INDEX - LECTURES SUMMARY

L1

- Examples of a ***‘Transportation or Transhipment problems’***
  - Earth Moving problem: (HW1) Which is the most efficient method to transport soil? Greedy Method? HW Find a counter example
  - Image retrieval??: create a signature using clustering…similarity measure
- Flow Problem: a more general problem

L2

- Linear problems: Transhipment & Flow problems
- Elementary network problem: Assignments of Jobs to Workers (HW2) Design a LP to minimize the cut of servicing all jobs given that each edge (w, j) has a cut Cwj
- Hall’s Matching Theorem: perfect matching

L3

- Bipartite Matching
- Koenig’s Theorem… by contradiction…
- MAX-FLOW ==== MIN CAPACITY of any CUT

L4

- Max-flow = Min cut → Weak-duality
- Algorithm for solving Maximum flow: residual capacities
- Complexity of the algorithm?
- Duality in Linear Programs

L5

- Visual representation of Linear Programs and its constraints
- HW: solve for alpha1 and alpha2, or lambda1 and lambda2.
- Dual program to linear program
- Convex set examples:
  - Ray
  - Lines
  - Line segments

- Cones
- Hyperplane
- Hypersphere & Ellipsoids

__L6__

- ➢ Do operations on convex sets preserve convexity? Is convexity preserved under intersections?
- ➢ Other transformations: convexity is preserved?
- ➢ Projection function: (HW: show that P preserves convexity)
- ➢ An interesting Property of Convex sets separating Hyperplane theorem of Convex sets: HW BONUS: Try to complete the proof
- ➢ Convex functions

- ➢ Other definitions of convexity
- ➢ Taylor series approximation
- ➢ Another condition convexity: Examples
- ➢ Operations on convex functions: multiplication? linear combination? max?

__L7__

- ➢ Why study convexity in optimization?
- ➢ Optimization problem
- ➢ Local optimum
- ➢ Global optimum
- ➢ For a convex function, optimization over a convex feasible set:: local optimum → global optimum. HW: show that is true

__L8__

- ➢ Solving convex programs:

- ➢ Lagrangian method
- ➢ Inequality constraints
- ➢ Dual function
- ➢ Linear programs
- ➢ Weak duality problem: inequality constraints
- ➢ Under certain conditions… strong duality holds

__L9__

- ➢ Condition for strong duality: Slater's condition
- ➢ Theorem: the convex program CP and its dual, has an optimum solution with strong duality
- ➢ Complementary slackness
- ➢ Flow problem: Lagrange method Dual problem… Max-Flow → Min-Cut

L10

- Geometry of Linear Programs
- Polytopes: intersection of hyperplanes in R
- Theorem: Polytopes
    - a facet is a face of dimension = (d-1)
    - an edge is a face of dimension 1
    - a vertex is a face of dimension 0

L11

- A BFS (basic feasible solution) in A*x = b, x>= 0 is obtained by…independent solution… set…rank..
- Theorem: the optimum solution to a LP defined by A*x = b, x>= 0 always exists at a BFS

