;:\; \sigma \in \mathcal{M}(\Omega, \mathbf{R}^n) \text{ et } \operatorname{div}\sigma = \delta_x - \delta_y\right\}$$

nous remplaçons le terme dépendant de la distance géodésique dans (D.12) par un terme dépendant du produit $\varphi|\sigma|$. La fonctionnelle proposée est définie pour les couples (σ, φ) est

$$\int_\Omega \frac{\varphi|\sigma|^2}{2\varepsilon}\, \mathrm{d}x + \int_\Omega \left[\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{(1-\varphi)^2}{2\varepsilon}\right] \mathrm{d}x,$$

oú σ est une fonction à valeur vectorielle corrélée à la contrainte

$$\operatorname{div}\sigma = (\mu_+ - \mu_-) * \rho_\varepsilon. \tag{D.13}$$

Dans l'equation précédente, ρ_ε est une suite régularisante et les fonctions de phase $\varphi \in L^1(\Omega)$ sont limitées par en bas par la quantité $\beta\,\varepsilon$, oú $\beta \geq 0$ un paramètre donné. Tout d'abord, nous montrons que cette fonctionelle Γ converge vers l'énergie $\mathscr{E}_h$, pour le choix

$$h(m) := \begin{cases} 1 + \beta m, & \text{if } m \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{D.14}$$

La preuve du résultat de convergence Γ est obtenue pour Ω des sous-ensembles convexes et ouverts de $\mathbf{R}^2$. L'avantage de choisir une pénalisation quadratique en σ est que le problème de Lagrange augmenté associé au fonctionnel peut àtre explicitement résolu dans la variable dual. Il est donc possible de concevoir un algorithme de minimisation alterné composé de deux fonctionnelles elliptiques et lisses résolubles par des méthodes d'éléments finis. The algorithm is proposed and studied at the end of the chapter. L'algorithme est proposé et étudié à la fin du chapitre. De plus nous présentons et étudions d'autres algorithmes qui profitent d'un concept de 'dérivée de forme' pour améliorer la qualité de l'approximation.

La généralisation à $\Omega \subset \mathbf{R}^n$ est traitée dans le **Deuxième Chapitre**. Pour obtenir le résultat dans une dimension supérieure, la composante Modica-Mortola de la fonctionnelle doit àtre recalibrée. Comme observé dans [Ghi14] cela conduit à l'introduction de certaines non linéarités dans le fonctionnelle comme suit

$$\int_\Omega \frac{\varphi|\sigma|^2}{\varepsilon}\,\mathrm{d}x + \int_\Omega \left[\varepsilon^{p-n+1}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-1}}\right]\,\mathrm{d}x, \tag{D.15}$$

pour quelques $p > n-1$. Encore une fois σ est corrélé avec la contrainte de divergence (D.13) pour un choix approprié de ρ_ε et il faut imposer une borne inférieure pour les fonctions de champ de phase, à savoir $\varphi \geq \beta\varepsilon^n$. Nous prouvons la Γ-convergence de la famille des fonctionnelles ci-dessus vers $\mathscr{E}_{h_\beta^{n-1}}$ oú la fonction de coàt h_β^{n-1} est la limite en ε d'un problème d'optimisation dépendant de la codimension $n-1$. Plus précisément pour une balle $B_r \subset \mathbf{R}^{n-1}$ nous définissons

$$h_{\varepsilon,\beta}^{n-1}(m) := \min \begin{cases} \displaystyle\int_{B_r} \left[\frac{\varphi|\theta|^2}{\varepsilon} + \varepsilon^{p-n+1}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-1}}\right]\,\mathrm{d}x, \\ (1-\varphi) \in W^{1,p}(B_r),\ \varphi = 1 \text{ sur } \partial B_r \text{ et } \displaystyle\int_{B_r} \theta\,\mathrm{d}x = m. \end{cases}$$

