

Gaussian Processes Part 2

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MSU

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Agenda

- ① Introduction
- ② GP approach
 - Introduction
 - Non-periodic part
 - Periodic part
 - The Model
- ③ Modelling
 - Priors
 - Parametrization
 - Seasonality
- ④ Stochastic Volatility

Time Series, Classical Approach

If data has seasonality, you usually use **STL** decomposition. However,

- Parameters are not interpretable, only decomposition is available
- No uncertainty estimates
- Quite strict on input values
- Significantly less flexible in modelling

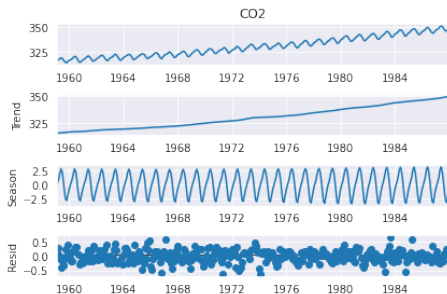


Figure: STL decomposition for CO2 data, Statsmodels

GP decomposition

A Gaussian process can handle a complicated set of assumptions in addition to what STL provides

- Granular seasonality (year + quarter + month + week)
- Changepoint models
- Flexible likelihood Function
- Panel regression models
- Missing values

Typical Model

Typical model is additive

$$x_t \sim \underbrace{g(t)}_{\text{non-periodic}} + \underbrace{s(t)}_{\text{periodic}} + \underbrace{h(t)}_{\text{holidays}}$$

Reference

See more in **Prophet** preprint [1]. Every time series model is unique

Reminder

$$x \in \mathbb{R}^n, y \in \mathbb{R}$$

$$Y \sim \mathcal{GP}(m(x), k(x, x'))$$

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- ① \mathcal{GP} Gaussian Process - simply, a normal distribution with special mean $m(x)$ and covariance $k(x, x')$

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 - Linear regression $\textcolor{red}{m}(x) = x^\top \beta$
 - Simply Constant or Zero $\textcolor{red}{m}(x) = c$
 - Other custom functions $\textcolor{red}{m}(x) = \sin(x)$

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 - Simply Constant or Zero $m(x) = c$
 - Other custom functions $m(x) = \sin(x)$
- ③ $k(x, x')$ - kernel function, simply - measure of similarity for x and x'
 - $[K]_{ij} = k(x_i, x_j)$ is an SPD matrix

Non-periodic Part (mean function)

- Growth models
- Linear trend models
- Changepoint models

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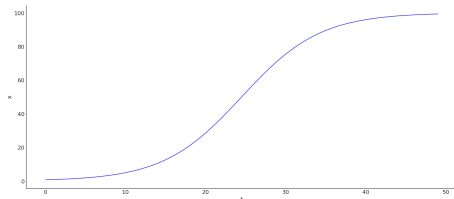


Figure: Growth Model

$$x = \frac{c}{1 + \exp(-k(t - m))}$$

Non-periodic Part (mean function)

- Growth models
- **Linear trend models**
- Changepoint models

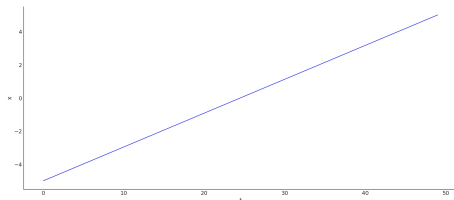


Figure: Linear Trend Model

$$x = \frac{c}{1 + \exp(-k(t - m))}$$

Non-periodic Part (mean function)

- Growth models
- Linear trend models
- **Changepoint models**

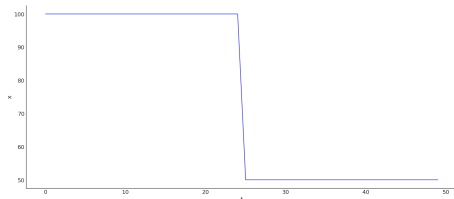


Figure: Changepoint Model

$$x = \begin{cases} c_1, & t < m \\ c_2, & t \geq m \end{cases}$$

Non-periodic Part (mean function)