SLACK VARIABLES: the leftover to completely fit the constraint equations

X + Y <= 20
X = 10
Y = 5
s1 = 5

X + Y + s1 = 20 convert it into equation

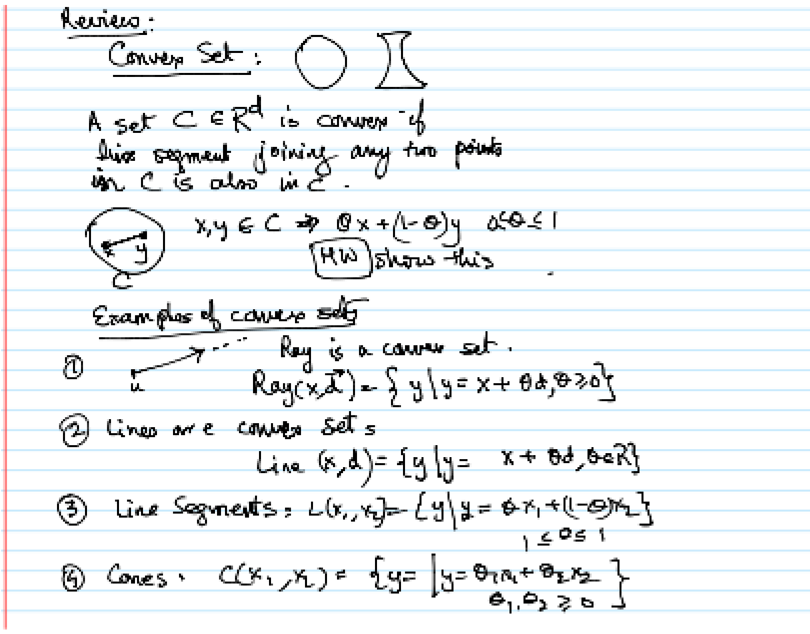
X + Y >= 20
X = 10
Y = 5
s1 = - 5

X + Y + s1 = 20 convert it into equation

COMPILATION OF HWs:

Special HW 3&4

HW3: L5 - page 3


Review:

Convex Set:

A set $C \in \mathbb{R}^d$ is convex if line segment joining any two points in C is also in C.

$x, y \in C \Rightarrow \theta x + (1-\theta) y \quad 0 \le \theta \le 1$

$\boxed{HW}$ show this

Examples of convex sets

① Ray is a convex set.
$Ray(x, \vec{d}) = \{ y \mid y = x + \theta d, \theta \ge 0 \}$

② Lines are convex sets
$Line(x, d) = \{ y \mid y = x + \theta d, \theta \in \mathbb{R} \}$

③ Line Segments: $L(x_1, x_2) = \{ y \mid y = \theta x_1 + (1-\theta) x_2 \}$, $0 \le \theta \le 1$

④ Cones: $C(x_1, x_2) = \{ y = \mid y = \theta_1 x_1 + \theta_2 x_2,\ \theta_1, \theta_2 \ge 0 \}$

Statement:

Let z be a point on the line segment joining x and y, where z = (1-t)x + ty for some scalar t between 0 and 1. Show that a set C in Rd is convex if the line segment joining any two points of C is also in C.

Solution:

In order to show this, we have to show that the line segment joining any two points of C is entirely in C. Therefore, we have to show that z is also in C. Since C is a convex set, we know that the line segment joining any two points p and q of C is entirely in C. For example, if we take p = x and q = y, we know that the line segment joining x and y is entirely in C.

Now let us think of the line segment joining x and z. From the definition of z, we know that the line segment joining x and y is entirely in C. For example, if we take p = x and q = y, we know that the line segment joining x and y is entirely in C.

Now let's think about the line segment joining x and z. From the definition of z, we know that z is in z. We can deduce that z is also in C, since z is in the line segment joining x and y, and both line segments joining x and z and y and z are entirely in C.

Therefore, C is a convex set. Since z lies on the line joining x and y and the lines joining x and z; and y and z are entirely in C, we can show that y and z are also entirely in C.

In conclusion, we have shown that C is a convex set if the line segment joining any two points of C in Rd is also in C.

HW4: L5 - page 4

4. Hyperplane: $H: \{ y \mid a^T y = b,\ y \in \mathbb{R}^d,\ a \in \mathbb{R}^d,\ b = \mathbb{R} \}$

5. Hyperspheres & Ellipsoids:

$$E = \{x \mid (x - x_c)^T A^{-1} (x - x_c) \leq 1\}$$

Hypersphere: $A^{-1} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

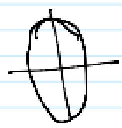
$$(x_1 - x_c^1,\ x_2 - x_c^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 - x_c^1 \\ x_2 - x_c^2 \end{pmatrix} \leq 1$$

$$(x_1 - x_c^1)^2 + (x_2 - x_c^2)^2 \leq 1$$

$$(x - x_c^1,\ x_2 - x_c^2)\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 - x_c^1 \\ x_2 - x_c^2 \end{pmatrix} \leq 1$$

