

Bayesian Modeling

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Lecture 2

Agenda

Sampling from a distribution

Conjugate models

- Limited set of applications
- Lack of flexibility
- They are scalable

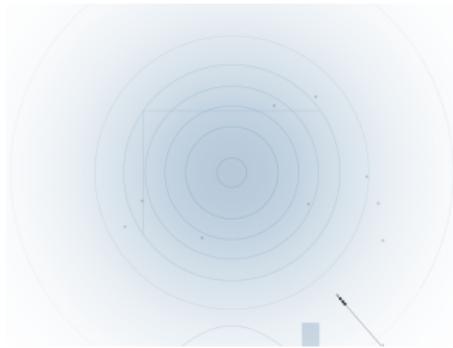


Figure: Easy distribution

Most models

- No closed form solution
- Posterior distributions is complicated
- Less scalable
- Flexible

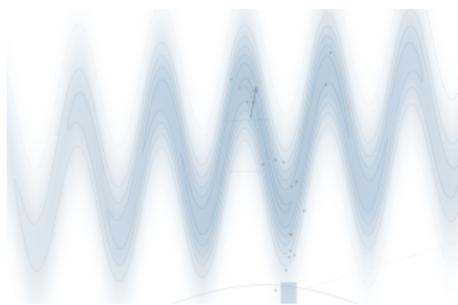


Figure: Complicated distribution

Hamiltonian Monte Carlo Intuition

HMC samples from a complicated distribution

- ① Ideas from physics
- ② Requires gradient
- ③ Requires numerical integration

Tuned HMC converges to the target distribution

Warning

I promised a not math heavy course.
But this is important for debugging
your models.

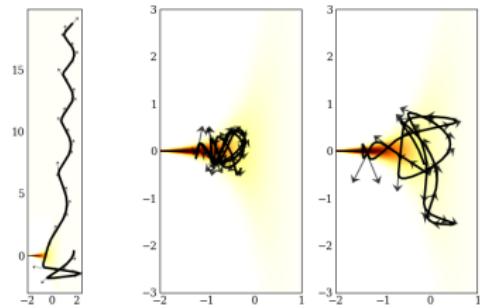


Figure: Leapfrog Integration

HMC Distributions

- $p(\Theta)$ - Target distribution, $\Theta \in \mathbb{R}^d$ (Θ aka **Position**)
- $p(\Delta | \Theta)$ - Momentum distribution, $\Delta \in \mathbb{R}^d$ (Δ aka **Velocity**)

Hamiltonian

$$H(\Delta, \Theta) = -\log p(\Delta, \Theta)$$

Notes

- $p(\Delta | \Theta) = \text{Normal}(0, M)$, usually a Normal distribution
- Δ and Θ have same dimensions

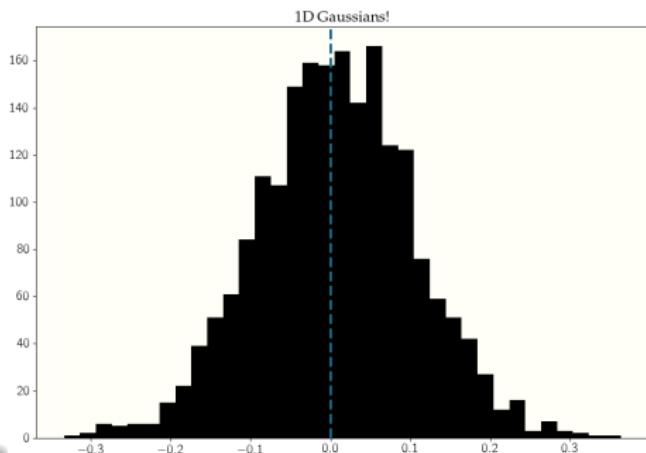


Figure: $p(\Theta) = \text{Normal}(0, 1)$

HMC Differential Equation

$$\begin{aligned}
 H(\Delta, \Theta) &= -\log p(\Delta, \Theta) \\
 &= -\log p(\Delta | \Theta) - \log p(\Theta) \\
 &= \underbrace{K(\Delta, \Theta)}_{\text{Kinetic E}} + \underbrace{V(\Theta)}_{\text{Potential E}}
 \end{aligned}$$

The Physical **motion** equation

$$\begin{aligned}
 \frac{\partial \Theta}{\partial t} &= \frac{\partial H}{\partial \Delta} \\
 \frac{\partial \Delta}{\partial t} &= -\frac{\partial H}{\partial \Theta}
 \end{aligned}$$

Motion preserves total energy
 $H(\Delta, \Theta)$



Figure: HMC analogy to skateboarding

HMC Divergences

A divergence is a huge integration error solving the differential equation.

