Phase-field approximations for branched transport, urban planning and Steiner problem

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GdT Calcul des Variations
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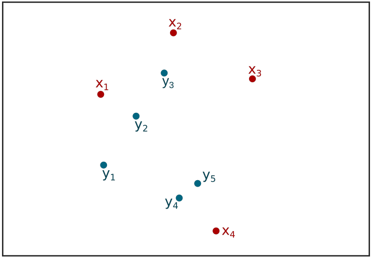
$\omega_+, \omega_- \in \mathcal{P}(\mathbf{R}^n)$ atomic probability measures

- *'Source' measure* $\omega_+ = \sum_i a_i \delta_{x_i}$,
- *'Sink' measure* $\omega_- = \sum_j b_j \delta_{x_j}$.

Polyhedral transport flux

$\sigma \in \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n)$ vector measure

$$\sigma = \sum_i \theta_i \nu_i \mathcal{H}^1 \llcorner \Sigma_i$$



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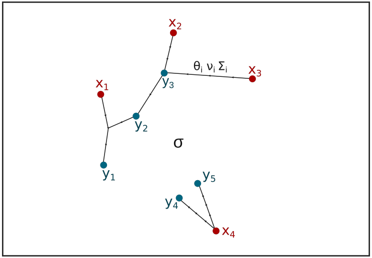

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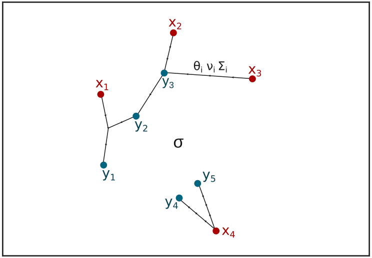
Action

for $\phi \in \mathcal{C}_0(\mathbf{R}^n, \mathbf{R}^n)$

$$(\sigma, \phi) = \sum_i \int_{\Sigma_i} \theta_i (\nu_i, \phi) \, \mathrm{d}\mathcal{H}^1$$



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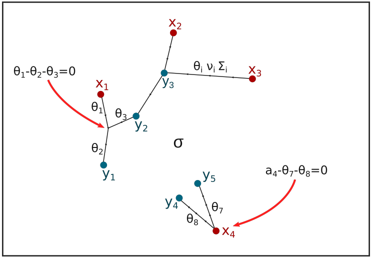
Constraint

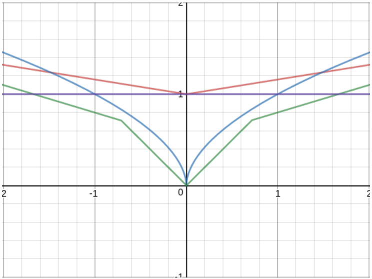
$$\operatorname{div} \sigma = \omega_+ - \omega_- \quad \text{in } \mathcal{D}'(\mathbf{R}^n)$$



Cost function

Let $f : \mathbf{R} \to \mathbf{R}^+$ be such that:

1. lower semicontinuous,
2. $f(0) = 0$,
3. sub-additive,
4. even.



- *Branched transport:*

$$f(\theta) = |\theta|^{\alpha} \quad \text{with } \alpha \in [0,1)$$

- *'Steiner' cost:*

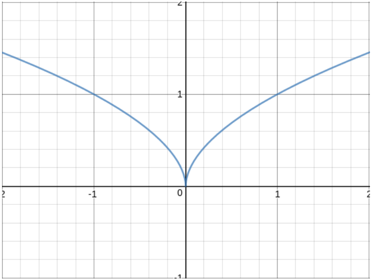
$$f(\theta) = \begin{cases} 0, & \text{if } \theta = 0, \\ 1, & \text{otherwise.} \end{cases}$$

- *Urban Planning cost:*

$$f(\theta) = \min\{\alpha_0|\theta|, \alpha_1|\theta| + \beta\}$$
$$\beta > 0 \text{ and } 0 < \alpha_1 < \alpha_0$$

- *'Our' cost functional:*

$$f(\theta) = \begin{cases} 0, & \text{if } \theta = 0, \\ (1 + \alpha|\theta|), & \text{otherwise.} \end{cases}$$



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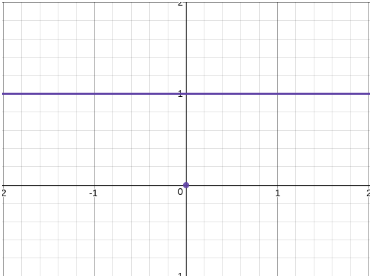

