A phase-field approximation of the Steiner problem in dimension two

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joint work with:
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Origin of our work

Given $S := \{x_0, \dots, x_N\} \subset \Omega$

Steiner problem $=$ $\operatorname{argmin}\{\mathcal{H}^1(K) : K$ is a compact, connected set $: S \subset K\}$.

[Bonnivard, Lemenant & Santambrogio (2015)]

$$\mathcal{BLS}_\varepsilon(\phi) := \underbrace{\frac{1}{c_\varepsilon}\sum_{i=1}^{N} d_\phi(x_0, x_i)}_{\text{Constraint term}} + \underbrace{\int_\Omega \left[\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{(1-\phi)^2}{2\varepsilon}\right] \mathrm{d}x}_{\text{Modica-Mortola}},$$

where

$$d_\phi(x, y) = \inf\left\{\int_\gamma \phi(x)\mathcal{H}^1(x) \; : \; \gamma \text{ curve in } \Omega \text{ connecting } x \text{ and } y\right\}.$$

Remark

$$\sum_{i=1}^{N} d_\phi(x_0, x_i) = \min\left\{\int_\Omega \phi|\sigma| \, \mathrm{d}x \; : \; \nabla\cdot\sigma = \delta_{x_0} - \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\right\}.$$

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Setting of the problem

Let Ω be an open, convex set, $\varepsilon \in (0,1]$ and $\omega_1, \omega_2 \in \mathcal{P}(\Omega)$ two probability measures such that

$$\mathrm{supp}(\omega_1) \cup \mathrm{supp}(\omega_2) \subset S := \{x_0, \dots, x_N\}, \quad (1)$$

$$\mathcal{M}_S(\Omega) := \{\sigma \in \mathcal{M}(\Omega, \mathbf{R}^2) \ : \ \nabla \cdot \sigma = \omega_1 - \omega_2 \text{ in } \mathcal{D}'(\overline{\Omega})\}. \quad (2)$$

Let ρ_ε be an approximation to the identity and $\eta = \eta(\varepsilon)$ we define the sets

$$W_\varepsilon(\Omega) = \left\{\phi \in W^{1,2}(\Omega) \ : \ \eta \leq \phi \leq 1 \text{ in } \Omega, \ \phi \equiv 1 \text{ on } \partial\Omega\right\},$$

$$V_\varepsilon(\Omega) := \{\sigma \in L^2(\Omega, \mathbf{R}^2) \ : \ \nabla \cdot \sigma = (\omega_1 - \omega_2) * \rho_\varepsilon \text{ in } \mathcal{D}'(\overline{\Omega})\}.$$

Functional

For couples $(\sigma, \phi) \in V_\varepsilon(\Omega) \times W_\varepsilon(\Omega)$ we define

$$\mathcal{F}_\varepsilon(\sigma, \phi) := \int_\Omega \frac{\phi^2|\sigma|^2}{2\varepsilon}\,\mathrm{d}x + \int_\Omega \left[\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{(1-\phi)^2}{2\varepsilon}\right]\mathrm{d}x.$$

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Main Results

Theorem: Equicoercivity [Chambolle, F., Merlet]

Let

$$\beta := \frac{\eta}{\varepsilon} \in (0, +\infty).$$

If $(\sigma_\varepsilon, \phi_\varepsilon) \in \mathcal{M}(\Omega) \times L^1(\Omega)$ such that $\mathcal{F}_\varepsilon(\sigma_\varepsilon, \phi_\varepsilon) \leq C < +\infty$. Then as $\varepsilon \downarrow 0$ we have $\phi_\varepsilon \to 1$ in $L^1(\Omega)$ and, up to a subsequence,

$$\sigma_\varepsilon \xrightarrow{*} \sigma = (\theta, \nu, \Sigma) \in \mathcal{M}_S(\Omega).$$

We write $\sigma = (\theta, \nu, \Sigma)$ for $\mathcal{H}^1$-rectifiable vector measures and denote

$$X := \{(\sigma, \phi) \in \mathcal{M}_S(\Omega) \times L^1(\Omega) : \sigma = (\theta, \nu, \Sigma) \text{ and } \phi \equiv 1\}.$$