When HMC Fails

Bad geometry for Hamiltonian

Bad geometry comes from a lot of things

- ① Strong correlations
- ② Narrow funnels in the posterior
- ③ Strong likelihood
- ④ Non homogeneous posterior

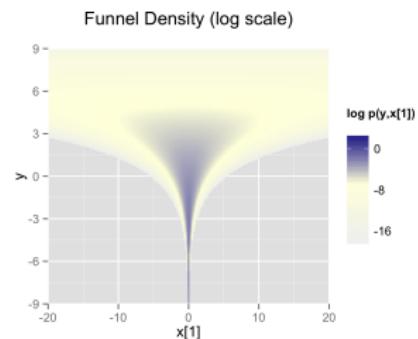


Figure: Neal's Funnel

HMC Reading Materials

Advanced Reading

- ① Interactive Demo
- ② A tutorial from Colin Carroll
- ③ A paper from Michael Betancourt
- ④ NUTS paper from Matthew D. Hoffman, Andrew Gelman

Example

Toy example - Cobb-Douglas

You should all know the Cobb-Douglas function

$$Y \approx A \cdot L^\beta$$

In our example:

- ① data has 6 groups (hierarchical)
- ② We know the groups
- ③ We know the total factor productivity A is different per group (different equipment)
- ④ Labour productivity β does not differ much

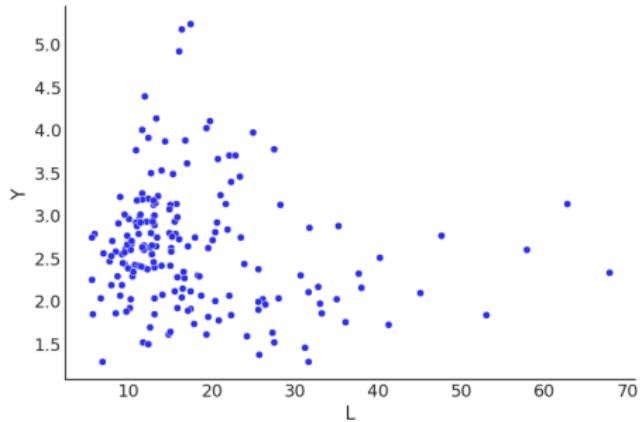


Figure: Example Data (aggregated)

Toy example - Carpet Knitters

Let's put more interpretation in the example

$$Y_g \approx A_g \cdot L^\beta$$

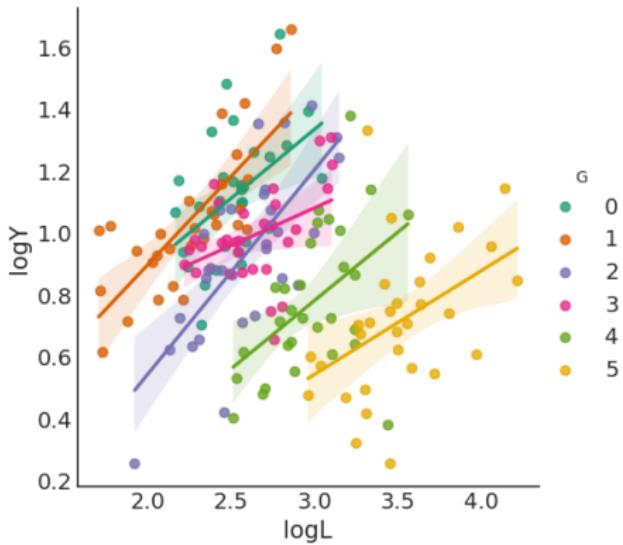
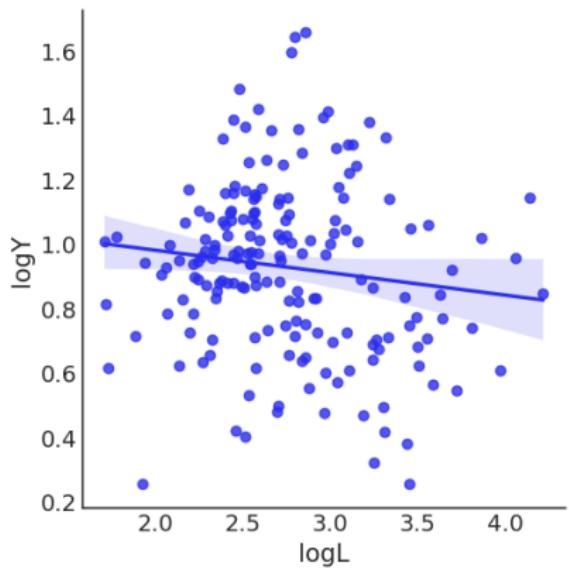
In our example we have a carpet manufacturing plant with 6 workers:

- ① Workers make different carpets, thus have total factor productivity A
- ② Labour productivity β is like concentration, the more you work the less productive you are
- ③ Workers produce carpets individually



Figure: Example Y

The Simpson Paradox



One group model

Best practices when you start.

- Start with a most simple model
- Make sure simple model converges well
- Write a more complex model

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- Start with a most simple model
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 - Make sure priors are well specified, do checks
- Make sure simple model converges well
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- Write a more complex model
 - Try several parametrizations
 - Check how model samples
 - Compare models (out of scope for now)

Starting with a simple model

To get an idea why we start simple

$$Y_{g=0} \approx A_{g=0} \cdot L^{\beta}$$

- ① What is prior for A ?
- ② What is prior for β ?
- ③ What is prior predictive for $Y_{g=0}$?

Writing a model

$$Y_{g=0} \approx A_{g=0} \cdot L^\beta$$

$$\log Y_{g=0} \approx \log A_{g=0} + \log L \cdot \beta$$

Introducing distributions

$$\log Y_{g=0} \sim \text{Normal}(\log A_{g=0} + \log L \cdot \beta, \epsilon)$$

$$\epsilon \sim ???$$

$$\beta \sim ???$$

$$A_{g=0} \sim ???$$

Prior for β

What is a reasonable prior for labour productivity (elasticity) β ? Questions to ask yourself

- ① Can it be < 0 ?
- ② Can it be large?
- ③ Can it be > 1 ?

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Conclusion: It is bounded by $(0, 1)$

The prior is subjective!

Who can argue these bounds do not make sense?

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Not yet a prior

To get a prior we need a distribution that fits the reasoning

Prior for β

What we know:

- $\beta \in (0, 1)$
- Less probable to be close to the boundary
- Nothing specific about exact value in the range.

In the mind

Enumerate possible distributions that fit the reasoning

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- ② LogitNormal(μ, σ) - always avoids boundaries

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- ③ Uniform(0, 1) - a special case of Beta(1, 1)

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- ② LogitNormal(μ, σ) - always avoids boundaries
- ③ Uniform(0, 1) - a special case of Beta(1, 1)
- ④ Kumaraswamy(a, b), $a > 0, b > 0$ you do not need to know that

Visualize your prior

Before writing a line of code, visualise your prior. What do you like more?

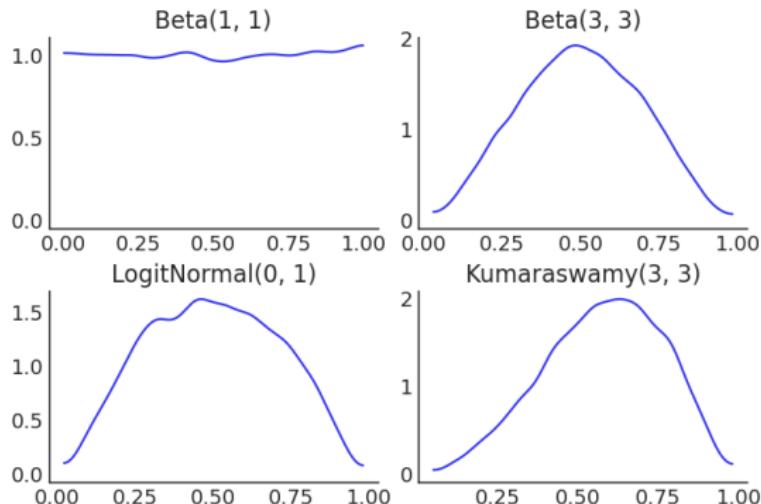


Figure: Visualized Priors

You can choose the form with theory in mind

Setting a prior

I prefer LogitNormal(0, 1) in this situation. It has a good functional form.