$x_c^1 = 0,\ x_c^2 = 0$

$$2x_1^2 + x_2^2 \leq 1$$



$\boxed{\text{HW}}$

Statement:

Show that the next spaces are convex:

- ***(i) Lines.***
- ***(ii) Line segments in Rd***
- ***(iii) Cones***
- ***(iv) Hyperspheres & Ellipsoids.***
- ***(v) Hyperplanes***

Solution:

General definition of convex:

A set S is convex if for every pair of points x, y in S, the line segment joining x and y is also in S.

(i) Lines.

To prove that straight lines are a convex set, we first have to define what we mean by a straight line. In geometry, a line is an infinitely long, straight collection of points extending in both directions without end. A line can be represented mathematically by an equation of the form $y = mx + b$ in two dimensions or by an equation of the form $ax + by + cz = d$ in three dimensions, where m, b, a, b, c and d are constants.

To prove that lines are a convex set, we have to show that given any two points on the line, every point on the line segment joining them lies on the line. Suppose we have two points P and Q on the line. We can represent these points by their coordinates as P = (x1, y1) and Q = (x2, y2) in two dimensions or P = (x1, y1, z1) and Q = (x2, y2, z2) in three dimensions.

Let R be a point on the line segment PQ, represented by the coordinates (tx1 + (1-t)x2, ty1 + (1-t)y2) in two dimensions or (tx1 + (1-t)x2, ty1 + (1-t)y2, tz1 + (1-t)z2) in three dimensions, where t is a scalar between 0 and 1. We need to show that R is also on the line.

To do this, we can substitute the coordinates of R into the equation of the line and see if the equation holds. For the two-dimensional case, the equation of the line is $y = mx + b$, so we have:

ty1 + (1-t)y2 = m(tx1 + (1-t)x2) + b.

Expanding the equation, we obtain

ty1 + y2 - ty2 - y1 = mtx1 + mx2 - mt x2 + mb - b

Simplifying, we obtain:

y1 - mx1 - b + y2 - mx2 - b = (y1 - mx1 - b)t + (y2 - mx2 - b)(1-t).

Since P and Q are on the line, we know that y1 = mx1 + b and y2 = mx2 + b. Substituting these values in the equation, we obtain:

0 = 0

The equation is valid for any value of t, so R is also on the line. The demonstration is similar for the three-dimensional case, where we can substitute the coordinates of R into the equation $ax + by + cz = d$ and see if it holds.

Thus, we have shown that the lines are a convex set, since every point on the line segment joining any two points on the line also lies on the line.

(ii) Line segments in Rd

To prove that the line segments in Rd are a convex set, we have to show that any point on the line segment joining two points of the set also belongs to the set.

Consider two points, x and y, in Rd. Then the line segment joining these two points can be expressed as

$z = (1-t)x + ty$

where $0 <= t <= 1$ is a scalar parameter.

Now, we have to show that any point on this line segment also belongs to the set. Let us take an arbitrary point p on this line segment. Then, p can be expressed as:

$p = (1-s)z + su$

where $0 <= s <= 1$ is a scalar parameter, and u is a point on Rd.

Substituting the value of z into the above equation, we obtain:

$p = (1-s)((1-t)x + ty) + su$
$= (1-s)(1-t)x + (1-s)ty + su$
$= [(1-s)(1-t)x + (1-s)t y] + su$
$= (1-st)x + st y + su$

Now we have to show that p is also a point of the line segment joining x and y. To do this, we have to show that $0 <= st <= 1$. Since $0 <= s, t <= 1$, p is a point on the line joining x and y.

Since $0 <= s, t <= 1$, we have:

$0 <= st <= 1$

Therefore, p belongs to the line segment joining x and y.

Since this is true for any point p of the line segment, we can conclude that the line segment in Rd is a convex set.