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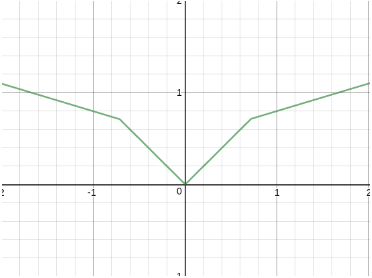

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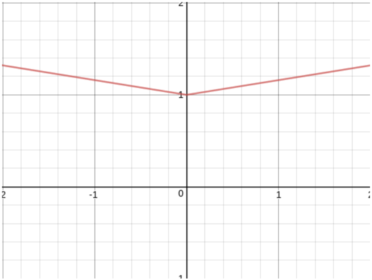

For polyhedral transport flux we introduce

$$\mathcal{F}(\sigma) = \sum_i f(\theta_i)\mathcal{H}^1(\Sigma_i) \qquad \text{if } \sigma = \sum_i \theta_i \nu_i \mathcal{H}^1 \llcorner \Sigma_i. \tag{1}$$

If f is:

1. lower semicontinuous,
2. $f(0) = 0$,
3. sub-additive,
4. even.

$\mathcal{F}$ extends on $X := \{\sigma \in \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n) \,:\, \operatorname{div} \sigma = \omega_+ - \omega_-\}$ via

Relaxation

$$\overline{\mathcal{F}}(\sigma) := \inf \left\{ \liminf_{j \to +\infty} \mathcal{F}(\sigma_j) : \sigma_j \in X \text{ of the form } (\textbf{??}) \text{ and } \sigma_j \overset{*}{\rightharpoonup} \sigma \right\}.$$

Rectifiable σ

We will say that σ is rectifiable if $\sigma = (\theta, \nu, \Sigma)$ with

1. Σ is $\mathcal{H}^1$-rectifiable,
2. $\theta \in L^1(\Sigma, \mathcal{H}^1 \llcorner \Sigma)$,
3. $\nu : \Sigma \to S^1$ is tangent to Σ, $\mathcal{H}^1 \llcorner \Sigma$-a.e..

Gilbert-Steiner energy

If $\sigma \in X$ is rectifiable then $\overline{\mathcal{F}}(\sigma)$ can be written as

$$\int_\Sigma f(|\theta|)\,\mathrm{d}\mathcal{H}^1.$$

Find approximation of: $\operatorname{argmin}\{\mathcal{F}(\sigma) : \sigma \in X\}$

WHY: Steiner Tree Problem is NP-hard. [Karp (1972)]

$$f(\theta) = \begin{cases} 0, & \text{if } \theta = 0, \\ 1, & \text{otherwise.} \end{cases} \qquad \omega_+ = \delta_{x_0}, \quad \omega_- = \frac{1}{N}\sum_1^N \delta_{y_i}$$

- A phase-field approximation of the Steiner problem in dimension two [A. Chambolle, F., B. Merlet (2016)]
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A phase-field approximation of the Steiner problem in dimension two

From now on $(\varepsilon) \subset \mathbf{R}$ will be a sequence such that $\varepsilon \downarrow 0$ and $\Omega \subset \mathbf{R}^2$ open and convex. Let ρ_ε be a convolution kernel we introduce

- Relaxed vector measure:

$$\sigma \in L^1(\Omega, \mathbf{R}^2) \quad \text{and} \quad \operatorname{div} \sigma = (\omega_+ - \omega_-) * \rho_\varepsilon \qquad \text{in } \mathcal{D}'(\mathbf{R}^n)$$

- Phase-field:

$$\varphi \in W^{1,2}(\Omega), \quad \varphi \geq \eta \quad \text{and} \quad \varphi \llcorner \partial\Omega \equiv 1$$

For such couples (σ, φ) we let

$$\mathcal{F}_\varepsilon(\sigma, \varphi) := \underbrace{\int_\Omega \frac{\varphi^2 |\sigma|^2}{2\,\varepsilon}\,\mathrm{d}x}_{\text{'Constraint'}} + \underbrace{\int_\Omega \frac{\varepsilon}{2}|\nabla \varphi|^2 + \frac{(1-\varphi)^2}{2\,\varepsilon}\,\mathrm{d}x}_{\text{'Modica-Mortola'}}$$

then we extend it on $\mathcal{M}(\Omega, R^2) \times L^2(\Omega)$ as $+\infty$.