To remember

- Prior is **your** modelling choice
- The choice has to be motivated
- The choice should make sense given practical constraints
- You should always be able to defend your choice
- **Prior is what you do not know, the uncertainty**

The model so far

$$\log Y_{g=0} \sim \text{Normal}(\log A_{g=0} + \log L \cdot \beta, \epsilon)$$

$$\epsilon \sim ???$$

$$\beta \sim \text{LogitNormal}(0, 1)$$

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Prior for ϵ

Rule of thumb

Error term is something small. Usually avoids zero.

In our case;

- small is "orders of 10-50%"

Let it be

$$\epsilon \sim \text{LogNormal}(-2, 1)$$

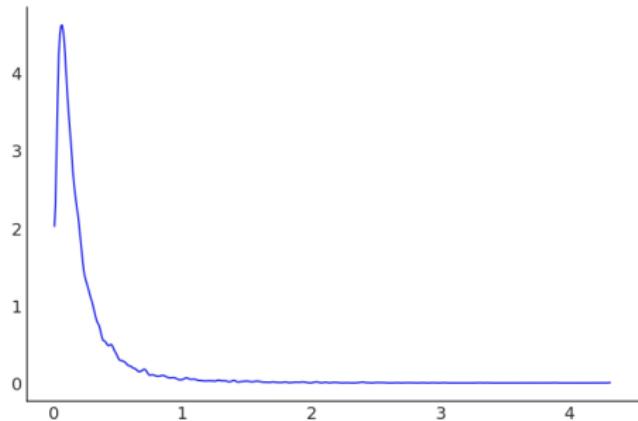


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Useful

In log-log models error term is on the relative scale

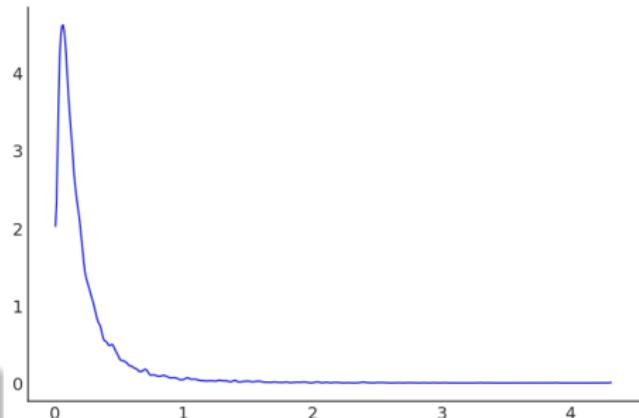


Figure: Prior for ϵ

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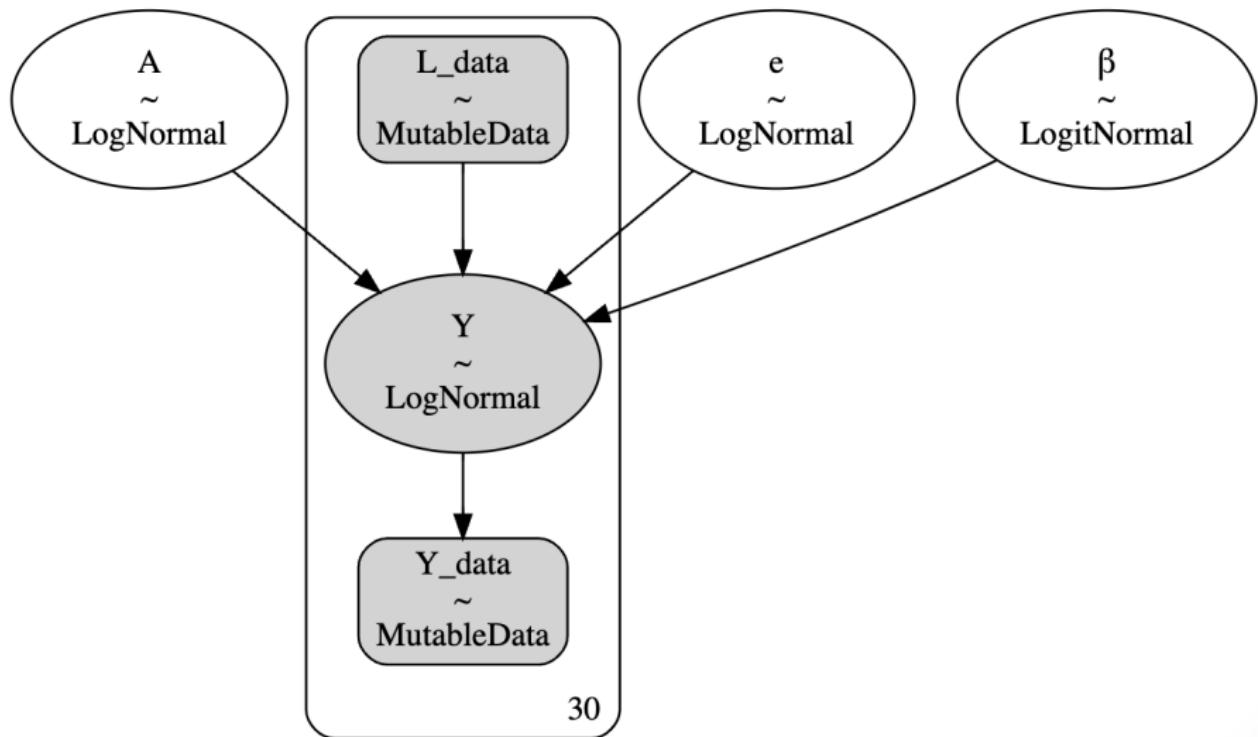
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Visual Model