- Growth models
- Linear trend models
- Changepoint models

Extentions

Extensions are possible, e.g. time dependent saturation in the growth model. See in [1]

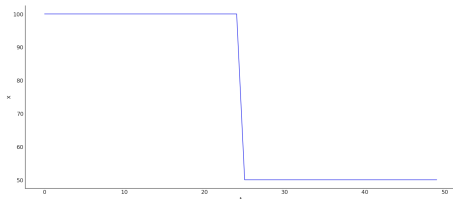


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Holidays

$$h(t) = \text{is-holiday}(t)$$

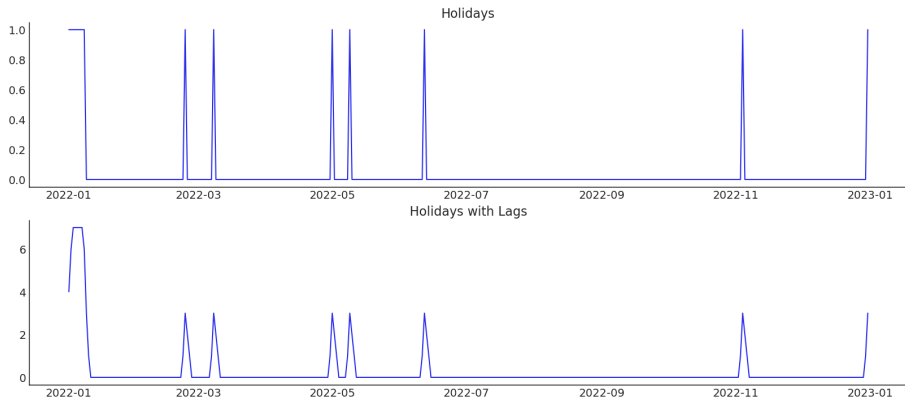
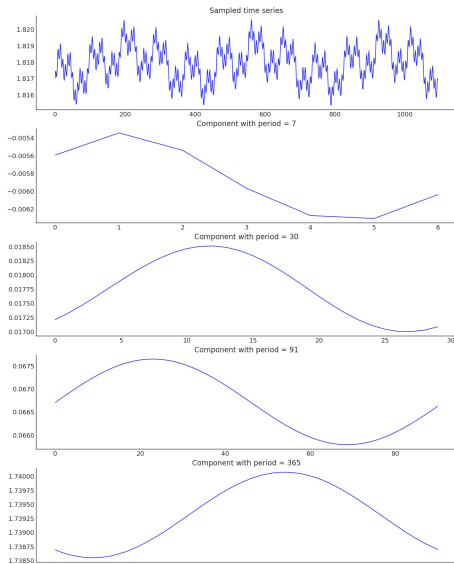


Figure: Holiday features

Periodic part (cov function)

Granularities are important here.
Multiple Periodic kernels can be used.

- Yearly
- Quarterly
- Monthly
- Weekly



Lengthscales for Periodic Part

Hyperparameters

Common sense driven lenthscale choice

- Week - couple of days make a change ($l_s \approx 3$)
- Month - week makes sense for a dramatic change ($l_s \approx 7$)
- Quarter - month makes sense for a dramatic change ($l_s \approx 30$)
- Year - quarter makes sense for a dramatic change ($l_s \approx 90$)

Lengthscales for Periodic Part

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- Month - week makes sense for a dramatic change ($l_s \approx 7$)
- Quarter - month makes sense for a dramatic change ($l_s \approx 30$)
- Year - quarter makes sense for a dramatic change ($l_s \approx 90$)