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[Bonnivard, Lemenant, Santambrogio (2015)] consider

$$\frac{1}{c_\varepsilon}\sum_{i=1}^{N} d_\varphi(x_0, x_i) \qquad \text{instead of} \qquad \int_\Omega \frac{\varphi^2|\sigma|^2}{2\,\varepsilon}\,\mathrm{d}x$$

where

$$d_\varphi(x,y) = \inf\left\{ \int_\gamma \varphi(x)\,\mathrm{d}\mathcal{H}^1(x) \;:\; \gamma \text{ curve in } \Omega \text{ connecting } x \text{ and } y \right\}.$$

Remark

$$\sum_{i=1}^{N} d_\varphi(x_0, x_i) = \min\left\{ \int_\Omega \varphi|\sigma|\,\mathrm{d}x \;:\; \nabla\cdot\sigma = \delta_{x_0} - \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \right\}.$$

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Equicoercivity

Consider a sequence $(\sigma_\varepsilon, \varphi_\varepsilon)$ such that

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Then $\varphi_\varepsilon \to 1$ in L^2

$$\int_\Omega (1-\varphi_\varepsilon)^2\,\mathrm{d}x \leq 2\,\varepsilon\,C.$$

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What about σ_ε? By Cauchy-Schwarz inequality we have

$$|\sigma_\varepsilon|(\Omega)^2 = \left(\int_\Omega |\sigma_\varepsilon|\,\mathrm{d}x\right)^2 \le \left(\int_\Omega \frac{\varphi_\varepsilon^2|\sigma_\varepsilon|^2}{2\varepsilon}\,\mathrm{d}x\right)\left(\int_\Omega \frac{2\varepsilon}{\varphi_\varepsilon^2}\,\mathrm{d}x\right)$$

Fix $\lambda \in (0,1)$ then

$$\begin{aligned}
\int_\Omega \frac{2\varepsilon}{\varphi_\varepsilon^2}\,\mathrm{d}x &= \int_{\{\varphi_\varepsilon \ge \lambda\}} \frac{2\varepsilon}{\varphi_\varepsilon^2}\,\mathrm{d}x + \int_{\{\varphi_\varepsilon \le \lambda\}} \frac{2\varepsilon}{\varphi_\varepsilon^2}\,\mathrm{d}x \\
&\le \frac{2\varepsilon}{\lambda^2}\mathcal{L}^2(\{\varphi_\varepsilon \ge \lambda\}) + \frac{2\varepsilon}{\eta^2}\frac{2\varepsilon}{(1-\lambda)^2}\int_{\{\varphi_\varepsilon \le \lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}\,\mathrm{d}x \\
&\le \frac{2\varepsilon}{\lambda^2}|\Omega| + \frac{4\varepsilon^2}{(1-\lambda)^2\,\eta^2}\int_{\{\varphi_\varepsilon \le \lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}\,\mathrm{d}x.
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Theorem [Chambolle, F., Merlet]

If $\eta = \beta\varepsilon$ with $\beta > 0$ the functional is equicoercive with respect to the weak-$*$ convergence of measures and the strong L^2 convergence.

This condition is not necessary for the Γ-convergence alternatively we could ask

$$\sigma \in L^1(\Omega, \mathbf{R}^2), \quad \operatorname{div} \sigma = (\omega_+ - \omega_-)\rho_\varepsilon \quad \text{and} \quad |\sigma| \leq M.$$

Γ-liminf (intuitively)

Consider $\sigma_\varepsilon \stackrel{*}{\rightharpoonup} \sigma$ and $\varphi \to 1$ in $L^2(\Omega)$. We split the functional on the two sets $\{\varphi_\varepsilon \leq \lambda\}$ and $\{\varphi_\varepsilon > \lambda\}$. First remark that

$$C > \int_{\{\varphi_\varepsilon > \lambda\}} \frac{\varphi_\varepsilon^2 |\sigma_\varepsilon|^2}{2\varepsilon} \,\mathrm{d}x \geq \frac{\lambda^2}{2\varepsilon |\{\varphi_\varepsilon > \lambda\}|} |\sigma_\varepsilon|(\{\varphi_\varepsilon > \lambda\})^2$$

