Gaussian Processes Part 2

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MSU

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Agenda

Time Series, Classical Approach

If data has seasonality, you usually use STL decomposition. However,

- Parameters are not interpretable, only decomposition is available
- No uncertainty estimates
- Quite strict on input values
- Significantly less flexible in modelling

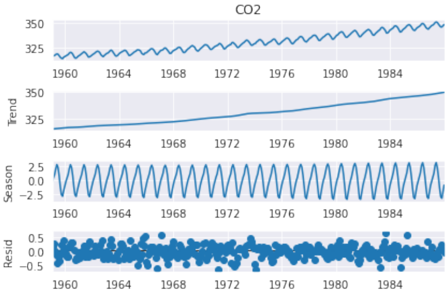

Figure: STL decomposition for CO2 data, Statsmodels

GP decomposition

A Gaussian process can handle a complicated set of assumptions in addition to what STL provides

- Granular seasonality (year + quarter + month + week)
- Changepoint models
- Flexible likelihood Function
- Panel regression models
- Missing values

Typical Model

Typical model is additive

$$x_t \sim \underbrace{g(t)}_{non-periodic} + \underbrace{s(t)}_{periodic} + \underbrace{h(t)}_{holidays}$$

Reference

See more in Prophet preprint [1]. Every time series model is unique

Reminder

$x \in \mathbb{R}^n$, $y \in \mathbb{R}$

$$Y \sim \mathcal{GP}(m(x), k(x, x'))$$

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2. $m(x)$ - mean function, e.g.
   - Linear regression $m(x) = x^\top \beta$
   - Simply Constant or Zero $m(x) = c$
   - Other custom functions $m(x) = \sin(x)$

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   - Other custom functions $m(x) = \sin(x)$
3. $k(x, x')$ - kernel function, simply - measure of similarity for x and x'
   - $[K]_{ij} = k(x_i, x_j)$ is an SPD matrix

Non-periodic Part (mean function)

- Growth models
- Linear trend models
- Changepoint models

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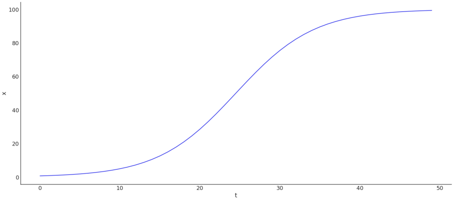

Figure: Growth Model

$$x = \frac{c}{1 + \exp(-k(t - m))}$$

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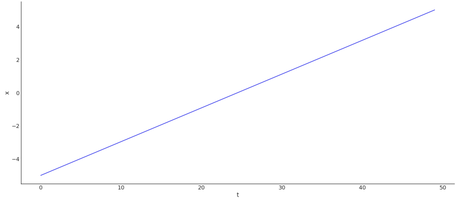

Figure: Linear Trend Model

$$x = \frac{c}{1 + \exp(-k(t - m))}$$

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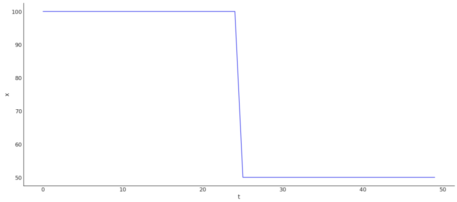

Figure: Changepoint Model

$$x = \begin{cases} c_1, & t < m \\ c_2, & t \geq m \end{cases}$$

Non-periodic Part (mean function)

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- Linear trend models
- Changepoint models

Extentions

Extensions are possible, e.g. time dependent saturation in the growth model. See in [1]