In practice

Hard to infer lengthscales, better just fix them if informed

Putting All Together

$$m(t) = \underbrace{g(t)}_{\text{non-periodic}} + \underbrace{h(t)}_{\text{holidays}}$$

$$\begin{aligned} k(*, *) = & \text{Periodic}(p = 365, l = 90) \\ & + \text{Periodic}(p = 90, l = 30) \\ & + \text{Periodic}(p = 30, l = 7) \\ & + \text{Periodic}(p = 7, l = 3) \end{aligned}$$

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Missing Parts

- 1 Weights for periodic components

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Missing Parts

- ① Weights for periodic components
- ② Trent violations from $g(t)$ that can't be explained by periodic component and observation noise

Example

CO2 data (Manua Loa, Hawaii)

- Monthly observations
- Clean data
- Unclear trend function

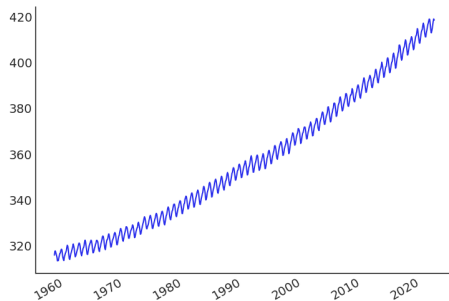


Figure: CO2 level

Example

CO2 data (Manua Loa, Hawaii)

- Monthly observations
- Clean data
- Unclear trend function

Where to start?

- 1 Decide on a trend model
- 2 Add periodic component

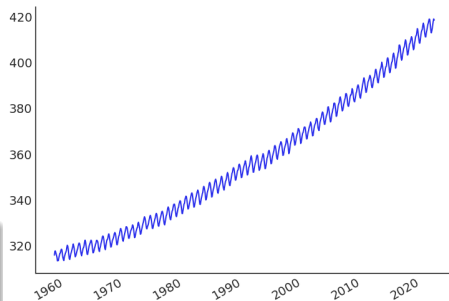


Figure: CO2 level

Priors

The exponential Growth Model

$$y_t = y_0 \cdot g^t$$

- ① The baseline level for CO2 (data tells it is ≈ 310)
- ② Growth rate for CO2 (some plausible value)
 - unlikely decay
 - no astronomic values

Prior Predictive Checks (Naive Attempt)

$$g \sim \text{Normal}(1.1, 0.1)$$

$$\epsilon \sim \text{LogNormal}(0, 1)$$

$$y_0 \sim \text{Normal}(310, 10)$$

$$y_t \sim \text{Normal}(y_0 \cdot g^t, \epsilon)$$

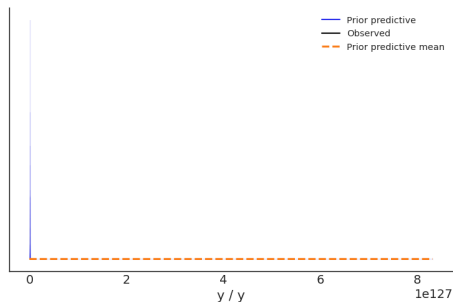


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PPC Observation

Model is unlikely to be correctly specified. Prior predictive ranges to 10^{127} **parts per million** of CO₂. Air has $2.5 \cdot 10^{25}$ molecules in m^3

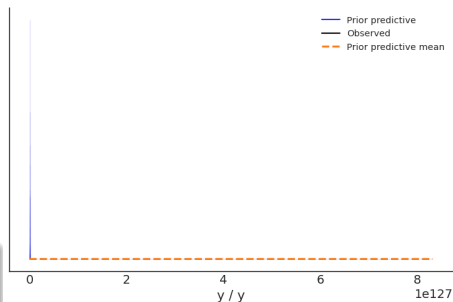


Figure: Prior Predictive Check

Prior Predictive (Problem Driven)

Thinking loudly

I do not believe in a century we can exceed 400% growth (we would have died already) and I suppose it should be at least 20% (it would not be a problem otherwise)

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- ① 400% in a century is 0.115% per month
- ② 20% in a century is 0.015% per month

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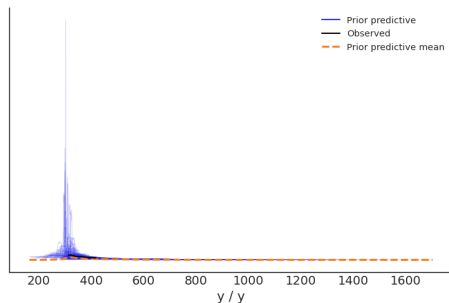


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PPC Observation

Model acts as expected ranging CO2 levels in plausible regions. It seems to be good enough to go to the next step.