$' \implies '$ $\mathrm{supp}(\sigma) \subset \{\varphi \neq 1\}$. Then

$$\mathcal{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \geq \underbrace{\int_\Omega \frac{\varphi_\varepsilon^2 |\sigma_\varepsilon|^2}{2\varepsilon} \,\mathrm{d}x + \int_{\{\varphi_\varepsilon < \lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon} \,\mathrm{d}x +}_{A_\varepsilon}$$

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$$\mathcal{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \geq \underbrace{\int_\Omega \frac{\varphi_\varepsilon^2 |\sigma_\varepsilon|^2}{2\varepsilon} \, \mathrm{d}x + \int_{\{\varphi_\varepsilon < \lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon} \, \mathrm{d}x}_{A_\varepsilon} +$$

$$+ \underbrace{\int_{\{\varphi_\varepsilon > \lambda\}} \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon|^2 + \frac{(1-\varphi_\varepsilon)^2}{2\,\varepsilon} \, \mathrm{d}x}_{B_\varepsilon}$$

$$A_\varepsilon \quad \geq \quad \frac{[|\sigma_\varepsilon|(\Omega)]^2}{\left(\beta^2 \, \frac{4}{(1-\lambda)^2} \, \int_{\{\varphi_\varepsilon < \lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon} + \frac{2\varepsilon}{\lambda^2} |\Omega|\right)} + \int_{\{\varphi_\varepsilon < \lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}$$

Γ-liminf (intuitively)

Optimize with respect to $\int_{\{\varphi_\varepsilon<\lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}$ we obtain

$$\int_\Omega \frac{\varphi_\varepsilon^2|\sigma_\varepsilon|^2}{2\varepsilon}\,\mathrm{d}x + \int_{\{\varphi_\varepsilon<\lambda\}} \frac{(1-\varphi_\varepsilon)^2}{2\varepsilon}\,\mathrm{d}x \geq \beta|\sigma_\varepsilon|(\Omega)$$

On the other hand similarly to [Modica-Mortola (1977)] .

$$\int_{\{\varphi_\varepsilon>\lambda\}} \frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{(1-\varphi_\varepsilon)^2}{2\,\varepsilon}\,\mathrm{d}x \geq \int_{\{\varphi_\varepsilon>\lambda\}} |\nabla\varphi_\varepsilon|(1-\varphi_\varepsilon)\,\mathrm{d}x$$
$$\geq \mathcal{H}^1(\{\phi\neq 1\})$$

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$$\liminf_{\varepsilon\downarrow 0} \mathcal{F}_\varepsilon(\sigma_\varepsilon,\varphi_\varepsilon) \geq \beta|\sigma|(\Omega) + \mathcal{H}^1(\mathrm{supp}(\sigma)).$$

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$\Gamma - \liminf$

Theorem: $\Gamma - \liminf$ [Chambolle, F., Merlet]

For any $(\sigma, \varphi) \in \mathcal{M}(\Omega, \mathbf{R}^2)$ and any $(\sigma_\varepsilon, \varphi_\varepsilon)$ such that $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ and $\varphi_\varepsilon \to \varphi$ it holds

$$\liminf_{\varepsilon \downarrow 0^+} \mathcal{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \geq \mathcal{F}_\beta(\sigma, \varphi)$$

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$$\mathcal{F}_\beta(\sigma, \varphi) := \begin{cases} \displaystyle\int_\Sigma (1 + \beta|\theta|) \, \mathrm{d}\mathcal{H}^1 & \text{if } \sigma = \theta\nu\mathcal{H}^1 \llcorner \Sigma \text{ and } \varphi \equiv 1 \\ +\infty & \text{otherwise.} \end{cases}$$

1. In $\Omega \setminus \operatorname{supp}((\omega_+ - \omega_-) * \rho_\varepsilon)$, $\operatorname{div} \sigma_\varepsilon = 0$ therefore
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2. $\Gamma - \liminf$ via slicing.