Ce dernier problème d'optimisation correspond à la version 0-dimensionnelle de (D.15). Nous introduisons et étudions $h_{\varepsilon,\beta}^d$ (obtenu en remplaçant $n-1$ par d dans cette dernière formule) dans l'annexe. Bouchitté, Dubs et Seppecher in [BDS96] ont étudié des problèmes similaires de transition de phase avec contrainte de masse qui conduisent à des mesures concentrées sur des atomes dans le contexte de l'équilibre des gouttelettes. En particulier, nous montrons que h_β^d est indépendant de r et qu'il s'agit d'une fonction de coàt de transport satisfaisant aux conditions (D.3). Nous prouvons également qu'il existe une constante $c > 0$ telle que

$$\frac{1}{c} \leq \frac{h_\beta^d(m)}{1+\sqrt{\beta}m} \leq c \quad \text{for } m > 0.$$

Remarquez que la composante Modica-Mortola de la composante fonctionnelle étudiée dans le deuxième chapitre dépend de $n-1$, la co-dimension du problème dans le cas de mesures rectifiables. Dans le **Chapitre Trois**, nous étudions un rééchelonnement différent pour approcher les minima de (D.10) définis pour k-courants, à savoir

$$\int_\Omega \frac{\varphi|\sigma|^2}{\varepsilon}\,\mathrm{d}x + \int_\Omega \left[\varepsilon^{p-n+k}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-k}}\right]\,\mathrm{d}x. \tag{D.16}$$

Dans ce contexte, σ n'est plus une mesure vectorielle, pour tenir compte de la frontière, la contrainte doit àtre modifiée de façon appropriée. Soit σ_0 une k-courant polyédrique donnée, pour ρ_ε une suite régularisante standard, nous laissons σ àtre un k-courant telle que

$$\partial\sigma = \partial\sigma_0 * \rho_\varepsilon.$$

(Dans l'équation (D.16), le courant est identifié avec sa densité.) Dans le chapitre nous introduisons formellement l'énergie et montrons qu'elle Γ-converge à l'énergie $\mathscr{E}_h$ définie dans (D.10) pour la fonction de coàt de transport $h = h_\beta^{n-k}$ étudié dans l'appendice.

Dans les **Quatrième et Cinquième Chapitres**, nous limitons à nouveau notre attention aux ensembles $\Omega \subset \mathbf{R}^2$ et développons deux fonctions pour l'approximation de toutes fonctions de coàt de transport h concave et continue. Remarquez que nous disons qu'une fonction de coàt de transport est concave si c'est une fonction paire dont la restriction à $[0, +\infty)$ est concave. Le premier résultat concerne les fonctions de coàt de transport h de la forme

$$h(m) = \min\{\alpha_i\,|m| + \beta_i \,:\, 0 \le i \le N\}.$$

pour $\alpha_0 > \alpha_1 > \ldots > \alpha_N \ge 0$ et $0 \le \beta_0 < \beta_1 < \ldots < \beta_N$. Notre approche profite du résultat du premier chapitre dans lequel nous avons récupéré dans la Γ-limite des fonctions de coàt de transport affine de la forme $1 + \beta|m|$. Dans le cas $N = 1$ et $\beta_0 > 0$ l'énergie de champ de phase proposée prend la forme suivante

$$\int_\Omega \left[\min\left\{\varphi_0^2 + \frac{\alpha_0^2\varepsilon^2}{\beta_0}\,;\,\varphi_1^2 + \frac{\alpha_1^2\varepsilon^2}{\beta_1}\right\}\frac{|\sigma|^2}{2\varepsilon}\right]\,\mathrm{d}x + \beta_0\mathscr{T}_\varepsilon(\varphi_1) + \beta_1\mathscr{T}_\varepsilon(\varphi_2)$$

oú $\mathscr{T}_\varepsilon$ est une énergie du type Modica-Mortola définie comme suit

$$\mathscr{T}_\varepsilon(\varphi) = \frac{1}{2}\int_\Omega \left[\varepsilon|\nabla\varphi(x)|^2 + \frac{(\varphi(x)-1)^2}{\varepsilon}\right]\,\mathrm{d}x.$$

Soulignons la présence de deux champs de phase qui interagissent dans la composante contrainte de la fonctionnelle. Idéalement, chaque $1 - \varphi_i$ est une fonction d'indicateur lisse d'un sous-ensemble du support de la mesure rectifiable limite σ. En particulier $\varphi_i = 0$ si le choix de la composante i-th dans la définition de h est optimal par rapport à l'intensité du flux de σ. Tout le **Quatrième Chapitre** est consacré à la preuve du résultat de Γ-convergence et à l'étude des méthodes numériques développées en collaboration avec Carolin Rossmanith et Benedikt Wirth de l'Université de Munster.