Prior Predictive

Prior for β was an easy one. We need one for $A_{g=0}$

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Definition

Prior predictive is simulated observation model given no data.

The truth

Nobody said setting priors is easy. It is the most work.

Random prior

Why not using e.g.

$$A \sim \text{LogNormal}(0, 1)$$

Random prior

Why not using e.g.

$$A \sim \text{LogNormal}(0, 1)$$

Nonsense

Workers do not produce 800 carpets per week.

That's why

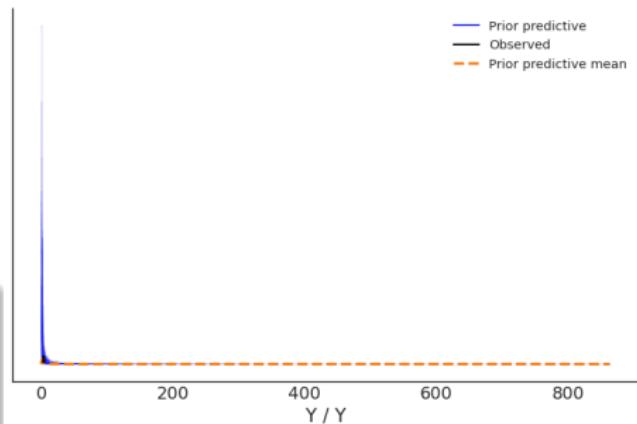


Figure: Prior predictive for Y vs data

Analysing the prior predictive

Getting back to a full model

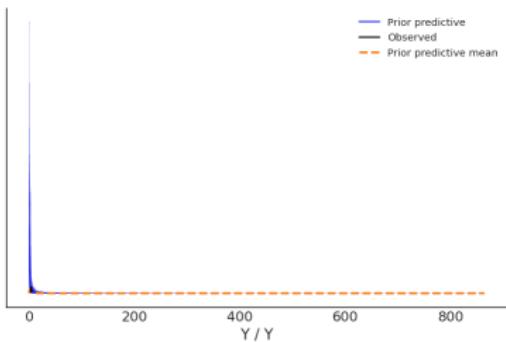
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$$A_{g=0} \sim \text{LogNormal}(0, 1)$$

What can we read here?



- We see over dispersion in predictions
- Variance may come from A or ϵ

Actions

- ➊ Try reducing A variance
- ➋ Try reducing ϵ variance

Figure: Prior predictive for Y vs data

Good prior predictive

Seminar

You will play with the example at the seminar.