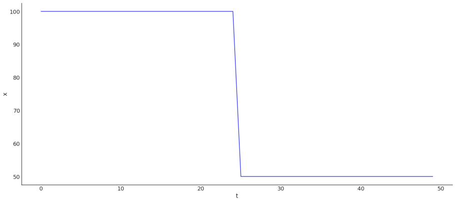

Figure: Changepoint Model

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Holidays

$$h(t) = \text{is-holiday}(t)$$

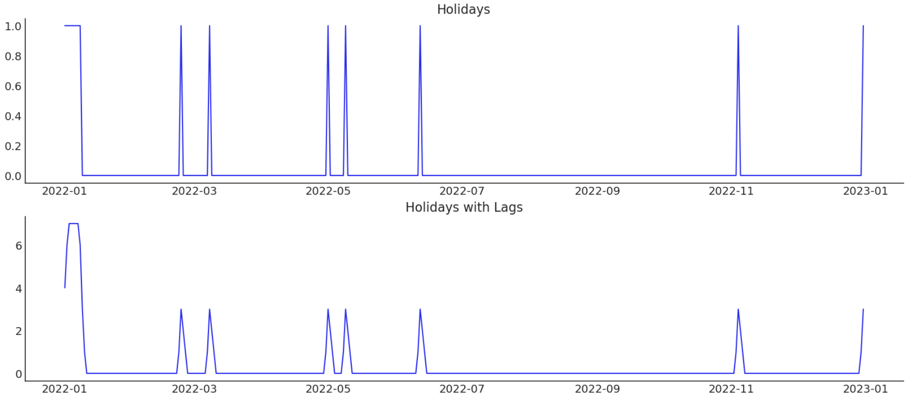
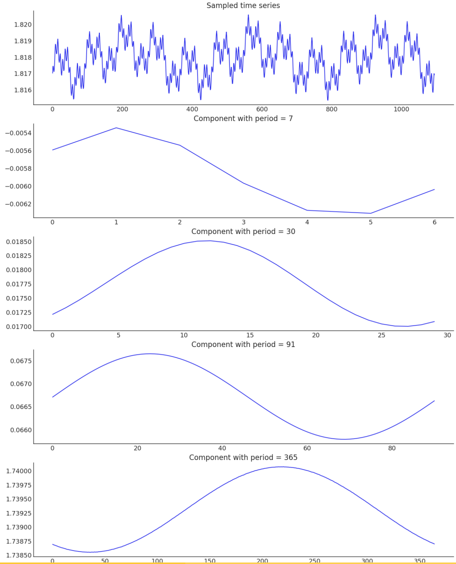


Figure: Holiday features

Periodic part (cov function)

Granularities are important here. Multiple Periodic kernels can be used.

- Yearly
- Quarterly
- Monthly
- Weekly



Lengthscales for Periodic Part

Hyperparameters

Common sense driven lenthscale choice

- Week - couple of days make a change ($ls \approx 3$)
- Month - week makes sense for a dramatic change ($ls \approx 7$)
- Quarter - month makes sense for a dramatic change ($ls \approx 30$)
- Year - quarter makes sense for a dramatic change ($ls \approx 90$)

Lengthscales for Periodic Part

Hyperparameters

Common sense driven lenthscale choice

- Week - couple of days make a change ($ls \approx 3$)
- Month - week makes sense for a dramatic change ($ls \approx 7$)
- Quarter - month makes sense for a dramatic change ($ls \approx 30$)
- Year - quarter makes sense for a dramatic change ($ls \approx 90$)

In practice

Hard to infer lengthscales, better just fix them if informed

Putting All Together

$$
\begin{aligned}
m(t) &= \underbrace{g(t)}_{non-periodic} + \underbrace{h(t)}_{holidays} \\
k(*,*) &= \mathrm{Periodic}(\mathrm{p} = 365, \mathrm{l} = 90) \\
&+ \mathrm{Periodic}(\mathrm{p} = 90, \mathrm{l} = 30) \\
&+ \mathrm{Periodic}(\mathrm{p} = 30, \mathrm{l} = 7) \\
&+ \mathrm{Periodic}(\mathrm{p} = 7, \mathrm{l} = 3)
\end{aligned}
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Putting All Together

$$m(t) = \underbrace{g(t)}_{non-periodic} + \underbrace{h(t)}_{holidays}$$

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Missing Parts

1. Weights for periodic components

Putting All Together

$$m(t) = \underbrace{g(t)}_{non-periodic} + \underbrace{h(t)}_{holidays}$$

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Missing Parts

1. Weights for periodic components
2. Trent violations from $g(t)$ that can't be explained by periodic component and observation noise

Example

CO_2 data (Manua Loa, Hawaii)

- Monthly observations
- Clean data
- Unclear trend function

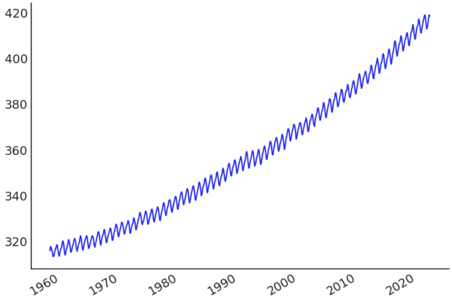

Figure: CO_2 level

Example

CO2 data (Manua Loa, Hawaii)

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Where to start?