(iii) Cones

Consider a cone and two points, P and Q, inside the cone. It is possible to express the line segment joining P and Q as a linear combination of the two points with a scalar parameter, t, varying between 0 and 1. Since P and Q are in the cone, they can be expressed as a linear combination of the vertex and the generators, denoted by v and G, respectively. Substituting these expressions into the equation of R, we see that R is also a linear combination of v and the generators, which proves that the cone is a convex set. This is because any point on the line segment connecting P and Q is also a linear combination of v and G, and therefore is also on the cone.

Mathematical development:

The cone is a convex set, which means that any two points inside the cone can be connected to a line segment that lies completely inside the cone. To demonstrate, P and Q are any two points on the cone. We can express the line segment joining them as R = tP + (1 - t)Q, where t is a scalar parameter between 0 and 1. Since P and Q are in the cone, we know that they can be expressed as linear combinations of the vertices and generators of the cone. That is, P = t1v + (1 - t1)g1 and Q = t2v + (1 - t2)g2, where t1, t2 ≥ 0 and g1, g2 are generators.

Substituting these expressions into the equation for R, we obtain:
R = t(t1v + (1 - t1)g1) + (1 - t)(t2v + (1 - t2)g2) = tt1v + (1 - tt1)g1 + t(1 - t1)g1 + (1 - t)g2 + t2v - t2(1 - t)g2 = (tt1 + t2(1 - t))v + ((1 - tt1)g1 + (1 - t2))g

Since v is fixed and G is a fixed set of rays emanating from v, the expression for R remains a linear combination of v and the generators: R = t3 ∗ v + (1 - t3)g3, where t3 ≥ 0 and g3 is a generator. Hence, the line segment joining P and Q is entirely inside the cone, as well as R. We can conclude that, the cone is a convex set.

(iv) Hyperspheres & Ellipsoids.

An ellipsoid is a set of points in which each point satisfies the equation:

$(x - c)^T A^{-1}(x - c) \leq 1$

where x is a coordinate vector, c is the center of the ellipsoid, A is a positive definite matrix and ^{T} denotes the transpose.

To prove that the ellipsoid is a convex set, we have to show that for any two points inside the ellipsoid, the line segment joining them is entirely inside the ellipsoid. Consider two points, x_1 and x_2, that satisfy the above inequality:

$(x_1 - c)^T A^{-1}(x_1 - c) \leq 1$

$(x_2 - c)^T A^{-1}(x_2 - c) \leq 1.$

Let us define a new point, y, as a convex combination of x_1 and x_2:

$y = \lambda x_1 + (1-\lambda)x_2$

where λ is a scalar parameter between 0 and 1.

We can express y as:

$y = \lambda(x_1 - c) + (1-\lambda)(x_2 - c) + c.$

Expanding this expression, we obtain:

$y - c = \lambda(x_1 - c) + (1-\lambda)(x_2 - c)$

$y - c = \lambda x_1 - \lambda c + x_2 - \lambda x_2 + \lambda c$

and $- c = \lambda(x_1 - c) + (1-\lambda)(x_2 - c)$

Multiplying both sides of the above inequality by A^{-1}, we obtain:

$(y - c)^T A^{-1}(y - c) = \lambda(x_1 - c)^T A^{-1}(x_1 - c) + (1-\lambda)(x_2 - c)^T A^{-1}(x_2 - c)$.

Since both x_1 and x_2 lie inside the ellipsoid, the left side of the inequality is less than or equal to 1. Furthermore, since A is positive definite, the right side of the inequality is a convex combination of two non-negative numbers, which is also less than or equal to 1. Therefore, y lies inside the ellipsoid.

Since this is true for any two points inside the ellipsoid, we have shown that the ellipsoid is a convex set. Therefore, as the condition "A set S is convex if for every pair of points x, y in S, the line segment joining x and y is also in S" is met, the Ellipsoids in Rd are convex.