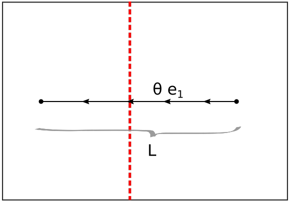
Consider a measure supported on a single segment $\sigma = \theta e_1 \mathcal{H}^1 \llcorner [0, L] \times \{0\}$. Let $a_\varepsilon := \frac{\theta \beta\, \varepsilon}{2}$ and $d_\Sigma(x)$ the distance function from Σ we set

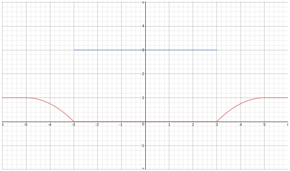
$$\sigma_\varepsilon := \frac{\theta}{2 a_\varepsilon} \cdot e_1 \chi_{\{d_\Sigma(x) \le a_\varepsilon\}}$$

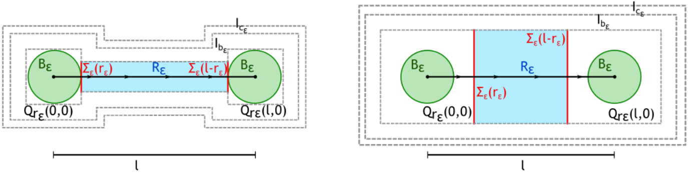
and

$$\phi_\varepsilon(x) := 1 - (1 - \eta) \exp\left(\frac{a_\varepsilon - d_\Sigma(x)}{\varepsilon} \right)$$

if $d_\Sigma(x) > a_\varepsilon$ and η otherwise.







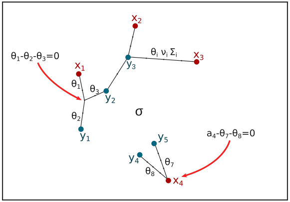
Idea: Solve the Poisson problem

$$\Delta u = \pm\theta\ \rho$$

with Neumann boundary conditions. Replace in a ball centered at branching points σ_ε with

$$\frac{\nabla u(x/\varepsilon)}{\varepsilon}.$$

Use Kirchoff's laws and linearity of divergence operator.



Since Polyhedral transport flux are dense in energy [Xia (1997)] we have

Theorem: $\Gamma - \limsup$ [Chambolle, F., Merlet]

For any $(\sigma, \varphi) \in \mathcal{M}(\Omega, \mathbf{R}^2)$ there exists a sequence $(\sigma_\varepsilon, \varphi_\varepsilon)$ such that $\varphi_\varepsilon \to \varphi$, $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$

$$\operatorname{div} \sigma_\varepsilon = (\omega_+ - \omega_-) * \rho_\varepsilon$$

and it holds

$$\limsup_{\varepsilon \downarrow 0^+} \mathcal{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \leq \mathcal{F}_\beta(\sigma, \varphi).$$

What happens in $\mathbf{R}^n$?

Let $\Omega \subset \mathbf{R}^n$ open and convex. Let ρ_ε be a convolution kernel we introduce

- Relaxed vector measure:

$$\sigma \in L^1(\Omega, \mathbf{R}^n) \quad \text{and} \quad \operatorname{div} \sigma = (\omega_+ - \omega_-) * \rho_\varepsilon \qquad \text{in } \mathcal{D}'(\mathbf{R}^n)$$

- Phase-field:

$$\varphi \in W^{1,2}(\Omega), \quad \varphi \geq \eta \quad \text{and} \quad \varphi \llcorner \partial\Omega \equiv 1$$

For such couples (σ, φ) we let

$$\mathcal{F}_\varepsilon(\sigma, \varphi) := \int_\Omega \left[\varepsilon^{p-n+1} |\nabla \varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-1}} + \frac{\varphi |\sigma|^2}{\varepsilon} \right] \mathrm{d}x$$

then we extend it on $\mathcal{M}(\Omega, R^n) \times L^2(\Omega)$ as $+\infty$.

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What happens in $\mathbf{R}^n$?

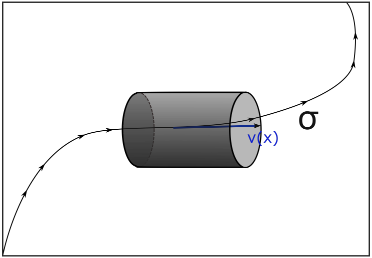
As before it holds

Theorem [Chambolle, F., Merlet]

If $\eta = \beta\varepsilon^n$ with $\beta > 0$ the functional is equicoercive with respect to the weak-$*$ convergence of measures and the strong L^2 convergence.