Theorem: Γ-convergence [Chambolle, F., Merlet]

$$\Gamma - \lim_{\varepsilon \downarrow 0} \mathcal{F}_\varepsilon = \mathcal{E}_\beta := \begin{cases} \int_\Sigma [1 + \beta|\theta|] \, \mathrm{d}\mathcal{H}^1, & \text{if } (\sigma, \phi) \in X, \\ +\infty, & \text{otherwise in } \mathcal{M}_S(\Omega) \times L^1(\Omega). \end{cases}$$

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Some comments on β

1. In the equicoercivity result we obtain

$$|\sigma_\varepsilon|(\Omega) \leq \frac{C}{\beta} \implies \text{We need } \beta > 0 \text{ to bound the mass.}$$

2. We could restrict our attention to $\sigma \in \mathcal{M}_S(\Omega)$ such that $|\sigma|(\Omega) < C$, where the constant depends on the constraint and drop the lower bound on ϕ.
3. In any case

$$\Gamma - \lim_{\beta \downarrow 0} \mathcal{E}_\beta = \mathcal{E}_0.$$

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Two main ingredients

- We can cover $\Omega \setminus S$ with countable many relatively open, simply connected sets.

Remark

For any relatively open, simply connected set $O \subset \Omega \setminus S$ it holds $\nabla \cdot \sigma = 0$, thus:

$$\sigma = \nabla u^{\perp} \text{ for some } u \in W^{1,2}(O).$$

On the set O, $\mathcal{F}_\varepsilon$ reads as

$$\mathcal{LF}_\varepsilon(u, \phi) := \int_\Omega \frac{\phi^2 |\nabla u|^2}{2\varepsilon} \,\mathrm{d}x + \int_\Omega \left[\frac{\varepsilon}{2}|\nabla \phi|^2 + \frac{(1-\phi)^2}{2\varepsilon}\right] \mathrm{d}x.$$

Strictly related to [Ambrosio, Tortorelli (1990)] and fracture models [Iurlano (2013)].

- Polyhedral vector measures are dense. [White (1999)], [Xia (2003)], [Colombo, De Rosa, Marchese, Stuvard (2017)].

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Main Inequality

Using Cauchy-Schwarz inequality we get

$$[|Du_\varepsilon|(O)]^2 = \left(\int_O |\nabla u_\varepsilon|\right)^2 \leq \left(2\,\varepsilon \int_O \frac{1}{\phi_\varepsilon^2}\right)\left(\frac{1}{\varepsilon}\int_O \phi_\varepsilon^2 |\nabla u_\varepsilon|^2\right).$$

We fix $\lambda \in (0,1)$ and observe that

$$[|Du_\varepsilon|(O)]^2 \leq \left(\frac{\varepsilon^2}{\eta^2}\,\frac{4}{(1-\lambda)^2}\int_{\{\phi_\varepsilon<\lambda\}}\frac{(1-\phi_\varepsilon)^2}{2\varepsilon} + \frac{2\varepsilon}{\lambda^2}|\{\phi_\varepsilon \geq \lambda\}|\right)\left(\frac{1}{\varepsilon}\int_O \phi_\varepsilon^2|\nabla u_\varepsilon|^2\right).$$

$$\mathcal{LF}_\varepsilon(u,\phi) \geq \frac{[|Du_\varepsilon|(O)]^2}{\left(\beta^2\,\frac{4}{(1-\lambda)^2}\int_{\{\phi_\varepsilon<\lambda\}}\frac{(1-\phi_\varepsilon)^2}{2\varepsilon} + \frac{2\varepsilon}{\lambda^2}|O|\right)} + \int_{\{\phi_\varepsilon<\lambda\}}\frac{(1-\phi_\varepsilon)^2}{2\varepsilon}$$

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Possible variations?

1. $$\mathcal{F}_\varepsilon(\sigma,\phi) := \int_\Omega \frac{\phi^2|\sigma|^2}{\varepsilon^a}\,\mathrm{d}x + \int_\Omega \left[\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{(1-\phi)^2}{2\varepsilon}\right]\,\mathrm{d}x.$$

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Numerical Setting

To introduce the numerical setting let $f_\varepsilon := \left(\delta_{x_0} - \frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right) * \rho_\varepsilon$

$$G_\varepsilon(\sigma,\phi) = \int_\Omega \left[\frac{1}{2\varepsilon}|\phi|^2|\sigma|^2\right]\,\mathrm{d}x \text{ and } \Lambda_\varepsilon(\phi) := \int_\Omega \left[\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{(1-\phi^2)}{2\varepsilon}\right]\,\mathrm{d}x$$

