Gaussian Processes Part 2

Max Kochurov

Moscow State University

Lecture 5

Agenda

Time Series, Classical Approach

If data has seasonality, you usually use STL decomposition. However,

- Parameters are not interpretable, only decomposition is available
- No uncertainty estimates
- Quite strict on input values
- Significantly less flexible in modelling

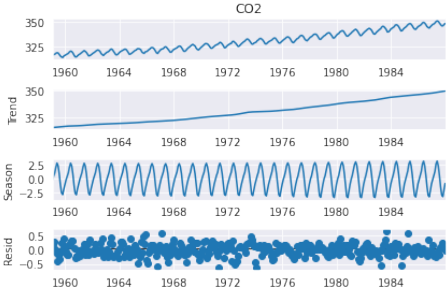

Figure: STL decomposition for $CO2$ data, Statsmodels

GP decomposition

A Gaussian process can handle a complicated set of assumptions in addition to what STL provides

- Granular seasonality (year + quarter + month + week)
- Changepoint models
- Flexible likelihood Function
- Panel regression models
- Missing values

Typical Model

Typical model is additive

$$x_t \sim \underbrace{g(t)}_{non-periodic} + \underbrace{s(t)}_{periodic} + \underbrace{h(t)}_{holidays}$$

Reference

See more in Prophet preprint [1]. Every time series model is unique

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3. $k(x, x')$ - kernel function, simply - measure of similarity for x and x'
   - $[K]_{ij} = k(x_i, x_j)$ is an SPD matrix

Non-periodic Part (mean function)

- Growth models
- Linear trend models
- Changepoint models

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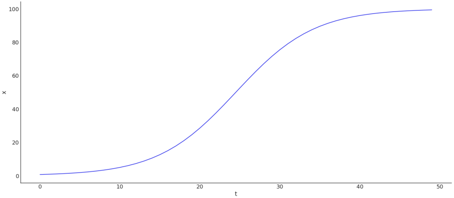

Figure: Growth Model

$$x = \frac{c}{1 + \exp(-k(t - m))}$$

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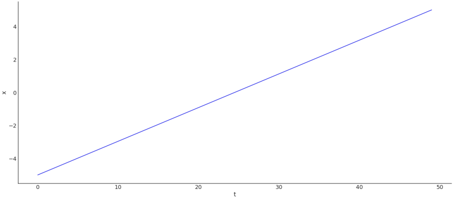

Figure: Linear Trend Model

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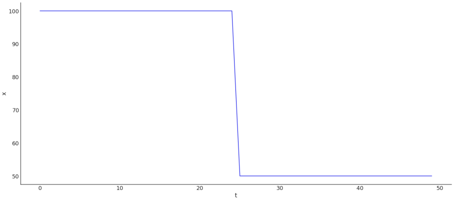

Figure: Changepoint Model

$$x = \begin{cases} c_1, & t < m \\ c_2, & t \geq m \end{cases}$$

Non-periodic Part (mean function)

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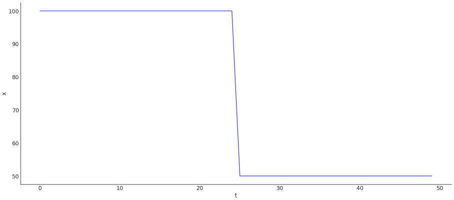
Extentions

Extensions are possible, e.g. time dependent saturation in the growth model. See in [1]



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Holidays

$$h(t) = \text{is-holiday}(t)$$

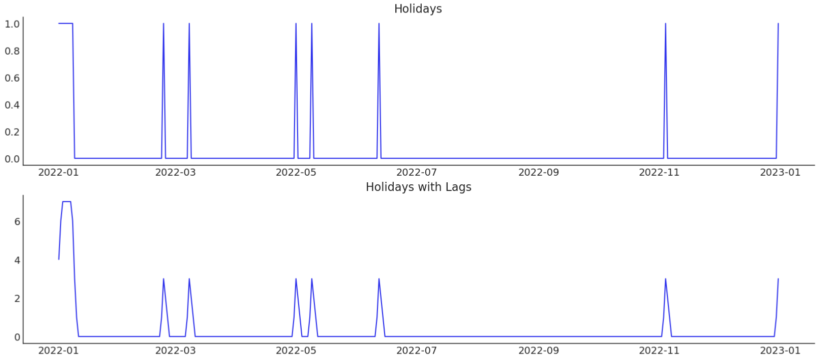

Figure: Holiday features

Periodic part (cov function)

Granularities are important here. Multiple Periodic kernels can be used.