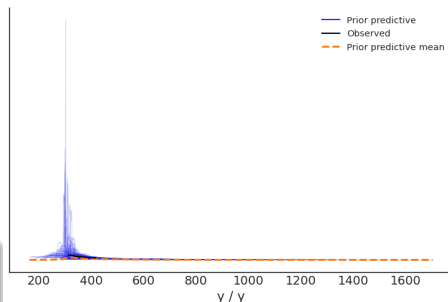


Figure: Prior Predictive Check

Inference

- 1 In sampling any value could appear
- 2 Here we got a very large value for some parameter
- 3 We can try to reparametrize

```
: with model:  
    trace = pm.sample()  
  
Auto-assigning NUTS sampler...  
Initializing NUTS using jitter+adapt_diag...  
/home/ferres/.miniconda3/envs/bayes-econ/lib/python3.10/site-packages/mcmcstatsmodels/core/mcmc.py:57: RuntimeWarning: overflow encountered in multiply  
    return bound(*args, **kwargs)  
Multiprocess sampling (4 chains in 4 jobs)  
NUTS: [initial, g, eps]
```

Figure: Overflow in sampling

Log Model

$$\log y_t = \log y_0 + g \cdot t$$

$$g \sim \text{Normal}(1.00065, 0.0005)$$

$$\epsilon \sim \text{LogNormal}(-2, 0.5)$$

$$\log y_0 \sim \text{Normal}(\log 310, 10/310)$$

$$y_t \sim \text{LogNormal}(\log y_0 + g \cdot t, \epsilon)$$

Important

ϵ has a different meaning in the model. It is now a relative error.

Prior Predictive (Reparametrized)

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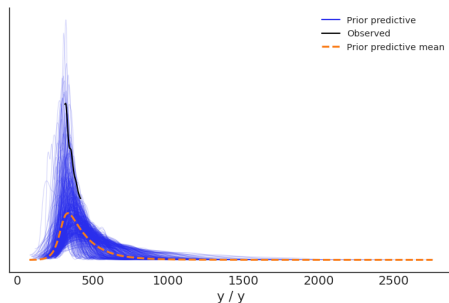


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PPC Observation

Model acts as expected ranging CO2 levels in plausible regions. It seems to be good enough to go to proceed with sampling.

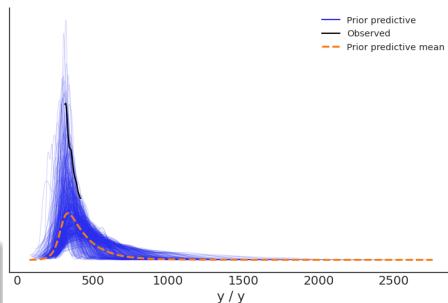


Figure: Prior Predictive Check

Inference

- ① Everything is good
- ② We can visualize

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 100.00% [8000/8000]
Sampling 4 chains, 0 divergences]

```
Sampling 4 chains for 1_000 tune and 1_000 draw iterations (4_000  
s total) took 6 seconds.
```

Figure: Good sampling

Inference

- 1 Everything is good
- 2 We can visualize
- 3 Trace plot is a good visual inspection if sampling went well

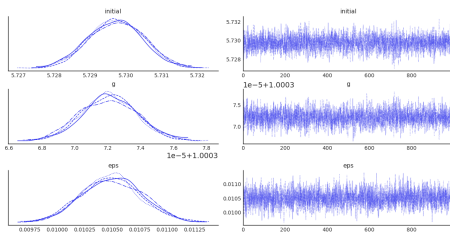


Figure: Good Trace Plot

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Posterior Predictive

Another way to validate your model is posterior predictive.