1. Decide on a trend model
2. Add periodic component

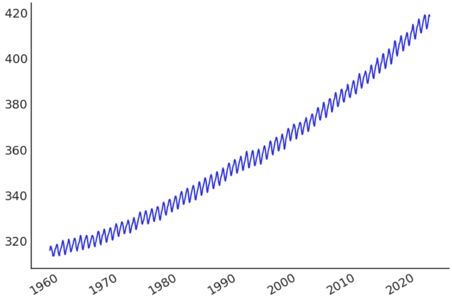

Figure: CO2 level

Priors

The exponential Growth Model

$$y_t = y_0 \cdot g^t$$

1. The baseline level for CO2 (data tells it is ≈ 310)
2. Growth rate for CO2 (some plausible value)
   - unlikely decay
   - no astronomic values

Prior Predictive Checks (Naive Attempt)

$$
\begin{aligned}
g &\sim \text{Normal}(1.1, 0.1)\\
\epsilon &\sim \text{LogNormal}(0, 1)\\
y_0 &\sim \text{Normal}(310, 10)\\
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\end{aligned}
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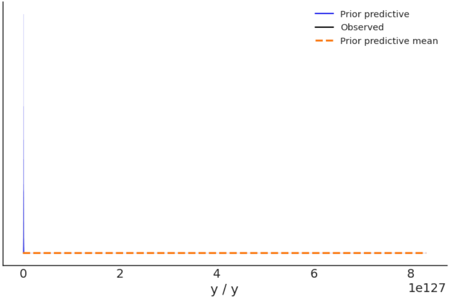

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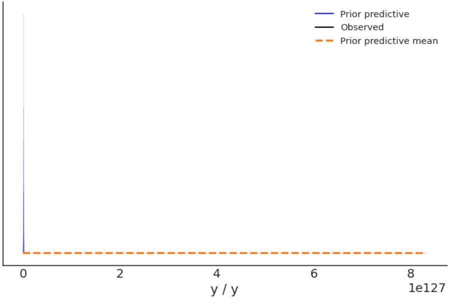

Figure: Prior Predictive Check

PPC Observation

Model is unlikely to be correctly specified. Prior predictive ranges to 10^{127} **parts per million** of CO2. Air has $2.5 \cdot 10^{25}$ molecules in m^3

Prior Predictive (Problem Driven)

Thinking loudly

I do not believe in a century we can exceed 400% growth (we would have died already) and I suppose it should be at least 20% (it would not be a problem otherwise)

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1. 400% in a century is 0.115% per month
2. 20% in a century is 0.015% per month

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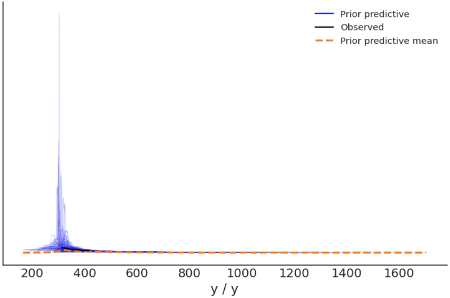

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PPC Observation

Model acts as expected ranging CO2 levels in plausible regions. It seems to be good enough to go to the next step.