- Yearly
- Quarterly
- Monthly
- Weekly

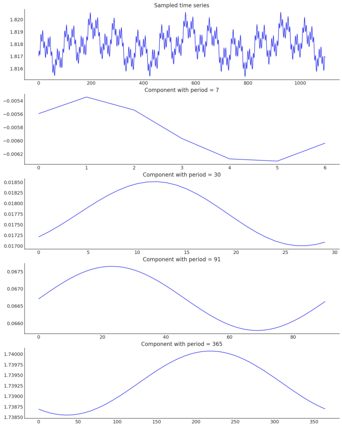

Figure: Seasonal Decomposition

Lengthscales for Periodic Part

Hyperparameters

Common sense driven lenthscale choice

- Week - couple of days make a change ($ls \approx 3$)
- Month - week makes sense for a dramatic change ($ls \approx 7$)
- Quarter - month makes sense for a dramatic change ($ls \approx 30$)
- Year - quarter makes sense for a dramatic change ($ls \approx 90$)

Lengthscales for Periodic Part

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- Quarter - month makes sense for a dramatic change ($ls \approx 30$)
- Year - quarter makes sense for a dramatic change ($ls \approx 90$)

In practice

Everybody is using Fourier features as a replacement for Periodic Kernel

Putting All Together

$$
\begin{aligned}
m(t) &= \underbrace{g(t)}_{non-periodic} + \underbrace{h(t)}_{holidays} \\
k(*,*) &= \mathrm{Periodic}(\mathrm{p}=365, \mathrm{l}=90) \\
&+ \mathrm{Periodic}(\mathrm{p}=90, \mathrm{l}=30) \\
&+ \mathrm{Periodic}(\mathrm{p}=30, \mathrm{l}=7) \\
&+ \mathrm{Periodic}(\mathrm{p}=7, \mathrm{l}=3)
\end{aligned}
$$

Putting All Together

$$m(t) = \underbrace{g(t)}_{non-periodic} + \underbrace{h(t)}_{holidays}$$

$$\begin{aligned} k(*, *) = & \ \alpha_{365} \, \mathrm{Periodic}(\mathrm{p} = 365, \mathrm{l} = 90) \\ & + \alpha_{90} \, \mathrm{Periodic}(\mathrm{p} = 90, \mathrm{l} = 30) \\ & + \alpha_{30} \, \mathrm{Periodic}(\mathrm{p} = 30, \mathrm{l} = 7) \\ & + \alpha_{7} \, \mathrm{Periodic}(\mathrm{p} = 7, \mathrm{l} = 3) \end{aligned}$$

Missing Parts

1. Weights for periodic components

Putting All Together

$$m(t) = \underbrace{g(t)}_{non-periodic} + \underbrace{h(t)}_{holidays}$$

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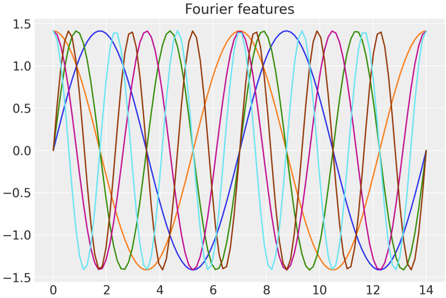
Missing Parts

1. Weights for periodic components
2. Trent violations from $g(t)$

More Efficient Periodic

Having at least one Periodic kernel in a large time series prevents from optimizations.

- Fourier features can be added as regressors
- This allows reasonable periodicity to be present in the model



Choosing Features

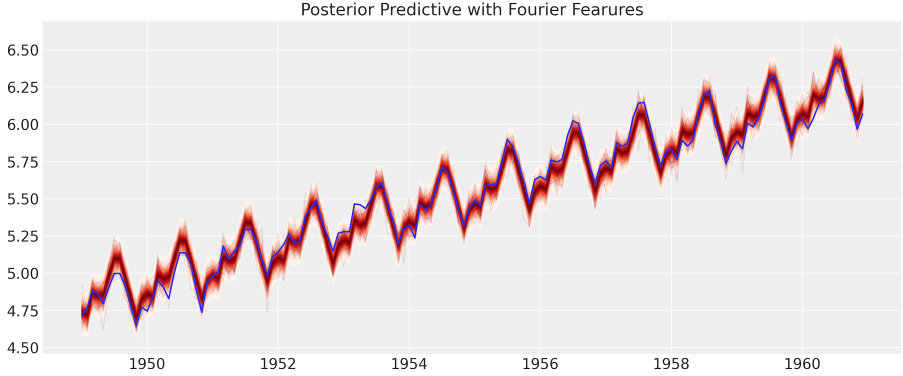
Every regular period has optimal number of fourier components (aka order)

- Weekly: 3
- Monthly: 10
- Yearly: 5

```
from collections import namedtuple
from enum import Enum

Season = namedtuple("Season", "period,order")

class Daily(Season, Enum):
    Week = 7.0, 3
    Year = 365.25, 5
    Month = 365.25 / 12, 10

class Monthly(Season, Enum):
    Year = 12, 5
    Month = 1, 10
```

Applications of Fourier Features

- E.g. Prophet uses Fourier Features (FF)
- It is VERY Fast, drop-in replacement for Periodic GP
- Proven to be useful for Bayesian models



Integrating into the model

```python
alpha = pm.Normal("alpha", 0, sigma, shape=Monthly.Year.order * 2)
features = fourier_series(months, season=Monthly.Year)
seasonality = features @ alpha

expectation = bias + trend + seasonality
```

References I

T. SJ and L. B.
Forecasting at scale.
2017.