(v) Hyperplanes: a^T x = b.

A hyperplane is a subspace of one dimension less than the ambient space. Let H be a hyperplane in Euclidean space, and let u and v be two points in H. The line segment joining u and v can be written as:

$w = (1 - t)u + tv$

where t is a scalar parameter between 0 and 1. Since H is a subspace, it is closed under linear combinations, so w is also in H. Therefore, the line segment connecting any two points in H lies entirely inside H, and H is a convex set.

Mathematical development:

Let H be a hyperplane in n-dimensional space. We can write the equation of H as

a1x1 + a2x2 + ... + anxn = b

where a1, a2, ..., an are constants, and b is a constant.

Suppose we have two points P and Q in H. We can write the coordinates of P as (p1, p2, ..., pn) and the coordinates of Q as (q1, q2, ..., qn). Then, the line segment connecting P and Q can be written as:

R = tP + (1-t)Q

where t is a scalar parameter between 0 and 1.

We can write the coordinates of R as:

$r_1 = tp_1 + (1-t)q_1$
$r_2 = tp_2 + (1-t)q_2$
...
$r_n = tp_n + (1-t)q_n$

Now, let us show that R is also in H. Substituting the coordinates of R into the equation of H, we obtain

$a_1r_1 + a_2r_2 + ... + a_nr_n = a_1(tp_1 + (1-t)q_1) + a_2(tp_2 + (1-t)q_2) + ... + a_n(tp_n + (1-t)q_n)$

$= t(a_1p_1 + a_2p_2 + ... + a_np_n) + (1-t)(a_1q_1 + a_2q_2 + ... + a_nq_n)$

$= tb + t(0) + (1-t)(0)$

$= b$

Therefore, R is also in H, which means that the line segment joining P and Q is entirely inside H. Therefore, H is a convex set.

This proof shows that any hyperplane in n-dimensional space is a convex set.

CS538-L6

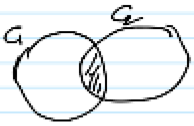
Monday, January 30, 2023 5:08 PM

Do operations on Convex sets preserve Convexity?

Is convexity preserved under intersections.

C_1 : convex set , C_2 : convex set

$C_1 \cap C_2$ is a convex set .



$x_1 \in C_1 \cap C_2$

$x_2 \in C_1 \cap C_2$

$\theta x_1 + (1-\theta) x_2 \in C_1 \cap C_2$.

$x_1, x_2 \in C_1 \Rightarrow y = \theta x_1 + (1-\theta) x_2 \in C_1$

$x_1, x_2 \in C_2 \Rightarrow y = \theta x_1 + (1-\theta) x_2 \in C_2$

Other transformations :

Affine function: $f(x) = Ax + b$

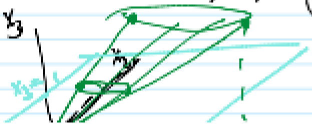
$x \in \mathbb{R}^n$ $A \in \mathbb{R}^{m \times n}$

$f(x) = m \underset{n}{\boxed{}} \begin{pmatrix} x \end{pmatrix} + \begin{pmatrix} b \end{pmatrix}$

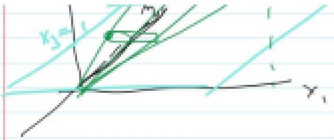
Convexity is preserved .

Projection function.

$P(x_1, \dots x_n, x_{n+1}) \to \left(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}} \dots, \frac{x_n}{x_{n+1}} \right)$



$x_1 = 5, x_2 = 4, x_3 = 5$



$x_1=5, x_2=4, x_3=5$

$x_1=1, x_2=4/5, x_3=1$

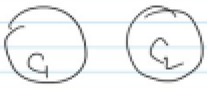
$x_1=5, x_2=4, x_3=6$

⋮

P preserves Convexity.
(HW: Show that P ↗)