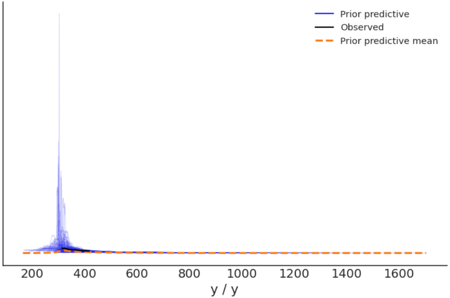

Figure: Prior Predictive Check

Inference

1. In sampling any value could appear
2. Here we got a very large value for some parameter
3. We can try to reparametrize

```
with model:
    trace = pm.sample()
```

```
Auto-assigning NUTS sampler...
Initializing NUTS using jitter+adapt_diag...
/home/ferres/.miniconda3/envs/bayes-econ/lib/python3.10/site-p
ric.py:57: RuntimeWarning: overflow encountered in multiply
  return bound(*args, **kwds)
Multiprocess sampling (4 chains in 4 jobs)
NUTS: [initial, g, eps]
```

Figure: Overflow in sampling

Log Model

$$\log y_t = \log y_0 + g \cdot t$$

$$\begin{aligned}
g &\sim \text{Normal}(1.00065, 0.0005) \\
\epsilon &\sim \text{LogNormal}(-2, 0.5) \\
\log y_0 &\sim \text{Normal}(\log 310, 10/310) \\
y_t &\sim \text{LogNormal}(\log y_0 + g \cdot t, \epsilon)
\end{aligned}$$

Important

ϵ has a different meaning in the model. It is now a relative error.

Prior Predictive (Reparametrized)

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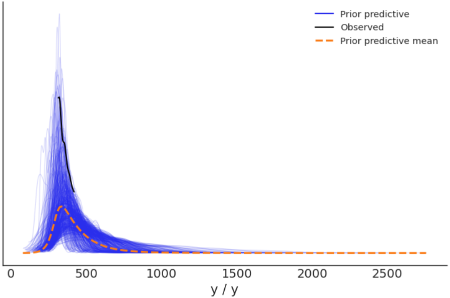

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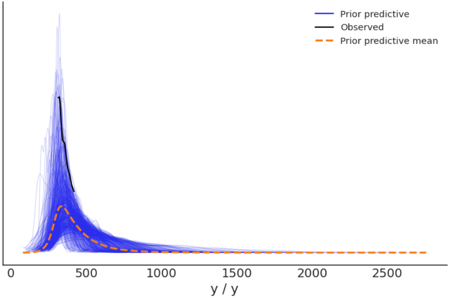

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1. Everything is good
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Sampling 4 chains, 0 divergences]

```
Sampling 4 chains for 1_000 tune and 1_000 draw iterations (4_
s total) took 6 seconds.
```

Figure: Good sampling

Inference

1. Everything is good
2. We can visualize
3. Trace plot is a good visual inspection if sampling went well

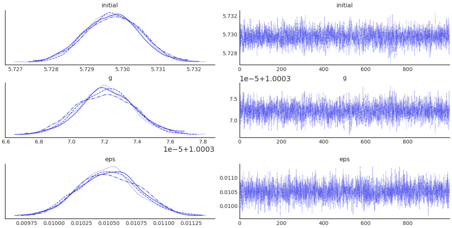

Figure: Good Trace Plot

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Posterior Predictive

Another way to validate your model is posterior predictive.

- How model performs vs observed data?
- Is there any missing effect to include?

Let's see

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Another way to validate your model is posterior predictive.

- How model performs vs observed data?
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Let's see

1. The fit is good
2. Model does not reflect seasonalities
3. We can fix it with a better model

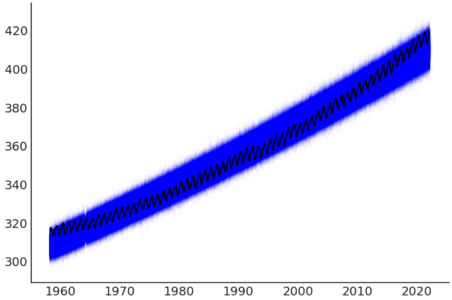

Figure: Posterior Predictive Plot

Conclusions so far

- The reparametrized model is decent
- It does not account for seasonality
- We have the mean function for the next modelling step

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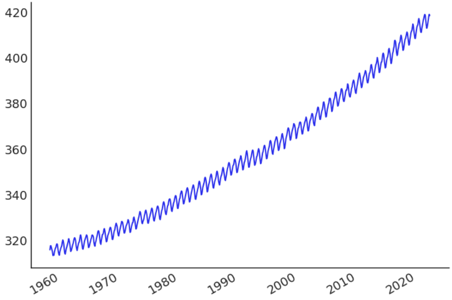
Adding Seasonality

CO2 data (Manua Loa, Hawaii)