ISSUE: for the $\Gamma - \liminf$ we cannot use anymore the techinques used in $\mathbf{R}^2$. We first prove that for any $(\sigma_\varepsilon, \varphi_\varepsilon)$ such that $\mathcal{F}_\varepsilon(\sigma_\varepsilon, \varphi_\varepsilon) \leq C < +\infty$ up to subsequence $\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma$ with σ rectifiable. [White (1999)]

We consider an infinitesimal cylinder oriented as the tangent and study the problem on each $n-1$ slice of the cylinder.



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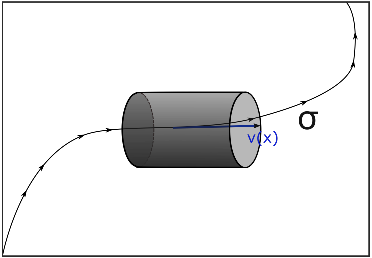

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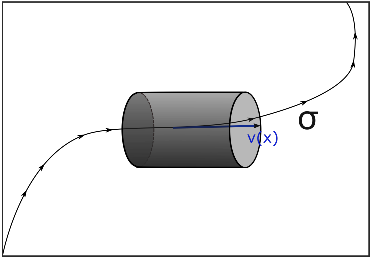

To recover the cost function we achieve for the limit functional we need to study the functional restricted on $n-1$ dimensional balls B'_r

$$f_\varepsilon(m,r) = \min\{\mathcal{F}_\varepsilon(\theta,\varphi) \;:\; \theta \in L^1(B'_r)\,\&\,\|\theta\| = m\}.$$

we show that it is equivalent to consider

$$\overline{f}_\varepsilon(m,r) = \min\{\mathcal{F}_\varepsilon(\theta,\varphi) \;:\; \theta \in L^1(B'_r)\,\&\,\|\theta\| = m, \varphi \llcorner \partial B'_r \equiv 1\}.$$

Pólya-Szegó $\implies$ minimizers are radially symmetric.

$$f^d_\beta(m) = \min_{\hat r > 0} \left\{ \frac{\beta\, m^2}{\omega_d\ \hat r^d} + \omega_d\ \hat r^d + (d-1)\,\omega_d\ q^d_\infty(0,\hat r) \right\}$$

Where

$$q^d_\infty(\xi,\hat r) := \inf\left\{ \int_{\hat r}^{+\infty} t^{d-1}\left[|v'|^p + (1-v)^2\right]\,\mathrm{d}t \;:\; v(\hat r) = \xi \text{ and } \lim_{t\to+\infty} v(t) = 1 \right\}.$$

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Result

Theorem: Γ-limit [Chambolle, F., Merlet]

$$\Gamma - \lim_{\varepsilon \downarrow 0} \mathcal{F}_\varepsilon = \mathcal{F}_\beta$$

where

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Drawback: No explicit cost function.

Advantages: The same approach can be used to approach similar problems with k-smooth manifold with the rescaling

$$\mathcal{F}_\varepsilon(\sigma,\varphi) := \int_\Omega \left[\varepsilon^{p-n+k}|\nabla\varphi|^p + \frac{(1-\varphi)^2}{\varepsilon^{n-k}} + \frac{\varphi|\sigma|^2}{\varepsilon}\right] \mathrm{d}x$$

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$$\mathcal{L}^i_\varepsilon(\varphi) = \int_\Omega \left[\frac{\beta_i^2\,|\nabla\varphi_i(x)|^2\,\varepsilon}{4\,\alpha_i} + \frac{\alpha_i(\varphi_i(x) - 1)^2}{4\,\varepsilon}\right]\,\mathrm{d}x\,.$$

Instead of the minimum in the latter consider the convex envelope of C_ε in $|\sigma|$.

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We no longer need to ask for $\omega_+, \omega_- \in \mathcal{P}(\Omega)$ to be atomic.

We prove that

Theorem [Chambolle, F., Rossmanith, Wirth]

The functional previously introduced are equicoercive in $\mathcal{M}(\Omega, \mathbf{R}^2) \times L^2(\Omega) \times L^2(\Omega)$ and

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where if $\sigma = \sigma^\perp + \theta\nu\mathcal{H}^1 \llcorner \Sigma$

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Remark: we can use any number of phase-fields.