A good looking prior predictive was with the definition below

$$\log Y_{g=0} \sim \text{Normal}(\log A_{g=0} + \log L \cdot \beta, \epsilon)$$

$$\epsilon \sim \text{LogNormal}(-2, 0.1)$$

$$\beta \sim \text{LogitNormal}(0, 1)$$

$$A_{g=0} \sim \text{LogNormal}(-0.5, 0.1)$$

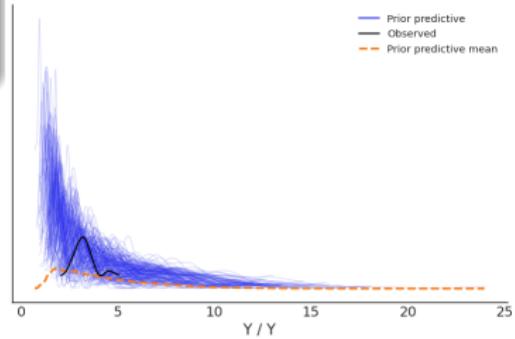


Figure: Prior predictive for Y vs data

What is a good prior predictive?

- Prior predictive covers **reasonable** range for observed data.

- Data is reference**, not your objective.

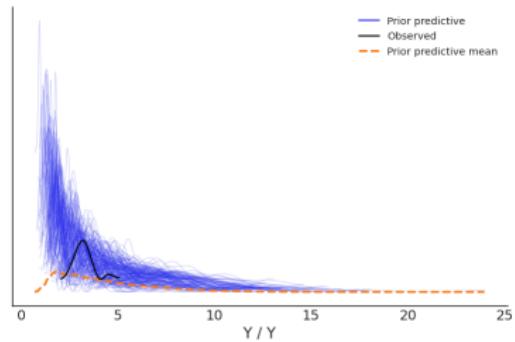


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 - no astronomic speeds
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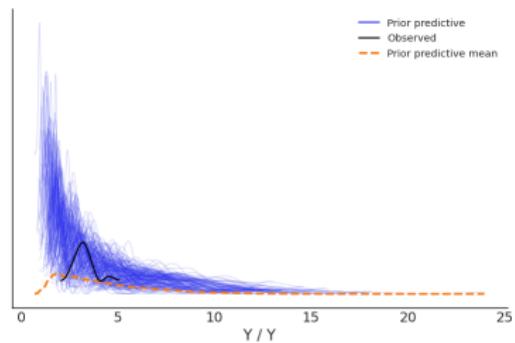


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What is a good prior predictive?

- Prior predictive covers **reasonable** range for observed data.
 - no astronomic speeds
 - no microscopic distances
 - no black hole densities
 - no superpower workers
- **Data is reference**, not your objective.
 - do not overfit priors on data.
 - in 90% cases you do not need data for prior predictive
 - in 90% cases common sense should work just fine
 - in 10% cases you can ask experts and adjust the priors
 - data is your last resort

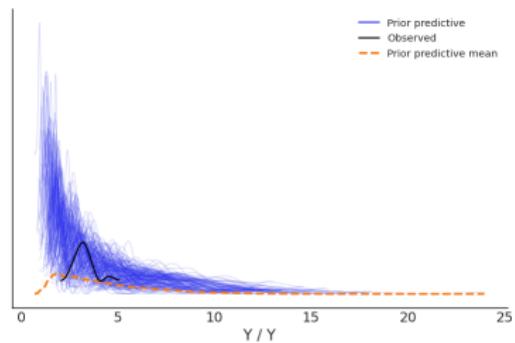


Figure: Prior predictive for Y vs data

HMC in action



Sampling

After we've checked the priors it is time to sample.

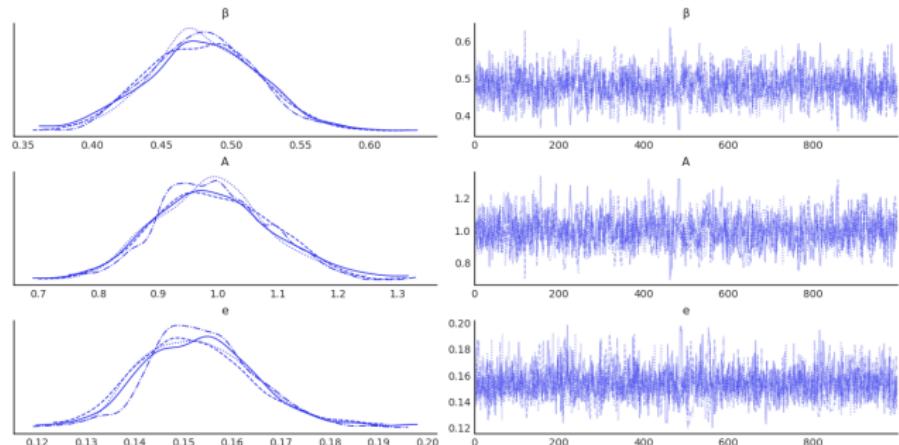


Figure: Posterior MCMC trace

Hierarchies

Hierarchies

Initial data has groups. How to take them in account?