- Monthly observations
- Yearly seasonality makes sense
- Trend function is clear from a simpler model



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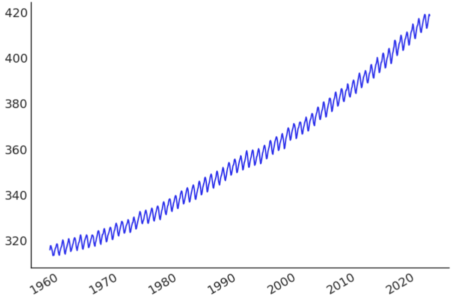

Figure: CO2 level

$$k(*,*) = \quad \text{Periodic}(\text{p} = \quad , \text{l} = \quad)$$

Adding Seasonality

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- Period is 12 months

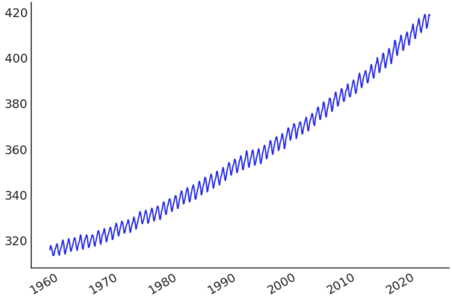

Figure: CO2 level

$$k(*,*) = \quad \mathrm{Periodic}(\mathrm{p} = 12, \mathrm{l} = \quad)$$

Adding Seasonality

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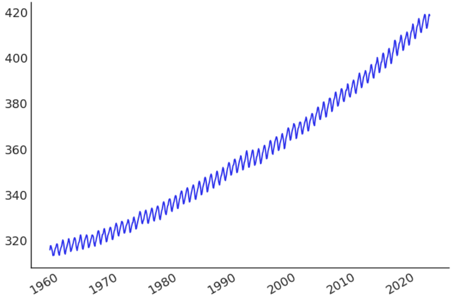

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Adding Seasonality

CO2 data (Manua Loa, Hawaii)

- Monthly observations
- Yearly seasonality makes sense
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- Seasonality can be added with a periodic kernel
- Period is 12 months
- Appropriate lengthscale is ≈ 3 months
- Do not forget to scale the covariance!

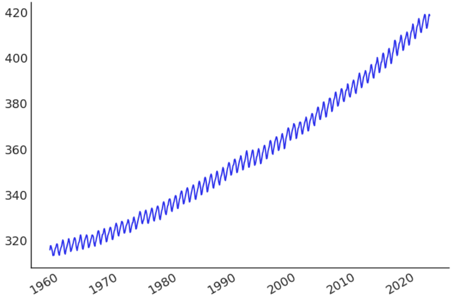

Figure: CO2 level

$$k(*, *) = \alpha_k \,\mathrm{Periodic}(\mathrm{p} = 12, \mathrm{l} = 3)$$

Seasonality Change

$$K(*, *) = \alpha_k \, \mathrm{Periodic}(\mathrm{p} = 12, \mathrm{l} = 3)$$

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- Seasonality pattern is assumed constant (what if ...)

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- We need a decay in correlation between distant year patterns

Seasonality Change

$$K(*,*) = \alpha_k \, \mathrm{Periodic(p=12, l=3)} \cdot \mathrm{Exponential(l=1200)}$$

- Periodic kernel is a strong prior.
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- Multiplication with the Exponential kernel should work just fine

Seasonality Change

$$K(*,*) = \alpha_k \, \mathrm{Periodic}(\mathrm{p} = 12, \mathrm{l} = 3) \cdot \mathrm{Exponential}(\mathrm{l} = 1200)$$

- Periodic kernel is a strong prior.
- Seasonality pattern is assumed constant (what if ...)
- We can make seasonality patterns change (kernel math!)
- We need a decay in correlation between distant year patterns
- Multiplication with the Exponential kernel should work just fine
- Lengthscale in the Exponential kernel is how strong patterns correlate across months
- I choose 1200, century long patterns