Set $f_\varepsilon = (\omega_+ - \omega_-) * \rho_\varepsilon$

$$\min_{\sigma,\ \nabla\cdot\sigma = f_\varepsilon} \int_\Omega \frac{(\varphi^2 + \alpha\varepsilon^2)|\sigma|^2}{2\varepsilon}\,\mathrm{d}x = \min_\sigma \max_\lambda \int_\Omega \frac{(\varphi^2 + \alpha\varepsilon^2)|\sigma|^2}{2\varepsilon} + \lambda\left(\nabla\cdot\sigma - f_\varepsilon\right)\,\mathrm{d}x$$
$$= \max_\lambda \min_\sigma \int_\Omega \frac{(\varphi^2 + \alpha\varepsilon^2)|\sigma|^2}{2\varepsilon} - \nabla\lambda\sigma - f_\varepsilon\lambda\,\mathrm{d}x$$
$$= \min_\lambda \int_\Omega \frac{\varepsilon|\nabla\lambda|^2}{2(\varphi^2 + \alpha\varepsilon^2)} + \lambda f_\varepsilon\,\mathrm{d}x.$$

Therefore we introduce an alternating minization algorithm. Given an initial guess φ_0

$$\begin{cases} \lambda_j = \operatorname{argmin} \int_\Omega \frac{\varepsilon|\nabla\lambda|^2}{2(\varphi_{j-1}^2 + \alpha\varepsilon^2)} + \lambda f_\varepsilon\,\mathrm{d}x, \\ \sigma_j = \frac{\varepsilon\nabla\lambda_j}{\varphi_{j-1}^2 + \alpha\varepsilon^2}, \\ \varphi_j = \operatorname{argmin} \mathcal{F}_\varepsilon(\sigma_j, \varphi). \end{cases}$$

Set $f_\varepsilon = (\omega_+ - \omega_-) * \rho_\varepsilon$

$$\min_{\sigma,\ \nabla\cdot\sigma = f_\varepsilon} \int_\Omega \frac{(\varphi^2 + \alpha\varepsilon^2)|\sigma|^2}{2\varepsilon}\,\mathrm{d}x = \min_\sigma \max_\lambda \int_\Omega \frac{(\varphi^2 + \alpha\varepsilon^2)|\sigma|^2}{2\varepsilon} + \lambda\left(\nabla\cdot\sigma - f_\varepsilon\right)\,\mathrm{d}x$$
$$= \max_\lambda \min_\sigma \int_\Omega \frac{(\varphi^2 + \alpha\varepsilon^2)|\sigma|^2}{2\varepsilon} - \nabla\lambda\sigma - f_\varepsilon\lambda\,\mathrm{d}x$$
$$= \min_\lambda \int_\Omega \frac{\varepsilon|\nabla\lambda|^2}{2(\varphi^2 + \alpha\varepsilon^2)} + \lambda f_\varepsilon\,\mathrm{d}x.$$

Therefore we introduce an alternating minization algorithm. Given an initial guess φ_0

$$\begin{cases} \lambda_j = \operatorname{argmin} \int_\Omega \frac{\varepsilon|\nabla\lambda|^2}{2(\varphi_{j-1}^2 + \alpha\varepsilon^2)} + \lambda f_\varepsilon\,\mathrm{d}x, \\ \sigma_j = \frac{\varepsilon\nabla\lambda_j}{\varphi_{j-1}^2 + \alpha\varepsilon^2}, \\ \varphi_j = \operatorname{argmin} \mathcal{F}_\varepsilon(\sigma_j, \varphi). \end{cases}$$

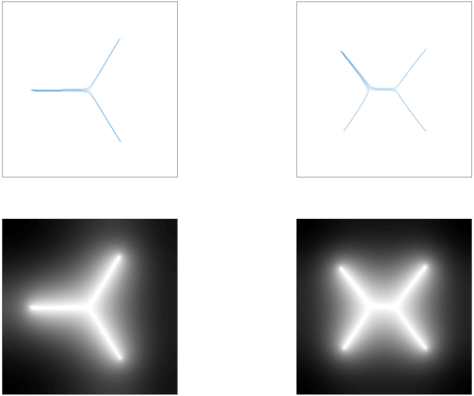

Figure: One source and two/three sinks, one phase field without diffuse component, $\alpha_1 = 0.05$, $\beta_1 = 1$.