$$\log Y_{\mathbf{g}} \sim \text{Normal}(\log A_{\mathbf{g}} + \log L \cdot \beta, \epsilon)$$

$$\epsilon \sim \text{LogNormal}(-2, 0.1)$$

$$\beta \sim \text{LogitNormal}(0, 1)$$

$$A_{\mathbf{g}} \sim ???$$

What is Hierarchy?

Hierarchy

Once you have similar groups in your data, you have hierarchy.

Examples:

- ① Countries, Regions
- ② User groups: by age, by profession, etc
- ③ Treatment groups
- ④ Time dependent effects
- ⑤ Panel Data

Our Example

Workers make different carpets and have total factor productivity A

Treating Hierarchy

Classical Econometrics view:

- ① All the groups are independent. **Pooled Model**

$$y_{k,i} = \alpha + \beta x_{k,i} + \varepsilon_{i,k}$$

- ② Groups have significant differences. **Fixed Effect Model**

$$y_{k,i} = \alpha_k + \beta x_{k,i} + \varepsilon_{i,k}$$

- ③ Groups have non significant, random differences. **Random Effects Model**

$$y_{k,i} = \alpha + \beta x_{k,i} + u_k + \varepsilon_{i,k}$$

Where

$$\mathbb{E} u_{k,i} = 0, \quad \mathbb{E} \varepsilon_{i,k} = 0$$

Bayesian Hierarchy

In

$$y_{k,i} = \alpha + \beta x_{k,i} + u_k + \varepsilon_{i,k}$$

Let's rearrange terms

$$y_{k,i} = (\alpha + u_k) + \beta x_{k,i} + \varepsilon_{i,k}$$

- α - population mean
- $\alpha_k = \alpha + u_k$ - group mean

In a Bayesian analysis we need priors. There is more than one way

$$\alpha \sim \text{Normal}(\bar{\mu}, \bar{\sigma})$$

$$u_k \sim \text{Normal}(0, 1)$$

$$\alpha_k = \alpha + u_k \cdot \sigma$$

$$\alpha \sim \text{Normal}(\bar{\mu}, \bar{\sigma})$$

$$\alpha_k \sim \text{Normal}(\alpha, \sigma)$$

More on priors

Non centered parametrization

$$\alpha \sim \text{Normal}(\bar{\mu}, \bar{\sigma})$$

$$u_k \sim \text{Normal}(0, 1)$$

$$\alpha_k = \alpha + u_k \cdot \sigma$$

Centered parametrization

$$\alpha \sim \text{Normal}(\bar{\mu}, \bar{\sigma})$$

$$\alpha_k \sim \text{Normal}(\alpha, \sigma)$$

Group specific parameter u_k is disentangled

σ is a measure of group differences

- ① $\sigma \rightarrow 0$: Pooled Model
- ② Small σ : Random Effects / Partial Pooling
- ③ Large σ : Fixed Effects / Unpooled Model

σ interpolates between the models

Degeneracy

Centered parametrization

$$\alpha \sim \text{Normal}(\bar{\mu}, \bar{\sigma})$$

$$\alpha_k \sim \text{Normal}(\alpha, \sigma)$$

Warning

Centered parametrization creates funnel geometry with few data

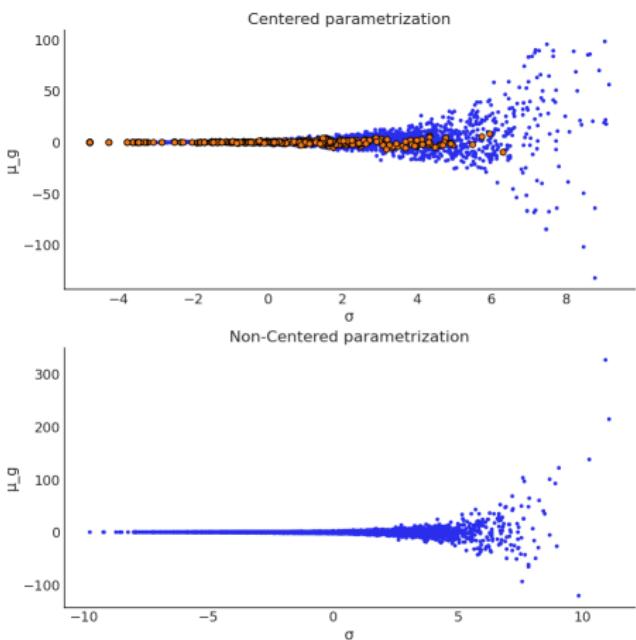


Figure: Divergences appear in the Centered Parametrization

Why Funnel is created?