Decay Visualized

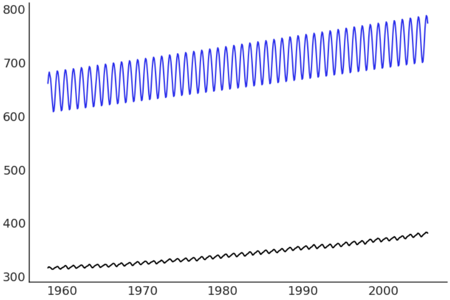

Figure: Mean without decay

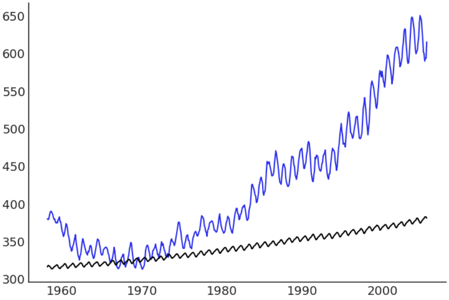

Figure: Mean with decay

Decay model

No decay model is the limit case of decay model with lengthscale $\to \infty$

The Model

$$
\begin{aligned}
g &\sim \text{Normal}(1.00065, 0.0005)\\
\epsilon &\sim \text{LogNormal}(-2, 0.5)\\
\log y_0 &\sim \text{Normal}(\log 310, 10/310)\\
\alpha_k &\sim \text{LogNormal}(-2, 0.5)\\
K(*,*) &= \alpha_k \,\text{Periodic}(\text{p}=12, \text{l}=3)\cdot \text{Exponential}(\text{l}=1200)\\
\log \bar{y}_t &= \mathcal{GP}(\log y_0 + g\cdot t, K)\\
y_t &\sim \text{LogNormal}(\log \bar{y}_t, \epsilon)
\end{aligned}
$$

Improve the model

1. Figure out a better ϵ prior
2. Infer better lengthscales (hyperpriors)
3. Infer structural medium term shift (additive kernel)

Prior Predictive Check

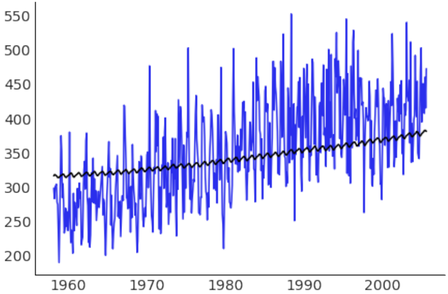

Figure: $\epsilon \sim \text{LogNormal}(-2, 0.5)$

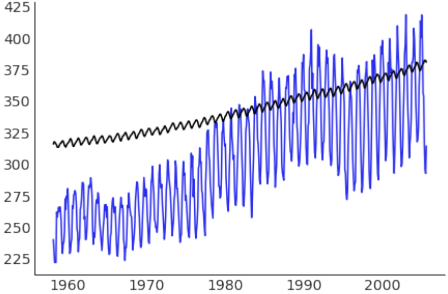

Figure: $\epsilon \sim \text{LogNormal}(-4, 0.5)$

Inference

- `pm.sample()` is slow
- GPU sampling is also slow
- `pm.find_MAP()` makes sense and fast

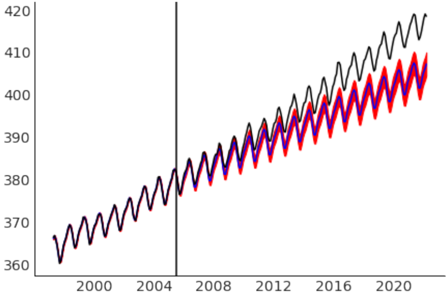

Figure: Predictions with MAP

Key takeaways

- Look at prior predictive to validate initial model
- Interpretable building blocks create interpretable model
- Sasonality model is done with Periodic kernel
- Seasonality might be flexible

Stochastic Volatility Model

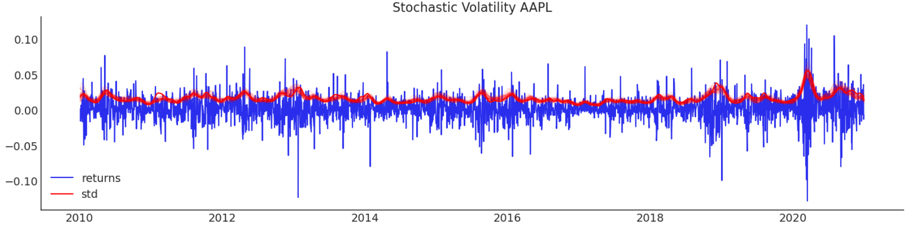

Figure: Stochastic Volatility Estimation for AAPL

Read More

https://github.com/quantopian/bayesalpha

Key Ideas

- Returns are constant
- Volatility is not constant
- Model volatility as a Gaussian process
- Use approximations to speed up the model
- Use GPU to do fast inference