- How model performs vs observed data?
- Is there any missing effect to include?

Let's see

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Another way to validate your model is posterior predictive.

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Let's see

- 1 The fit is good
- 2 Model does not reflect seasonalities
- 3 We can fix it with a better model

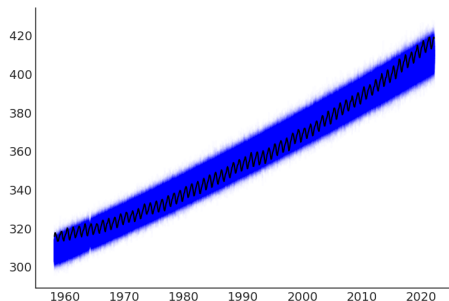


Figure: Posterior Predictive Plot

Conclusions so far

- The reparametrized model is decent
- It does not account for seasonality
- We have the mean function for the next modelling step

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$$\log y_0 \sim \text{Normal}(\log 310, 10/310)$$

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Adding Seasonality

CO2 data (Manua Loa, Hawaii)

- Monthly observations
- Yearly seasonality makes sense
- Trend function is clear from a simpler model

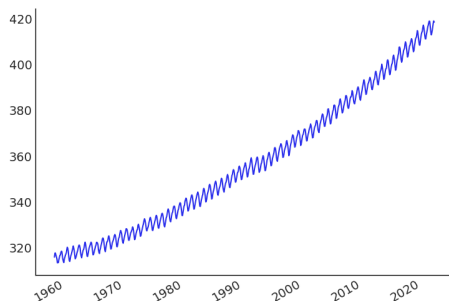


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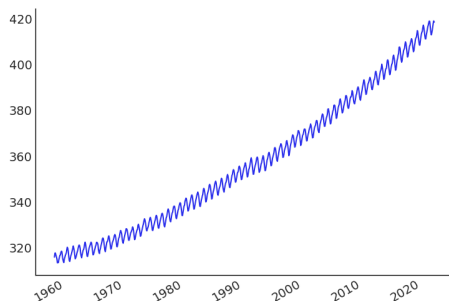


Figure: CO2 level

$$k(*,*) = \text{Periodic}(p = \quad, l = \quad)$$

Adding Seasonality

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- Period is 12 months

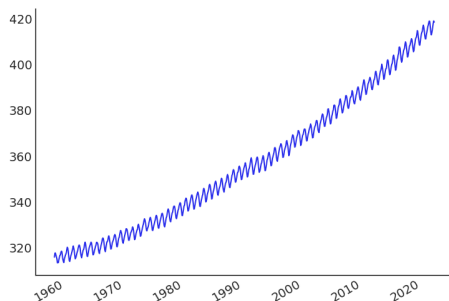


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$$k(*,*) = \text{Periodic}(p = 12, l =)$$

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- Period is 12 months
- Appropriate lengthscale is ≈ 3 months

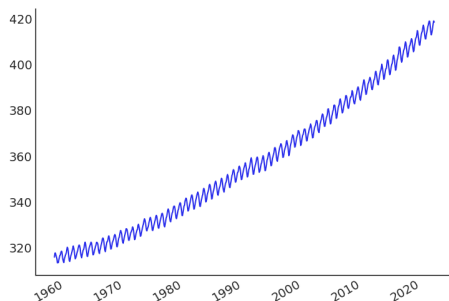


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- Yearly seasonality makes sense
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- Seasonality can be added with a periodic kernel
- Period is 12 months
- Appropriate lengthscale is ≈ 3 months
- Do not forget to scale the covariance!