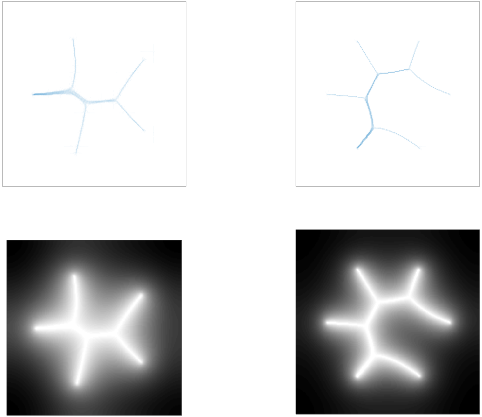

Figure: One source and four/five sinks, one phase field without diffuse component, $\alpha_1 = 0.05$, $\beta_1 = 1$.

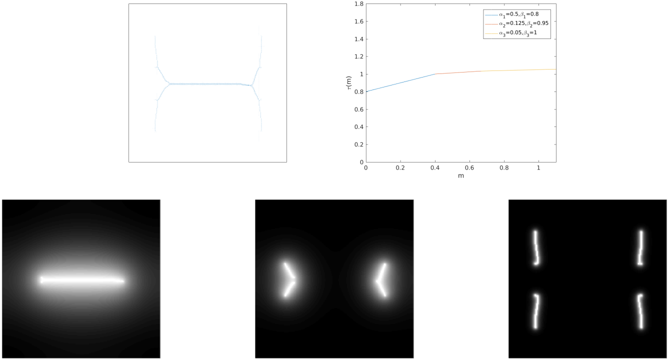

Figure: Four sources and four sinks, three phase fields without diffuse component, $\alpha_1 = 0.5$, $\beta_1 = 0.8$, $\alpha_2 = 0.125$, $\beta_2 = 0.95$, $\alpha_3 = 0.05$, $\beta_3 = 1$.

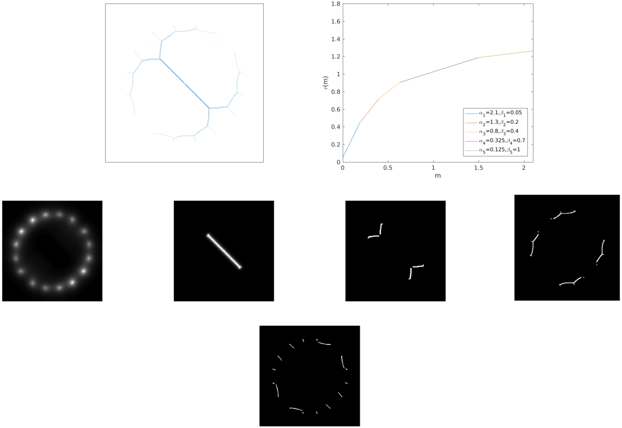

Figure: One source in the middle and 16 sinks on the circle boundaries, five phase fields without diffuse component, $\alpha_1 = 4.2$, $\beta_1 = 0.05$, $\alpha_2 = 2.6$, $\beta_2 = 0.2$, $\alpha_3 = 1.6$, $\beta_3 = 0.4$, $\alpha_4 = 0.65$, $\beta_4 = 0.7$, $\alpha_5 = 0.25$, $\beta_5 = 1$.

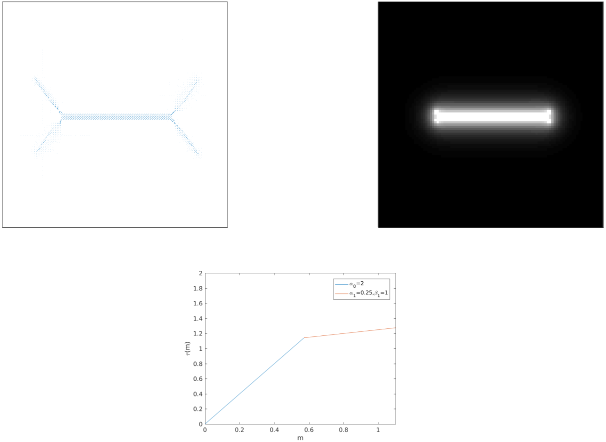

Figure: Two sources and two sinks, one phase field with a diffuse component, $\alpha_0 = 2$, $\alpha_1 = 0.25$, $\beta_1 = 1$.

Conclusion

Recall:

1) Introduced a phase-field approximation to the branched transport with $f(\theta) = 1 + \beta|\theta|$,
2) Generalized this approach to deal with
   a. ambient space $\mathbf{R}^n$
   b. k-currents problem
   c. Urban planning-type costs in $\mathbf{R}^2$.

THANKS FOR YOUR ATTENTION