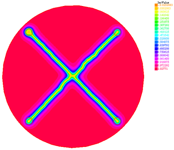
By duality we have

$$\begin{aligned}\min_\sigma G_\varepsilon(\sigma,\phi) &= \sup_u \inf_\sigma \int_\Omega \frac{1}{2\varepsilon}|\phi|^2|\sigma|^2 - (\langle\nabla u,\sigma\rangle + uf_\varepsilon)\,\mathrm{d}x \\ &= -\min_u \int_\Omega \frac{\varepsilon|\nabla u|^2}{2|\phi|^2} + uf_\varepsilon\,\mathrm{d}x = -\min_u \overline{G}_\varepsilon(u,\phi),\end{aligned}$$

with $\sigma = \frac{\varepsilon\nabla u}{\phi^2}$. Given an initial guess ϕ_0 we define

$$u_j := \operatorname{argmin} \overline{G}_\varepsilon(u,\phi_j), \qquad \text{set } \sigma_j := \frac{\varepsilon\nabla u_j}{\phi_j^2}$$

$$\phi_{j+1} := \operatorname{argmin} G_\varepsilon(\sigma_j,\phi) + \Lambda_\varepsilon(\phi).$$



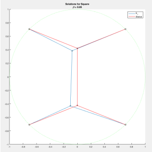
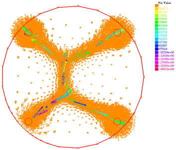


Figure: Left: Graph of the level sets of the function ϕ obtained via alternate minimization. Right: Set minimizing the energy $\mathcal{E}_\beta$ for $\beta = 0$ and $\beta = 0.05$.

Idea: (Assuming we have already obtained the correct topology.)

Consider a deformation $V : \Omega \to \Omega$ zero near the points of the constraint. Set $\mathcal{F}_\varepsilon(V) := \mathcal{F}_\varepsilon(\sigma \circ (Id + V), \phi \circ (Id + V))$ and

$$\langle V, W \rangle_{W^{1,2}} = \langle d\mathcal{F}_\varepsilon(Id), W \rangle$$



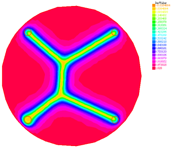


Figure: Graphs for the couple (σ, ϕ) obtained via the joint minimization.

Same idea only on the component Λ_ε

$$\langle V, W \rangle_{W^{1,2}} = \langle d\Lambda_\varepsilon(Id), W \rangle$$

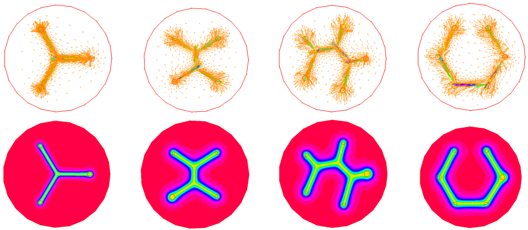


Figure: Graphs for the couple (σ, ϕ) obtained via the second joint minimization.

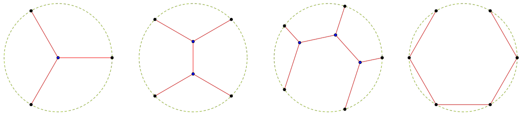


Figure: Exact solutions for the Steiner problem.

Undergoing generalization in $\mathbf{R}^n$

For couples $(\sigma, \phi) \in V_\varepsilon(\Omega) \times W_\varepsilon(\Omega)$ we define

$$\mathcal{F}_\varepsilon(\sigma\,,\,\phi) := \int_\Omega \left[\varepsilon^{p-n+1}|\nabla\phi|^p + \frac{(1-\phi)^2}{\varepsilon^{n-1}} + \frac{\phi|\sigma|^2}{\varepsilon^\beta}\right]\,\mathrm{d}x$$

and $+\infty$ otherwise in $\mathcal{M}(\Omega, \mathbf{R}^n) \times L^1(\Omega)$.

The cost function is given by the study of the above functional restricted on $n-1$ dimensional balls, namely

$$f_\varepsilon(m, r) = \min\{\mathcal{F}_\varepsilon(\theta\,,\,\phi)\ :\ \theta \in L^1(B_r') \,\&\, \|\theta\| = m\}.$$

Thank you for your attention!