PROBLEM 2

In a bipartite graph, it is known that in order to find a maximum matching, we repeatedly look for alternating chains, and when we cannot find one, it means that we have reached the maximum matching.

We then introduce the "vertex cover" concept, which is a set of vertices (S) where every edge is incident into a vertex. To prove that in a bipartite graph the maximum matching size is the same as the minimum cover size, we will use a constructive proof. It provides a way of constructing a minimum vertex cover from a maximum matching. Let $G=(V,E)$ be a bipartite graph and let W, U be the two parts of the vertex set V. Suppose that M is a maximum matching for G. No vertex in a vertex cover can cover more than one edge of M, so if a vertex cover with $|M|$ vertices can be constructed, it should be a minimum cover.

In order to construct a cover as mentioned, let K be the set of unmatched vertices in W (possibly empty) and let Z be the set of vertices that are either in K or are connected to K by alternating paths.

LET $S=(W\setminus Z)\cup(U\cap Z)$

Every edge in E either belongs to an alternating path (and has a right endpoint in S), or it has a left endpoint in S. If e is matched but not in an alternating path, then its left endpoint cannot be in an alternating path (because two matched edges cannot touch a vertex) and thus belongs to $W\setminus Z$. Alternatively, if e is unmatched but not in a alternating path, then its left endpoint cannot be in a alternating path, for such a path could be extended by adding e to it. Thus, S forms a vertex cover. Additionally, every vertex in S is an endpoint of a matched edge. For every in $W\setminus Z$ is matched because Z is a superset of K, the set of unmatched left vertices. And every verted in $U\cap Z$ must also be matched, for if there existed an alternating path to an unmatched vertex then changing the matching by removing the matched edges from this path and adding the unmatched edges in their place. would increase the size of the matching. However, no matched edge can have both of its endpoints in S. Thus, S is a vertex cover of cardinality equal to M, and must be a minimum vertex cover.