Priors

- Returns
- Volatility
- Time Component

Priors

- Returns
  - Expect year return at orders $\pm 100\%$
- Volatility
- Time Component

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- Returns
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  - Daily return $\pm(2^{\frac{1}{250}} - 1)$
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- Volatility
  - Expect signal to noise $\frac{\text{mean}}{\text{std}} \approx 1$
  - $\log \text{std} = \log(2^{\frac{1}{250}} - 1)$
- Time Component
  - After the model is framed

The Model

I did the modelling choice to make it as simple as possible

$$
\begin{aligned}
\text{return} &\sim \text{Normal}(0, 2^{\frac{1}{250}} - 1) \\
\log \text{std} &\sim \text{Normal}(\log(2^{\frac{1}{250}} - 1), 0.05) \\
\text{ls} &\sim \text{Gamma}(30, 5) \\
\alpha_{\text{vol}} &\sim \text{Exponential}(100) \\
K(*, *) &= \alpha_{\text{vol}} \, \text{Matern32}(\text{ls}) \\
\Delta_t^{\log \text{std}} &\sim \mathcal{GP}(0, K) \\
\text{obs}_t &\sim \text{Normal}(\text{return}, \exp(\log \text{std} + \Delta_t^{\log \text{std}}))
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\end{aligned}
$$

Splines

Make the model faster!

Spline interpolation

`pmx.utils.spline.bspline_interpolation`

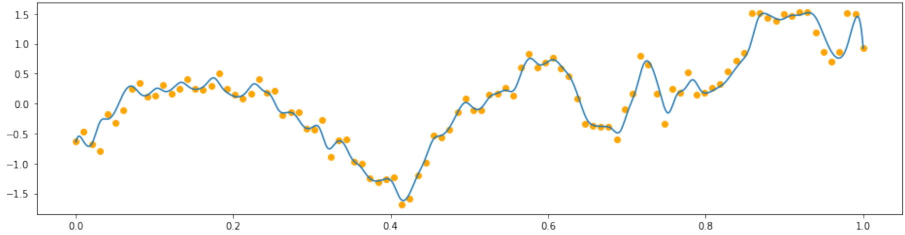

Figure: B-Spline interpolation

$$y' = B(x, x', d)y,$$

where $\{x, y\}$ original function, d - degree,
x' - new domain, y' interpolated function

Spline interpolation

`pmx.utils.spline.bspline_interpolation`

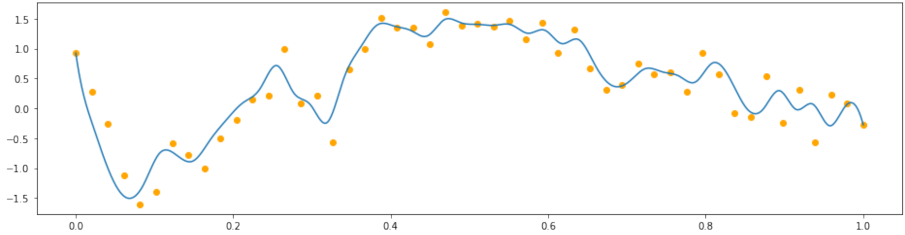

Figure: B-Spline interpolation (fewer points)

$$y' = B(x, x', d)y,$$

where $\{x, y\}$ original function, d - degree,
x' - new domain, y' interpolated function

Why B-Spline

$$y' = B(x, x', d)y,$$

1. x, x' are usually given
2. y - is random (Gaussian Process)
3. y' - is a smooth interpolation
4. $B(x, x', d)$ is static
5. $B(x, x', d)$ is sparse

To remember

1. Bad extrapolation when degree $d > 2$

Spline Interpolation

- $d = 1$ - linear interpolation
- $d = 2$ - quadratic interpolation
- $d = 3$ - cubic interpolation