Dans le dernier chapitre de la thèse, nous étudions des fonctionnelles de la forme

$$\mathscr{F}_\varepsilon(\sigma,\varphi) := \int_\Omega f(\varphi)|\sigma| + \frac{1}{2}\left[\varepsilon|\nabla\varphi|^2 + \frac{\varphi^2}{\varepsilon}\right]\,\mathrm{d}x$$

Les deux principales différences par rapport aux modèles précédents sont la pénalisation linéaire du terme en $|\sigma|$ et la présence du terme φ^2 au lieu de $(1-\varphi)^2$. Des modèles analogues avec une pénalisation linéaire de la composante $|\sigma|$ ont été étudiés récemment dans le cas de la théorie de la fracture et de la fonctionnelle de Mumford-Shah. [ABS99,

Figure D.6: Computed mass flux σ and phase fields $\varphi_1, \varphi_2, \varphi_3$ for the cost function shown on the right, $\varepsilon = 0.005$. The color in σ indicates which phase field is active. The result is obtained by optimizing the functional defined in **Chapter Three**.

DMOT16]. Notre principale contribution est de trouver une forme explicite de la fonction de poids f pour obtenir dans la limite l'énergie $\mathscr{E}_h$. FPour une fonction de coàt de transport continu et concave h, nous définissons f comme suit

$$f(t) = (-h_*)^{-1}(t^2).$$

La fonction h_* est la transformation de Legendre (concave) de h. Dans ce modèle, φ prend la valeur 0 et non 1 en dehors du support de la mesure limite σ.En raison de ce résultat général, nous abordons le problème de l'approximation numérique de l'approximation fonctionnelle $\mathscr{F}_\varepsilon$.La pénalisation linéaire en σ peut àtre considérée comme un inconvénient par rapport aux méthodes étudiées précédemment qui étaient profondément basées sur le coàt quadratique $|\sigma|^2$. En raison de cette différence, nous avons commencé à étudier de nouvelles méthodes numériques basées sur le modèle de transport introduit par Beckman dans [Bec52]. Le màme résultat peut àtre obtenu avec un choix différent du potentiel du puits. A savoir, étant donné un potentiel W qui est une fonction paire, croissant sur $[0, +\infty)$ et nulle dans 0 nous introduisons l'énergie de transition

$$c_W(t) := \int_0^{|t|} 2\sqrt{W(s)}\, \mathrm{d}s.$$

Ensuite, en choisissant $f(t) = (-h_*)^{-1} \circ c_W(t)$ le màme résultat de Γ-convergence peut àtre obtenu avec une famille de fonctionnels définie comme suit

$$\mathscr{F}_\varepsilon(\sigma, \varphi) := \int_\Omega f(\varphi)|\sigma| + \frac{1}{2}\left[\varepsilon|\nabla\varphi|^2 + \frac{W(\varphi)}{\varepsilon}\right] \mathrm{d}x.$$

En vigueur de ce degré de liberté dans le choix du potentiel W nous commençons à analyser quel serait le meilleur choix. Ces questions, ainsi que d'autres, sont le sujet de la section finale qui examine les développements possibles des méthodes proposées.

Bibliography

[ABM14] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces*, volume 17 of *MOS-SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, second edition, 2014. doi:10.1137/1.9781611973488. Applications to PDEs and optimization.

[ABS99] R. Alicandro, A. Braides, and J. Shah. Free-discontinuity problems via functionals involving the L^1-norm of the gradient and their approximations. *Interfaces Free Bound.*, 1(1):17–37, 1999. URL `https://doi.org/10.4171/IFB/2`.

