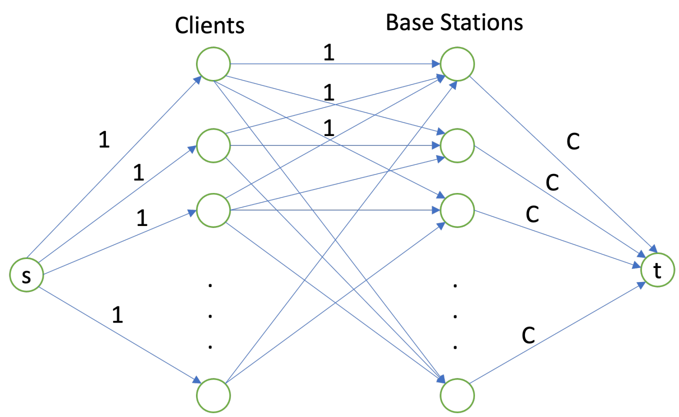
CS538 Spring 2023
HW 1 Solutions

1. Consider a mobile company that wants to connect wireless clients to base stations. Further suppose that each base station has a capacity of C clients. A client can connect to a base station only if the base station set is less than or equal to distance r away. Given a collection of base stations, wireless clients and the geographic location of base stations and wireless clients, design a method to determine an assignment of clients to base stations such that the number of clients serviced is maximized.

   We can formulate this as a max-flow problem in a bipartite graph. Create a bipartite graph $G(U, V, E)$ where U represents the set of clients We let u_i be the node corresponding to client i, v_j corresponding to the jth base station. $E = (u_i, v_j$ represent an edge directed from client u_i to base station v_j iff u_i is within distance r from v_j. Create a super source and connect to $u_i, \forall i$ and connect $v_j, \forall j$ to t. Let all edges have capacity 1 and let the edge from v_j to t have capacity C. This capacity will ensure that no base station will get more than C client. Finding the maximum flow in this graph will give an optimum assignment.



2. There are n doctors that need to be scheduled for duty in a hospital during holiday periods. There are m vacation periods (Thanksgiving, Christmas etc.) each of period p_i. However, there must be at least one doctor on call at the hospital every day during the vacation period. Each doctor D_i specifies the set of vacation days W_i that he can work. How will you determine a schedule such that each doctor is not working more than c days in any vacation period.

   Again construct a flow network using a variation of a bipartite graph $G = (V, U, E)$ where U represents the doctors, V the holiday days and an edge between u_i and v_j if the doctor u_i can work on holiday v_j. Each edge has capacity 1. However we need to make sure that no more than c days are scheduled for every doctor. So we add a third layer of nodes in between U and V such that for every holiday period h and for every doctor there is a node h_i and now every node corresponding to a doctor u_i is connected to h_i with capacity c and every node h_i is then connected to all the nodes corresponding to the days in the holiday period h with capacity 1. Add a super source s with edges to each doctor and of unbounded capacity and a super sink t with edges from v_j to t with capacity lower bound of 1 and upper bound of 1.

3. Describe the proof of Hall's theorem discussed in class for a bipartite graph $G = (U, V, E)$. In particular, in the induction step show that if there is a subset $S \subseteq U$ with neighbor set $N(S)$, where $|N(S)| = |S|$ then S and T satisfy the hypothesis of the the theorem, i.e., for every subset in S(or T) the neighborhood set is of size greater than or equal to S (or T).

   When $|A| = 1$, then an A-perfect matching is trivial. When $|A| > 1$, we can consider $x \in A$ and $y \in B$

   Next, for any vertex $x \in A$ it must have at least one neighbor $y \in B$. We can try to pair x and y and find a matching of size $|A| - 1$ in the graph induced by $A - \{x\}, B - \{y\}$. Such a matching will not be possible if there is some set S in the induced graph where $S < N(S)$ (Observe that the total neighborhood size of A has only decreased by 1, invalidating Hall's condition.) Therefore, S has exactly $|S|$ neighbors in B.

   Let T denote the neighborhood of any such S in B, i.e., $|S| = |T|$. Extract S and T and Let B' be the induced graph (B is the original graph.) B' is definitely smaller in size than B, and therefore, there exists a matching for each vertex of S to a vertex in B. For any set $S' = A \setminus S$. Since S has $|T|$ neighbors in B, S' must have at least $|S'|$ neighbors in $B \setminus T$.

   Ref: https://homes.cs.washington.edu/˜anuprao/pubs/CSE599sExtremal/lecture6.pdf

4. Show that $y(\theta) = \theta x + (1 - \theta)y, 0 \leq \theta \leq 1$ represents all the points on the line segment defined by two endpoints x and y. This is true in $\mathcal{R}^d$ for any dimension d.

   All points on the line segment can be expressed in terms of the vector $x + t(y - x)$ where $0 \leq t \leq 1$.

   Manipulating the expression in the question, we have $y(\theta) = y + \theta(x - y), 0 \leq \theta \leq 1$, which matches the definition.

5. Show that the following sets are convex:

   (i) Lines and Line segments in $\mathcal{R}^d$.

   Using parametric definition of a line: A line is defined by a point p, a vector v and is the set $\{y = p + \alpha v\}$, for any α. Given $y_1 = p + \alpha_1 v$ and $y_2 = p + \alpha_2 v$ on the line consider $z = \theta y_1 + (1 - \theta) y_2$.

   $$z = p + (\alpha_1 v)(\theta) + (\alpha_2) v(1 - \theta)$$

   or $z = p + (\alpha')v$ where $\alpha' = (\alpha_1 \theta + \alpha_2(1 - \theta))$

   (ii) Ellipsoids in $\mathcal{R}^d$.

   Ellipsoids in d dimensions are defined by

   $$(x - c)^T B(x - c) \leq 1$$

   Note that B can be expressed as $A^T A$ and thus the above is equivalent to $||(x - c)^t A||_2 \leq 1$. Then checking for $z = \theta x + (1 - \theta)y$ gives

   $$(z - c)^T B(z - c) = \theta^2 (x - c)^T B(x - c) + 2\theta(1 - \theta)(x - c)^T B(y - c) + (1 - \theta)^2 (y - c)^T B(y - c)$$

   $$\leq \theta^2 + (1 - \theta)^2 + 2\theta(1 - \theta)((x - c)A)^T (A(y - c))$$

Using Cauchy Schwartz's inequality ($x^t y \leq ||x||||y||$) we get $((x-c)A)^T(y-c) \leq 1$

(iii) Halfspaces defined by the hyperplane $a^T x = b$.

This is easy to see because of the linearity.

6. Let $C \subseteq R^n$ be a convex set, with $x_1 \dots x_k \in C$. Show that $\theta_1 x_1 + \dots + \theta_k x_k \in C$ where $\theta_1, \dots \theta_k \in R$ satisfy $\theta_i \geq 0, \theta_1 + \cdots + \theta_k = 1$.

   Use induction and the fact that $\theta_1 x_1 + \dots + \theta_k x_k = \theta_1 + (1-\theta_1)y$ where $y = \sum_{i=2}^{k} \frac{\theta_i x_i}{\sum_{j=2}^{k} \theta j}$

7. Show that if C is a convex set then $C' = T(C)$ is a convex set, where $T(C) = Ax + b, x \in C$.

   For two points $z_1 = Ax_1 + b, z_2 = Ax_2 + b$ we get

$$\theta(Ax_1 + b) + (1-\theta)(Ax_2 + b) = A(\theta x_1 + (1-\theta)x_2) + b$$