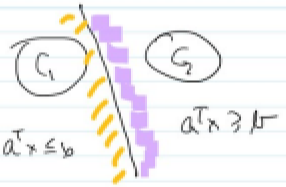
An interesting Property of Convex Sets

Separating Hyperplane theorem of Convex Sets



Theorem: Let C_1 & C_2 be 2 disjoint convex sets. Then $\exists$ a separating hyperplane $H = a^T x = b$ such that

$$C_1 \subseteq \{a^T x \le b\} \ \& \ C_2 \subseteq \{a^T x \geq b\}$$



Statement:

- ➢ Do operations on convex sets preserve convexity?
- ➢ Show that the 'Projection function' operation preserves convexity

Solution:

- ➢ ***Do operations on convex sets preserve convexity?***

We must demonstrate that if C and D are two convex sets in Rd, and the following operations result in a convex set, then operations on convex sets preserve convexity.

Intersection: A convex set is formed when C and D meet at their intersection.

If x and y are two points at the intersection of C and D, then they are convex sets in both C and D. Because C and D are convex, the line segment that connects any two points in C or D is also in C or D. As a result, the line segment that connects x and y lies entirely in both C and D, and as a result, it intersects them. As a result, both sets' intersections are convex.

Union: A convex set is formed when two convex sets, C and D, are joined together, and only if C and D are distinct.

Consider the two convex sets C and D. Since C and D are both convex, their union is clearly convex if they are disjoint.

Their union may not be convex if they are not disjoint. Take, for instance, two unit circles in R2 whose centers are located at (0,0) and (2,0). The circles do not convex when they join at two points.

In a nutshell, the intersection of two convex sets keeps being convex, whereas the union of two convex sets keeps being convex as long as they are disjoint. As a result, convexity is maintained by operations on convex sets.

- ***Show that the 'Projection function' operation preserves convexity***

In order to prove that the projection operation (P_y) preserves convexity, let C be a convex set in Rd and P_y(x) the projection of x onto the subspace spanned by y. To this end, note that for any x, z and y in Rd, we have ||P_y(x) - P_y(z)|| and ||x - z||. We can show that "λP_y(x) + (1 -λ)P_y(z)" also exists in C as follows using this fact:

Let y be any vector in Rd, and let x, z be two points in C. Then they exist:

When we add and subtract "λP_y(x) + (1 -λ)P_y(z)" from both sides, we get: P_y(x) - x ⊥ y, and P_y(z) - z ⊥ y.

Rearranging the terms we obtain: P_y(x) - P_y(x) - (1 -λ)P_y(z) + P_y(z) - x + λ x + (1 - λ)z - λP_y(x) - (1 - λ)P_y(z) ⊥ y

Using the fact that:

|| P_y(x) - P_y(z) || <= || x -z || the distances between points in the subset of projected subspaces are always smaller or equal than the distances in the whole dimensional space

we can show that:

λP_y(x) + (1 - λ) P_y(z) - λx - (1 - λ) z ⊥ y

Since y is arbitrary, this means that λP_y(x) + (1 - λ)P_y(z) - λx - (1 - λ)z lies in the orthogonal complement of R^d, which is nothing but the singleton {0}. Therefore, we have:

λP_y(x) + (1 - λ)P_y(z) = λx + (1 - λ)z + (λP_y(x) + (1 - λ)P_y(z) - λx - (1 - λ)z) ∈ C

This shows that P_y maintains convexity

Bonus HW: L6 - page 3

$f: a^T x - b$ is non-negative for all points in C_2. $\left| f: (c_2 - c_1)^T \left(x - \frac{(c_2 + c_1)}{2}\right) \right.$ — (A)

By contradiction:

Suppose $\exists$ point $u \in C_2$ s.t.

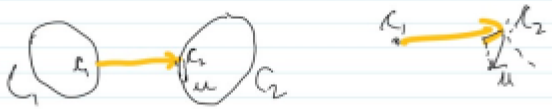
$f(u) < 0$.