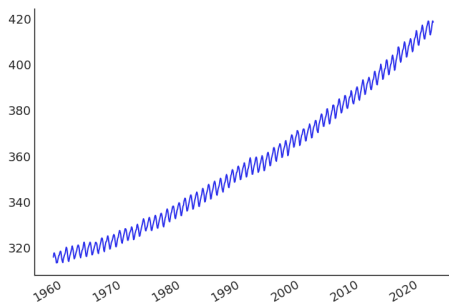


Figure: CO2 level

$$k(*,*) = \alpha_k \text{Periodic}(p = 12, l = 3)$$

Seasonality Change

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Seasonality Change

$$K(*, *) = \alpha_k \text{Periodic}(p = 12, l = 3) \cdot \text{Exponential}(l = 1200)$$

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- Multiplication with the Exponential kernel should work just fine

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- We need a decay in correlation between distant year patterns
- Multiplication with the Exponential kernel should work just fine
- Lengthscale in the Exponential kernel is how strong patterns correlate across months
- I choose 1200, century long patterns

Decay Visualized

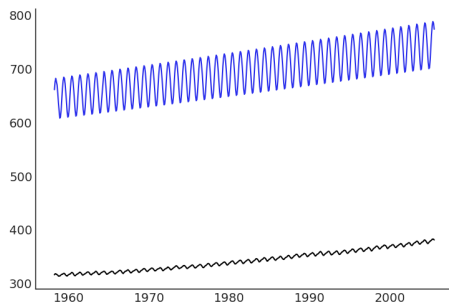


Figure: Mean without decay

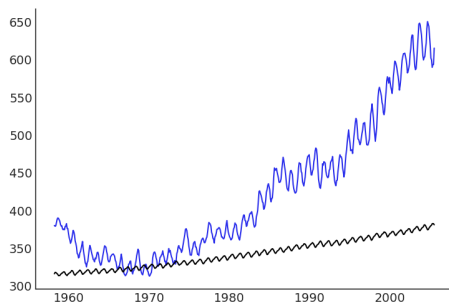


Figure: Mean with decay

Decay model

No decay model is the limit case of decay model with lengthscale $\rightarrow \infty$

The Model

$$g \sim \text{Normal}(1.00065, 0.0005)$$

$$\epsilon \sim \text{LogNormal}(-2, 0.5)$$

$$\log y_0 \sim \text{Normal}(\log 310, 10/310)$$

$$\alpha_k \sim \text{LogNormal}(-2, 0.5)$$

$$K(*, *) = \alpha_k \text{Periodic}(p = 12, l = 3) \cdot \text{Exponential}(l = 1200)$$

$$\log \bar{y}_t = \mathcal{GP}(\log y_0 + g \cdot t, K)$$

$$y_t \sim \text{LogNormal}(\log \bar{y}_t, \epsilon)$$

Improve the model

- 1 Figure out a better ϵ prior
- 2 Infer better lengthscales (hyperpriors)
- 3 Infer structural medium term shift (additive kernel)

Prior Predictive Check

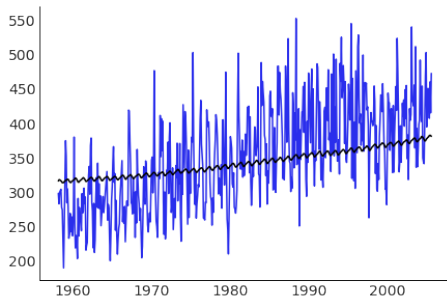


Figure: $\epsilon \sim \text{LogNormal}(-2, 0.5)$

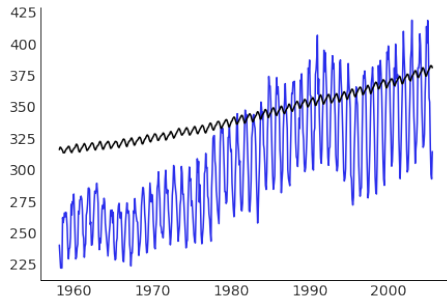


Figure: $\epsilon \sim \text{LogNormal}(-4, 0.5)$

Inference

- `pm.sample()` is slow
- GPU sampling is also slow
- `pm.find_MAP()` makes sense and fast