[AD00] L. Ambrosio and N. Dancer. *Calculus of variations and partial differential equations*. Springer-Verlag, Berlin, 2000. doi:10.1007/978-3-642-57186-2. Topics on geometrical evolution problems and degree theory, Papers from the Summer School held in Pisa, September 1996, Edited by G. Buttazzo, A. Marino and M. K. V. Murthy.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. Oxford University Press, New York, 2000.

[AT90] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Comm. Pure Appl. Math.*, 43(8):999–1036, 1990. doi:10.1002/cpa.3160430805.

[AT92] L. Ambrosio and V. M. Tortorelli. On the approximation of free discontinuity problems. *Boll. Un. Mat. Ital. B (7)*, 6(1):105–123, 1992.

[AT04] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[BBB95] G. Bouchitté, A. Braides, and G. Buttazzo. Relaxation results for some free discontinuity problems. *J. Reine Angew. Math.*, 458:1–18, 1995. doi:10.1515/crll.1995.458.1.

[BBL18] M. Bonnivard, E. Bretin, and A. Lemenant. Numerical approximation of the steiner problem in dimension 2 and 3. 04 2018.

[BCG14] G. Bellettini, A. Chambolle, and M. Goldman. The Γ-limit for singularly perturbed functionals of Perona-Malik type in arbitrary dimension. *Math. Models Methods Appl. Sci.*, 24(6):1091–1113, 2014. doi:10.1142/S0218202513500772.

[BCM09] M. Bernot, V. Caselles, and J.-M. Morel. *Optimal transportation networks*, volume 1955 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Models and theory.

[BDS96] G. Bouchitté, C. Dubs, and P. Seppecher. Transitions de phases avec un potentiel dégénéré à l'infini, application à l'équilibre de petites gouttes. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(9):1103–1108, 1996.

[Bec52] M. Beckmann. A continuous model of transportation. *Econometrica*, 20:643–660, 1952. doi:10.2307/1907646.

[BEZ15] M. Burger, T. Esposito, and C. I. Zeppieri. Second-order edge-penalization in the Ambrosio-Tortorelli functional. *Multiscale Model. Simul.*, 13(4):1354–1389, 2015. doi:10.1137/15M1020848.

[BLS15] M. Bonnivard, A. Lemenant, and F. Santambrogio. Approximation of length minimization problems among compact connected sets. *SIAM J. Math. Anal.*, 47(2):1489–1529, 2015. doi:10.1137/14096061X.

[Bra98] A. Braides. *Approximation of free-discontinuity problems*, volume 1694 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. doi:10.1007/BFb0097344.

[Bra02] A. Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002. doi:10.1093/acprof:oso/9780198507840.001.0001.

[BW17] A. Brancolini and B. Wirth. General transport problems with branched minimizers as functionals of 1-currents with prescribed boundary. preprint on https://arxiv.org/abs/1705.00162, 2017.

[BW18] A. Brancolini and B. Wirth. General transport problems with branched minimizers as functionals of 1-currents with prescribed boundary. *Calculus of Variations and Partial Differential Equations*, 57(3):82, Apr 2018. doi:10.1007/s00526-018-1364-4.

[CDRMS17] M. Colombo, A. De Rosa, A. Marchese, and S. Stuvard. On the lower semicontinuous envelope of functionals defined on polyhedral chains. *Nonlinear Anal.*, 163:201–215, 2017. URL `https://doi.org/10.1016/j.na.2017.08.002`.

[CFI16] S. Conti, M. Focardi, and F. Iurlano. Phase field approximation of cohesive fracture models. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(4):1033–1067, 2016. doi:10.1016/j.anihpc.2015.02.001.

[CFM17a] A. Chambolle, L. Ferrari, and B. Merlet. A phase-field approximation of the steiner problem in dimension two. *Advances in Calculus of Variations*, 2017. doi:10.1515/acv-2016-0034.

[CFM17b] A. Chambolle, L. Ferrari, and B. Merlet. Variational approximation of size-mass energies for k-dimensional currents. Accepted with minor revisions, 2017.