Using (A)

$$= (c_2 - c_1)^T \left(u - \frac{(c_2 + c_1)}{2}\right) = (c_2 - c_1)^T \left(u - c_2 + \frac{1}{2}(c_2 - c_1)\right)$$

$$= (c_2 - c_1)^T (u - c_2) + \underbrace{\frac{1}{2}(c_2 - c_1)^T (c_2 - c_1)}_{+ve}$$

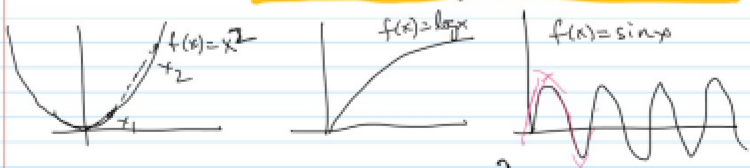
$$\Rightarrow (c_2 - c_1)^T (u - c_2) < 0.$$



HW: BONUS^F : Try to Complete the proof.

CONVEX FUNCTIONS:

A function $f(x)$ is convex if the domain $Dom(f)$ is a convex set & $\forall x_1, x_2 \in Dom(f)$: $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$



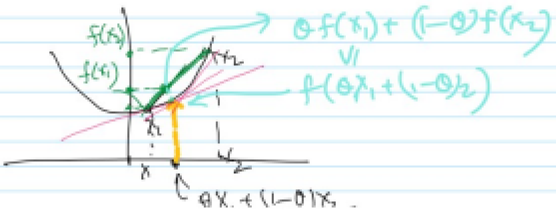
$$f(\theta x_1 + (1-\theta)x_2) = (\theta x_1 + (1-\theta)x_2)^2 \overset{?}{\leq} \theta x_1^2 + (1-\theta) x_2^2$$

$$\overset{?}{\leq} \theta f(x_1) + (1-\theta) f(x_2)$$

$$\theta^2 x_1^2 + 2\theta(1-\theta) x_1 x_2 + (1-\theta)^2 x_2^2$$

$f(x) = x^2$

$\frac{\partial f}{\partial x} = 2x$



$\frac{d^2 f}{dx^2} = 2$

Statement:

In this BONUS point, we have been asked to prove by contradiction that given points c1, c2 such that they belong to two different convex sets C1 and C2, there is a point that contradicts that you have chosen c1 and c2 as the closest of C1 and C2. Therefore, complete the proof by contradiction of the fact 'there is another point in the C1 convex set that is closer to c2, than c1 of C1 actually is.

Solution:

In order to prove by contradiction that there exists another point in the convex set C1 that is closer to c2 than c1 actually is to C1, we will assume the opposite.

Let c1 and c2 be points of the convex sets C1 and C2 respectively, such that c1 is the point of C1 closest to c2. We also assume that there is no other point in C1 that is closer to c2 than c1.

Let us now consider the line segment joining c1 and c2. This line segment lies entirely in the space between C1 and C2, since C1 and C2 are convex sets. Therefore, any point on this line is a convex combination of c1 and c2.

Let x be a point on this line segment such that x is closer to c2 than to c1, i.e., the distance from x to c2 is less than the distance from c1 to c2. Then, we can express x as a convex combination of c1 and c2 such that the weight of c2 in this combination is greater than the weight of c1. Let α be the weight of c2 in this convex combination, then we have:

$x = \alpha c2 + (1-\alpha)c1$

Since $\alpha > 1-\alpha$, we can rewrite the above equation as:

$x = c2 + [(1-\alpha)/(\alpha)](c1 - c2)$.

Since C1 is a convex set, any convex combination of points in C1 also belongs to C1. Therefore, the point $y = c2 + [(1-\alpha)/(\alpha)](c1 - c2)$ is in C1. However, the distance from y to c2 is less than the distance from c1 to c2, which contradicts our assumption that c1 is the point in C1 closest to c2.

Therefore, our assumption that there is no other point in C1 that is closer to c2 than c1 is false, and there exists another point in C1 that is closer to c2 than c1 is to C1 is in reality.