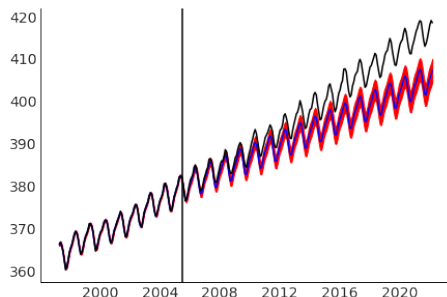


Figure: Predictions with MAP

Key takeaways

- Look at prior predictive to validate initial model
- Interpretable building blocks create interpretable model
- Seasonality model is done with Periodic kernel
- Seasonality might be flexible

Stochastic Volatility Model

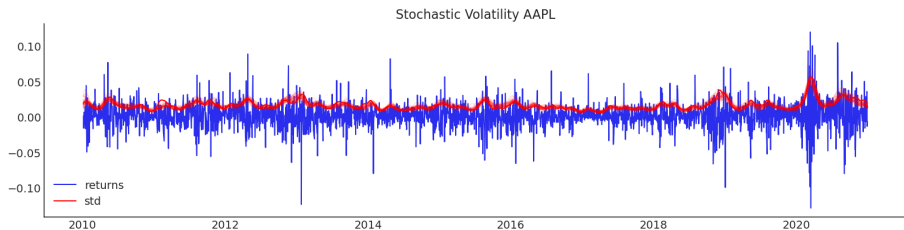


Figure: Stochastic Volatility Estimation for AAPL

Read More

<https://github.com/quantopian/bayesalpha>

Key Ideas

- Returns are constant
- Volatility is not constant
- Model volatility as a Gaussian process
- Use approximations to speed up the model
- Use GPU to do fast inference

Priors

- Returns
- Volatility
- Time Component

Priors

- Returns
 - Expect year return at orders $\pm 100\%$
- Volatility
- Time Component

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The Model

I did the modelling choice to make it as simple as possible

$$\text{return} \sim \text{Normal}(0, 2^{\frac{1}{250}} - 1)$$

$$\text{log std} \sim \text{Normal}(\log(2^{\frac{1}{250}} - 1), 0.05)$$

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Splines

Make the model faster!

Spline interpolation

```
pmx.utils.spline.bspline_interpolation
```

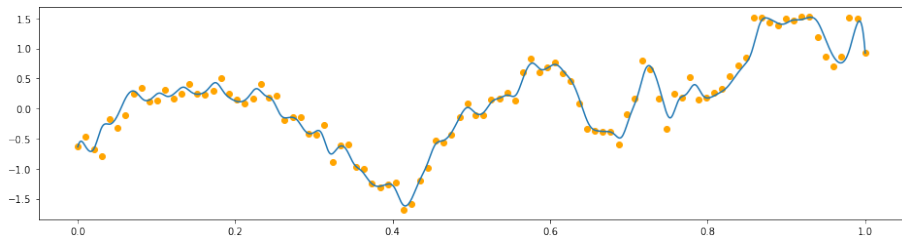


Figure: B-Spline interpolation

$$y' = B(x, x', d)y,$$

where $\{x, y\}$ original function, d - degree,
 x' - new domain, y' interpolated function

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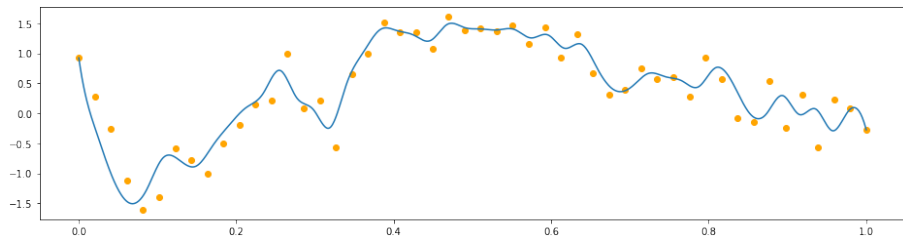