[CFM18] A. Chambolle, L. Ferrari, and B. Merlet. Strong approximation in h-mass of rectifiable currents under homological constraint. *In preparation*, 2018. To appear.

[CH58] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. i. interfacial free energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958, https://doi.org/10.1063/1.1744102. doi:10.1063/1.1744102.

[CR79] R. Courant and H. Robbins. *What is mathematics?* Oxford University Press, New York, 1979. An elementary approach to ideas and methods.

[CS11] G. Carlier and F. Santambrogio. A continuous theory of traffic congestion and Wardrop equilibria. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 390(Teoriya Predstavleniĭ, Dinamicheskie Sistemy, Kombinatornye Metody. XX):69–91, 307–308, 2011. doi:10.1007/s10958-012-0715-5.

[DG75] E. De Giorgi. Sulla convergenza di alcune successioni d'integrali del tipo dell'area. *Rend. Mat. (6)*, 8:277–294, 1975. Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday.

[DM93] G. Dal Maso. *An introduction to* Γ*-convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1993. doi:10.1007/978-1-4612-0327-8.

[DMOT16] G. Dal Maso, G. Orlando, and R. Toader. Fracture models for elasto-plastic materials as limits of gradient damage models coupled with plasticity: the antiplane case. *Calculus of Variations and Partial Differential Equations*, 55(3):45, Apr 2016. doi:10.1007/s00526-016-0981-z.

[DPH03] T. De Pauw and R. Hardt. Size minimization and approximating problems. *Calc. Var. Partial Differential Equations*, 17(4):405–442, 2003. doi:10.1007/s00526-002-0177-6.

[EG15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[Fed69] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[FF60] H. Federer and W. H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960. doi:10.2307/1970227.

[Fle66] W. H. Fleming. Flat chains over a finite coefficient group. *Trans. Amer. Math. Soc.*, 121:160–186, 1966. doi:10.2307/1994337.

[FMBM16] V. L. do Forte, F. Montenegro, J. A. de Moura Brito, and N. Maculan. Iterated local search algorithms for the euclidean steiner tree problem in n dimensions. *ITOR*, 23(6):1185–1199, 2016. doi:10.1111/itor.12168.

[Ghi14] F. Ghiraldin. Variational approximation of a functional of Mumford-Shah type in codimension higher than one. *ESAIM Control Optim. Calc. Var.*, 20(1):190–221, 2014. doi:10.1051/cocv/2013061.

[Gil67] E. N. Gilbert. Minimum cost communication networks. *Bell System Technical Journal*, 46(9):2209–2227, 1967. doi:10.1002/j.1538-7305.1967.tb04250.x.

[Iur13] F. Iurlano. Fracture and plastic models as Γ-limits of damage models under different regimes. *Adv. Calc. Var.*, 6(2):165–189, 2013. doi:10.1515/acv-2011-0011.

[Kar72] R. M. Karp. *Reducibility among combinatorial problems.* Plenum, New York, 1972.

[KP08] S. G. Krantz and H. R. Parks. *Geometric integration theory.* Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2008. doi:10.1007/978-0-8176-4679-0.

[Kru56] J. B. Kruskal, Jr. On the shortest spanning subtree of a graph and the traveling salesman problem. *Proc. Amer. Math. Soc.*, 7:48–50, 1956. doi:10.2307/2033241.

[LL97] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[LLSV14] J. Lellmann, D. A. Lorenz, C. Schönlieb, and T. Valkonen. Imaging with Kantorovich-Rubinstein discrepancy. *SIAM J. Imaging Sci.*, 7(4):2833–2859, 2014. URL `https://doi.org/10.1137/140975528`.

[LS14] A. Lemenant and F. Santambrogio. A Modica-Mortola approximation for the Steiner problem. *C. R. Math. Acad. Sci. Paris*, 352(5):451–454, 2014. doi:10.1016/j.crma.2014.03.008.

[MM77a] L. Modica and S. Mortola. Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. *Boll. Un. Mat. Ital. A (5)*, 14(3):526–529, 1977.