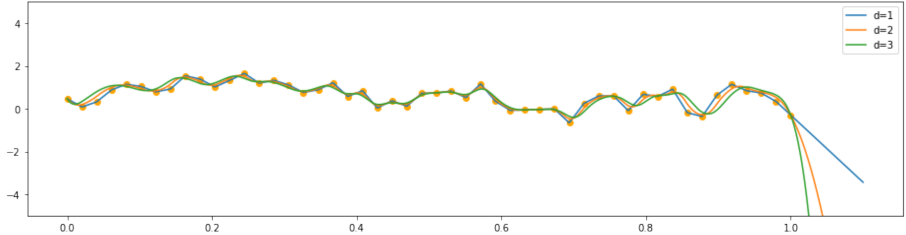

Figure: Spline interpolation

Splines for Stochastic volatility

$$
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\bar{\Delta}_{t'}^{\log \text{std}} &\sim \mathcal{GP}(0, K) \\
\Delta_{t}^{\log \text{std}} &= \text{B-Spline}(\bar{\Delta}_{t'}^{\log \text{std}}, \text{d} = 3) \\
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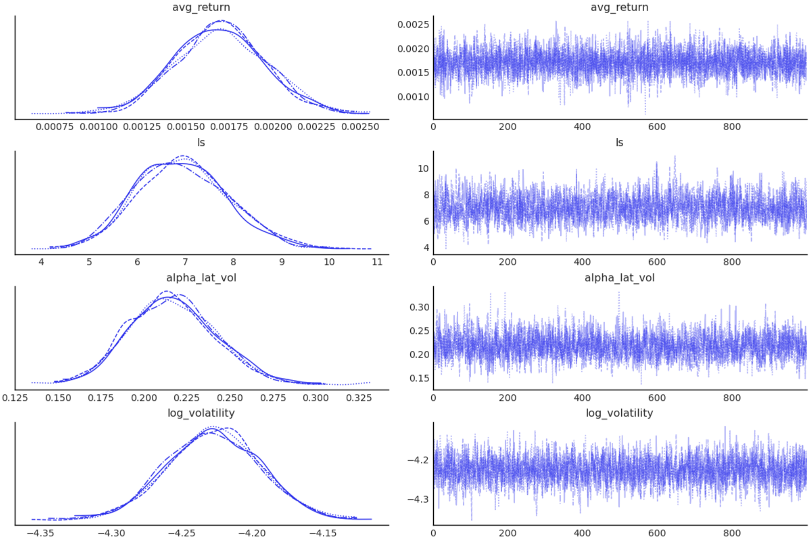
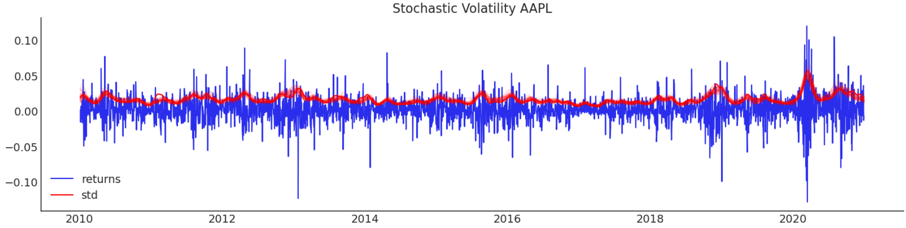
$$

Results


avg_return
0.00075 0.00100 0.00125 0.00150 0.00175 0.00200 0.00225 0.00250
avg_return
0.0025 0.0020 0.0015 0.0010
0 200 400 600 800
ls
4 5 6 7 8 9 10 11
ls
10 8 6 4
0 200 400 600 800
alpha_lat_vol
0.125 0.150 0.175 0.200 0.225 0.250 0.275 0.300 0.325
alpha_lat_vol
0.30 0.25 0.20 0.15
0 200 400 600 800
log_volatility
−4.35 −4.30 −4.25 −4.20 −4.15
log_volatility
−4.2 −4.3
0 200 400 600 800

Takeaways



- Interpretable parameters
- Splines make model much faster
- Uncertainties for
  - average volatility
  - average returns
  - stochastic variation (α_{vol})

References I

T. SJ and L. B.
Forecasting at scale.
2017.