Figure: B-Spline interpolation (fewer points)

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where $\{x, y\}$ original function, d - degree,
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Why B-Spline

$$y' = B(x, x', d)y,$$

- ① x, x' are usually given
- ② y - is random (Gaussian Process)
- ③ y' - is a smooth interpolation
- ④ $B(x, x', d)$ is static
- ⑤ $B(x, x', d)$ is sparse

To remember

- ① Bad extrapolation when degree $d > 2$

Spline Interpolation

- $d = 1$ - linear interpolation
- $d = 2$ - quadratic interpolation
- $d = 3$ - cubic interpolation

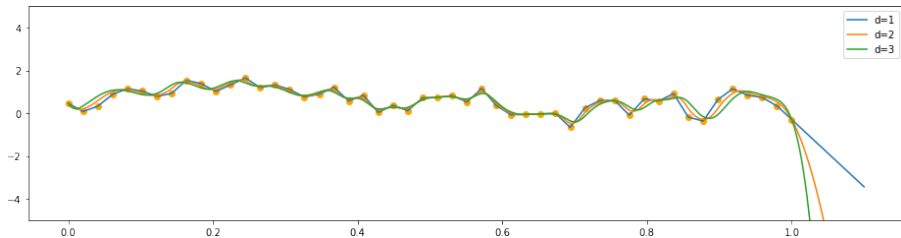


Figure: Spline interpolation

Splines for Stochastic volatility

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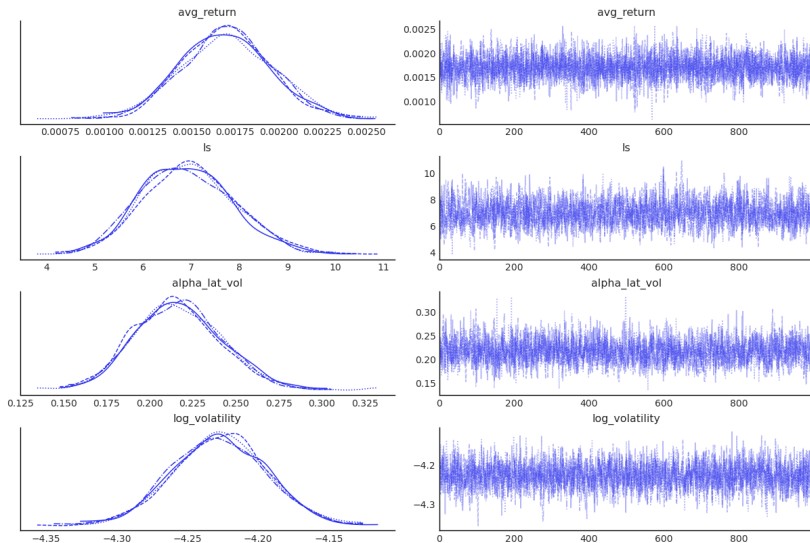
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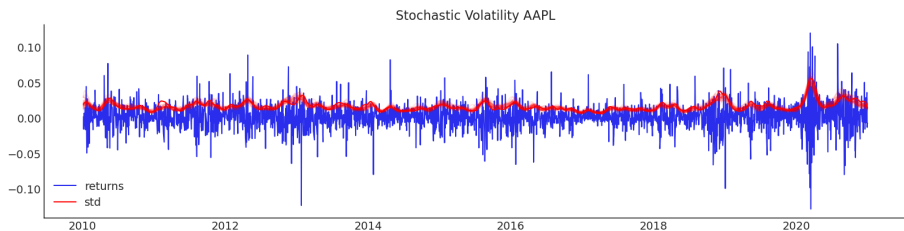
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Results



Takeaways



- Interpretable parameters
- Splines make model much faster
- Uncertainties for
 - average volatility
 - average returns
 - stochastic variation (α_{vol})

References I



T. SJ and L. B.
Forecasting at scale.
2017.