[MM77b] L. Modica and S. Mortola. Un esempio di Γ^--convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.

[Mon15] A. Monteil. *Approximations elliptiques d'nergies singulires sous contrainte de divergence*. PhD thesis, 2015. URL `http://www.theses.fr/2015SACLS135`. Thse de doctorat dirige par Ignat, Radu Mathmatiques appliques Paris Saclay 2015.

[Mon17] A. Monteil. Uniform estimates for a Modica-Mortola type approximation of branched transportation. *ESAIM Control Optim. Calc. Var.*, 23(1):309–335, 2017. doi:10.1051/cocv/2015049.

[Mor89] F. Morgan. Size-minimizing rectifiable currents. *Invent. Math.*, 96(2):333–348, 1989. doi:10.1007/BF01393966.

[MS89] D. Mumford and J. Shah. Optimal approximation by piecewise smooth functions and associated variational problems. *Communications on Pure and Applied Mathematics*, 42(5):577–685, 1989. doi:10.1002/cpa.3160420503.

[OS11] E. Oudet and F. Santambrogio. A Modica-Mortola approximation for branched transport and applications. *Arch. Ration. Mech. Anal.*, 201(1):115–142, 2011. doi:10.1007/s00205-011-0402-6.

[PS13] E. Paolini and E. Stepanov. Existence and regularity results for the Steiner problem. *Calc. Var. Partial Differential Equations*, 46(3-4):837–860, 2013. doi:10.1007/s00526-012-0505-4.

[PST15] E. Paolini, E. Stepanov, and Y. Teplitskaya. An example of an infinite Steiner tree connecting an uncountable set. *Adv. Calc. Var.*, 8(3):267–290, 2015. URL `https://doi.org/10.1515/acv-2013-0025`.

[San14] F. Santambrogio. A Dacorogna-Moser approach to flow decomposition and minimal flow problems. In *Congrès SMAI 2013*, volume 45 of *ESAIM Proc. Surveys*, pages 265–274. EDP Sci., Les Ulis, 2014. doi:10.1051/proc/201445027.

[Set99] J. A. Sethian. Fast marching methods. *SIAM Rev.*, 41(2):199–235, 1999. doi:10.1137/S0036144598347059.

[Smi93] S. K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. *Algebra i Analiz*, 5(4):206–238, 1993.

[Tal76] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976. doi:10.1007/BF02418013.

[Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003. doi:10.1007/b12016.

[Š07] M. Šilhavý. Divergence measure vectorfields: their structure and the divergence theorem. In *Mathematical modelling of bodies with complicated* *bulk and boundary behavior*, volume 20 of *Quad. Mat.*, pages 217–237. Dept. Math., Seconda Univ. Napoli, Caserta, 2007.

[Whi57] H. Whitney. *Geometric integration theory.* Princeton University Press, Princeton, N. J., 1957.

[Whi99a] B. White. The deformation theorem for flat chains. *Acta Math.*, 183(2):255–271, 1999. doi:10.1007/BF02392829.

[Whi99b] B. White. Rectifiability of flat chains. *Annals of Mathematics*, 150(1):165–184, 1999. URL `http://www.jstor.org/stable/121100`.

[WZ97] P. Winter and M. Zachariasen. Euclidean steiner minimum trees: An improved exact algorithm. *Networks*, 30(3):149–166, 1997, https://onlinelibrary.wiley.com/doi/pdf/10.1002/. doi:10.1002/(SICI)1097-0037(199710)30:3¡149::AID-NET1¿3.0.CO;2-L.

[Xia03] Q. Xia. Optimal paths related to transport problems. *Commun. Contemp. Math.*, 5(2):251–279, 2003. doi:10.1142/S021919970300094X.

[Xia04] Q. Xia. Interior regularity of optimal transport paths. *Calc. Var. Partial Differential Equations*, 20(3):283–299, 2004. doi:10.1007/s00526-003-0237-6.

université PARIS-SACLAY

ÉCOLE DOCTORALE
INTERFACES
Approches Interdisciplinaires : Fondements, Applications et Innovation

Titre : Approximations par champs de phases pour des problemes de transport branché

Mots clés : transport branché, calcul des variations, théorie géométrique de la mesure, Γ-convergence.