Geometry is important

- ① Sampler has adaptive step size
- ② With bad geometry Sampler can't find a good one

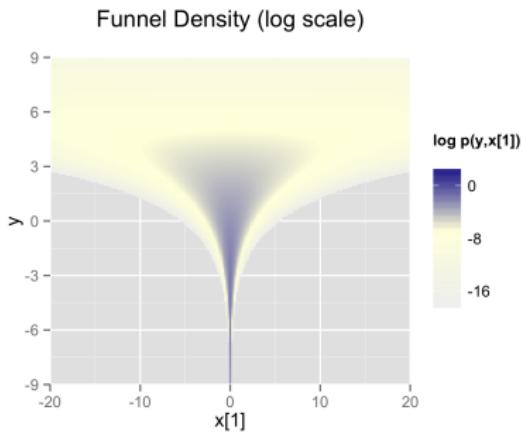


Figure: Funnel Geometry

Suggested reading

Read more on reparametrization in [Stan's Guide](#)

Inverted Funnel degeneracy

A "nice" parametrization does have issues as well.

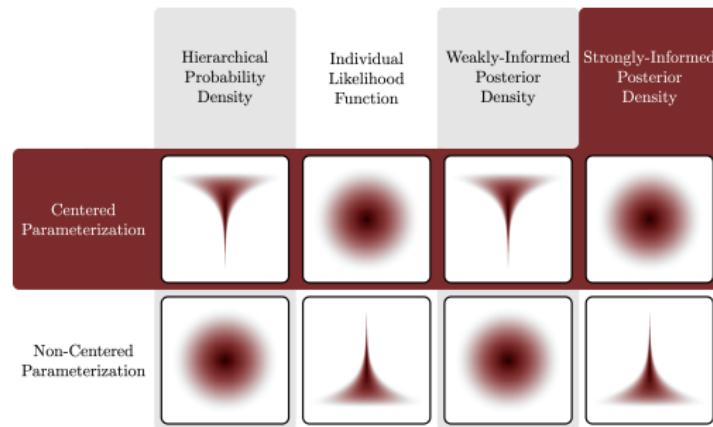


Figure: Inverted Funnel Degeneracy

Advanced Reading

Read more from Michael Betancourt

Setting a Hierarchical Prior

- ① Start with a Pooled or Single group model
- ② Add Hierarchy

Setting a Hierarchical Prior

- ① Start with a Pooled or Single group model
 - You get an idea of prior parameter scales
 - You get a decent model structure
 - Do not care about predictions
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- Decide on which parameters to share
- Decide on allowed variability for the rest parameters
- Debug divergences, reparametrize if required

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Best Practice

Do not hard-code the parametrization, toggle it in the code

The Cobb-Douglas Case

Single group model

$$\log Y_0 \sim \text{Normal}(\log A_0 + \log L \cdot \beta, \epsilon)$$

$$\epsilon \sim \text{LogNormal}(-2, 0.1)$$

$$\beta \sim \text{LogitNormal}(0, 1)$$

$$A_0 \sim \text{LogNormal}(-0.5, 0.1)$$

Hierarchical model

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$$A_k \sim \text{LogNormal}(\log A_{\text{pop}}, \sigma_A)$$

$$A_{\text{pop}} \sim \text{LogNormal}(-0.5, 0.1)$$

$$\sigma_A \sim \text{LogNormal}(-2, 0.1)$$

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$$\sigma_A \sim \text{LogNormal}(-2, 0.1)$$

Hint

You can reuse some parameters, just add reasonable variability σ_A

Discussion Time

Setting priors

- Sometimes you do not have expert knowledge
- Sometimes parametrization does not allow you to set a good prior
- Sometimes prior predictive depends on many parameters
- You are limited in time
- Using hyperpriors