Résumé : Dans cette thèse, nous concevons des approximations par champ de phase de certains problèmes de Transport Branché. Le Transport Branché est un cadre mathématique pour modéliser des réseaux de distribution offre-demande qui présentent une structure d'arbre. En particulier, le réseau, les usines d'approvisionnement et le lieu de la demande sont modélisés en tant que mesures et le probléme est présenté comme un probléme d'optimisation sous contrainte. Le coût de transport d'une masse m le long d'un bord de longueur ℓ est $h(m)\ell$ et le coût total d'un réseau est défini comme la somme de la contribution sur tous ses arcs. Le cas du Transport Branché correspond avec la choix $h(m) = \alpha|m|$ où α est dans $[0, 1)$. La sous-additivité de la fonction cout s'assure que déplacer deux masses conjointement est moins cher que de le faire séparément. Dans ce travail, nous introduisons diverses approximations variationnelles du problème du transport branché. Les fonctionnelles que on vais utiliser sont basées sur une représentation par champ de phase du réseau et sont plus lisses que le problème original, ce qui permet des méthodes d'optimisation numérique efficaces. Nous introduisons une famille des fonctionnelles inspirées par le fonctionnelle de Ambrosio et Tortorelli pour modéliser une fonction de coût h affine dans l'espace $\mathbf{R}^2$. Pour ce cas, nous produisons un résultat complet de Γ-convergence et nous le corrélons avec une procédure de minimisation alternée pour obtenir des approximations numériques des minimiseurs. Puis nous généralisons cette approche à n'importe quel espace $\mathbf{R}^n$ et obtenons un résultat complet de Γ-convergence dans le cas de surfaces k-dimensionnelles avec $k < n$. En particulier, nous obtenons une approximation variationnelle du problème du Plateau dans n'importe quelle dimension et co-dimension. Dans la dernière partie de la thèse, nous proposons deux approches générales pour des fonctions de coût concave. Dans le premier, nous introduisons une approche par plusieurs champs de phase et récupérons n'importe quelle fonction de coût affine par morceaux. Enfin, nous proposons et étudions une famille de fonctions permettant d'obtenir dans la limite toutes fonction de coût concave h.

Title : Phase-field approximations for some branched transportation problems

Keywords : branched transport, calculus of variations, geometric measure theory, Γ-convergence.

Abstract : In this thesis we devise phase field approximations of some Branched Transportation problems. Branched Transportation is a mathematical framework for modeling supply-demand distribution networks which exhibit tree like structures. In particular the network, the supply factories and the demand location are modeled as measures and the problem is cast as a constrained optimization problem. The transport cost of a mass m along an edge with length ℓ is $h(m)\,\ell$ and the total cost of a network is defined as the sum of the contribution on all its edges. The branched transportation case consists with the specific choice $h(m) = |m|^\alpha$ where α is a value in $[0, 1)$. The sub-additivity of the cost function ensures that transporting two masses jointly is cheaper than doing it separately. In this work we introduce various variational approximations of the branched transport optimization problem. The approximating functionals are based on a phase field representation of the network and are smoother than the original problem which allows for efficient numerical optimization methods. We introduce a family of functionals inspired by the Ambrosio and Tortorelli one to model an affine transport cost functions. This approach is firstly used to study the problem any affine cost function h in the ambient space $\mathbf{R}^2$. For this case we produce a full Γ-convergence result and correlate it with an alternate minimization procedure to obtain numerical approximations of the minimizers. We then generalize this approach to any ambient space and obtain a full Γ-convergence result in the case of k-dimensional surfaces. In particular, we obtain a variational approximation of the Plateau problem in any dimension and co-dimension. In the last part of the thesis we propose two models for general concave cost functions. In the first one we introduce a multiphase field approach and recover any piecewise affine cost function. Finally we propose and study a family of functionals allowing to recover in the limit any concave cost function h.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l'Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France